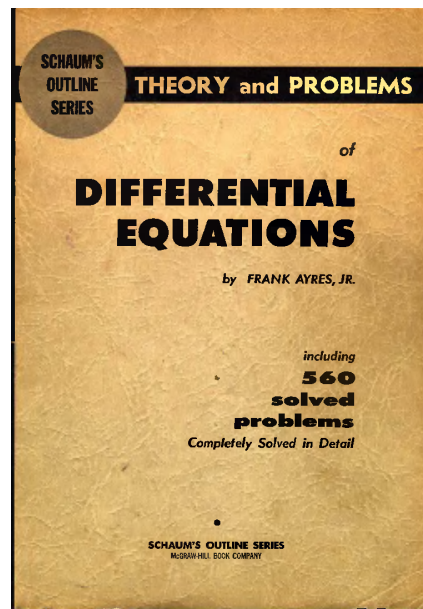


A Solution Manual For

**Schaums Outline. Theory and problems
of Differential Equations, 1st edition.
Frank Ayres. McGraw Hill 1952**



Nasser M. Abbasi

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1.1 problem 13

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Internal problem ID [5226]

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Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 13.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$xy' - 2y = 0$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{2y}{x}\end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= \frac{2}{x} dx \\ \int \frac{1}{y} dy &= \int \frac{2}{x} dx \\ \ln(y) &= 2 \ln(x) + c_1 \\ y &= e^{2 \ln(x) + c_1} \\ &= c_1 x^2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

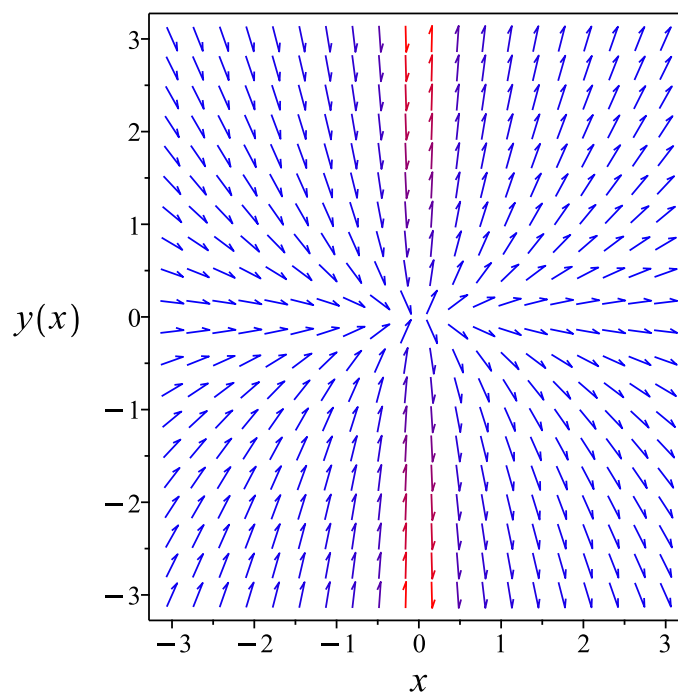


Figure 1: Slope field plot

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$

$$q(x) = 0$$

Hence the ode is

$$y' - \frac{2y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx} \mu y &= 0 \\ \frac{d}{dx} \left(\frac{y}{x^2} \right) &= 0\end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = c_1 x^2$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

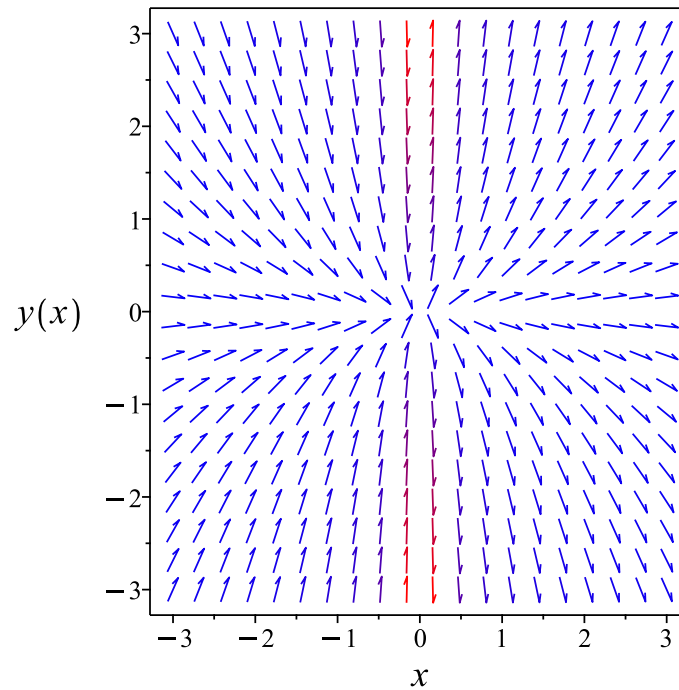


Figure 2: Slope field plot

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)x + u(x)) - 2u(x)x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{1}{x} dx \\ \int \frac{1}{u} du &= \int \frac{1}{x} dx \\ \ln(u) &= \ln(x) + c_2 \\ u &= e^{\ln(x)+c_2} \\ &= c_2 x\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= ux \\ &= c_2 x^2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 \tag{1}$$

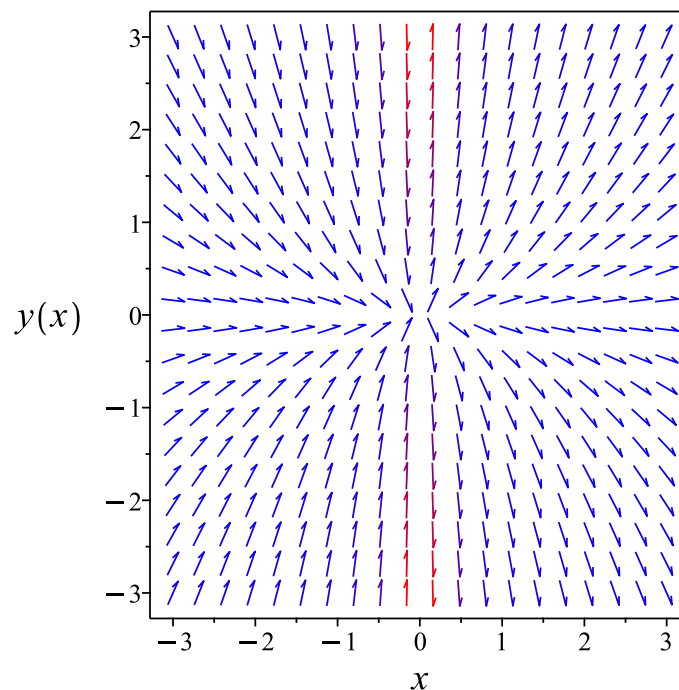


Figure 3: Slope field plot

Verification of solutions

$$y = c_2 x^2$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x^2\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = c_1$$

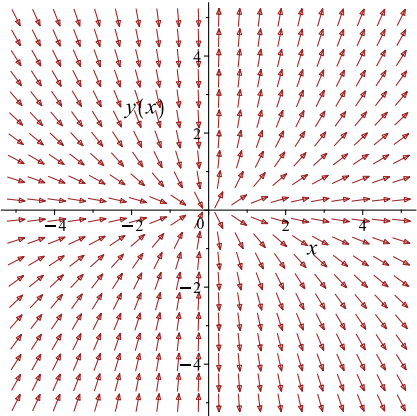
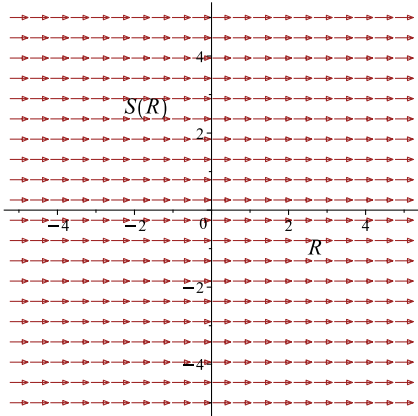
Which simplifies to

$$\frac{y}{x^2} = c_1$$

Which gives

$$y = c_1 x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

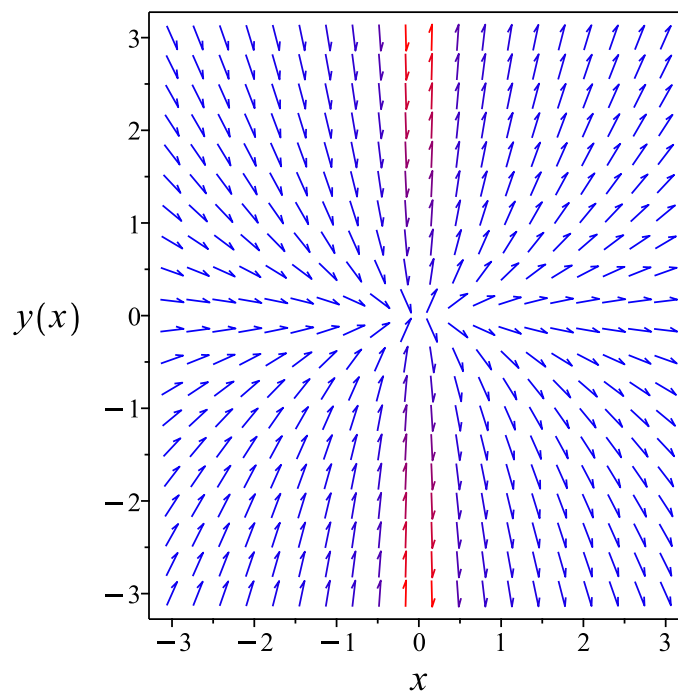


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{2y}\right) dy &= \left(\frac{1}{x}\right) dx \\ \left(-\frac{1}{x}\right) dx + \left(\frac{1}{2y}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x} \\ N(x, y) &= \frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{2y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x} dx \\ \phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{1}{2y} \right) dy \\ f(y) &= \frac{\ln(y)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{\ln(y)}{2}$$

The solution becomes

$$y = e^{2c_1} x^2$$

Summary

The solution(s) found are the following

$$y = e^{2c_1} x^2 \tag{1}$$

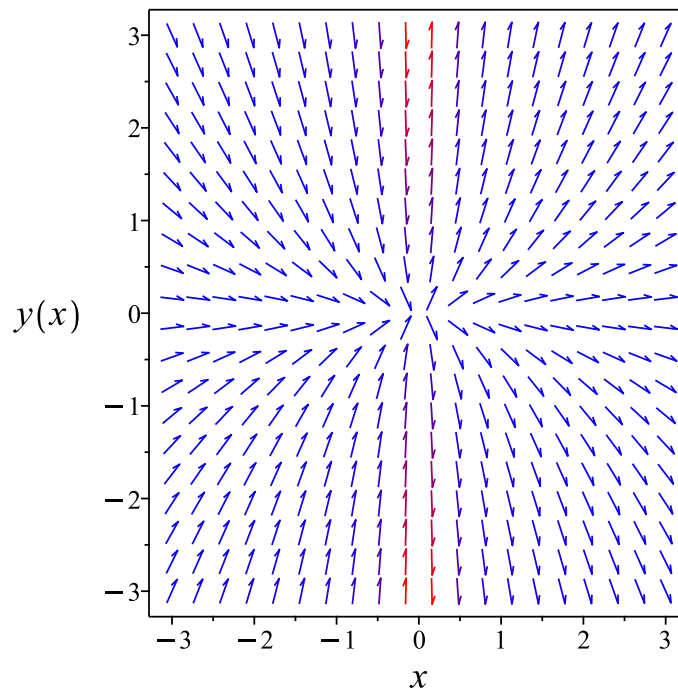


Figure 5: Slope field plot

Verification of solutions

$$y = e^{2c_1} x^2$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$$xy' - 2y = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = \frac{2}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int \frac{2}{x} dx + c_1$$

- Evaluate integral

$$\ln(y) = 2 \ln(x) + c_1$$

- Solve for y

$$y = x^2 e^{c_1}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)=2*y(x),y(x), singsol=all)
```

$$y(x) = c_1 x^2$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[x*y'[x]==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 x^2$$

$$y(x) \rightarrow 0$$

1.2 problem 14

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Internal problem ID [5227]

Internal file name [OUTPUT/4718_Friday_February_02_2024_05_10_31_AM_68847962/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 14.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "separable", "differential-Type", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$y'y = -x$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{x}{y}\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{y} dy = -x dx$$

$$\int \frac{1}{y} dy = \int -x dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{-x^2 + 2c_1}$$

$$y = -\sqrt{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{-x^2 + 2c_1} \tag{2}$$

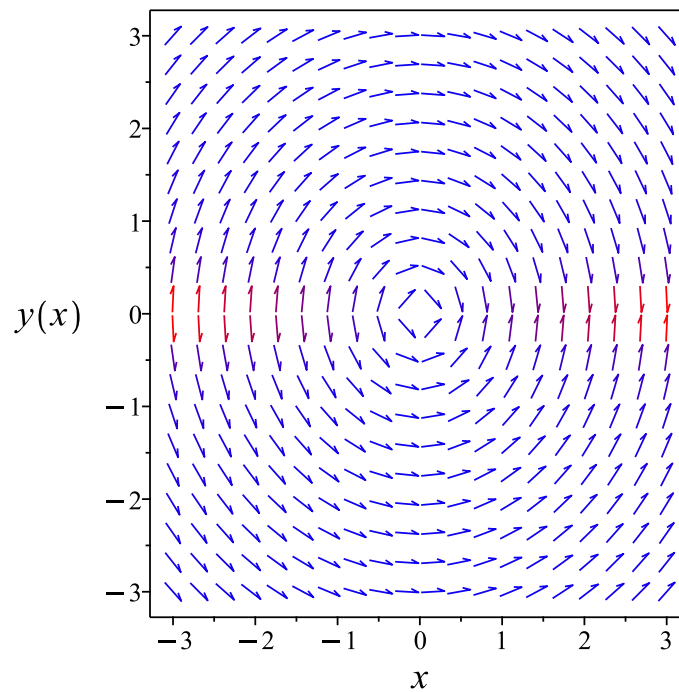


Figure 6: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1}$$

Verified OK.

1.2.2 Solving as homogeneous Type D2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$(u'(x)x + u(x))u(x)x = -x$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u^2 + 1}{ux}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2+1}{u}} du &= -\frac{1}{x} dx \\ \int \frac{1}{\frac{u^2+1}{u}} du &= \int -\frac{1}{x} dx \\ \frac{\ln(u^2 + 1)}{2} &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 + 1} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2 + 1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2 + 1} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2 + 1} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}\sqrt{\frac{y^2}{x^2} + 1} &= \frac{c_3 e^{c_2}}{x} \\ \sqrt{\frac{x^2 + y^2}{x^2}} &= \frac{c_3 e^{c_2}}{x}\end{aligned}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^2 + y^2}{x^2}} = \frac{c_3 e^{c_2}}{x} \quad (1)$$

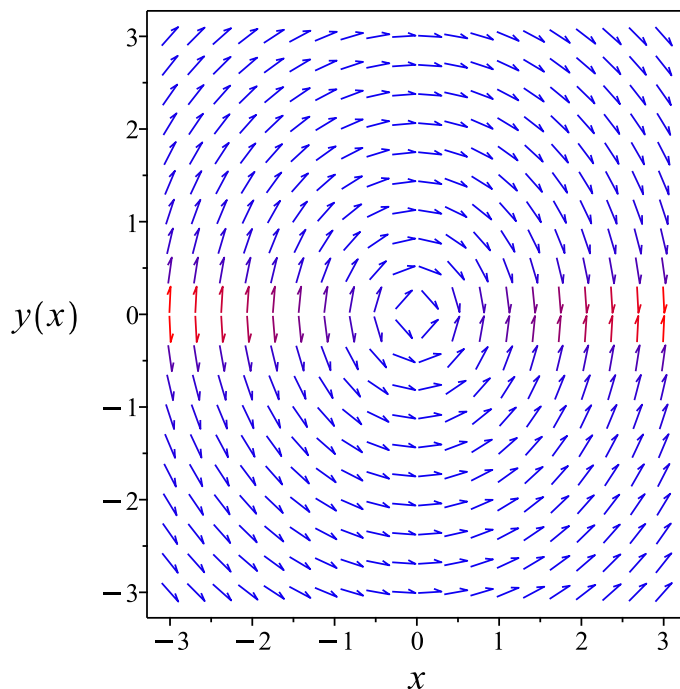


Figure 7: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^2 + y^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

1.2.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x}{y} \quad (1)$$

Which becomes

$$(y) dy = (-x) dx \quad (2)$$

But the RHS is complete differential because

$$(-x) dx = d\left(-\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y) dy = d\left(-\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$

$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} + c_1 \quad (1)$$

$$y = -\sqrt{-x^2 + 2c_1} + c_1 \quad (2)$$

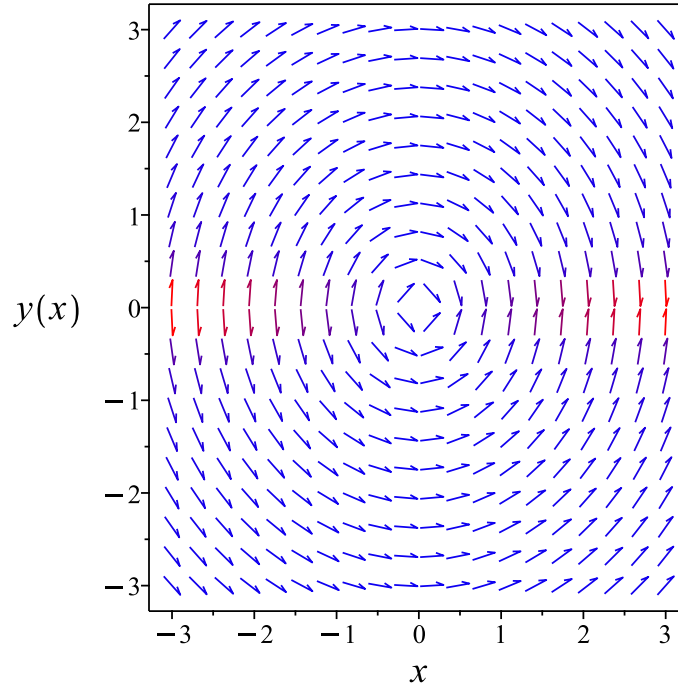


Figure 8: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x}{y}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= -\frac{1}{x} \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \quad (4)$$

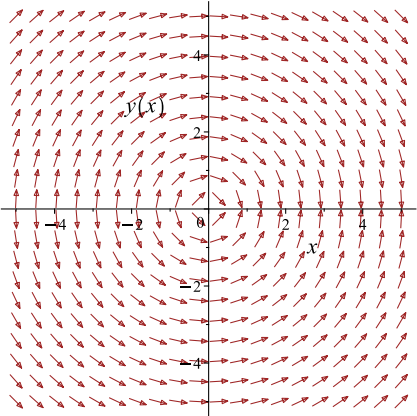
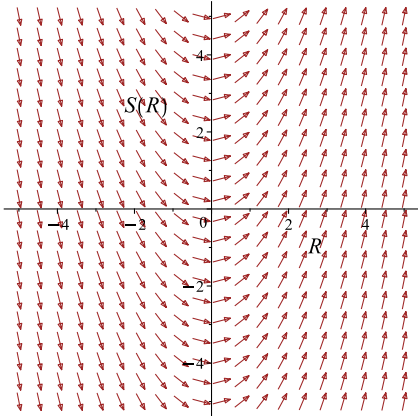
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1 \quad (1)$$

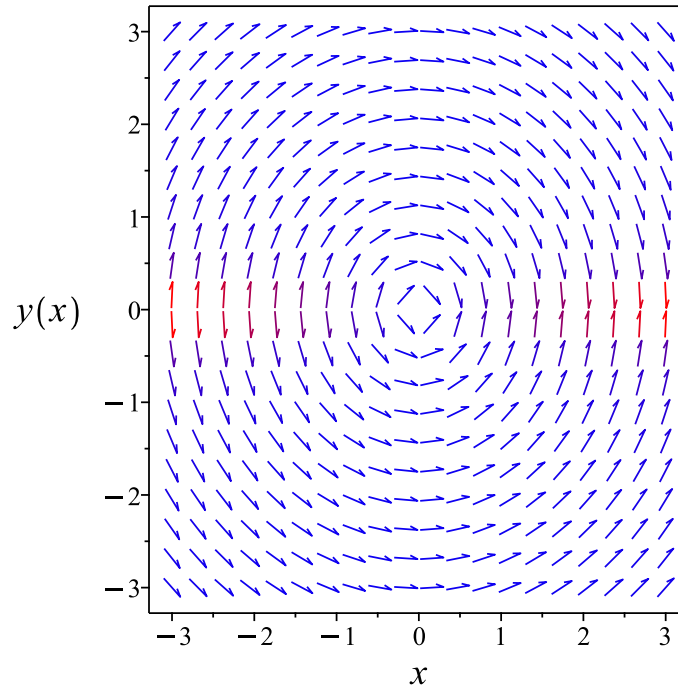


Figure 9: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y) dy &= (x) dx \\ (-x) dx + (-y) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= -y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int -x dx$$

$$\phi = -\frac{x^2}{2} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int (-y) dy$$

$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{y^2}{2} - \frac{x^2}{2} = c_1 \tag{1}$$

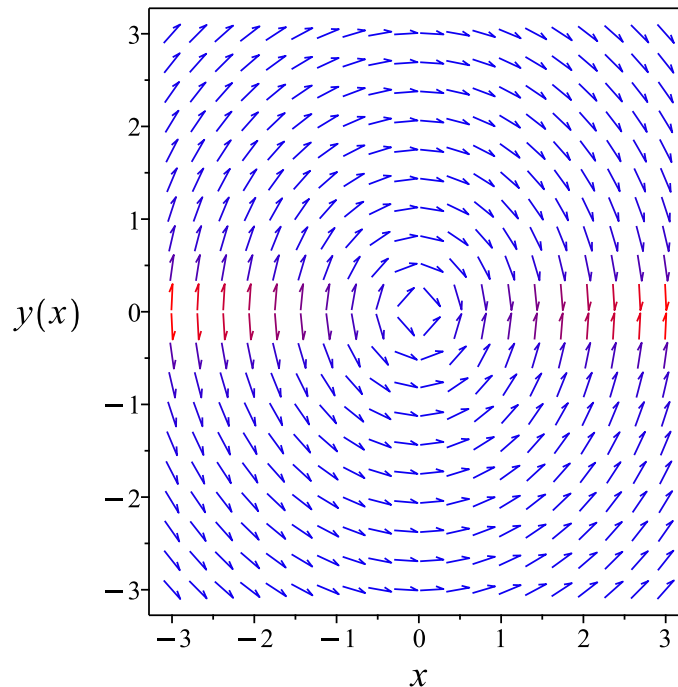


Figure 10: Slope field plot

Verification of solutions

$$-\frac{y^2}{2} - \frac{x^2}{2} = c_1$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$$y'y = -x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Integrate both sides with respect to x

$$\int y'y dx = \int -x dx + c_1$$

- Evaluate integral

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

- Solve for y

$$\{y = \sqrt{-x^2 + 2c_1}, y = -\sqrt{-x^2 + 2c_1}\}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2 + c_1}$$
$$y(x) = -\sqrt{-x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 39

```
DSolve[y[x]*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{-x^2 + 2c_1}$$

$$y(x) \rightarrow \sqrt{-x^2 + 2c_1}$$

1.3 problem 15

1.3.1 Solving as clairaut ode 34

Internal problem ID [5228]

Internal file name [OUTPUT/4719_Friday_February_02_2024_05_10_32_AM_47063498/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 15.

ODE order: 1.

ODE degree: 4.

The type(s) of ODE detected by this program : "**clairaut**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _Clairaut]
```

$$y - xy' - y'^4 = 0$$

1.3.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^4 - xp + y = 0$$

Solving for y from the above results in

$$y = p^4 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^4 + xp \\ &= p^4 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x . Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^4$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x \frac{dp}{dx} \right) + \left(g' \frac{dp}{dx} \right) \\ p &= p + (x + g') \frac{dp}{dx} \\ 0 &= (x + g') \frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p . The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^4 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^4$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= 4p^3 + x \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{(-2x)^{\frac{1}{3}}}{2}$$

$$p_2 = -\frac{(-2x)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}(-2x)^{\frac{1}{3}}}{4}$$

$$p_3 = -\frac{(-2x)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}(-2x)^{\frac{1}{3}}}{4}$$

Substituting the above back in (1) results in

$$y_1 = -\frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}}{8}$$

$$y_2 = \frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}(1 + i\sqrt{3})}{16}$$

$$y_3 = -\frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}(i\sqrt{3} - 1)}{16}$$

Summary

The solution(s) found are the following

$$y = c_1^4 + c_1 x \tag{1}$$

$$y = -\frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}}{8} \tag{2}$$

$$y = \frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}(1 + i\sqrt{3})}{16} \tag{3}$$

$$y = -\frac{3 \cdot 2^{\frac{1}{3}}(-x)^{\frac{4}{3}}(i\sqrt{3} - 1)}{16} \tag{4}$$

Verification of solutions

$$y = c_1^4 + c_1 x$$

Verified OK.

$$y = -\frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}}}{8}$$

Verified OK.

$$y = \frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}} (1 + i\sqrt{3})}{16}$$

Verified OK.

$$y = -\frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}} (i\sqrt{3} - 1)}{16}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 66

```
dsolve(y(x)=x*diff(y(x),x)+diff(y(x),x)^4,y(x), singsol=all)
```

$$y(x) = -\frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}}}{8}$$
$$y(x) = \frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}} (1 + i\sqrt{3})}{16}$$
$$y(x) = -\frac{3 \cdot 2^{\frac{1}{3}} (-x)^{\frac{4}{3}} (i\sqrt{3} - 1)}{16}$$
$$y(x) = c_1(c_1^3 + x)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 75

```
DSolve[y[x]==x*y'[x]+(y'[x])^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1(x + c_1^3)$$
$$y(x) \rightarrow -\frac{3}{4} \left(-\frac{1}{2}\right)^{2/3} x^{4/3}$$
$$y(x) \rightarrow -\frac{3x^{4/3}}{4 \cdot 2^{2/3}}$$
$$y(x) \rightarrow \frac{3\sqrt[3]{-1}x^{4/3}}{4 \cdot 2^{2/3}}$$

1.4 problem 16

1.4.1	Solving as homogeneousTypeD2 ode	39
1.4.2	Solving as first order ode lie symmetry lookup ode	41
1.4.3	Solving as bernoulli ode	45
1.4.4	Solving as exact ode	49

Internal problem ID [5229]

Internal file name [OUTPUT/4720_Friday_February_02_2024_05_10_33_AM_57195971/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 16.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**bernoulli**", "**homogeneousTypeD2**", "**exactWithIntegrationFactor**", "**first_order_ode_lie_symmetry_lookup**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$2y'x^3 - y(y^2 + 3x^2) = 0$$

1.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$2(u'(x)x + u(x))x^3 - u(x)x(u(x)^2x^2 + 3x^2) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(u^2 + 1)}{2x}\end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u(u^2 + 1)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u(u^2 + 1)} du &= \frac{1}{2x} dx \\ \int \frac{1}{u(u^2 + 1)} du &= \int \frac{1}{2x} dx \\ -\frac{\ln(u^2 + 1)}{2} + \ln(u) &= \frac{\ln(x)}{2} + c_2\end{aligned}$$

Raising both side to exponential gives

$$e^{-\frac{\ln(u^2+1)}{2}+\ln(u)} = e^{\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\frac{u}{\sqrt{u^2 + 1}} = c_3 \sqrt{x}$$

Therefore the solution y is

$$\begin{aligned}y &= xu \\ &= x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x} - 1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x} - 1} \tag{1}$$

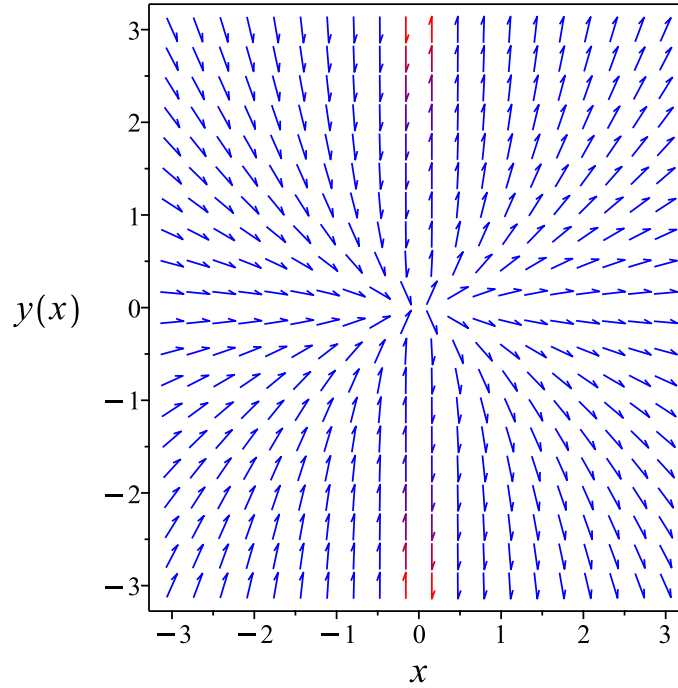


Figure 11: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x - 1}}$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(3x^2 + y^2)}{2x^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{y^3}{x^3}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^3}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^3}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(3x^2 + y^2)}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3x^2}{2y^2} \\ S_y &= \frac{x^3}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \quad (4)$$

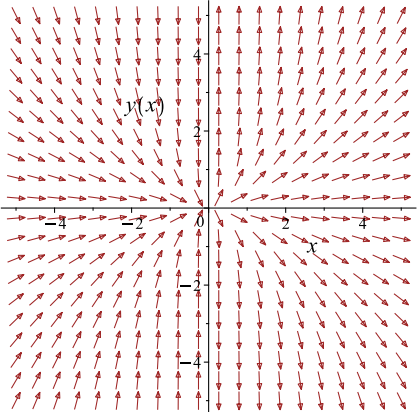
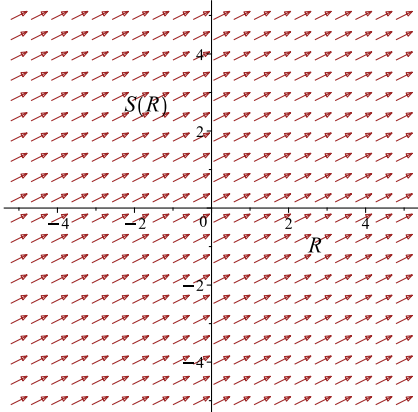
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

Which simplifies to

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(3x^2 + y^2)}{2x^3}$ 	$R = x$ $S = -\frac{x^3}{2y^2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary

The solution(s) found are the following

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1 \quad (1)$$

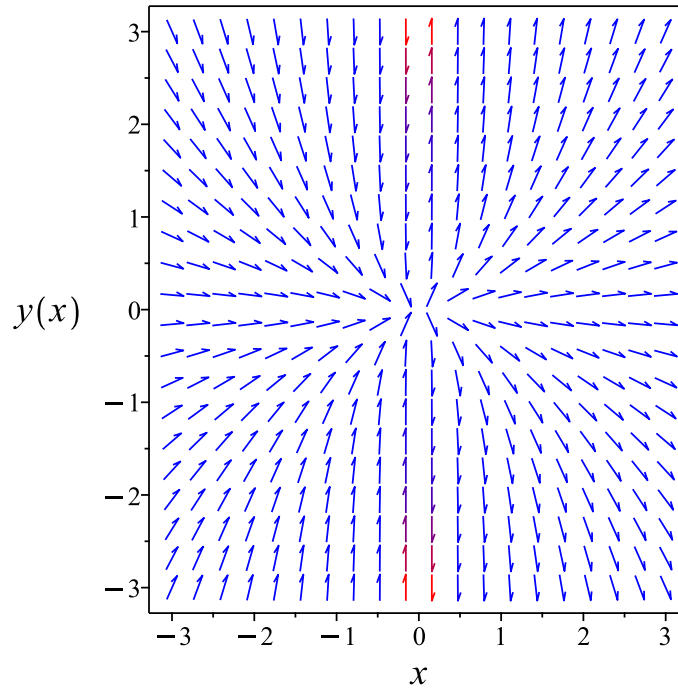


Figure 12: Slope field plot

Verification of solutions

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

Verified OK.

1.4.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(3x^2 + y^2)}{2x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x}y + \frac{1}{2x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= \frac{3}{2x} \\f_1(x) &= \frac{1}{2x^3} \\n &= 3\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{3}{2y^2x} + \frac{1}{2x^3} \quad (4)$$

Let

$$\begin{aligned}w &= y^{1-n} \\&= \frac{1}{y^2}\end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}-\frac{w'(x)}{2} &= \frac{3w(x)}{2x} + \frac{1}{2x^3} \\w' &= -\frac{3w}{x} - \frac{1}{x^3}\end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{3}{x} \\q(x) &= -\frac{1}{x^3}\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = -\frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x} dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu) \left(-\frac{1}{x^3}\right) \\ \frac{d}{dx}(x^3 w) &= (x^3) \left(-\frac{1}{x^3}\right) \\ d(x^3 w) &= -1 dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int -1 dx \\ x^3 w &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = -\frac{1}{x^2} + \frac{c_1}{x^3}$$

which simplifies to

$$w(x) = \frac{-x + c_1}{x^3}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{-x + c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{(-x + c_1) x x}}{-x + c_1} \\ y(x) &= \frac{\sqrt{(-x + c_1) x x}}{-c_1 + x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(-x + c_1) x x}}{-x + c_1} \quad (1)$$

$$y = \frac{\sqrt{(-x + c_1) x x}}{-c_1 + x} \quad (2)$$

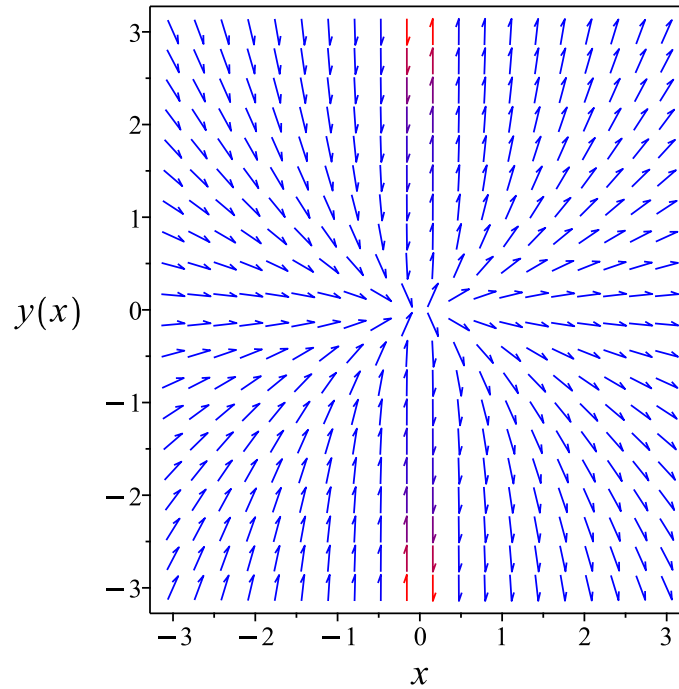


Figure 13: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(-x + c_1) x x}}{-x + c_1}$$

Verified OK.

$$y = \frac{\sqrt{(-x + c_1) x x}}{-c_1 + x}$$

Verified OK.

1.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (2x^3) dy &= (y(3x^2 + y^2)) dx \\ (-y(3x^2 + y^2)) dx &+ (2x^3) dy = 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -y(3x^2 + y^2) \\ N(x, y) &= 2x^3 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(3x^2 + y^2)) \\ &= -3x^2 - 3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x^3) \\ &= 6x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{2x^3} ((-3x^2 - 3y^2) - (6x^2)) \\ &= \frac{-9x^2 - 3y^2}{2x^3}\end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= -\frac{1}{3yx^2 + y^3} ((6x^2) - (-3x^2 - 3y^2)) \\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned}\mu &= e^{\int B \, dy} \\ &= e^{\int -\frac{3}{y} \, dy}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(y)} \\ &= \frac{1}{y^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^3}(-y(3x^2 + y^2)) \\ &= \frac{-3x^2 - y^2}{y^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^3}(2x^3) \\ &= \frac{2x^3}{y^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-3x^2 - y^2}{y^2} \right) + \left(\frac{2x^3}{y^3} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-3x^2 - y^2}{y^2} dx \\ \phi &= -\frac{x(x^2 + y^2)}{y^2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{2x}{y} + \frac{2x(x^2 + y^2)}{y^3} + f'(y) \\ &= \frac{2x^3}{y^3} + f'(y)\end{aligned}\tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{2x^3}{y^3}$. Therefore equation (4) becomes

$$\frac{2x^3}{y^3} = \frac{2x^3}{y^3} + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x^2 + y^2)}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x^2 + y^2)}{y^2}$$

Summary

The solution(s) found are the following

$$-\frac{x(x^2 + y^2)}{y^2} = c_1\tag{1}$$

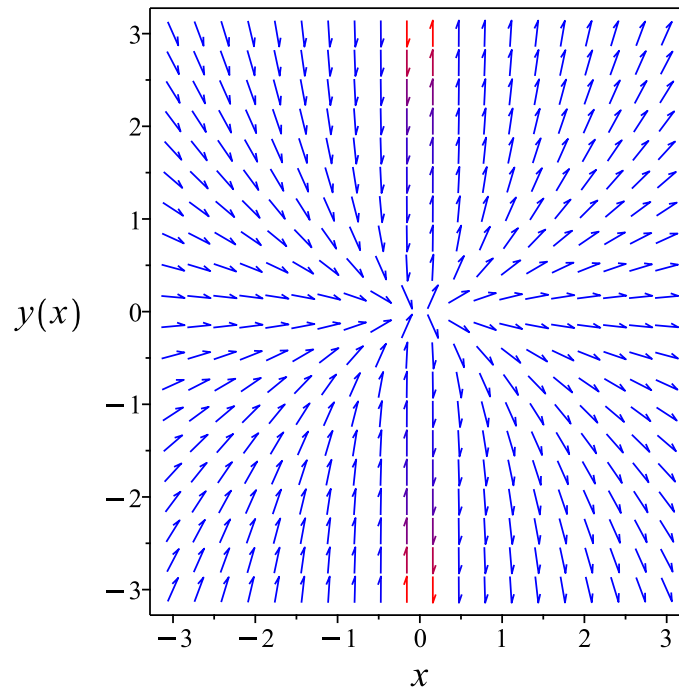


Figure 14: Slope field plot

Verification of solutions

$$-\frac{x(x^2 + y^2)}{y^2} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(2*x^3*diff(y(x),x)=y(x)*(y(x)^2+3*x^2),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(-x + c_1) x x}}{-x + c_1}$$
$$y(x) = \frac{\sqrt{(-x + c_1) x x}}{x - c_1}$$

✓ Solution by Mathematica

Time used: 0.179 (sec). Leaf size: 47

```
DSolve[2*x^3*y'[x]==y[x]*(y[x]^2+3*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x^{3/2}}{\sqrt{-x + c_1}}$$
$$y(x) \rightarrow \frac{x^{3/2}}{\sqrt{-x + c_1}}$$
$$y(x) \rightarrow 0$$

1.5 problem 17

1.5.1	Solving as second order linear constant coeff ode	55
1.5.2	Solving as linear second order ode solved by an integrating factor ode	57
1.5.3	Solving using Kovacic algorithm	58
1.5.4	Maple step by step solution	62

Internal problem ID [5230]

Internal file name [OUTPUT/4721_Friday_February_02_2024_05_10_37_AM_91798504/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' - 2y' + y = 0$$

1.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

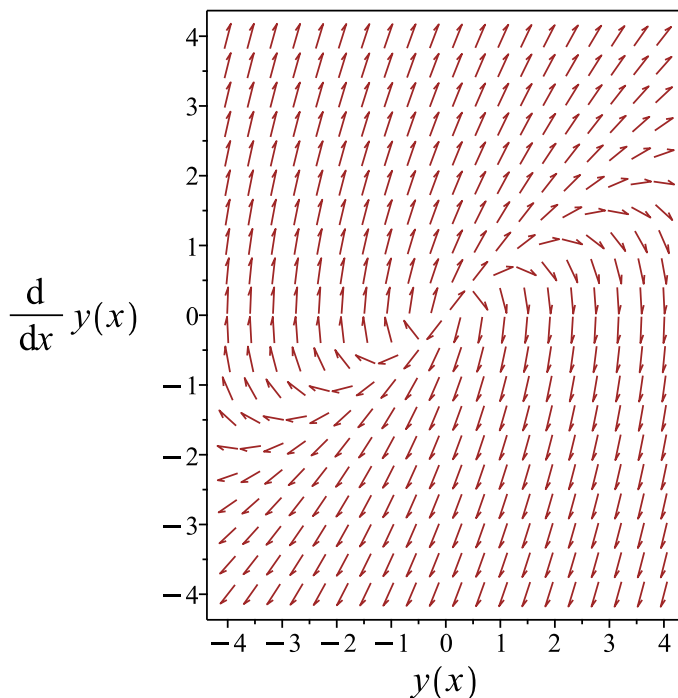


Figure 15: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 x e^x$$

Verified OK.

1.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x)^2 + p'(x))y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p dx} \\ &= e^{\int -2 dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ (e^{-x}y)'' &= 0 \end{aligned}$$

Integrating once gives

$$(e^{-x}y)' = c_1$$

Integrating again gives

$$(e^{-x}y) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{-x}}$$

Or

$$y = c_1x e^x + c_2e^x$$

Summary

The solution(s) found are the following

$$y = c_1x e^x + c_2e^x \tag{1}$$

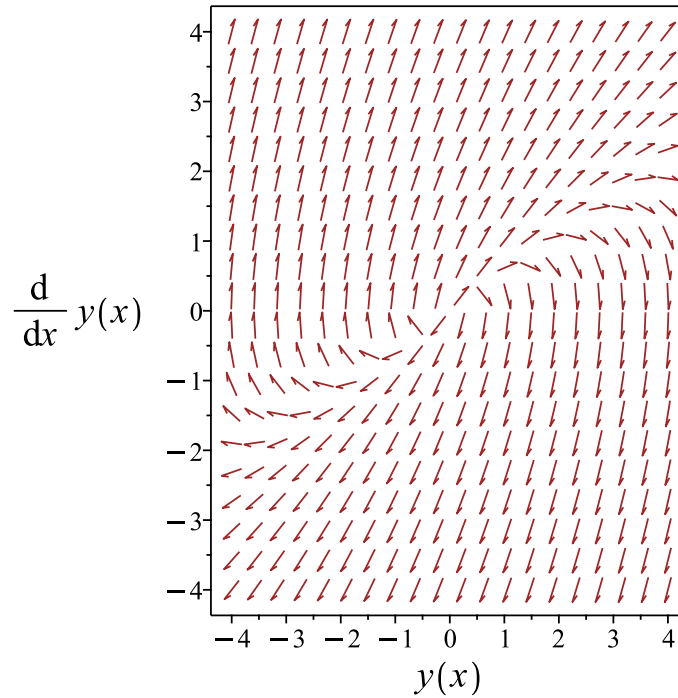


Figure 16: Slope field plot

Verification of solutions

$$y = c_1 x e^x + c_2 e^x$$

Verified OK.

1.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} dx} \\ &= z_1 e^x \\ &= z_1(e^x) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2} dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(x))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x e^x \tag{1}$$

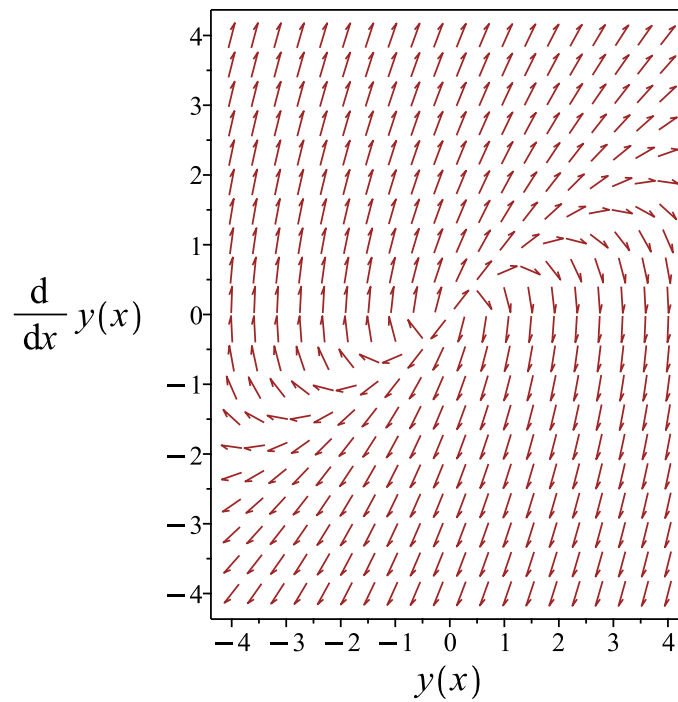


Figure 17: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 x e^x$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$$y'' - 2y' + y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 - 2r + 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)^2 = 0$$

- Root of the characteristic polynomial

$$r = 1$$

- 1st solution of the ODE

$$y_1(x) = e^x$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = x e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^x + c_2 x e^x$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x(c_2x + c_1)$$

1.6 problem 18

1.6.1	Solving as second order change of variable on y method 2 ode .	64
1.6.2	Solving as second order ode non constant coeff transformation on B ode	67
1.6.3	Solving using Kovacic algorithm	69
1.6.4	Maple step by step solution	75

Internal problem ID [5231]

Internal file name [OUTPUT/4722_Friday_February_02_2024_05_10_38_AM_86621537/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$(x - 1)y'' - xy' + y = 0$$

1.6.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x - 1)y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{x}{x-1}$$
$$q(x) = \frac{1}{x-1}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x-1} + \frac{1}{x-1} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right)v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right)v'(x) &= 0 \end{aligned} \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right)u(x) = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 2x + 2)}{x(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{x^2 - 2x + 2}{x(x-1)} dx \\ \int \frac{1}{u} du &= \int \frac{x^2 - 2x + 2}{x(x-1)} dx \\ \ln(u) &= x + \ln(x-1) - 2 \ln(x) + c_1 \\ u &= e^{x+\ln(x-1)-2\ln(x)+c_1} \\ &= c_1 e^{x+\ln(x-1)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= \frac{e^x c_1}{x} + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x) x^n \\ &= \left(\frac{e^x c_1}{x} + c_2 \right) x \\ &= c_1 e^x + c_2 x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(\frac{e^x c_1}{x} + c_2 \right) x \tag{1}$$

Verification of solutions

$$y = \left(\frac{e^x c_1}{x} + c_2 \right) x$$

Verified OK.

1.6.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}y' &= B'v + v'B \\y'' &= B''v + B'v' + v''B + v'B' \\&= v''B + 2v' + B' + B''v\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0\end{aligned}\tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= x - 1 \\B &= -x \\C &= 1 \\F &= 0\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (x - 1)(0) + (-x)(-1) + (1)(-x) \\&= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x(x-1)v'' + (x^2 - 2x + 2)v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-x^2 + x)u'(x) + (x^2 - 2x + 2)u(x) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 - 2x + 2)}{x(x-1)}\end{aligned}$$

Where $f(x) = \frac{x^2 - 2x + 2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{x^2 - 2x + 2}{x(x-1)} dx \\ \int \frac{1}{u} du &= \int \frac{x^2 - 2x + 2}{x(x-1)} dx \\ \ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\ u &= e^{x + \ln(x-1) - 2\ln(x) + c_1} \\ &= c_1 e^{x + \ln(x-1) - 2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 \left(\frac{e^x}{x} - \frac{e^x}{x^2} \right)\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{(x-1)c_1 e^x}{x^2} dx \\ &= \frac{e^x c_1}{x} + c_2\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y(x) &= Bv \\ &= (-x) \left(\frac{e^x c_1}{x} + c_2 \right) \\ &= -c_1 e^x - c_2 x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -c_1 e^x - c_2 x \quad (1)$$

Verification of solutions

$$y = -c_1 e^x - c_2 x$$

Verified OK.

1.6.3 Solving using Kovacic algorithm

Writing the ode as

$$(x - 1)y'' - xy' + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x - 1 \\ B &= -x \\ C &= 1\end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x - 1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for \sqrt{r} at ∞ . Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^v a_i x^i \\ &= \sum_{i=0}^0 a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of \sqrt{r} at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_{\infty} &= \sum_{i=0}^0 a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b , where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $([\sqrt{r}]_{\infty})^2$ where $[\sqrt{r}]_{\infty}$ was found above in Eq (10). Hence

$$([\sqrt{r}]_{\infty})^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r . How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t . Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2 . Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2} \left(\frac{b}{a} - v \right) = \frac{1}{2} \left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0 \right) = -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} \left(-\frac{b}{a} - v \right) = \frac{1}{2} \left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0 \right) = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2} \right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
 \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
 &= -\frac{1}{2(x-1)} + \left(\frac{1}{2} \right) \\
 &= -\frac{1}{2(x-1)} + \frac{1}{2} \\
 &= \frac{-2+x}{2x-2}
 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
 (0) + 2 \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right) (0) + \left(\left(\frac{1}{2(x-1)^2} \right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2} \right) \right) &= 0 \\
 0 &= 0
 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
 z_1(x) &= p e^{\int \omega dx} \\
 &= e^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2} \right) dx} \\
 &= \frac{e^{\frac{x}{2}}}{\sqrt{x-1}}
 \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
 y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
 &= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} dx} \\
 &= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
 &= z_1 (\sqrt{x-1} e^{\frac{x}{2}})
 \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2} dx \\ &= y_1(-x e^{-x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(-x e^{-x})) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 x$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$$(x - 1) y'' - x y' + y = 0$$

- Highest derivative means the order of the ODE is 2
 y''
- Isolate 2nd derivative

$$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$$

- Check to see if $x_0 = 1$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}]$$

- $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

$$((x-1) \cdot P_2(x)) \Big|_{x=1} = -1$$

- $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

$$((x-1)^2 \cdot P_3(x)) \Big|_{x=1} = 0$$

- $x = 1$ is a regular singular point

Check to see if $x_0 = 1$ is a regular singular point

$$x_0 = 1$$

- Multiply by denominators

$$(x-1)y'' - xy' + y = 0$$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

$$u \left(\frac{d^2}{du^2} y(u) \right) + (-u-1) \left(\frac{d}{du} y(u) \right) + y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u) \right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) u^{k+r}$$

- Convert $u \cdot \left(\frac{d^2}{du^2} y(u) \right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-1}$$

- Shift index using $k \rightarrow k+1$

$$u \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) u^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)) u^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation
 $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
 $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation
 $(k+r-1)(a_{k+1}(k+1+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1} \right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1} \right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3} \right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3} \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k \right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2} \right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3} \right]$$

Maple trace **Kovacic algorithm successful**

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Reducible group (found another exponential solution)
<- Kovacics algorithm successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 17

```
DSolve[(x-1)*y'[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x - c_2 x$$

1.7 problem 19

1.7.1	Solving as second order linear constant coeff ode	79
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Internal problem ID [5232]

Internal file name [OUTPUT/4723_Friday_February_02_2024_05_10_39_AM_67853078/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 19.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' - y = 0$$

1.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} \tag{1}$$

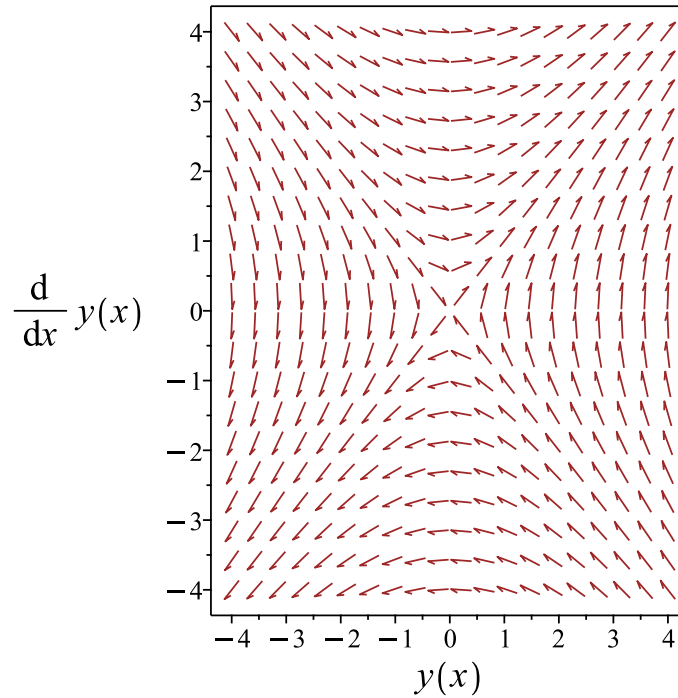


Figure 18: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x}$$

Verified OK.

1.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y' y'' - y' y = 0$$

Integrating the above w.r.t x gives

$$\int (y' y'' - y' y) dx = 0$$

$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y . Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln \left(y + \sqrt{y^2 + 2c_1} \right) = c_2 + x$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{c_2+x}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$-\ln \left(y + \sqrt{y^2 + 2c_1} \right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = c_5 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{2x} c_3^2 - 2c_1) e^{-x}}{2c_3} \quad (1)$$

$$y = -\frac{(2c_1 c_5^2 e^{2x} - 1) e^{-x}}{2c_5} \quad (2)$$

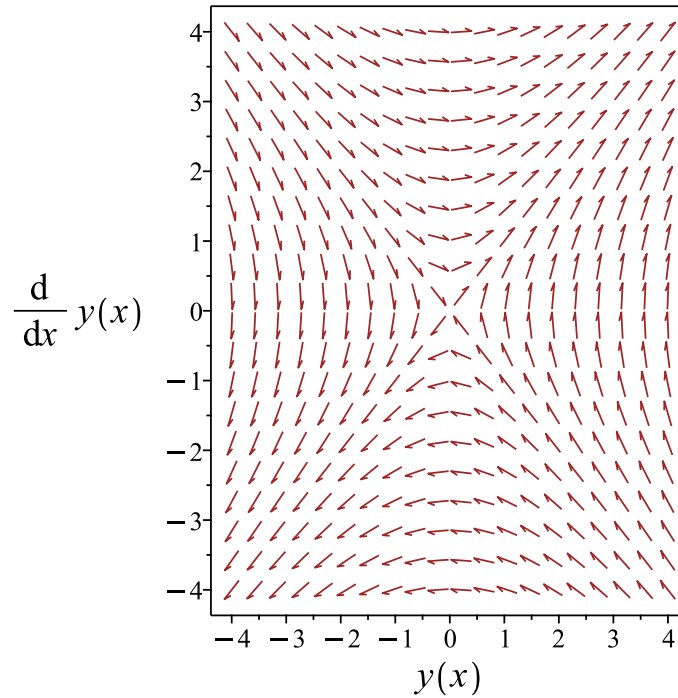


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)e^{-x}}{2c_5}$$

Verified OK.

1.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 0 \tag{3}$$

$$C = -1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} \tag{1}$$

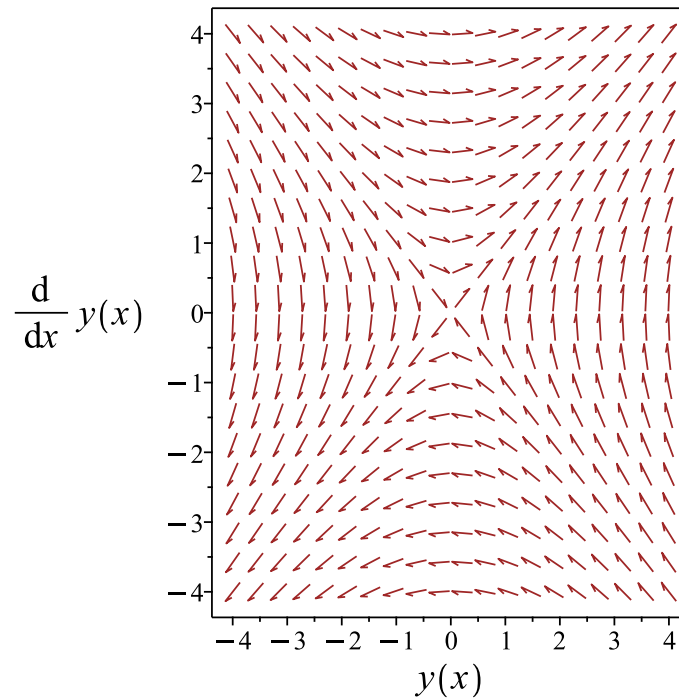


Figure 20: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$$y'' - y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

- $r = (-1, 1)$
 - 1st solution of the ODE
 $y_1(x) = e^{-x}$
 - 2nd solution of the ODE
 $y_2(x) = e^x$
 - General solution of the ODE
 $y = c_1 y_1(x) + c_2 y_2(x)$
 - Substitute in solutions
 $y = c_1 e^{-x} + c_2 e^x$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x} + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x + c_2 e^{-x}$$

1.8 problem 20

1.8.1	Solving as second order linear constant coeff ode	89
1.8.2	Solving using Kovacic algorithm	92
1.8.3	Maple step by step solution	97

Internal problem ID [5233]

Internal file name [OUTPUT/4724_Friday_February_02_2024_05_10_40_AM_27468358/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - y = 4 - x$$

1.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 4 - x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2x - A_1 = 4 - x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -4 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^x + c_2e^{-x}) + (-4 + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{-x} - 4 + x \tag{1}$$

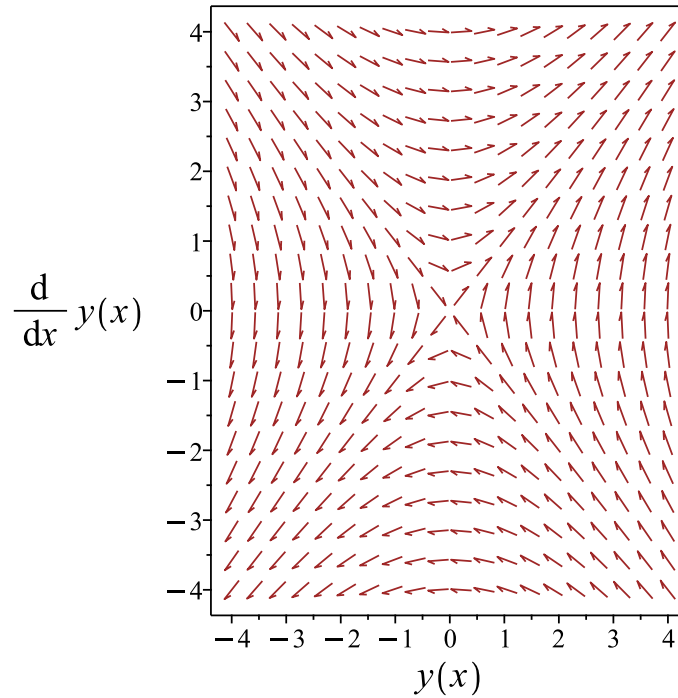


Figure 21: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - 4 + x$$

Verified OK.

1.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 0 \tag{3}$$

$$C = -1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 15: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{2}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2x - A_1 = 4 - x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -4 + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + (-4 + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - 4 + x \tag{1}$$

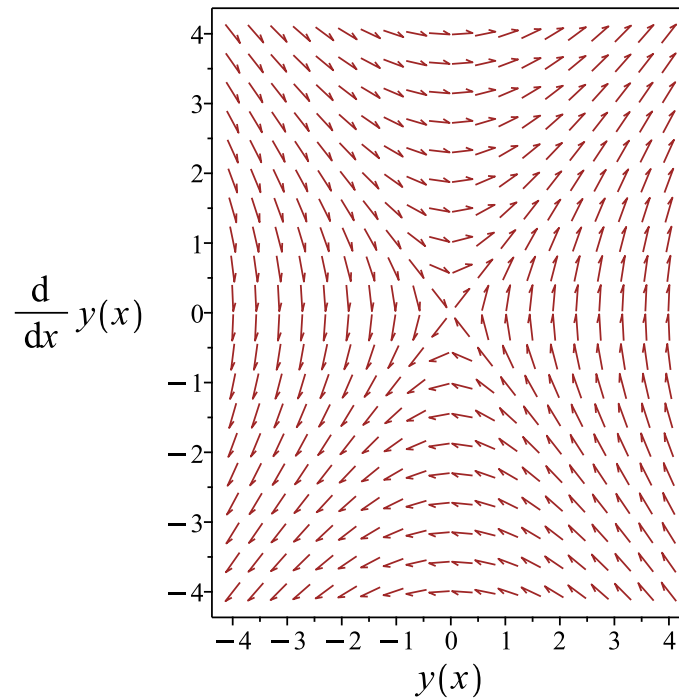


Figure 22: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - 4 + x$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$$y'' - y = 4 - x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = 4 - x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{e^{-x} (\int e^x (-4+x) dx)}{2} - \frac{e^x (\int e^{-x} (-4+x) dx)}{2}$$

- Compute integrals

$$y_p(x) = -4 + x$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x - 4 + x$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-y(x)=4-x,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 + x - 4$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[y''[x]-y[x]==4-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x + c_1 e^x + c_2 e^{-x} - 4$$

1.9 problem 21

1.9.1	Solving as second order linear constant coeff ode	100
1.9.2	Solving using Kovacic algorithm	102
1.9.3	Maple step by step solution	106

Internal problem ID [5234]

Internal file name [OUTPUT/4725_Friday_February_02_2024_05_10_40_AM_9370276/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 21.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' - 3y' + 2y = 0$$

1.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^x$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x \tag{1}$$

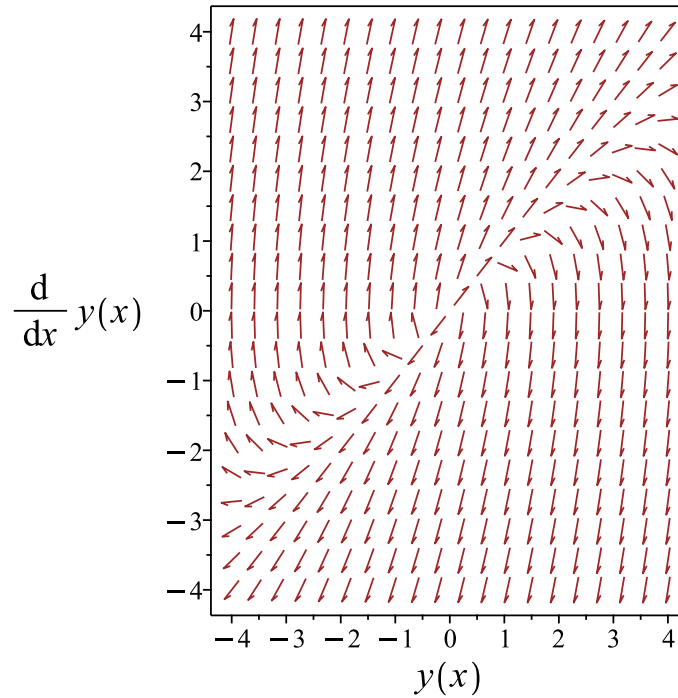


Figure 23: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^x$$

Verified OK.

1.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(e^{\frac{3x}{2}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2} dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(e^x))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} \tag{1}$$

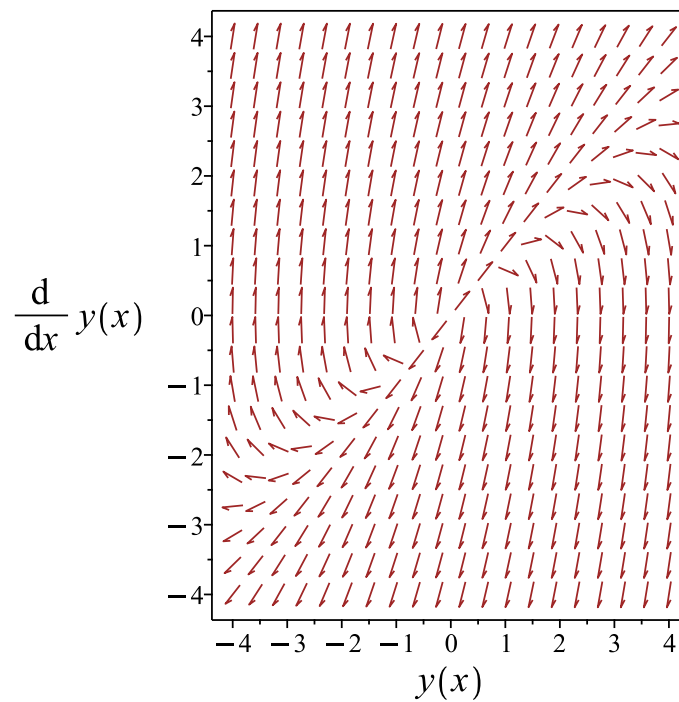


Figure 24: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x}$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$$y'' - 3y' + 2y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 - 3r + 2 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 2) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 2)$$

- 1st solution of the ODE

$$y_1(x) = e^x$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x)$$

- Substitute in solutions

$$y = c_1e^x + c_2e^{2x}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x (c_2 e^x + c_1)$$

1.10 problem 22

1.10.1 Solving as second order linear constant coeff ode	108
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Internal problem ID [5235]

Internal file name [OUTPUT/4726_Friday_February_02_2024_05_10_40_AM_17761984/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 2. Solutions of differential equations. Supplementary problems. Page 11

Problem number: 22.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 3y' + 2y = 2e^x(1 - x)$$

1.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = -2(x - 1)e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x} \end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^x$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2(x - 1) e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x e^x, x^2 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x + 2A_2 e^x - 2A_2 x e^x = -2(x - 1) e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2 e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^x) + (x^2 e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + x^2 e^x \quad (1)$$

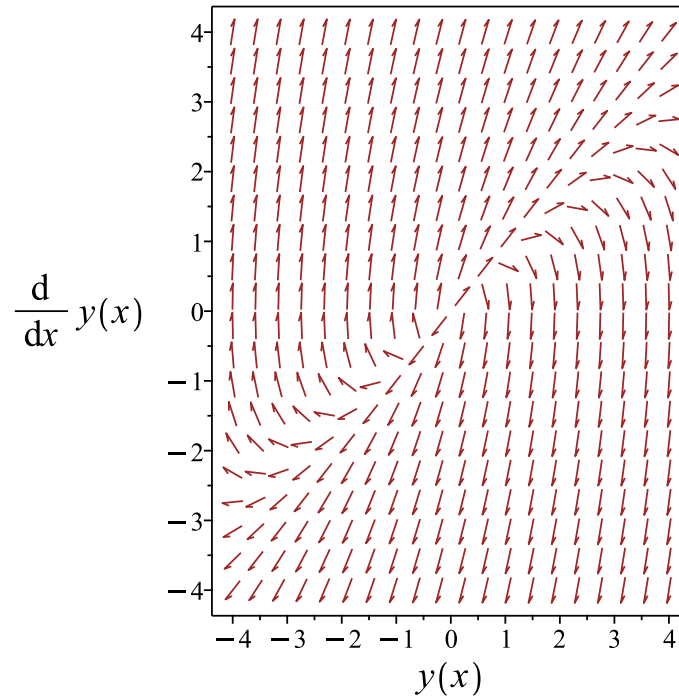


Figure 25: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^x + x^2 e^x$$

Verified OK.

1.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = -3 \quad (3)$$

$$C = 2$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
 y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
 &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} dx} \\
 &= z_1 e^{\frac{3x}{2}} \\
 &= z_1 \left(e^{\frac{3x}{2}} \right)
 \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2} dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2(e^x(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2(x - 1)e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x e^x, x^2 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^x + 2A_2 e^x - 2A_2 x e^x = -2(x - 1) e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2 e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{2x}) + (x^2 e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} + x^2 e^x \quad (1)$$

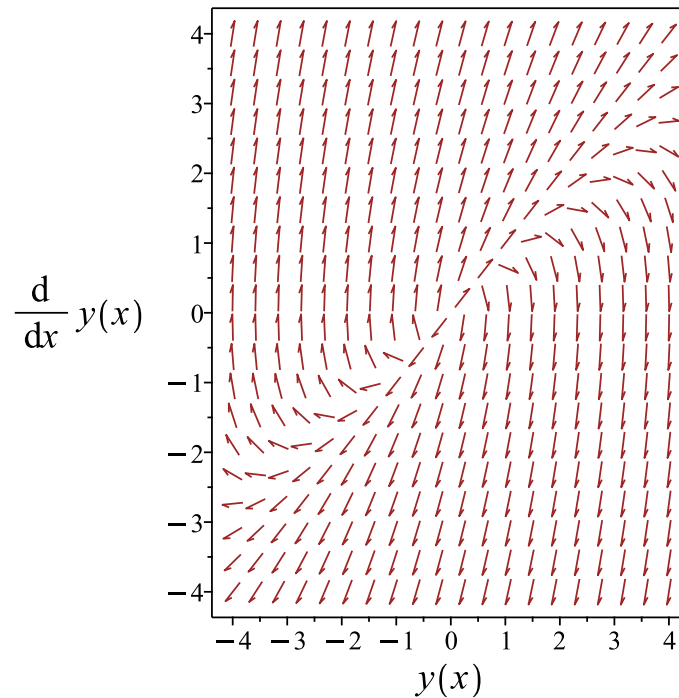


Figure 26: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x} + x^2 e^x$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$$y'' - 3y' + 2y = -2(x - 1)e^x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 3r + 2 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 2) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 2)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^x$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{2x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = -2(x-1)e^x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{3x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = 2e^x \left(\int (x-1) dx \right) - 2e^{2x} \left(\int (x-1)e^{-x} dx \right)$$

- Compute integrals

$$y_p(x) = x^2 e^x$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{2x} + x^2 e^x$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=2*exp(x)*(1-x),y(x), singsol=all)
```

$$y(x) = (e^x c_1 + x^2 + c_2) e^x$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 21

```
DSolve[y''[x]-3*y'[x]+2*y[x]==2*Exp[x]*(1-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x (x^2 + c_2 e^x + c_1)$$

2 Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

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2.1 problem 24

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Internal problem ID [5236]

Internal file name [OUTPUT/4727_Friday_February_02_2024_05_10_41_AM_53066185/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 24.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$xy' + 4y = 0$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{4y}{x}\end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= -\frac{4}{x} dx \\ \int \frac{1}{y} dy &= \int -\frac{4}{x} dx \\ \ln(y) &= -4 \ln(x) + c_1 \\ y &= e^{-4 \ln(x) + c_1} \\ &= \frac{c_1}{x^4}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} \tag{1}$$

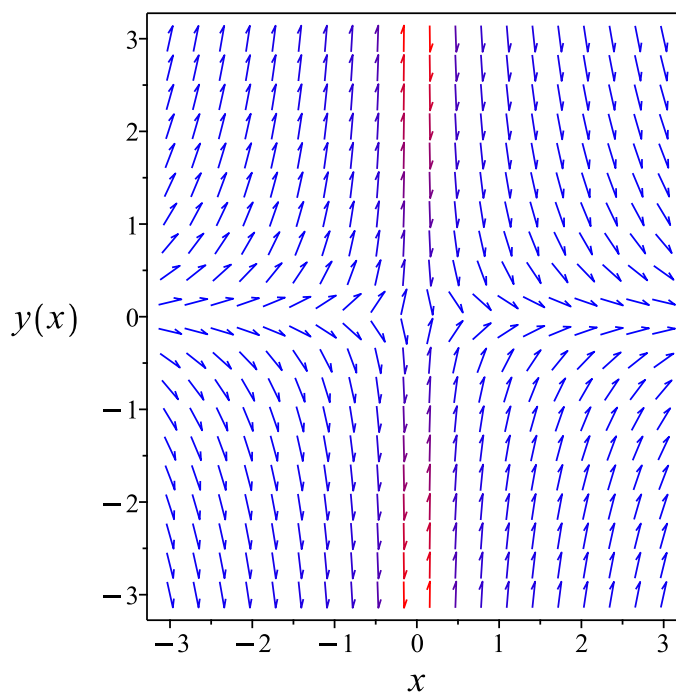


Figure 27: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^4}$$

Verified OK.

2.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{4}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' + \frac{4y}{x} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{x} dx}$$
$$= x^4$$

The ode becomes

$$\frac{d}{dx} \mu y = 0$$
$$\frac{d}{dx} (y x^4) = 0$$

Integrating gives

$$y x^4 = c_1$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$y = \frac{c_1}{x^4}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} \tag{1}$$

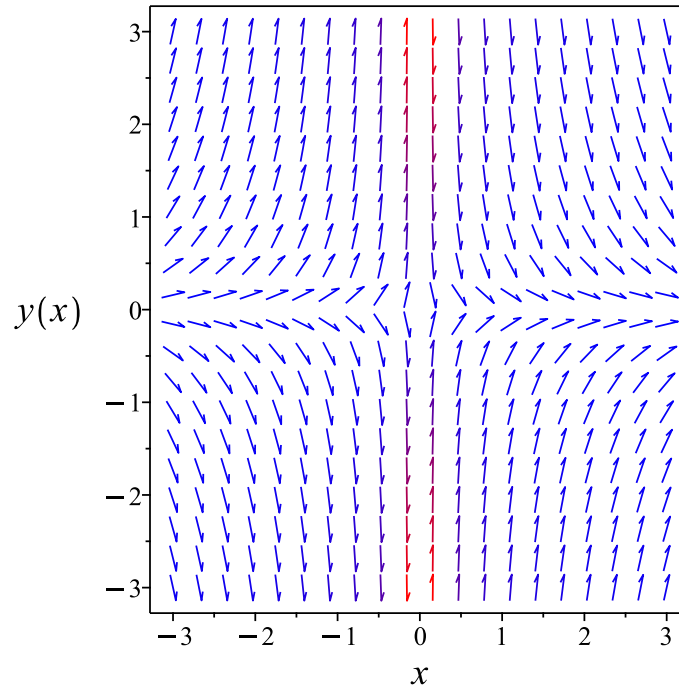


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^4}$$

Verified OK.

2.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)x + u(x)) + 4u(x)x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{5}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{5}{x} dx \\ \ln(u) &= -5 \ln(x) + c_2 \\ u &= e^{-5 \ln(x) + c_2} \\ &= \frac{c_2}{x^5}\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= ux \\ &= \frac{c_2}{x^4}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{x^4} \tag{1}$$

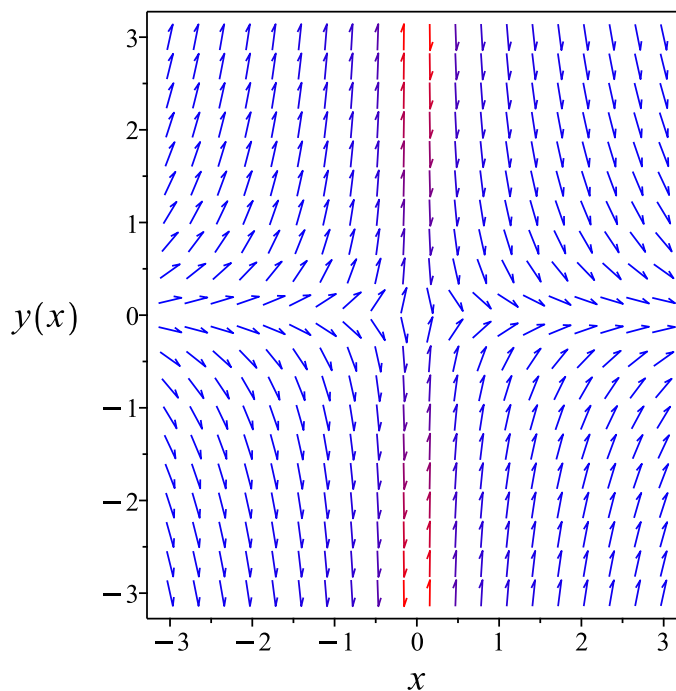


Figure 29: Slope field plot

Verification of solutions

$$y = \frac{c_2}{x^4}$$

Verified OK.

2.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{4y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{x^4}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^4}} dy \end{aligned}$$

Which results in

$$S = y x^4$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4y x^3 \\ S_y &= x^4 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^4 = c_1$$

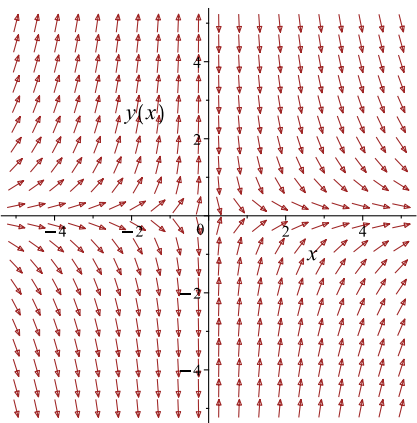
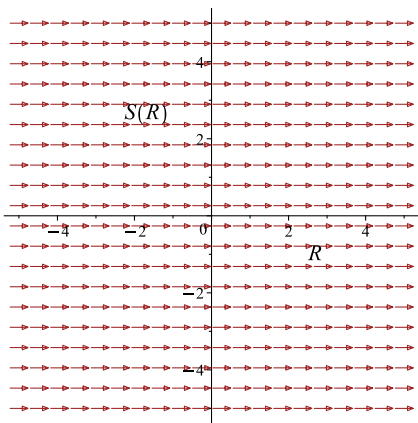
Which simplifies to

$$yx^4 = c_1$$

Which gives

$$y = \frac{c_1}{x^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4y}{x}$ 	$R = x$ $S = yx^4$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} \quad (1)$$

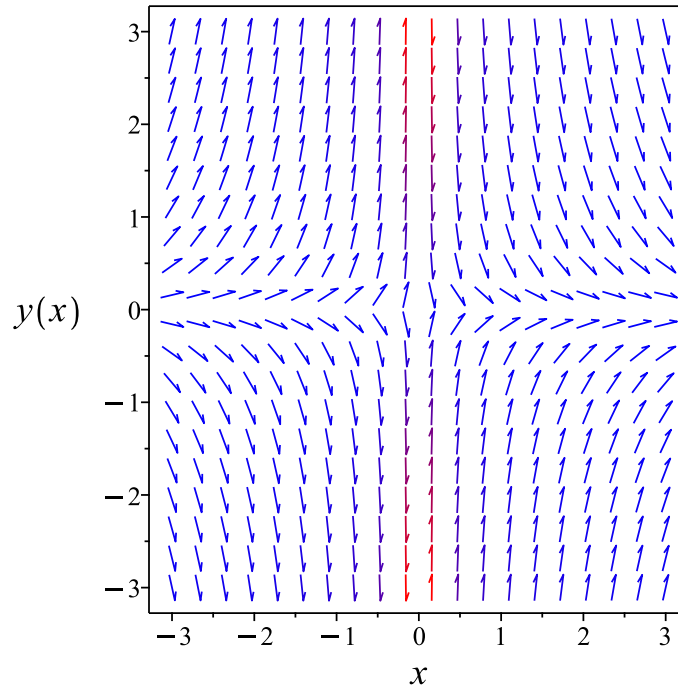


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^4}$$

Verified OK.

2.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{4y}\right) dy &= \left(\frac{1}{x}\right) dx \\ \left(-\frac{1}{x}\right) dx + \left(-\frac{1}{4y}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x} \\ N(x, y) &= -\frac{1}{4y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{1}{4y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x} dx \\ \phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{4y}$. Therefore equation (4) becomes

$$-\frac{1}{4y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{4y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{1}{4y} \right) dy \\ f(y) &= -\frac{\ln(y)}{4} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{4}$$

The solution becomes

$$y = \frac{e^{-4c_1}}{x^4}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-4c_1}}{x^4} \tag{1}$$

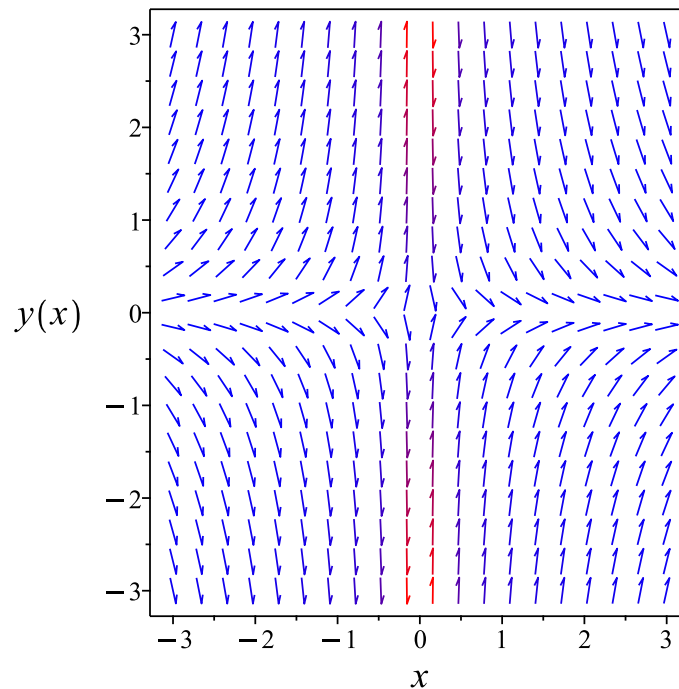


Figure 31: Slope field plot

Verification of solutions

$$y = \frac{e^{-4c_1}}{x^4}$$

Verified OK.

2.1.6 Maple step by step solution

Let's solve

$$xy' + 4y = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = -\frac{4}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int -\frac{4}{x} dx + c_1$$

- Evaluate integral

$$\ln(y) = -4 \ln(x) + c_1$$

- Solve for y

$$y = \frac{e^{c_1}}{x^4}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(4*y(x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x^4}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[4*y[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_1}{x^4}$$

$$y(x) \rightarrow 0$$

2.2 problem 25

2.2.1	Solving as separable ode	135
2.2.2	Solving as linear ode	137
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Internal problem ID [5237]

Internal file name [OUTPUT/4728_Friday_February_02_2024_05_10_42_AM_71893101/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 25.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$2y + (-x^2 + 4)y' = -1$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{1 + 2y}{x^2 - 4}\end{aligned}$$

Where $f(x) = \frac{1}{x^2-4}$ and $g(y) = 1 + 2y$. Integrating both sides gives

$$\frac{1}{1 + 2y} dy = \frac{1}{x^2 - 4} dx$$

$$\int \frac{1}{1+2y} dy = \int \frac{1}{x^2-4} dx$$

$$\frac{\ln(1+2y)}{2} = -\frac{\ln(x+2)}{4} + \frac{\ln(-2+x)}{4} + c_1$$

Raising both side to exponential gives

$$\sqrt{1+2y} = e^{-\frac{\ln(x+2)}{4} + \frac{\ln(-2+x)}{4} + c_1}$$

Which simplifies to

$$\sqrt{1+2y} = c_2 e^{-\frac{\ln(x+2)}{4} + \frac{\ln(-2+x)}{4}}$$

Which simplifies to

$$y = \frac{c_2^2 \sqrt{-2+x} e^{2c_1}}{2\sqrt{x+2}} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2^2 \sqrt{-2+x} e^{2c_1}}{2\sqrt{x+2}} - \frac{1}{2} \tag{1}$$

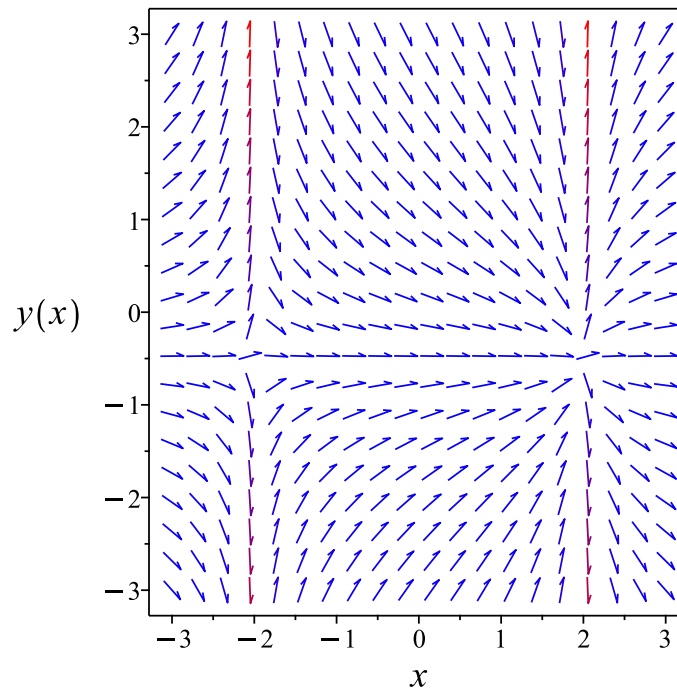


Figure 32: Slope field plot

Verification of solutions

$$y = \frac{c_2^2 \sqrt{-2+x} e^{2c_1}}{2\sqrt{x+2}} - \frac{1}{2}$$

Verified OK.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x^2 - 4}$$
$$q(x) = \frac{1}{x^2 - 4}$$

Hence the ode is

$$y' - \frac{2y}{x^2 - 4} = \frac{1}{x^2 - 4}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x^2-4} dx}$$
$$= e^{\frac{\ln(x+2)}{2} - \frac{\ln(-2+x)}{2}}$$

Which simplifies to

$$\mu = \frac{\sqrt{x+2}}{\sqrt{-2+x}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{1}{x^2 - 4} \right)$$
$$\frac{d}{dx} \left(\frac{\sqrt{x+2} y}{\sqrt{-2+x}} \right) = \left(\frac{\sqrt{x+2}}{\sqrt{-2+x}} \right) \left(\frac{1}{x^2 - 4} \right)$$
$$d \left(\frac{\sqrt{x+2} y}{\sqrt{-2+x}} \right) = \left(\frac{\sqrt{x+2}}{(x^2 - 4) \sqrt{-2+x}} \right) dx$$

Integrating gives

$$\frac{\sqrt{x+2}y}{\sqrt{-2+x}} = \int \frac{\sqrt{x+2}}{(x^2-4)\sqrt{-2+x}} dx$$

$$\frac{\sqrt{x+2}y}{\sqrt{-2+x}} = -\frac{\sqrt{-2+x}(x+2)^{\frac{3}{2}}}{2(x^2-4)} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{x+2}}{\sqrt{-2+x}}$ results in

$$y = -\frac{(x+2)(-2+x)}{2(x^2-4)} + \frac{c_1\sqrt{-2+x}}{\sqrt{x+2}}$$

which simplifies to

$$y = -\frac{1}{2} + \frac{c_1\sqrt{-2+x}}{\sqrt{x+2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + \frac{c_1\sqrt{-2+x}}{\sqrt{x+2}} \tag{1}$$

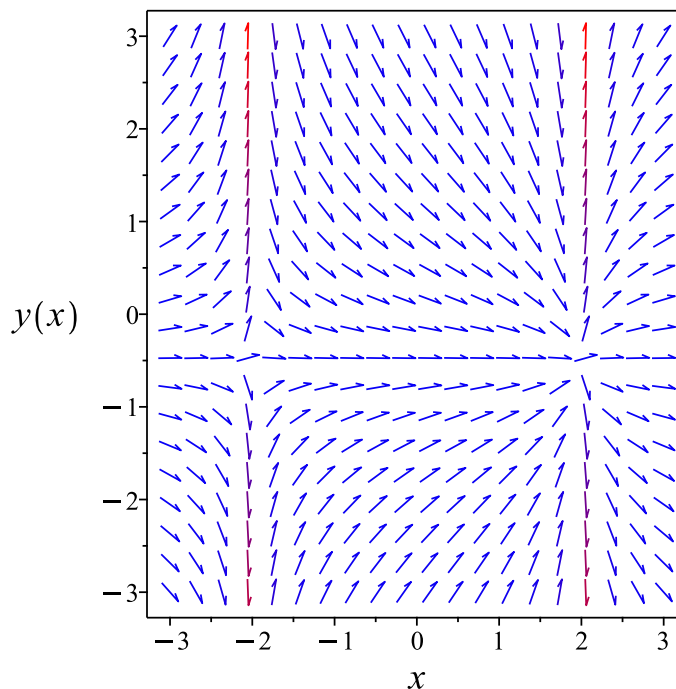


Figure 33: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + \frac{c_1\sqrt{-2+x}}{\sqrt{x+2}}$$

Verified OK.

2.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1+2y}{x^2-4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{-\frac{\ln(x+2)}{2} + \frac{\ln(-2+x)}{2}}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{\ln(x+2)}{2} + \frac{\ln(-2+x)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln(\sqrt{x+2}) + \ln\left(\frac{1}{\sqrt{-2+x}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + 2y}{x^2 - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{(-2+x)^{\frac{3}{2}}\sqrt{x+2}} \\ S_y &= \frac{\sqrt{x+2}}{\sqrt{-2+x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(-2+x)^{\frac{3}{2}}\sqrt{x+2}} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(-2+R)^{\frac{3}{2}}\sqrt{R+2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{\sqrt{R+2}}{2\sqrt{-2+R}} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x+2}y}{\sqrt{-2+x}} = -\frac{\sqrt{x+2}}{2\sqrt{-2+x}} + c_1$$

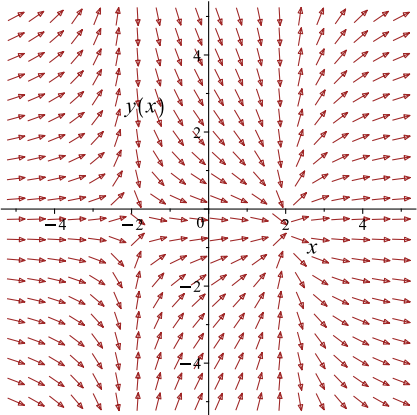
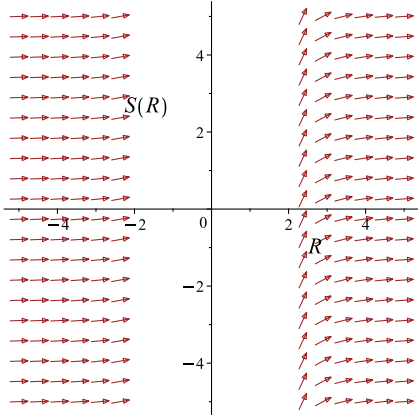
Which simplifies to

$$\frac{\sqrt{x+2}y}{\sqrt{-2+x}} = -\frac{\sqrt{x+2}}{2\sqrt{-2+x}} + c_1$$

Which gives

$$y = -\frac{-2c_1\sqrt{-2+x} + \sqrt{x+2}}{2\sqrt{x+2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1+2y}{x^2-4}$ 	$R = x$ $S = \frac{\sqrt{x+2}y}{\sqrt{-2+x}}$	$\frac{dS}{dR} = \frac{1}{(-2+R)^{\frac{3}{2}}\sqrt{R+2}}$ 

Summary

The solution(s) found are the following

$$y = -\frac{-2c_1\sqrt{-2+x} + \sqrt{x+2}}{2\sqrt{x+2}} \quad (1)$$

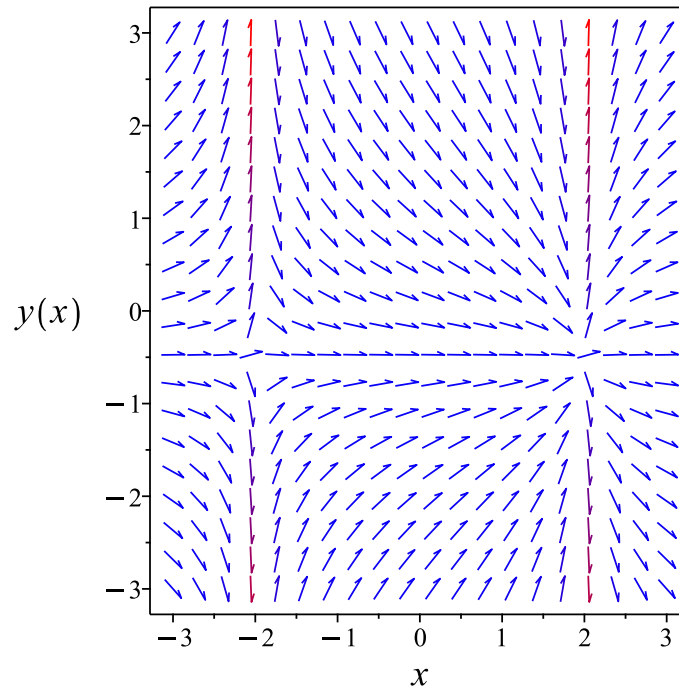


Figure 34: Slope field plot

Verification of solutions

$$y = -\frac{-2c_1\sqrt{-2+x} + \sqrt{x+2}}{2\sqrt{x+2}}$$

Verified OK.

2.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (A)$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{1}{1+2y}\right) dy &= \left(\frac{1}{x^2-4}\right) dx \\ \left(-\frac{1}{x^2-4}\right) dx + \left(\frac{1}{1+2y}\right) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\frac{1}{x^2-4} \\ N(x, y) &= \frac{1}{1+2y} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x^2 - 4} \right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{1 + 2y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x^2 - 4} dx \\ \phi &= \frac{\ln(x + 2)}{4} - \frac{\ln(-2 + x)}{4} + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+2y}$. Therefore equation (4) becomes

$$\frac{1}{1 + 2y} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{1 + 2y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(\frac{1}{1+2y} \right) dy$$
$$f(y) = \frac{\ln(1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x+2)}{4} - \frac{\ln(-2+x)}{4} + \frac{\ln(1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x+2)}{4} - \frac{\ln(-2+x)}{4} + \frac{\ln(1+2y)}{2}$$

The solution becomes

$$y = \frac{e^{-\frac{\ln(x+2)}{2} + \frac{\ln(-2+x)}{2} + 2c_1}}{2} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{\ln(x+2)}{2} + \frac{\ln(-2+x)}{2} + 2c_1}}{2} - \frac{1}{2} \quad (1)$$

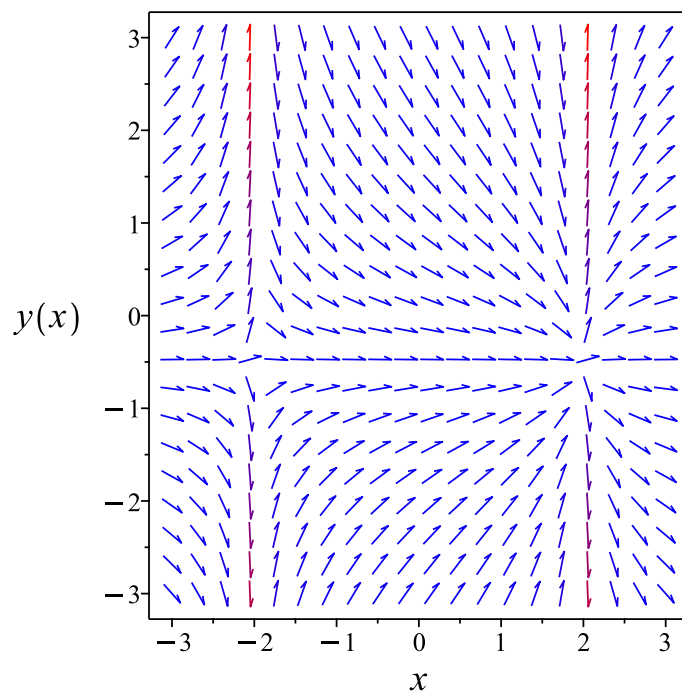


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{\ln(x+2)}{2} + \frac{\ln(-2+x)}{2} + 2c_1}}{2} - \frac{1}{2}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$$2y + (-x^2 + 4)y' = -1$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'}{-2y-1} = \frac{1}{-x^2+4}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-2y-1} dx = \int \frac{1}{-x^2+4} dx + c_1$$

- Evaluate integral

$$-\frac{\ln(-2y-1)}{2} = \frac{\ln(x+2)}{4} - \frac{\ln(-2+x)}{4} + C_1$$

- Solve for y

$$\left\{ y = \frac{-x-2+\sqrt{e^{-4c_1}x^2-4e^{-4c_1}}}{2(x+2)}, y = -\frac{x+2+\sqrt{e^{-4c_1}x^2-4e^{-4c_1}}}{2(x+2)} \right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((1+2*y(x))+(4-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + \frac{\sqrt{-2+x} c_1}{\sqrt{x+2}}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 35

```
DSolve[(1+2*y[x])+(4-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2} + \frac{c_1 \sqrt{2-x}}{\sqrt{x+2}}$$

$$y(x) \rightarrow -\frac{1}{2}$$

2.3 problem 26

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Internal problem ID [5238]

Internal file name [OUTPUT/4729_Friday_February_02_2024_05_10_42_AM_66874428/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 26.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$y^2 - x^2y' = 0$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2}{x^2}\end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2} dy &= \frac{1}{x^2} dx \\ \int \frac{1}{y^2} dy &= \int \frac{1}{x^2} dx \\ -\frac{1}{y} &= c_1 - \frac{1}{x}\end{aligned}$$

Which results in

$$y = -\frac{x}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - 1} \tag{1}$$

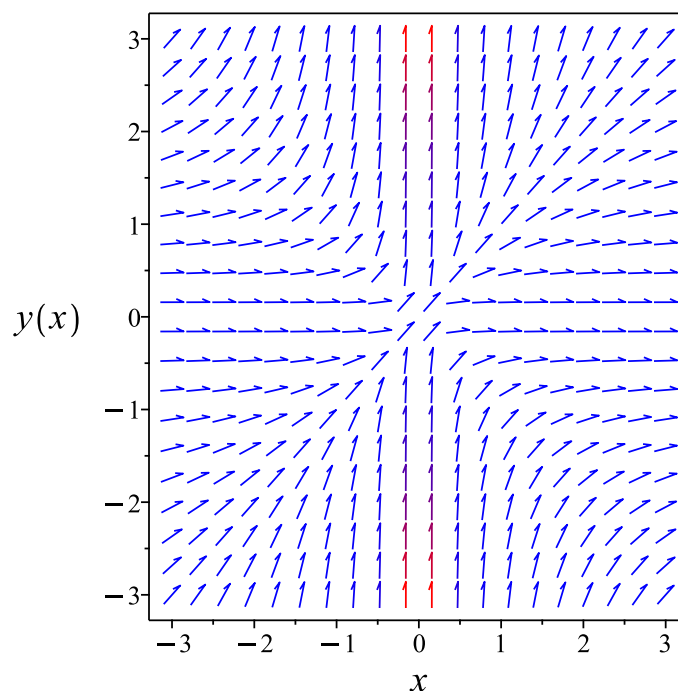


Figure 36: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1x - 1}$$

Verified OK.

2.3.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 - x^2(u'(x)x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(u-1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u(u-1)} du &= \frac{1}{x} dx \\ \int \frac{1}{u(u-1)} du &= \int \frac{1}{x} dx \\ \ln(u-1) - \ln(u) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(u-1)-\ln(u)} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u-1}{u} = c_3x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= -\frac{x}{c_3x - 1} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_3x - 1} \quad (1)$$

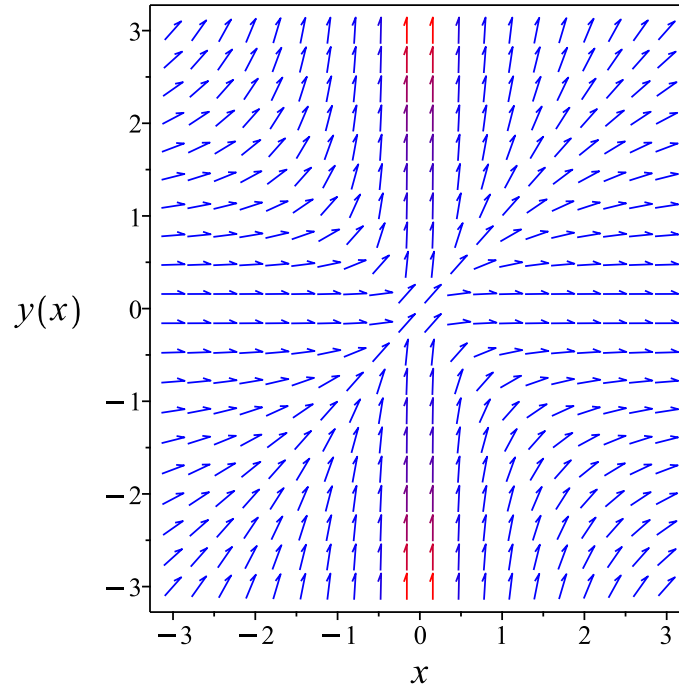


Figure 37: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_3x - 1}$$

Verified OK.

2.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= x^2 \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$

$$R_y = 1$$

$$S_x = \frac{1}{x^2}$$

$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

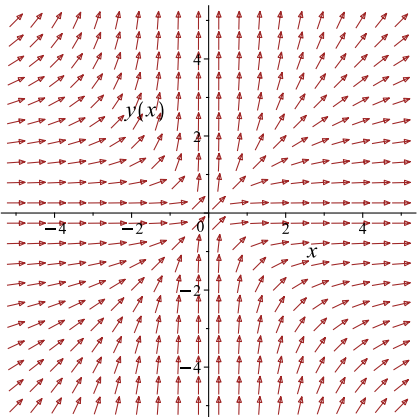
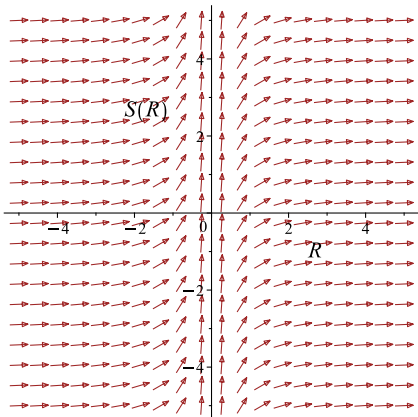
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{c_1 x + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x^2}$ 	$R = y$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{c_1x + 1} \quad (1)$$

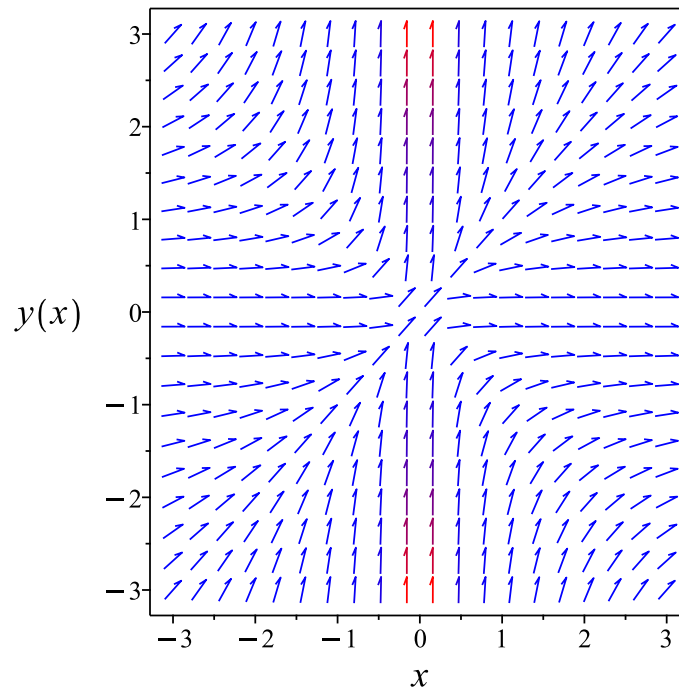


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{x}{c_1x + 1}$$

Verified OK.

2.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{1}{y^2}\right) dy &= \left(\frac{1}{x^2}\right) dx \\ \left(-\frac{1}{x^2}\right) dx + \left(\frac{1}{y^2}\right) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\frac{1}{x^2} \\ N(x, y) &= \frac{1}{y^2} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x^2}\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{y^2} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x^2} dx \\ \phi &= \frac{1}{x} + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(\frac{1}{y^2} \right) dy$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{x}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - 1} \tag{1}$$

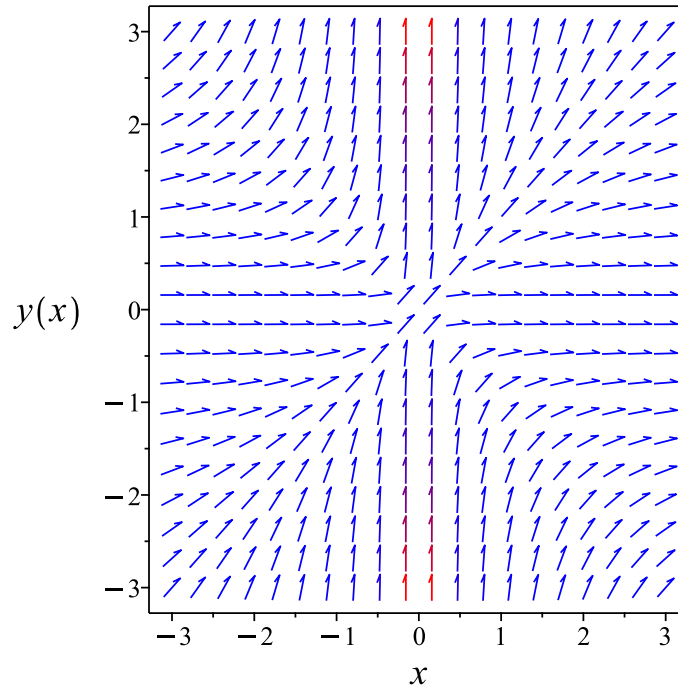


Figure 39: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1x - 1}$$

Verified OK.

2.3.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{c_1 + \frac{c_2}{x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{c_3 + \frac{1}{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_3 + \frac{1}{x}} \quad (1)$$

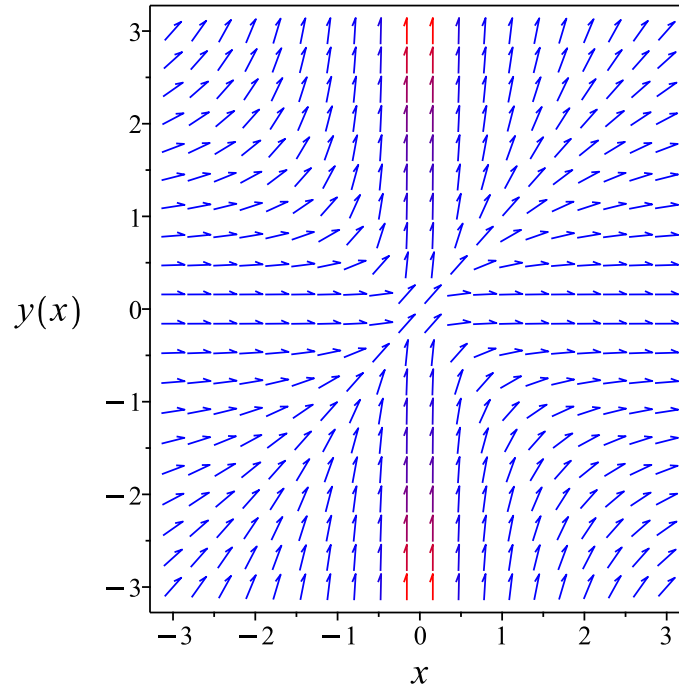


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{1}{c_3 + \frac{1}{x}}$$

Verified OK.

2.3.6 Maple step by step solution

Let's solve

$$y^2 - x^2 y' = 0$$

- Highest derivative means the order of the ODE is 1
- y'
- Separate variables

$$\frac{y'}{y^2} = \frac{1}{x^2}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y^2} dx = \int \frac{1}{x^2} dx + c_1$$

- Evaluate integral

$$-\frac{1}{y} = c_1 - \frac{1}{x}$$

- Solve for y

$$y = -\frac{x}{c_1 x - 1}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(y(x)^2-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 x + 1}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 21

```
DSolve[y[x]^2-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x}{1 - c_1 x}$$

$$y(x) \rightarrow 0$$

2.4 problem 27

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Internal problem ID [5239]

Internal file name [OUTPUT/4730_Friday_February_02_2024_05_10_43_AM_16493042/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 27.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$-(x + 1)y' + y = -1$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{1 + y}{x + 1}\end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = 1 + y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{1+y} dy &= \frac{1}{x+1} dx \\ \int \frac{1}{1+y} dy &= \int \frac{1}{x+1} dx \\ \ln(1+y) &= \ln(x+1) + c_1\end{aligned}$$

Raising both side to exponential gives

$$1 + y = e^{\ln(x+1)+c_1}$$

Which simplifies to

$$1 + y = c_2(x + 1)$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\ln(x+1)+c_1} - 1 \tag{1}$$

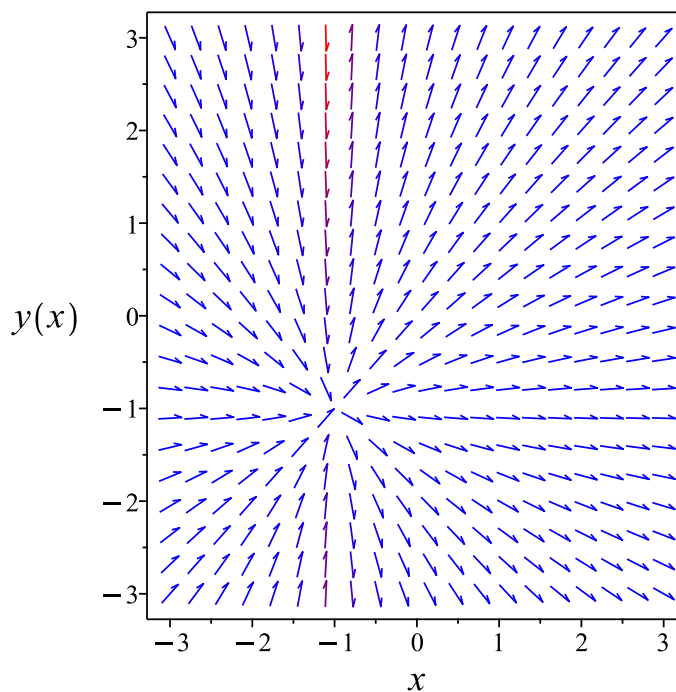


Figure 41: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(x+1)+c_1} - 1$$

Verified OK.

2.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x+1}$$
$$q(x) = \frac{1}{x+1}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{1}{x+1}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x+1} dx}$$
$$= \frac{1}{x+1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{1}{x+1} \right)$$
$$\frac{d}{dx} \left(\frac{y}{x+1} \right) = \left(\frac{1}{x+1} \right) \left(\frac{1}{x+1} \right)$$
$$d \left(\frac{y}{x+1} \right) = \frac{1}{(x+1)^2} dx$$

Integrating gives

$$\frac{y}{x+1} = \int \frac{1}{(x+1)^2} dx$$
$$\frac{y}{x+1} = -\frac{1}{x+1} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = -1 + (x + 1) c_1$$

which simplifies to

$$y = c_1 x + c_1 - 1$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 1 \tag{1}$$

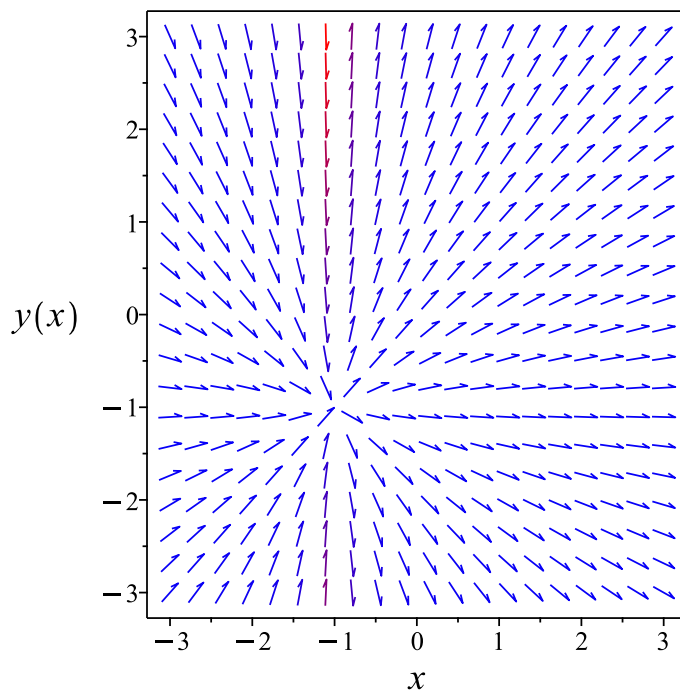


Figure 42: Slope field plot

Verification of solutions

$$y = c_1 x + c_1 - 1$$

Verified OK.

2.4.3 Solving as homogeneous Type D2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$-(x+1)(u'(x)x + u(x)) + u(x)x = -1$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{-u+1}{x(x+1)}\end{aligned}$$

Where $f(x) = \frac{1}{x(x+1)}$ and $g(u) = -u+1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-u+1} du &= \frac{1}{x(x+1)} dx \\ \int \frac{1}{-u+1} du &= \int \frac{1}{x(x+1)} dx \\ -\ln(u-1) &= -\ln(x+1) + \ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u-1} = e^{-\ln(x+1)+\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{u-1} = c_3 e^{-\ln(x+1)+\ln(x)}$$

Which simplifies to

$$u(x) = \frac{\left(\frac{c_3 e^{c_2} x}{x+1} + 1\right)(x+1)e^{-c_2}}{c_3 x}$$

Therefore the solution y is

$$\begin{aligned}y &= ux \\ &= \frac{\left(\frac{c_3 e^{c_2} x}{x+1} + 1\right)(x+1)e^{-c_2}}{c_3}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{c_3 e^{c_2 x}}{x+1} + 1\right) (x+1) e^{-c_2}}{c_3} \quad (1)$$

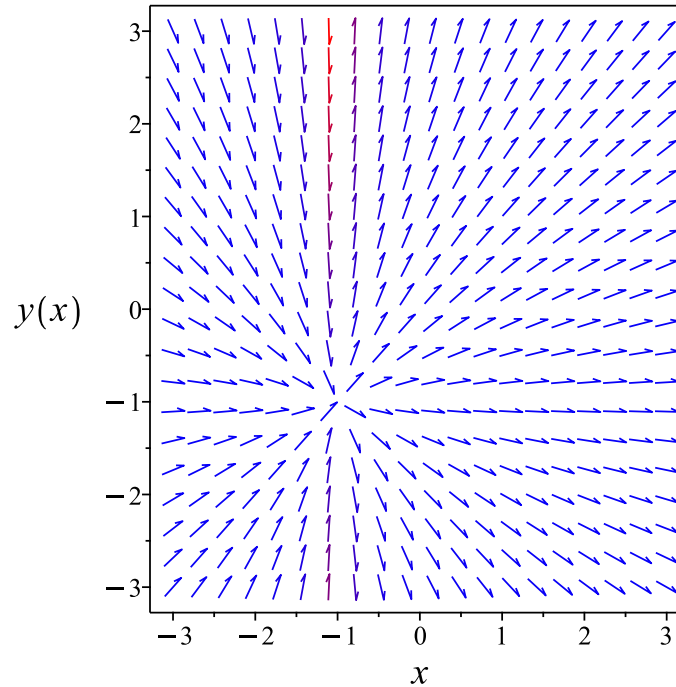


Figure 43: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{c_3 e^{c_2 x}}{x+1} + 1\right) (x+1) e^{-c_2}}{c_3}$$

Verified OK.

2.4.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX} Y(X) = \frac{1 + Y(X) + y_0}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= -1 \\y_0 &= -1\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X, Y) \\ &= \frac{Y}{X}\end{aligned}\tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dX}X + u &= u \\ \frac{du}{dX} &= 0\end{aligned}$$

Or

$$\frac{d}{dX}u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$\begin{aligned}u(X) &= \int 0 \, dX \\ &= c_2\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = Xc_2$$

Using the solution for $Y(X)$

$$Y(X) = Xc_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$

$$X = x + x_0$$

Or

$$Y = y - 1$$

$$X = x - 1$$

Then the solution in y becomes

$$1 + y = c_2(x + 1)$$

Summary

The solution(s) found are the following

$$1 + y = c_2(x + 1) \tag{1}$$

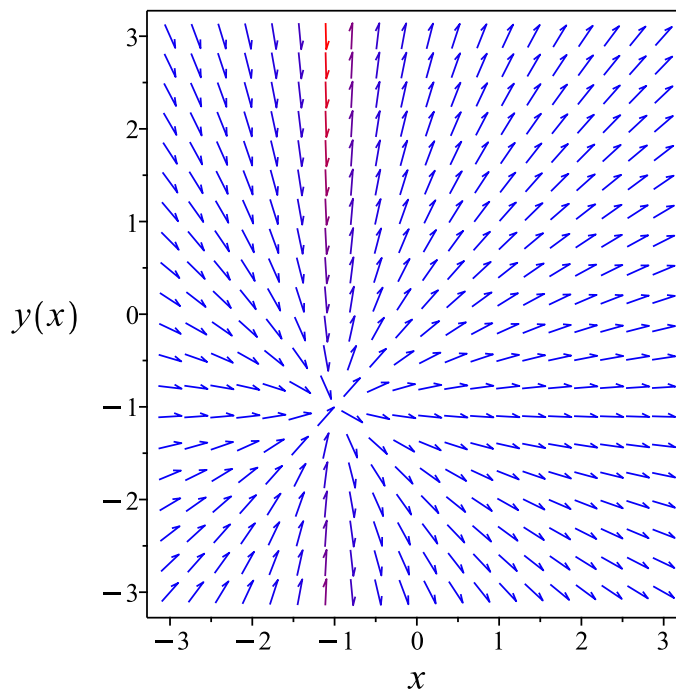


Figure 44: Slope field plot

Verification of solutions

$$1 + y = c_2(x + 1)$$

Verified OK.

2.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1 + y}{x + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x + 1\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1+y}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)^2} \\ S_y &= \frac{1}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(x+1)^2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{R+1} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x+1} = -\frac{1}{x+1} + c_1$$

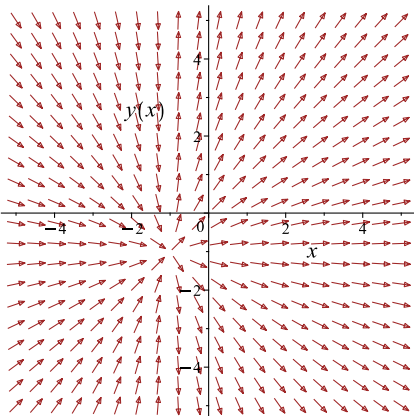
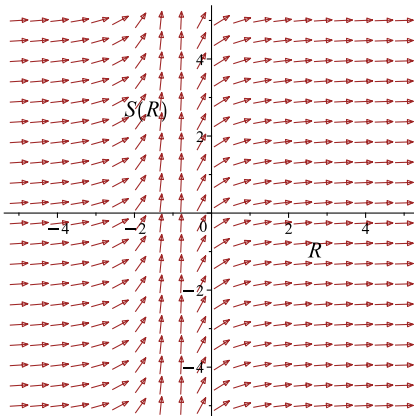
Which simplifies to

$$\frac{y}{x+1} = -\frac{1}{x+1} + c_1$$

Which gives

$$y = c_1 x + c_1 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1+y}{x+1}$ 	$R = x$ $S = \frac{y}{x+1}$	$\frac{dS}{dR} = \frac{1}{(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 1 \quad (1)$$

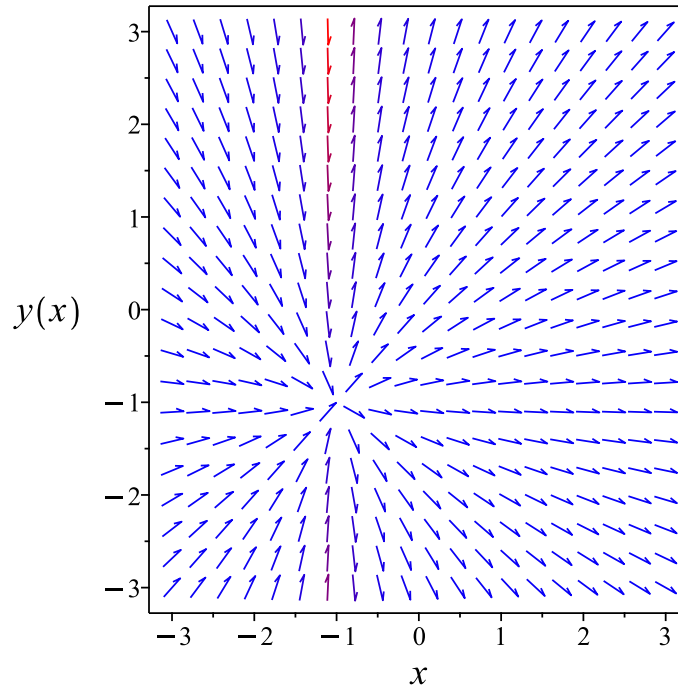


Figure 45: Slope field plot

Verification of solutions

$$y = c_1x + c_1 - 1$$

Verified OK.

2.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}\left(\frac{1}{1+y}\right) dy &= \left(\frac{1}{x+1}\right) dx \\ \left(-\frac{1}{x+1}\right) dx + \left(\frac{1}{1+y}\right) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x+1} \\ N(x, y) &= \frac{1}{1+y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{1+y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x+1} dx \\ \phi &= -\ln(x+1) + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+y}$. Therefore equation (4) becomes

$$\frac{1}{1+y} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{1+y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{1}{1+y} \right) dy \\ f(y) &= \ln(1+y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \ln(1+y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \ln(1+y)$$

The solution becomes

$$y = x e^{c_1} + e^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = x e^{c_1} + e^{c_1} - 1 \tag{1}$$

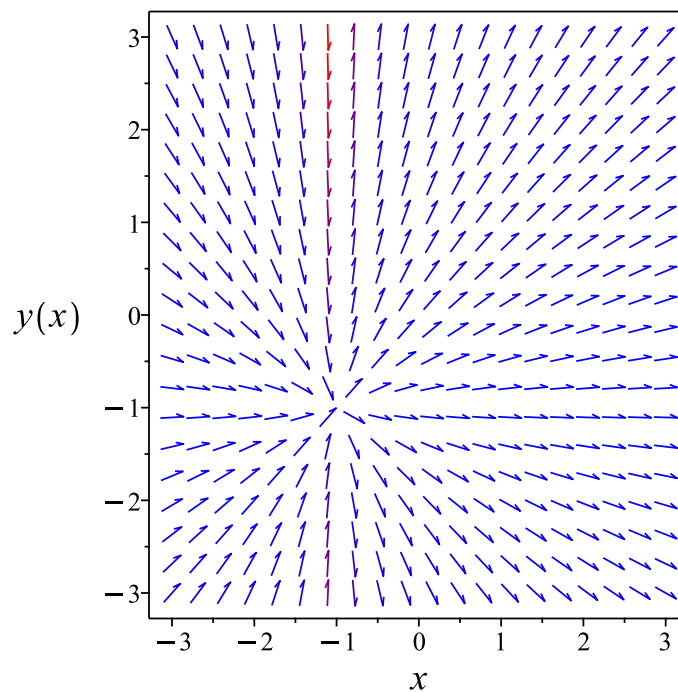


Figure 46: Slope field plot

Verification of solutions

$$y = x e^{c_1} + e^{c_1} - 1$$

Verified OK.

2.4.7 Maple step by step solution

Let's solve

$$-(x+1)y' + y = -1$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{-1-y} = -\frac{1}{x+1}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-1-y} dx = \int -\frac{1}{x+1} dx + c_1$$

- Evaluate integral

$$-\ln(-1-y) = -\ln(x+1) + c_1$$

- Solve for y

$$y = -\frac{e^{c_1+x+1}}{e^{c_1}}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((1+y(x))-(1+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + c_1 - 1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 18

```
DSolve[(1+y[x])-(1+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -1 + c_1(x + 1)$$

$$y(x) \rightarrow -1$$

2.5 problem 28

2.5.1 Solving as first order ode lie symmetry calculated ode	182
2.5.2 Solving as exact ode	188

Internal problem ID [5240]

Internal file name [OUTPUT/4731_Friday_February_02_2024_05_10_44_AM_120439/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 28.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$xy^2 + y + (yx^2 - x)y' = 0$$

2.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(xy+1)}{x(xy-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1\text{E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2\text{E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{y(xy+1)(b_3-a_2)}{x(xy-1)} - \frac{y^2(xy+1)^2 a_3}{x^2(xy-1)^2} \\ - \left(-\frac{y^2}{x(xy-1)} + \frac{y(xy+1)}{x^2(xy-1)} + \frac{y^2(xy+1)}{x(xy-1)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{xy+1}{x(xy-1)} - \frac{y}{xy-1} + \frac{y(xy+1)}{(xy-1)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{2x^4y^2b_2 - 2x^2y^4a_3 + x^3y^2b_1 - x^2y^3a_1 - 4x^3yb_2 - 2x^2y^2a_2 - 2x^2y^2b_3 - 4xy^3a_3 - 2x^2yb_1 - 2xy^2a_1 - xb_1 - ya_1}{x^2(xy-1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} 2x^4y^2b_2 - 2x^2y^4a_3 + x^3y^2b_1 - x^2y^3a_1 - 4x^3yb_2 - 2x^2y^2a_2 \\ - 2x^2y^2b_3 - 4xy^3a_3 - 2x^2yb_1 - 2xy^2a_1 - xb_1 + ya_1 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -2a_3v_1^2v_2^4 + 2b_2v_1^4v_2^2 - a_1v_1^2v_2^3 + b_1v_1^3v_2^2 - 2a_2v_1^2v_2^2 - 4a_3v_1v_2^3 \\ - 4b_2v_1^3v_2 - 2b_3v_1^2v_2^2 - 2a_1v_1v_2^2 - 2b_1v_1^2v_2 + a_1v_2 - b_1v_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$2b_2v_1^4v_2^2 + b_1v_1^3v_2^2 - 4b_2v_1^3v_2 - 2a_3v_1^2v_2^4 - a_1v_1^2v_2^3 + (-2a_2 - 2b_3)v_1^2v_2^2 - 2b_1v_1^2v_2 - 4a_3v_1v_2^3 - 2a_1v_1v_2^2 - b_1v_1 + a_1v_2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ -b_1 &= 0 \\ -4b_2 &= 0 \\ 2b_2 &= 0 \\ -2a_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(-\frac{y(xy + 1)}{x(xy - 1)} \right) (-x) \\ &= -\frac{2y}{xy - 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{2y}{xy-1}} dy\end{aligned}$$

Which results in

$$S = -\frac{xy}{2} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(xy + 1)}{x(xy - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2} \\ S_y &= \frac{-xy + 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{xy}{2} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1$$

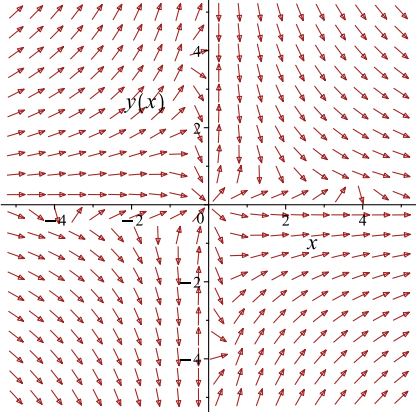
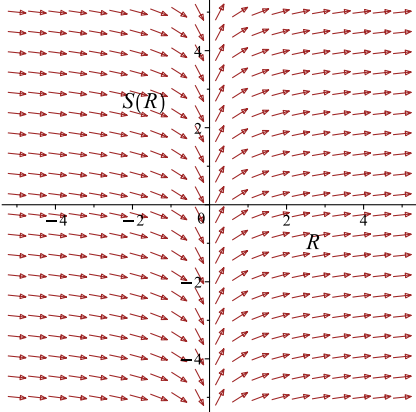
Which simplifies to

$$-\frac{xy}{2} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1$$

Which gives

$$y = -\frac{\text{LambertW}(-e^{2c_1}x^2)}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(xy+1)}{x(xy-1)}$ 	$R = x$ $S = -\frac{xy}{2} + \frac{\ln(y)}{2}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\text{LambertW}(-e^{2c_1}x^2)}{x} \tag{1}$$

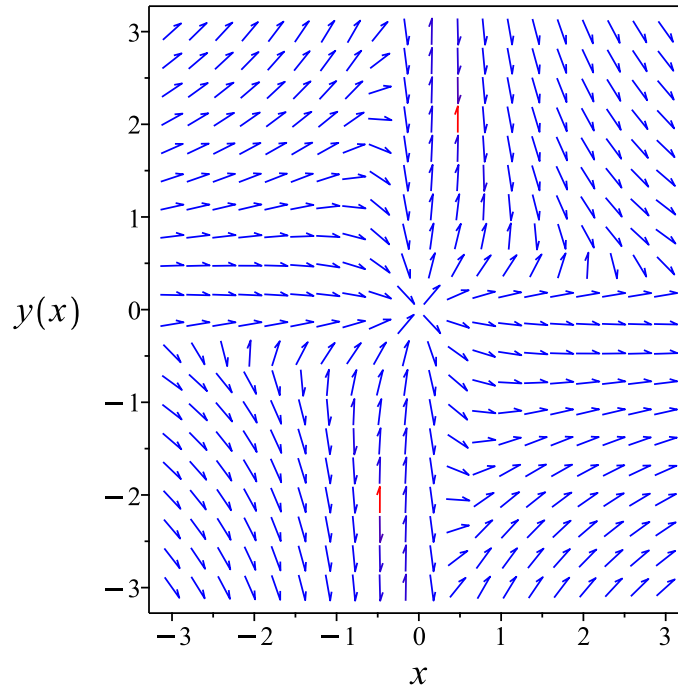


Figure 47: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}(-e^{2c_1 x^2})}{x}$$

Verified OK.

2.5.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(y x^2 - x) dy &= (-y^2 x - y) dx \\ (y^2 x + y) dx + (y x^2 - x) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 x + y \\ N(x, y) &= y x^2 - x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2 x + y) \\ &= 2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y x^2 - x) \\ &= 2xy - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x(xy-1)} ((2xy+1) - (2xy-1)) \\ &= \frac{2}{x(xy-1)} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y^2x+y} ((2xy-1) - (2xy+1)) \\ &= -\frac{2}{y(xy+1)} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(2xy-1) - (2xy+1)}{x(y^2x+y) - y(yx^2-x)} \\ &= -\frac{1}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int (-\frac{1}{t}) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)} \\ &= \frac{1}{t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{yx}$$

Multiplying M and N by this integrating factor gives new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{yx}(y^2x + y) \\ &= \frac{xy + 1}{x}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{yx}(yx^2 - x) \\ &= \frac{xy - 1}{y}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{xy + 1}{x}\right) + \left(\frac{xy - 1}{y}\right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{xy + 1}{x} dx \\ \phi &= xy + \ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{xy-1}{y}$. Therefore equation (4) becomes

$$\frac{xy-1}{y} = x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{1}{y}\right) dy$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy + \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + \ln(x) - \ln(y)$$

The solution becomes

$$y = -\frac{\text{LambertW}(-e^{-c_1}x^2)}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{LambertW}(-e^{-c_1}x^2)}{x} \quad (1)$$

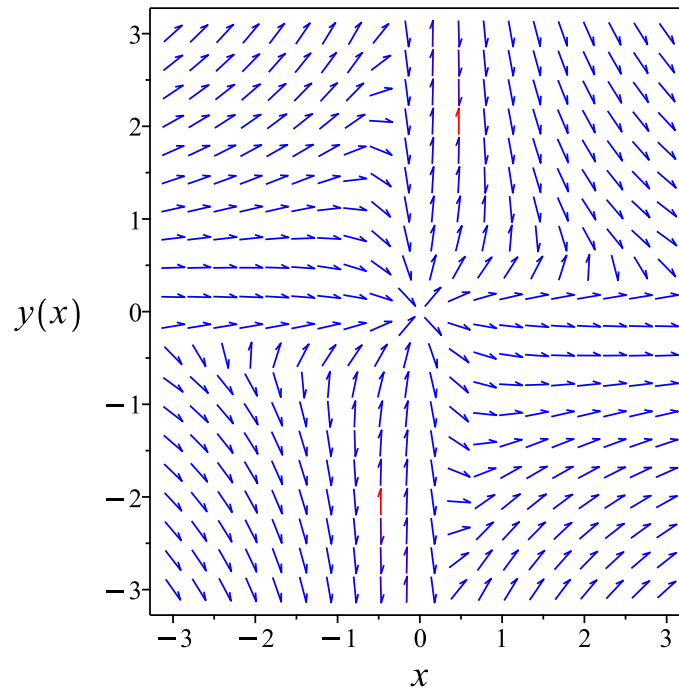


Figure 48: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}(-e^{-c_1 x^2})}{x}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 19

```
dsolve((x*y(x)^2+y(x))+(x^2*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\text{LambertW}(-x^2e^{-2c_1})}{x}$$

✓ Solution by Mathematica

Time used: 13.386 (sec). Leaf size: 33

```
DSolve[(x*y[x]^2+y[x])+(x^2*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{W\left(e^{-1+\frac{9c_1}{2^{2/3}}}x^2\right)}{x}$$
$$y(x) \rightarrow 0$$

2.6 problem 29

2.6.1	Solving as homogeneousTypeD2 ode	195
2.6.2	Solving as first order ode lie symmetry calculated ode	197
2.6.3	Solving as exact ode	204

Internal problem ID [5241]

Internal file name [OUTPUT/4732_Friday_February_02_2024_05_10_45_AM_26690807/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 29.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**homogeneousTypeD2**", "**exactWithIntegrationFactor**", "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right) + x \cos\left(\frac{y}{x}\right) y' = 0$$

2.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$x \sin(u(x)) - u(x)x \cos(u(x)) + x \cos(u(x))(u'(x)x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\tan(u)} du &= -\frac{1}{x} dx \\ \int \frac{1}{\tan(u)} du &= \int -\frac{1}{x} dx \\ \ln(\sin(u)) &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\sin(u) = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned}y &= xu \\ &= x \arcsin\left(\frac{c_3 e^{c_2}}{x}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x \arcsin\left(\frac{c_3 e^{c_2}}{x}\right) \tag{1}$$

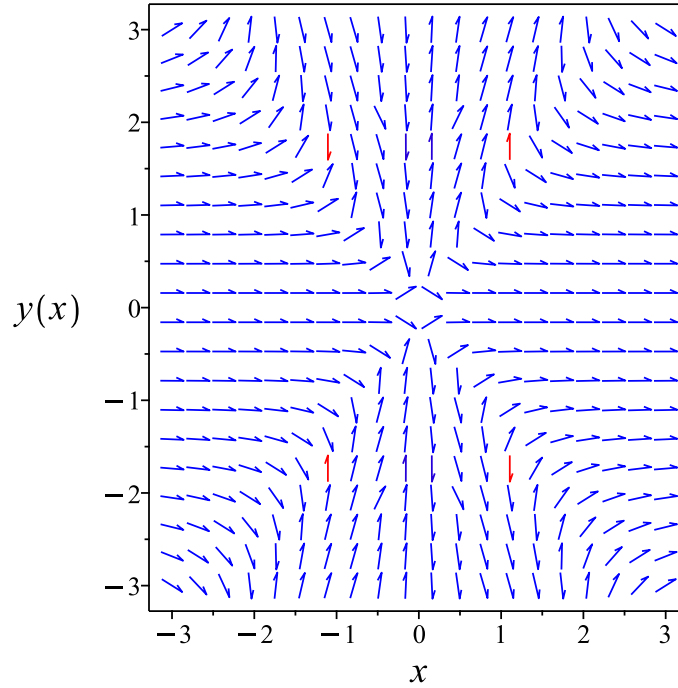


Figure 49: Slope field plot

Verification of solutions

$$y = x \arcsin \left(\frac{c_3 e^{c_2}}{x} \right)$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-x \sin \left(\frac{y}{x} \right) + y \cos \left(\frac{y}{x} \right)}{x \cos \left(\frac{y}{x} \right)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(-x \sin(\frac{y}{x}) + y \cos(\frac{y}{x})) (b_3 - a_2)}{x \cos(\frac{y}{x})} - \frac{(-x \sin(\frac{y}{x}) + y \cos(\frac{y}{x}))^2 a_3}{x^2 \cos(\frac{y}{x})^2} \\ - \left(\frac{-\sin(\frac{y}{x}) + \frac{y \cos(\frac{y}{x})}{x} + \frac{y^2 \sin(\frac{y}{x})}{x^2}}{x \cos(\frac{y}{x})} - \frac{-x \sin(\frac{y}{x}) + y \cos(\frac{y}{x})}{x^2 \cos(\frac{y}{x})} \right. \\ \left. - \frac{(-x \sin(\frac{y}{x}) + y \cos(\frac{y}{x})) y \sin(\frac{y}{x})}{x^3 \cos(\frac{y}{x})^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{y \sin(\frac{y}{x})}{x^2 \cos(\frac{y}{x})} + \frac{(-x \sin(\frac{y}{x}) + y \cos(\frac{y}{x})) \sin(\frac{y}{x})}{x^2 \cos(\frac{y}{x})^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{\sin(\frac{y}{x})^2 x^2 a_3 - \sin(\frac{y}{x})^2 x^2 b_2 + \sin(\frac{y}{x})^2 xy a_2 - \sin(\frac{y}{x})^2 xy b_3 + \sin(\frac{y}{x})^2 y^2 a_3 - \sin(\frac{y}{x}) \cos(\frac{y}{x}) x^2 a_2 + \sin(\frac{y}{x}) \cos(\frac{y}{x}) x^2 b_3}{2} \\ = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -\sin(\frac{y}{x})^2 x^2 a_3 + \sin(\frac{y}{x})^2 x^2 b_2 - \sin(\frac{y}{x})^2 xy a_2 + \sin(\frac{y}{x})^2 xy b_3 \\ - \sin(\frac{y}{x})^2 y^2 a_3 + \sin(\frac{y}{x}) \cos(\frac{y}{x}) x^2 a_2 - \sin(\frac{y}{x}) \cos(\frac{y}{x}) x^2 b_3 \\ + 2 \sin(\frac{y}{x}) \cos(\frac{y}{x}) xy a_3 + b_2 x^2 \cos(\frac{y}{x})^2 - \cos(\frac{y}{x})^2 xy a_2 \\ + \cos(\frac{y}{x})^2 xy b_3 - \cos(\frac{y}{x})^2 y^2 a_3 + \sin(\frac{y}{x})^2 xb_1 - \sin(\frac{y}{x})^2 ya_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\frac{x(x^2 a_2 \sin(\frac{2y}{x}) - x^2 b_3 \sin(\frac{2y}{x}) + 2xy a_3 \sin(\frac{2y}{x}) + x^2 a_3 \cos(\frac{2y}{x}) - xb_1 \cos(\frac{2y}{x}) + ya_1 \cos(\frac{2y}{x}) - x^2 a_3 + 2x^2 b_3)}{2} \\ = 0 \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \cos\left(\frac{2y}{x}\right), \sin\left(\frac{2y}{x}\right) \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \cos\left(\frac{2y}{x}\right) = v_3, \sin\left(\frac{2y}{x}\right) = v_4 \right\}$$

The above PDE (6E) now becomes

$$\frac{v_1(v_1^2 a_2 v_4 + v_1^2 a_3 v_3 + 2v_1 v_2 a_3 v_4 - v_1^2 b_3 v_4 + v_2 a_1 v_3 - 2v_1 v_2 a_2 - v_1^2 a_3 - 2v_2^2 a_3 - v_1 b_1 v_3 + 2v_1^2 b_2 + 2v_1 v_2 b_3 - v_1^2 b_3)}{2} = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} & \left(-\frac{a_3}{2} + b_2\right) v_1^3 + \frac{a_3 v_3 v_1^3}{2} + \left(\frac{a_2}{2} - \frac{b_3}{2}\right) v_4 v_1^3 + \frac{b_1 v_1^2}{2} + (b_3 - a_2) v_2 v_1^2 \\ & - \frac{b_1 v_3 v_1^2}{2} + a_3 v_4 v_2 v_1^2 - a_3 v_2^2 v_1 - \frac{a_1 v_2 v_1}{2} + \frac{a_1 v_2 v_3 v_1}{2} = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 a_3 &= 0 \\
 -\frac{a_1}{2} &= 0 \\
 \frac{a_1}{2} &= 0 \\
 -a_3 &= 0 \\
 \frac{a_3}{2} &= 0 \\
 -\frac{b_1}{2} &= 0 \\
 \frac{b_1}{2} &= 0 \\
 \frac{a_2}{2} - \frac{b_3}{2} &= 0 \\
 -\frac{a_3}{2} + b_2 &= 0 \\
 b_3 - a_2 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= b_3 \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= b_3
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= x \\
 \eta &= y
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= y - \left(\frac{-x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)} \right) (x) \\
 &= \frac{\sin\left(\frac{y}{x}\right) x}{\cos\left(\frac{y}{x}\right)} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sin(\frac{y}{x})x}{\cos(\frac{y}{x})}} dy \end{aligned}$$

Which results in

$$S = \ln \left(\sin \left(\frac{y}{x} \right) \right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x \sin \left(\frac{y}{x} \right) + y \cos \left(\frac{y}{x} \right)}{x \cos \left(\frac{y}{x} \right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y \cot \left(\frac{y}{x} \right)}{x^2} \\ S_y &= \frac{\cot \left(\frac{y}{x} \right)}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\ln(R) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

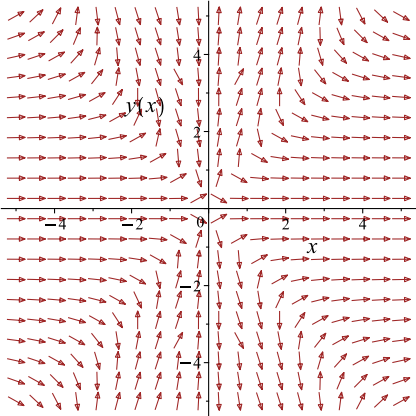
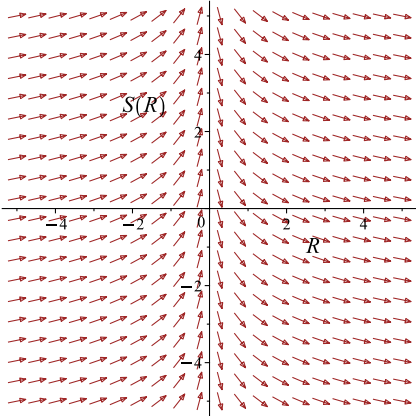
Which simplifies to

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

Which gives

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)}{x \cos\left(\frac{y}{x}\right)}$ 	$R = x$ $S = \ln\left(\sin\left(\frac{y}{x}\right)\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right) \tag{1}$$

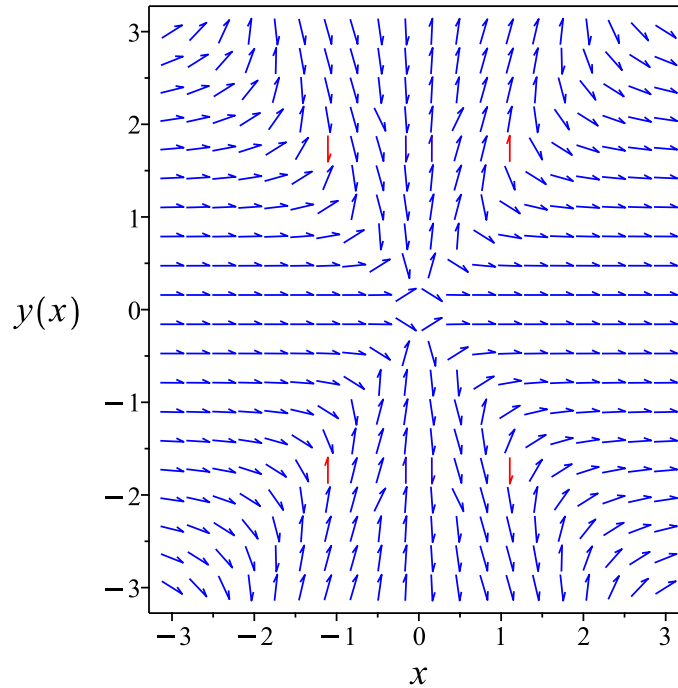


Figure 50: Slope field plot

Verification of solutions

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} & \left(\cos \left(\frac{y}{x} \right) x \right) dy = \left(-x \sin \left(\frac{y}{x} \right) + y \cos \left(\frac{y}{x} \right) \right) dx \\ \left(x \sin \left(\frac{y}{x} \right) - y \cos \left(\frac{y}{x} \right) \right) dx + \left(\cos \left(\frac{y}{x} \right) x \right) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x \sin \left(\frac{y}{x} \right) - y \cos \left(\frac{y}{x} \right) \\ N(x, y) &= \cos \left(\frac{y}{x} \right) x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(x \sin \left(\frac{y}{x} \right) - y \cos \left(\frac{y}{x} \right) \right) \\ &= \frac{y \sin \left(\frac{y}{x} \right)}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\cos \left(\frac{y}{x} \right) x \right) \\ &= \frac{y \sin \left(\frac{y}{x} \right)}{x} + \cos \left(\frac{y}{x} \right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{\sec \left(\frac{y}{x} \right)}{x} \left(\left(\frac{y \sin \left(\frac{y}{x} \right)}{x} \right) - \left(\frac{y \sin \left(\frac{y}{x} \right)}{x} + \cos \left(\frac{y}{x} \right) \right) \right) \\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int -\frac{1}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)} \\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x} \left(x \sin \left(\frac{y}{x} \right) - y \cos \left(\frac{y}{x} \right) \right) \\ &= \frac{x \sin \left(\frac{y}{x} \right) - y \cos \left(\frac{y}{x} \right)}{x}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x} \left(\cos \left(\frac{y}{x} \right) x \right) \\ &= \cos \left(\frac{y}{x} \right)\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x} \right) + \left(\cos\left(\frac{y}{x}\right) \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{x} dx \\ \phi &= x \sin\left(\frac{y}{x}\right) + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos\left(\frac{y}{x}\right) + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos\left(\frac{y}{x}\right)$. Therefore equation (4) becomes

$$\cos\left(\frac{y}{x}\right) = \cos\left(\frac{y}{x}\right) + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x \sin\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x \sin\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$x \sin\left(\frac{y}{x}\right) = c_1 \tag{1}$$

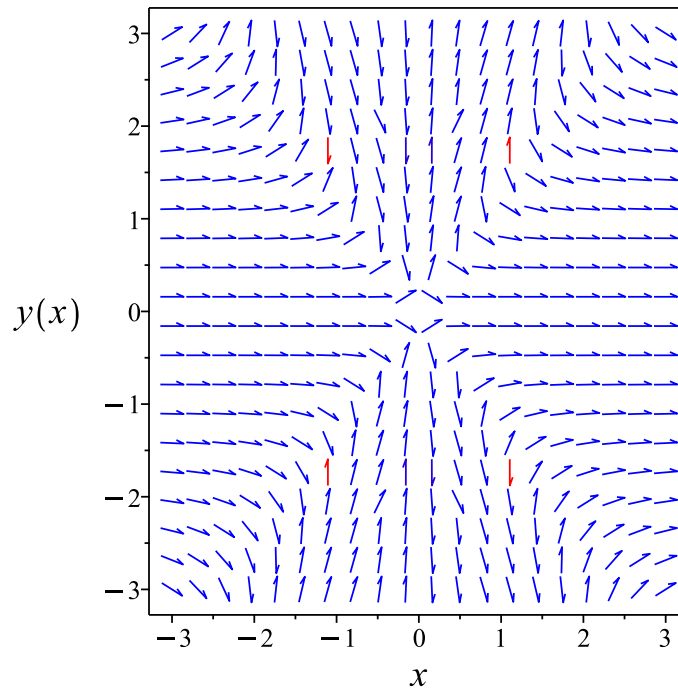


Figure 51: Slope field plot

Verification of solutions

$$x \sin\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous D  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((x*sin(y(x)/x)-y(x)*cos(y(x)/x))+(x*cos(y(x)/x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x \arcsin\left(\frac{1}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 12.962 (sec). Leaf size: 21

```
DSolve[(x*Sin[y[x]/x]-y[x]*Cos[y[x]/x])+(x*Cos[y[x]/x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \rightarrow x \arcsin\left(\frac{e^{c_1}}{x}\right)$$
$$y(x) \rightarrow 0$$

2.7 problem 30

2.7.1 Solving as homogeneousTypeD2 ode 210

Internal problem ID [5242]

Internal file name [OUTPUT/4733_Friday_February_02_2024_05_10_46_AM_84485799/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 30.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational]
```

$$y^2(x^2 + 2) + (x^3 + y^3)(-xy' + y) = 0$$

2.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2(x^2 + 2) + (x^3 + u(x)^3 x^3)(-x(u'(x)x + u(x)) + u(x)x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u^2(x^2 + 2)}{(u^3 + 1)x^3} \end{aligned}$$

Where $f(x) = \frac{x^2+2}{x^3}$ and $g(u) = \frac{u^2}{u^3+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u^3+1}} du = \frac{x^2 + 2}{x^3} dx$$

$$\int \frac{1}{\frac{u^2}{u^3+1}} du = \int \frac{x^2+2}{x^3} dx$$

$$\frac{u^2}{2} - \frac{1}{u} = \ln(x) - \frac{1}{x^2} + c_2$$

The solution is

$$\frac{u(x)^2}{2} - \frac{1}{u(x)} - \ln(x) + \frac{1}{x^2} - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{2x^2} - \frac{x}{y} - \ln(x) + \frac{1}{x^2} - c_2 = 0$$

$$\frac{y^2}{2x^2} - \frac{x}{y} - \ln(x) + \frac{1}{x^2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} - \frac{x}{y} - \ln(x) + \frac{1}{x^2} - c_2 = 0 \tag{1}$$

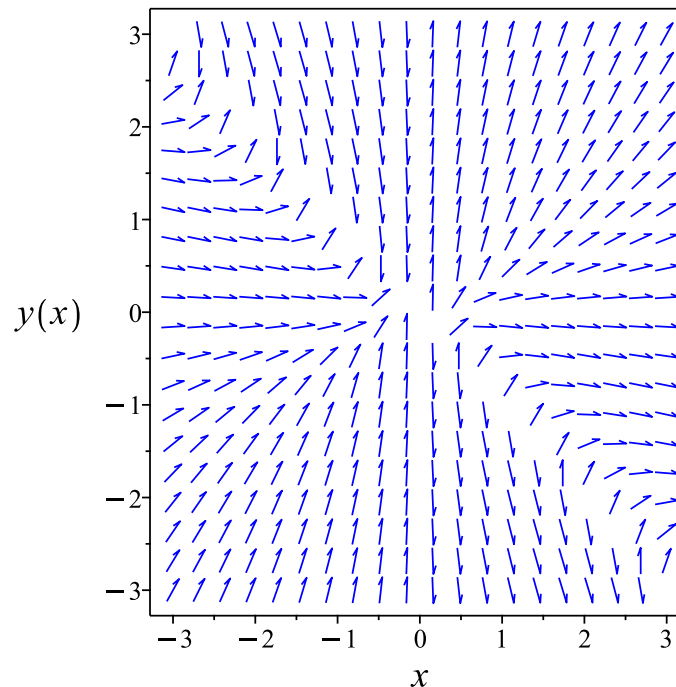


Figure 52: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - \frac{x}{y} - \ln(x) + \frac{1}{x^2} - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous D  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 659

```
dsolve(y(x)^2*(x^2+2)+(x^3+y(x)^3)*(y(x)-x*diff(y(x),x))=0,y(x), singsol=all)
```

$y(x)$

$$= \frac{6 \ln(x) x^2 + 6c_1 x^2 + \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 \right)}{3 \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 \right)}$$

$y(x)$

$$= \frac{(-1-i\sqrt{3}) \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 - 24 \ln(x)^3 x^6 + 72 \ln(x)^2 x^4 - 72 \ln(x) x^2 + 24 \right)}{6 \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 - 24 \ln(x)^3 x^6 + 72 \ln(x)^2 x^4 - 72 \ln(x) x^2 + 24 \right)}$$

$y(x)$

$$= \frac{(i\sqrt{3}-1) \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 - 24 \ln(x)^3 x^6 + 72 \ln(x)^2 x^4 - 72 \ln(x) x^2 + 24 \right)}{6 \left(27x^3 + 3\sqrt{-24c_1^3 x^6 - 72c_1^2 x^6 \ln(x) + 72c_1^2 x^4 - 72c_1 x^6 \ln(x)^2 + 144c_1 x^4 \ln(x)} - 72c_1 x^2 - 24 \ln(x)^3 x^6 + 72 \ln(x)^2 x^4 - 72 \ln(x) x^2 + 24 \right)}$$

✓ Solution by Mathematica

Time used: 54.35 (sec). Leaf size: 396

`DSolve[y[x]^2*(x^2+2)+(x^3+y[x]^3)*(y[x]-x*y'[x])==0,y[x],x,IncludeSingularSolutions -> True`

$y(x)$

$$\rightarrow \frac{6x^2 \log(x) + 6c_1x^2 + 3^{2/3} \left(9x^3 + \frac{1}{3} \sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3} \right)^{2/3} - 6}{3\sqrt[3]{3} \sqrt[3]{9x^3 + \frac{1}{3} \sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3}}}$$

$$y(x) \rightarrow \frac{i(\sqrt{3} + i) \sqrt[3]{9x^3 + \frac{1}{3} \sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3}}}{2 \cdot 3^{2/3}} - \frac{i\sqrt[3]{2}(\sqrt{3} - i) (x^2 \log(x) + c_1x^2 - 1)}{\sqrt[3]{54x^3 + 2\sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3}}}$$

$$y(x) \rightarrow \frac{i\sqrt[3]{2}(\sqrt{3} + i) (x^2 \log(x) + c_1x^2 - 1)}{\sqrt[3]{54x^3 + 2\sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3}}} - \frac{(1 + i\sqrt{3}) \sqrt[3]{54x^3 + 2\sqrt{729x^6 + (-6x^2 \log(x) - 6c_1x^2 + 6)^3}}}{6\sqrt[3]{2}}$$

2.8 problem 31

2.8.1 Solving as first order ode lie symmetry calculated ode	214
2.8.2 Solving as exact ode	220

Internal problem ID [5243]

Internal file name [OUTPUT/4734_Friday_February_02_2024_05_10_48_AM_20890870/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 31.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exactByInspection", "first_order_ode_lie_symmetry_calculated"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _dAlembert]
```

$$\sqrt{x^2 + y^2}y - x(x + \sqrt{x^2 + y^2})y' = 0$$

2.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{\sqrt{x^2 + y^2}y}{x(x + \sqrt{x^2 + y^2})}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{\sqrt{x^2 + y^2} y(b_3 - a_2)}{x(x + \sqrt{x^2 + y^2})} - \frac{(x^2 + y^2) y^2 a_3}{x^2 (x + \sqrt{x^2 + y^2})^2} \\ - \left(\frac{y}{\sqrt{x^2 + y^2} (x + \sqrt{x^2 + y^2})} - \frac{\sqrt{x^2 + y^2} y}{x^2 (x + \sqrt{x^2 + y^2})} \right. \\ \left. - \frac{\sqrt{x^2 + y^2} y \left(1 + \frac{x}{\sqrt{x^2 + y^2}}\right)}{x (x + \sqrt{x^2 + y^2})^2} \right) (x a_2 + y a_3 + a_1) \\ - \left(\frac{y^2}{\sqrt{x^2 + y^2} x (x + \sqrt{x^2 + y^2})} + \frac{\sqrt{x^2 + y^2}}{x (x + \sqrt{x^2 + y^2})} \right. \\ \left. - \frac{y^2}{x (x + \sqrt{x^2 + y^2})^2} \right) (x b_2 + y b_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{-(x^2 + y^2)^{\frac{3}{2}} y^2 a_3 - \sqrt{x^2 + y^2} x^4 b_2 + y^2 a_3 \sqrt{x^2 + y^2} x^2 + y^4 a_3 \sqrt{x^2 + y^2} - x^5 b_2 - x^3 y^2 a_3 - x^2 y^3 a_2 + x^2 y^3}{\sqrt{x^2 + y^2} (x + \sqrt{x^2 + y^2})} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} (x^2 + y^2)^{\frac{3}{2}} y^2 a_3 + \sqrt{x^2 + y^2} x^4 b_2 - y^2 a_3 \sqrt{x^2 + y^2} x^2 - y^4 a_3 \sqrt{x^2 + y^2} \\ + x^5 b_2 + x^3 y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 + 2x y^4 a_3 - (x^2 + y^2)^{\frac{3}{2}} x b_1 \\ + (x^2 + y^2)^{\frac{3}{2}} y a_1 - x^4 b_1 + x^3 y a_1 - 2x^2 y^2 b_1 + 2x y^3 a_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} (x^2 + y^2)^{\frac{3}{2}} y^2 a_3 + (x^2 + y^2) x^3 b_2 + (x^2 + y^2) x^2 y a_2 + 2(x^2 + y^2) x y^2 a_3 \\ + \sqrt{x^2 + y^2} x^4 b_2 - y^2 a_3 \sqrt{x^2 + y^2} x^2 - y^4 a_3 \sqrt{x^2 + y^2} - x^4 y a_2 \\ - x^3 y^2 a_3 - x^3 y^2 b_2 - x^2 y^3 b_3 - (x^2 + y^2)^{\frac{3}{2}} x b_1 + (x^2 + y^2)^{\frac{3}{2}} y a_1 \\ - (x^2 + y^2) x^2 b_1 + 2(x^2 + y^2) x y a_1 - x^3 y a_1 - x^2 y^2 b_1 = 0 \end{aligned} \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$x^5 b_2 + \sqrt{x^2 + y^2} x^4 b_2 + x^3 y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 + 2x y^4 a_3 - x^4 b_1 - \sqrt{x^2 + y^2} x^3 b_1 \\ + x^3 y a_1 + \sqrt{x^2 + y^2} y a_1 x^2 - 2x^2 y^2 b_1 - \sqrt{x^2 + y^2} x b_1 y^2 + 2x y^3 a_1 + \sqrt{x^2 + y^2} y^3 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{x^2 + y^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\}$$

The above PDE (6E) now becomes

$$v_1^2 v_2^3 a_2 + v_1^3 v_2^2 a_3 + 2v_1 v_2^4 a_3 + v_1^5 b_2 + v_3 v_1^4 b_2 - v_1^2 v_2^3 b_3 + v_1^3 v_2 a_1 + v_3 v_2 a_1 v_1^2 \quad (7E) \\ + 2v_1 v_2^3 a_1 + v_3 v_2^3 a_1 - v_1^4 b_1 - v_3 v_1^3 b_1 - 2v_1^2 v_2^2 b_1 - v_3 v_1 b_1 v_2^2 = 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_1^5 b_2 + v_3 v_1^4 b_2 - v_1^4 b_1 + v_1^3 v_2^2 a_3 + v_1^3 v_2 a_1 - v_3 v_1^3 b_1 + (-b_3 + a_2) v_1^2 v_2^3 \quad (8E) \\ - 2v_1^2 v_2^2 b_1 + v_3 v_2 a_1 v_1^2 + 2v_1 v_2^4 a_3 + 2v_1 v_2^3 a_1 - v_3 v_1 b_1 v_2^2 + v_3 v_2^3 a_1 = 0$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$a_1 = 0 \\ a_3 = 0 \\ b_2 = 0 \\ 2a_1 = 0 \\ 2a_3 = 0 \\ -2b_1 = 0 \\ -b_1 = 0 \\ -b_3 + a_2 = 0$$

Solving the above equations for the unknowns gives

$$a_1 = 0$$

$$a_2 = b_3$$

$$a_3 = 0$$

$$b_1 = 0$$

$$b_2 = 0$$

$$b_3 = b_3$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$

$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{\sqrt{x^2 + y^2} y}{x(x + \sqrt{x^2 + y^2})} \right) (x) \\ &= \frac{yx}{x + \sqrt{x^2 + y^2}} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{yx}{x + \sqrt{x^2 + y^2}}} dy \end{aligned}$$

Which results in

$$S = \frac{\sqrt{x^2 + y^2}}{x} - \frac{x \ln \left(\frac{2x^2 + 2\sqrt{x^2} \sqrt{x^2 + y^2}}{y} \right)}{\sqrt{x^2}} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\sqrt{x^2 + y^2} y}{x (x + \sqrt{x^2 + y^2})}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-x - \sqrt{x^2 + y^2}}{x^2} \\ S_y &= \frac{2x\sqrt{x^2 + y^2} + 2x^2 + y^2}{y (x + \sqrt{x^2 + y^2}) x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-2x\sqrt{x^2 + y^2} - 2x^2 - y^2}{x (x + \sqrt{x^2 + y^2})^2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\ln(R) + c_1 \quad (4)$$

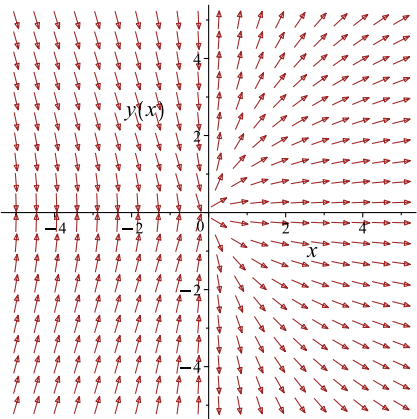
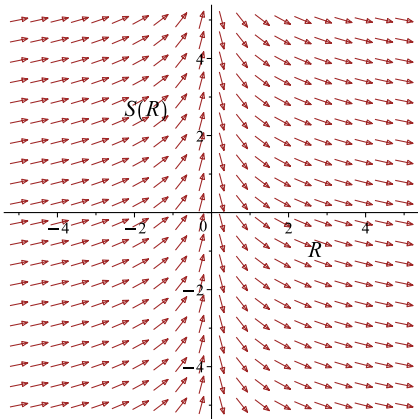
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-\ln(2)x + 2x \ln(y) - \ln(x)x - \ln(x + \sqrt{x^2 + y^2})x + \sqrt{x^2 + y^2}}{x} = -\ln(x) + c_1$$

Which simplifies to

$$\frac{-\ln(2)x + 2x \ln(y) - \ln(x + \sqrt{x^2 + y^2})x - c_1x + \sqrt{x^2 + y^2}}{x} = 0$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} y}{x(x + \sqrt{x^2 + y^2})}$ 	$R = x$ $S = \frac{-\ln(2)x + 2 \ln(y)x}{x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{-\ln(2)x + 2x \ln(y) - \ln(x + \sqrt{x^2 + y^2})x - c_1x + \sqrt{x^2 + y^2}}{x} = 0 \quad (1)$$

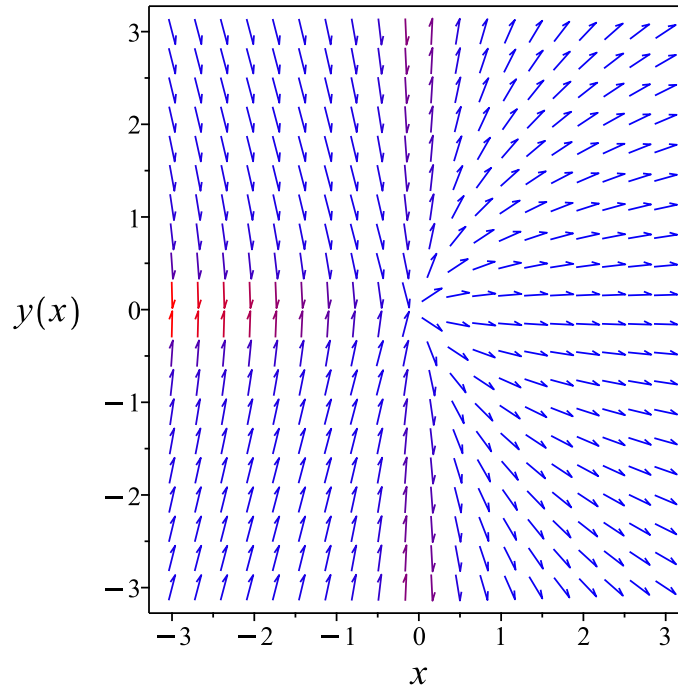


Figure 53: Slope field plot

Verification of solutions

$$\frac{-\ln(2)x + 2x \ln(y) - \ln(x + \sqrt{x^2 + y^2})x - c_1x + \sqrt{x^2 + y^2}}{x} = 0$$

Verified OK.

2.8.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x(x + \sqrt{x^2 + y^2})) dy &= (-\sqrt{x^2 + y^2} y) dx \\ (\sqrt{x^2 + y^2} y) dx + (-x(x + \sqrt{x^2 + y^2})) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \sqrt{x^2 + y^2} y \\ N(x, y) &= -x(x + \sqrt{x^2 + y^2})\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (\sqrt{x^2 + y^2} y) \\ &= \frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-x \left(x + \sqrt{x^2 + y^2} \right) \right) \\ &= - \frac{2 \left(x^2 + x \sqrt{x^2 + y^2} + \frac{y^2}{2} \right)}{\sqrt{x^2 + y^2}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = \sqrt{x^2 + y^2} y$ and $N = -x(x + \sqrt{x^2 + y^2})$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{\sqrt{x^2 + y^2}}{x^2} \\ N &= - \frac{x + \sqrt{x^2 + y^2}}{xy}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} \left(-\frac{x + \sqrt{x^2 + y^2}}{xy} \right) dy &= \left(-\frac{\sqrt{x^2 + y^2}}{x^2} \right) dx \\ \left(\frac{\sqrt{x^2 + y^2}}{x^2} \right) dx + \left(-\frac{x + \sqrt{x^2 + y^2}}{xy} \right) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{\sqrt{x^2 + y^2}}{x^2} \\ N(x, y) &= -\frac{x + \sqrt{x^2 + y^2}}{xy} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{\sqrt{x^2 + y^2}}{x^2} \right) \\ &= \frac{y}{x^2 \sqrt{x^2 + y^2}} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{x + \sqrt{x^2 + y^2}}{xy} \right) \\ &= \frac{y}{x^2 \sqrt{x^2 + y^2}} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{\sqrt{x^2 + y^2}}{x^2} dx \\ \phi &= \frac{\ln(x + \sqrt{x^2 + y^2}) x - \sqrt{x^2 + y^2}}{x} + f(y)\end{aligned}\quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{\frac{yx}{\sqrt{x^2 + y^2}(x + \sqrt{x^2 + y^2})} - \frac{y}{\sqrt{x^2 + y^2}}}{x} + f'(y) \\ &= -\frac{y}{x(x + \sqrt{x^2 + y^2})} + f'(y)\end{aligned}\quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x + \sqrt{x^2 + y^2}}{xy}$. Therefore equation (4) becomes

$$-\frac{x + \sqrt{x^2 + y^2}}{xy} = -\frac{y}{x(x + \sqrt{x^2 + y^2})} + f'(y)\quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{2}{y}\right) dy \\ f(y) &= -2 \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x + \sqrt{x^2 + y^2}) x - \sqrt{x^2 + y^2}}{x} - 2 \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x + \sqrt{x^2 + y^2}) x - \sqrt{x^2 + y^2}}{x} - 2 \ln(y)$$

Summary

The solution(s) found are the following

$$\frac{\ln(x + \sqrt{x^2 + y^2}) x - \sqrt{x^2 + y^2}}{x} - 2 \ln(y) = c_1 \quad (1)$$

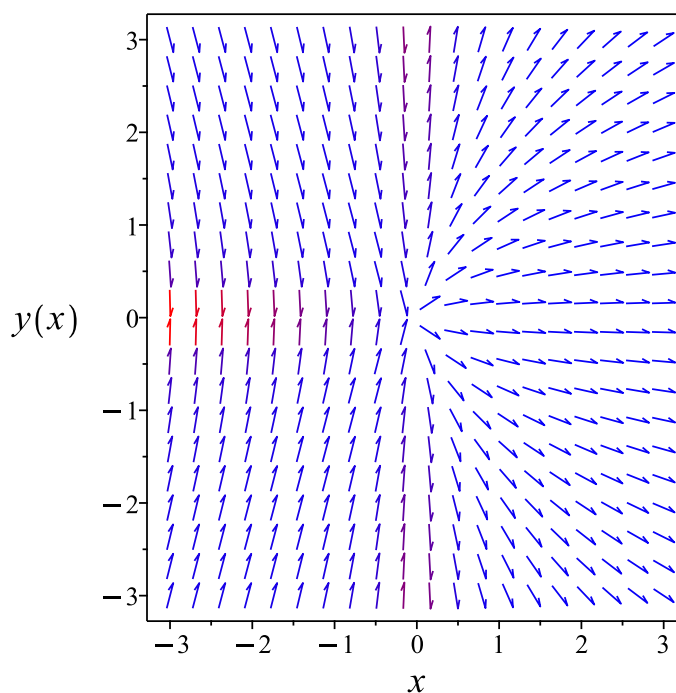


Figure 54: Slope field plot

Verification of solutions

$$\frac{\ln(x + \sqrt{x^2 + y^2}) x - \sqrt{x^2 + y^2}}{x} - 2 \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying homogeneous types:  
trying homogeneous G  
1st order, trying the canonical coordinates of the invariance group  
<- 1st order, canonical coordinates successful  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 60

```
dsolve(y(x)*sqrt(x^2+y(x)^2)-x*(x+sqrt(x^2+y(x)^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x \ln(2) + x \ln\left(\frac{x(x + \sqrt{x^2 + y(x)^2})}{y(x)}\right) - \ln(y(x))x - \ln(x)x - c_1x - \sqrt{x^2 + y(x)^2}}{x} = 0$$

✓ Solution by Mathematica

Time used: 0.319 (sec). Leaf size: 43

```
DSolve[y[x]*Sqrt[x^2+y[x]^2]-x*(x+Sqrt[x^2+y[x]^2])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[\sqrt{\frac{y(x)^2}{x^2} + 1} + \log\left(\sqrt{\frac{y(x)^2}{x^2} + 1} - 1\right) = -\log(x) + c_1, y(x)\right]$$

2.9 problem 32

2.9.1 Solving as first order ode lie symmetry calculated ode 227

Internal problem ID [5244]

Internal file name [OUTPUT/4735_Friday_February_02_2024_05_10_50_AM_13746180/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 32.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `class A`]]
```

$$y + (2x + 2y + 1)y' = -1 - x$$

2.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x + y + 1}{2x + 2y + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(x+y+1)(b_3 - a_2)}{2x+2y+1} - \frac{(x+y+1)^2 a_3}{(2x+2y+1)^2} \\ - \left(-\frac{1}{2x+2y+1} + \frac{2x+2y+2}{(2x+2y+1)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{1}{2x+2y+1} + \frac{2x+2y+2}{(2x+2y+1)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 + 2xa_2 - 2xa_3 + 3xb_2 - 3xb_3 + 3ya_2 - 3ya_3 + 4yb_2 - 4yb_3 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3}{(2x+2y+1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} 2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 \\ + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 + 2xa_2 - 2xa_3 + 3xb_2 - 3xb_3 \\ + 3ya_2 - 3ya_3 + 4yb_2 - 4yb_3 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} 2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 \\ + 4b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 + 2a_2v_1 + 3a_2v_2 - 2a_3v_1 - 3a_3v_2 \\ + 3b_2v_1 + 4b_2v_2 - 3b_3v_1 - 4b_3v_2 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (2a_2 - a_3 + 4b_2 - 2b_3)v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3)v_1v_2 \\ & + (2a_2 - 2a_3 + 3b_2 - 3b_3)v_1 + (2a_2 - a_3 + 4b_2 - 2b_3)v_2^2 \\ & + (3a_2 - 3a_3 + 4b_2 - 4b_3)v_2 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_2 - 2a_3 + 3b_2 - 3b_3 &= 0 \\ 2a_2 - a_3 + 4b_2 - 2b_3 &= 0 \\ 3a_2 - 3a_3 + 4b_2 - 4b_3 &= 0 \\ 4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0 \\ -a_1 + a_2 - a_3 - b_1 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_1 \\ a_2 &= -2b_3 \\ a_3 &= -2b_3 \\ b_1 &= b_1 \\ b_2 &= b_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -1 \\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y) \xi \\ &= 1 - \left(-\frac{x + y + 1}{2x + 2y + 1} \right) (-1) \\ &= \frac{x + y}{2x + 2y + 1} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+y}{2x+2y+1}} dy \end{aligned}$$

Which results in

$$S = 2y + \ln(x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y + 1}{2x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y} \\ S_y &= 2 + \frac{1}{x + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -R + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + \ln(x + y) = -x + c_1$$

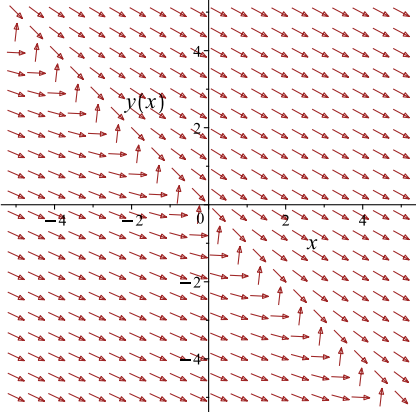
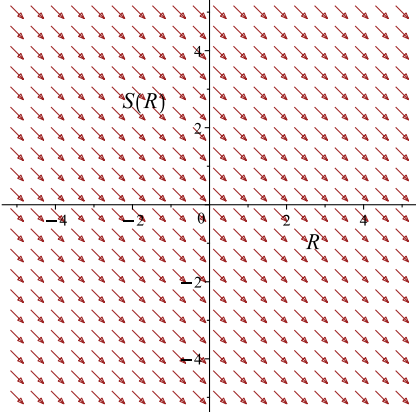
Which simplifies to

$$2y + \ln(x + y) = -x + c_1$$

Which gives

$$y = \frac{\text{LambertW}(2e^{x+c_1})}{2} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+1}{2x+2y+1}$ 	$R = x$ $S = 2y + \ln(x + y)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}(2e^{x+c_1})}{2} - x \tag{1}$$

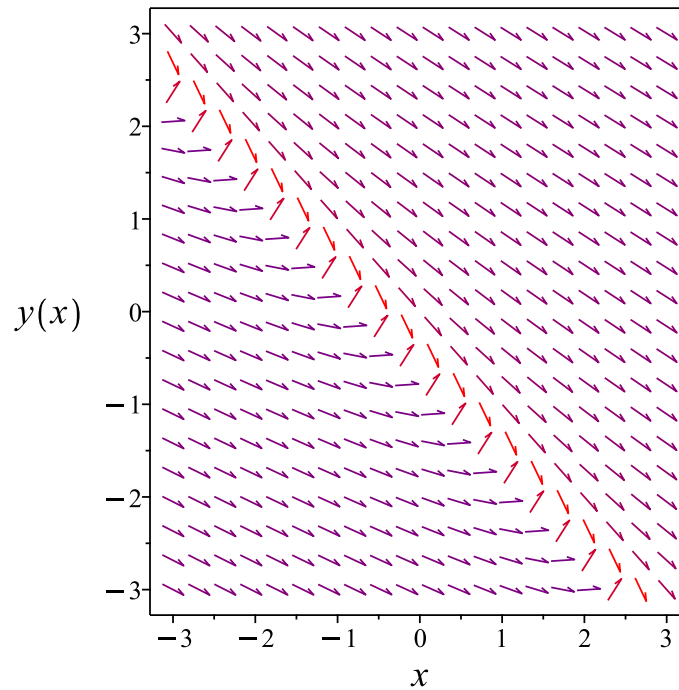


Figure 55: Slope field plot

Verification of solutions

$$y = \frac{\text{LambertW}(2e^{x+c_1})}{2} - x$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((x+y(x)+1)+(2*x+2*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\text{LambertW}(2e^{x-c_1})}{2} - x$$

✓ Solution by Mathematica

Time used: 4.251 (sec). Leaf size: 30

```
DSolve[(x+y[x]+1)+(2*x+2*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}(-2x + W(-e^{x-1+c_1}))$$
$$y(x) \rightarrow -x$$

2.10 problem 34

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Internal problem ID [5245]

Internal file name [OUTPUT/4736_Friday_February_02_2024_05_10_51_AM_38539046/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 34.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$2y - y'(4 - x) = -1$$

2.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{-2y - 1}{-4 + x}\end{aligned}$$

Where $f(x) = \frac{1}{-4+x}$ and $g(y) = -2y - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-2y-1} dy &= \frac{1}{-4+x} dx \\ \int \frac{1}{-2y-1} dy &= \int \frac{1}{-4+x} dx \\ -\frac{\ln(-2y-1)}{2} &= \ln(-4+x) + c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y-1}} = e^{\ln(-4+x)+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y-1}} = c_2(-4+x)$$

Summary

The solution(s) found are the following

$$y = -\frac{(c_2^2 x^2 e^{2c_1} - 8 e^{2c_1} c_2^2 x + 16 e^{2c_1} c_2^2 + 1) e^{-2c_1}}{2c_2^2 (-4+x)^2} \quad (1)$$

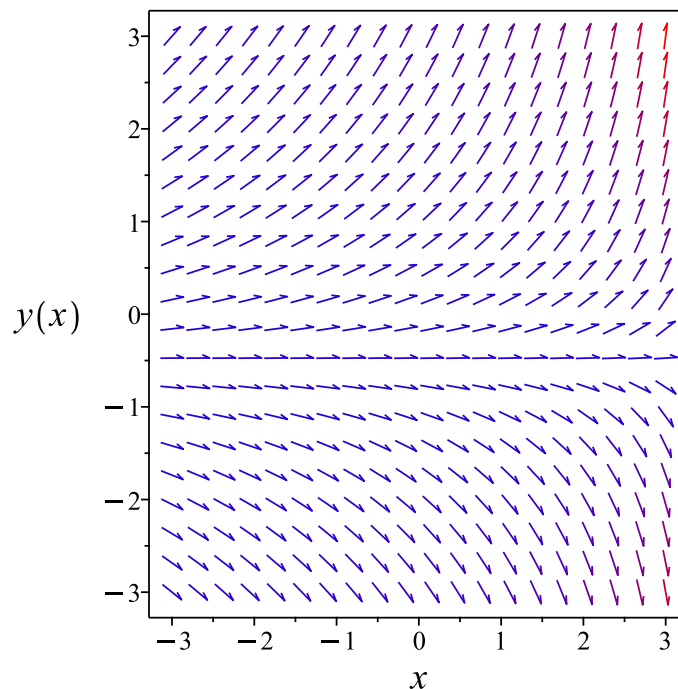


Figure 56: Slope field plot

Verification of solutions

$$y = -\frac{(c_2^2 x^2 e^{2c_1} - 8 e^{2c_1} c_2^2 x + 16 e^{2c_1} c_2^2 + 1) e^{-2c_1}}{2c_2^2 (-4 + x)^2}$$

Verified OK.

2.10.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{-4 + x}$$
$$q(x) = -\frac{1}{-4 + x}$$

Hence the ode is

$$y' + \frac{2y}{-4 + x} = -\frac{1}{-4 + x}$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{-4+x} dx}$$
$$= (-4 + x)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(-\frac{1}{-4 + x} \right)$$
$$\frac{d}{dx}((-4 + x)^2 y) = ((-4 + x)^2) \left(-\frac{1}{-4 + x} \right)$$
$$d((-4 + x)^2 y) = (4 - x) dx$$

Integrating gives

$$(-4 + x)^2 y = \int 4 - x dx$$
$$(-4 + x)^2 y = 4x - \frac{1}{2}x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = (-4 + x)^2$ results in

$$y = \frac{4x - \frac{1}{2}x^2}{(-4 + x)^2} + \frac{c_1}{(-4 + x)^2}$$

which simplifies to

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2} \tag{1}$$

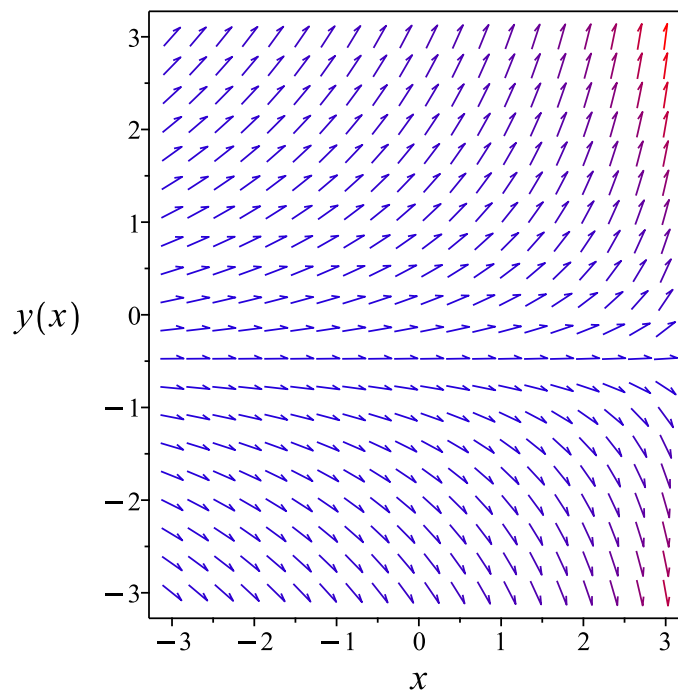


Figure 57: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2}$$

Verified OK.

2.10.3 Solving as homogeneous Type MapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{1 + 2Y(X) + 2y_0}{-4 + X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 4 \\y_0 &= -\frac{1}{2}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X, Y) \\ &= -\frac{2Y}{X}\end{aligned}\tag{1}$$

An ode of the form $Y' = \frac{M(X, Y)}{N(X, Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dX}X + u &= -2u \\ \frac{du}{dX} &= -\frac{3u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) + \frac{3u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{3u}{X}\end{aligned}$$

Where $f(X) = -\frac{3}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{3}{X} dX \\ \int \frac{1}{u} du &= \int -\frac{3}{X} dX \\ \ln(u) &= -3 \ln(X) + c_2 \\ u &= e^{-3 \ln(X) + c_2} \\ &= \frac{c_2}{X^3}\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{c_2}{X^2}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{c_2}{X^2}$$

And replacing back terms in the above solution using

$$\begin{aligned}Y &= y + y_0 \\ X &= x + x_0\end{aligned}$$

Or

$$\begin{aligned}Y &= y - \frac{1}{2} \\ X &= x + 4\end{aligned}$$

Then the solution in y becomes

$$\frac{1}{2} + y = \frac{c_2}{(-4 + x)^2}$$

Summary

The solution(s) found are the following

$$\frac{1}{2} + y = \frac{c_2}{(-4 + x)^2} \quad (1)$$

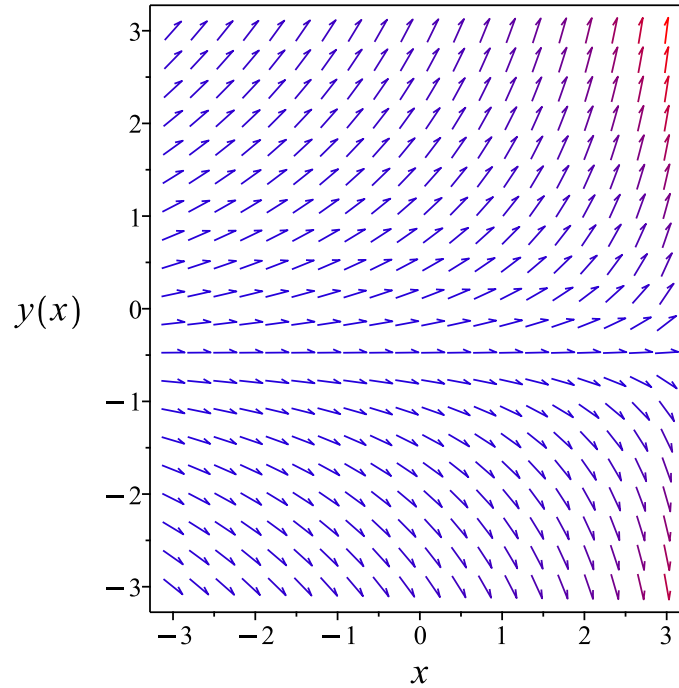


Figure 58: Slope field plot

Verification of solutions

$$\frac{1}{2} + y = \frac{c_2}{(-4 + x)^2}$$

Verified OK.

2.10.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{1 + 2y}{-4 + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{(-4+x)^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(-4+x)^2}} dy \end{aligned}$$

Which results in

$$S = (-4 + x)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{1 + 2y}{-4 + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2(-4 + x) y \\ S_y &= (-4 + x)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4 - x \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4 - R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{2}R^2 + 4R + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(-4 + x)^2 y = -\frac{1}{2}x^2 + 4x + c_1$$

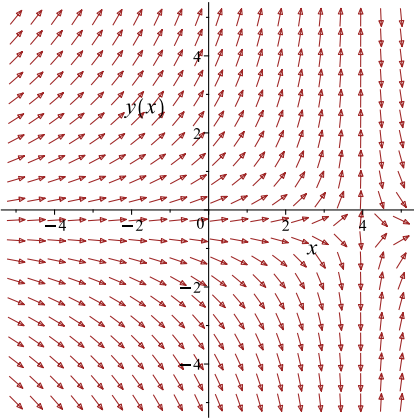
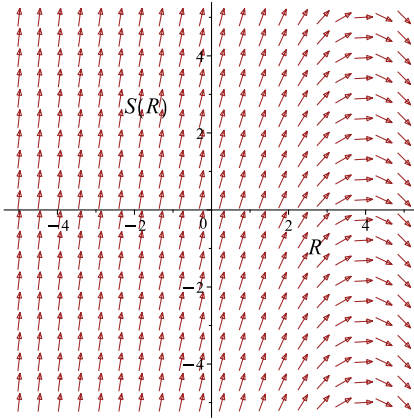
Which simplifies to

$$(-4 + x)^2 y = -\frac{1}{2}x^2 + 4x + c_1$$

Which gives

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{1+2y}{-4+x}$ 	$R = x$ $S = (-4 + x)^2 y$	$\frac{dS}{dR} = 4 - R$ 

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2} \quad (1)$$

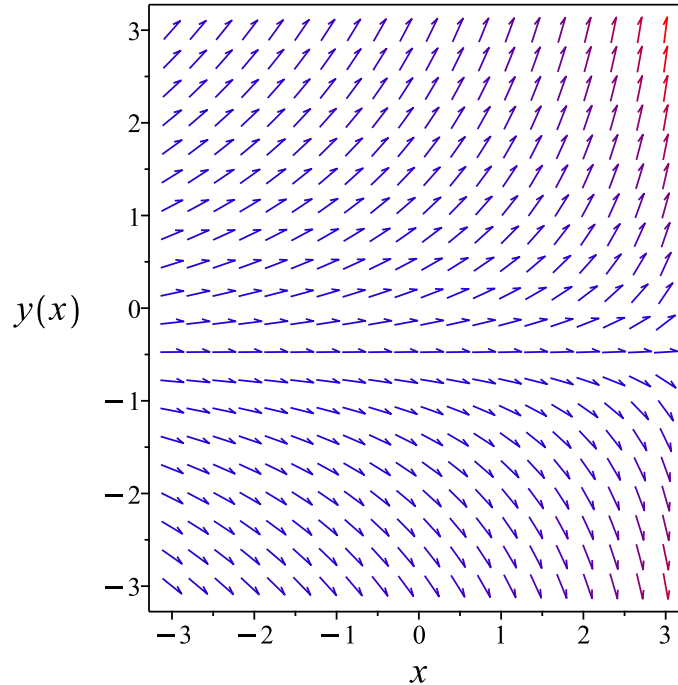


Figure 59: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1 + 8x}{2(-4 + x)^2}$$

Verified OK.

2.10.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (A)$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{1}{-2y-1}\right) dy &= \left(\frac{1}{-4+x}\right) dx \\ \left(-\frac{1}{-4+x}\right) dx + \left(\frac{1}{-2y-1}\right) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\frac{1}{-4+x} \\ N(x, y) &= \frac{1}{-2y-1} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{-4+x} \right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{-2y-1} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{-4+x} dx \\ \phi &= -\ln(-4+x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-2y-1}$. Therefore equation (4) becomes

$$\frac{1}{-2y-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{1+2y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{1}{1+2y} \right) dy$$
$$f(y) = -\frac{\ln(1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(-4+x) - \frac{\ln(1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(-4+x) - \frac{\ln(1+2y)}{2}$$

The solution becomes

$$y = \frac{-x^2 + e^{-2c_1} + 8x - 16}{2x^2 - 16x + 32}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + e^{-2c_1} + 8x - 16}{2x^2 - 16x + 32} \quad (1)$$

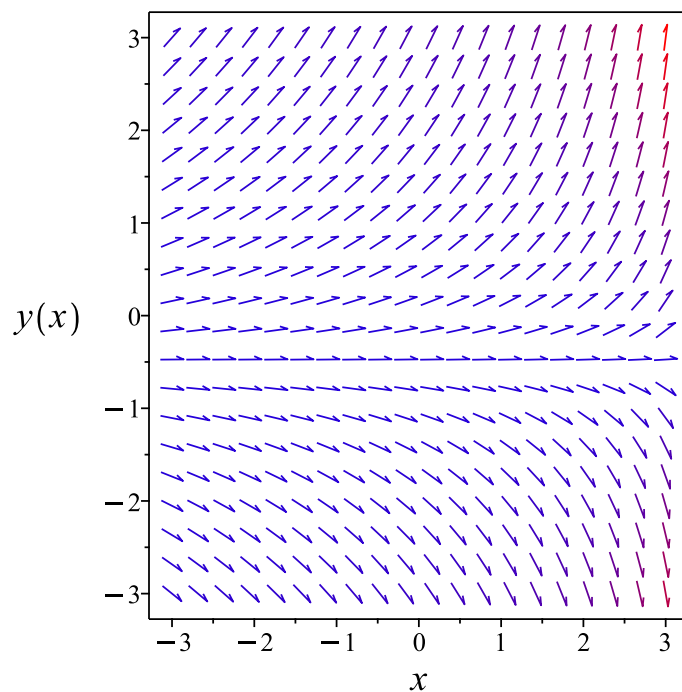


Figure 60: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + e^{-2c_1} + 8x - 16}{2x^2 - 16x + 32}$$

Verified OK.

2.10.6 Maple step by step solution

Let's solve

$$2y - y'(4 - x) = -1$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'}{-2y-1} = -\frac{1}{4-x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-2y-1} dx = \int -\frac{1}{4-x} dx + c_1$$

- Evaluate integral

$$-\frac{\ln(-2y-1)}{2} = \ln(4-x) + c_1$$

- Solve for y

$$y = -\frac{x^2 + e^{-2c_1} - 8x + 16}{2(x^2 - 8x + 16)}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((1+2*y(x))-(4-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{4x - \frac{1}{2}x^2 + c_1}{(x-4)^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 34

```
DSolve[(1+2*y[x])-(4-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{-x^2 + 8x + 2c_1}{2(x-4)^2}$$

$$y(x) \rightarrow -\frac{1}{2}$$

2.11 problem 35

2.11.1 Solving as separable ode	251
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Internal problem ID [5246]

Internal file name [OUTPUT/4737_Friday_February_02_2024_05_10_52_AM_63507069/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 35.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$(x^2 + 1) y' + xy = 0$$

2.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{xy}{x^2 + 1} \end{aligned}$$

Where $f(x) = -\frac{x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= -\frac{x}{x^2+1} dx \\ \int \frac{1}{y} dy &= \int -\frac{x}{x^2+1} dx \\ \ln(y) &= -\frac{\ln(x^2+1)}{2} + c_1 \\ y &= e^{-\frac{\ln(x^2+1)}{2} + c_1} \\ &= \frac{c_1}{\sqrt{x^2+1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

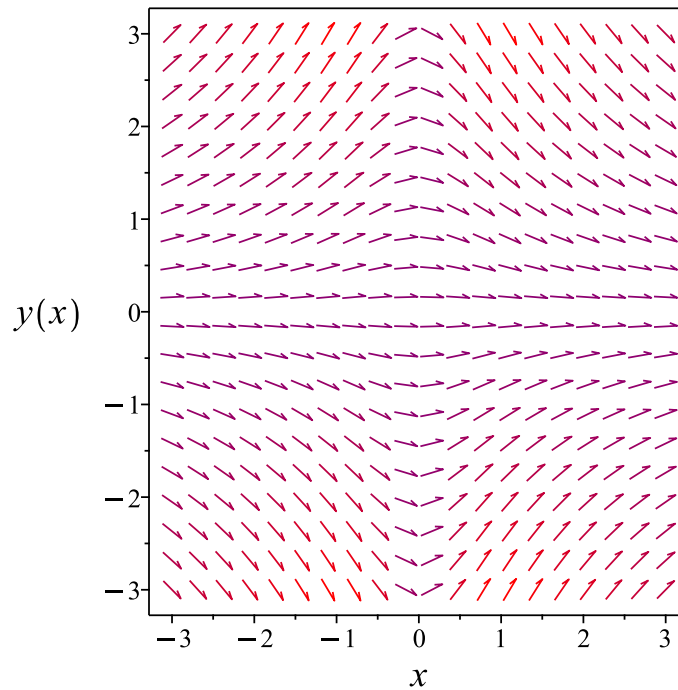


Figure 61: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

2.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x}{x^2 + 1}$$
$$q(x) = 0$$

Hence the ode is

$$y' + \frac{xy}{x^2 + 1} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{x}{x^2+1} dx}$$
$$= \sqrt{x^2 + 1}$$

The ode becomes

$$\frac{d}{dx} \mu y = 0$$
$$\frac{d}{dx} (\sqrt{x^2 + 1} y) = 0$$

Integrating gives

$$\sqrt{x^2 + 1} y = c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2 + 1}$ results in

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2 + 1}} \tag{1}$$

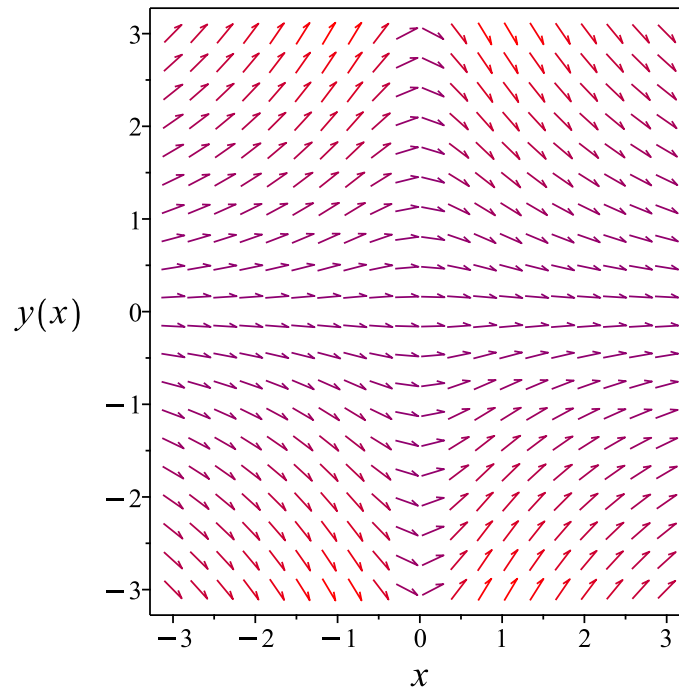


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

2.11.3 Solving as homogeneous TypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$(x^2 + 1)(u'(x)x + u(x)) + x^2u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(2x^2 + 1)}{x(x^2 + 1)} \end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{2x^2+1}{x(x^2+1)} dx \\ \int \frac{1}{u} du &= \int -\frac{2x^2+1}{x(x^2+1)} dx \\ \ln(u) &= -\frac{\ln(x^2+1)}{2} - \ln(x) + c_2 \\ u &= e^{-\frac{\ln(x^2+1)}{2} - \ln(x) + c_2} \\ &= c_2 e^{-\frac{\ln(x^2+1)}{2} - \ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{\sqrt{x^2+1}x}$$

Therefore the solution y is

$$\begin{aligned}y &= ux \\ &= \frac{c_2}{\sqrt{x^2+1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2}{\sqrt{x^2+1}} \tag{1}$$

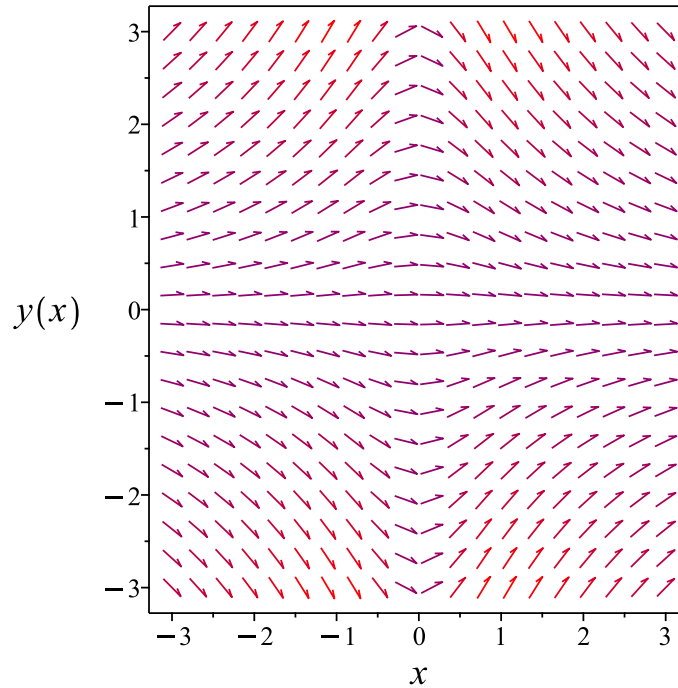


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{c_2}{\sqrt{x^2 + 1}}$$

Verified OK.

2.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy}{x^2 + 1}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{\sqrt{x^2 + 1}}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x^2 + 1} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2 + 1}} \\ S_y &= \sqrt{x^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2 + 1} y = c_1$$

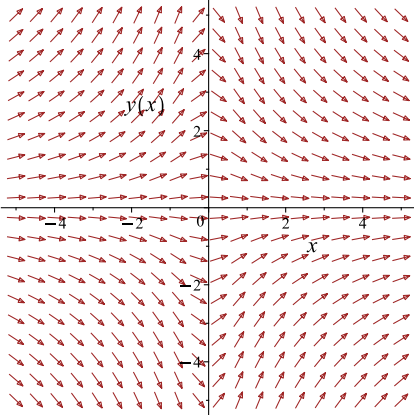
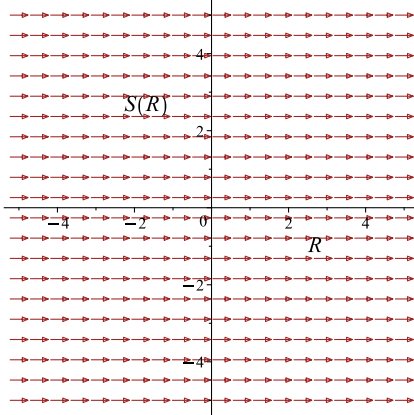
Which simplifies to

$$\sqrt{x^2 + 1} y = c_1$$

Which gives

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy}{x^2+1}$ 	$R = x$ $S = \sqrt{x^2 + 1} y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2 + 1}} \tag{1}$$

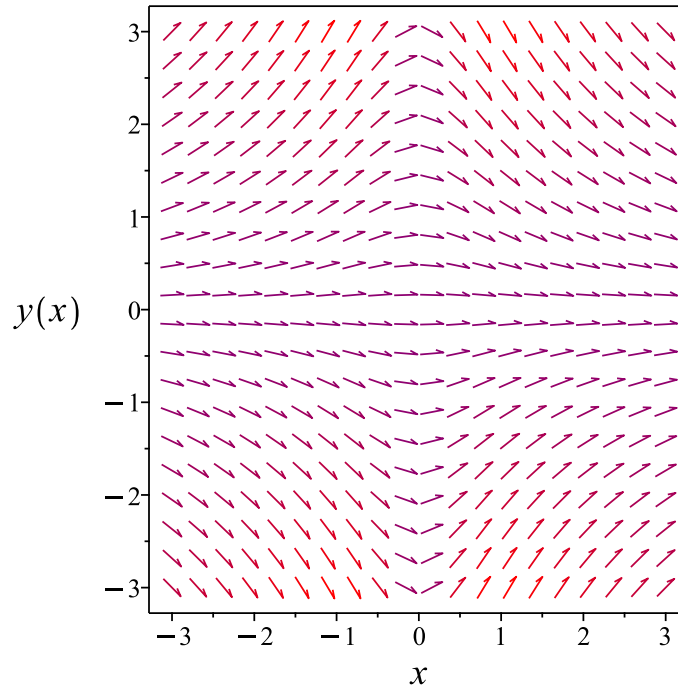


Figure 64: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

2.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right) dy &= \left(\frac{x}{x^2+1}\right) dx \\ \left(-\frac{x}{x^2+1}\right) dx + \left(-\frac{1}{y}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x}{x^2+1} \\ N(x, y) &= -\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{x}{x^2+1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{1}{y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{x}{x^2 + 1} dx \\ \phi &= -\frac{\ln(x^2 + 1)}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{1}{y} \right) dy \\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2 + 1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2 + 1)}{2} - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\ln(x^2+1)}{2} - c_1} \tag{1}$$

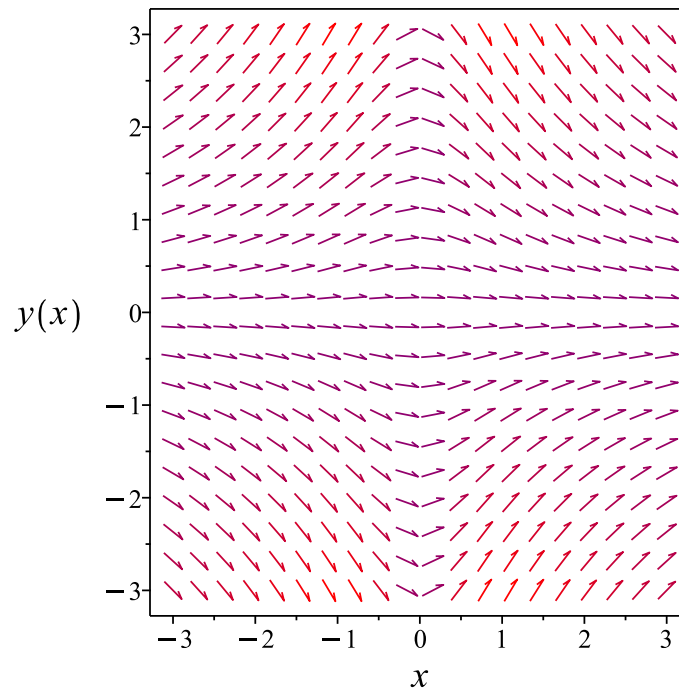


Figure 65: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln(x^2+1)}{2}-c_1}$$

Verified OK.

2.11.6 Maple step by step solution

Let's solve

$$(x^2 + 1)y' + xy = 0$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'}{y} = -\frac{x}{x^2+1}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int -\frac{x}{x^2+1} dx + c_1$$

- Evaluate integral

$$\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$$

- Solve for y

$$y = e^{-\frac{\ln(x^2+1)}{2}+c_1}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve((x*y(x))+(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2 + 1}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 22

```
DSolve[(x*y[x])+(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_1}{\sqrt{x^2 + 1}}$$
$$y(x) \rightarrow 0$$

2.12 problem 37

Internal problem ID [5247]

Internal file name [OUTPUT/4738_Friday_February_02_2024_05_10_53_AM_14454408/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 37.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$2xy + (3y + 2x)y' = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

X Solution by Maple

```
dsolve((x*2*y(x))+(2*x+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

X Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x*2*y[x])+(2*x+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.13 problem 38

2.13.1 Solving as first order ode lie symmetry calculated ode 269

Internal problem ID [5248]

Internal file name [OUTPUT/4739_Friday_February_02_2024_05_10_53_AM_118797/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 38.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$2xy' - 2y - \sqrt{x^2 + 4y^2} = 0$$

2.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2y + \sqrt{x^2 + 4y^2}}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(2y + \sqrt{x^2 + 4y^2})(b_3 - a_2)}{2x} - \frac{(2y + \sqrt{x^2 + 4y^2})^2 a_3}{4x^2} \\ - \left(\frac{1}{2\sqrt{x^2 + 4y^2}} - \frac{2y + \sqrt{x^2 + 4y^2}}{2x^2} \right) (xa_2 + ya_3 + a_1) \\ - \frac{\left(2 + \frac{4y}{\sqrt{x^2 + 4y^2}} \right) (xb_2 + yb_3 + b_1)}{2x} = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{(x^2 + 4y^2)^{\frac{3}{2}} a_3 + 2x^3 a_2 - 2x^3 b_3 + 4x^2 y a_3 + 8x^2 y b_2 + 8y^3 a_3 + 4\sqrt{x^2 + 4y^2} x b_1 - 4\sqrt{x^2 + 4y^2} y a_1 + 8xy b_1 - 8y^2 a_1}{4\sqrt{x^2 + 4y^2} x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -(x^2 + 4y^2)^{\frac{3}{2}} a_3 - 2x^3 a_2 + 2x^3 b_3 - 4x^2 y a_3 - 8x^2 y b_2 - 8y^3 a_3 \\ - 4\sqrt{x^2 + 4y^2} x b_1 + 4\sqrt{x^2 + 4y^2} y a_1 - 8xy b_1 + 8y^2 a_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} -(x^2 + 4y^2)^{\frac{3}{2}} a_3 + 2(x^2 + 4y^2) x b_3 - 2(x^2 + 4y^2) y a_3 \\ - 2x^3 a_2 - 2x^2 y a_3 - 8x^2 y b_2 - 8x y^2 b_3 + 2(x^2 + 4y^2) a_1 \\ - 4\sqrt{x^2 + 4y^2} x b_1 + 4\sqrt{x^2 + 4y^2} y a_1 - 2x^2 a_1 - 8xy b_1 = 0 \end{aligned} \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} -2x^3 a_2 + 2x^3 b_3 - x^2 \sqrt{x^2 + 4y^2} a_3 - 4x^2 y a_3 - 8x^2 y b_2 - 4\sqrt{x^2 + 4y^2} y^2 a_3 \\ - 8y^3 a_3 - 4\sqrt{x^2 + 4y^2} x b_1 - 8xy b_1 + 4\sqrt{x^2 + 4y^2} y a_1 + 8y^2 a_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{x^2 + 4y^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{x^2 + 4y^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -2v_1^3a_2 - 4v_1^2v_2a_3 - v_1^2v_3a_3 - 8v_2^3a_3 - 4v_3v_2^2a_3 - 8v_1^2v_2b_2 \\ + 2v_1^3b_3 + 8v_2^2a_1 + 4v_3v_2a_1 - 8v_1v_2b_1 - 4v_3v_1b_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} (-2a_2 + 2b_3)v_1^3 + (-4a_3 - 8b_2)v_1^2v_2 - v_1^2v_3a_3 - 8v_1v_2b_1 \\ - 4v_3v_1b_1 - 8v_2^3a_3 - 4v_3v_2^2a_3 + 8v_2^2a_1 + 4v_3v_2a_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 4a_1 &= 0 \\ 8a_1 &= 0 \\ -8a_3 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -8b_1 &= 0 \\ -4b_1 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -4a_3 - 8b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$a_1 = 0$$

$$a_2 = b_3$$

$$a_3 = 0$$

$$b_1 = 0$$

$$b_2 = 0$$

$$b_3 = b_3$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$

$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{2y + \sqrt{x^2 + 4y^2}}{2x} \right) (x) \\ &= -\frac{\sqrt{x^2 + 4y^2}}{2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{\sqrt{x^2 + 4y^2}}{2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(\sqrt{4}y + \sqrt{x^2 + 4y^2})\sqrt{4}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y + \sqrt{x^2 + 4y^2}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{\sqrt{x^2 + 4y^2} (2y + \sqrt{x^2 + 4y^2})} \\ S_y &= -\frac{2}{\sqrt{x^2 + 4y^2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2(2\sqrt{x^2 + 4y^2}y + x^2 + 4y^2)}{x\sqrt{x^2 + 4y^2} (2y + \sqrt{x^2 + 4y^2})} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(2y + \sqrt{x^2 + 4y^2}) = -2\ln(x) + c_1$$

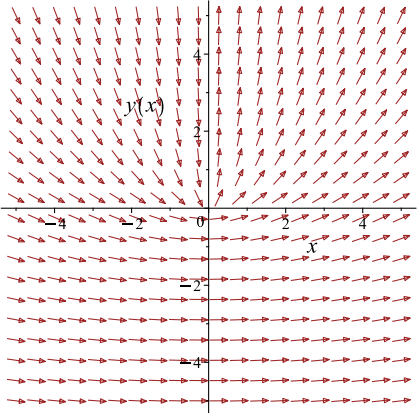
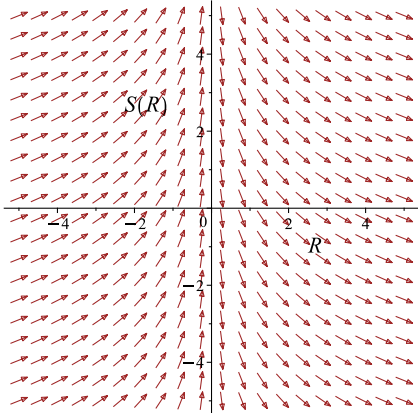
Which simplifies to

$$-\ln\left(2y + \sqrt{x^2 + 4y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y + \sqrt{x^2 + 4y^2}}{2x}$ 	$R = x$ $S = -\ln\left(2y + \sqrt{x^2 + 4y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{4} \tag{1}$$

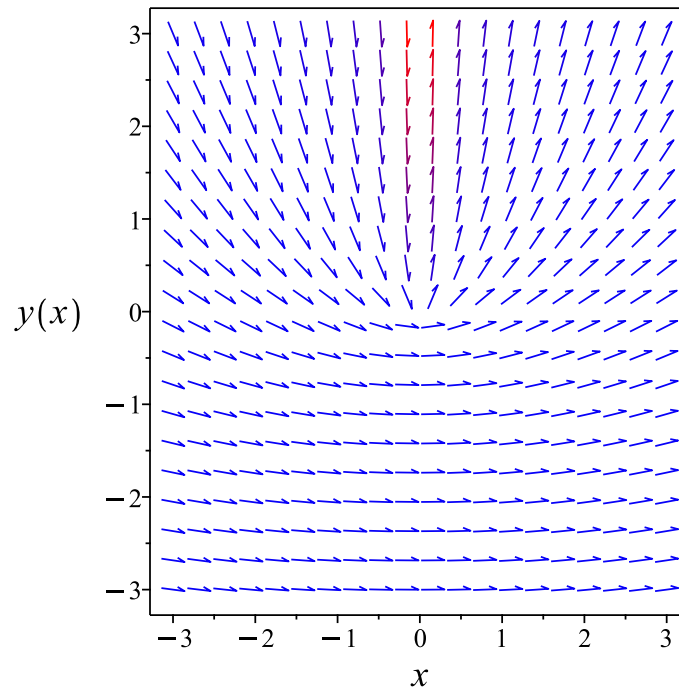


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{4}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve(2*x*diff(y(x),x)-2*y(x)= sqrt(x^2+4*y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + 2y(x) + \sqrt{x^2 + 4y(x)^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 27

```
DSolve[2*x*y'[x]-2*y[x]== Sqrt[x^2+4*y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4} e^{-2c_1} (-1 + e^{4c_1 x^2})$$

2.14 problem 39

- 2.14.1 Solving as homogeneousTypeMapleC ode 277
- 2.14.2 Solving as first order ode lie symmetry calculated ode 281

Internal problem ID [5249]

Internal file name [OUTPUT/4740_Friday_February_02_2024_05_10_56_AM_45477591/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 39.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `class A`]]
```

$$3y + (7y - 3x + 3)y' = 7x - 7$$

2.14.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 - 7X - 7x_0 + 7}{7Y(X) + 7y_0 - 3X - 3x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1 \\y_0 &= 0\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) - 7X}{7Y(X) - 3X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{3Y - 7X}{7Y - 3X} \end{aligned} \quad (1)$$

An ode of the form $Y' = \frac{M(X, Y)}{N(X, Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = -7Y + 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{-3u + 7}{7u - 3} \\ \frac{du}{dX} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0$$

Or

$$7\left(\frac{d}{dX}u(X)\right)Xu(X) - 3\left(\frac{d}{dX}u(X)\right)X + 7u(X)^2 - 7 = 0$$

Or

$$-7 + X(7u(X) - 3)\left(\frac{d}{dX}u(X)\right) + 7u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{7(u^2 - 1)}{X(7u - 3)} \end{aligned}$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}} du = -\frac{7}{X} dX$$

$$\int \frac{1}{\frac{u^2-1}{7u-3}} du = \int -\frac{7}{X} dX$$

$$2 \ln(u-1) + 5 \ln(u+1) = -7 \ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2 \ln(u-1) + 5 \ln(u+1)} = e^{-7 \ln(X) + c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \text{RootOf}(X^7 + 3X^6_Z + X^5_Z^2 - 5X^4_Z^3 - 5X^3_Z^4 + X^2_Z^5 + 3X_Z^6 +_Z^7 - c_3)$$

Using the solution for $Y(X)$

$$Y(X) = \text{RootOf}(X^7 + 3X^6_Z + X^5_Z^2 - 5X^4_Z^3 - 5X^3_Z^4 + X^2_Z^5 + 3X_Z^6 +_Z^7 - c_3)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$

$$X = x + x_0$$

Or

$$Y = y$$

$$X = x + 1$$

Then the solution in y becomes

$$y = \text{RootOf}(_Z^7 + (-3 + 3x)_Z^6 + (x^2 - 2x + 1)_Z^5 + (-5x^3 + 15x^2 - 15x + 5)_Z^4 + (-5x^4 + 20x^3 - 15x^2 + 5x - 3)_Z^3 + (5x^5 - 15x^4 + 10x^3 - 5x^2)_Z^2 + (-5x^6 + 15x^5 - 10x^4 + 5x^3)_Z + 5x^7 - c_3)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \text{RootOf} (& _Z^7 + (-3 + 3x)_Z^6 + (x^2 - 2x + 1)_Z^5 + (-5x^3 + 15x^2 - 15x + 5)_Z^4 \\ & + (-5x^4 + 20x^3 - 30x^2 + 20x - 5)_Z^3 + (x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1)_Z^2 \\ & + (3x^6 - 18x^5 + 45x^4 - 60x^3 + 45x^2 - 18x + 3)_Z + x^7 - 7x^6 + 21x^5 - 35x^4 \\ & + 35x^3 - 21x^2 - c_3 + 7x - 1) \end{aligned} \quad (1)$$

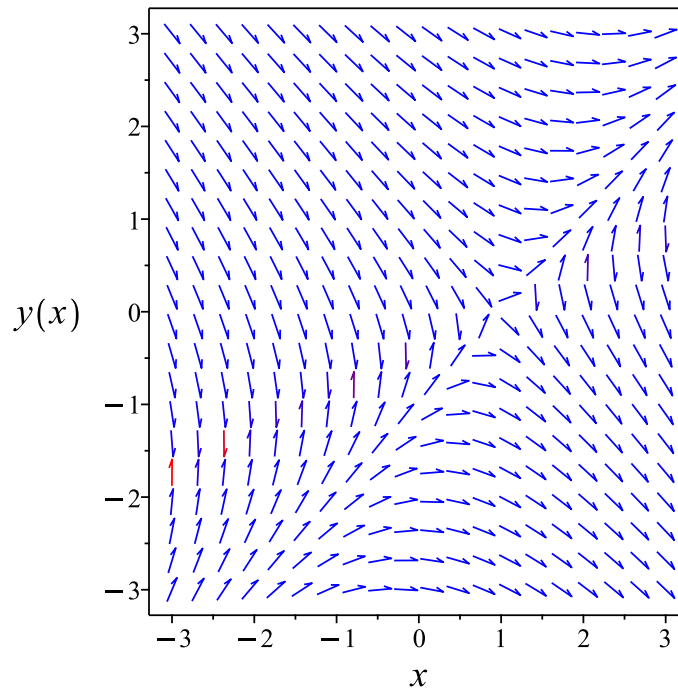


Figure 67: Slope field plot

Verification of solutions

$$\begin{aligned} y = \text{RootOf} (& _Z^7 + (-3 + 3x)_Z^6 + (x^2 - 2x + 1)_Z^5 + (-5x^3 + 15x^2 - 15x + 5)_Z^4 \\ & + (-5x^4 + 20x^3 - 30x^2 + 20x - 5)_Z^3 + (x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1)_Z^2 \\ & + (3x^6 - 18x^5 + 45x^4 - 60x^3 + 45x^2 - 18x + 3)_Z + x^7 - 7x^6 + 21x^5 - 35x^4 \\ & + 35x^3 - 21x^2 - c_3 + 7x - 1) \end{aligned}$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(3y - 7x + 7)(b_3 - a_2)}{7y - 3x + 3} - \frac{(3y - 7x + 7)^2 a_3}{(7y - 3x + 3)^2}$$

$$- \left(\frac{7}{7y - 3x + 3} - \frac{3(3y - 7x + 7)}{(7y - 3x + 3)^2} \right) (xa_2 + ya_3 + a_1) \quad (\text{5E})$$

$$- \left(-\frac{3}{7y - 3x + 3} + \frac{21y - 49x + 49}{(7y - 3x + 3)^2} \right) (xb_2 + yb_3 + b_1) = 0$$

Putting the above in normal form gives

$$\underline{21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3 - 40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0}$$

Setting the numerator to zero gives

$$21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 \quad (\text{6E})$$

$$+ 21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3$$

$$- 40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\ &\quad - 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\ &\quad - 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\ &\quad + 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(21a_2 - 49a_3 + 49b_2 - 21b_3)v_1^2 + (-98a_2 + 42a_3 - 42b_2 + 98b_3)v_1v_2 \\ &\quad + (-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3)v_1 + (21a_2 - 49a_3 + 49b_2 - 21b_3)v_2^2 \\ &\quad + (-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3)v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -98a_2 + 42a_3 - 42b_2 + 98b_3 &= 0 \\ 21a_2 - 49a_3 + 49b_2 - 21b_3 &= 0 \\ -40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3 &= 0 \\ -42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3 &= 0 \\ 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_3 \\ a_2 &= b_3 \\ a_3 &= b_2 \\ b_1 &= -b_2 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= y \\ \eta &= x - 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= x - 1 - \left(-\frac{3y - 7x + 7}{7y - 3x + 3} \right) (y) \\ &= \frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3}} dy\end{aligned}$$

Which results in

$$S = \frac{5 \ln(x + y - 1)}{3} + \frac{2 \ln(-x + y + 1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{5}{3x - 3 + 3y} + \frac{2}{3x - 3 - 3y} \\ S_y &= \frac{5}{3x - 3 + 3y} - \frac{2}{3x - 3 - 3y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

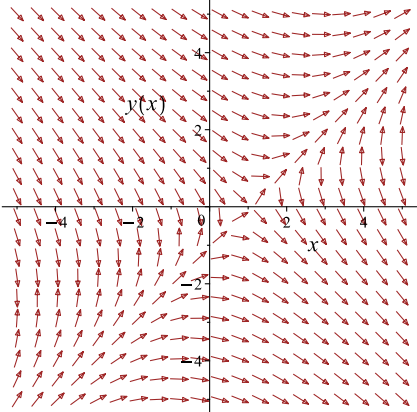
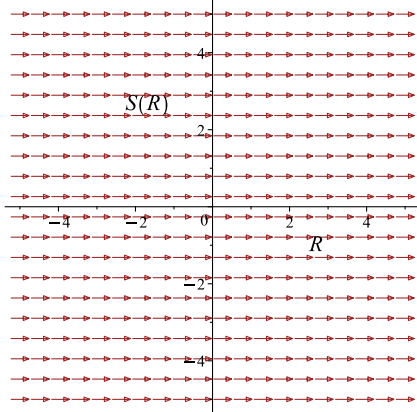
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{5 \ln(x - 1 + y)}{3} + \frac{2 \ln(-x + y + 1)}{3} = c_1$$

Which simplifies to

$$\frac{5 \ln(x - 1 + y)}{3} + \frac{2 \ln(-x + y + 1)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ 	$R = x$ $S = \frac{5 \ln(x + y - 1)}{3} + \frac{21}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{5 \ln(x - 1 + y)}{3} + \frac{2 \ln(-x + y + 1)}{3} = c_1 \tag{1}$$

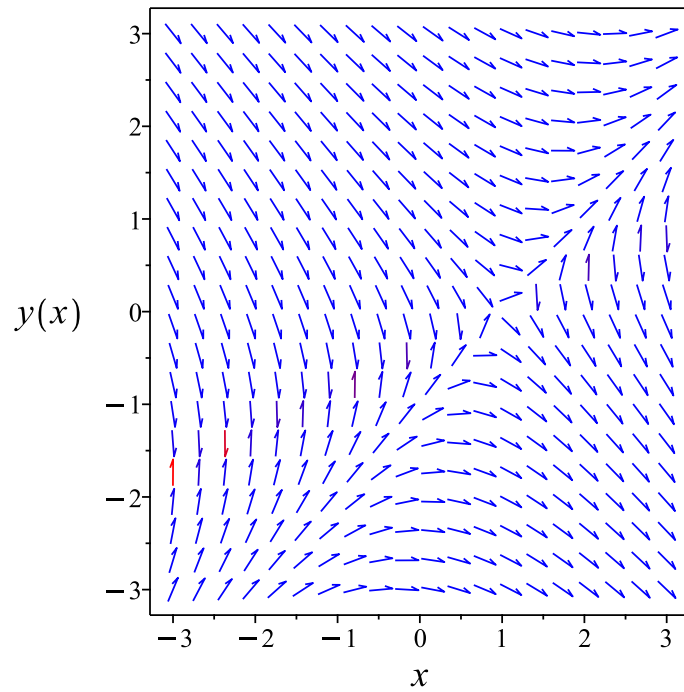


Figure 68: Slope field plot

Verification of solutions

$$\frac{5 \ln(x - 1 + y)}{3} + \frac{2 \ln(-x + y + 1)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous C  
trying homogeneous types:  
trying homogeneous D  
<- homogeneous successful  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 1814

```
dsolve((3*y(x)-7*x+7)+(7*y(x)-3*x+3)*diff(y(x),x)= 0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.746 (sec). Leaf size: 7785

```
DSolve[(3*y[x]-7*x+7)+(7*y[x]-3*x+3)*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.15 problem 40

2.15.1 Solving as separable ode	288
2.15.2 Solving as first order ode lie symmetry lookup ode	290
2.15.3 Solving as exact ode	294
2.15.4 Maple step by step solution	298

Internal problem ID [5250]

Internal file name [OUTPUT/4741_Friday_February_02_2024_05_10_58_AM_59247198/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 40.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "separable", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$xyy' - (1 + y)(1 - x) = 0$$

2.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{(x - 1)(-y - 1)}{xy}\end{aligned}$$

Where $f(x) = \frac{x-1}{x}$ and $g(y) = \frac{-y-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{-y-1}{y}} dy = \frac{x-1}{x} dx$$

$$\int \frac{1}{\frac{-y-1}{y}} dy = \int \frac{x-1}{x} dx$$

$$-y + \ln(1+y) = x - \ln(x) + c_1$$

Which results in

$$y = -\text{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\text{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

gives

$$y = -\text{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\text{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1 \tag{1}$$

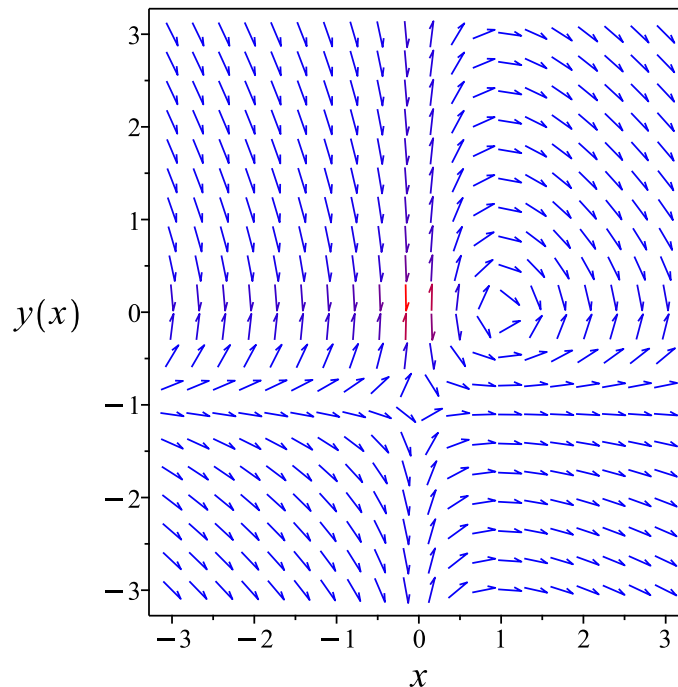


Figure 69: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1$$

Verified OK.

2.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy + x - y - 1}{xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= \frac{x}{x-1} \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{x-1}} dx \end{aligned}$$

Which results in

$$S = x - \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy + x - y - 1}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x-1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y}{1+y} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{1+R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -R + \ln(1 + R) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x - \ln(x) = -y + \ln(1 + y) + c_1$$

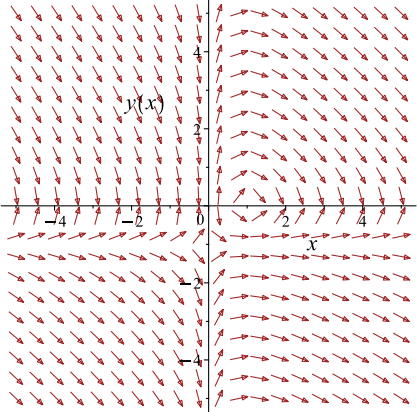
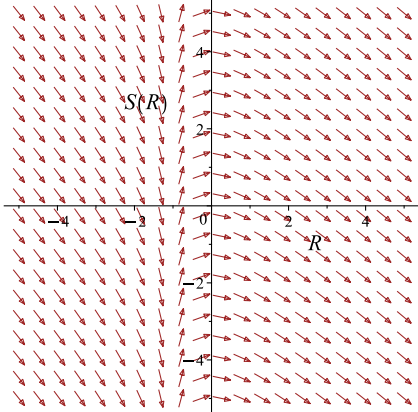
Which simplifies to

$$x - \ln(x) = -y + \ln(1 + y) + c_1$$

Which gives

$$y = -\text{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+x-y-1}{xy}$ 	$R = y$ $S = x - \ln(x)$	$\frac{dS}{dR} = -\frac{R}{1+R}$ 

Summary

The solution(s) found are the following

$$y = -\text{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1 \quad (1)$$

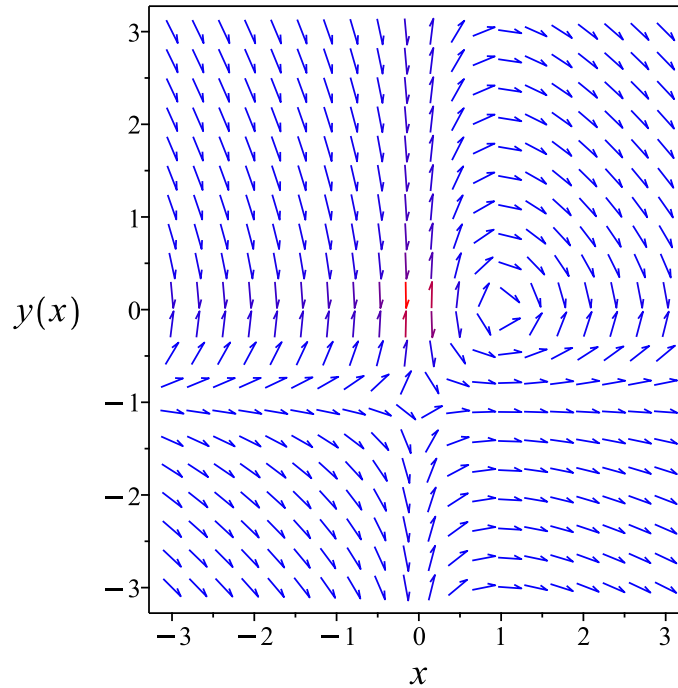


Figure 70: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$$

Verified OK.

2.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{-y-1}\right) dy &= \left(\frac{x-1}{x}\right) dx \\ \left(-\frac{x-1}{x}\right) dx + \left(\frac{y}{-y-1}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x-1}{x} \\ N(x, y) &= \frac{y}{-y-1}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{x-1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{y}{-y-1} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{x-1}{x} dx \\ \phi &= -x + \ln(x) + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{-y-1}$. Therefore equation (4) becomes

$$\frac{y}{-y-1} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{1+y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{y}{1+y} \right) dy \\ f(y) &= -y + \ln(1+y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \ln(x) - y + \ln(1 + y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \ln(x) - y + \ln(1 + y)$$

The solution becomes

$$y = -\text{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\text{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1 \quad (1)$$

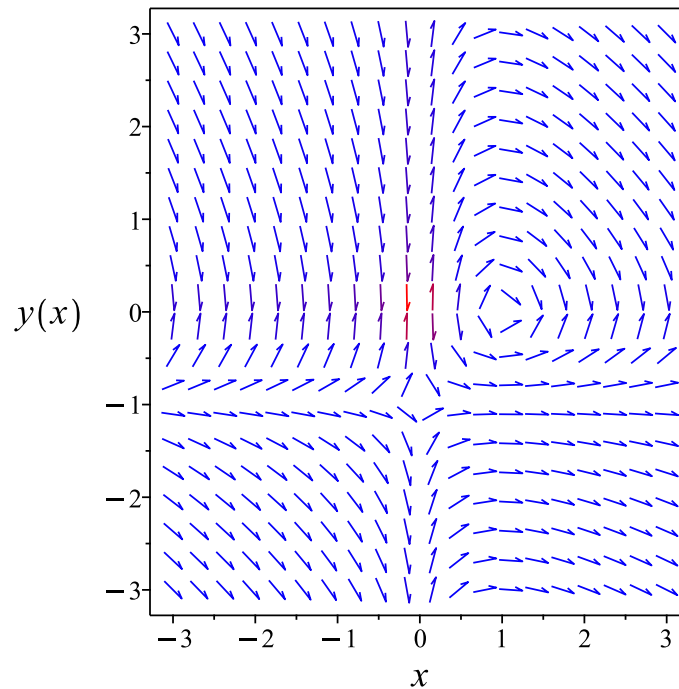


Figure 71: Slope field plot

Verification of solutions

$$y = -\text{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

Verified OK.

2.15.4 Maple step by step solution

Let's solve

$$xyy' - (1 + y)(1 - x) = 0$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'y}{1+y} = \frac{1-x}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{1+y} dx = \int \frac{1-x}{x} dx + c_1$$

- Evaluate integral

$$y - \ln(1 + y) = -x + \ln(x) + c_1$$

- Solve for y

$$y = -\text{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*y(x)*diff(y(x),x)= (y(x)+1)*(1-x),y(x), singsol=all)
```

$$y(x) = -\text{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1$$

✓ Solution by Mathematica

Time used: 6.202 (sec). Leaf size: 29

```
DSolve[x*y[x]*y'[x]== (y[x]+1)*(1-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -1 - W\left(-\frac{e^{x-1-c_1}}{x}\right)$$
$$y(x) \rightarrow -1$$

2.16 problem 41

2.16.1 Solving as homogeneousTypeD2 ode	300
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2.16.4 Solving as exact ode	309

Internal problem ID [5251]

Internal file name [OUTPUT/4742_Friday_February_02_2024_05_10_59_AM_6697373/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 41.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli"**, **"homogeneousTypeD2"**, **"exactWithIntegrationFactor"**, **"first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y^2 + xyy' = x^2$$

2.16.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 + x^2 u(x) (u'(x)x + u(x)) = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 - 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{2u^2-1}{u}} du &= -\frac{1}{x} dx \\ \int \frac{1}{\frac{2u^2-1}{u}} du &= \int -\frac{1}{x} dx \\ \frac{\ln(2u^2-1)}{4} &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$(2u^2-1)^{\frac{1}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^2-1)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^2-1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$(2u(x)^2-1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}\left(\frac{2y^2}{x^2}-1\right)^{\frac{1}{4}} &= \frac{c_3 e^{c_2}}{x} \\ \left(\frac{2y^2-x^2}{x^2}\right)^{\frac{1}{4}} &= \frac{c_3 e^{c_2}}{x}\end{aligned}$$

Which simplifies to

$$\left(-\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\left(-\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

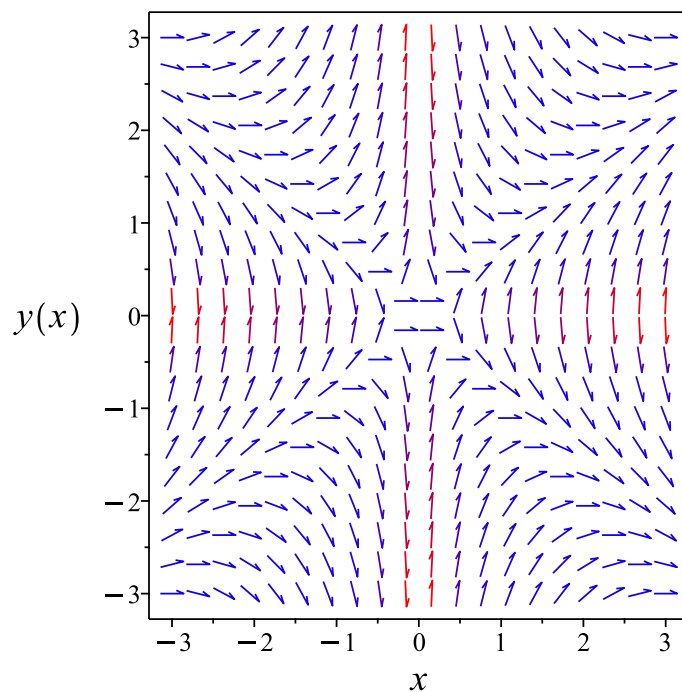


Figure 72: Slope field plot

Verification of solutions

$$\left(-\frac{2y^2 + x^2}{x^2} \right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

2.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2 + y^2}{xy}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{yx^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$

$$R_y = 0$$

$$S_x = y^2 x$$

$$S_y = y x^2$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \quad (4)$$

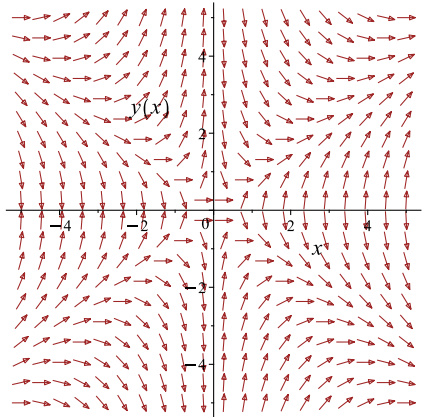
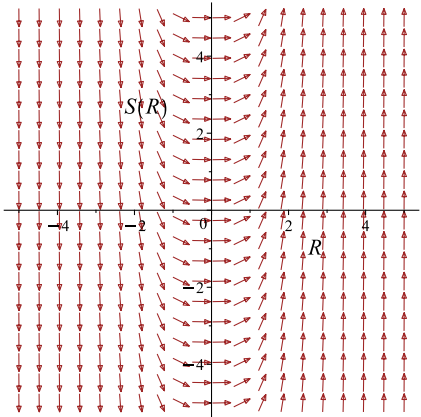
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x^2}{2} = \frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{y^2 x^2}{2} = \frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y^2}{xy}$ 	$R = x$ $S = \frac{y^2 x^2}{2}$	$\frac{dS}{dR} = R^3$ 

Summary

The solution(s) found are the following

$$\frac{y^2 x^2}{2} = \frac{x^4}{4} + c_1 \quad (1)$$

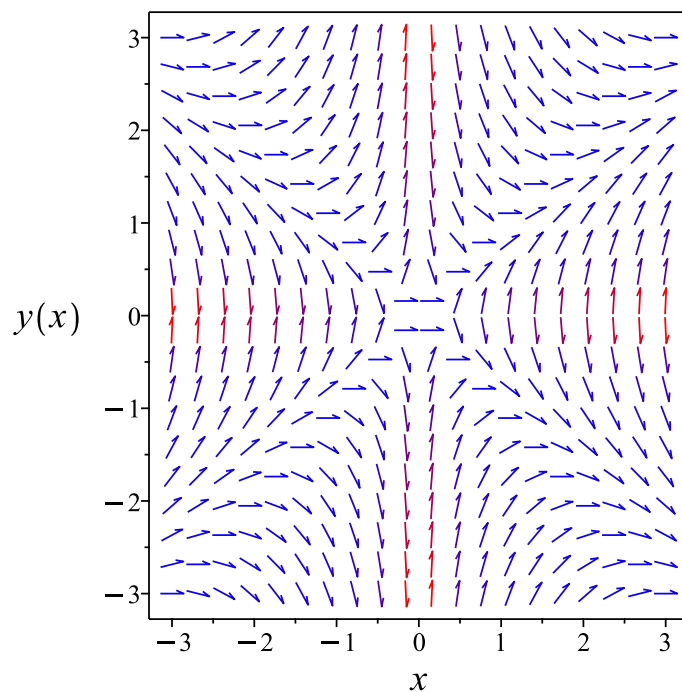


Figure 73: Slope field plot

Verification of solutions

$$\frac{y^2 x^2}{2} = \frac{x^4}{4} + c_1$$

Verified OK.

2.16.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{-x^2 + y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= -\frac{1}{x} \\f_1(x) &= x \\n &= -1\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} + x \quad (4)$$

Let

$$\begin{aligned}w &= y^{1-n} \\&= y^2\end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}\frac{w'(x)}{2} &= -\frac{w(x)}{x} + x \\w' &= -\frac{2w}{x} + 2x\end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2}{x} \\q(x) &= 2x\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(2x) \\ \frac{d}{dx}(x^2 w) &= (x^2)(2x) \\ d(x^2 w) &= (2x^3) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int 2x^3 dx \\ x^2 w &= \frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{2}$$

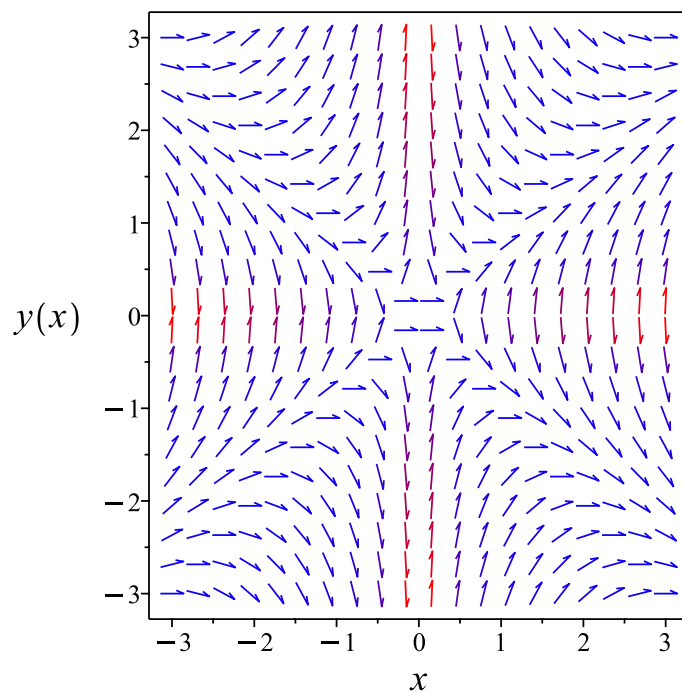


Figure 74: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

2.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (xy) dy &= (x^2 - y^2) dx \\ (-x^2 + y^2) dx + (xy) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x^2 + y^2 \\ N(x, y) &= xy \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-x^2 + y^2) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{yx} ((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= x(-x^2 + y^2) \\ &= -x^3 + y^2x\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= x(xy) \\ &= yx^2\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (-x^3 + y^2x) + (yx^2) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$
$$\int \frac{\partial \phi}{\partial x} dx = \int -x^3 + y^2 x dx$$
$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}y^2 x^2 + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y x^2 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y x^2$. Therefore equation (4) becomes

$$y x^2 = y x^2 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}y^2 x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^4 + \frac{1}{2}y^2 x^2$$

Summary

The solution(s) found are the following

$$\frac{y^2 x^2}{2} - \frac{x^4}{4} = c_1 \quad (1)$$

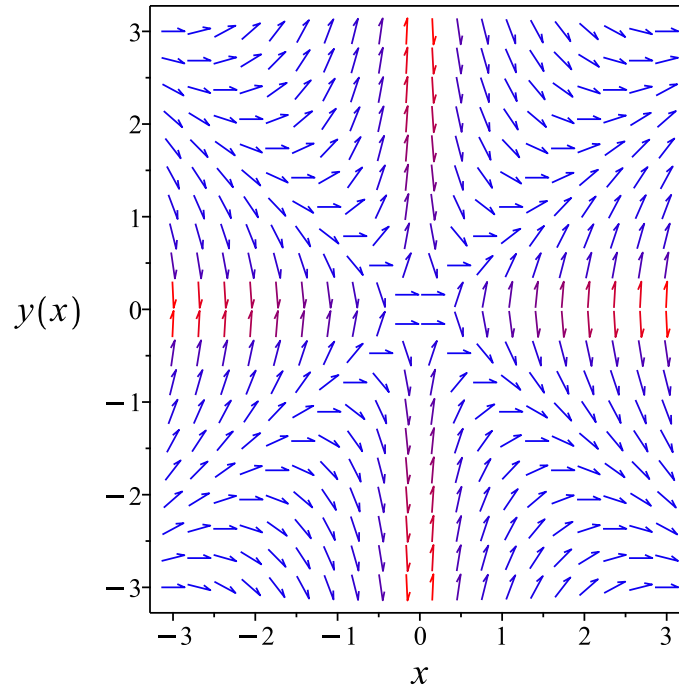


Figure 75: Slope field plot

Verification of solutions

$$\frac{y^2 x^2}{2} - \frac{x^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((y(x)^2-x^2)+x*y(x)*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2x^4 + 4c_1}}{2x}$$

$$y(x) = \frac{\sqrt{2x^4 + 4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 46

```
DSolve[(y[x]^2-x^2)+x*y[x]*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{\frac{x^4}{2} + c_1}}{x}$$

$$y(x) \rightarrow \frac{\sqrt{\frac{x^4}{2} + c_1}}{x}$$

2.17 problem 42

2.17.1 Solving as first order ode lie symmetry calculated ode 315

2.17.2 Solving as exact ode 321

Internal problem ID [5252]

Internal file name [OUTPUT/4743_Friday_February_02_2024_05_11_00_AM_59621178/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 42.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$y(1 + 2xy) + x(1 - xy)y' = 0$$

2.17.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(2xy + 1)}{x(xy - 1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{y(2xy+1)(b_3-a_2)}{x(xy-1)} - \frac{y^2(2xy+1)^2 a_3}{x^2(xy-1)^2} \\ - \left(\frac{2y^2}{x(xy-1)} - \frac{y(2xy+1)}{x^2(xy-1)} - \frac{y^2(2xy+1)}{x(xy-1)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(\frac{2xy+1}{x(xy-1)} + \frac{2y}{xy-1} - \frac{y(2xy+1)}{(xy-1)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{x^4 y^2 b_2 + 2x^2 y^4 a_3 + 2x^3 y^2 b_1 - 2x^2 y^3 a_1 - 2x^3 y b_2 - 3x^2 y^2 a_2 - 3x^2 y^2 b_3 + 2x y^3 a_3 - 4x^2 y b_1 - 2x y^2 a_1 - 2x^2 y^2 b_3}{x^2 (xy-1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -x^4 y^2 b_2 - 2x^2 y^4 a_3 - 2x^3 y^2 b_1 + 2x^2 y^3 a_1 + 2x^3 y b_2 + 3x^2 y^2 a_2 + 3x^2 y^2 b_3 \\ - 2x y^3 a_3 + 4x^2 y b_1 + 2x y^2 a_1 + 2b_2 x^2 - 2y^2 a_3 + x b_1 - y a_1 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -2a_3 v_1^2 v_2^4 - b_2 v_1^4 v_2^2 + 2a_1 v_1^2 v_2^3 - 2b_1 v_1^3 v_2^2 + 3a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 + 2b_2 v_1^3 v_2 \\ + 3b_3 v_1^2 v_2^2 + 2a_1 v_1 v_2^2 + 4b_1 v_1^2 v_2 - 2a_3 v_2^2 + 2b_2 v_1^2 - a_1 v_2 + b_1 v_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} -b_2v_1^4v_2^2 - 2b_1v_1^3v_2^2 + 2b_2v_1^3v_2 - 2a_3v_1^2v_2^4 + 2a_1v_1^2v_2^3 + (3a_2 + 3b_3)v_1^2v_2^2 \\ + 4b_1v_1^2v_2 + 2b_2v_1^2 - 2a_3v_1v_2^3 + 2a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ 4b_1 &= 0 \\ -b_2 &= 0 \\ 2b_2 &= 0 \\ 3a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{y(2xy + 1)}{x(xy - 1)} \right) (-x) \\ &= \frac{3y^2x}{xy - 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{3y^2x}{xy-1}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{3} + \frac{1}{3yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(2xy + 1)}{x(xy - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\R_y &= 0 \\S_x &= -\frac{1}{3y x^2} \\S_y &= \frac{xy - 1}{3y^2 x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{2 \ln(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy \ln(y) + 1}{3xy} = \frac{2 \ln(x)}{3} + c_1$$

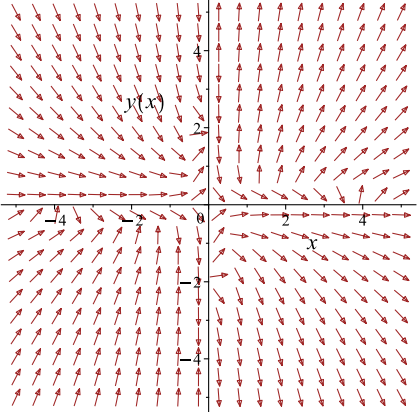
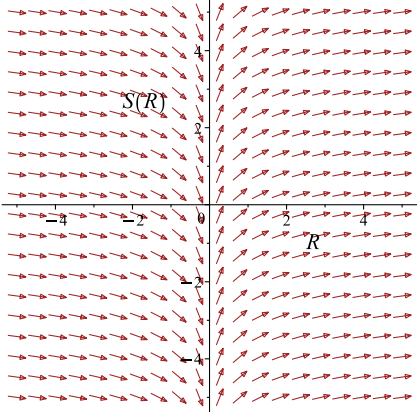
Which simplifies to

$$\frac{xy \ln(y) + 1}{3xy} = \frac{2 \ln(x)}{3} + c_1$$

Which gives

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-3c_1}}{x^3}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2xy+1)}{x(xy-1)}$ 	$R = x$ $S = \frac{\ln(y)xy + 1}{3xy}$	$\frac{dS}{dR} = \frac{2}{3R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \text{ LambertW}\left(-\frac{e^{-3c_1}}{x^3}\right)} \tag{1}$$

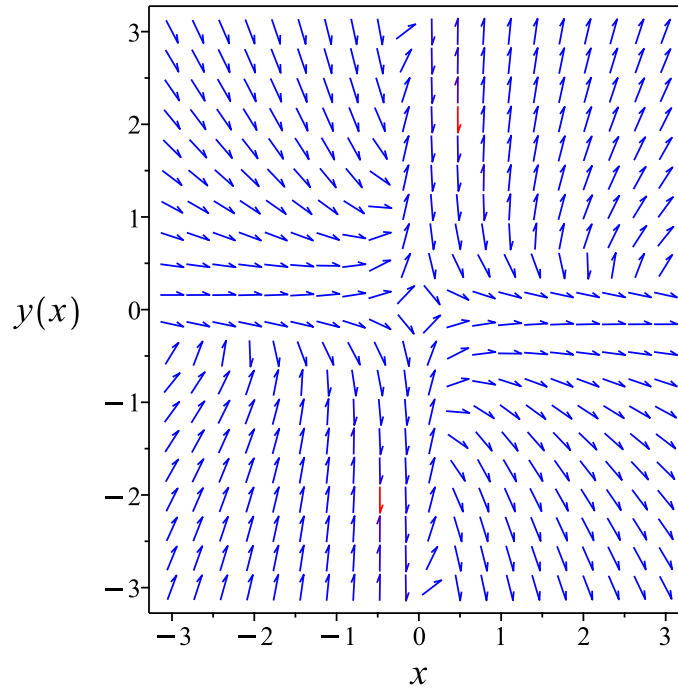


Figure 76: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-3c_1}}{x^3}\right)}$$

Verified OK.

2.17.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(-xy + 1)) dy &= (-y(2xy + 1)) dx \\ (y(2xy + 1)) dx + (x(-xy + 1)) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(2xy + 1) \\ N(x, y) &= x(-xy + 1)\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(2xy + 1)) \\ &= 4xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(-xy + 1)) \\ &= -2xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{x(xy-1)} ((4xy+1) - (-2xy+1)) \\ &= -\frac{6y}{xy-1} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{2y^2x+y} ((-2xy+1) - (4xy+1)) \\ &= -\frac{6x}{2xy+1} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(-2xy+1) - (4xy+1)}{x(y(2xy+1)) - y(x(-xy+1))} \\ &= -\frac{2}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{2}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int (-\frac{2}{t}) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(t)} \\ &= \frac{1}{t^2}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^2x^2}$$

Multiplying M and N by this integrating factor gives new \bar{M} and new \bar{N} which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{y^2x^2}(y(2xy + 1)) \\ &= \frac{2xy + 1}{y x^2}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{y^2x^2}(x(-xy + 1)) \\ &= \frac{-xy + 1}{x y^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{2xy + 1}{y x^2}\right) + \left(\frac{-xy + 1}{x y^2}\right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{2xy + 1}{y x^2} dx \\ \phi &= 2 \ln(x) - \frac{1}{yx} + f(y)\end{aligned}\quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x} + f'(y)\quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-xy+1}{x y^2}$. Therefore equation (4) becomes

$$\frac{-xy + 1}{x y^2} = \frac{1}{y^2 x} + f'(y)\quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{1}{y}\right) dy \\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2 \ln(x) - \frac{1}{yx} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2 \ln(x) - \frac{1}{yx} - \ln(y)$$

The solution becomes

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{e_1}}{x^3}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{e_1}}{x^3}\right)} \quad (1)$$

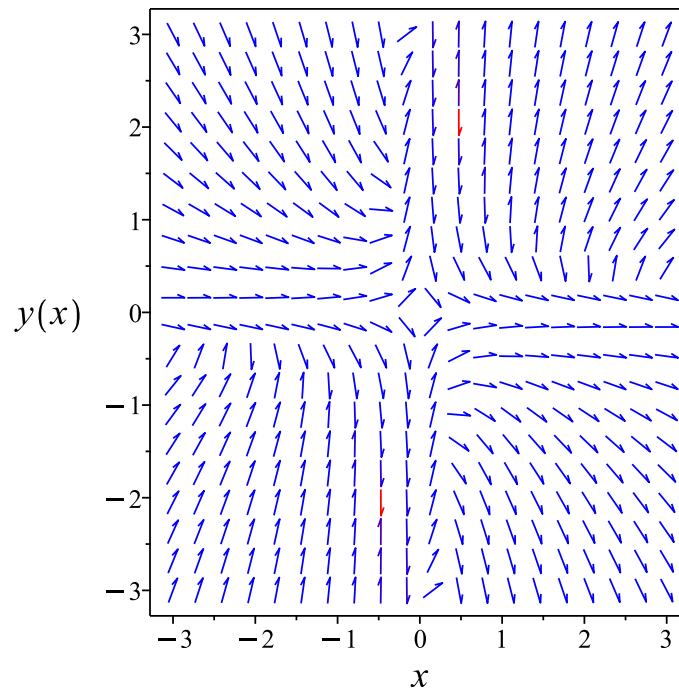


Figure 77: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{e_1}}{x^3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous G  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve(y(x)*(1+2*x*y(x))+x*(1-x*y(x))*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\text{LambertW}\left(-\frac{c_1}{x^3}\right)x}$$

✓ Solution by Mathematica

Time used: 6.645 (sec). Leaf size: 35

```
DSolve[y[x]*(1+2*x*y[x])+x*(1-x*y[x])*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{xW\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x^3}\right)}$$
$$y(x) \rightarrow 0$$

2.18 problem 43

2.18.1 Solving as separable ode	328
2.18.2 Solving as first order ode lie symmetry lookup ode	330
2.18.3 Solving as exact ode	334
2.18.4 Maple step by step solution	338

Internal problem ID [5253]

Internal file name [OUTPUT/4744_Friday_February_02_2024_05_11_02_AM_57669137/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 43.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "separable", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$(-x^2 + 1) \cot(y) y' = -1$$

2.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{1}{\cot(y)(x^2 - 1)} \end{aligned}$$

Where $f(x) = \frac{1}{x^2-1}$ and $g(y) = \frac{1}{\cot(y)}$. Integrating both sides gives

$$\frac{1}{\cot(y)} dy = \frac{1}{x^2 - 1} dx$$

$$\int \frac{1}{\frac{1}{\cot(y)}} dy = \int \frac{1}{x^2 - 1} dx$$

$$\ln(\sin(y)) = -\operatorname{arctanh}(x) + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{-\operatorname{arctanh}(x)+c_1}$$

Which simplifies to

$$\sin(y) = \frac{c_2 \sqrt{-x^2 + 1}}{x + 1}$$

Summary

The solution(s) found are the following

$$y = \arcsin(c_2 e^{-\operatorname{arctanh}(x)+c_1}) \tag{1}$$

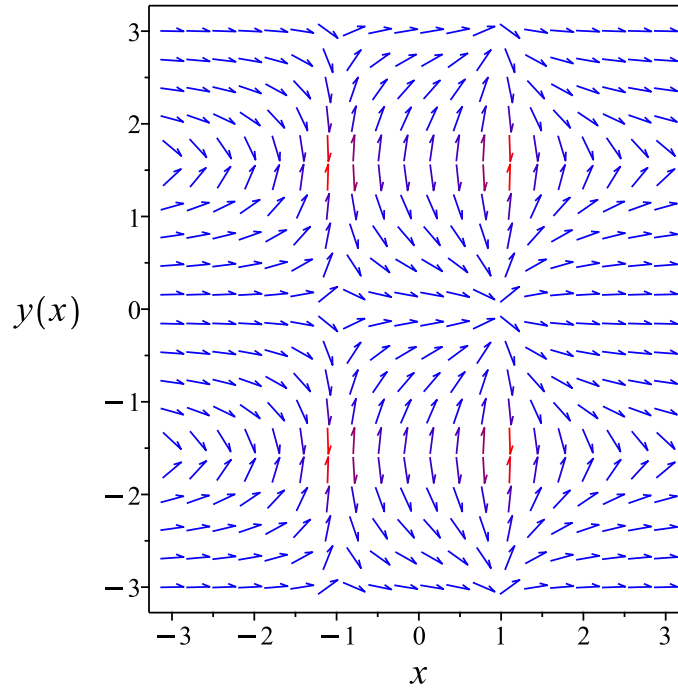


Figure 78: Slope field plot

Verification of solutions

$$y = \arcsin(c_2 e^{-\operatorname{arctanh}(x)+c_1})$$

Verified OK.

2.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1}{\cot(y)(x^2 - 1)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= x^2 - 1 \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2 - 1} dx\end{aligned}$$

Which results in

$$S = -\operatorname{arctanh}(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y}\tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1}{\cot(y)(x^2 - 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\R_y &= 1 \\S_x &= \frac{1}{x^2 - 1} \\S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cot(y) \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \ln(\sin(R)) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\operatorname{arctanh}(x) = \ln(\sin(y)) + c_1$$

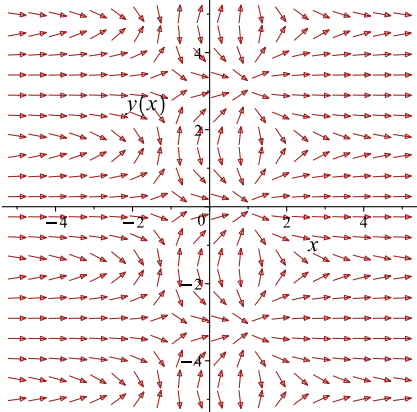
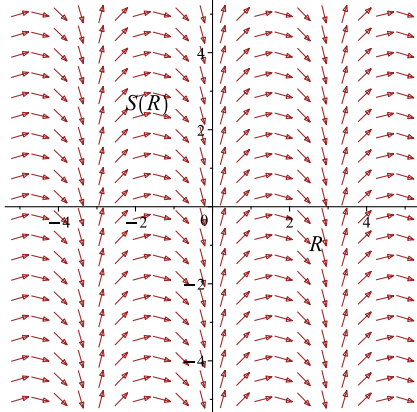
Which simplifies to

$$-\operatorname{arctanh}(x) = \ln(\sin(y)) + c_1$$

Which gives

$$y = \arcsin\left(e^{-\operatorname{arctanh}(x)-c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{\cot(y)(x^2-1)}$ 	$R = y$ $S = -\operatorname{arctanh}(x)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin \left(e^{-\operatorname{arctanh}(x)-c_1} \right) \tag{1}$$

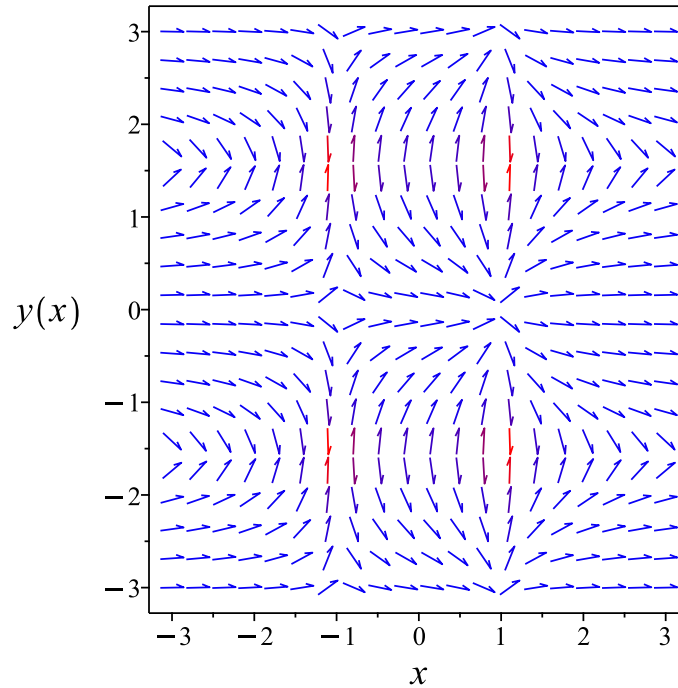


Figure 79: Slope field plot

Verification of solutions

$$y = \arcsin \left(e^{-\operatorname{arctanh}(x)-c_1} \right)$$

Verified OK.

2.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(\cot(y)) dy &= \left(\frac{1}{x^2 - 1}\right) dx \\ \left(-\frac{1}{x^2 - 1}\right) dx + (\cot(y)) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x^2 - 1} \\ N(x, y) &= \cot(y)\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x^2 - 1}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(y)) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x^2 - 1} dx \\ \phi &= \operatorname{arctanh}(x) + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cot(y)$. Therefore equation (4) becomes

$$\cot(y) = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \cot(y)$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int (\cot(y)) dy \\ f(y) &= \ln(\sin(y)) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \operatorname{arctanh}(x) + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \operatorname{arctanh}(x) + \ln(\sin(y))$$

Summary

The solution(s) found are the following

$$\operatorname{arctanh}(x) + \ln(\sin(y)) = c_1 \quad (1)$$

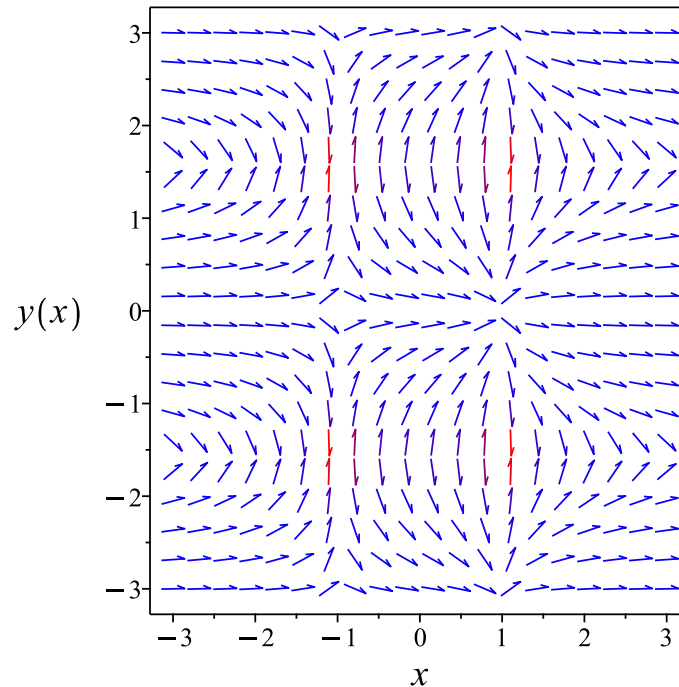


Figure 80: Slope field plot

Verification of solutions

$$\operatorname{arctanh}(x) + \ln(\sin(y)) = c_1$$

Verified OK.

2.18.4 Maple step by step solution

Let's solve

$$(-x^2 + 1) \cot(y) y' = -1$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\cot(y) y' = -\frac{1}{-x^2+1}$$

- Integrate both sides with respect to x

$$\int \cot(y) y' dx = \int -\frac{1}{-x^2+1} dx + c_1$$

- Evaluate integral

$$\ln(\sin(y)) = -\operatorname{arctanh}(x) + c_1$$

- Solve for y

$$y = \arcsin(e^{-\operatorname{arctanh}(x)+c_1})$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(1+(1-x^2)*cot(y(x))*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{\sqrt{-x^2+1} c_1}{x+1}\right)$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 27

```
DSolve[1+(1-x^2)*Cot[y[x]]*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \arcsin\left(\frac{e^{c_1}\sqrt{1-x}}{\sqrt{x+1}}\right)$$

2.19 problem 44

2.19.1 Solving as homogeneousTypeD2 ode	340
2.19.2 Solving as first order ode lie symmetry lookup ode	342
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Internal problem ID [5254]

Internal file name [OUTPUT/4745_Friday_February_02_2024_05_11_02_AM_13641118/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 44.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$y^3 + 3y'y^2x = -x^3$$

2.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^3 x^3 + 3(u'(x)x + u(x))u(x)^2 x^3 = -x^3$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{4u^3 + 1}{3u^2x}\end{aligned}$$

Where $f(x) = -\frac{1}{3x}$ and $g(u) = \frac{4u^3+1}{u^2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{4u^3+1}{u^2}} du &= -\frac{1}{3x} dx \\ \int \frac{1}{\frac{4u^3+1}{u^2}} du &= \int -\frac{1}{3x} dx \\ \frac{\ln(4u^3+1)}{12} &= -\frac{\ln(x)}{3} + c_2\end{aligned}$$

Raising both side to exponential gives

$$(4u^3+1)^{\frac{1}{12}} = e^{-\frac{\ln(x)}{3} + c_2}$$

Which simplifies to

$$(4u^3+1)^{\frac{1}{12}} = \frac{c_3}{x^{\frac{1}{3}}}$$

Which simplifies to

$$(4u(x)^3+1)^{\frac{1}{12}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}$$

The solution is

$$(4u(x)^3+1)^{\frac{1}{12}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}\left(\frac{4y^3}{x^3} + 1\right)^{\frac{1}{12}} &= \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}} \\ \left(\frac{4y^3 + x^3}{x^3}\right)^{\frac{1}{12}} &= \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$\left(\frac{4y^3 + x^3}{x^3}\right)^{\frac{1}{12}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}} \quad (1)$$

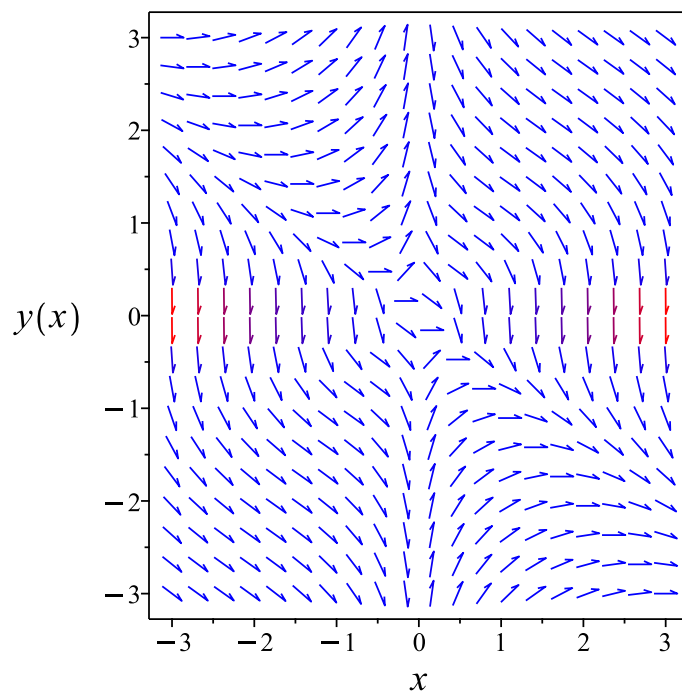


Figure 81: Slope field plot

Verification of solutions

$$\left(\frac{4y^3 + x^3}{x^3}\right)^{\frac{1}{12}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}$$

Verified OK.

2.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^3 + y^3}{3y^2x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{y^2x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x}} dy \end{aligned}$$

Which results in

$$S = \frac{x y^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^3 + y^3}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3}{3} \\ S_y &= y^2 x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^3}{3} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^3}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{R^4}{12} + c_1 \quad (4)$$

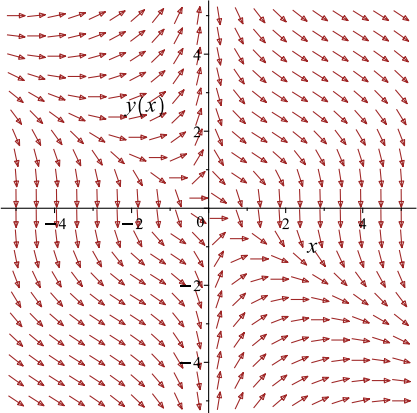
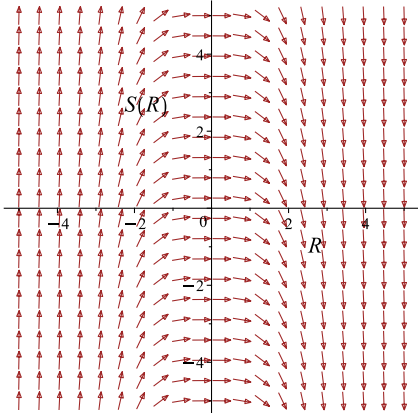
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 x}{3} = -\frac{x^4}{12} + c_1$$

Which simplifies to

$$\frac{y^3 x}{3} = -\frac{x^4}{12} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^3+y^3}{3y^2x}$ 	$R = x$ $S = \frac{xy^3}{3}$	$\frac{dS}{dR} = -\frac{R^3}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3 x}{3} = -\frac{x^4}{12} + c_1 \quad (1)$$

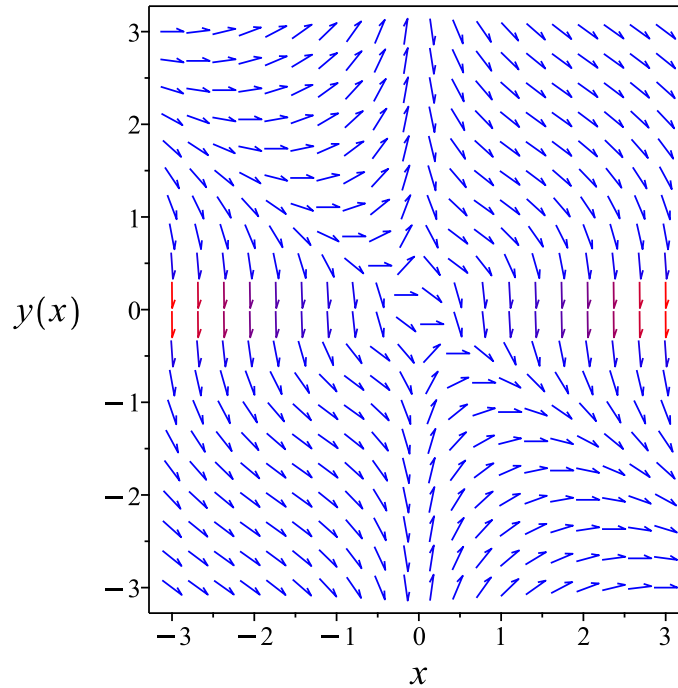


Figure 82: Slope field plot

Verification of solutions

$$\frac{y^3 x}{3} = -\frac{x^4}{12} + c_1$$

Verified OK.

2.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^3 + y^3}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{3x}y - \frac{x^2}{3} \frac{1}{y^2} \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{3x} \\ f_1(x) &= -\frac{x^2}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{y^3}{3x} - \frac{x^2}{3} \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)}{3x} - \frac{x^2}{3} \\ w' &= -\frac{w}{x} - x^2 \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -x^2 \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x^2) \\ \frac{d}{dx}(xw) &= (x)(-x^2) \\ d(xw) &= (-x^3) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^3 dx \\ xw &= -\frac{x^4}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^3}{4} + \frac{c_1}{x}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -\frac{x^3}{4} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}}{2x} \\ y(x) &= \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(i\sqrt{3} - 1)}{4x} \\ y(x) &= -\frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(1 + i\sqrt{3})}{4x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}}{2x} \quad (1)$$

$$y = \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(i\sqrt{3} - 1)}{4x} \quad (2)$$

$$y = -\frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(1 + i\sqrt{3})}{4x} \quad (3)$$

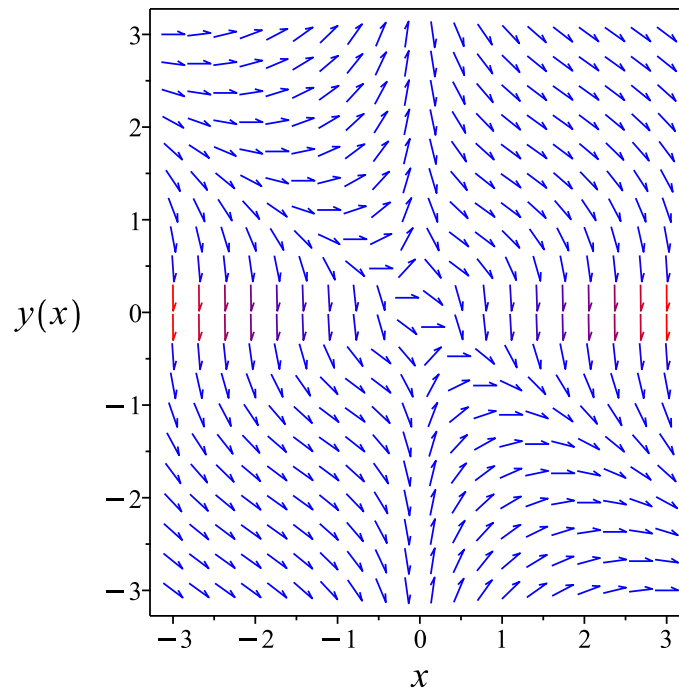


Figure 83: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}}{2x}$$

Verified OK.

$$y = \frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(i\sqrt{3} - 1)}{4x}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}}(-(x^4 - 4c_1)x^2)^{\frac{1}{3}}(1 + i\sqrt{3})}{4x}$$

Verified OK.

2.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (3y^2 x) dy &= (-x^3 - y^3) dx \\ (x^3 + y^3) dx + (3y^2 x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^3 + y^3 \\ N(x, y) &= 3y^2 x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (x^3 + y^3) \\ &= 3y^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (3y^2 x) \\ &= 3y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x^3 + y^3 dx \\ \phi &= \frac{1}{4}x^4 + xy^3 + f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2x + f'(y)\tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2x$. Therefore equation (4) becomes

$$3y^2x = 3y^2x + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{4}x^4 + xy^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{4}x^4 + xy^3$$

Summary

The solution(s) found are the following

$$\frac{x^4}{4} + y^3x = c_1\tag{1}$$

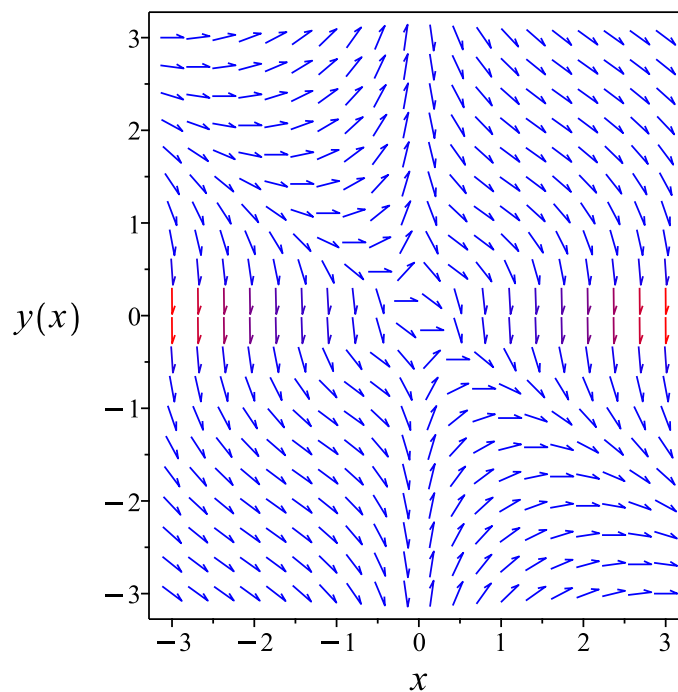


Figure 84: Slope field plot

Verification of solutions

$$\frac{x^4}{4} + y^3 x = c_1$$

Verified OK.

2.19.5 Maple step by step solution

Let's solve

$$y^3 + 3y'y^2x = -x^3$$

- Highest derivative means the order of the ODE is 1
 y'
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function
 $F'(x, y) = 0$
 - Compute derivative of lhs
 $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

$$3y^2 = 3y^2$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (x^3 + y^3) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = \frac{x^4}{4} + x y^3 + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$3y^2 x = 3y^2 x + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{1}{4} x^4 + x y^3$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{1}{4} x^4 + x y^3 = c_1$$

- Solve for y

$$\left\{ y = \frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{2x}, y = -\frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x} - \frac{\sqrt[3]{3}((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x}, y = -\frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x} + \frac{\sqrt[3]{3}((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x} \right.$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 92

```
dsolve((x^3+y(x)^3)+3*x*y(x)^2*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{1}{3}}(-x^4 - 4c_1)x^2)^{\frac{1}{3}}}{2x}$$
$$y(x) = -\frac{2^{\frac{1}{3}}(-x^4 - 4c_1)x^2)^{\frac{1}{3}}(1 + i\sqrt{3})}{4x}$$
$$y(x) = \frac{2^{\frac{1}{3}}(-x^4 - 4c_1)x^2)^{\frac{1}{3}}(i\sqrt{3} - 1)}{4x}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 99

```
DSolve[(x^3+y[x]^3)+3*x*y[x]^2*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{\sqrt[3]{-x^4 + 4c_1}}{2^{2/3}\sqrt[3]{x}}$$
$$y(x) \rightarrow -\frac{\sqrt[3]{-1}\sqrt[3]{-x^4 + 4c_1}}{2^{2/3}\sqrt[3]{x}}$$
$$y(x) \rightarrow \frac{(-1)^{2/3}\sqrt[3]{-x^4 + 4c_1}}{2^{2/3}\sqrt[3]{x}}$$

2.20 problem 45

2.20.1 Solving as first order ode lie symmetry calculated ode 356

Internal problem ID [5255]

Internal file name [OUTPUT/4746_Friday_February_02_2024_05_11_10_AM_65773297/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 45.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `class A`]]
```

$$2y - (3x + 2y - 1)y' = -3x - 1$$

2.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{3x + 2y + 1}{3x + 2y - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(3x + 2y + 1)(b_3 - a_2)}{3x + 2y - 1} - \frac{(3x + 2y + 1)^2 a_3}{(3x + 2y - 1)^2} \\ - \left(\frac{3}{3x + 2y - 1} - \frac{3(3x + 2y + 1)}{(3x + 2y - 1)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(\frac{2}{3x + 2y - 1} - \frac{2(3x + 2y + 1)}{(3x + 2y - 1)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{9x^2a_2 + 9x^2a_3 - 9x^2b_2 - 9x^2b_3 + 12xya_2 + 12xya_3 - 12xyb_2 - 12xyb_3 + 4y^2a_2 + 4y^2a_3 - 4y^2b_2 - 4y^2b_3}{(3x + 2y - 1)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -9x^2a_2 - 9x^2a_3 + 9x^2b_2 + 9x^2b_3 - 12xya_2 - 12xya_3 + 12xyb_2 \\ + 12xyb_3 - 4y^2a_2 - 4y^2a_3 + 4y^2b_2 + 4y^2b_3 + 6xa_2 - 6xa_3 - 2xb_2 \\ + 2ya_3 - 4yb_2 + 4yb_3 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -9a_2v_1^2 - 12a_2v_1v_2 - 4a_2v_2^2 - 9a_3v_1^2 - 12a_3v_1v_2 - 4a_3v_2^2 + 9b_2v_1^2 \\ + 12b_2v_1v_2 + 4b_2v_2^2 + 9b_3v_1^2 + 12b_3v_1v_2 + 4b_3v_2^2 + 6a_2v_1 - 6a_3v_1 \\ + 2a_3v_2 - 2b_2v_1 - 4b_2v_2 + 4b_3v_2 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-9a_2 - 9a_3 + 9b_2 + 9b_3)v_1^2 + (-12a_2 - 12a_3 + 12b_2 + 12b_3)v_1v_2 \\ &+ (6a_2 - 6a_3 - 2b_2)v_1 + (-4a_2 - 4a_3 + 4b_2 + 4b_3)v_2^2 \\ &+ (2a_3 - 4b_2 + 4b_3)v_2 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 6a_2 - 6a_3 - 2b_2 &= 0 \\ 2a_3 - 4b_2 + 4b_3 &= 0 \\ -12a_2 - 12a_3 + 12b_2 + 12b_3 &= 0 \\ -9a_2 - 9a_3 + 9b_2 + 9b_3 &= 0 \\ -4a_2 - 4a_3 + 4b_2 + 4b_3 &= 0 \\ 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= -9a_1 - 6b_1 \\ a_3 &= -6a_1 - 4b_1 \\ b_1 &= b_1 \\ b_2 &= -9a_1 - 6b_1 \\ b_3 &= -6a_1 - 4b_1 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -6x - 4y \\ \eta &= -6x - 4y + 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= -6x - 4y + 1 - \left(\frac{3x + 2y + 1}{3x + 2y - 1} \right) (-6x - 4y) \\ &= \frac{15x + 10y - 1}{3x + 2y - 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{15x+10y-1}{3x+2y-1}} dy\end{aligned}$$

Which results in

$$S = \frac{y}{5} - \frac{2 \ln(15x + 10y - 1)}{25}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3x + 2y + 1}{3x + 2y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6}{75x + 50y - 5} \\ S_y &= \frac{3x + 2y - 1}{15x + 10y - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{5} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{R}{5} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{5} - \frac{2 \ln(15x + 10y - 1)}{25} = \frac{x}{5} + c_1$$

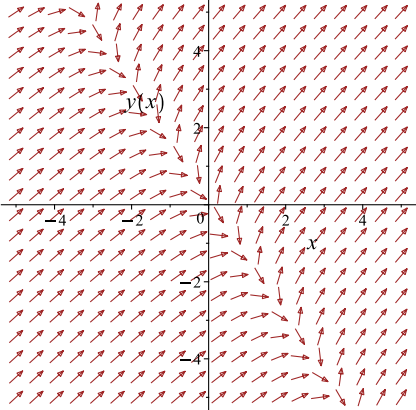
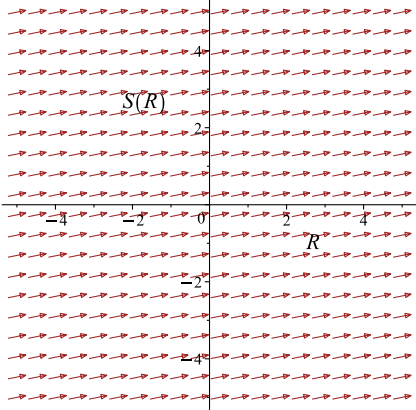
Which simplifies to

$$\frac{y}{5} - \frac{2 \ln(15x + 10y - 1)}{25} = \frac{x}{5} + c_1$$

Which gives

$$y = -\frac{3x}{2} - \frac{2 \text{LambertW}\left(-\frac{e^{-\frac{25x}{4} + \frac{1}{4} - \frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+2y+1}{3x+2y-1}$ 	$R = x$ $S = \frac{y}{5} - \frac{2 \ln(15x + 10y - 1)}{25}$	$\frac{dS}{dR} = \frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = -\frac{3x}{2} - \frac{2 \operatorname{LambertW}\left(-\frac{e^{-\frac{25x}{4} + \frac{1}{4} - \frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10} \quad (1)$$

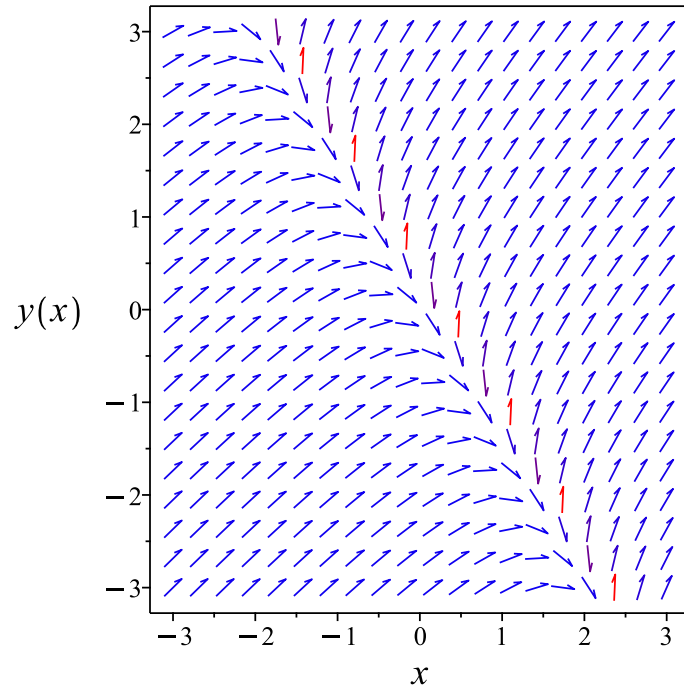


Figure 85: Slope field plot

Verification of solutions

$$y = -\frac{3x}{2} - \frac{2 \operatorname{LambertW}\left(-\frac{e^{-\frac{25x}{4} + \frac{1}{4}} - \frac{25c_1}{2}}{4}\right)}{5} + \frac{1}{10}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
  -> Calling odsolve with the ODE`, diff(y(x), x) = -3/2, y(x)` *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve((3*x+2*y(x)+1)-(3*x+2*y(x)-1)*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{3x}{2} - \frac{2 \operatorname{LambertW}\left(-\frac{c_1 e^{\frac{1}{4} - \frac{25x}{4}}}{4}\right)}{5} + \frac{1}{10}$$

✓ Solution by Mathematica

Time used: 4.841 (sec). Leaf size: 43

```
DSolve[(3*x+2*y[x]+1)-(3*x+2*y[x]-1)*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{10} \left(-4W\left(-e^{-\frac{25x}{4}-1+c_1}\right) - 15x + 1 \right)$$
$$y(x) \rightarrow \frac{1}{10} - \frac{3x}{2}$$

2.21 problem 46

2.21.1 Existence and uniqueness analysis	365
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2.21.3 Solving as linear ode	366
2.21.4 Solving as homogeneousTypeD2 ode	368
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2.21.6 Solving as exact ode	373
2.21.7 Maple step by step solution	377

Internal problem ID [5256]

Internal file name [OUTPUT/4747_Friday_February_02_2024_05_11_12_AM_99165738/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 46.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

`[_separable]`

$$xy' + 2y = 0$$

With initial conditions

$$[y(2) = 1]$$

2.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' + \frac{2y}{x} = 0$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. Hence solution exists and is unique.

2.21.2 Solving as separable ode

In canonical form the ODE is

$$y' = F(x, y)$$
$$= f(x)g(y)$$
$$= -\frac{2y}{x}$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\frac{1}{y} dy = -\frac{2}{x} dx$$
$$\int \frac{1}{y} dy = \int -\frac{2}{x} dx$$
$$\ln(y) = -2 \ln(x) + c_1$$
$$y = e^{-2 \ln(x) + c_1}$$
$$= \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1 . Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

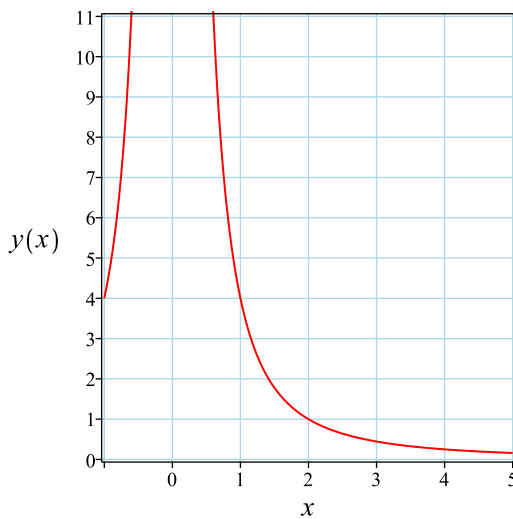
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

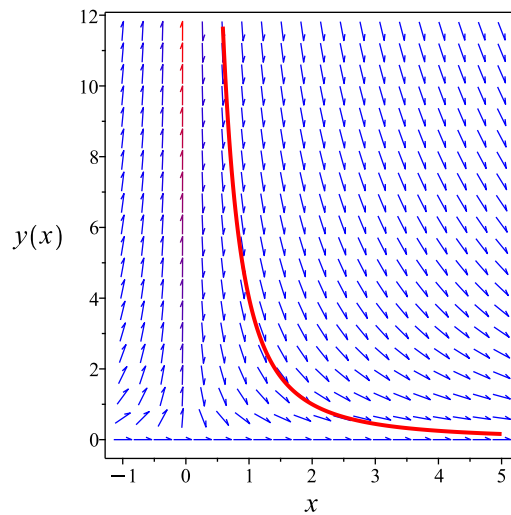
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{x} dx} \\ &= x^2 \end{aligned}$$

The ode becomes

$$\frac{d}{dx} \mu y = 0$$
$$\frac{d}{dx} (y x^2) = 0$$

Integrating gives

$$y x^2 = c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1 . Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

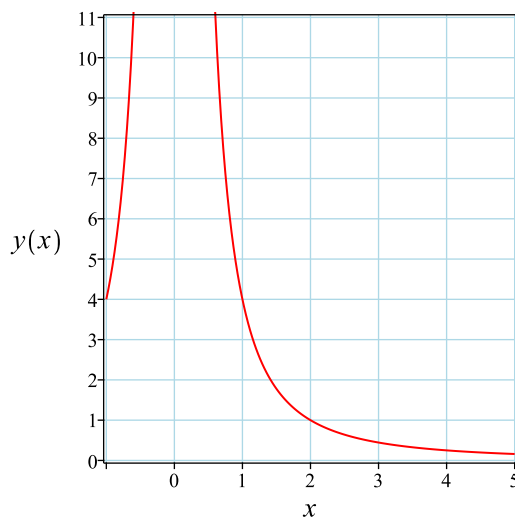
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

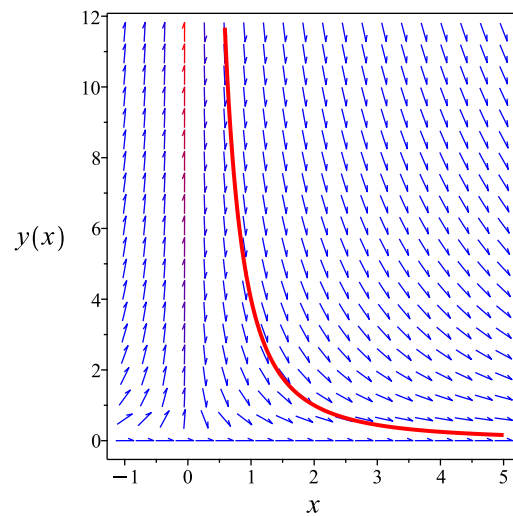
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.4 Solving as homogeneous Type D2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)x + u(x)) + 2u(x)x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{3}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{3}{x} dx \\ \ln(u) &= -3 \ln(x) + c_2 \\ u &= e^{-3 \ln(x) + c_2} \\ &= \frac{c_2}{x^3} \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{c_2}{x^2} \end{aligned}$$

Initial conditions are used to solve for c_2 . Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2}{4}$$

$$c_2 = 4$$

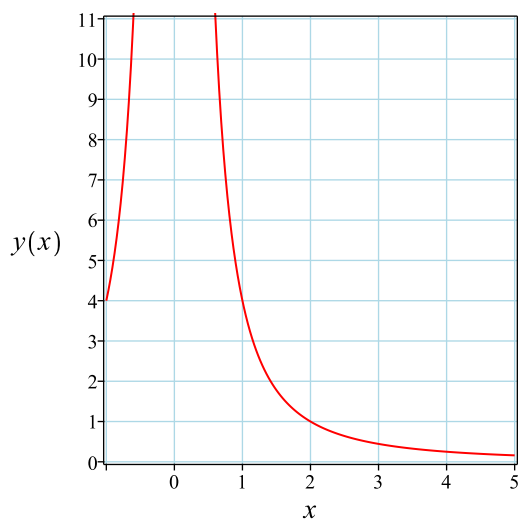
Substituting c_2 found above in the general solution gives

$$y = \frac{4}{x^2}$$

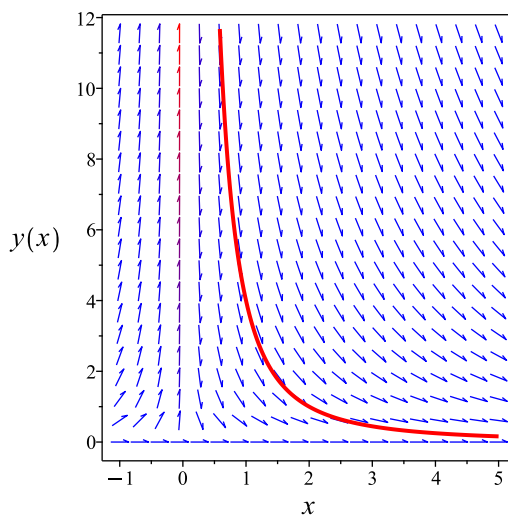
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{x^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^2 = c_1$$

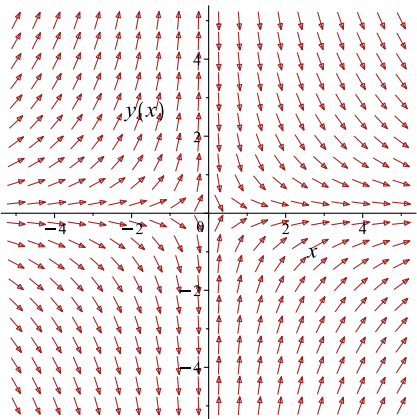
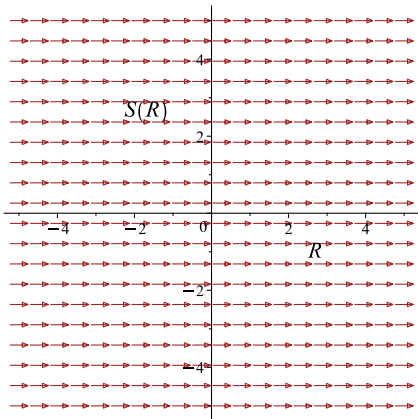
Which simplifies to

$$yx^2 = c_1$$

Which gives

$$y = \frac{c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{x}$ 	$R = x$ $S = yx^2$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1 . Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

$$c_1 = 4$$

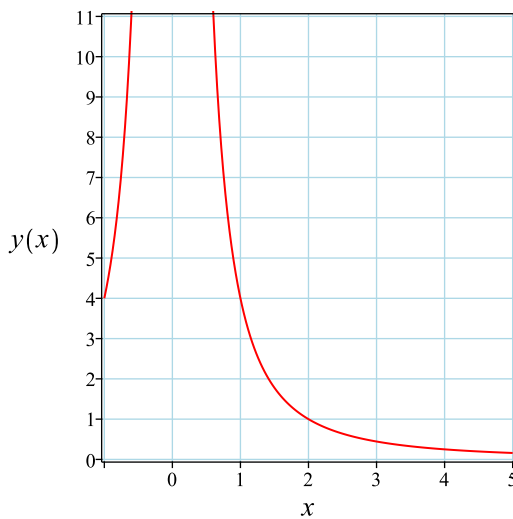
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

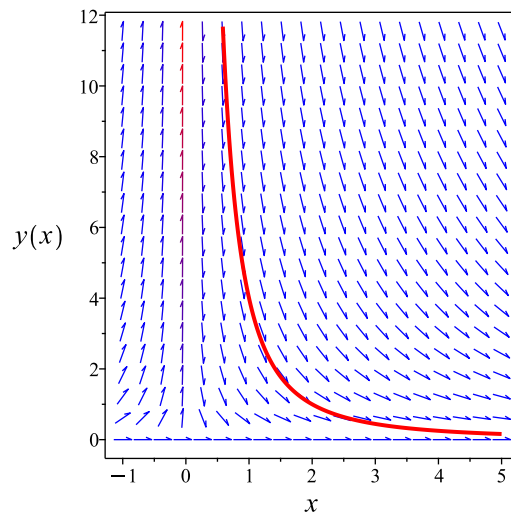
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(-\frac{1}{2y}\right) dy &= \left(\frac{1}{x}\right) dx \\ \left(-\frac{1}{x}\right) dx + \left(-\frac{1}{2y}\right) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\frac{1}{x} \\ N(x, y) &= -\frac{1}{2y} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x} \right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{1}{2y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int -\frac{1}{x} dx$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$-\frac{1}{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{1}{2y}\right) dy$$
$$f(y) = -\frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \frac{e^{-2c_1}}{x^2}$$

Initial conditions are used to solve for c_1 . Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{e^{-2c_1}}{4}$$

$$c_1 = -\ln(2)$$

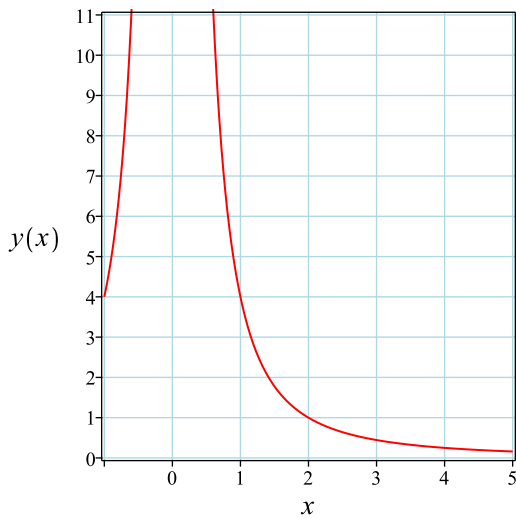
Substituting c_1 found above in the general solution gives

$$y = \frac{4}{x^2}$$

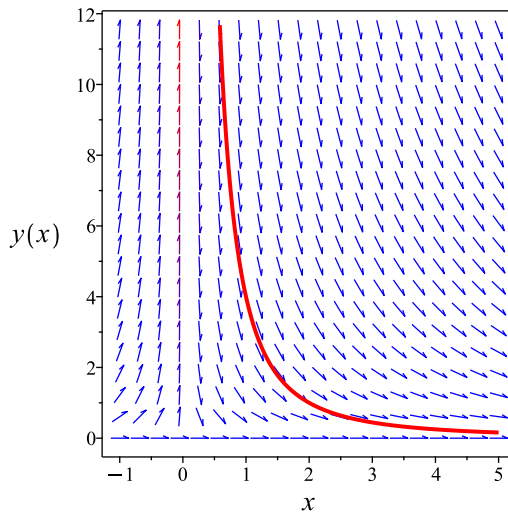
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.7 Maple step by step solution

Let's solve

$$[xy' + 2y = 0, y(2) = 1]$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = -\frac{2}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int -\frac{2}{x} dx + c_1$$

- Evaluate integral

$$\ln(y) = -2 \ln(x) + c_1$$

- Solve for y

$$y = \frac{e^{c_1}}{x^2}$$

- Use initial condition $y(2) = 1$
 $1 = \frac{e^{c_1}}{4}$
- Solve for c_1
 $c_1 = 2 \ln(2)$
- Substitute $c_1 = 2 \ln(2)$ into general solution and simplify
 $y = \frac{4}{x^2}$
- Solution to the IVP
 $y = \frac{4}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)+2*y(x)= 0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{4}{x^2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 10

```
DSolve[{x*y'[x]+2*y[x]==0,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{4}{x^2}$$

2.22 problem 47

2.22.1 Existence and uniqueness analysis	379
2.22.2 Solving as homogeneousTypeD2 ode	380
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Internal problem ID [5257]

Internal file name [OUTPUT/4748_Friday_February_02_2024_05_11_13_AM_48587445/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 47.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y^2 + xy' = -x^2$$

With initial conditions

$$[y(1) = -1]$$

2.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{x^2 + y^2}{xy} \right) \\ &= -\frac{2}{x} + \frac{x^2 + y^2}{xy^2} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

2.22.2 Solving as homogeneous Type D2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 + x^2 u(x) (u'(x)x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 + 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{2u^2+1}{u}} du &= -\frac{1}{x} dx \\ \int \frac{1}{\frac{2u^2+1}{u}} du &= \int -\frac{1}{x} dx \\ \frac{\ln(2u^2 + 1)}{4} &= -\ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$(2u^2 + 1)^{\frac{1}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^2 + 1)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^2 + 1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$(2u(x)^2 + 1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2} + 1\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

$$\left(\frac{2y^2 + x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{3}{c_3^4}\right)}{4}$. Hence the solution becomes Initial conditions are used to solve for c_3 . Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$3^{\frac{1}{4}} = c_3 \sqrt{\sqrt{3} \sqrt{\frac{1}{c_3^4}}}$$

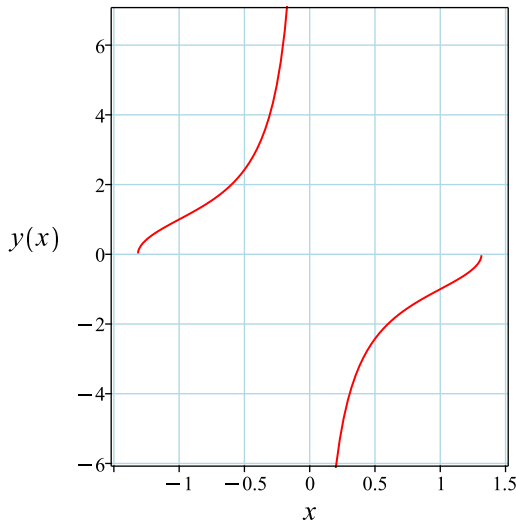
Since $\lim_{c_1 \rightarrow \infty}$ gives $\left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \sqrt{\sqrt{3} \sqrt{\frac{1}{c_3^4}}}}{x} = \left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{3^{\frac{1}{4}}}{x}$ and this result satisfies the given initial condition. Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

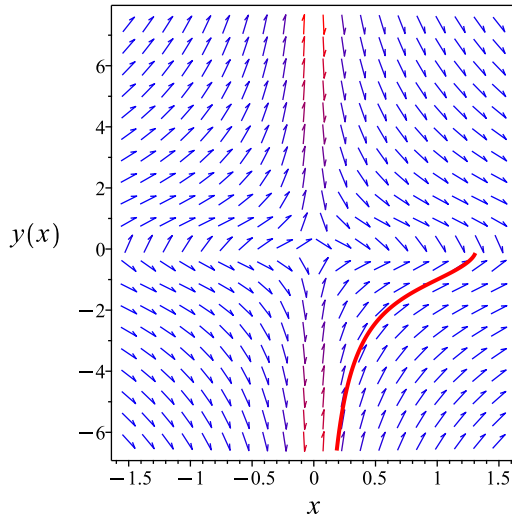
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Verified OK.

2.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 + y^2}{xy}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{yx^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$

$$R_y = 0$$

$$S_x = y^2 x$$

$$S_y = y x^2$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \quad (4)$$

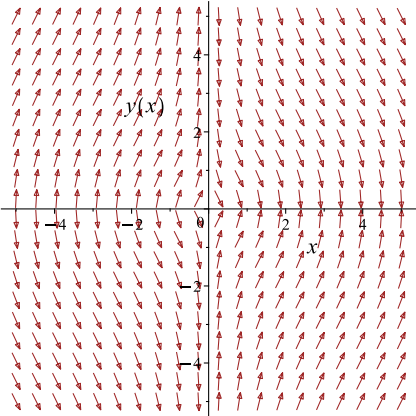
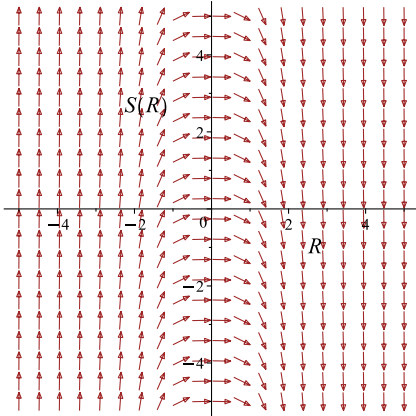
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{y^2 x^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{4} + c_1$$

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + \frac{3}{4}$$

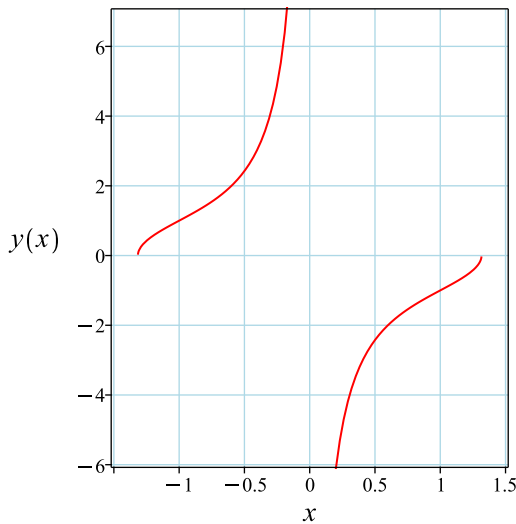
Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

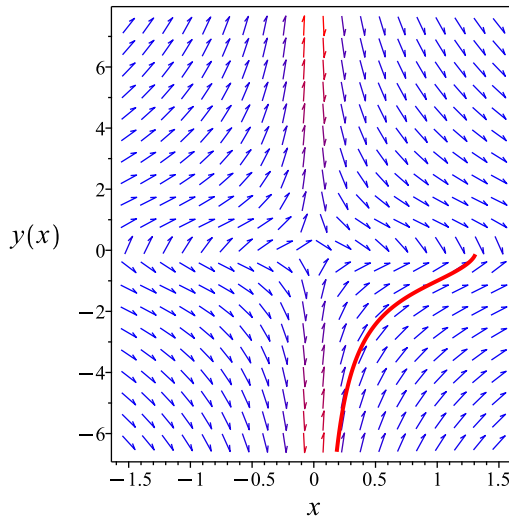
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Verified OK.

2.22.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{xy}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -x \\ n &= -1\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - x \quad (4)$$

Let

$$\begin{aligned}w &= y^{1-n} \\ &= y^2\end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\ w' &= -\frac{2w}{x} - 2x\end{aligned}\tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2}{x} \\ q(x) &= -2x\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\ \frac{d}{dx}(x^2 w) &= (x^2)(-2x) \\ d(x^2 w) &= (-2x^3) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int -2x^3 dx \\ x^2 w &= -\frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2} + c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y^2 = -\frac{x^4 - 3}{2x^2}$$

The above simplifies to

$$x^4 + 2y^2x^2 - 3 = 0$$

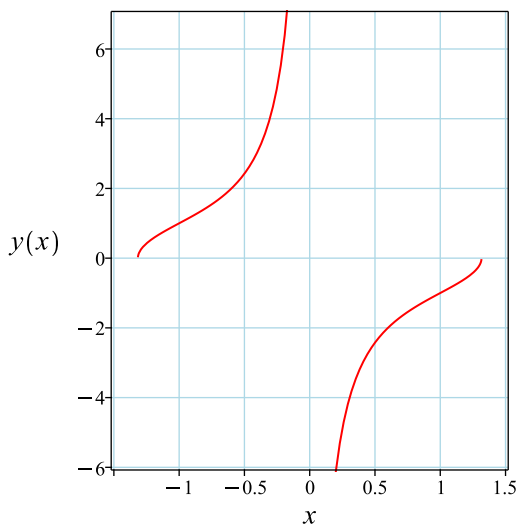
Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

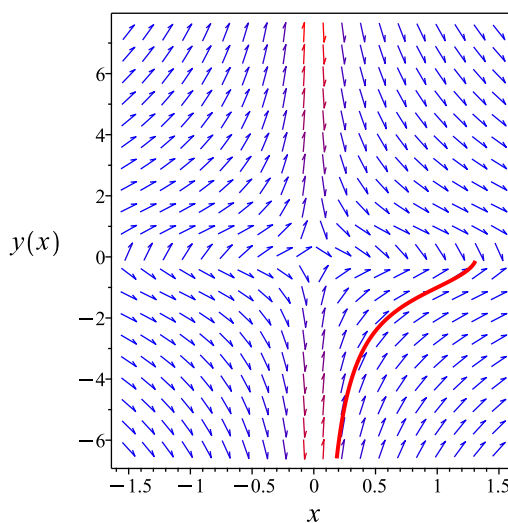
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Verified OK.

2.22.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (xy) dy &= (-x^2 - y^2) dx \\ (x^2 + y^2) dx + (xy) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2 \\N(x, y) &= xy\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{yx} ((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= x(x^2 + y^2) \\ &= x(x^2 + y^2)\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= x(xy) \\ &= yx^2\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (x(x^2 + y^2)) + (yx^2) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x(x^2 + y^2) dx \\ \phi &= \frac{(x^2 + y^2)^2}{4} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x^2 + y^2) y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y x^2$. Therefore equation (4) becomes

$$y x^2 = (x^2 + y^2) y + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int (-y^3) dy$$
$$f(y) = -\frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4}$$

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{4} = c_1$$

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = \frac{3}{4}$$

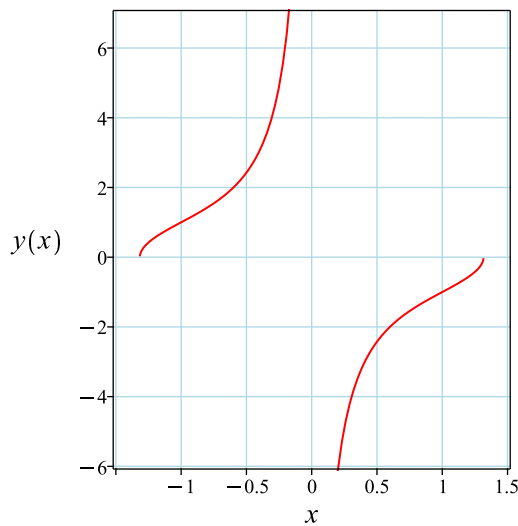
Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

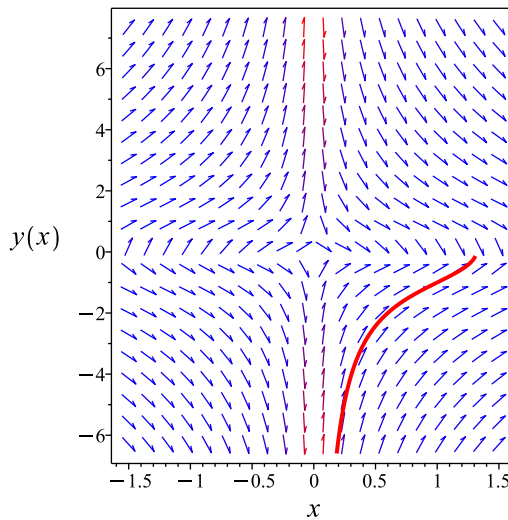
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 18

```
dsolve([(x^2+y(x)^2)+x*y(x)*diff(y(x),x)= 0,y(1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 26

```
DSolve[{(x^2+y[x]^2)+x*y[x]*y'[x]==0,{y[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{3 - x^4}}{\sqrt{2}x}$$

2.23 problem 48

2.23.1 Existence and uniqueness analysis	396
2.23.2 Solving as separable ode	397
2.23.3 Solving as first order ode lie symmetry lookup ode	399
2.23.4 Solving as exact ode	403
2.23.5 Maple step by step solution	407

Internal problem ID [5258]

Internal file name [OUTPUT/4749_Friday_February_02_2024_05_11_15_AM_62043747/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 48.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "separable", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$\cos(y) + (1 + e^{-x}) \sin(y) y' = 0$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4} \right]$$

2.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{\cos(y)}{(1 + e^{-x}) \sin(y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{\pi}{4}$ is

$$\{-2i\pi - i\pi < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{\cot(y)}{1 + e^{-x}} \end{aligned}$$

Where $f(x) = -\frac{1}{1+e^{-x}}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(y)} dy &= -\frac{1}{1 + e^{-x}} dx \\ \int \frac{1}{\cot(y)} dy &= \int -\frac{1}{1 + e^{-x}} dx \\ -\ln(\cos(y)) &= -\ln(1 + e^{-x}) + \ln(e^{-x}) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{-\ln(1+e^{-x})+\ln(e^{-x})+c_1}$$

Which simplifies to

$$\sec(y) = c_2 e^{-\ln(1+e^{-x})+\ln(e^{-x})}$$

Initial conditions are used to solve for c_1 . Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\frac{2e^{-c_1}}{c_2}\right)$$

$$c_1 = -\frac{\ln\left(\frac{c_2^2}{8}\right)}{2}$$

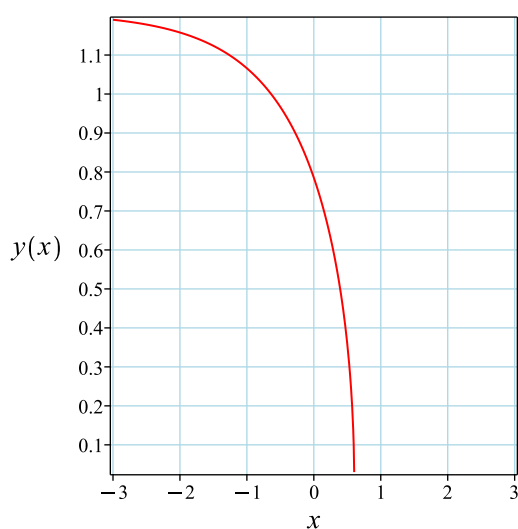
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

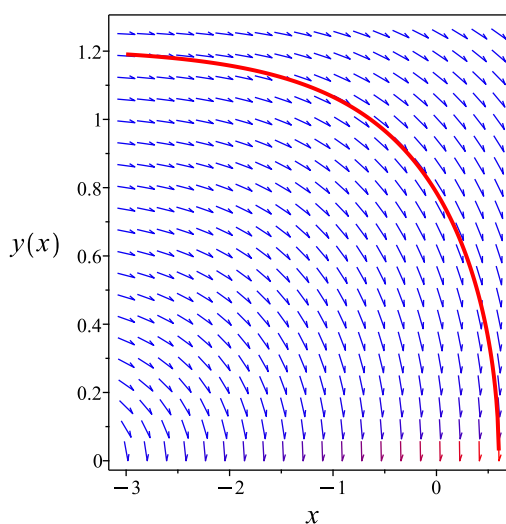
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right) \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

Verified OK. {positive}

2.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{\cos(y)}{(1 + e^{-x}) \sin(y)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= -1 - e^{-x} \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-1 - e^{-x}} dx\end{aligned}$$

Which results in

$$S = -\ln(1 + e^{-x}) + \ln(e^{-x})$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y}\tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\cos(y)}{(1 + e^{-x}) \sin(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0 \\R_y &= 1 \\S_x &= -\frac{1}{1 + e^{-x}} \\S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(y) \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(1 + e^{-x}) - x = -\ln(\cos(y)) + c_1$$

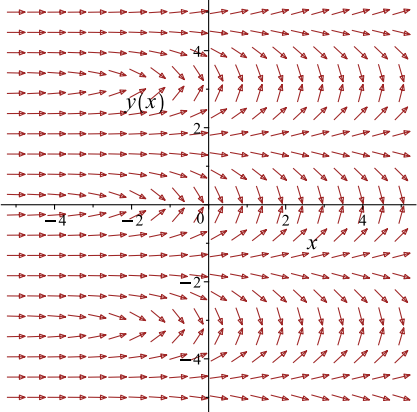
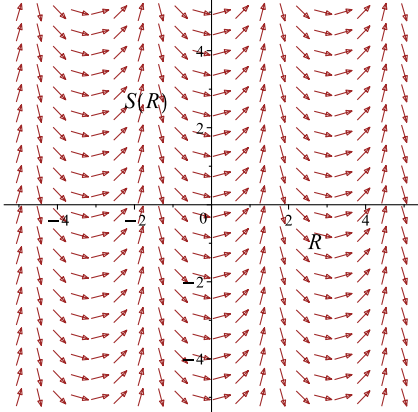
Which simplifies to

$$-\ln(1 + e^{-x}) - x = -\ln(\cos(y)) + c_1$$

Which gives

$$y = \arccos(e^{x+c_1}(1 + e^x)e^{-x})$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\cos(y)}{(1+e^{-x})\sin(y)}$ 	$R = y$ $S = -\ln(1 + e^{-x}) - x$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1 . Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin(2e^{c_1})$$

$$c_1 = -\frac{3 \ln(2)}{2}$$

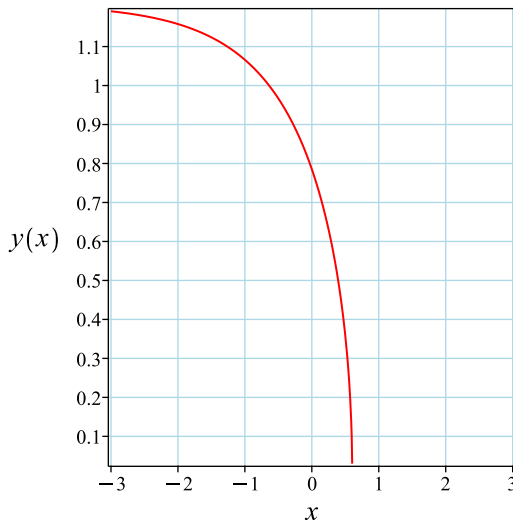
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

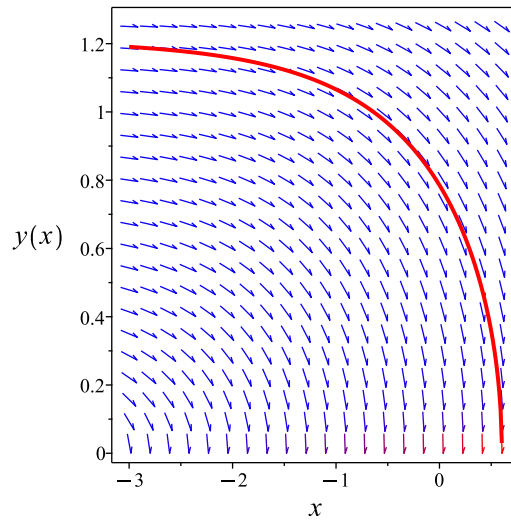
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

Verified OK. {positive}

2.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}\left(-\frac{\sin(y)}{\cos(y)}\right) dy &= \left(\frac{1}{1+e^{-x}}\right) dx \\ \left(-\frac{1}{1+e^{-x}}\right) dx + \left(-\frac{\sin(y)}{\cos(y)}\right) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{1+e^{-x}} \\ N(x, y) &= -\frac{\sin(y)}{\cos(y)}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{1+e^{-x}}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{\sin(y)}{\cos(y)} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{1 + e^{-x}} dx \\ \phi &= -\ln(1 + e^{-x}) + \ln(e^{-x}) + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\frac{\sin(y)}{\cos(y)} \\ &= -\tan(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y) dy = \int (-\tan(y)) dy$$

$$f(y) = \ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(1 + e^{-x}) + \ln(e^{-x}) + \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(1 + e^{-x}) + \ln(e^{-x}) + \ln(\cos(y))$$

Initial conditions are used to solve for c_1 . Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{3 \ln(2)}{2} = c_1$$

$$c_1 = -\frac{3 \ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\ln(1 + e^{-x}) + \ln(e^{-x}) + \ln(\cos(y)) = -\frac{3 \ln(2)}{2}$$

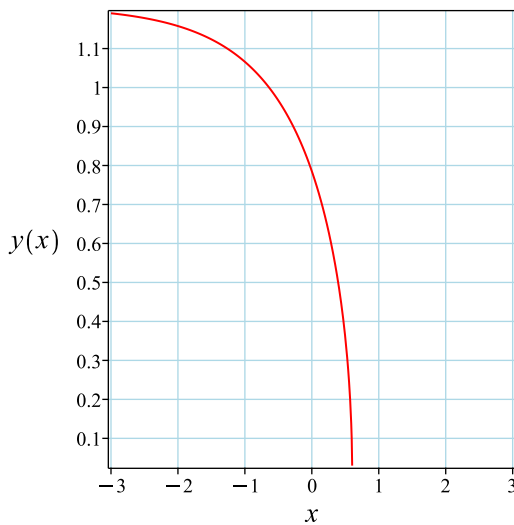
Solving for y from the above gives

$$y = \arccos\left(\frac{(1 + e^x) \sqrt{2}}{4}\right)$$

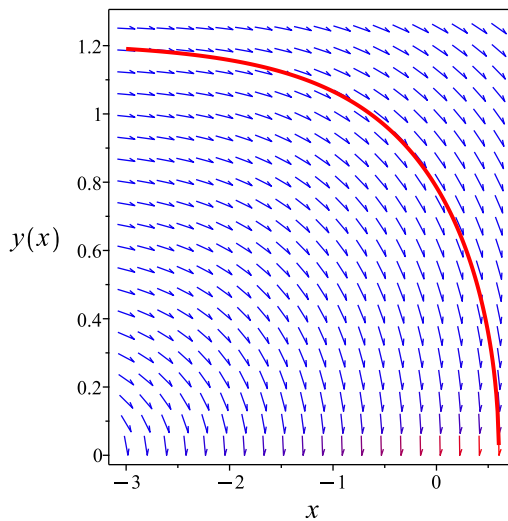
Summary

The solution(s) found are the following

$$y = \arccos\left(\frac{(1 + e^x) \sqrt{2}}{4}\right) \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

Verified OK. {positive}

2.23.5 Maple step by step solution

Let's solve

$$[\cos(y) + (1 + e^{-x}) \sin(y) y' = 0, y(0) = \frac{\pi}{4}]$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y' \sin(y)}{\cos(y)} = -\frac{1}{1+e^{-x}}$$

- Integrate both sides with respect to x

$$\int \frac{y' \sin(y)}{\cos(y)} dx = \int -\frac{1}{1+e^{-x}} dx + c_1$$

- Evaluate integral

$$-\ln(\cos(y)) = -\ln(1 + e^{-x}) + \ln(e^{-x}) + c_1$$

- Solve for y

$$y = \arccos\left(\frac{e^{-c_1+x}(1+e^x)}{e^x}\right)$$

- Use initial condition $y(0) = \frac{\pi}{4}$

$$\frac{\pi}{4} = \arccos(2e^{-c_1})$$

- Solve for c_1

$$c_1 = \frac{3\ln(2)}{2}$$

- Substitute $c_1 = \frac{3\ln(2)}{2}$ into general solution and simplify

$$y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$$

- Solution to the IVP

$$y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`

```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 14

```
dsolve([cos(y(x))+(1+exp(-x))*sin(y(x))*diff(y(x),x)= 0,y(0) = 1/4*Pi],y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\sqrt{2}(e^x + 1)}{4}\right)$$

✓ Solution by Mathematica

Time used: 46.229 (sec). Leaf size: 20

```
DSolve[{Cos[y[x]]+(1+Exp[-x])*Sin[y[x]]*y'[x]== 0,{y[0]==Pi/4}},y[x],x,IncludeSingularSoluti
```

$$y(x) \rightarrow \arccos\left(\frac{e^x + 1}{2\sqrt{2}}\right)$$

2.24 problem 49

2.24.1 Existence and uniqueness analysis	410
2.24.2 Solving as first order ode lie symmetry lookup ode	411
2.24.3 Solving as bernoulli ode	416
2.24.4 Solving as riccati ode	419

Internal problem ID [5259]

Internal file name [OUTPUT/4750_Friday_February_02_2024_05_11_16_AM_8396201/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 49.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$y^2 + xy - xy' = 0$$

With initial conditions

$$[y(1) = 1]$$

2.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(x+y)}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{y(x+y)}{x} \right) \\ &= \frac{x+y}{x} + \frac{y}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y(x+y)}{x} \\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{-x}y^2\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{e^x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(x + y)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{e^x}{y} \\ S_y &= \frac{e^x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^x}{x} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^R}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\exp\text{Integral}_1(-R) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^x}{y} = -\exp\text{Integral}_1(-x) + c_1$$

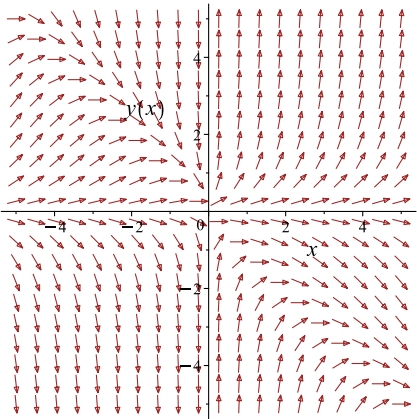
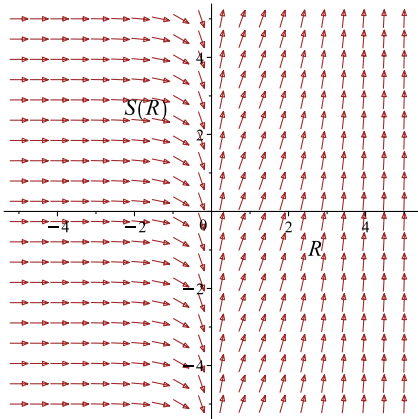
Which simplifies to

$$-\frac{e^x}{y} = -\exp\text{Integral}_1(-x) + c_1$$

Which gives

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x+y)}{x}$ 	$R = x$ $S = -\frac{e^x}{y}$	$\frac{dS}{dR} = \frac{e^R}{R}$ 

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{e}{\exp\text{Integral}_1(-1) - c_1}$$

$$c_1 = \exp\text{Integral}_1(-1) - e$$

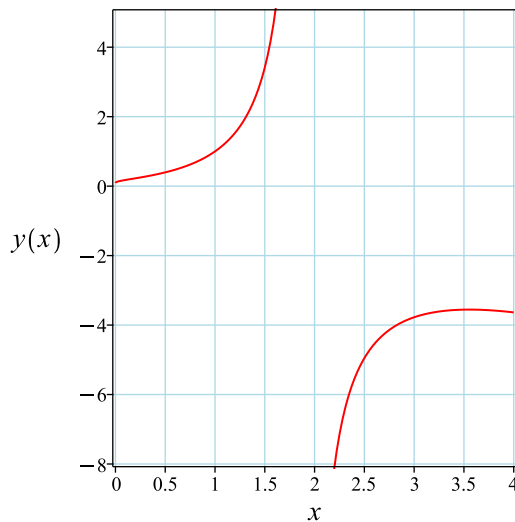
Substituting c_1 found above in the general solution gives

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e}$$

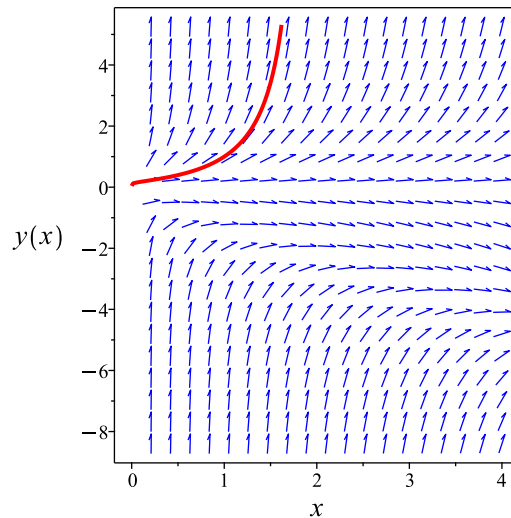
Summary

The solution(s) found are the following

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e} \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e}$$

Verified OK.

2.24.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= \frac{y(x+y)}{x}\end{aligned}$$

This is a Bernoulli ODE.

$$y' = y + \frac{1}{x}y^2 \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= 1 \\ f_1(x) &= \frac{1}{x} \\ n &= 2\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{1}{y} + \frac{1}{x} \quad (4)$$

Let

$$\begin{aligned}w &= y^{1-n} \\ &= \frac{1}{y}\end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= w(x) + \frac{1}{x} \\ w' &= -w - \frac{1}{x} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -\frac{1}{x} \end{aligned}$$

Hence the ode is

$$w'(x) + w(x) = -\frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu) \left(-\frac{1}{x} \right) \\ \frac{d}{dx}(e^x w) &= (e^x) \left(-\frac{1}{x} \right) \\ d(e^x w) &= \left(-\frac{e^x}{x} \right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^x w &= \int -\frac{e^x}{x} dx \\ e^x w &= \text{expIntegral}_1(-x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$w(x) = e^{-x} \text{expIntegral}_1(-x) + c_1 e^{-x}$$

which simplifies to

$$w(x) = e^{-x}(\text{expIntegral}_1(-x) + c_1)$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = e^{-x}(\text{expIntegral}_1(-x) + c_1)$$

Or

$$y = \frac{e^x}{\text{expIntegral}_1(-x) + c_1}$$

Which is simplified to

$$y = \frac{e^x}{\text{expIntegral}_1(-x) + c_1}$$

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{e}{\text{expIntegral}_1(-1) + c_1}$$

$$c_1 = -\text{expIntegral}_1(-1) + e$$

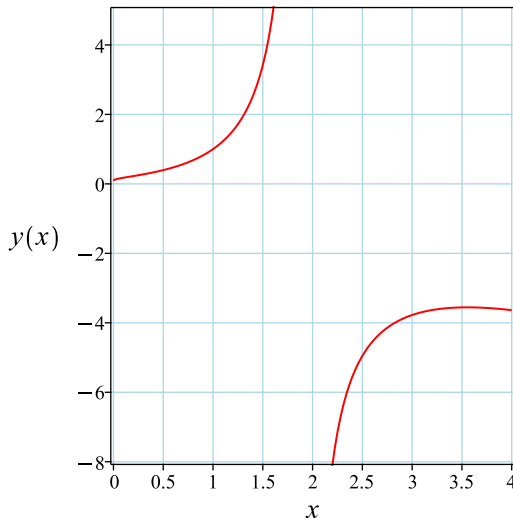
Substituting c_1 found above in the general solution gives

$$y = \frac{e^x}{\text{expIntegral}_1(-x) - \text{expIntegral}_1(-1) + e}$$

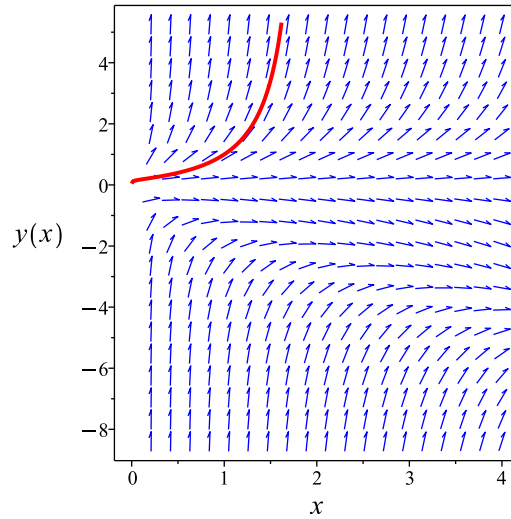
Summary

The solution(s) found are the following

$$y = \frac{e^x}{\text{expIntegral}_1(-x) - \text{expIntegral}_1(-1) + e} \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{\expIntegral_1(-x) - \expIntegral_1(-1) + e}$$

Verified OK.

2.24.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x+y)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y + \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 1$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \quad (2)$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} - \left(-\frac{1}{x^2} + \frac{1}{x} \right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \text{expIntegral}_1(-x) c_2$$

The above shows that

$$u'(x) = -\frac{e^x c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{e^x c_2}{c_1 + \text{expIntegral}_1(-x) c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{e^x}{c_3 + \text{expIntegral}_1(-x)}$$

Initial conditions are used to solve for c_3 . Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{e}{c_3 + \text{expIntegral}_1(-1)}$$

$$c_3 = -\exp\text{Integral}_1(-1) + e$$

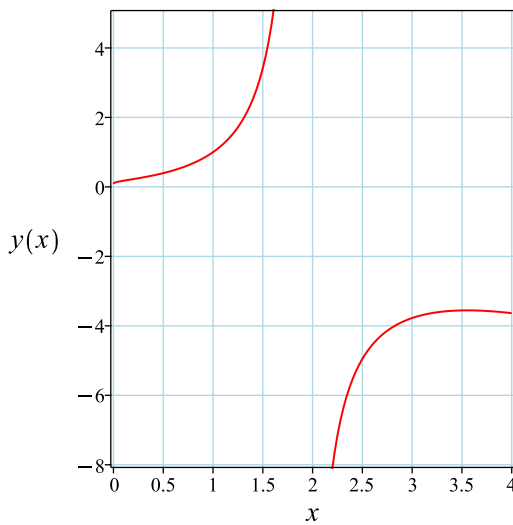
Substituting c_3 found above in the general solution gives

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e}$$

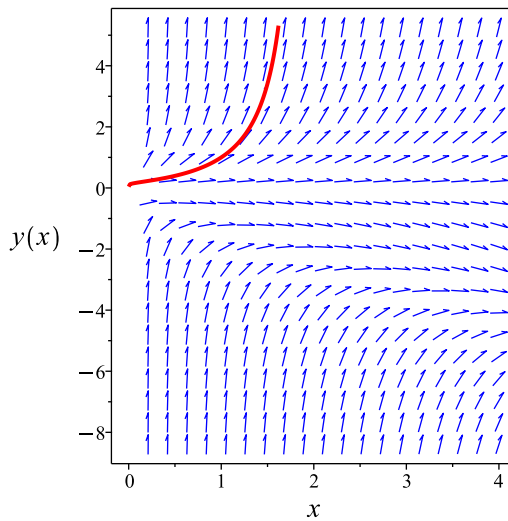
Summary

The solution(s) found are the following

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e} \quad (1)$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{\exp\text{Integral}_1(-x) - \exp\text{Integral}_1(-1) + e}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 22

```
dsolve([(y(x)^2+x*y(x))-x*diff(y(x),x)= 0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^x}{\expIntegral_1(-x) + e - \expIntegral_1(-1)}$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 19

```
DSolve[{(y[x]^2+x*y[x])-x*y'[x]== 0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^x}{- \text{ExpIntegralEi}(x) + \text{ExpIntegralEi}(1) + e}$$

2.25 problem 51

2.25.1 Solving as first order ode lie symmetry calculated ode 423

2.25.2 Solving as riccati ode 429

Internal problem ID [5260]

Internal file name [OUTPUT/4751_Friday_February_02_2024_05_11_32_AM_46710413/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 51.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**riccati**", "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$y' + 2(3y + 2x)^2 = 0$$

2.25.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -2(3y + 2x)^2$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1\text{E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2\text{E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - 2(3y + 2x)^2 (b_3 - a_2) - 4(3y + 2x)^4 a_3 \\ - (-16x - 24y)(xa_2 + ya_3 + a_1) - (-36y - 24x)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\begin{aligned} -64x^4 a_3 - 384x^3 y a_3 - 864x^2 y^2 a_3 - 864x y^3 a_3 - 324y^4 a_3 + 24x^2 a_2 \\ + 24x^2 b_2 - 8x^2 b_3 + 48x y a_2 + 16x y a_3 + 36x y b_2 + 18y^2 a_2 \\ + 24y^2 a_3 + 18y^2 b_3 + 16x a_1 + 24x b_1 + 24y a_1 + 36y b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} -64x^4 a_3 - 384x^3 y a_3 - 864x^2 y^2 a_3 - 864x y^3 a_3 - 324y^4 a_3 + 24x^2 a_2 \\ + 24x^2 b_2 - 8x^2 b_3 + 48x y a_2 + 16x y a_3 + 36x y b_2 + 18y^2 a_2 \\ + 24y^2 a_3 + 18y^2 b_3 + 16x a_1 + 24x b_1 + 24y a_1 + 36y b_1 + b_2 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -64a_3 v_1^4 - 384a_3 v_1^3 v_2 - 864a_3 v_1^2 v_2^2 - 864a_3 v_1 v_2^3 - 324a_3 v_2^4 + 24a_2 v_1^2 \\ + 48a_2 v_1 v_2 + 18a_2 v_2^2 + 16a_3 v_1 v_2 + 24a_3 v_2^2 + 24b_2 v_1^2 + 36b_2 v_1 v_2 \\ - 8b_3 v_1^2 + 18b_3 v_2^2 + 16a_1 v_1 + 24a_1 v_2 + 24b_1 v_1 + 36b_1 v_2 + b_2 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
 & -64a_3v_1^4 - 384a_3v_1^3v_2 - 864a_3v_1^2v_2^2 + (24a_2 + 24b_2 - 8b_3)v_1^2 \\
 & - 864a_3v_1v_2^3 + (48a_2 + 16a_3 + 36b_2)v_1v_2 + (16a_1 + 24b_1)v_1 \\
 & - 324a_3v_2^4 + (18a_2 + 24a_3 + 18b_3)v_2^2 + (24a_1 + 36b_1)v_2 + b_2 = 0
 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 b_2 &= 0 \\
 -864a_3 &= 0 \\
 -384a_3 &= 0 \\
 -324a_3 &= 0 \\
 -64a_3 &= 0 \\
 16a_1 + 24b_1 &= 0 \\
 24a_1 + 36b_1 &= 0 \\
 18a_2 + 24a_3 + 18b_3 &= 0 \\
 24a_2 + 24b_2 - 8b_3 &= 0 \\
 48a_2 + 16a_3 + 36b_2 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= -\frac{3b_1}{2} \\
 a_2 &= 0 \\
 a_3 &= 0 \\
 b_1 &= b_1 \\
 b_2 &= 0 \\
 b_3 &= 0
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= -\frac{3}{2} \\
 \eta &= 1
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= 1 - (-2(3y + 2x)^2) \left(-\frac{3}{2}\right) \\ &= -12x^2 - 36xy - 27y^2 + 1 \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-12x^2 - 36xy - 27y^2 + 1} dy\end{aligned}$$

Which results in

$$S = \frac{\sqrt{3} \operatorname{arctanh}\left(\frac{(54y+36x)\sqrt{3}}{18}\right)}{9}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2(3y + 2x)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{3(1 - 3(3y + 2x)^2)} \\ S_y &= \frac{1}{1 - 3(3y + 2x)^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{3} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{2R}{3} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{3} \operatorname{arctanh}((3y + 2x)\sqrt{3})}{9} = \frac{2x}{3} + c_1$$

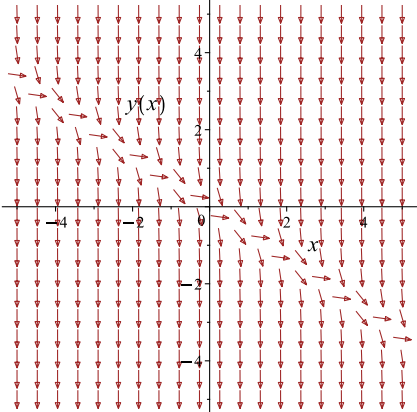
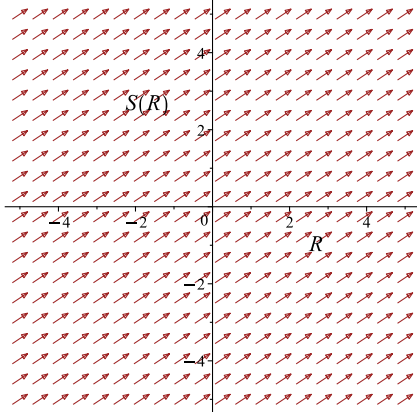
Which simplifies to

$$\frac{\sqrt{3} \operatorname{arctanh}((3y + 2x)\sqrt{3})}{9} = \frac{2x}{3} + c_1$$

Which gives

$$y = -\frac{(2\sqrt{3}x - \tanh((3c_1 + 2x)\sqrt{3}))\sqrt{3}}{9}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2(3y + 2x)^2$ 	$R = x$ $S = \frac{\sqrt{3} \operatorname{arctanh}((3y + 2x)\sqrt{3})}{9}$	$\frac{dS}{dR} = \frac{2}{3}$ 

Summary

The solution(s) found are the following

$$y = -\frac{(2\sqrt{3}x - \tanh((3c_1 + 2x)\sqrt{3}))\sqrt{3}}{9} \quad (1)$$

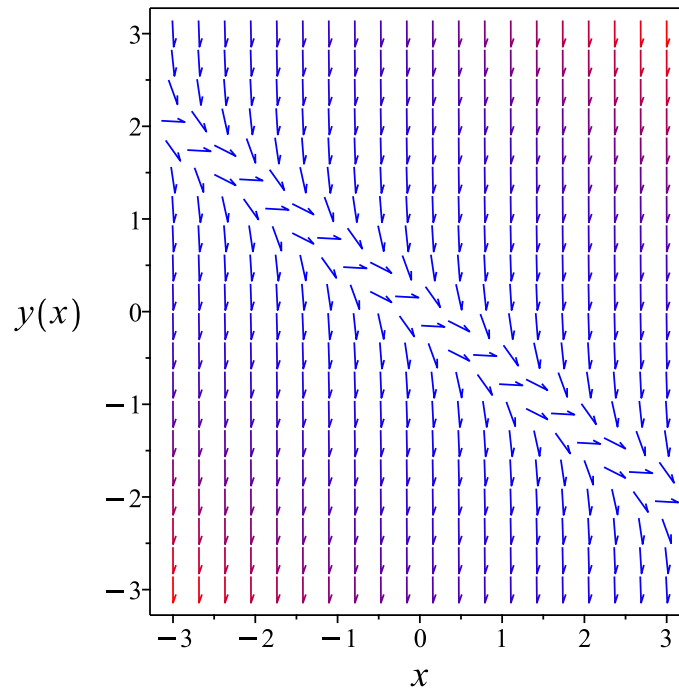


Figure 101: Slope field plot

Verification of solutions

$$y = -\frac{(2\sqrt{3}x - \tanh((3c_1 + 2x)\sqrt{3}))\sqrt{3}}{9}$$

Verified OK.

2.25.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -2(3y + 2x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -8x^2 - 24xy - 18y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -8x^2$, $f_1(x) = -24x$ and $f_2(x) = -18$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-18u} \end{aligned} \quad (1)$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \quad (2)$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 432x \\ f_2^2 f_0 &= -2592x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-18u''(x) - 432xu'(x) - 2592x^2u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 e^{-6x^2+2\sqrt{3}x} + c_2 e^{-6x^2-2\sqrt{3}x}$$

The above shows that

$$u'(x) = -12 \left(x - \frac{\sqrt{3}}{6} \right) c_1 e^{-6x^2+2\sqrt{3}x} - 12c_2 \left(x + \frac{\sqrt{3}}{6} \right) e^{-6x^2-2\sqrt{3}x}$$

Using the above in (1) gives the solution

$$y = \frac{-12 \left(x - \frac{\sqrt{3}}{6} \right) c_1 e^{-6x^2+2\sqrt{3}x} - 12c_2 \left(x + \frac{\sqrt{3}}{6} \right) e^{-6x^2-2\sqrt{3}x}}{18c_1 e^{-6x^2+2\sqrt{3}x} + 18c_2 e^{-6x^2-2\sqrt{3}x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-6x - \sqrt{3}) e^{-6x^2-2\sqrt{3}x} - 6 \left(x - \frac{\sqrt{3}}{6} \right) c_3 e^{-6x^2+2\sqrt{3}x}}{9c_3 e^{-6x^2+2\sqrt{3}x} + 9 e^{-6x^2-2\sqrt{3}x}}$$

Summary

The solution(s) found are the following

$$y = \frac{(-6x - \sqrt{3}) e^{-6x^2 - 2\sqrt{3}x} - 6\left(x - \frac{\sqrt{3}}{6}\right) c_3 e^{-6x^2 + 2\sqrt{3}x}}{9c_3 e^{-6x^2 + 2\sqrt{3}x} + 9e^{-6x^2 - 2\sqrt{3}x}} \quad (1)$$

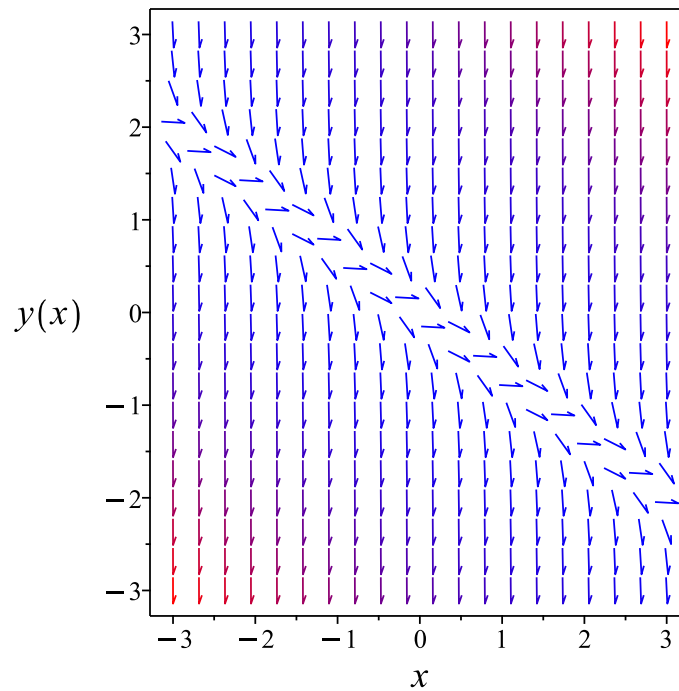


Figure 102: Slope field plot

Verification of solutions

$$y = \frac{(-6x - \sqrt{3}) e^{-6x^2 - 2\sqrt{3}x} - 6\left(x - \frac{\sqrt{3}}{6}\right) c_3 e^{-6x^2 + 2\sqrt{3}x}}{9c_3 e^{-6x^2 + 2\sqrt{3}x} + 9e^{-6x^2 - 2\sqrt{3}x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
  -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)` *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)= -2*(2*x+3*y(x))^2,y(x), singsol=all)
```

$$y(x) = -\frac{2x}{3} - \frac{\sqrt{3} \tanh(2(-x + c_1)\sqrt{3})}{9}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 59

```
DSolve[y'[x]==-2*(2*x+3*y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{9} \left(-6x - \frac{6}{\sqrt{3} + 12c_1 e^{4\sqrt{3}x}} + \sqrt{3} \right)$$
$$y(x) \rightarrow \frac{1}{9} (\sqrt{3} - 6x)$$

2.26 problem 52

Internal problem ID [5261]

Internal file name [OUTPUT/4752_Friday_February_02_2024_05_11_34_AM_63088358/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplementary problems. Page 22

Problem number: 52.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

Unable to solve or complete the solution.

$$-2 \sin(y) + (2x - 4 \sin(y) - 3) \cos(y) y' = -x - 3$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5` [0, (-3+8*sin(y)-4*x)/(-2*x+4*sin(y)+3)/cos(y)]
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 22

```
dsolve((x-2*sin(y(x))+3)+(2*x-4*sin(y(x))-3)*cos(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin \left(\frac{9 \operatorname{LambertW} \left(\frac{c_1 e^{-\frac{1}{3} - \frac{8x}{9}}}{9} \right)}{8} + \frac{3}{8} + \frac{x}{2} \right)$$

✓ Solution by Mathematica

Time used: 60.95 (sec). Leaf size: 73

```
DSolve[(x-2*Sin[y[x]]+3)+(2*x-4*Sin[y[x]]-3)*Cos[y[x]]*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \rightarrow \arcsin \left(\frac{1}{8} \left(9W \left(-\frac{1}{9} e^{-\frac{2}{9}(4x+3-8c_1)} \right) + 4x + 3 \right) \right)$$

$$y(x) \rightarrow \arcsin \left(\frac{1}{8} \left(9W \left(-\frac{1}{9} e^{-\frac{2}{9}(4x+3-8c_1)} \right) + 4x + 3 \right) \right)$$

3 Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

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3.1 problem 23 (a)

3.1.1 Solving as exact ode	438
3.1.2 Maple step by step solution	442

Internal problem ID [5262]

Internal file name [OUTPUT/4753_Friday_February_02_2024_05_11_37_AM_91689604/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

`[_linear]`

$$-y - xy' = -x^2$$

3.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x) dy &= (-x^2 + y) dx \\ (x^2 - y) dx + (-x) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 - y \\ N(x, y) &= -x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int x^2 - y dx$$

$$\phi = \frac{1}{3}x^3 - xy + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -x$. Therefore equation (4) becomes

$$-x = -x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 - xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 - xy$$

The solution becomes

$$y = -\frac{-x^3 + 3c_1}{3x}$$

Summary

The solution(s) found are the following

$$y = -\frac{-x^3 + 3c_1}{3x} \tag{1}$$

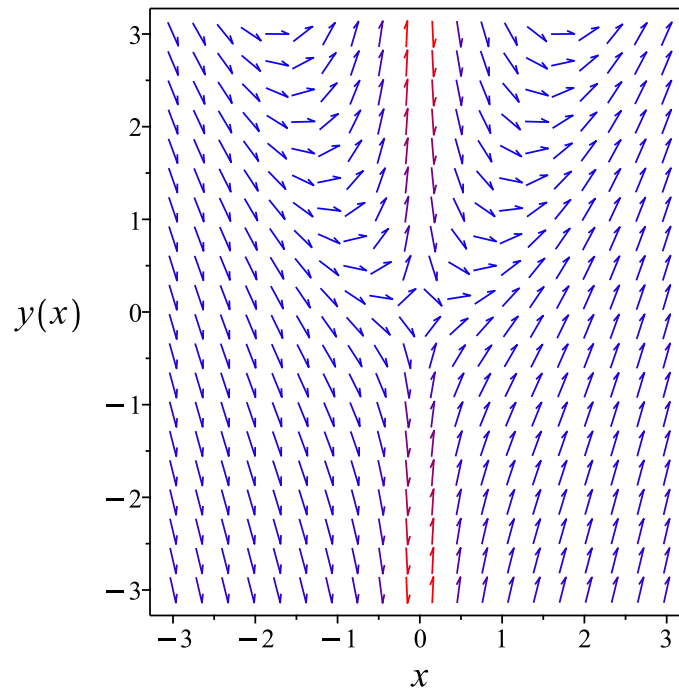


Figure 103: Slope field plot

Verification of solutions

$$y = -\frac{-x^3 + 3c_1}{3x}$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$$-y - xy' = -x^2$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = -\frac{y}{x} + x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{y}{x} = x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{y}{x} \right) = \mu(x) x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) \left(y' + \frac{y}{x} \right) = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int \mu(x) x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int \mu(x) x dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x$

$$y = \frac{\int x^2 dx + c_1}{x}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{x^3}{3} + c_1}{x}$$

- Simplify

$$y = \frac{x^3 + 3c_1}{3x}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve((x^2-y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1}{3x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[(x^2-y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^2}{3} + \frac{c_1}{x}$$

3.2 problem 23 (d)

3.2.1 Solving as exact ode	444
3.2.2 Maple step by step solution	447

Internal problem ID [5263]

Internal file name [OUTPUT/4754_Friday_February_02_2024_05_11_38_AM_91193230/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$y^2 + 2xyy' = -x^2$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy) dy &= (-x^2 - y^2) dx \\ (x^2 + y^2) dx + (2xy) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2 \\ N(x, y) &= 2xy\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x^2 + y^2) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int x^2 + y^2 dx$$

$$\phi = \frac{1}{3}x^3 + y^2x + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y^2x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y^2x$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + xy^2 = c_1 \quad (1)$$

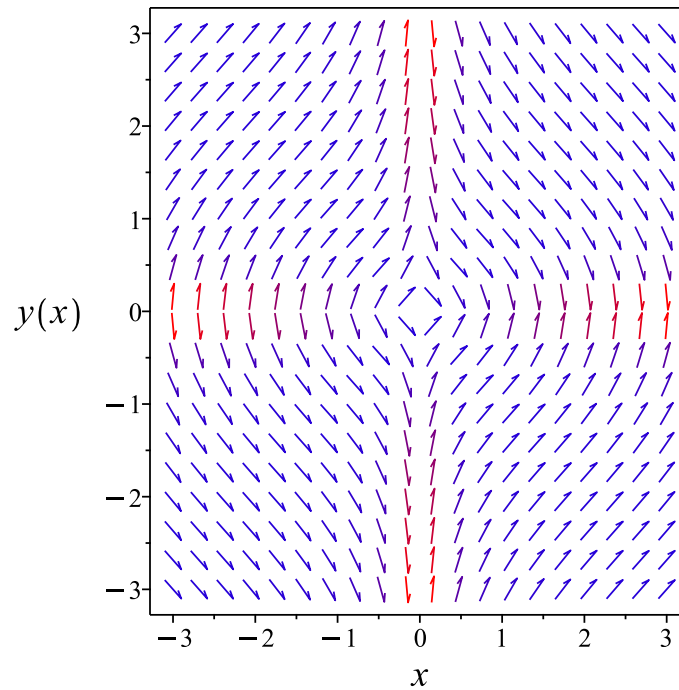


Figure 104: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + xy^2 = c_1$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$$y^2 + 2xyy' = -x^2$$

- Highest derivative means the order of the ODE is 1
 y'
- Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$
- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$$
- Evaluate derivatives

$$2y = 2y$$
- Condition met, ODE is exact
- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (x^2 + y^2) dx + f_1(y)$$
- Evaluate integral

$$F(x, y) = \frac{x^3}{3} + y^2x + f_1(y)$$
- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$
- Compute derivative

$$2xy = 2xy + \frac{d}{dy} f_1(y)$$
- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = 0$$
- Solve for $f_1(y)$

$$f_1(y) = 0$$
- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{1}{3}x^3 + y^2x$$
- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{1}{3}x^3 + y^2x = c_1$$
- Solve for y

$$\left\{ y = -\frac{\sqrt{3} \sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3} \sqrt{x(-x^3+3c_1)}}{3x} \right\}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x}$$
$$y(x) = \frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{-x^3 + 3c_1}}{\sqrt{3}\sqrt{x}}$$
$$y(x) \rightarrow \frac{\sqrt{-x^3 + 3c_1}}{\sqrt{3}\sqrt{x}}$$

3.3 problem 23 (e)

3.3.1 Solving as exact ode	450
3.3.2 Maple step by step solution	454

Internal problem ID [5264]

Internal file name [OUTPUT/4755_Friday_February_02_2024_05_11_38_AM_77911395/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (e).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

`[_linear]`

$$y \cos(x) + \sin(x) y' = -x$$

3.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(\sin(x)) dy &= (-y \cos(x) - x) dx \\ (x + y \cos(x)) dx + (\sin(x)) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + y \cos(x) \\ N(x, y) &= \sin(x)\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y \cos(x)) \\ &= \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\ &= \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int x + y \cos(x) dx$$

$$\phi = \frac{x^2}{2} + y \sin(x) + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} + y \sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} + y \sin(x)$$

The solution becomes

$$y = \frac{-x^2 + 2c_1}{2 \sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2 \sin(x)} \tag{1}$$

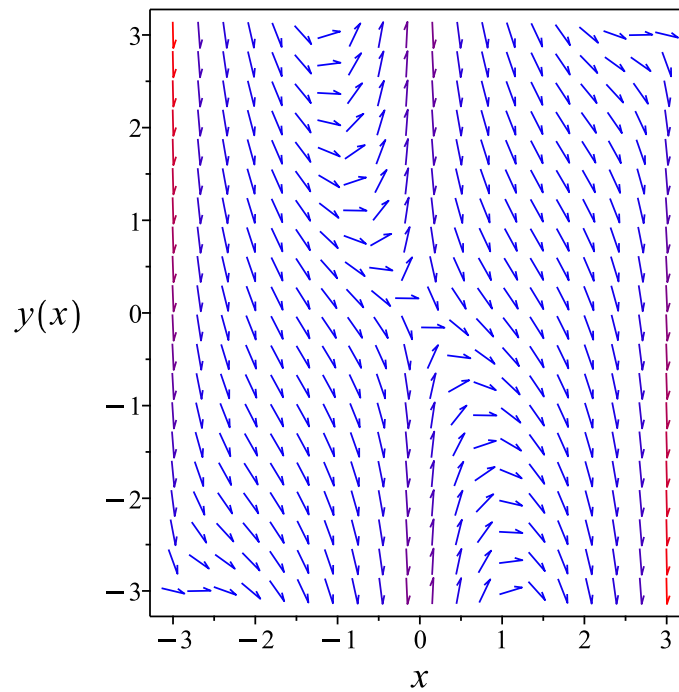


Figure 105: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2 \sin(x)}$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$$y \cos(x) + \sin(x) y' = -x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = -\frac{\cos(x)y}{\sin(x)} - \frac{x}{\sin(x)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{\cos(x)y}{\sin(x)} = -\frac{x}{\sin(x)}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{\cos(x)y}{\sin(x)} \right) = -\frac{\mu(x)x}{\sin(x)}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' + \frac{\cos(x)y}{\sin(x)} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)\cos(x)}{\sin(x)}$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int -\frac{\mu(x)x}{\sin(x)} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int -\frac{\mu(x)x}{\sin(x)} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)x}{\sin(x)} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int -x dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{x^2}{2} + c_1}{\sin(x)}$$

- Simplify

$$y = -\frac{(x^2 - 2c_1) \csc(x)}{2}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x+y(x)*cos(x))+sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{(x^2 - 2c_1) \csc(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 19

```
DSolve[(x+y[x]*Cos[x])+Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2}(x^2 - 2c_1) \csc(x)$$

3.4 problem 23 (h)

3.4.1 Solving as exact ode	456
3.4.2 Maple step by step solution	460

Internal problem ID [5265]

Internal file name [OUTPUT/4756_Friday_February_02_2024_05_11_38_AM_9973994/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (h).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
  type`, `class A`]]
```

$3y + (3x + 4y + 5)y' = -2x - 4$

3.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x + 4y + 5) dy &= (-2x - 3y - 4) dx \\ (2x + 3y + 4) dx + (3x + 4y + 5) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2x + 3y + 4 \\ N(x, y) &= 3x + 4y + 5\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + 3y + 4) \\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x + 4y + 5) \\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int 2x + 3y + 4 dx \\ \phi &= x(x + 3y + 4) + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x + 4y + 5$. Therefore equation (4) becomes

$$3x + 4y + 5 = 3x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y + 5$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) dy &= \int (4y + 5) dy \\ f(y) &= 2y^2 + 5y + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x + 3y + 4) + 2y^2 + 5y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x + 3y + 4) + 2y^2 + 5y$$

Summary

The solution(s) found are the following

$$x(x + 3y + 4) + 2y^2 + 5y = c_1 \tag{1}$$

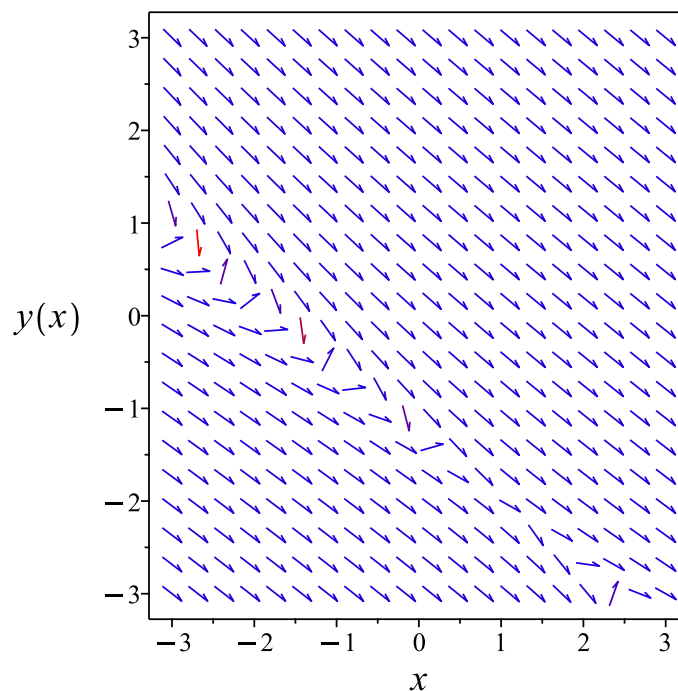


Figure 106: Slope field plot

Verification of solutions

$$x(x + 3y + 4) + 2y^2 + 5y = c_1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$$3y + (3x + 4y + 5) y' = -2x - 4$$

- Highest derivative means the order of the ODE is 1
 y'
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function
 $F'(x, y) = 0$
 - Compute derivative of lhs
 $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
 - Evaluate derivatives
 $3 = 3$
 - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
 $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x
 $F(x, y) = \int (2x + 3y + 4) dx + f_1(y)$
- Evaluate integral
 $F(x, y) = x^2 + 3xy + 4x + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y
 $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative
 $3x + 4y + 5 = 3x + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$
 $\frac{d}{dy} f_1(y) = 4y + 5$
- Solve for $f_1(y)$
 $f_1(y) = 2y^2 + 5y$
- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = x^2 + 3xy + 2y^2 + 4x + 5y$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x^2 + 3xy + 2y^2 + 4x + 5y = c_1$$

- Solve for y

$$\left\{ y = -\frac{3x}{4} - \frac{5}{4} - \frac{\sqrt{x^2 + 8c_1 - 2x + 25}}{4}, y = -\frac{3x}{4} - \frac{5}{4} + \frac{\sqrt{x^2 + 8c_1 - 2x + 25}}{4} \right\}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 32

```
dsolve((2*x+3*y(x)+4)+(3*x+4*y(x)+5)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{(x-1)^2 c_1^2 + 8} + (-3x - 5) c_1}{4c_1}$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 61

```
DSolve[(2*x+3*y[x]+4)+(3*x+4*y[x]+5)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4} \left(-\sqrt{x^2 - 2x + 25 + 16c_1} - 3x - 5 \right)$$

$$y(x) \rightarrow \frac{1}{4} \left(\sqrt{x^2 - 2x + 25 + 16c_1} - 3x - 5 \right)$$

3.5 problem 23 (i)

3.5.1 Solving as exact ode	463
3.5.2 Maple step by step solution	467

Internal problem ID [5266]

Internal file name [OUTPUT/4757_Friday_February_02_2024_05_11_39_AM_99484607/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (i).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational]
```

$$4x^3y^3 + \left(3y^2x^4 - \frac{1}{y}\right) y' = -\frac{1}{x}$$

3.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}\left(3y^2x^4 - \frac{1}{y}\right) dy &= \left(-4y^3x^3 - \frac{1}{x}\right) dx \\ \left(4y^3x^3 + \frac{1}{x}\right) dx + \left(3y^2x^4 - \frac{1}{y}\right) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 4y^3x^3 + \frac{1}{x} \\ N(x, y) &= 3y^2x^4 - \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(4y^3x^3 + \frac{1}{x}\right) \\ &= 12y^2x^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(3y^2x^4 - \frac{1}{y} \right) \\ &= 12y^2x^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int 4y^3x^3 + \frac{1}{x} dx \\ \phi &= \ln(x) + y^3x^4 + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2x^4 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2x^4 - \frac{1}{y}$. Therefore equation (4) becomes

$$3y^2x^4 - \frac{1}{y} = 3y^2x^4 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(-\frac{1}{y} \right) dy \\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + y^3 x^4 - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + y^3 x^4 - \ln(y)$$

The solution becomes

$$y = e^{-\frac{\text{LambertW}(-3x^7 e^{-3c_1})}{3} - c_1 x}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{\text{LambertW}(-3x^7 e^{-3c_1})}{3} - c_1 x} \tag{1}$$

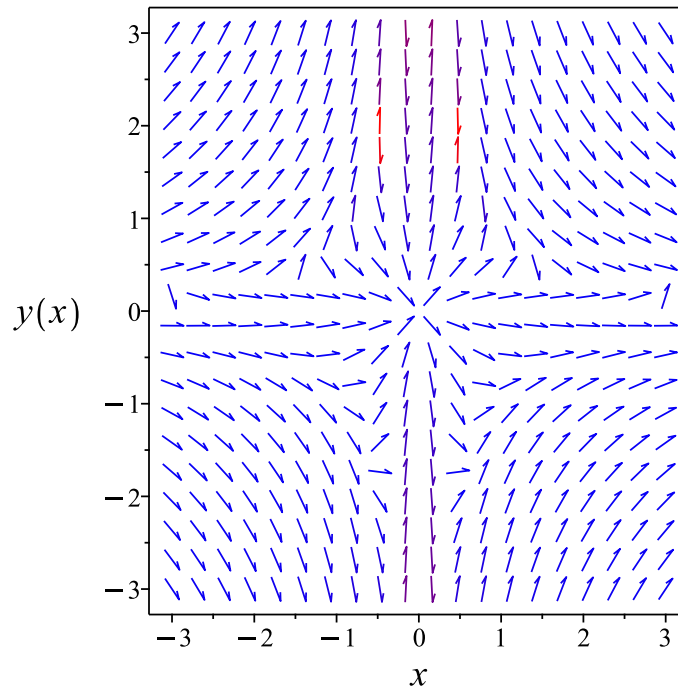


Figure 107: Slope field plot

Verification of solutions

$$y = e^{-\frac{\text{LambertW}\left(-3x^7 e^{-3c_1}\right)}{3} - c_1} x$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$$4x^3y^3 + \left(3y^2x^4 - \frac{1}{y}\right) y' = -\frac{1}{x}$$

- Highest derivative means the order of the ODE is 1
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function
 - $F'(x, y) = 0$
 - Compute derivative of lhs
 - $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
 - Evaluate derivatives
 - $12y^2x^3 = 12y^2x^3$
 - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
- $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x
- $F(x, y) = \int \left(4y^3x^3 + \frac{1}{x}\right) dx + f_1(y)$
- Evaluate integral
- $F(x, y) = \ln(x) + y^3x^4 + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y
- $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative
- $3y^2x^4 - \frac{1}{y} = 3y^2x^4 + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = -\frac{1}{y}$$

- Solve for $f_1(y)$

$$f_1(y) = -\ln(y)$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \ln(x) + y^3 x^4 - \ln(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\ln(x) + y^3 x^4 - \ln(y) = c_1$$

- Solve for y

$$y = \frac{x}{e^{\frac{\text{LambertW}(-3x^7 e^{-3c_1})}{3} + c_1}}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 25

```
dsolve((4*x^3*y(x)^3+1/x)+(3*x^4*y(x)^2-1/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{2}{3}}}{3 \left(-\frac{x^4}{\text{LambertW}(-3c_1 x^7)} \right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.154 (sec). Leaf size: 108

```
DSolve[(4*x^3*y[x]^3+1/x)+(3*x^4*y[x]^2-1/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow \frac{\sqrt[3]{-\frac{1}{3}} \sqrt[3]{W(-3e^{-3c_1}x^7)}}{x^{4/3}}$$
$$y(x) \rightarrow -\frac{\sqrt[3]{W(-3e^{-3c_1}x^7)}}{\sqrt[3]{3}x^{4/3}}$$
$$y(x) \rightarrow -\frac{(-1)^{2/3} \sqrt[3]{W(-3e^{-3c_1}x^7)}}{\sqrt[3]{3}x^{4/3}}$$
$$y(x) \rightarrow 0$$

3.6 problem 23 (j)

3.6.1 Solving as exact ode	470
3.6.2 Maple step by step solution	474

Internal problem ID [5267]

Internal file name [OUTPUT/4758_Friday_February_02_2024_05_11_39_AM_54772160/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (j).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$2uv + (u^2 + v^2) v' = -2u^2$$

3.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(u, v) du + N(u, v) dv = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(u^2 + v^2) dv &= (-2u^2 - 2uv) du \\ (2u^2 + 2uv) du + (u^2 + v^2) dv &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(u, v) &= 2u^2 + 2uv \\ N(u, v) &= u^2 + v^2\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}(2u^2 + 2uv) \\ &= 2u\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}(u^2 + v^2) \\ &= 2u\end{aligned}$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial \phi}{\partial u} = M \quad (1)$$

$$\frac{\partial \phi}{\partial v} = N \quad (2)$$

Integrating (1) w.r.t. u gives

$$\int \frac{\partial \phi}{\partial u} du = \int M du$$

$$\int \frac{\partial \phi}{\partial u} du = \int 2u^2 + 2uv du$$

$$\phi = \frac{u^2(2u + 3v)}{3} + f(v) \quad (3)$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both u and v . Taking derivative of equation (3) w.r.t v gives

$$\frac{\partial \phi}{\partial v} = u^2 + f'(v) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = u^2 + v^2$. Therefore equation (4) becomes

$$u^2 + v^2 = u^2 + f'(v) \quad (5)$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = v^2$$

Integrating the above w.r.t v gives

$$\int f'(v) dv = \int (v^2) dv$$

$$f(v) = \frac{v^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = \frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{u^2(2u + 3v)}{3} + \frac{v^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} = c_1 \quad (1)$$

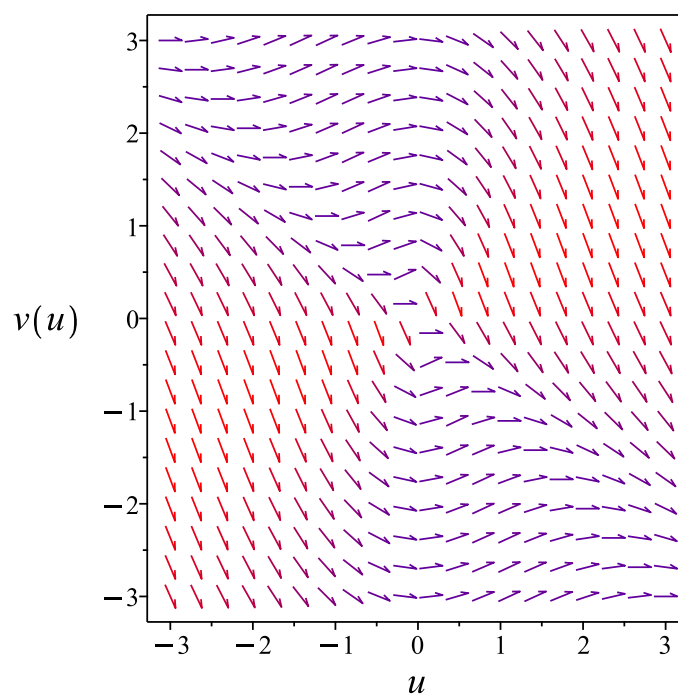


Figure 108: Slope field plot

Verification of solutions

$$\frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} = c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$$2uv + (u^2 + v^2) v' = -2u^2$$

- Highest derivative means the order of the ODE is 1
 v'
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function
 $F'(u, v) = 0$
 - Compute derivative of lhs
 $F'(u, v) + \left(\frac{\partial}{\partial v} F(u, v)\right) v' = 0$
 - Evaluate derivatives
 $2u = 2u$
 - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
 $[F(u, v) = c_1, M(u, v) = F'(u, v), N(u, v) = \frac{\partial}{\partial v} F(u, v)]$
- Solve for $F(u, v)$ by integrating $M(u, v)$ with respect to u
 $F(u, v) = \int (2u^2 + 2uv) du + f_1(v)$
- Evaluate integral
 $F(u, v) = \frac{2u^3}{3} + u^2v + f_1(v)$
- Take derivative of $F(u, v)$ with respect to v
 $N(u, v) = \frac{\partial}{\partial v} F(u, v)$
- Compute derivative
 $u^2 + v^2 = u^2 + \frac{d}{dv} f_1(v)$
- Isolate for $\frac{d}{dv} f_1(v)$
 $\frac{d}{dv} f_1(v) = v^2$
- Solve for $f_1(v)$
 $f_1(v) = \frac{v^3}{3}$
- Substitute $f_1(v)$ into equation for $F(u, v)$
 $F(u, v) = \frac{2}{3}u^3 + u^2v + \frac{1}{3}v^3$

- Substitute $F(u, v)$ into the solution of the ODE

$$\frac{2}{3}u^3 + u^2v + \frac{1}{3}v^3 = c_1$$

- Solve for v

$$\left\{ v = \frac{\left(-8u^3+12c_1+4\sqrt{8u^6-12u^3c_1+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2u^2}{\left(-8u^3+12c_1+4\sqrt{8u^6-12u^3c_1+9c_1^2}\right)^{\frac{1}{3}}}, v = -\frac{\left(-8u^3+12c_1+4\sqrt{8u^6-12u^3c_1+9c_1^2}\right)^{\frac{1}{3}}}{4} \right.$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 321

`dsolve(2*(u^2+u*v(u))+(u^2+v(u)^2)*diff(v(u),u)=0,v(u), singsol=all)`

$$v(u) = -\frac{2\left(u^2c_1 - \frac{\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}}{4}\right)}{\sqrt{c_1}\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$v(u) = -\frac{(1+i\sqrt{3})\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}}$$

$$-\frac{u^2\sqrt{c_1}(i\sqrt{3}-1)}{\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$v(u)$$

$$= \frac{4i\sqrt{3}c_1u^2 + i\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}\sqrt{3} + 4u^2c_1 - \left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}{4\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 15.565 (sec). Leaf size: 593

`DSolve [2*(u^2+u*v[u])+(u^2+v[u]^2)*v'[u]==0,v[u],u,IncludeSingularSolutions -> True]`

$$v(u) \rightarrow \frac{\sqrt[3]{-2u^3 + \sqrt{8u^6 - 4e^{3c_1}u^3 + e^{6c_1}} + e^{3c_1}}}{\sqrt[3]{2} \sqrt[3]{2u^2}}$$

$$v(u) \rightarrow \frac{\sqrt[3]{2}(2 + 2i\sqrt{3})u^2 + i2^{2/3}(\sqrt{3} + i)(-2u^3 + \sqrt{8u^6 - 4e^{3c_1}u^3 + e^{6c_1}} + e^{3c_1})^{2/3}}{4\sqrt[3]{-2u^3 + \sqrt{8u^6 - 4e^{3c_1}u^3 + e^{6c_1}} + e^{3c_1}}(1 - i\sqrt{3})u^2}$$

$$v(u) \rightarrow \frac{2^{2/3}\sqrt[3]{-2u^3 + \sqrt{8u^6 - 4e^{3c_1}u^3 + e^{6c_1}} + e^{3c_1}}}{(1 + i\sqrt{3})\sqrt[3]{-2u^3 + \sqrt{8u^6 - 4e^{3c_1}u^3 + e^{6c_1}} + e^{3c_1}} - \frac{2\sqrt[3]{2}}{u^2}}$$

$$v(u) \rightarrow \sqrt[3]{\sqrt{2}\sqrt{u^6} - u^3} - \frac{2\sqrt[3]{2}}{u^2}$$

$$v(u) \rightarrow \frac{(1 - i\sqrt{3})u^2 + (-1 - i\sqrt{3})(\sqrt{2}\sqrt{u^6} - u^3)^{2/3}}{2\sqrt[3]{\sqrt{2}\sqrt{u^6} - u^3}}$$

$$v(u) \rightarrow \frac{(1 + i\sqrt{3})u^2 + i(\sqrt{3} + i)(\sqrt{2}\sqrt{u^6} - u^3)^{2/3}}{2\sqrt[3]{\sqrt{2}\sqrt{u^6} - u^3}}$$

3.7 problem 23 (k)

3.7.1 Solving as exact ode	478
3.7.2 Maple step by step solution	482

Internal problem ID [5268]

Internal file name [OUTPUT/4759_Friday_February_02_2024_05_11_39_AM_80796220/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (k).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

[_exact]

$$x\sqrt{x^2 + y^2} - y + \left(\sqrt{x^2 + y^2} y - x\right) y' = 0$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(\sqrt{x^2 + y^2} y - x) dy &= (-x\sqrt{x^2 + y^2} + y) dx \\ (x\sqrt{x^2 + y^2} - y) dx + (\sqrt{x^2 + y^2} y - x) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x\sqrt{x^2 + y^2} - y \\ N(x, y) &= \sqrt{x^2 + y^2} y - x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (x\sqrt{x^2 + y^2} - y) \\ &= \frac{yx}{\sqrt{x^2 + y^2}} - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\sqrt{x^2 + y^2} y - x \right) \\ &= \frac{yx}{\sqrt{x^2 + y^2}} - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x\sqrt{x^2 + y^2} - y dx \\ \phi &= \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sqrt{x^2 + y^2} y - x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{x^2 + y^2} y - x$. Therefore equation (4) becomes

$$\sqrt{x^2 + y^2} y - x = \sqrt{x^2 + y^2} y - x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy = c_1 \quad (1)$$

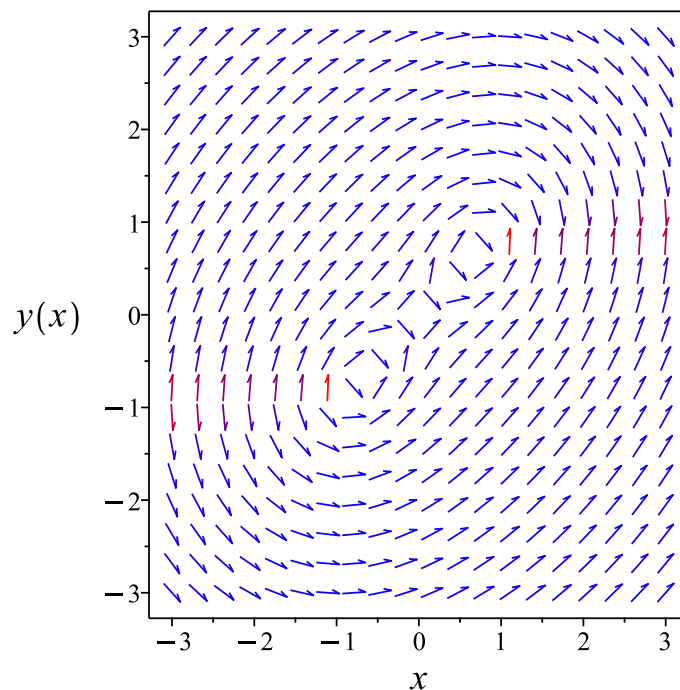


Figure 109: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy = c_1$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$$x\sqrt{x^2 + y^2} - y + (\sqrt{x^2 + y^2} y - x) y' = 0$$

- Highest derivative means the order of the ODE is 1
- y'
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$$

- Evaluate derivatives

$$\frac{yx}{\sqrt{x^2+y^2}} - 1 = \frac{yx}{\sqrt{x^2+y^2}} - 1$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (x\sqrt{x^2 + y^2} - y) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$\sqrt{x^2 + y^2} y - x = \sqrt{x^2 + y^2} y - x + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy = c_1$$

- Solve for y

$$y = \frac{\text{RootOf}(_Z^6 - 6c_1_Z^3 - 9x^2_Z^2 + 9x^4 + 9c_1^2)^3 - 3c_1}{3x}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve((x*sqrt(x^2+y(x)^2)-y(x))+(y(x)*sqrt(x^2+y(x)^2)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{(x^2 + y(x)^2)^{\frac{3}{2}}}{3} - xy(x) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 30.753 (sec). Leaf size: 319

```
DSolve[(x*Sqrt[x^2+y[x]^2]-y[x])+(y[x]*Sqrt[x^2+y[x]^2]-x)*y'[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 1]$$

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 2]$$

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 3]$$

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 4]$$

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 5]$$

$$y(x) \rightarrow \text{Root}[\#1^6 + 3\#1^4x^2 + \#1^2(3x^4 - 9x^2) - 18\#1c_1x + x^6 - 9c_1^2\&, 6]$$

3.8 problem 23 (m)

3.8.1 Solving as exact ode	484
3.8.2 Maple step by step solution	488

Internal problem ID [5269]

Internal file name [OUTPUT/4760_Friday_February_02_2024_05_11_40_AM_79254509/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (m).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd  
type`, `class A`]]
```

$$y - (y - x + 3)y' = -1 - x$$

3.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y + x - 3) dy &= (-y - x - 1) dx \\ (x + y + 1) dx + (-y + x - 3) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + y + 1 \\ N(x, y) &= -y + x - 3\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y + 1) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y + x - 3) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x + y + 1 dx \\ \phi &= \frac{x(x + 2y + 2)}{2} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y + x - 3$. Therefore equation (4) becomes

$$-y + x - 3 = x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3 - y$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) dy &= \int (-3 - y) dy \\ f(y) &= -3y - \frac{1}{2}y^2 + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x + 2y + 2)}{2} - 3y - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x + 2y + 2)}{2} - 3y - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(x + 2y + 2)}{2} - 3y - \frac{y^2}{2} = c_1 \quad (1)$$

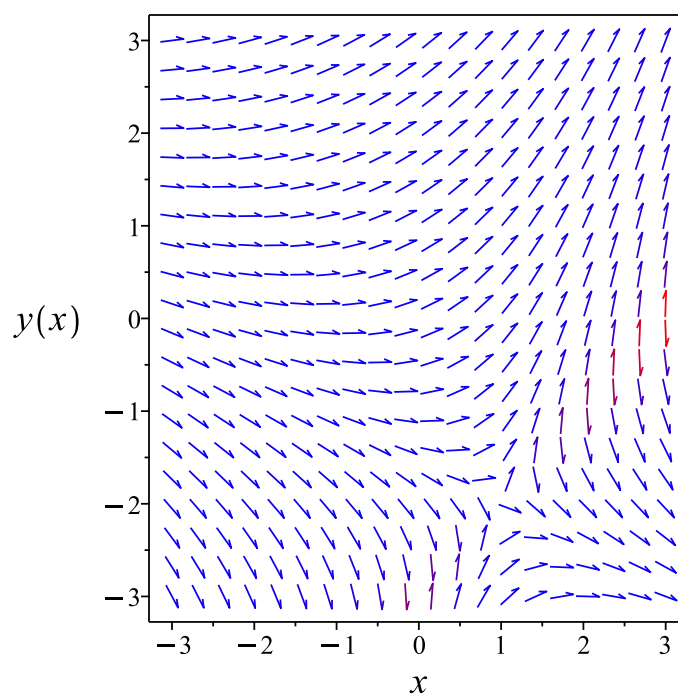


Figure 110: Slope field plot

Verification of solutions

$$\frac{x(x + 2y + 2)}{2} - 3y - \frac{y^2}{2} = c_1$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$$y - (y - x + 3) y' = -1 - x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y) \right) y' = 0$$

- Evaluate derivatives

$$1 = 1$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (x + y + 1) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = \frac{x^2}{2} + xy + x + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$-y + x - 3 = x + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = -3 - y$$

- Solve for $f_1(y)$

$$f_1(y) = -3y - \frac{1}{2}y^2$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{1}{2}x^2 + xy + x - 3y - \frac{1}{2}y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{1}{2}x^2 + xy + x - 3y - \frac{1}{2}y^2 = c_1$$

- Solve for y

$$\{y = x - 3 - \sqrt{2x^2 - 2c_1 - 4x + 9}, y = x - 3 + \sqrt{2x^2 - 2c_1 - 4x + 9}\}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 30

```
dsolve((x+y(x)+1)-(y(x)-x+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{2(x-1)^2 c_1^2 + 1} + (x-3)c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 59

```
DSolve[(x+y[x]+1)-(y[x]-x+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -i\sqrt{-2x^2 + 4x - 9 - c_1} + x - 3$$

$$y(x) \rightarrow i\sqrt{-2x^2 + 4x - 9 - c_1} + x - 3$$

3.9 problem 23 (o)

3.9.1 Solving as exact ode	491
3.9.2 Maple step by step solution	495

Internal problem ID [5270]

Internal file name [OUTPUT/4761_Friday_February_02_2024_05_11_40_AM_31601424/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (o).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

`[_exact , _rational]`

$$y^2 - \frac{y}{x(x+y)} + \left(\frac{1}{x+y} + 2(x+1)y \right) y' = -2$$

3.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}\left(\frac{1}{x+y} + 2(x+1)y\right) dy &= \left(-y^2 + \frac{y}{x(x+y)} - 2\right) dx \\ \left(y^2 - \frac{y}{x(x+y)} + 2\right) dx &+ \left(\frac{1}{x+y} + 2(x+1)y\right) dy = 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 - \frac{y}{x(x+y)} + 2 \\ N(x, y) &= \frac{1}{x+y} + 2(x+1)y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(y^2 - \frac{y}{x(x+y)} + 2 \right) \\ &= 2y - \frac{1}{x(x+y)} + \frac{y}{x(x+y)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{x+y} + 2(x+1)y \right) \\ &= -\frac{1}{(x+y)^2} + 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int y^2 - \frac{y}{x(x+y)} + 2 dx \\ \phi &= y^2x + 2x + \ln(x+y) - \ln(x) + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + \frac{1}{x+y} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x+y} + 2(x+1)y$. Therefore equation (4) becomes

$$\frac{1}{x+y} + 2(x+1)y = 2xy + \frac{1}{x+y} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int (2y) dy \\ f(y) &= y^2 + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2x + 2x + \ln(x + y) - \ln(x) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2x + 2x + \ln(x + y) - \ln(x) + y^2$$

Summary

The solution(s) found are the following

$$xy^2 + 2x + \ln(x + y) - \ln(x) + y^2 = c_1 \tag{1}$$

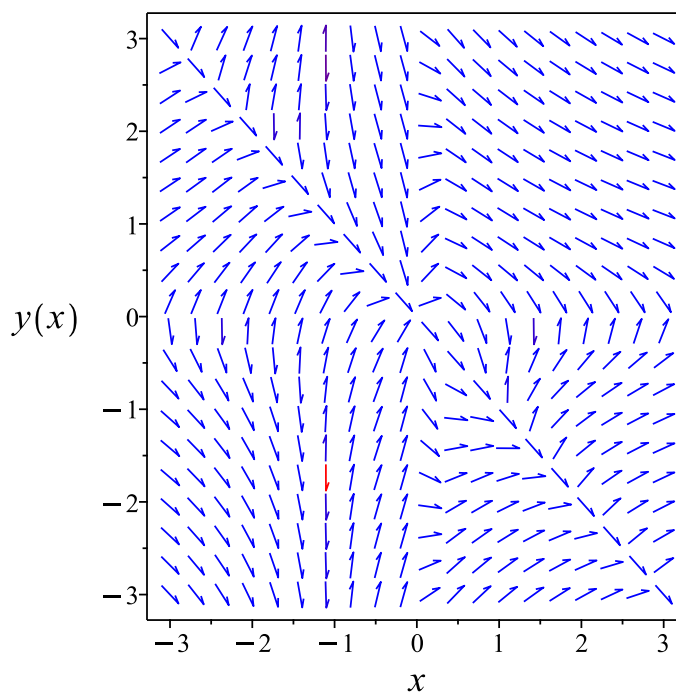


Figure 111: Slope field plot

Verification of solutions

$$xy^2 + 2x + \ln(x + y) - \ln(x) + y^2 = c_1$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$$y^2 - \frac{y}{x(x+y)} + \left(\frac{1}{x+y} + 2(x+1)y \right) y' = -2$$

- Highest derivative means the order of the ODE is 1

y'

- Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y) \right) y' = 0$$

- Evaluate derivatives

$$2y - \frac{1}{x(x+y)} + \frac{y}{x(x+y)^2} = -\frac{1}{(x+y)^2} + 2y$$

- Simplify

$$2y - \frac{1}{x(x+y)} + \frac{y}{x(x+y)^2} = -\frac{1}{(x+y)^2} + 2y$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int \left(y^2 - \frac{y}{x(x+y)} + 2 \right) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = y^2 x + 2x + \ln(x+y) - \ln(x) + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$\frac{1}{x+y} + 2(x+1)y = 2xy + \frac{1}{x+y} + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = -2xy + 2(x+1)y$$

- Solve for $f_1(y)$
 $f_1(y) = y^2$
- Substitute $f_1(y)$ into equation for $F(x, y)$
 $F(x, y) = y^2x + 2x + \ln(x + y) - \ln(x) + y^2$
- Substitute $F(x, y)$ into the solution of the ODE
 $y^2x + 2x + \ln(x + y) - \ln(x) + y^2 = c_1$
- Solve for y

$$y = \frac{-xe^{\frac{\text{RootOf}(-x^3(e^{-Z})^2 - (e^{-Z})^2x^2 + 2x^3e^{-Z} + (e^{-Z})^2c_1 - Z(e^{-Z})^2 - 2x(e^{-Z})^2 + 2x^2e^{-Z} - x^3 - x^2)}{e}} + x}{\text{RootOf}(-x^3(e^{-Z})^2 - (e^{-Z})^2x^2 + 2x^3e^{-Z} + (e^{-Z})^2c_1 - Z(e^{-Z})^2 - 2x(e^{-Z})^2 + 2x^2e^{-Z} - x^3 - x^2)}}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 129

```
dsolve((y(x)^2- y(x)/(x*(x+y(x))))+2)+( 1/(x+y(x)) + 2*y(x)*(1+x))*diff(y(x),x)=0,y(x), sings
```

$$y(x) = -x \left(e^{\text{RootOf}(x^3e^{2-Z} + x^2e^{2-Z} - 2x^3e^{-Z} + c_1e^{2-Z} - Ze^{2-Z} + 2xe^{2-Z} - 2x^2e^{-Z} + x^3 + x^2)} - 1 \right) e^{-\text{RootOf}(x^3e^{2-Z} + x^2e^{2-Z} - 2x^3e^{-Z} + c_1e^{2-Z} - Ze^{2-Z} + 2xe^{2-Z} - 2x^2e^{-Z} + x^3 + x^2)}$$

✓ Solution by Mathematica

Time used: 0.43 (sec). Leaf size: 29

```
DSolve[(y[x]^2 - y[x]/(x*(x+y[x]))+2)+(1/(x+y[x]) + 2*y[x]*(1+x))*y'[x]==0,y[x],x,IncludeSin
```

$$\text{Solve}[xy(x)^2 + y(x)^2 + \log(y(x) + x) + 2x - \log(x) = c_1, y(x)]$$

3.10 problem 23 (p)

3.10.1 Solving as exact ode	498
3.10.2 Maple step by step solution	502

Internal problem ID [5271]

Internal file name [OUTPUT/4762_Friday_February_02_2024_05_11_40_AM_81253565/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 23 (p).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

[_exact]

$$2xy e^{yx^2} + y^2 e^{xy^2} + \left(x^2 e^{yx^2} + 2xy e^{xy^2} - 2y\right) y' = -1$$

3.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(x^2 e^{y x^2} + 2xy e^{y^2 x} - 2y) dy &= (-2xy e^{y x^2} - y^2 e^{y^2 x} - 1) dx \\ (2xy e^{y x^2} + y^2 e^{y^2 x} + 1) dx &+ (x^2 e^{y x^2} + 2xy e^{y^2 x} - 2y) dy = 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2xy e^{y x^2} + y^2 e^{y^2 x} + 1 \\ N(x, y) &= x^2 e^{y x^2} + 2xy e^{y^2 x} - 2y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (2xy e^{y x^2} + y^2 e^{y^2 x} + 1) \\ &= (2x y^3 + 2y) e^{y^2 x} + 2x e^{y x^2} (y x^2 + 1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x^2 e^{y x^2} + 2xy e^{y^2 x} - 2y) \\ &= (2x y^3 + 2y) e^{y^2 x} + 2x e^{y x^2} (y x^2 + 1)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int 2xy e^{y^2 x} + y^2 e^{y^2 x} + 1 dx \\ \phi &= x + e^{y^2 x} + e^{y^2 x} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy e^{y^2 x} + x^2 e^{y^2 x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 e^{y^2 x} + 2xy e^{y^2 x} - 2y$. Therefore equation (4) becomes

$$x^2 e^{y^2 x} + 2xy e^{y^2 x} - 2y = 2xy e^{y^2 x} + x^2 e^{y^2 x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) dy &= \int (-2y) dy \\ f(y) &= -y^2 + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x + e^{y^2 x} + e^{y^2 x} - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + e^{y^2x} + e^{yx^2} - y^2$$

Summary

The solution(s) found are the following

$$x + e^{xy^2} + e^{yx^2} - y^2 = c_1 \tag{1}$$

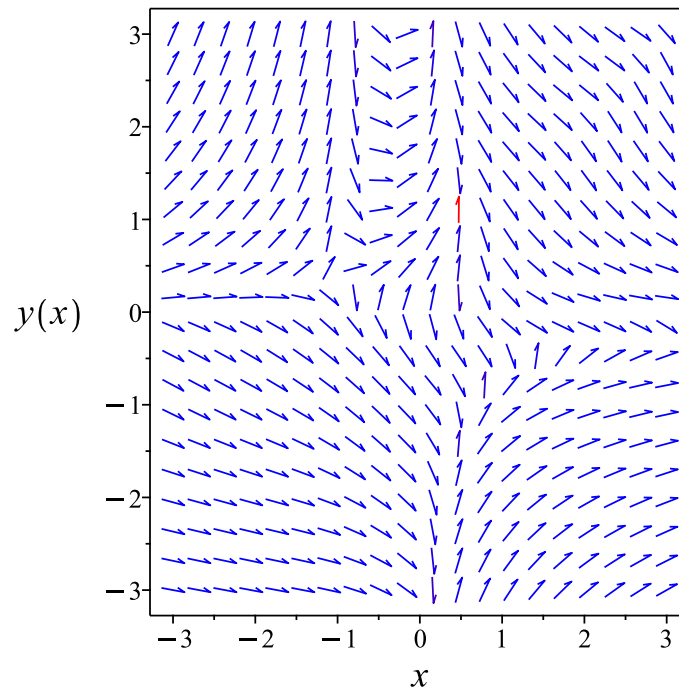


Figure 112: Slope field plot

Verification of solutions

$$x + e^{xy^2} + e^{yx^2} - y^2 = c_1$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$$2xy e^{yx^2} + y^2 e^{xy^2} + \left(x^2 e^{yx^2} + 2xy e^{xy^2} - 2y\right) y' = -1$$

- Highest derivative means the order of the ODE is 1

y'

- Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$$

- Evaluate derivatives

$$2x e^{yx^2} + 2x^3 y e^{yx^2} + 2y e^{y^2 x} + 2y^3 x e^{y^2 x} = 2x e^{yx^2} + 2x^3 y e^{yx^2} + 2y e^{y^2 x} + 2y^3 x e^{y^2 x}$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int \left(2xy e^{yx^2} + y^2 e^{y^2 x} + 1 \right) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = x + e^{y^2 x} + e^{yx^2} + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$x^2 e^{yx^2} + 2xy e^{y^2 x} - 2y = 2xy e^{y^2 x} + x^2 e^{yx^2} + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = -2y$$

- Solve for $f_1(y)$

$$f_1(y) = -y^2$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = x + e^{y^2 x} + e^{y x^2} - y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x + e^{y^2 x} + e^{y x^2} - y^2 = c_1$$

- Solve for y

$$y = \text{RootOf}\left(-Z^2 x - \text{RootOf}\left(x^3 Z - \ln\left(-\frac{e^{-Zx - c_1 x + x^2} - Z}{x}\right)^2\right)\right)$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve((2*x*y(x)*exp(x^2*y(x))+ y(x)^2*exp(x*y(x)^2)+1)+(x^2*exp(x^2*y(x))+ 2*x*y(x)*exp(x*y(x)^2)+1)+x^2*exp(x^2*y(x))+ 2*x*y(x)*exp(x*y(x)^2)+1)+x^2*exp(x^2*y(x))+ 2*x*y(x)*exp(x*y(x)^2)+1)
```

$$y(x) = \frac{\text{RootOf}\left(e^{-Z}x^4 + e^{\frac{Z^2}{x^3}}x^4 + c_1x^4 + x^5 - Z^2\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.385 (sec). Leaf size: 30

```
DSolve[(2*x*y[x]*Exp[x^2*y[x]]+ y[x]^2*Exp[x*y[x]^2]+1)+(x^2*Exp[x^2*y[x]]+ 2*x*y[x]*Exp[x*y[x]^2]+1)+x^2*Exp[x^2*y[x]]+ 2*x*y[x]*Exp[x*y[x]^2]+1)+x^2*Exp[x^2*y[x]]+ 2*x*y[x]*Exp[x*y[x]^2]+1)
```

$$\text{Solve}\left[e^{x^2 y(x)} - y(x)^2 + e^{x y(x)^2} + x = c_1, y(x)\right]$$

3.11 problem 24 (p)

3.11.1 Solving as homogeneousTypeD2 ode	504
3.11.2 Solving as first order ode lie symmetry lookup ode	506
3.11.3 Solving as bernoulli ode	510
3.11.4 Solving as exact ode	514
3.11.5 Solving as riccati ode	519

Internal problem ID [5272]

Internal file name [OUTPUT/4763_Friday_February_02_2024_05_11_41_AM_4963611/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 24 (p).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y(x - 2y) - x^2y' = 0$$

3.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)x(x - 2u(x)x) - x^2(u'(x)x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u^2}{x}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u^2} du &= -\frac{2}{x} dx \\ \int \frac{1}{u^2} du &= \int -\frac{2}{x} dx \\ -\frac{1}{u} &= -2 \ln(x) + c_2\end{aligned}$$

The solution is

$$-\frac{1}{u(x)} + 2 \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}-\frac{x}{y} + 2 \ln(x) - c_2 &= 0 \\ -\frac{x}{y} + 2 \ln(x) - c_2 &= 0\end{aligned}$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} + 2 \ln(x) - c_2 = 0 \tag{1}$$

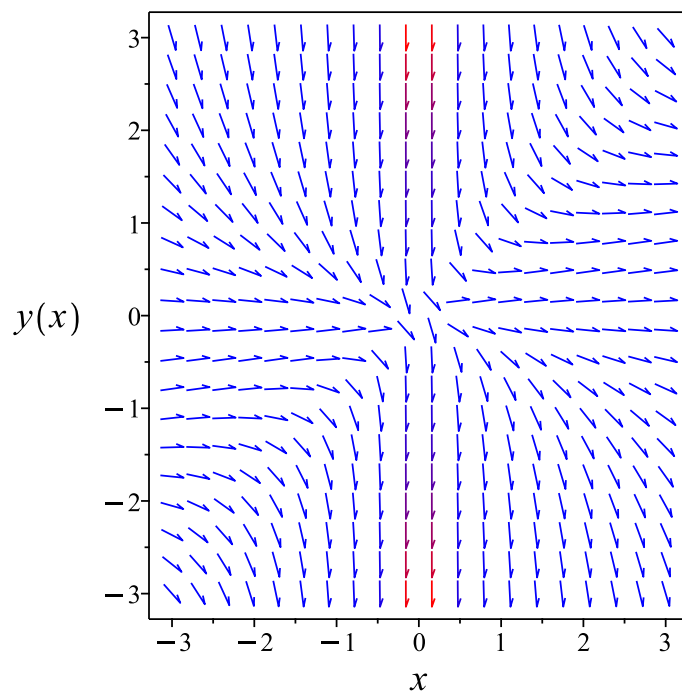


Figure 113: Slope field plot

Verification of solutions

$$-\frac{x}{y} + 2 \ln(x) - c_2 = 0$$

Verified OK.

3.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(-x + 2y)}{x^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{y^2}{x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x + 2y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -2 \ln(R) + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -2 \ln(x) + c_1$$

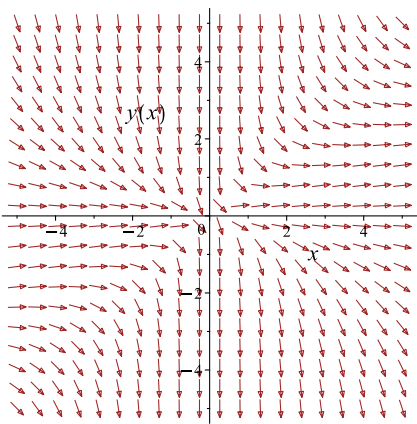
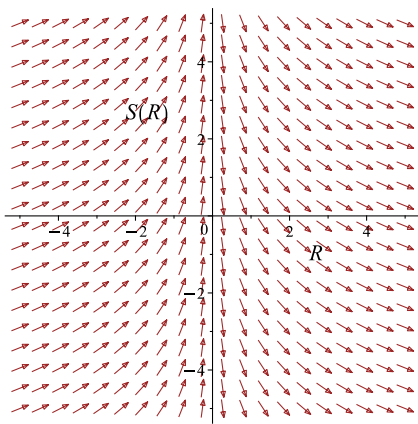
Which simplifies to

$$-\frac{x}{y} = -2 \ln(x) + c_1$$

Which gives

$$y = \frac{x}{2 \ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+2y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{2 \ln(x) - c_1} \quad (1)$$

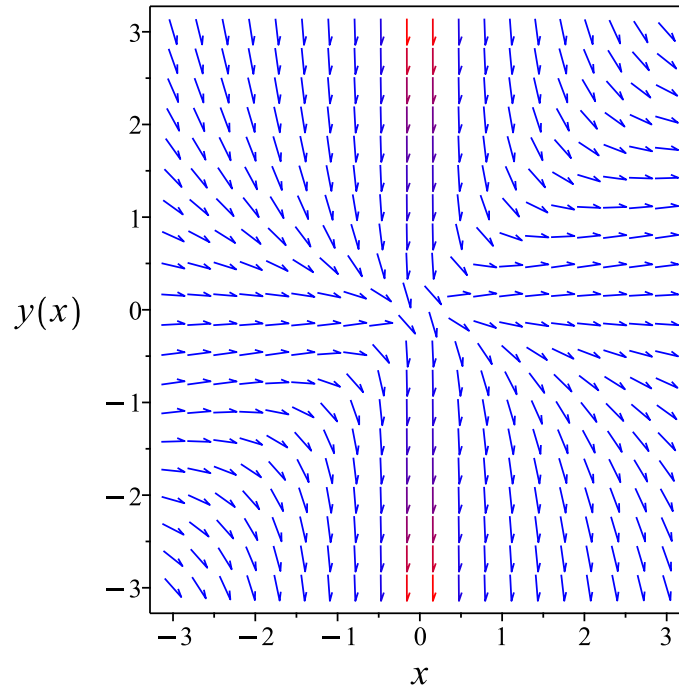


Figure 114: Slope field plot

Verification of solutions

$$y = \frac{x}{2 \ln(x) - c_1}$$

Verified OK.

3.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x + 2y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{2}{x^2}y^2 \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{2}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{1}{yx} - \frac{2}{x^2} \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{2}{x^2} \\ w' &= -\frac{w}{x} + \frac{2}{x^2} \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2}{x^2}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{2}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x} dx}$$
$$= x$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu) \left(\frac{2}{x^2} \right)$$
$$\frac{d}{dx}(xw) = (x) \left(\frac{2}{x^2} \right)$$
$$d(xw) = \left(\frac{2}{x} \right) dx$$

Integrating gives

$$xw = \int \frac{2}{x} dx$$
$$xw = 2 \ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{2 \ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{2 \ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{2 \ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{2 \ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2 \ln(x) + c_1} \tag{1}$$

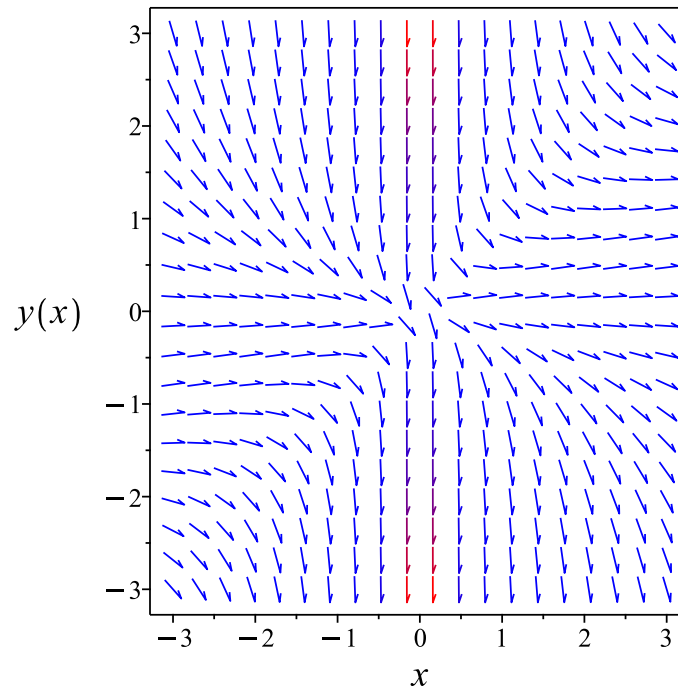


Figure 115: Slope field plot

Verification of solutions

$$y = \frac{x}{2 \ln(x) + c_1}$$

Verified OK.

3.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (-x^2) dy &= (-y(x - 2y)) dx \\ (y(x - 2y)) dx + (-x^2) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y(x - 2y) \\ N(x, y) &= -x^2 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(x - 2y)) \\ &= x - 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2) \\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y(x - 2y)$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x - 2y}{xy} \\ N &= -\frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} \left(-\frac{x}{y^2}\right) dy &= \left(-\frac{x-2y}{xy}\right) dx \\ \left(\frac{x-2y}{xy}\right) dx + \left(-\frac{x}{y^2}\right) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{x-2y}{xy} \\ N(x, y) &= -\frac{x}{y^2} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x-2y}{xy} \right) \\ &= -\frac{1}{y^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(-\frac{x}{y^2} \right) \\ &= -\frac{1}{y^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{x - 2y}{xy} dx \\ \phi &= -2 \ln(x) + \frac{x}{y} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2 \ln(x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2 \ln(x) + \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{2 \ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2 \ln(x) + c_1} \tag{1}$$

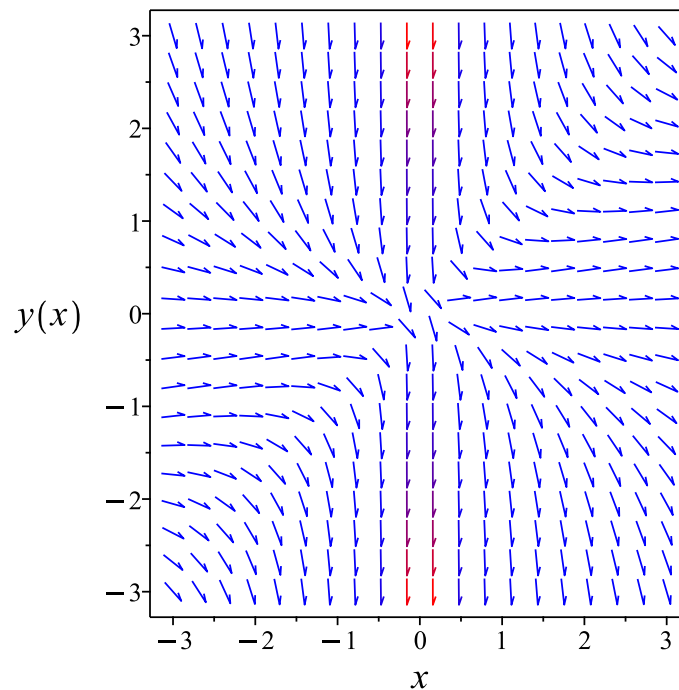


Figure 116: Slope field plot

Verification of solutions

$$y = \frac{x}{2 \ln(x) + c_1}$$

Verified OK.

3.11.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= -\frac{y(-x + 2y)}{x^2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} - \frac{2y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{2}{x^2}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{2u}{x^2}}\end{aligned}\tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0\tag{2}$$

But

$$\begin{aligned}f_2' &= \frac{4}{x^3} \\ f_1 f_2 &= -\frac{2}{x^3} \\ f_2^2 f_0 &= 0\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{2u''(x)}{x^2} - \frac{2u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{2c_2 \ln(x) + 2c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{2 \ln(x) + 2c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2 \ln(x) + 2c_3} \tag{1}$$

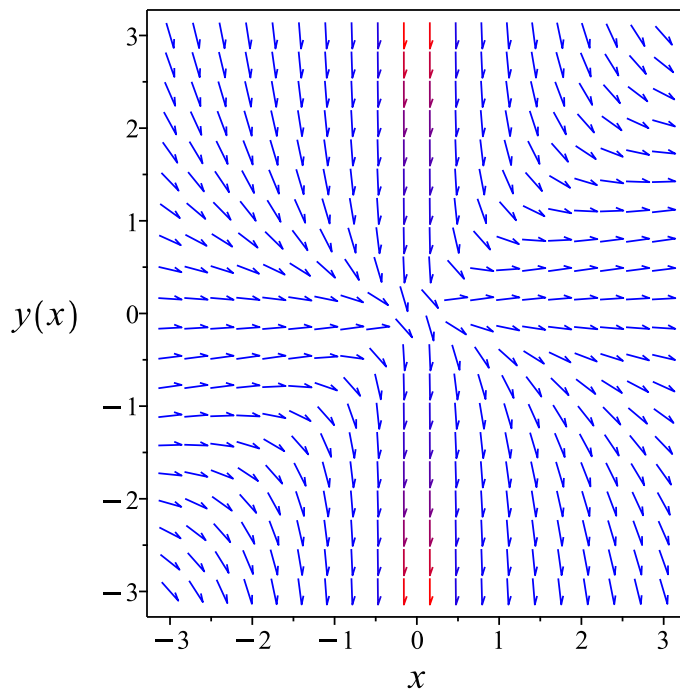


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{x}{2 \ln(x) + 2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)*(x-2*y(x))-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{2 \ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 21

```
DSolve[y[x]*(x-2*y[x])-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x}{2 \log(x) + c_1}$$
$$y(x) \rightarrow 0$$

3.12 problem 24 (c)

3.12.1 Solving as homogeneousTypeD2 ode	522
3.12.2 Solving as first order ode lie symmetry lookup ode	524
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3.12.4 Solving as exact ode	531

Internal problem ID [5273]

Internal file name [OUTPUT/4764_Friday_February_02_2024_05_11_41_AM_81589728/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 24 (c).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y^2 + xyy' = -x^2$$

3.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 + x^2 u(x) (u'(x)x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 + 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{2u^2+1}{u}} du &= -\frac{1}{x} dx \\ \int \frac{1}{\frac{2u^2+1}{u}} du &= \int -\frac{1}{x} dx \\ \frac{\ln(2u^2+1)}{4} &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$(2u^2+1)^{\frac{1}{4}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$(2u^2+1)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$(2u(x)^2+1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$(2u(x)^2+1)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} &= \frac{c_3 e^{c_2}}{x} \\ \left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} &= \frac{c_3 e^{c_2}}{x}\end{aligned}$$

Summary

The solution(s) found are the following

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x} \quad (1)$$

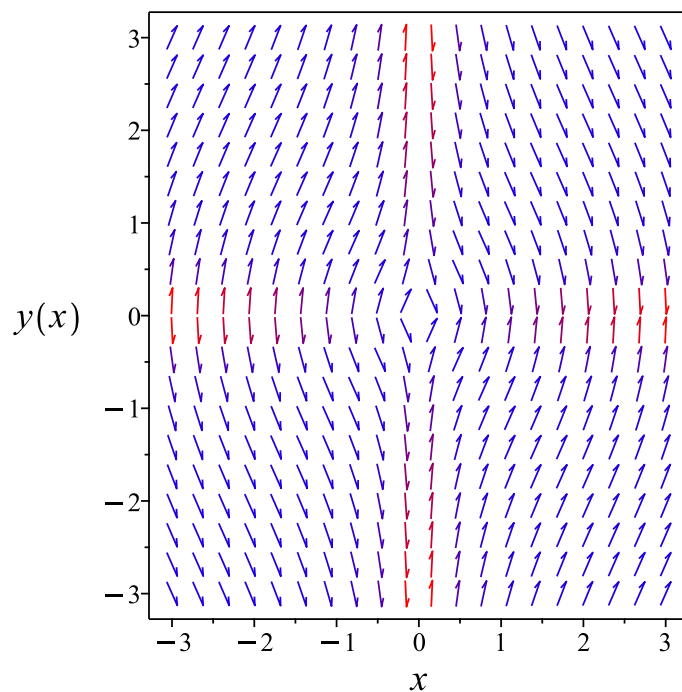


Figure 118: Slope field plot

Verification of solutions

$$\left(\frac{2y^2 + x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

3.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 + y^2}{xy}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{yx^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$

$$R_y = 0$$

$$S_x = y^2 x$$

$$S_y = y x^2$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \quad (4)$$

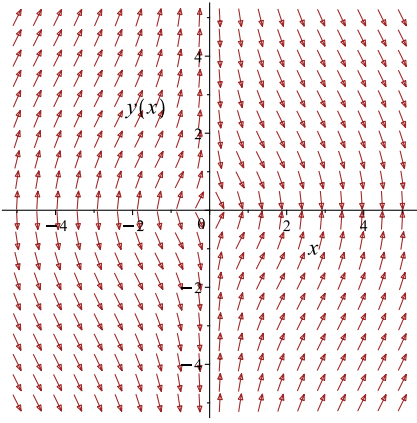
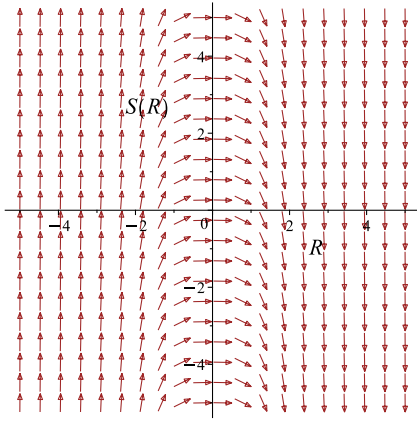
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{y^2 x^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Summary

The solution(s) found are the following

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1 \quad (1)$$

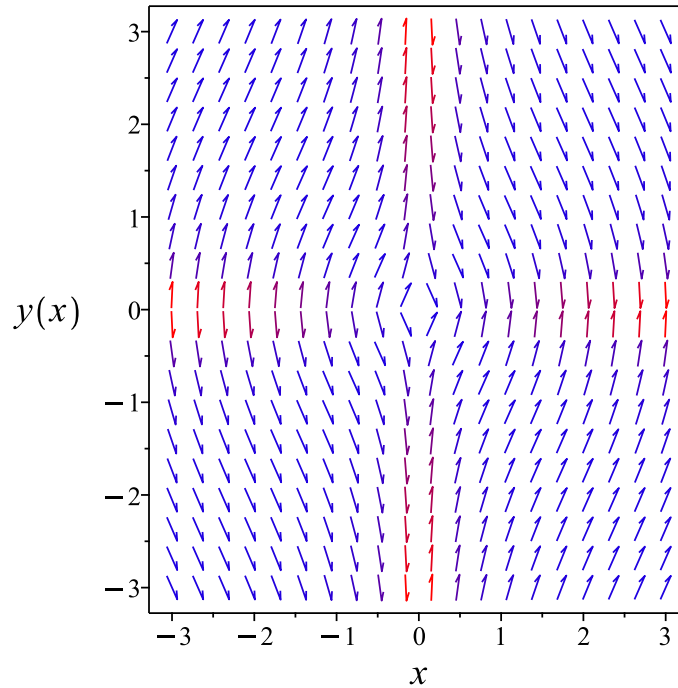


Figure 119: Slope field plot

Verification of solutions

$$\frac{y^2 x^2}{2} = -\frac{x^4}{4} + c_1$$

Verified OK.

3.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= -\frac{1}{x} \\f_1(x) &= -x \\n &= -1\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - x \quad (4)$$

Let

$$\begin{aligned}w &= y^{1-n} \\&= y^2\end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\w' &= -\frac{2w}{x} - 2x\end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2}{x} \\q(x) &= -2x\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\ \frac{d}{dx}(x^2 w) &= (x^2)(-2x) \\ d(x^2 w) &= (-2x^3) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int -2x^3 dx \\ x^2 w &= -\frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{-2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{-2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{2}$$

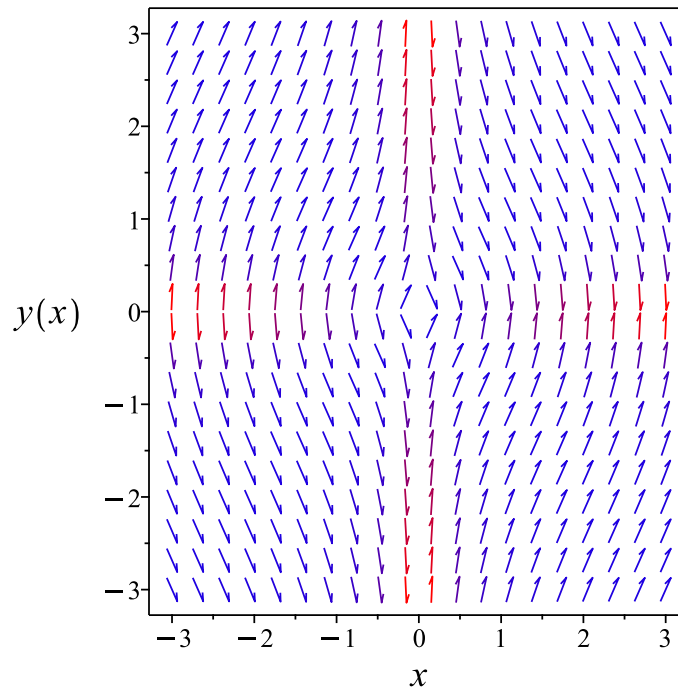


Figure 120: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

3.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (xy) dy &= (-x^2 - y^2) dx \\ (x^2 + y^2) dx + (xy) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^2 + y^2 \\ N(x, y) &= xy \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (x^2 + y^2) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{yx} ((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= x(x^2 + y^2) \\ &= x(x^2 + y^2)\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= x(xy) \\ &= yx^2\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (x(x^2 + y^2)) + (yx^2) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x(x^2 + y^2) dx \\ \phi &= \frac{(x^2 + y^2)^2}{4} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x^2 + y^2) y + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y x^2$. Therefore equation (4) becomes

$$y x^2 = (x^2 + y^2) y + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) dy &= \int (-y^3) dy \\ f(y) &= -\frac{y^4}{4} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1 \quad (1)$$

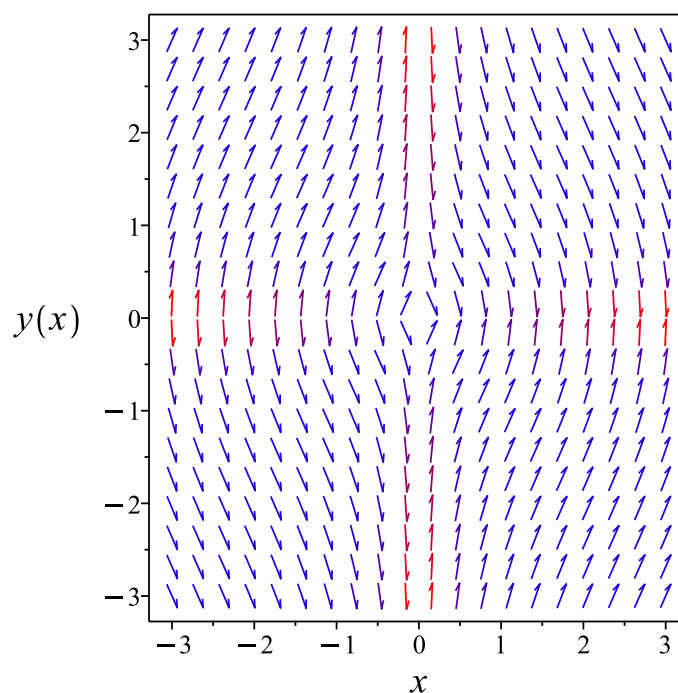


Figure 121: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((x^2+y(x)^2)+x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$
$$y(x) = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 46

```
DSolve[(x^2+y[x]^2)+x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$
$$y(x) \rightarrow \frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$

3.13 problem 24 (d)

3.13.1 Solving as homogeneousTypeD2 ode	537
3.13.2 Solving as first order ode lie symmetry lookup ode	539
3.13.3 Solving as bernoulli ode	543
3.13.4 Solving as exact ode	546
3.13.5 Maple step by step solution	550

Internal problem ID [5274]

Internal file name [OUTPUT/4765_Friday_February_02_2024_05_11_43_AM_34495175/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 24 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$y^2 + 2xyy' = -x^2$$

3.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 + 2x^2 u(x) (u'(x)x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u^2 + 1}{2ux} \end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{3u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{3u^2+1}{u}} du &= -\frac{1}{2x} dx \\ \int \frac{1}{\frac{3u^2+1}{u}} du &= \int -\frac{1}{2x} dx \\ \frac{\ln(3u^2+1)}{6} &= -\frac{\ln(x)}{2} + c_2\end{aligned}$$

Raising both side to exponential gives

$$(3u^2+1)^{\frac{1}{6}} = e^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$(3u^2+1)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$(3u(x)^2+1)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

The solution is

$$(3u(x)^2+1)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}\left(\frac{3y^2}{x^2}+1\right)^{\frac{1}{6}} &= \frac{c_3 e^{c_2}}{\sqrt{x}} \\ \left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} &= \frac{c_3 e^{c_2}}{\sqrt{x}}\end{aligned}$$

Summary

The solution(s) found are the following

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}} \quad (1)$$

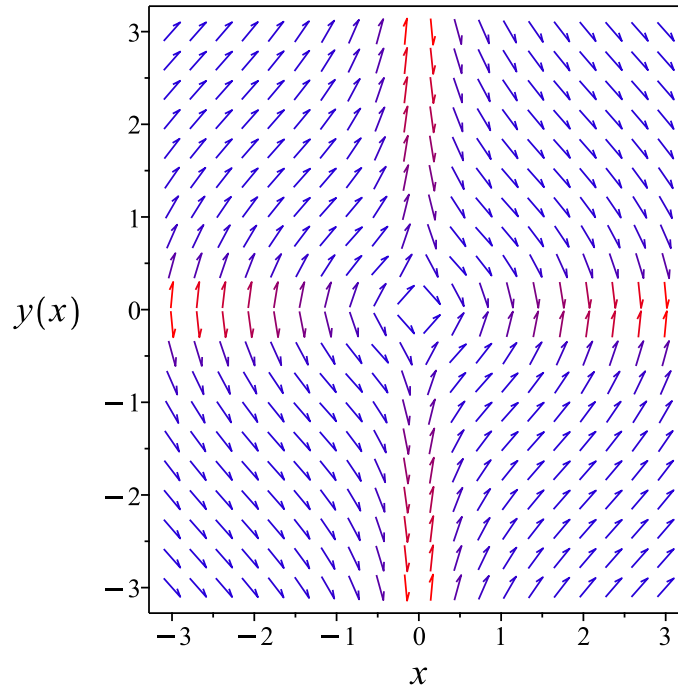


Figure 122: Slope field plot

Verification of solutions

$$\left(\frac{3y^2 + x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Verified OK.

3.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2 + y^2}{2xy}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{yx}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= xy \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^2}{2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{R^3}{6} + c_1 \quad (4)$$

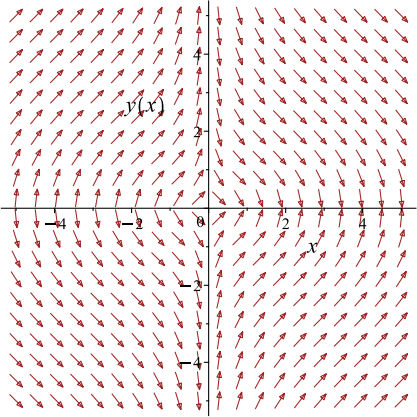
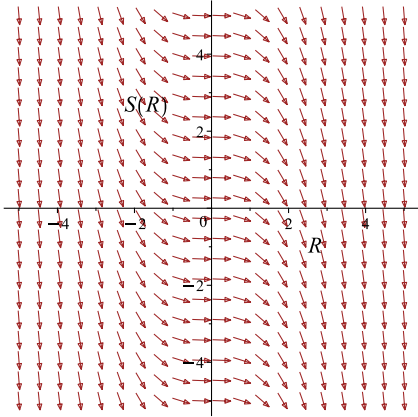
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1$$

Which simplifies to

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2 x}{2}$	$\frac{dS}{dR} = -\frac{R^2}{2}$ 

Summary

The solution(s) found are the following

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1 \quad (1)$$

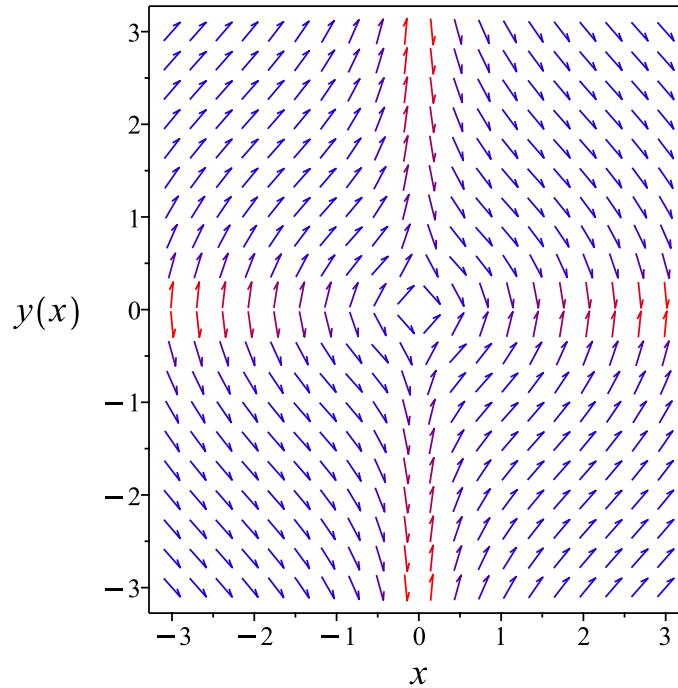


Figure 123: Slope field plot

Verification of solutions

$$\frac{xy^2}{2} = -\frac{x^3}{6} + c_1$$

Verified OK.

3.13.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y - \frac{x}{2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x} \\ f_1(x) &= -\frac{x}{2} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x} - \frac{x}{2} \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{x}{2} \\ w' &= -\frac{w}{x} - x \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}(xw) &= (x)(-x) \\ d(xw) &= (-x^2) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^2 dx \\ xw &= -\frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^2}{3} + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{3} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x} \\ y(x) &= -\frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x} \tag{2}$$

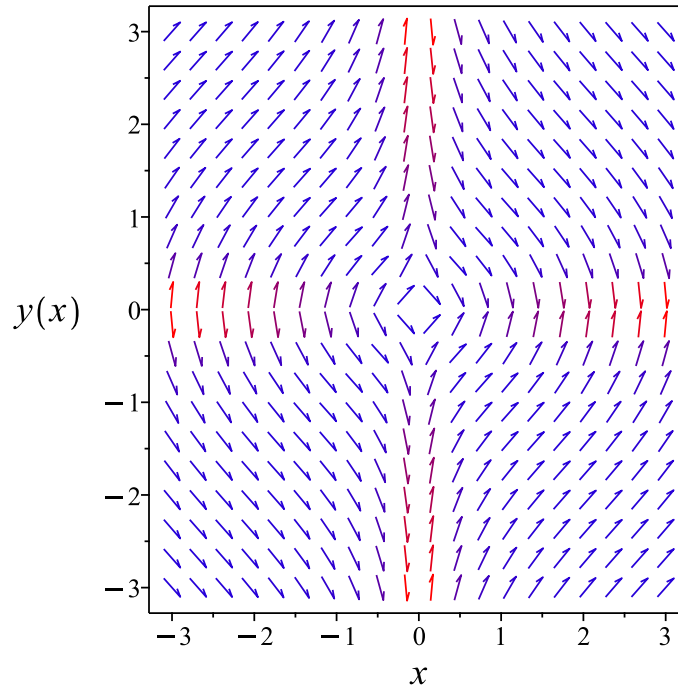


Figure 124: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{3} \sqrt{-x(x^3 - 3c_1)}}{3x}$$

Verified OK.

3.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (2xy) dy &= (-x^2 - y^2) dx \\ (x^2 + y^2) dx + (2xy) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^2 + y^2 \\ N(x, y) &= 2xy \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (x^2 + y^2) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int x^2 + y^2 dx \\ \phi &= \frac{1}{3}x^3 + y^2x + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y^2x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y^2x$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + xy^2 = c_1 \quad (1)$$

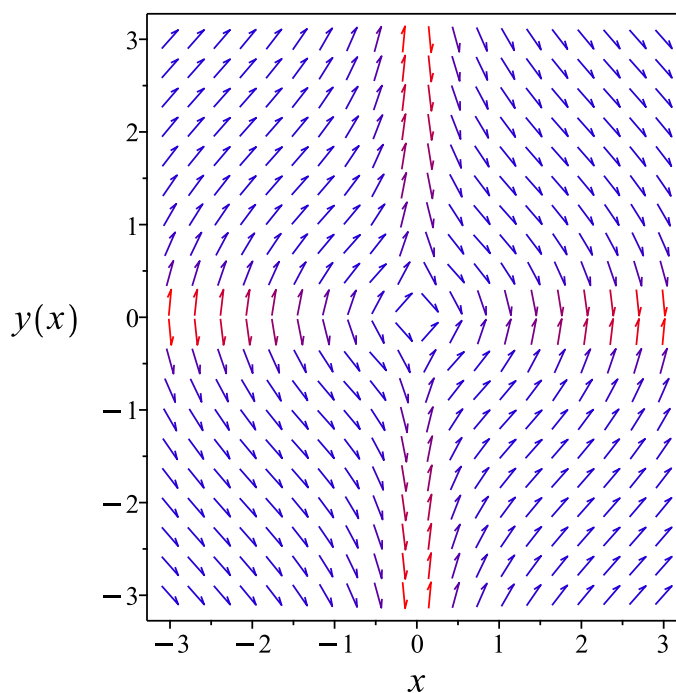


Figure 125: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + xy^2 = c_1$$

Verified OK.

3.13.5 Maple step by step solution

Let's solve

$$y^2 + 2xyy' = -x^2$$

- Highest derivative means the order of the ODE is 1

y'

- Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$$

- Evaluate derivatives

$$2y = 2y$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y) \right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (x^2 + y^2) dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = \frac{x^3}{3} + y^2x + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y} F(x, y)$$

- Compute derivative

$$2xy = 2xy + \frac{d}{dy} f_1(y)$$

- Isolate for $\frac{d}{dy} f_1(y)$

$$\frac{d}{dy} f_1(y) = 0$$

- Solve for $f_1(y)$

$$f_1(y) = 0$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \frac{1}{3}x^3 + y^2x$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{1}{3}x^3 + y^2x = c_1$$

- Solve for y

$$\left\{ y = -\frac{\sqrt{3}\sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\sqrt{x(-x^3+3c_1)}}{3x} \right\}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\sqrt{-x(x^3-3c_1)}}{3x}$$

$$y(x) = \frac{\sqrt{3}\sqrt{-x(x^3-3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$

$$y(x) \rightarrow \frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$

3.14 problem 24 (g)

3.14.1 Solving as quadrature ode	552
3.14.2 Maple step by step solution	553

Internal problem ID [5275]

Internal file name [OUTPUT/4766_Friday_February_02_2024_05_11_44_AM_93727827/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 24 (g).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**quadrature**"

Maple gives the following as the ode type

[_quadrature]

$$-\sqrt{a^2 - x^2} y' = -1$$

3.14.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{a^2 - x^2}} dx \\ &= \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1 \tag{1}$$

Verification of solutions

$$y = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$$-\sqrt{a^2 - x^2} y' = -1$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$y' = \frac{1}{\sqrt{a^2 - x^2}}$$

- Integrate both sides with respect to x

$$\int y' dx = \int \frac{1}{\sqrt{a^2 - x^2}} dx + c_1$$

- Evaluate integral

$$y = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

- Solve for y

$$y = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(1-(sqrt(a^2-x^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 24

```
DSolve[1-(Sqrt[a^2-x^2])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

3.15 problem 24 (L)

3.15.1 Solving as homogeneousTypeMapleC ode 555

3.15.2 Solving as first order ode lie symmetry calculated ode 558

Internal problem ID [5276]

Internal file name [OUTPUT/4767_Friday_February_02_2024_05_11_45_AM_93566786/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 24 (L).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `class A`]]
```

$$y - (-y + x - 3)y' = -1 - x$$

3.15.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + y_0 + X + x_0 + 1}{Y(X) + y_0 - X - x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1 \\y_0 &= -2\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{Y(X) + X}{Y(X) - X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{Y + X}{Y - X} \end{aligned} \quad (1)$$

An ode of the form $Y' = \frac{M(X, Y)}{N(X, Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = Y + X$ and $N = -Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{-u - 1}{u - 1} \\ \frac{du}{dX} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) - \left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 1 = 0$$

Or

$$X(u(X) - 1) \left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= -\frac{u^2 + 1}{X(u - 1)} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}} du = -\frac{1}{X} dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}} du = \int -\frac{1}{X} dX$$

$$\frac{\ln(u^2 + 1)}{2} - \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln(u(X)^2 + 1)}{2} - \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2} + 1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2} + 1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$

$$X = x + x_0$$

Or

$$Y = y - 2$$

$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(2+y)^2}{(x-1)^2} + 1\right)}{2} - \arctan\left(\frac{2+y}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(2+y)^2}{(x-1)^2} + 1\right)}{2} - \arctan\left(\frac{2+y}{x-1}\right) + \ln(x-1) - c_2 = 0 \quad (1)$$

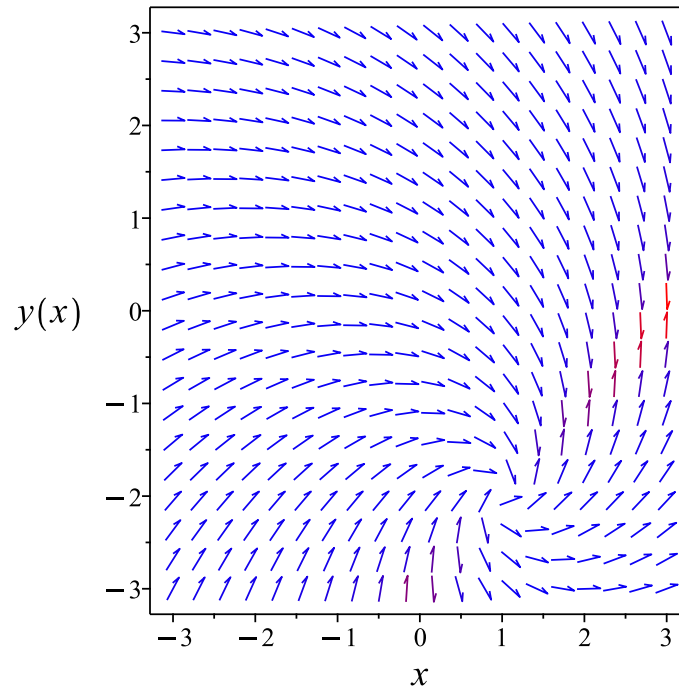


Figure 126: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(2+y)^2}{(x-1)^2} + 1\right)}{2} - \arctan\left(\frac{2+y}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Verified OK.

3.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y+1}{y-x+3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(x+y+1)(b_3-a_2)}{y-x+3} - \frac{(x+y+1)^2 a_3}{(y-x+3)^2} \\ - \left(-\frac{1}{y-x+3} - \frac{x+y+1}{(y-x+3)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{1}{y-x+3} + \frac{x+y+1}{(y-x+3)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (\text{5E})$$

Putting the above in normal form gives

$$\frac{x^2 a_2 + x^2 a_3 + x^2 b_2 - x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 - y^2 a_2 - y^2 a_3 - y^2 b_2 + y^2 b_3 - 6xa_2 + 2xa_3 - 6yb_2 + 2yb_3 - 6xa_2 + 2xa_3 - 6yb_2 + 2yb_3}{(-y+x-3)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -x^2 a_2 - x^2 a_3 - x^2 b_2 + x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 + y^2 a_2 \\ + y^2 a_3 + y^2 b_2 - y^2 b_3 + 6xa_2 - 2xa_3 - 2xb_1 - 4xb_2 - 2xb_3 + 2ya_1 \\ + 4ya_2 + 2ya_3 + 6yb_2 - 2yb_3 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \quad (\text{6E})$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 - 2b_2v_1v_2 + b_2v_2^2 \\ & + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 + 6a_2v_1 + 4a_2v_2 - 2a_3v_1 + 2a_3v_2 - 2b_1v_1 \\ & - 4b_2v_1 + 6b_2v_2 - 2b_3v_1 - 2b_3v_2 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (-a_2 - a_3 - b_2 + b_3)v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)v_1v_2 \\ & + (6a_2 - 2a_3 - 2b_1 - 4b_2 - 2b_3)v_1 + (a_2 + a_3 + b_2 - b_3)v_2^2 \\ & + (2a_1 + 4a_2 + 2a_3 + 6b_2 - 2b_3)v_2 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\ 2a_1 + 4a_2 + 2a_3 + 6b_2 - 2b_3 &= 0 \\ 6a_2 - 2a_3 - 2b_1 - 4b_2 - 2b_3 &= 0 \\ 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -2b_2 - b_3 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= -b_2 + 2b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x - 1 \\ \eta &= y + 2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y + 2 - \left(-\frac{x + y + 1}{y - x + 3} \right) (x - 1) \\ &= \frac{-x^2 - y^2 + 2x - 4y - 5}{-y + x - 3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - y^2 + 2x - 4y - 5}{-y + x - 3}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(x^2 + y^2 - 2x + 4y + 5)}{2} + \frac{2(1 - x) \arctan\left(\frac{2y+4}{2x-2}\right)}{2x - 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y + 1}{y - x + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y + 1}{x^2 + y^2 - 2x + 4y + 5} \\ S_y &= \frac{y - x + 3}{x^2 + y^2 - 2x + 4y + 5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

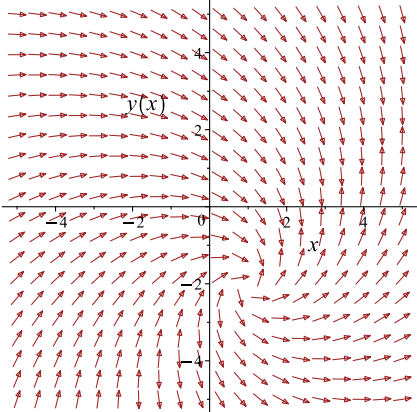
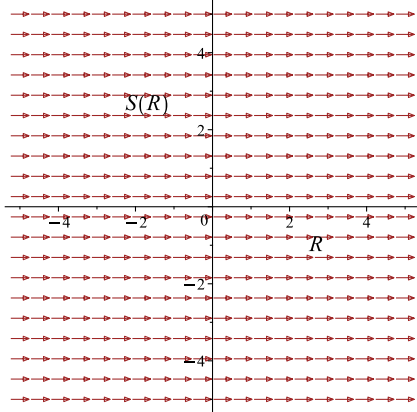
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} - \arctan\left(\frac{2 + y}{x - 1}\right) = c_1$$

Which simplifies to

$$\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} - \arctan\left(\frac{2 + y}{x - 1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+1}{y-x+3}$ 	$R = x$ $S = \frac{\ln(x^2 + y^2 - 2x + 4)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} - \arctan\left(\frac{2+y}{x-1}\right) = c_1 \quad (1)$$

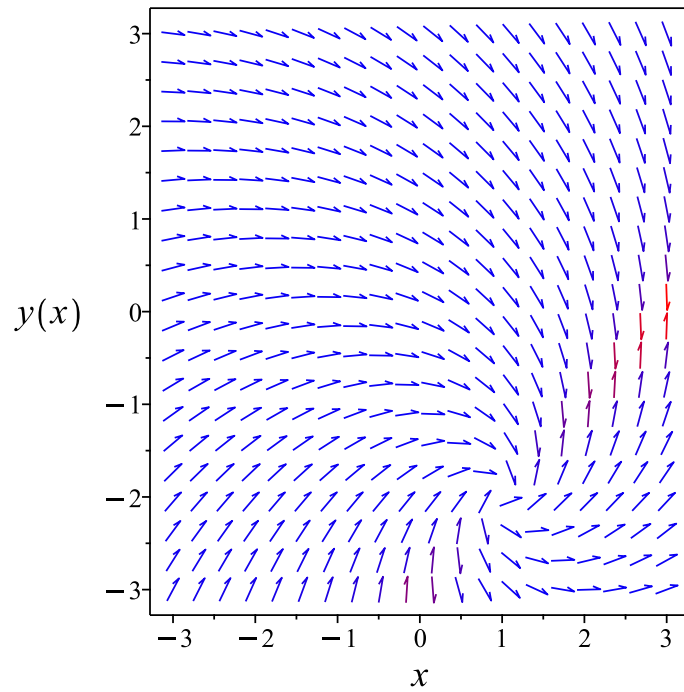


Figure 127: Slope field plot

Verification of solutions

$$\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} - \arctan\left(\frac{2 + y}{x - 1}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous C  
trying homogeneous types:  
trying homogeneous D  
<- homogeneous successful  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve((x+y(x)+1)-(x-y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -2 - \tan(\text{RootOf}(2_Z + \ln(\sec(_Z)^2) + 2\ln(x-1) + 2c_1))(x-1)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 58

```
DSolve[(x+y[x]+1)-(x-y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[2 \arctan\left(\frac{y(x) + x + 1}{y(x) - x + 3}\right) + \log\left(\frac{x^2 + y(x)^2 + 4y(x) - 2x + 5}{2(x-1)^2}\right) + 2 \log(x-1) + c_1 = 0, y(x)\right]$$

3.16 problem 25 (a)

3.16.1 Solving as exact ode 566

Internal problem ID [5277]

Internal file name [OUTPUT/4768_Friday_February_02_2024_05_11_47_AM_88750982/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$-y^2 + y'y = x^2 - x$$

3.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (y) dy &= (x^2 + y^2 - x) dx \\ (-x^2 - y^2 + x) dx + (y) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x^2 - y^2 + x \\ N(x, y) &= y \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2 - y^2 + x) \\ &= -2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{y} ((-2y) - (0)) \\ &= -2 \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int -2 dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2x} \\ &= e^{-2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= e^{-2x}(-x^2 - y^2 + x) \\ &= -(x^2 + y^2 - x)e^{-2x}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= e^{-2x}(y) \\ &= ye^{-2x}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ -(x^2 + y^2 - x)e^{-2x} + (ye^{-2x}) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -(x^2 + y^2 - x)e^{-2x} dx \\ \phi &= \frac{(x^2 + y^2)e^{-2x}}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y e^{-2x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y e^{-2x}$. Therefore equation (4) becomes

$$y e^{-2x} = y e^{-2x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2) e^{-2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2) e^{-2x}}{2}$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2) e^{-2x}}{2} = c_1 \quad (1)$$

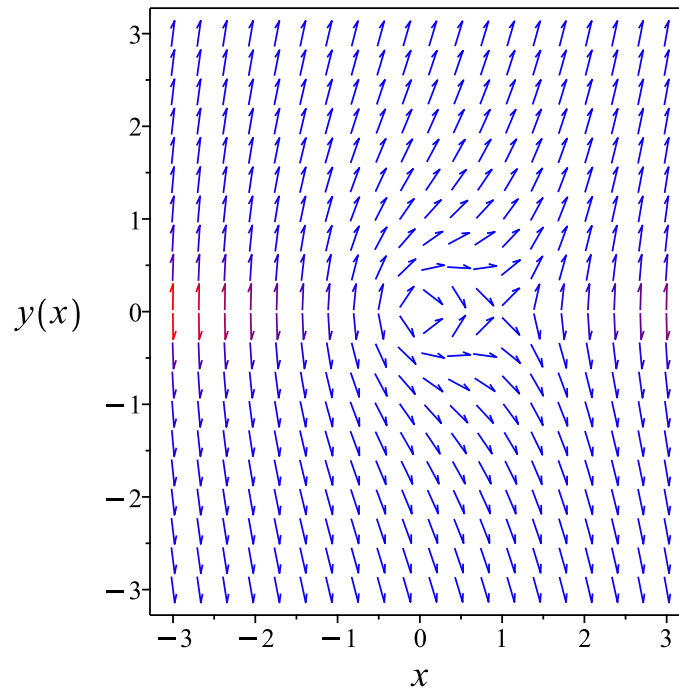


Figure 128: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2) e^{-2x}}{2} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve((x-x^2-y(x)^2)+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{2x}c_1 - x^2}$$
$$y(x) = -\sqrt{e^{2x}c_1 - x^2}$$

✓ Solution by Mathematica

Time used: 4.613 (sec). Leaf size: 47

```
DSolve[(x-x^2-y[x]^2)+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{-x^2 + c_1 e^{2x}}$$
$$y(x) \rightarrow \sqrt{-x^2 + c_1 e^{2x}}$$

3.17 problem 25 (b)

3.17.1 Solving as exact ode	572
3.17.2 Maple step by step solution	577

Internal problem ID [5278]

Internal file name [OUTPUT/4769_Friday_February_02_2024_05_11_47_AM_41865652/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (b).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

`[_linear]`

$$xy' + 2y = 3x$$

3.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) dy &= (3x - 2y) dx \\ (-3x + 2y) dx + (x) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -3x + 2y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3x + 2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((2) - (1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= x(-3x + 2y) \\ &= (-3x + 2y)x \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= x(x) \\ &= x^2 \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ ((-3x + 2y)x) + (x^2) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \bar{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int (-3x + 2y) x dx$$

$$\phi = -x^2(x - y) + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2(x - y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2(x - y)$$

The solution becomes

$$y = \frac{x^3 + c_1}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + c_1}{x^2} \tag{1}$$

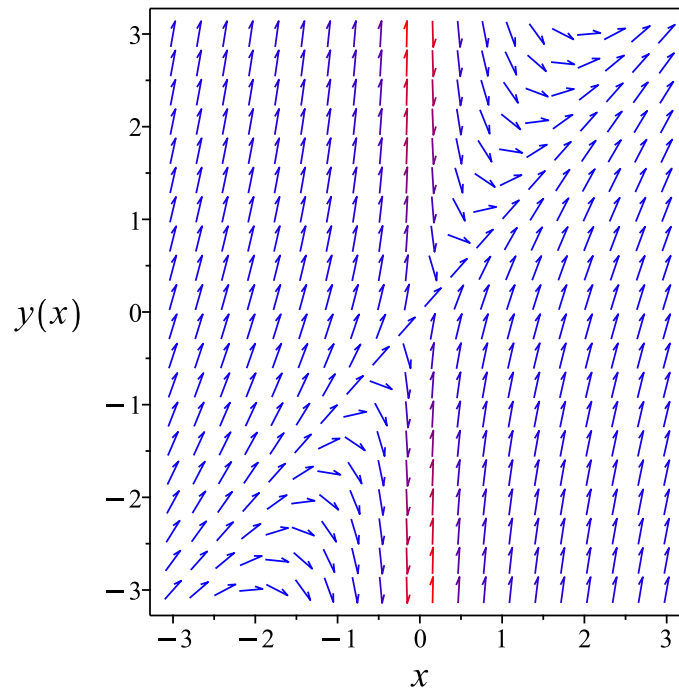


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{x^3 + c_1}{x^2}$$

Verified OK.

3.17.2 Maple step by step solution

Let's solve

$$xy' + 2y = 3x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = 3 - \frac{2y}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{2y}{x} = 3$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{2y}{x} \right) = 3\mu(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' + \frac{2y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x^2$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int 3\mu(x) dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int 3\mu(x) dx + c_1$$

- Solve for y

$$y = \frac{\int 3\mu(x)dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^2$

$$y = \frac{\int 3x^2 dx + c_1}{x^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{x^3 + c_1}{x^2}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((2*y(x)-3*x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x + \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 13

```
DSolve[(2*y[x]-3*x)+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x + \frac{c_1}{x^2}$$

3.18 problem 25 (c)

3.18.1 Solving as exact ode 579

Internal problem ID [5279]

Internal file name [OUTPUT/4770_Friday_February_02_2024_05_11_47_AM_45671391/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (c).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$-y^2 + 2xyy' = -x$$

3.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (2xy) dy &= (y^2 - x) dx \\ (-y^2 + x) dx + (2xy) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -y^2 + x \\ N(x, y) &= 2xy \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^2 + x) \\ &= -2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{2yx} ((-2y) - (2y)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{2}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2 \ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x^2}(-y^2 + x) \\ &= \frac{-y^2 + x}{x^2}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x^2}(2xy) \\ &= \frac{2y}{x}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-y^2 + x}{x^2} \right) + \left(\frac{2y}{x} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-y^2 + x}{x^2} dx \\ \phi &= \ln(x) + \frac{y^2}{x} + f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y)\tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$\frac{2y}{x} = \frac{2y}{x} + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$\ln(x) + \frac{y^2}{x} = c_1\tag{1}$$

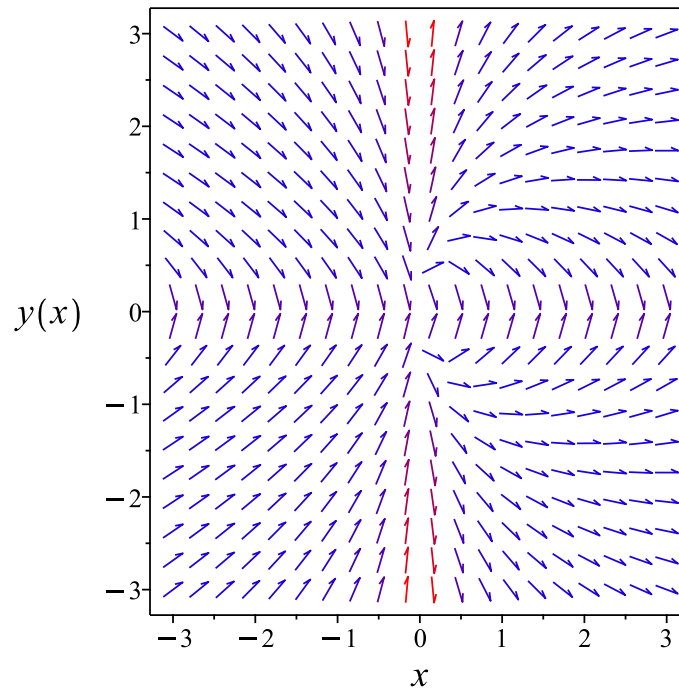


Figure 130: Slope field plot

Verification of solutions

$$\ln(x) + \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```


✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((x-y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x(\ln(x) - c_1)}$$
$$y(x) = -\sqrt{x(-\ln(x) + c_1)}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 44

```
DSolve[(x-y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{x}\sqrt{-\log(x) + c_1}$$
$$y(x) \rightarrow \sqrt{x}\sqrt{-\log(x) + c_1}$$

3.19 problem 25 (d)

3.19.1 Solving as exact ode 585

Internal problem ID [5280]

Internal file name [OUTPUT/4771_Friday_February_02_2024_05_11_47_AM_57323726/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$-y - 3x^2(x^2 + y^2) + xy' = 0$$

3.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x) dy &= (y + 3x^2(x^2 + y^2)) dx \\ (-y - 3x^2(x^2 + y^2)) dx + (x) dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -y - 3x^2(x^2 + y^2) \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-y - 3x^2(x^2 + y^2)) \\ &= -6yx^2 - 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y - 3x^2(x^2 + y^2)$ and $N = x$ by this integrating factor

the ode becomes exact. The new M, N are

$$M = \frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2}$$

$$N = \frac{x}{x^2 + y^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\left(\frac{x}{x^2 + y^2} \right) dy = \left(-\frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2} \right) dx$$

$$\left(\frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy = 0 \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2}$$

$$N(x, y) = \frac{x}{x^2 + y^2}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2} \right) \\ &= \frac{-x^2 + y^2}{(x^2 + y^2)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) \\ &= \frac{-x^2 + y^2}{(x^2 + y^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-y - 3x^2(x^2 + y^2)}{x^2 + y^2} dx \\ \phi &= -x^3 - \arctan \left(\frac{x}{y} \right) + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{x}{y^2\left(\frac{x^2}{y^2} + 1\right)} + f'(y) \\ &= \frac{x}{x^2 + y^2} + f'(y)\end{aligned}\tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x}{x^2 + y^2} = \frac{x}{x^2 + y^2} + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 - \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 - \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = -\frac{x}{\tan(x^3 + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\tan(x^3 + c_1)}\tag{1}$$

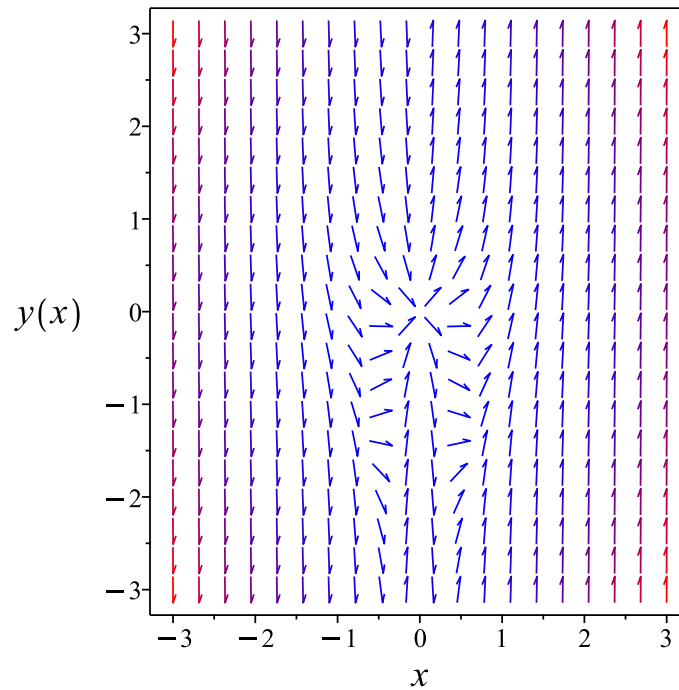


Figure 131: Slope field plot

Verification of solutions

$$y = -\frac{x}{\tan(x^3 + c_1)}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve((-y(x)-3*x^2*(x^2+y(x)^2))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan(x^3 + 3c_1) x$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 14

```
DSolve[(-y[x]-3*x^2*(x^2+y[x]^2))+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x \tan(x^3 + c_1)$$

3.20 problem 25 (e)

3.20.1 Solving as exact ode	592
3.20.2 Maple step by step solution	597

Internal problem ID [5281]

Internal file name [OUTPUT/4772_Friday_February_02_2024_05_11_48_AM_79599042/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (e).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

`[_linear]`

$$-xy' + y = \ln(x)$$

3.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(-x) dy &= (-y + \ln(x)) dx \\ (y - \ln(x)) dx + (-x) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - \ln(x) \\ N(x, y) &= -x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \ln(x)) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{x} ((1) - (-1)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int -\frac{2}{x} dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2 \ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \frac{1}{x^2} (y - \ln(x)) \\ &= \frac{y - \ln(x)}{x^2} \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= \frac{1}{x^2} (-x) \\ &= -\frac{1}{x} \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{y - \ln(x)}{x^2} \right) + \left(-\frac{1}{x} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{y - \ln(x)}{x^2} dx$$

$$\phi = \frac{-y + \ln(x) + 1}{x} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-y + \ln(x) + 1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-y + \ln(x) + 1}{x}$$

The solution becomes

$$y = -c_1x + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = -c_1x + \ln(x) + 1 \tag{1}$$

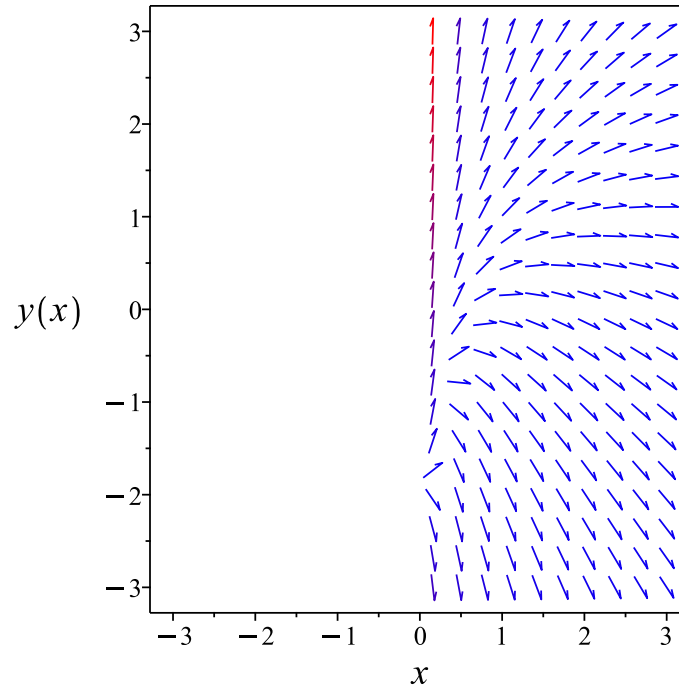


Figure 132: Slope field plot

Verification of solutions

$$y = -c_1x + \ln(x) + 1$$

Verified OK.

3.20.2 Maple step by step solution

Let's solve

$$-xy' + y = \ln(x)$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{y}{x} - \frac{\ln(x)}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{y}{x} = -\frac{\ln(x)}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{y}{x} \right) = -\frac{\mu(x)\ln(x)}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)\ln(x)}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x}$

$$y = x \left(\int -\frac{\ln(x)}{x^2} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x \left(\frac{\ln(x)}{x} + \frac{1}{x} + c_1 \right)$$

- Simplify

$$y = c_1x + \ln(x) + 1$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((y(x)-ln(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1x + \ln(x) + 1$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 13

```
DSolve[(y[x]-Log[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \log(x) + c_1x + 1$$

3.21 problem 25 (f)

3.21.1 Solving as exact ode 599

Internal problem ID [5282]

Internal file name [OUTPUT/4773_Friday_February_02_2024_05_11_48_AM_87394506/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (f).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y^2 - 2xyy' = -3x^2$$

3.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (-2xy) dy &= (-3x^2 - y^2) dx \\ (3x^2 + y^2) dx + (-2xy) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 3x^2 + y^2 \\ N(x, y) &= -2xy \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2 + y^2) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2xy) \\ &= -2y \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{2yx} ((2y) - (-2y)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{2}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}(3x^2 + y^2) \\ &= \frac{3x^2 + y^2}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(-2xy) \\ &= -\frac{2y}{x}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{3x^2 + y^2}{x^2}\right) + \left(-\frac{2y}{x}\right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{3x^2 + y^2}{x^2} dx \\ \phi &= 3x - \frac{y^2}{x} + f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y)\tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$-\frac{2y}{x} = -\frac{2y}{x} + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 3x - \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 3x - \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$3x - \frac{y^2}{x} = c_1\tag{1}$$

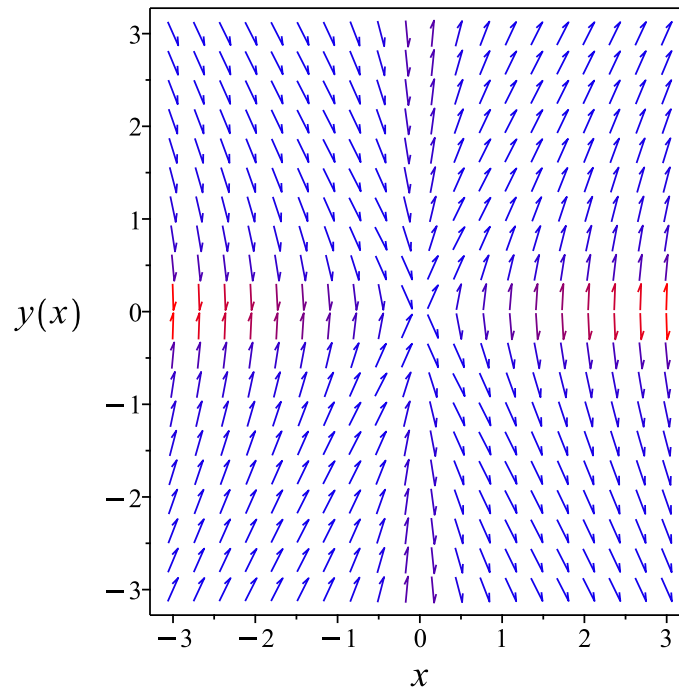


Figure 133: Slope field plot

Verification of solutions

$$3x - \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve((3*x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(3x + c_1)x}$$
$$y(x) = -\sqrt{(3x + c_1)x}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 42

```
DSolve[(3*x^2+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{x}\sqrt{3x + c_1}$$
$$y(x) \rightarrow \sqrt{x}\sqrt{3x + c_1}$$

3.22 problem 25 (g)

3.22.1 Solving as exact ode 605

Internal problem ID [5283]

Internal file name [OUTPUT/4774_Friday_February_02_2024_05_11_48_AM_23473528/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (g).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactByInspection**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$xy - 2y^2 - (x^2 - 3xy) y' = 0$$

3.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(-x^2 + 3xy) dy &= (-xy + 2y^2) dx \\ (xy - 2y^2) dx + (-x^2 + 3xy) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= xy - 2y^2 \\ N(x, y) &= -x^2 + 3xy\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(xy - 2y^2) \\ &= x - 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + 3xy) \\ &= -2x + 3y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = xy - 2y^2$ and $N = -x^2 + 3xy$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{xy - 2y^2}{xy^2}$$

$$N = \frac{-x^2 + 3xy}{xy^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\left(\frac{-x^2 + 3xy}{x y^2}\right) dy = \left(-\frac{xy - 2y^2}{x y^2}\right) dx$$

$$\left(\frac{xy - 2y^2}{x y^2}\right) dx + \left(\frac{-x^2 + 3xy}{x y^2}\right) dy = 0 \quad (2A)$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{xy - 2y^2}{x y^2}$$

$$N(x, y) = \frac{-x^2 + 3xy}{x y^2}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{xy - 2y^2}{x y^2}\right)$$

$$= -\frac{1}{y^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{-x^2 + 3xy}{x y^2}\right)$$

$$= -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{xy - 2y^2}{x y^2} dx$$

$$\phi = -2 \ln(x) + \frac{x}{y} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+3xy}{xy^2}$. Therefore equation (4) becomes

$$\frac{-x^2 + 3xy}{xy^2} = -\frac{x}{y^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(\frac{3}{y}\right) dy$$

$$f(y) = 3 \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2 \ln(x) + \frac{x}{y} + 3 \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2 \ln(x) + \frac{x}{y} + 3 \ln(y)$$

The solution becomes

$$y = e^{\text{LambertW}\left(-\frac{x e^{-\frac{2 \ln(x)}{3} - \frac{c_1}{3}}}{3}\right) + \frac{2 \ln(x)}{3} + \frac{c_1}{3}}$$

Summary

The solution(s) found are the following

$$y = e^{\text{LambertW}\left(-x e^{-\frac{2\ln(x) - c_1}{3}}\right) + \frac{2\ln(x) + c_1}{3}} \quad (1)$$

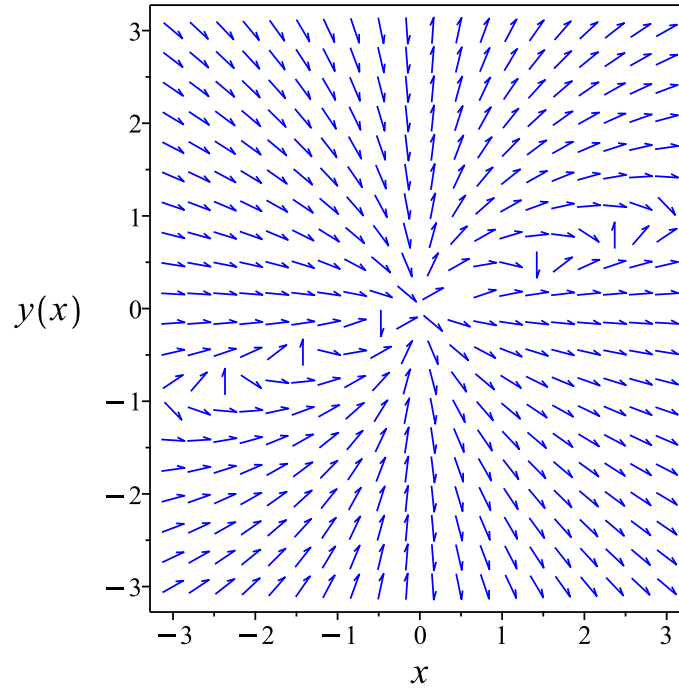


Figure 134: Slope field plot

Verification of solutions

$$y = e^{\text{LambertW}\left(-x e^{-\frac{2\ln(x) - c_1}{3}}\right) + \frac{2\ln(x) + c_1}{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous D  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve((x*y(x)-2*y(x)^2)-(x^2-3*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{3 \operatorname{LambertW}\left(-\frac{e^{\frac{c_1}{3}} x^{\frac{1}{3}}}{3}\right)}$$

✓ Solution by Mathematica

Time used: 4.722 (sec). Leaf size: 35

```
DSolve[(x*y[x]-2*y[x]^2)-(x^2-3*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x}{3W\left(-\frac{1}{3}e^{-\frac{c_1}{3}}\sqrt[3]{x}\right)}$$
$$y(x) \rightarrow 0$$

3.23 problem 25 (h)

3.23.1 Solving as exact ode 612

Internal problem ID [5284]

Internal file name [OUTPUT/4775_Friday_February_02_2024_05_11_49_AM_67211904/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (h).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `class A`]]
```

$$y - (x - y)y' = -x$$

3.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(-x + y) dy &= (-y - x) dx \\ (x + y) dx + (-x + y) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + y \\ N(x, y) &= -x + y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x + y) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x + y}{x^2 + y^2}$$

$$N = \frac{-x + y}{x^2 + y^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{-x+y}{x^2+y^2}\right) dy &= \left(-\frac{x+y}{x^2+y^2}\right) dx \\ \left(\frac{x+y}{x^2+y^2}\right) dx + \left(\frac{-x+y}{x^2+y^2}\right) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{x+y}{x^2+y^2} \\ N(x, y) &= \frac{-x+y}{x^2+y^2} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{x+y}{x^2+y^2} \right) \\ &= \frac{x^2 - 2xy - y^2}{(x^2+y^2)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{-x+y}{x^2+y^2} \right) \\ &= \frac{x^2 - 2xy - y^2}{(x^2+y^2)^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{x+y}{x^2+y^2} dx \\ \phi &= \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= \frac{y}{x^2 + y^2} - \frac{x}{y^2 \left(\frac{x^2}{y^2} + 1\right)} + f'(y) \\ &= \frac{-x + y}{x^2 + y^2} + f'(y)\end{aligned}\tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{-x + y}{x^2 + y^2} = \frac{-x + y}{x^2 + y^2} + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right) = c_1\tag{1}$$

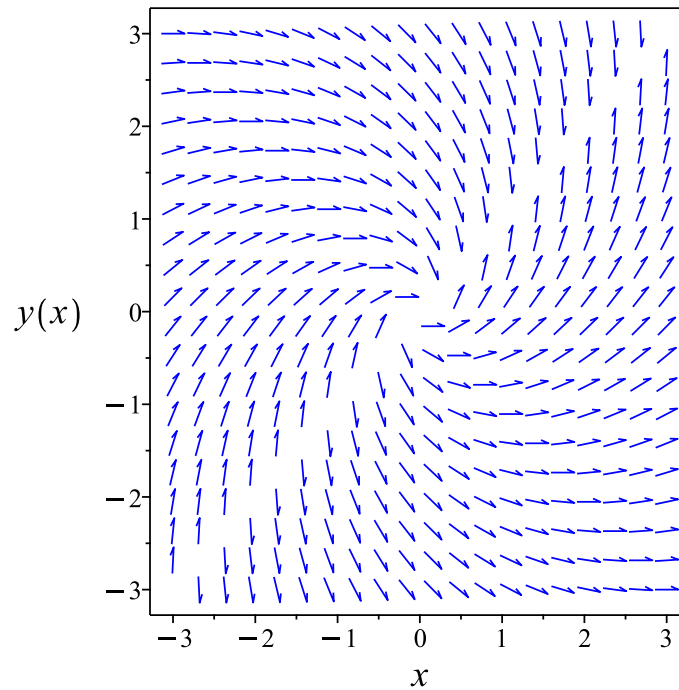


Figure 135: Slope field plot

Verification of solutions

$$\frac{\ln(x^2 + y^2)}{2} + \arctan\left(\frac{x}{y}\right) = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan(\text{RootOf}(-2_Z + \ln(\sec(_Z)^2) + 2\ln(x) + 2c_1)) x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2} + 1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

3.24 problem 25 (L)

3.24.1 Solving as exact ode 619

Internal problem ID [5285]

Internal file name [OUTPUT/4776_Friday_February_02_2024_05_11_49_AM_91658592/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (L).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$2y - 3xy^2 - xy' = 0$$

3.24.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (-x) dy &= (3y^2x - 2y) dx \\ (-3y^2x + 2y) dx + (-x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -3y^2x + 2y \\ N(x, y) &= -x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3y^2x + 2y) \\ &= -6xy + 2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{x} ((-6xy + 2) - (-1)) \\ &= \frac{6xy - 3}{x} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{-3y^2x + 2y} ((-1) - (-6xy + 2)) \\ &= \frac{-6xy + 3}{3y^2x - 2y} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(-1) - (-6xy + 2)}{x(-3y^2x + 2y) - y(-x)} \\ &= \frac{-2xy + 1}{xy(xy - 1)} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t + 1}{t(t - 1)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int \left(\frac{-2t+1}{t(t-1)} \right) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(t(t-1))} \\ &= \frac{1}{t(t-1)} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy(xy - 1)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{xy(xy-1)}(-3y^2x + 2y) \\ &= \frac{-3xy + 2}{x(xy-1)}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{xy(xy-1)}(-x) \\ &= -\frac{1}{y(xy-1)}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-3xy + 2}{x(xy-1)} \right) + \left(-\frac{1}{y(xy-1)} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-3xy + 2}{x(xy-1)} dx \\ \phi &= -\ln(xy-1) - 2\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{xy-1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(xy-1)}$. Therefore equation (4) becomes

$$-\frac{1}{y(xy-1)} = -\frac{x}{xy-1} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(\frac{1}{y}\right) dy$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(xy-1) - 2\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(xy-1) - 2\ln(x) + \ln(y)$$

The solution becomes

$$y = \frac{x^2 e^{c_1}}{e^{c_1} x^3 - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 e^{c_1}}{e^{c_1} x^3 - 1} \quad (1)$$

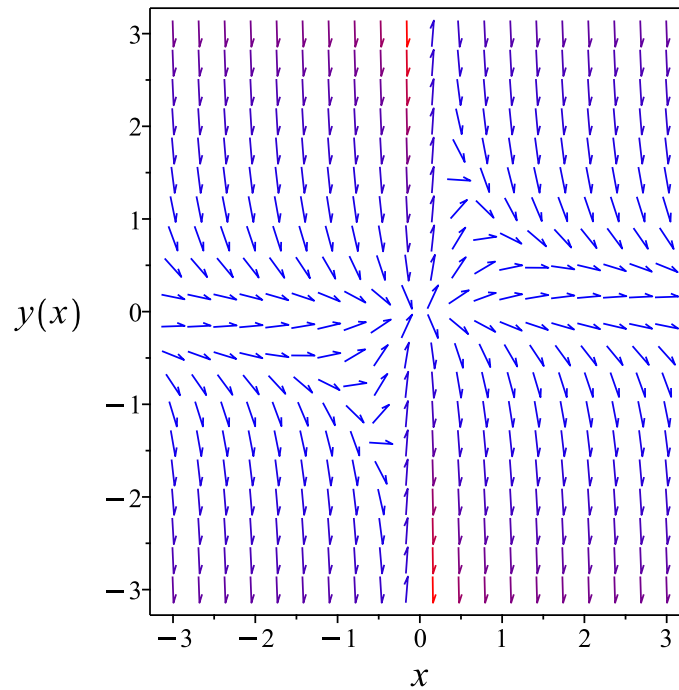


Figure 136: Slope field plot

Verification of solutions

$$y = \frac{x^2 e^{c_1}}{e^{c_1} x^3 - 1}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((2*y(x)-3*x*y(x)^2)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{x^3 + c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 22

```
DSolve[(2*y[x]-3*x*y[x]^2)-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^2}{x^3 + c_1}$$
$$y(x) \rightarrow 0$$

3.25 problem 25 (j)

3.25.1 Solving as exact ode 626

Internal problem ID [5286]

Internal file name [OUTPUT/4777_Friday_February_02_2024_05_11_50_AM_54804854/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (j).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactByInspection**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$y + x(yx^2 - 1)y' = 0$$

3.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(x(y x^2 - 1)) dy &= (-y) dx \\ (y) dx + (x(y x^2 - 1)) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y \\ N(x, y) &= x(y x^2 - 1)\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(y x^2 - 1)) \\ &= 3y x^2 - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{y}{x^3}$ is an integrating factor. Therefore by multiplying $M = y$ and $N = x(yx^2 - 1)$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{y^2}{x^3}$$

$$N = \frac{y(yx^2 - 1)}{x^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{y(yx^2-1)}{x^2}\right) dy &= \left(-\frac{y^2}{x^3}\right) dx \\ \left(\frac{y^2}{x^3}\right) dx + \left(\frac{y(yx^2-1)}{x^2}\right) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{y^2}{x^3} \\ N(x, y) &= \frac{y(yx^2-1)}{x^2} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{y^2}{x^3}\right) \\ &= \frac{2y}{x^3} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{y(yx^2-1)}{x^2}\right) \\ &= \frac{2y}{x^3} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{y^2}{x^3} dx \\ \phi &= -\frac{y^2}{2x^2} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{y}{x^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{y(yx^2-1)}{x^2}$. Therefore equation (4) becomes

$$\frac{y(yx^2-1)}{x^2} = -\frac{y}{x^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int (y^2) dy$$

$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{2x^2} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{2x^2} + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c_1 \quad (1)$$

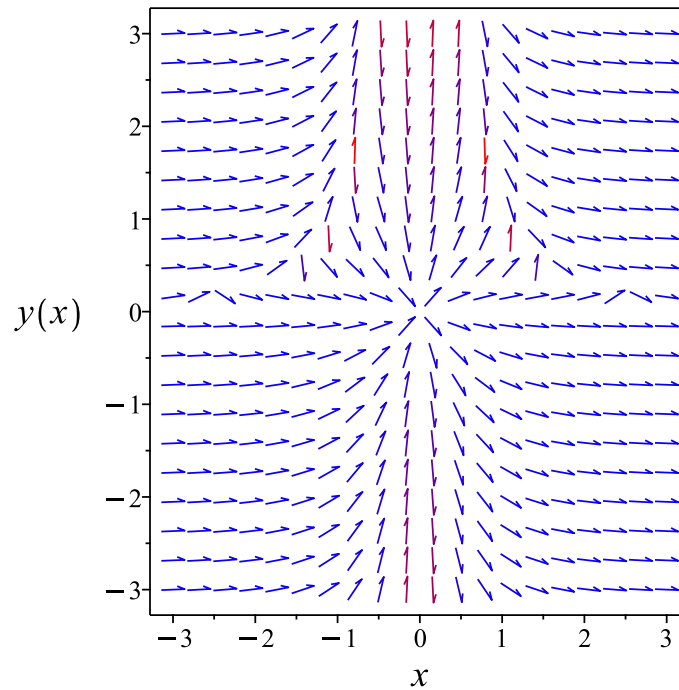


Figure 137: Slope field plot

Verification of solutions

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`

```


✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 685

`dsolve(y(x)+x*(x^2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)`

$$y(x) = \frac{3 + \frac{\left((x^3 + \sqrt{c_1^6 + x^6})^{\frac{2}{3}} - c_1^2 \right)^2}{c_1^2 (x^3 + \sqrt{c_1^6 + x^6})^{\frac{2}{3}}}}{2x^2}$$

$$y(x) = \frac{\left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} + c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + c_1^4}{2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2}$$

$$y(x) = \frac{\left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} + c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + c_1^4}{2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2}$$

$$y(x) = \frac{2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (i\sqrt{3} - 1) - (1 + i\sqrt{3}) c_1^4}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (1 + i\sqrt{3}) - c_1^4 (i\sqrt{3} - 1)}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = \frac{2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (i\sqrt{3} - 1) - (1 + i\sqrt{3}) c_1^4}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (1 + i\sqrt{3}) - c_1^4 (i\sqrt{3} - 1)}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = \frac{2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (i\sqrt{3} - 1) - (1 + i\sqrt{3}) c_1^4}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} + \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{4}{3}} (1 + i\sqrt{3}) - c_1^4 (i\sqrt{3} - 1)}{4 \left(x^3 + \sqrt{c_1^6 + x^6} \right)^{\frac{2}{3}} x^2 c_1^2}$$

✓ Solution by Mathematica

Time used: 56.665 (sec). Leaf size: 452

`DSolve[y[x]+x*(x^2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$y(x) \rightarrow \frac{e^{-6c_1} \sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1} + \frac{e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1}}}{2x^2}$$

$$y(x) \rightarrow \frac{i(\sqrt{3} + i) e^{-6c_1} \sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1} - \frac{(1+i\sqrt{3})e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1}}}{4x^2}$$

$$y(x) \rightarrow \frac{-\left((1 + i\sqrt{3}) e^{-6c_1} \sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1} \right) + \frac{i(\sqrt{3}+i)e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6 + 2\sqrt{-e^{24c_1}x^6(-x^6 + e^{6c_1})}} + e^{18c_1}}}{4x^2}$$

$$y(x) \rightarrow 0$$

$$y(x) \rightarrow \frac{3}{2x^2}$$

3.26 problem 25 (k)

3.26.1 Solving as exact ode 634

Internal problem ID [5287]

Internal file name [OUTPUT/4778_Friday_February_02_2024_05_11_50_AM_68593338/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 25 (k).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

[_rational]

$$yx^3 + y + (x + 4xy^4 + 8y^3) y' = -2x^2$$

3.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (4y^4x + 8y^3 + x) dy &= (-y x^3 - 2x^2 - y) dx \\ (y x^3 + 2x^2 + y) dx + (4y^4x + 8y^3 + x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y x^3 + 2x^2 + y \\ N(x, y) &= 4y^4x + 8y^3 + x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (y x^3 + 2x^2 + y) \\ &= x^3 + 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (4y^4x + 8y^3 + x) \\ &= 4y^4 + 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{4y^4x + 8y^3 + x} ((x^3 + 1) - (4y^4 + 1)) \\ &= \frac{-4y^4 + x^3}{4y^4x + 8y^3 + x} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y x^3 + 2x^2 + y} ((4y^4 + 1) - (x^3 + 1)) \\ &= \frac{4y^4 - x^3}{y x^3 + 2x^2 + y} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(4y^4 + 1) - (x^3 + 1)}{x(y x^3 + 2x^2 + y) - y(4y^4 x + 8y^3 + x)} \\ &= -\frac{1}{xy + 2} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{t + 2}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int \left(-\frac{1}{t+2}\right) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(t+2)} \\ &= \frac{1}{t + 2} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy + 2}$$

Multiplying M and N by this integrating factor gives new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{xy+2}(yx^3 + 2x^2 + y) \\ &= \frac{yx^3 + 2x^2 + y}{xy+2}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{xy+2}(4y^4x + 8y^3 + x) \\ &= \frac{4y^4x + 8y^3 + x}{xy+2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{yx^3 + 2x^2 + y}{xy+2} \right) + \left(\frac{4y^4x + 8y^3 + x}{xy+2} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{yx^3 + 2x^2 + y}{xy+2} dx \\ \phi &= \frac{x^3}{3} + \ln(xy+2) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{xy+2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{4y^4x + 8y^3 + x}{xy + 2}$. Therefore equation (4) becomes

$$\frac{4y^4x + 8y^3 + x}{xy + 2} = \frac{x}{xy + 2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) \, dy &= \int (4y^3) \, dy \\ f(y) &= y^4 + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + \ln(xy + 2) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + \ln(xy + 2) + y^4$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + \ln(xy + 2) + y^4 = c_1 \quad (1)$$

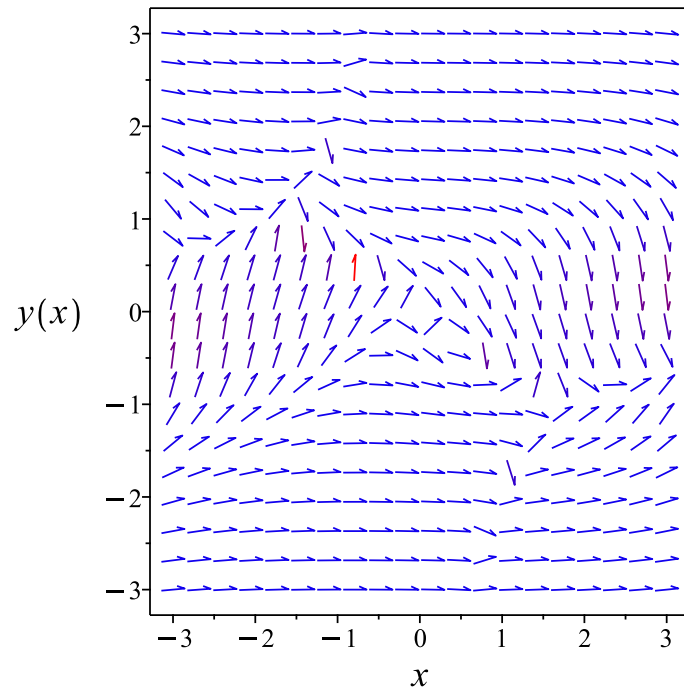


Figure 138: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + \ln(xy + 2) + y^4 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, (x*y+2)/(4*x*y^4+8*y^3+x)]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 25

```
dsolve((y(x)+x^3*y(x)+2*x^2)+(x+4*x*y(x)^4+8*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{x^3}{3} - \ln(xy(x) + 2) - y(x)^4 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 25

```
DSolve[(y[x]+x^3*y[x]+2*x^2)+(x+4*x*y[x]^4+8*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[\frac{x^3}{3} + y(x)^4 + \log(xy(x) + 2) = c_1, y(x)\right]$$

3.27 problem 26 (a)

3.27.1 Solving as exact ode	641
3.27.2 Maple step by step solution	646

Internal problem ID [5288]

Internal file name [OUTPUT/4779_Friday_February_02_2024_05_11_50_AM_53460436/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

`[_linear]`

$$-y + xy' = x^2e^x$$

3.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(x) dy &= (x^2 e^x + y) dx \\ (-x^2 e^x - y) dx + (x) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 e^x - y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-x^2 e^x - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((-1) - (1)) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{2}{x} \, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2 \ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \frac{1}{x^2} (-x^2 e^x - y) \\ &= \frac{-x^2 e^x - y}{x^2} \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= \frac{1}{x^2} (x) \\ &= \frac{1}{x} \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-x^2 e^x - y}{x^2} \right) + \left(\frac{1}{x} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-x^2 e^x - y}{x^2} dx \\ \phi &= \frac{-x e^x + y}{x} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x e^x + y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x e^x + y}{x}$$

The solution becomes

$$y = x(e^x + c_1)$$

Summary

The solution(s) found are the following

$$y = x(e^x + c_1) \tag{1}$$

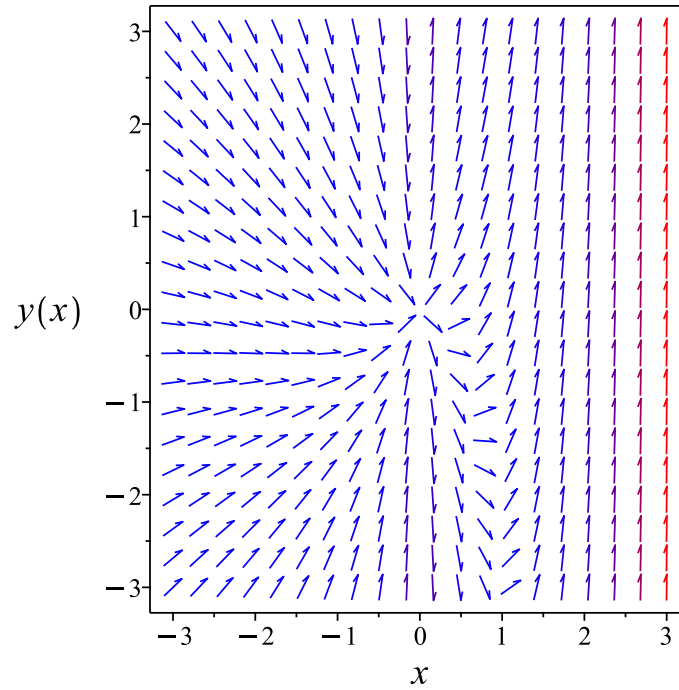


Figure 139: Slope field plot

Verification of solutions

$$y = x(e^x + c_1)$$

Verified OK.

3.27.2 Maple step by step solution

Let's solve

$$-y + xy' = x^2 e^x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{y}{x} + x e^x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{y}{x} = x e^x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{y}{x} \right) = \mu(x) x e^x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) \left(y' - \frac{y}{x} \right) = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int \mu(x) x e^x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int \mu(x) x e^x dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) x e^x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x}$

$$y = x \left(\int e^x dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x(e^x + c_1)$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((-y(x)-x^2*exp(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (e^x + c_1)x$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 13

```
DSolve[(-y[x]-x^2*Exp[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x(e^x + c_1)$$

3.28 problem 26 (b)

3.28.1 Solving as exact ode	648
3.28.2 Maple step by step solution	652

Internal problem ID [5289]

Internal file name [OUTPUT/4780_Friday_February_02_2024_05_11_50_AM_83324793/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (b).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$y^2 - (x^2 + x) y' = -1$$

3.28.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2 + 1}\right) dy &= \left(\frac{1}{x(x+1)}\right) dx \\ \left(-\frac{1}{x(x+1)}\right) dx + \left(\frac{1}{y^2 + 1}\right) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x(x+1)} \\ N(x, y) &= \frac{1}{y^2 + 1}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x(x+1)}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{y^2 + 1} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x(x+1)} dx \\ \phi &= \ln(x+1) - \ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{1}{y^2+1} \right) dy \\ f(y) &= \arctan(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x + 1) - \ln(x) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x + 1) - \ln(x) + \arctan(y)$$

The solution becomes

$$y = \tan(-\ln(x + 1) + \ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(-\ln(x + 1) + \ln(x) + c_1) \tag{1}$$

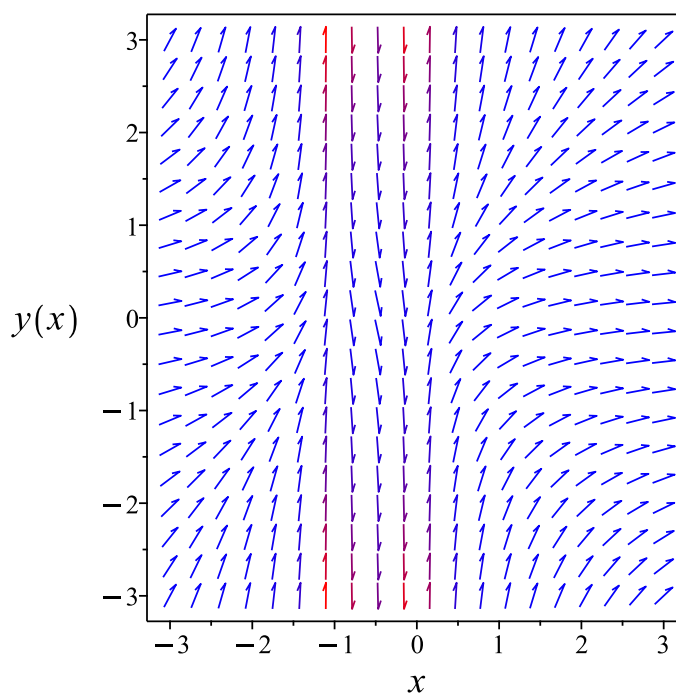


Figure 140: Slope field plot

Verification of solutions

$$y = \tan(-\ln(x + 1) + \ln(x) + c_1)$$

Verified OK.

3.28.2 Maple step by step solution

Let's solve

$$y^2 - (x^2 + x)y' = -1$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'}{-1-y^2} = -\frac{1}{x^2+x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-1-y^2} dx = \int -\frac{1}{x^2+x} dx + c_1$$

- Evaluate integral

$$-\arctan(y) = \ln(x+1) - \ln(x) + c_1$$

- Solve for y

$$y = -\tan(\ln(x+1) - \ln(x) + c_1)$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(1+y(x)^2=(x+x^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan(-\ln(x+1) + \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 31

```
DSolve[1+y[x]^2==(x+x^2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \tan(\log(x) - \log(x + 1) + c_1)$$

$$y(x) \rightarrow -i$$

$$y(x) \rightarrow i$$

3.29 problem 26 (c)

3.29.1 Solving as exact ode	654
3.29.2 Maple step by step solution	659

Internal problem ID [5290]

Internal file name [OUTPUT/4781_Friday_February_02_2024_05_11_51_AM_84981706/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (c).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

`[_linear]`

$$xy' + 2y = x^3$$

3.29.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(x) dy &= (x^3 - 2y) dx \\ (-x^3 + 2y) dx + (x) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^3 + 2y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3 + 2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((2) - (1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= x(-x^3 + 2y) \\ &= -x(x^3 - 2y) \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= x(x) \\ &= x^2 \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (-x(x^3 - 2y)) + (x^2) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$
$$\int \frac{\partial \phi}{\partial x} dx = \int -x(x^3 - 2y) dx$$
$$\phi = -\frac{1}{5}x^5 + yx^2 + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{5}x^5 + yx^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{5}x^5 + yx^2$$

The solution becomes

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^5 + 5c_1}{5x^2} \tag{1}$$

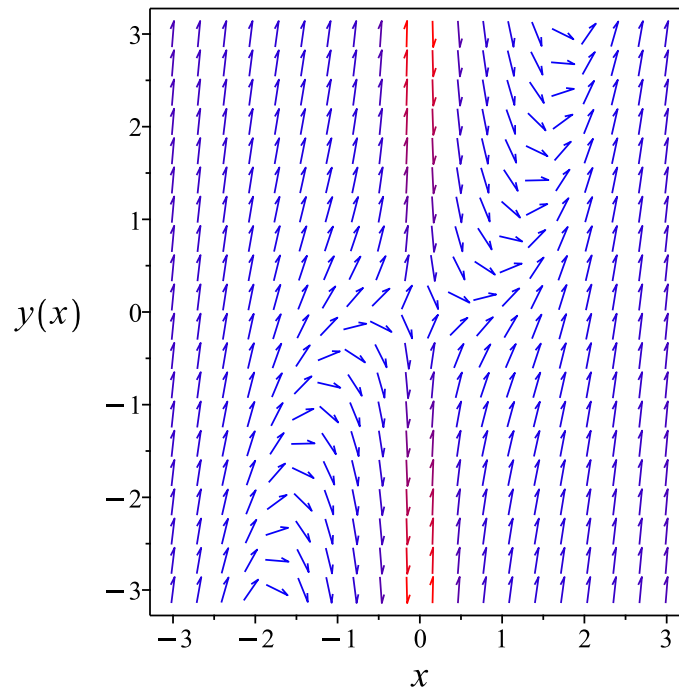


Figure 141: Slope field plot

Verification of solutions

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Verified OK.

3.29.2 Maple step by step solution

Let's solve

$$xy' + 2y = x^3$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = -\frac{2y}{x} + x^2$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{2y}{x} = x^2$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{2y}{x} \right) = \mu(x) x^2$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' + \frac{2y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = x^2$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int \mu(x) x^2 dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int \mu(x) x^2 dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^2$

$$y = \frac{\int x^4 dx + c_1}{x^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{c_1 + \frac{x^5}{5}}{x^2}$$

- Simplify

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2*y(x)-x^3)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^5 + 5c_1}{5x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[(2*y[x]-x^3)+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^3}{5} + \frac{c_1}{x^2}$$

3.30 problem 26 (d)

3.30.1 Solving as exact ode 661

Internal problem ID [5291]

Internal file name [OUTPUT/4782_Friday_February_02_2024_05_11_51_AM_82988294/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$y + (-x + y^2) y' = 0$$

3.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (y^2 - x) dy &= (-y) dx \\ (y) dx + (y^2 - x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y \\ N(x, y) &= y^2 - x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 - x) \\ &= -1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{y^2 - x} ((1) - (-1)) \\ &= -\frac{2}{-y^2 + x} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y} ((-1) - (1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int -\frac{2}{y} \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2 \ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} so not to confuse them with the original M and N .

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2} (y) \\ &= \frac{1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2} (y^2 - x) \\ &= \frac{y^2 - x}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{1}{y} \right) + \left(\frac{y^2 - x}{y^2} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{1}{y} dx$$

$$\phi = \frac{x}{y} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2 - x}{y^2}$. Therefore equation (4) becomes

$$\frac{y^2 - x}{y^2} = -\frac{x}{y^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int (1) dy$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y} + y$$

Summary

The solution(s) found are the following

$$\frac{x}{y} + y = c_1 \tag{1}$$

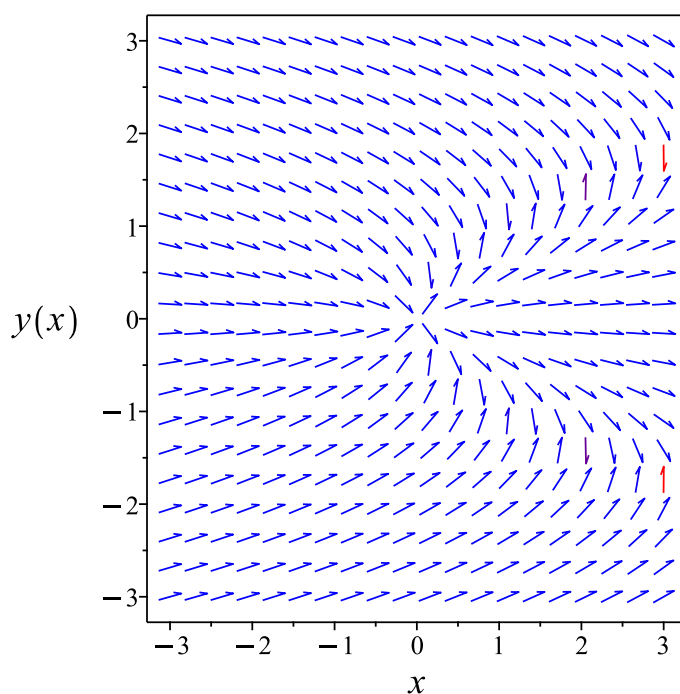


Figure 142: Slope field plot

Verification of solutions

$$\frac{x}{y} + y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
<- 1st order linear successful  
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(y(x)+(-x+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{2} - \frac{\sqrt{c_1^2 - 4x}}{2}$$
$$y(x) = \frac{c_1}{2} + \frac{\sqrt{c_1^2 - 4x}}{2}$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 54

```
DSolve[y[x]+(-x+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2} \left(c_1 - \sqrt{-4x + c_1^2} \right)$$
$$y(x) \rightarrow \frac{1}{2} \left(\sqrt{-4x + c_1^2} + c_1 \right)$$
$$y(x) \rightarrow 0$$

3.31 problem 26 (e)

3.31.1 Solving as exact ode 667

Internal problem ID [5292]

Internal file name [OUTPUT/4783_Friday_February_02_2024_05_11_51_AM_10904012/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (e).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$3y^3 - xy - (x^2 + 6xy^2) y' = 0$$

3.31.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (-6y^2x - x^2) dy &= (-3y^3 + xy) dx \\ (3y^3 - xy) dx + (-6y^2x - x^2) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 3y^3 - xy \\ N(x, y) &= -6y^2x - x^2 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y^3 - xy) \\ &= 9y^2 - x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-6y^2x - x^2) \\ &= -6y^2 - 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2y}$ is an integrating factor. Therefore by multiplying $M = 3y^3 - xy$ and $N = -x^2 - 6xy^2$ by this integrating factor

the ode becomes exact. The new M, N are

$$M = \frac{3y^3 - xy}{x^2y}$$

$$N = \frac{-x^2 - 6xy^2}{x^2y}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$

$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\left(\frac{-6y^2x - x^2}{x^2y}\right) dy = \left(-\frac{3y^3 - xy}{x^2y}\right) dx$$

$$\left(\frac{3y^3 - xy}{x^2y}\right) dx + \left(\frac{-6y^2x - x^2}{x^2y}\right) dy = 0 \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{3y^3 - xy}{x^2y}$$
$$N(x, y) = \frac{-6y^2x - x^2}{x^2y}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{3y^3 - xy}{x^2y} \right)$$
$$= \frac{6y}{x^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(\frac{-6y^2x - x^2}{x^2y} \right)$$
$$= \frac{6y}{x^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$
$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{3y^3 - xy}{x^2y} dx$$
$$\phi = -\ln(x) - \frac{3y^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{6y}{x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-6y^2x - x^2}{x^2y}$. Therefore equation (4) becomes

$$\frac{-6y^2x - x^2}{x^2y} = -\frac{6y}{x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{1}{y}\right) dy$$

$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{3y^2}{x} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{3y^2}{x} - \ln(y)$$

The solution becomes

$$y = \frac{e^{-\frac{\text{LambertW}\left(\frac{6e^{-2c_1}}{x^3}\right) - c_1}{2}}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{\text{LambertW}\left(\frac{6e^{-2c_1}}{x^3}\right)}{2} - c_1}}{x} \quad (1)$$

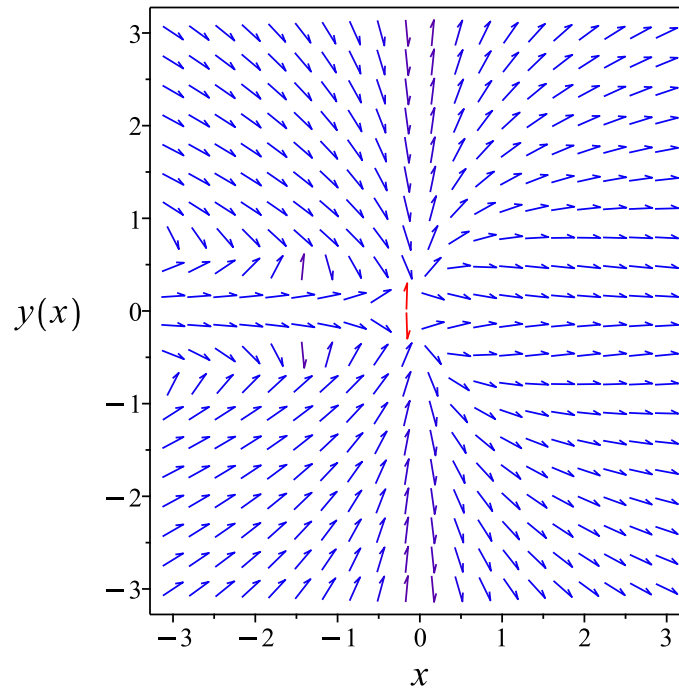


Figure 143: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{\text{LambertW}\left(\frac{6e^{-2c_1}}{x^3}\right)}{2} - c_1}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous G  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 38

```
dsolve((3*y(x)^3-x*y(x))-(x^2+6*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{3c_1}{2}} \sqrt{6}}{6x \sqrt{\frac{e^{3c_1}}{x^3 \text{LambertW}\left(\frac{6e^{3c_1}}{x^3}\right)}}}$$

✓ Solution by Mathematica

Time used: 3.943 (sec). Leaf size: 69

```
DSolve[(3*y[x]^3-x*y[x])-(x^2+6*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{x} \sqrt{W\left(\frac{6e^{3c_1}}{x^3}\right)}}{\sqrt{6}}$$

$$y(x) \rightarrow \frac{\sqrt{x} \sqrt{W\left(\frac{6e^{3c_1}}{x^3}\right)}}{\sqrt{6}}$$

$$y(x) \rightarrow 0$$

3.32 problem 26 (f)

3.32.1 Solving as exact ode 674

Internal problem ID [5293]

Internal file name [OUTPUT/4784_Friday_February_02_2024_05_11_52_AM_97656390/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (f).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$3y^2x^2 + 4(yx^3 - 3)y' = 0$$

3.32.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4y x^3 - 12) dy &= (-3y^2 x^2) dx \\ (3y^2 x^2) dx + (4y x^3 - 12) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3y^2 x^2 \\ N(x, y) &= 4y x^3 - 12\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (3y^2 x^2) \\ &= 6y x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (4y x^3 - 12) \\ &= 12y x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{4y x^3 - 12} ((6y x^2) - (12y x^2)) \\ &= -\frac{3y x^2}{2y x^3 - 6} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{3y^2 x^2} ((12y x^2) - (6y x^2)) \\ &= \frac{2}{y} \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int \frac{2}{y} \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2 \ln(y)} \\ &= y^2 \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} so not to confuse them with the original M and N .

$$\begin{aligned} \overline{M} &= \mu M \\ &= y^2 (3y^2 x^2) \\ &= 3y^4 x^2 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= y^2 (4y x^3 - 12) \\ &= 4y^2 (y x^3 - 3) \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ (3y^4x^2) + (4y^2(yx^3 - 3)) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int 3y^4x^2 dx \\ \phi &= x^3y^4 + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4y^3x^3 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4y^2(yx^3 - 3)$. Therefore equation (4) becomes

$$4y^2(yx^3 - 3) = 4y^3x^3 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -12y^2$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int (-12y^2) dy \\ f(y) &= -4y^3 + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3y^4 - 4y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3y^4 - 4y^3$$

Summary

The solution(s) found are the following

$$y^4x^3 - 4y^3 = c_1 \tag{1}$$

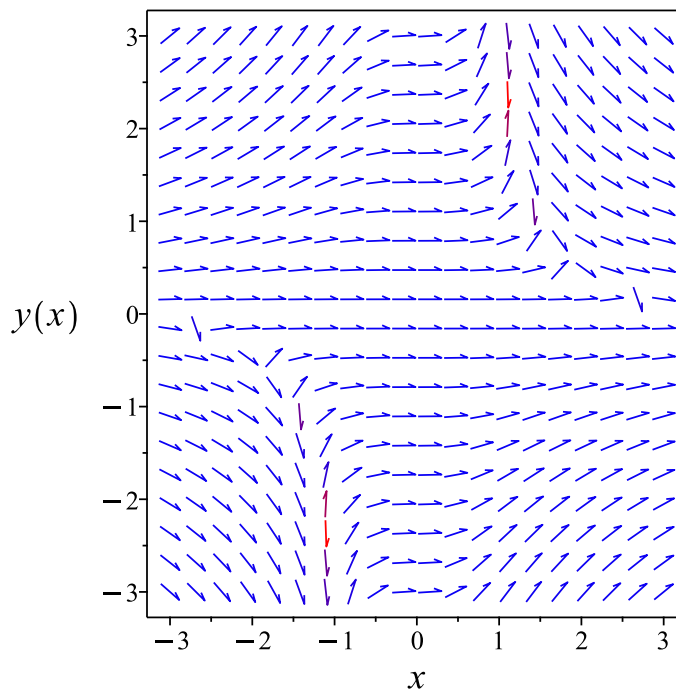


Figure 144: Slope field plot

Verification of solutions

$$y^4x^3 - 4y^3 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous G  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.297 (sec). Leaf size: 30

```
dsolve((3*x^2*y(x)^2)+4*(x^3*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\text{RootOf}(-Z^{12}c_1 + 4Z^3c_1 - x^3)^9 + 4}{x^3}$$

✓ Solution by Mathematica

Time used: 60.296 (sec). Leaf size: 1175

`DSolve[(3*x^2*y[x]^2)+4*(x^3*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$\begin{aligned}
 y(x) &\rightarrow \frac{1}{x^3} - \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1} + \frac{6}{x^6} + \frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 27c_1}{x^3}}{\sqrt{6}} \\
 &- \frac{1}{2} \sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 27c_1} + \frac{8}{x^6} - \frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 9c_1}{3^{2/3}x^3} - \frac{x^9 \sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1}}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} \\
 y(x) &\rightarrow \frac{1}{x^3} - \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1} + \frac{6}{x^6} + \frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 27c_1}{x^3}}{\sqrt{6}} \\
 &+ \frac{1}{2} \sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 27c_1} + \frac{8}{x^6} - \frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 9c_1}{3^{2/3}x^3} - \frac{x^9 \sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1}}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} \\
 y(x) &\rightarrow \frac{1}{x^3} + \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1} + \frac{6}{x^6} + \frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 27c_1}{x^3}}{\sqrt{6}} \\
 &- \frac{1}{2} \sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 27c_1} + \frac{8}{x^6} - \frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 9c_1}{3^{2/3}x^3} + \frac{x^9 \sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1}}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} \\
 y(x) &\rightarrow \frac{1}{x^3} + \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1} + \frac{6}{x^6} + \frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 27c_1}{x^3}}{\sqrt{6}} \\
 &+ \frac{1}{2} \sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 27c_1} + \frac{8}{680x^6} - \frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}} - 9c_1}{3^{2/3}x^3} + \frac{x^9 \sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}} - 9c_1}}{\sqrt[3]{\sqrt{3}\sqrt{c_1^2(27+c_1x^9)}}}
 \end{aligned}$$

3.33 problem 26 (g)

3.33.1 Solving as exact ode 681

Internal problem ID [5294]

Internal file name [OUTPUT/4785_Friday_February_02_2024_05_11_52_AM_36040598/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (g).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$y(x + y) - x^2y' = 0$$

3.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-x^2) dy &= (-y(x + y)) dx \\ (y(x + y)) dx + (-x^2) dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y(x + y) \\ N(x, y) &= -x^2 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(x + y)) \\ &= 2y + x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2) \\ &= -2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y(x + y)$ and $N = -x^2$ by this integrating factor the

ode becomes exact. The new M, N are

$$M = \frac{x+y}{xy}$$

$$N = -\frac{x}{y^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$

$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\left(-\frac{x}{y^2}\right) dy = \left(-\frac{x+y}{xy}\right) dx$$

$$\left(\frac{x+y}{xy}\right) dx + \left(-\frac{x}{y^2}\right) dy = 0 \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{x + y}{xy}$$
$$N(x, y) = -\frac{x}{y^2}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(\frac{x + y}{xy} \right)$$
$$= -\frac{1}{y^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \left(-\frac{x}{y^2} \right)$$
$$= -\frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int M dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{x + y}{xy} dx$$

$$\phi = \ln(x) + \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) - c_1} \quad (1)$$

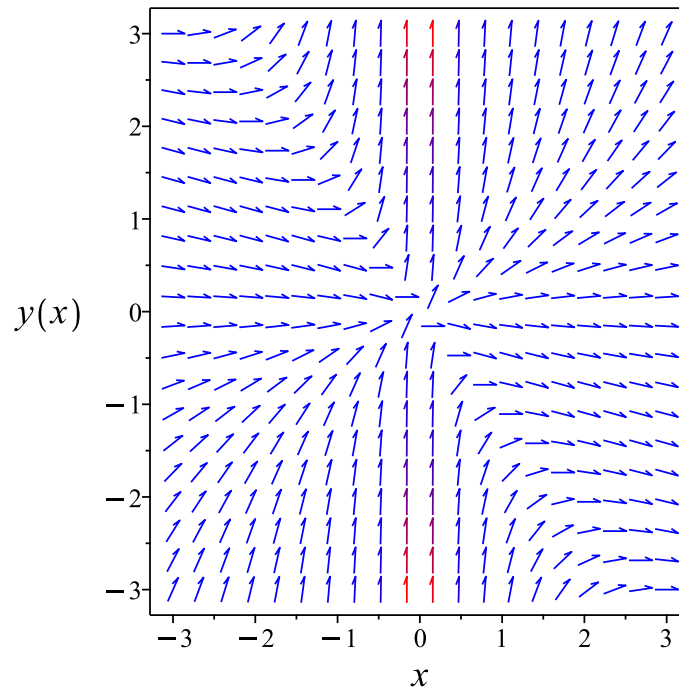


Figure 145: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) - c_1}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)*(x+y(x))-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{-\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 21

```
DSolve[y[x]*(x+y[x])-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x}{-\log(x) + c_1}$$
$$y(x) \rightarrow 0$$

3.34 problem 26 (h)

3.34.1 Solving as exact ode 688

Internal problem ID [5295]

Internal file name [OUTPUT/4786_Friday_February_02_2024_05_11_53_AM_24430299/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (h).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `class B`]]
```

$$2y + 3xy^2 + (x + 2yx^2) y' = 0$$

3.34.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y x^2 + x) dy &= (-3y^2 x - 2y) dx \\ (3y^2 x + 2y) dx + (2y x^2 + x) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3y^2 x + 2y \\ N(x, y) &= 2y x^2 + x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (3y^2 x + 2y) \\ &= 6xy + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (2y x^2 + x) \\ &= 4xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{2y x^2 + x} ((6xy + 2) - (4xy + 1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int \frac{1}{x} dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= x(3y^2 x + 2y) \\ &= 3y^2 x^2 + 2xy \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= x(2y x^2 + x) \\ &= 2y x^3 + x^2 \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (3y^2 x^2 + 2xy) + (2y x^3 + x^2) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int 3y^2x^2 + 2xy dx$$

$$\phi = yx^2(xy + 1) + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2(xy + 1) + yx^3 + f'(y) \quad (4)$$

$$= 2yx^3 + x^2 + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx^3 + x^2$. Therefore equation (4) becomes

$$2yx^3 + x^2 = 2yx^3 + x^2 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = yx^2(xy + 1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = yx^2(xy + 1)$$

Summary

The solution(s) found are the following

$$yx^2(xy + 1) = c_1 \tag{1}$$

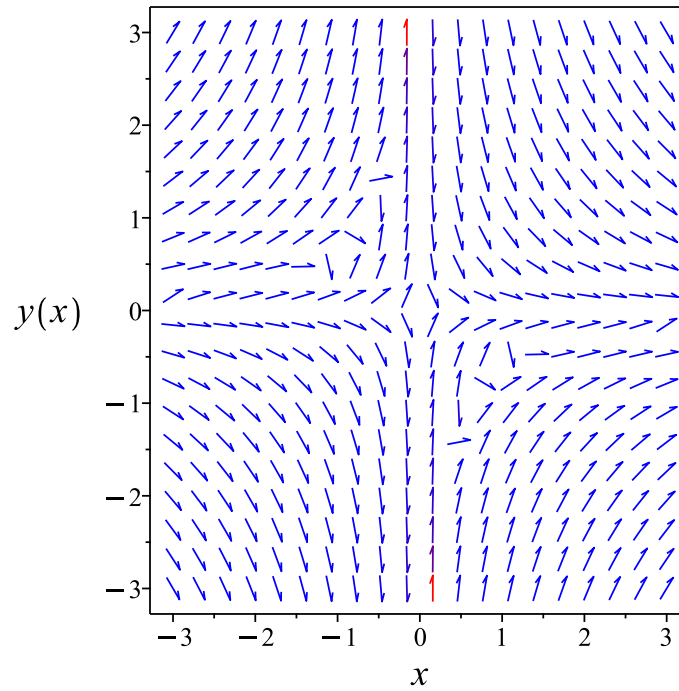


Figure 146: Slope field plot

Verification of solutions

$$yx^2(xy + 1) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying homogeneous G  
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve((2*y(x)+3*x*y(x)^2)+(x+2*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x + \sqrt{x(4c_1 + x)}}{2x^2}$$
$$y(x) = \frac{-x - \sqrt{x(4c_1 + x)}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.526 (sec). Leaf size: 69

```
DSolve[(2*y[x]+3*x*y[x]^2)+(x+2*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$
$$y(x) \rightarrow \frac{-x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$

3.35 problem 26 (i)

3.35.1 Solving as exact ode 694

Internal problem ID [5296]

Internal file name [OUTPUT/4787_Friday_February_02_2024_05_11_53_AM_15451902/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 26 (i).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$y(y^2 - 2x^2) + x(2y^2 - x^2) y' = 0$$

3.35.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (x(-x^2 + 2y^2)) dy &= (-y(-2x^2 + y^2)) dx \\ (y(-2x^2 + y^2)) dx &+ (x(-x^2 + 2y^2)) dy = 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y(-2x^2 + y^2) \\ N(x, y) &= x(-x^2 + 2y^2) \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (y(-2x^2 + y^2)) \\ &= -2x^2 + 3y^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x(-x^2 + 2y^2)) \\ &= -3x^2 + 2y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{(x^2 - 2y^2)x} ((-2x^2 + 3y^2) - (-3x^2 + 2y^2)) \\ &= \frac{-x^2 - y^2}{x(x^2 - 2y^2)} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y(-2x^2 + y^2)} ((-3x^2 + 2y^2) - (-2x^2 + 3y^2)) \\ &= \frac{x^2 + y^2}{2yx^2 - y^3} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(-3x^2 + 2y^2) - (-2x^2 + 3y^2)}{x(y(-2x^2 + y^2)) - y(x(-x^2 + 2y^2))} \\ &= \frac{1}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int (\frac{1}{t}) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(t)} \\ &= t\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = xy$$

Multiplying M and N by this integrating factor gives new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= xy(y(-2x^2 + y^2)) \\ &= y^2(-2x^2 + y^2)x\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= xy(x(-x^2 + 2y^2)) \\ &= -y(x^2 - 2y^2)x^2\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (y^2(-2x^2 + y^2)x) + (-y(x^2 - 2y^2)x^2) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int y^2(-2x^2 + y^2)x dx \\ \phi &= -\frac{y^2(2x^2 - y^2)^2}{8} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{y(2x^2 - y^2)^2}{4} + \frac{y^3(2x^2 - y^2)}{2} + f'(y) \\ &= -yx^4 + 2y^3x^2 - \frac{3}{4}y^5 + f'(y)\end{aligned}\quad (4)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = -y(x^2 - 2y^2)x^2$. Therefore equation (4) becomes

$$-y(x^2 - 2y^2)x^2 = -yx^4 + 2y^3x^2 - \frac{3}{4}y^5 + f'(y)\quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3y^5}{4}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{3y^5}{4}\right) dy \\ f(y) &= \frac{y^6}{8} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8}$$

Summary

The solution(s) found are the following

$$-\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8} = c_1\quad (1)$$

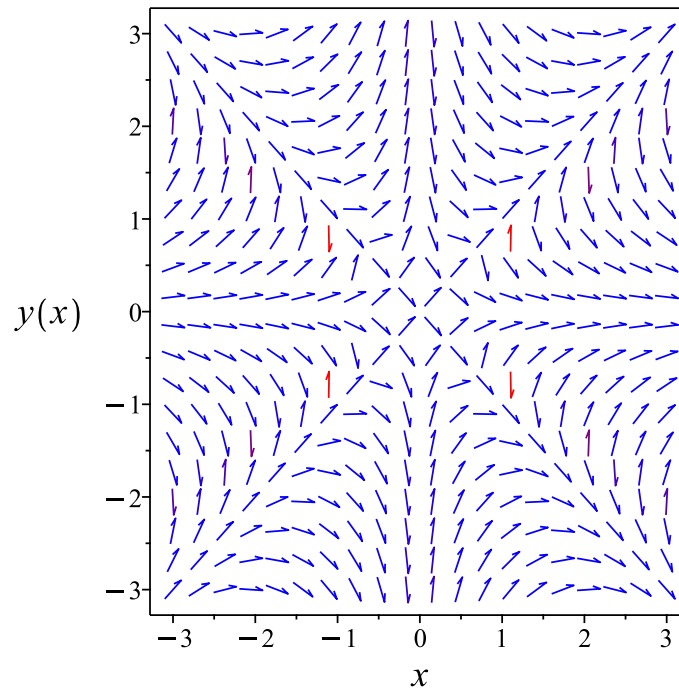


Figure 147: Slope field plot

Verification of solutions

$$-\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`

```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 71

`dsolve(y(x)*(y(x)^2-2*x^2)+x*(2*y(x)^2-x^2)*diff(y(x),x)=0,y(x), singsol=all)`

$$y(x) = \frac{\sqrt{\frac{2c_1x^3 - 2\sqrt{c_1^2x^6 + 4}}{c_1x^3}} x}{2}$$

$$y(x) = \frac{\sqrt{2} \sqrt{\frac{c_1x^3 + \sqrt{c_1^2x^6 + 4}}{c_1x^3}} x}{2}$$

✓ Solution by Mathematica

Time used: 11.861 (sec). Leaf size: 2777

`DSolve[y[x]*(y[x]^2-2*x^2)+x*(2*y[x]^2-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$y(x) \rightarrow -\frac{\sqrt{x^2 - \frac{\sqrt{x^6 - 4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{x^2 - \frac{\sqrt{x^6 - 4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow -\frac{\sqrt{\frac{x^3 + \sqrt{x^6 - 4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{\frac{x^3 + \sqrt{x^6 - 4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow -\frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow -\frac{\sqrt{\frac{\sqrt{x^6 + x^3}}{x}}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{\frac{\sqrt{x^6 + x^3}}{x}}}{\sqrt{2}}$$

3.36 problem 27

3.36.1 Solving as exact ode	701
3.36.2 Maple step by step solution	705

Internal problem ID [5297]

Internal file name [OUTPUT/4788_Friday_February_02_2024_05_11_53_AM_53657820/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplementary problems. Page 33

Problem number: 27.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

`[_separable]`

$$-y + xy' = 0$$

3.36.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right) dy &= \left(\frac{1}{x}\right) dx \\ \left(-\frac{1}{x}\right) dx + \left(\frac{1}{y}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x} \\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-\frac{1}{x}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x} dx \\ \phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{1}{y} \right) dy \\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = x e^{c_1}$$

Summary

The solution(s) found are the following

$$y = x e^{c_1} \tag{1}$$

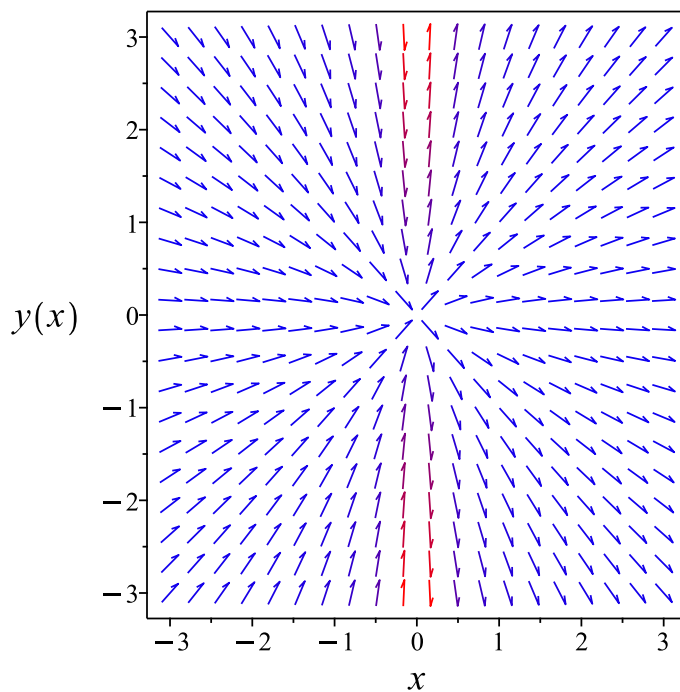


Figure 148: Slope field plot

Verification of solutions

$$y = x e^{c_1}$$

Verified OK.

3.36.2 Maple step by step solution

Let's solve

$$-y + xy' = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = \frac{1}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$$

- Evaluate integral

$$\ln(y) = \ln(x) + c_1$$

- Solve for y

$$y = x e^{c_1}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(-y(x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[-y[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 x$$

$$y(x) \rightarrow 0$$

4 Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

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4.1 problem 19 (a)

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4.1.2	Solving as homogeneousTypeD2 ode	710
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4.1.5	Maple step by step solution	720

Internal problem ID [5298]

Internal file name [OUTPUT/4789_Friday_February_02_2024_05_11_54_AM_71699313/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"linear"**, **"homogeneousTypeD2"**, **"exactWithIntegrationFactor"**, **"first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$y + y' = 2 + 2x$$

4.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$

$$q(x) = 2 + 2x$$

Hence the ode is

$$y + y' = 2 + 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dx} \\ &= e^x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(2 + 2x) \\ \frac{d}{dx}(y e^x) &= (e^x)(2 + 2x) \\ d(y e^x) &= (2 e^x(x + 1)) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y e^x &= \int 2 e^x(x + 1) dx \\ y e^x &= 2x e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = 2 e^{-x} e^x x + c_1 e^{-x}$$

which simplifies to

$$y = 2x + c_1 e^{-x}$$

Summary

The solution(s) found are the following

$$y = 2x + c_1 e^{-x} \tag{1}$$

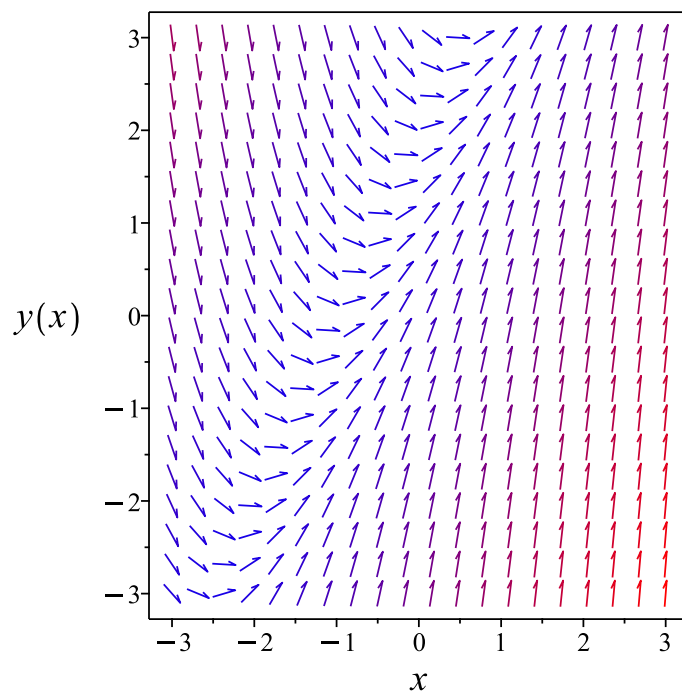


Figure 149: Slope field plot

Verification of solutions

$$y = 2x + c_1 e^{-x}$$

Verified OK.

4.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u(x)x + u'(x)x + u(x) = 2 + 2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{(x+1)(-u+2)}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(u) = -u + 2$. Integrating both sides gives

$$\frac{1}{-u+2} du = \frac{x+1}{x} dx$$

$$\int \frac{1}{-u+2} du = \int \frac{x+1}{x} dx$$

$$-\ln(u-2) = x + \ln(x) + c_2$$

Raising both side to exponential gives

$$\frac{1}{u-2} = e^{x+\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{u-2} = c_3 e^{x+\ln(x)}$$

Which simplifies to

$$u(x) = \frac{(2c_3 e^x x e^{c_2} + 1) e^{-x} e^{-c_2}}{c_3 x}$$

Therefore the solution y is

$$y = ux$$

$$= \frac{(2c_3 e^x x e^{c_2} + 1) e^{-x} e^{-c_2}}{c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{(2c_3 e^x x e^{c_2} + 1) e^{-x} e^{-c_2}}{c_3} \tag{1}$$

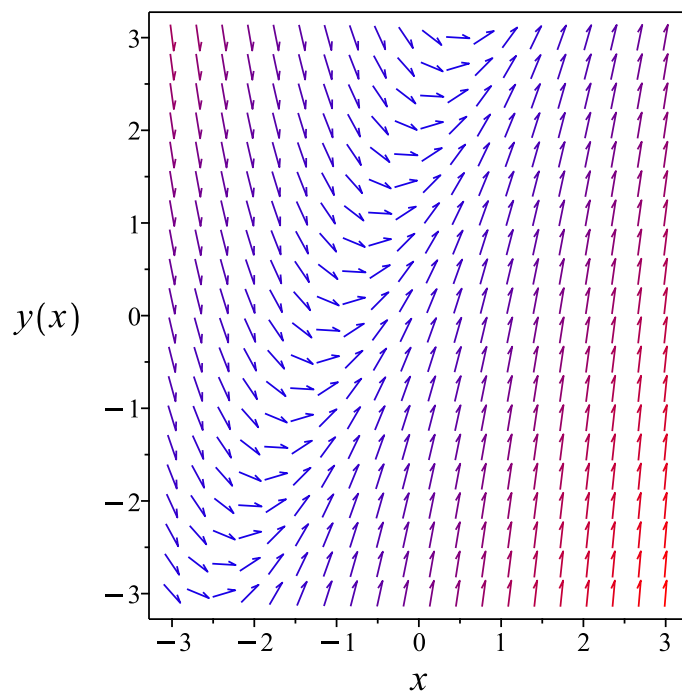


Figure 150: Slope field plot

Verification of solutions

$$y = \frac{(2c_3 e^x x e^{c_2} + 1) e^{-x} e^{-c_2}}{c_3}$$

Verified OK.

4.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + 2 + 2x$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{-x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = y e^x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + 2 + 2x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y e^x \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2 e^x (x + 1) \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2 e^R (R + 1)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = 2e^R R + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = 2x e^x + c_1$$

Which simplifies to

$$e^x y = 2x e^x + c_1$$

Which gives

$$y = (2x e^x + c_1) e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 2 + 2x$	$R = x$ $S = y e^x$	$\frac{dS}{dR} = 2e^R(R + 1)$

Summary

The solution(s) found are the following

$$y = (2x e^x + c_1) e^{-x} \quad (1)$$

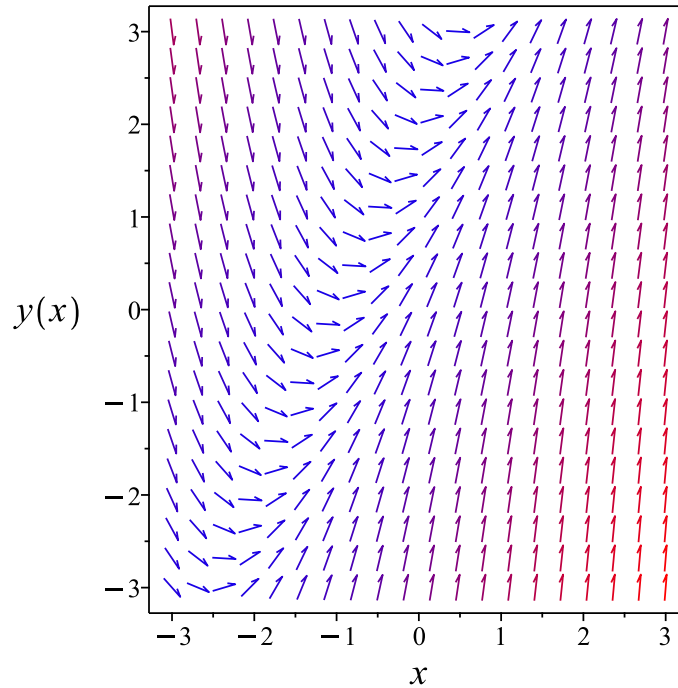


Figure 151: Slope field plot

Verification of solutions

$$y = (2x e^x + c_1) e^{-x}$$

Verified OK.

4.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}dy &= (-y + 2 + 2x) dx \\ (y - 2 - 2x) dx + dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - 2 - 2x \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 2 - 2x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int 1 dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^x \\ &= e^x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= e^x(y - 2 - 2x) \\ &= (y - 2 - 2x) e^x \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= e^x(1) \\ &= e^x \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ ((y - 2 - 2x) e^x) + (e^x) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int (y - 2 - 2x) e^x dx \\ \phi &= (-2x + y) e^x + f(y)\end{aligned}\tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^x + f'(y)\tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^x$. Therefore equation (4) becomes

$$e^x = e^x + f'(y)\tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-2x + y) e^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-2x + y) e^x$$

The solution becomes

$$y = (2x e^x + c_1) e^{-x}$$

Summary

The solution(s) found are the following

$$y = (2x e^x + c_1) e^{-x}\tag{1}$$

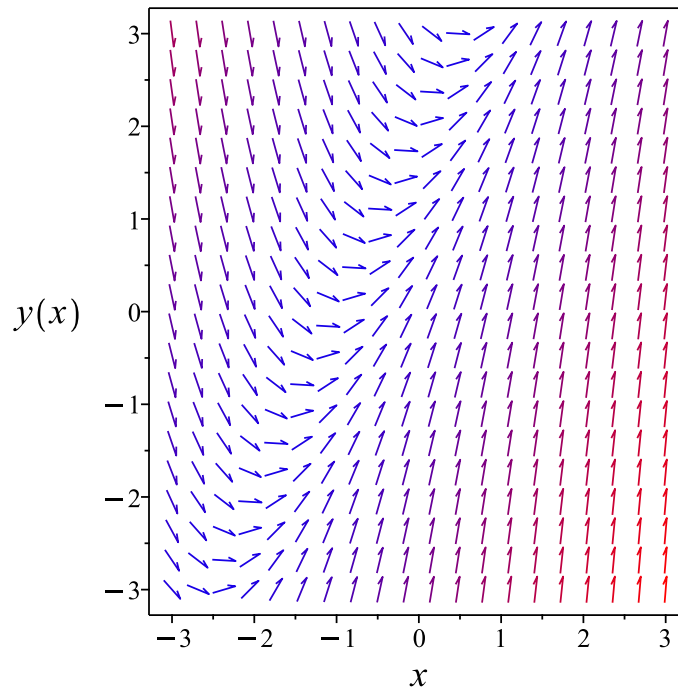


Figure 152: Slope field plot

Verification of solutions

$$y = (2x e^x + c_1) e^{-x}$$

Verified OK.

4.1.5 Maple step by step solution

Let's solve

$$y + y' = 2 + 2x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = -y + 2 + 2x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y + y' = 2 + 2x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) (y + y') = \mu(x) (2 + 2x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) (y + y') = \mu'(x) y + \mu(x) y'$$
- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x)$$
- Solve to find the integrating factor

$$\mu(x) = e^x$$
- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int \mu(x) (2 + 2x) dx + c_1$$
- Evaluate the integral on the lhs

$$\mu(x) y = \int \mu(x) (2 + 2x) dx + c_1$$
- Solve for y

$$y = \frac{\int \mu(x)(2+2x)dx+c_1}{\mu(x)}$$
- Substitute $\mu(x) = e^x$

$$y = \frac{\int (2+2x)e^x dx+c_1}{e^x}$$
- Evaluate the integrals on the rhs

$$y = \frac{2xe^x+c_1}{e^x}$$
- Simplify

$$y = 2x + c_1e^{-x}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)=2+2*x,y(x), singsol=all)
```

$$y(x) = 2x + c_1e^{-x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[y'[x]+y[x]==2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 2x + c_1e^{-x}$$

4.2 problem 19 (c)

4.2.1	Solving as separable ode	723
4.2.2	Solving as linear ode	725
4.2.3	Solving as homogeneousTypeD2 ode	726
4.2.4	Solving as first order ode lie symmetry lookup ode	728
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4.2.6	Maple step by step solution	736

Internal problem ID [5299]

Internal file name [OUTPUT/4790_Friday_February_02_2024_05_11_55_AM_66438453/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (c).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$y' - y - xy = 0$$

4.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= (x + 1)y\end{aligned}$$

Where $f(x) = x + 1$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= x + 1 dx \\ \int \frac{1}{y} dy &= \int x + 1 dx \\ \ln(y) &= \frac{1}{2}x^2 + x + c_1 \\ y &= e^{\frac{1}{2}x^2 + x + c_1} \\ &= c_1 e^{\frac{1}{2}x^2 + x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{1}{2}x^2 + x} \quad (1)$$

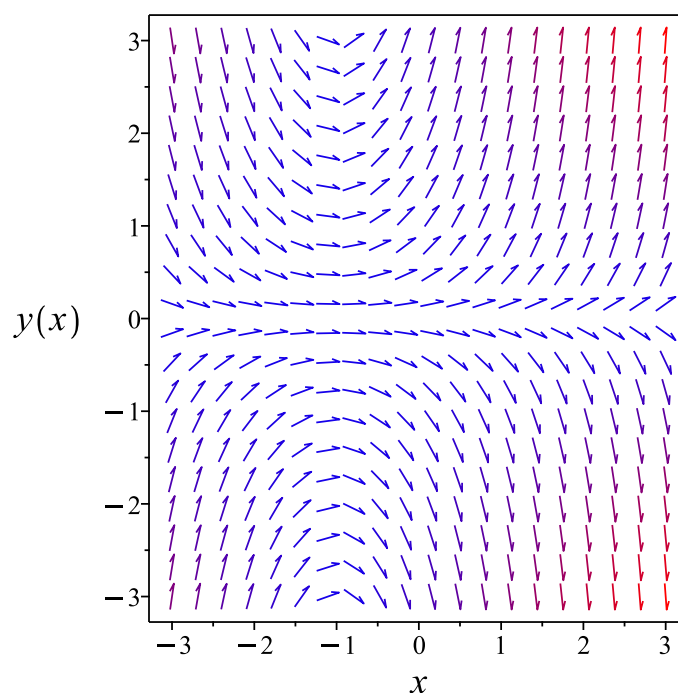


Figure 153: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{1}{2}x^2 + x}$$

Verified OK.

4.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1 - x$$

$$q(x) = 0$$

Hence the ode is

$$y' + (-1 - x)y = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int(-1-x)dx} \\ &= e^{-x-\frac{1}{2}x^2}\end{aligned}$$

Which simplifies to

$$\mu = e^{-\frac{x(x+2)}{2}}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-\frac{x(x+2)}{2}}y\right) &= 0\end{aligned}$$

Integrating gives

$$e^{-\frac{x(x+2)}{2}}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x(x+2)}{2}}$ results in

$$y = c_1 e^{\frac{x(x+2)}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x(x+2)}{2}} \tag{1}$$

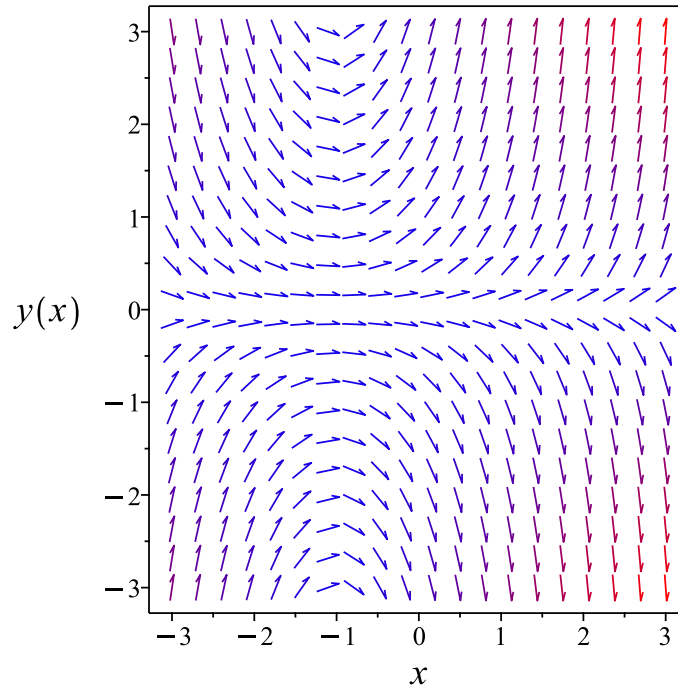


Figure 154: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x(x+2)}{2}}$$

Verified OK.

4.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)x$ on the above ode results in new ode in $u(x)$

$$u'(x)x + u(x) - u(x)x - x^2u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 + x - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{x^2+x-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{x^2 + x - 1}{x} dx \\ \int \frac{1}{u} du &= \int \frac{x^2 + x - 1}{x} dx \\ \ln(u) &= \frac{x^2}{2} + x - \ln(x) + c_2 \\ u &= e^{\frac{x^2}{2} + x - \ln(x) + c_2} \\ &= c_2 e^{\frac{x^2}{2} + x - \ln(x)}\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= ux \\ &= xc_2 e^{\frac{x^2}{2} + x - \ln(x)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = xc_2 e^{\frac{x^2}{2} + x - \ln(x)} \tag{1}$$

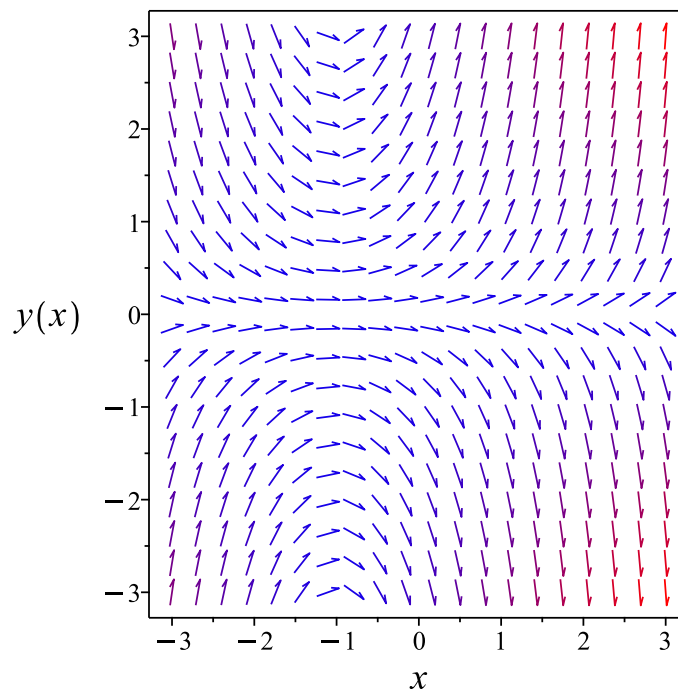


Figure 155: Slope field plot

Verification of solutions

$$y = xc_2 e^{\frac{x^2}{2} + x - \ln(x)}$$

Verified OK.

4.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= xy + y \\y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{\frac{1}{2}x^2+x}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right)S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{2}x^2+x}} dy \end{aligned}$$

Which results in

$$S = e^{-x-\frac{1}{2}x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = xy + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -(x+1)e^{-\frac{x(x+2)}{2}} y \\ S_y &= e^{-\frac{x(x+2)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-\frac{x(x+2)}{2}} y = c_1$$

Which simplifies to

$$e^{-\frac{x(x+2)}{2}} y = c_1$$

Which gives

$$y = c_1 e^{\frac{x(x+2)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy + y$	$R = x$ $S = e^{-\frac{x(x+2)}{2}} y$	$\frac{dS}{dR} = 0$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{x(x+2)}{2}} \tag{1}$$

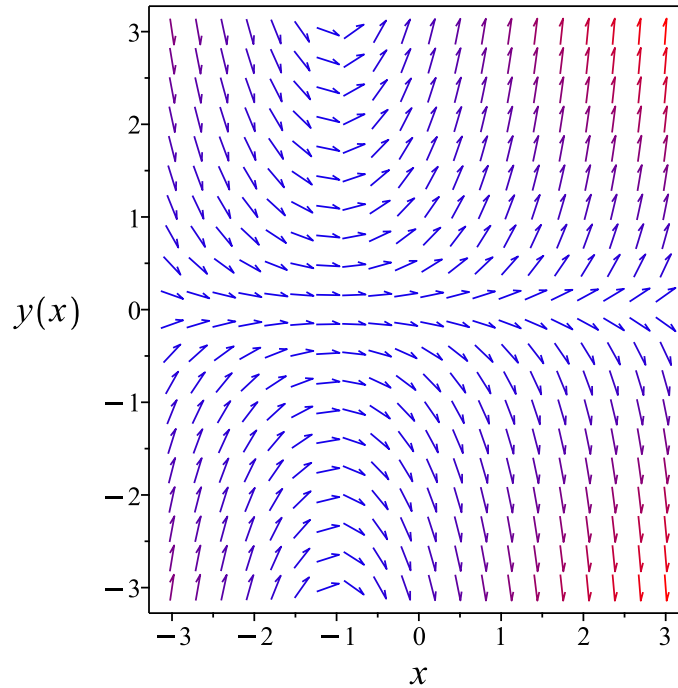


Figure 156: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x(x+2)}{2}}$$

Verified OK.

4.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right) dy &= (x + 1) dx \\ (-1 - x) dx + \left(\frac{1}{y}\right) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -1 - x \\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 - x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{y} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -1 - x dx \\ \phi &= -x - \frac{1}{2}x^2 + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{1}{y} \right) dy \\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{x^2}{2} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \frac{x^2}{2} + \ln(y)$$

The solution becomes

$$y = e^{\frac{1}{2}x^2 + x + c_1}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{1}{2}x^2 + x + c_1} \tag{1}$$

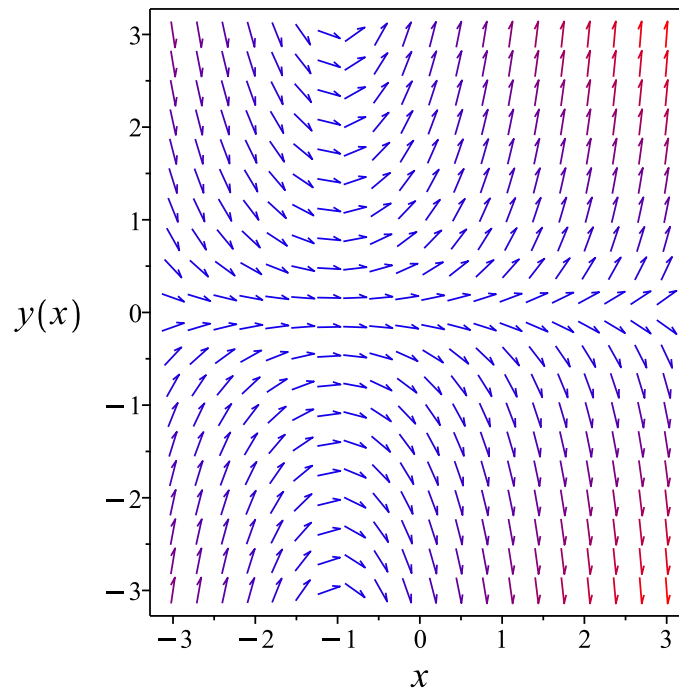


Figure 157: Slope field plot

Verification of solutions

$$y = e^{\frac{1}{2}x^2+x+c_1}$$

Verified OK.

4.2.6 Maple step by step solution

Let's solve

$$y' - y - xy = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = x + 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int (x + 1) dx + c_1$$

- Evaluate integral

$$\ln(y) = \frac{1}{2}x^2 + x + c_1$$

- Solve for y

$$y = e^{\frac{1}{2}x^2+x+c_1}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)-y(x)=x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x(x+2)}{2}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[y'[x]-y[x]==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^{\frac{1}{2}x(x+2)}$$

$$y(x) \rightarrow 0$$

4.3 problem 19 (d)

4.3.1	Solving as linear ode	738
4.3.2	Solving as first order ode lie symmetry lookup ode	740
4.3.3	Solving as exact ode	744
4.3.4	Maple step by step solution	749

Internal problem ID [5300]

Internal file name [OUTPUT/4791_Friday_February_02_2024_05_11_55_AM_71430434/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_linear]

$$-3y + xy' = (-2 + x)e^x$$

4.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{e^x(-2 + x)}{x}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{e^x(-2 + x)}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{e^x(-2+x)}{x} \right) \\ \frac{d}{dx} \left(\frac{y}{x^3} \right) &= \left(\frac{1}{x^3} \right) \left(\frac{e^x(-2+x)}{x} \right) \\ d \left(\frac{y}{x^3} \right) &= \left(\frac{e^x(-2+x)}{x^4} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^3} &= \int \frac{e^x(-2+x)}{x^4} dx \\ \frac{y}{x^3} &= -\frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\text{expIntegral}_1(-x)}{6} + \frac{2e^x}{3x^3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3 \left(-\frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\text{expIntegral}_1(-x)}{6} + \frac{2e^x}{3x^3} \right) + c_1 x^3$$

which simplifies to

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} + \frac{(-x^2 - x + 4) e^x}{6} + c_1 x^3$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} + \frac{(-x^2 - x + 4) e^x}{6} + c_1 x^3 \quad (1)$$

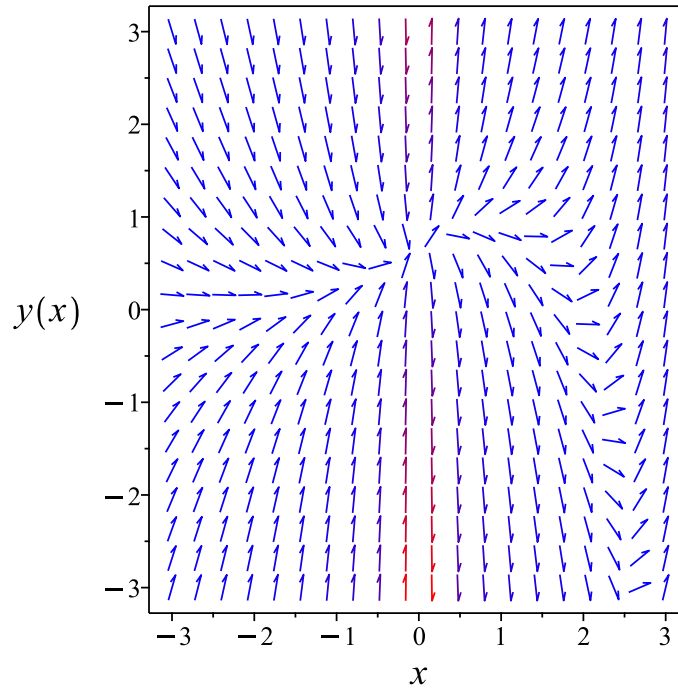


Figure 158: Slope field plot

Verification of solutions

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} + \frac{(-x^2 - x + 4) e^x}{6} + c_1 x^3$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x e^x - 2 e^x + 3y}{x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 90: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x^3\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x e^x - 2 e^x + 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{x^4} \\ S_y &= \frac{1}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^x(-2 + x)}{x^4} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^R(-2 + R)}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{e^R}{6R^2} - \frac{e^R}{6R} - \frac{\text{expIntegral}_1(-R)}{6} + \frac{2e^R}{3R^3} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^3} = -\frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\text{expIntegral}_1(-x)}{6} + \frac{2e^x}{3x^3} + c_1$$

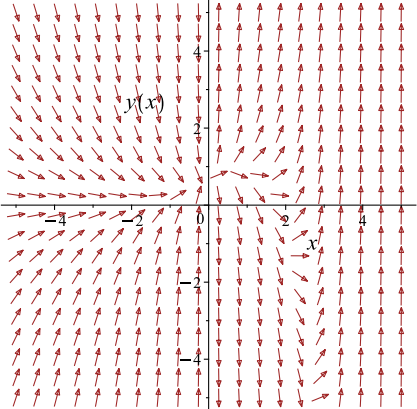
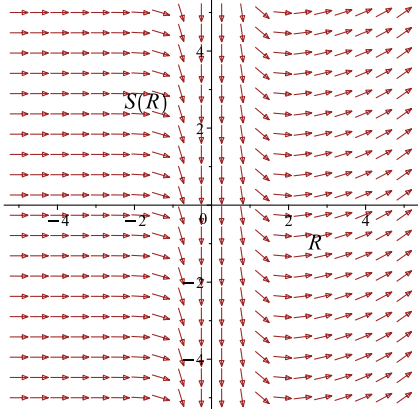
Which simplifies to

$$\frac{\text{expIntegral}_1(-x) x^3 + (x^2 + x - 4) e^x - 6c_1 x^3 + 6y}{6x^3} = 0$$

Which gives

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x e^x - 2 e^x + 3y}{x}$ 	$R = x$ $S = \frac{y}{x^3}$	$\frac{dS}{dR} = \frac{e^R(-2+R)}{R^4}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3 \quad (1)$$

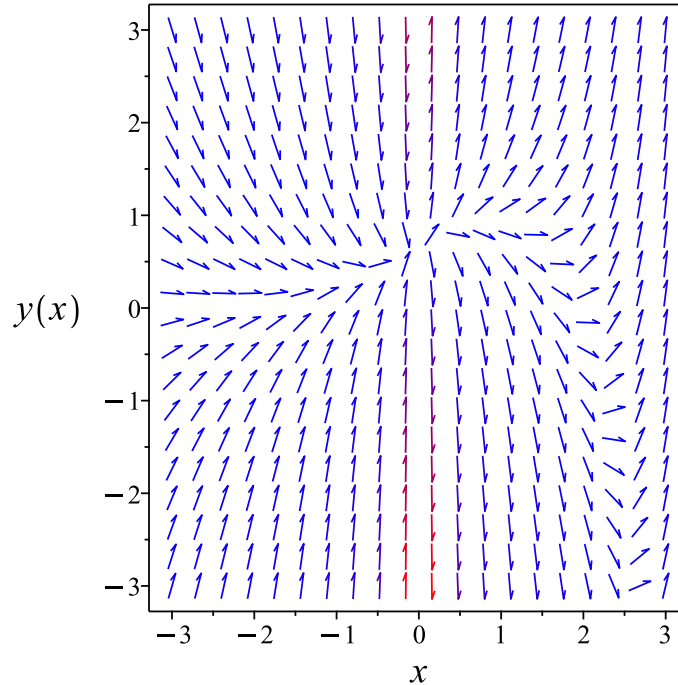


Figure 159: Slope field plot

Verification of solutions

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (A)$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (x) dy &= (3y + (-2 + x) e^x) dx \\ (-3y - (-2 + x) e^x) dx + (x) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -3y - (-2 + x) e^x \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-3y - (-2 + x) e^x) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((-3) - (1)) \\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{4}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4 \ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x^4} (-3y - (-2 + x) e^x) \\ &= \frac{(2 - x) e^x - 3y}{x^4}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x^4}(x) \\ &= \frac{1}{x^3}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{(2-x)e^x - 3y}{x^4} \right) + \left(\frac{1}{x^3} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{(2-x)e^x - 3y}{x^4} dx \\ \phi &= \frac{\text{expIntegral}_1(-x)x^3 + (x^2 + x - 4)e^x + 6y}{6x^3} + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^3} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^3}$. Therefore equation (4) becomes

$$\frac{1}{x^3} = \frac{1}{x^3} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\text{expIntegral}_1(-x)x^3 + (x^2 + x - 4)e^x + 6y}{6x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\text{expIntegral}_1(-x)x^3 + (x^2 + x - 4)e^x + 6y}{6x^3}$$

The solution becomes

$$y = -\frac{\text{expIntegral}_1(-x)x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{expIntegral}_1(-x)x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3 \quad (1)$$

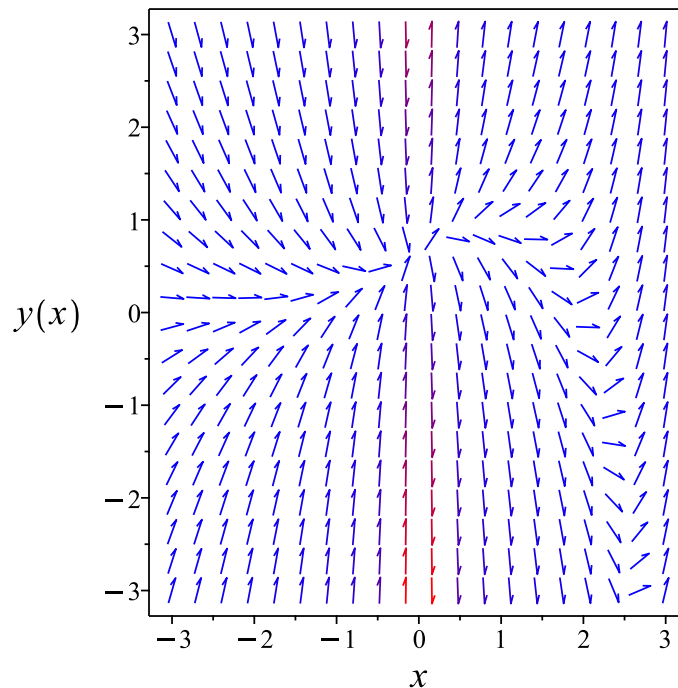


Figure 160: Slope field plot

Verification of solutions

$$y = -\frac{\text{expIntegral}_1(-x) x^3}{6} - \frac{x^2 e^x}{6} - \frac{x e^x}{6} + \frac{2 e^x}{3} + c_1 x^3$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$$-3y + xy' = (-2 + x) e^x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{3y}{x} + \frac{e^x(-2+x)}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{3y}{x} = \frac{e^x(-2+x)}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{3y}{x} \right) = \frac{\mu(x)e^x(-2+x)}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) \left(y' - \frac{3y}{x} \right) = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{3\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x^3}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int \frac{\mu(x)e^x(-2+x)}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int \frac{\mu(x)e^x(-2+x)}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)e^x(-2+x)}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^3}$

$$y = x^3 \left(\int \frac{e^x(-2+x)}{x^4} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x^3 \left(-\frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\text{Ei}_1(-x)}{6} + \frac{2e^x}{3x^3} + c_1 \right)$$

- Simplify

$$y = -\frac{\text{Ei}_1(-x)x^3}{6} + \frac{(-x^2-x+4)e^x}{6} + c_1x^3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve((-2*y(x)-(x-2)*exp(x))+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^3 \text{expIntegral}_1(-x)}{6} + \frac{(-x^2 - x + 4)e^x}{6} + c_1x^3$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 33

```
DSolve[(-2*y[x]-(x-2)*Exp[x])+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6}x^3 \left(\text{ExpIntegralEi}(x) - \frac{e^x(x^2 + x - 4)}{x^3} + 6c_1 \right)$$

4.4 problem 19 (e)

4.4.1	Solving as linear ode	751
4.4.2	Solving as first order ode lie symmetry lookup ode	753
4.4.3	Solving as exact ode	757
4.4.4	Maple step by step solution	761

Internal problem ID [5301]

Internal file name [OUTPUT/4792_Friday_February_02_2024_05_11_56_AM_13116245/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (e).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**linear**", "**exactWithIntegrationFactor**", "**first_order_ode_lie_symmetry_lookup**"

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$i' - 6i = 10 \sin(2t)$$

4.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$i' + p(t)i = q(t)$$

Where here

$$p(t) = -6$$

$$q(t) = 10 \sin(2t)$$

Hence the ode is

$$i' - 6i = 10 \sin(2t)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int(-6)dt} \\ &= e^{-6t}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu i) &= (\mu) (10 \sin(2t)) \\ \frac{d}{dt}(e^{-6t}i) &= (e^{-6t}) (10 \sin(2t)) \\ d(e^{-6t}i) &= (10 \sin(2t) e^{-6t}) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-6t}i &= \int 10 \sin(2t) e^{-6t} dt \\ e^{-6t}i &= -\frac{\cos(2t) e^{-6t}}{2} - \frac{3 \sin(2t) e^{-6t}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-6t}$ results in

$$i = e^{6t} \left(-\frac{\cos(2t) e^{-6t}}{2} - \frac{3 \sin(2t) e^{-6t}}{2} \right) + c_1 e^{6t}$$

which simplifies to

$$i = c_1 e^{6t} - \frac{3 \sin(2t)}{2} - \frac{\cos(2t)}{2}$$

Summary

The solution(s) found are the following

$$i = c_1 e^{6t} - \frac{3 \sin(2t)}{2} - \frac{\cos(2t)}{2} \tag{1}$$

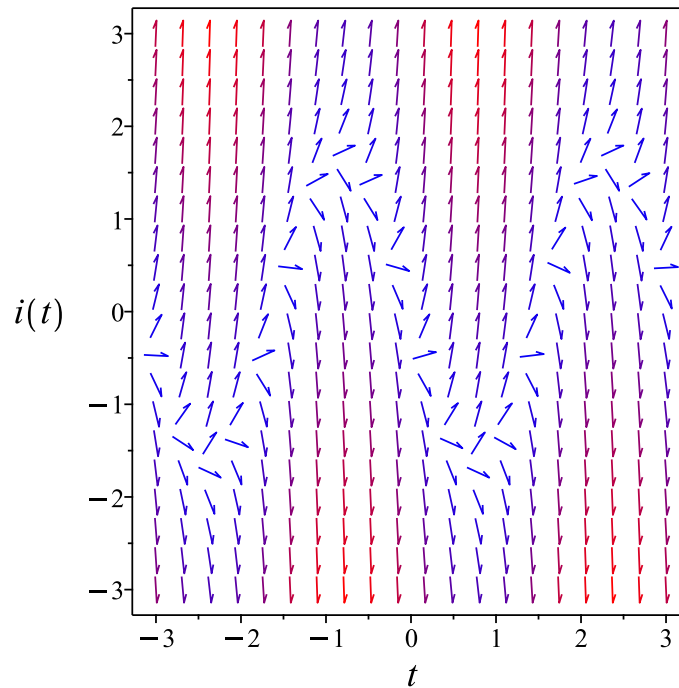


Figure 161: Slope field plot

Verification of solutions

$$i = c_1 e^{6t} - \frac{3 \sin(2t)}{2} - \frac{\cos(2t)}{2}$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} i' &= 6i + 10 \sin(2t) \\ i' &= \omega(t, i) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_i - \xi_t) - \omega^2 \xi_i - \omega_t \xi - \omega_i \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, i) &= 0 \\ \eta(t, i) &= e^{6t}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(t, i) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{di}{\eta} = dS\tag{1}$$

The above comes from the requirements that $(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial i}) S(t, i) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{6t}} dy \end{aligned}$$

Which results in

$$S = e^{-6t}i$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, i)S_i}{R_t + \omega(t, i)R_i} \quad (2)$$

Where in the above R_t, R_i, S_t, S_i are all partial derivatives and $\omega(t, i)$ is the right hand side of the original ode given by

$$\omega(t, i) = 6i + 10 \sin(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_i &= 0 \\ S_t &= -6e^{-6t}i \\ S_i &= e^{-6t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 10 \sin(2t) e^{-6t} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for t, i in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 10 \sin(2R) e^{-6R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 - \frac{e^{-6R}(\cos(2R) + 3\sin(2R))}{2} \quad (4)$$

To complete the solution, we just need to transform (4) back to t, i coordinates. This results in

$$e^{-6t}i = -\frac{(\cos(2t) + 3\sin(2t))e^{-6t}}{2} + c_1$$

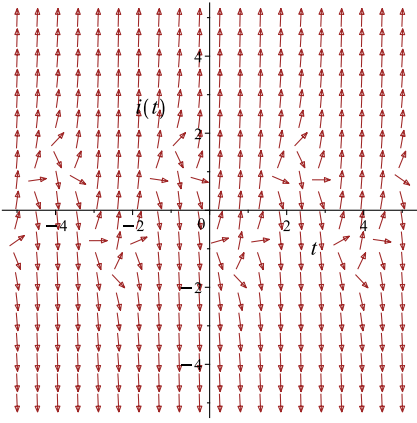
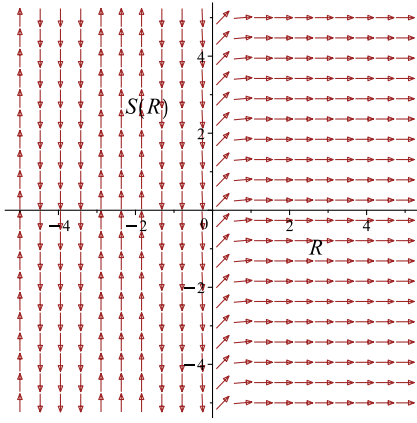
Which simplifies to

$$e^{-6t}i = -\frac{(\cos(2t) + 3\sin(2t))e^{-6t}}{2} + c_1$$

Which gives

$$i = -\frac{e^{6t}(3\sin(2t)e^{-6t} + \cos(2t)e^{-6t} - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, i coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{di}{dt} = 6i + 10\sin(2t)$ 	$R = t$ $S = e^{-6t}i$	$\frac{dS}{dR} = 10\sin(2R)e^{-6R}$ 

Summary

The solution(s) found are the following

$$i = -\frac{e^{6t}(3\sin(2t)e^{-6t} + \cos(2t)e^{-6t} - 2c_1)}{2} \quad (1)$$

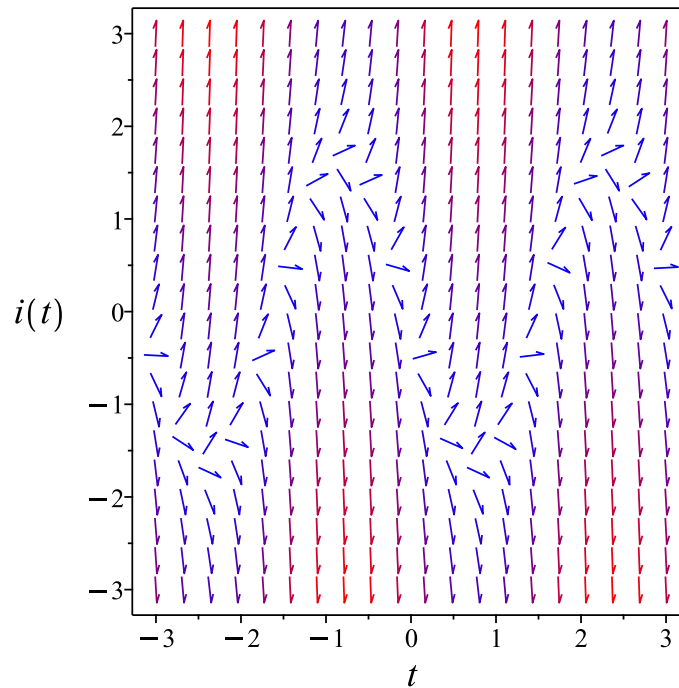


Figure 162: Slope field plot

Verification of solutions

$$i = -\frac{e^{6t}(3 \sin(2t) e^{-6t} + \cos(2t) e^{-6t} - 2c_1)}{2}$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, i) dt + N(t, i) di = 0 \tag{1A}$$

Therefore

$$\begin{aligned}di &= (6i + 10 \sin(2t)) dt \\ (-6i - 10 \sin(2t)) dt + di &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, i) &= -6i - 10 \sin(2t) \\ N(t, i) &= 1\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial i} &= \frac{\partial}{\partial i}(-6i - 10 \sin(2t)) \\ &= -6\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial i} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial i} - \frac{\partial N}{\partial t} \right) \\ &= 1((-6) - (0)) \\ &= -6 \end{aligned}$$

Since A does not depend on i , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dt} \\ &= e^{\int -6 dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-6t} \\ &= e^{-6t} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= e^{-6t}(-6i - 10 \sin(2t)) \\ &= (-6i - 10 \sin(2t)) e^{-6t} \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= e^{-6t}(1) \\ &= e^{-6t} \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{di}{dt} &= 0 \\ ((-6i - 10 \sin(2t)) e^{-6t}) + (e^{-6t}) \frac{di}{dt} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, i)$

$$\frac{\partial \phi}{\partial t} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial i} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t} dt = \int \bar{M} dt$$

$$\int \frac{\partial \phi}{\partial t} dt = \int (-6i - 10 \sin(2t)) e^{-6t} dt$$

$$\phi = \frac{(2i + \cos(2t) + 3 \sin(2t)) e^{-6t}}{2} + f(i) \quad (3)$$

Where $f(i)$ is used for the constant of integration since ϕ is a function of both t and i . Taking derivative of equation (3) w.r.t i gives

$$\frac{\partial \phi}{\partial i} = e^{-6t} + f'(i) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial i} = e^{-6t}$. Therefore equation (4) becomes

$$e^{-6t} = e^{-6t} + f'(i) \quad (5)$$

Solving equation (5) for $f'(i)$ gives

$$f'(i) = 0$$

Therefore

$$f(i) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(i)$ into equation (3) gives ϕ

$$\phi = \frac{(2i + \cos(2t) + 3 \sin(2t)) e^{-6t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2i + \cos(2t) + 3 \sin(2t)) e^{-6t}}{2}$$

The solution becomes

$$i = -\frac{e^{6t}(3 \sin(2t) e^{-6t} + \cos(2t) e^{-6t} - 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$i = -\frac{e^{6t}(3 \sin(2t) e^{-6t} + \cos(2t) e^{-6t} - 2c_1)}{2} \quad (1)$$

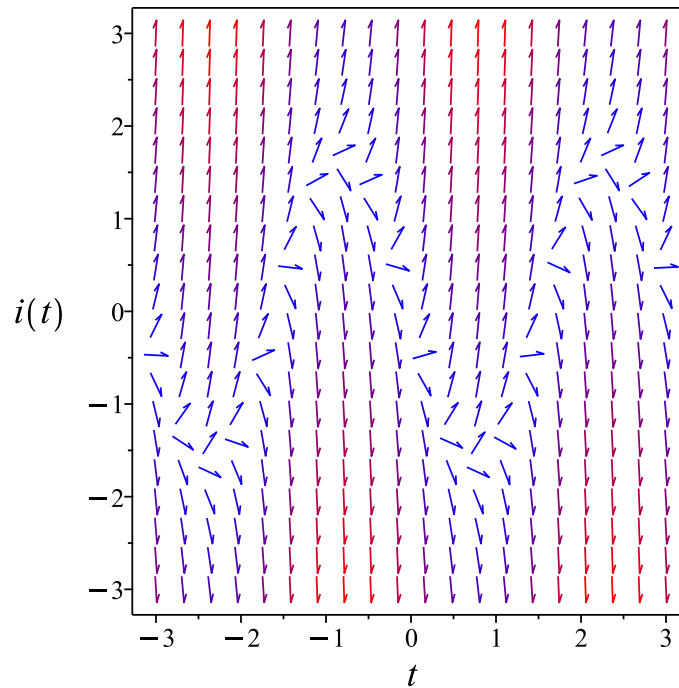


Figure 163: Slope field plot

Verification of solutions

$$i = -\frac{e^{6t}(3 \sin(2t) e^{-6t} + \cos(2t) e^{-6t} - 2c_1)}{2}$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$$i' - 6i = 10 \sin(2t)$$

- Highest derivative means the order of the ODE is 1

i'

- Isolate the derivative

$$i' = 6i + 10 \sin(2t)$$

- Group terms with i on the lhs of the ODE and the rest on the rhs of the ODE

$$i' - 6i = 10 \sin(2t)$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t) (i' - 6i) = 10\mu(t) \sin(2t)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) i)$

$$\mu(t) (i' - 6i) = \mu'(t) i + \mu(t) i'$$

- Isolate $\mu'(t)$

$$\mu'(t) = -6\mu(t)$$

- Solve to find the integrating factor

$$\mu(t) = e^{-6t}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t) i) \right) dt = \int 10\mu(t) \sin(2t) dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t) i = \int 10\mu(t) \sin(2t) dt + c_1$$

- Solve for i

$$i = \frac{\int 10\mu(t) \sin(2t) dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{-6t}$

$$i = \frac{\int 10 \sin(2t) e^{-6t} dt + c_1}{e^{-6t}}$$

- Evaluate the integrals on the rhs

$$i = \frac{-\frac{3 \sin(2t) e^{-6t}}{2} - \frac{\cos(2t) e^{-6t}}{2} + c_1}{e^{-6t}}$$

- Simplify

$$i = c_1 e^{6t} - \frac{3 \sin(2t)}{2} - \frac{\cos(2t)}{2}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(i(t),t)-6*i(t)=10*sin(2*t),i(t), singsol=all)
```

$$i(t) = -\frac{\cos(2t)}{2} - \frac{3 \sin(2t)}{2} + e^{6t} c_1$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 28

```
DSolve[i'[t]-6*i[t]==10*Sin[2*t],i[t],t,IncludeSingularSolutions -> True]
```

$$i(t) \rightarrow -\frac{1}{2} \cos(2t) + c_1 e^{6t} - 3 \sin(t) \cos(t)$$

4.5 problem 19 (f)

4.5.1	Solving as first order ode lie symmetry lookup ode	764
4.5.2	Solving as bernoulli ode	768
4.5.3	Solving as riccati ode	771

Internal problem ID [5302]

Internal file name [OUTPUT/4793_Friday_February_02_2024_05_11_57_AM_30202892/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (f).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _Bernoulli]
```

$$y + y' - e^x y^2 = 0$$

4.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + y^2 e^x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= y^2 e^x\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 e^x} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-x}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + y^2 e^x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{-x}}{y} \\ S_y &= \frac{e^{-x}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = R + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{-x}}{y} = x + c_1$$

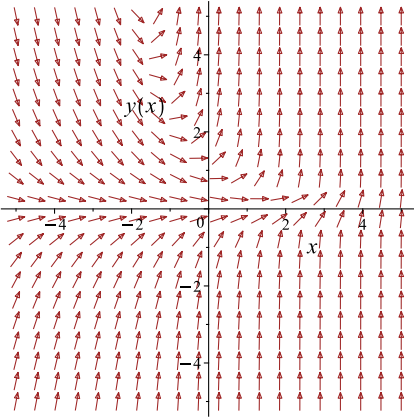
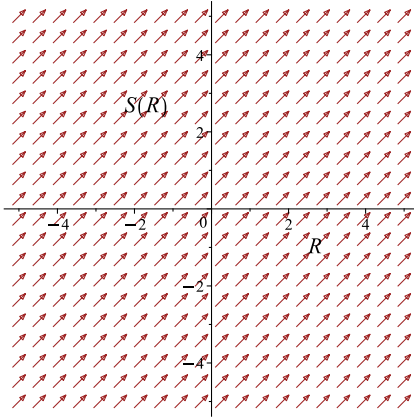
Which simplifies to

$$-\frac{e^{-x}}{y} = x + c_1$$

Which gives

$$y = -\frac{e^{-x}}{x + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + y^2 e^x$ 	$R = x$ $S = -\frac{e^{-x}}{y}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{x + c_1} \quad (1)$$

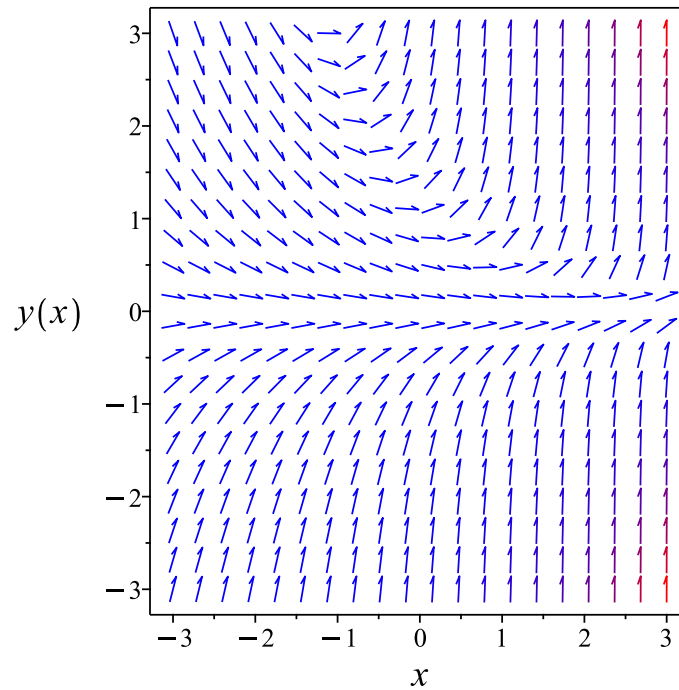


Figure 164: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{x + c_1}$$

Verified OK.

4.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + y^2 e^x \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + e^x y^2 \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -1 \\ f_1(x) &= e^x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = -\frac{1}{y} + e^x \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -w(x) + e^x \\ w' &= w - e^x \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -1 \\q(x) &= -e^x\end{aligned}$$

Hence the ode is

$$w'(x) - w(x) = -e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-e^x) \\ \frac{d}{dx}(e^{-x}w) &= (e^{-x})(-e^x) \\ d(e^{-x}w) &= -1 dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x}w &= \int -1 dx \\ e^{-x}w &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$w(x) = -x e^x + c_1 e^x$$

which simplifies to

$$w(x) = (-x + c_1) e^x$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = (-x + c_1) e^x$$

Or

$$y = \frac{e^{-x}}{-x + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{-x + c_1} \quad (1)$$

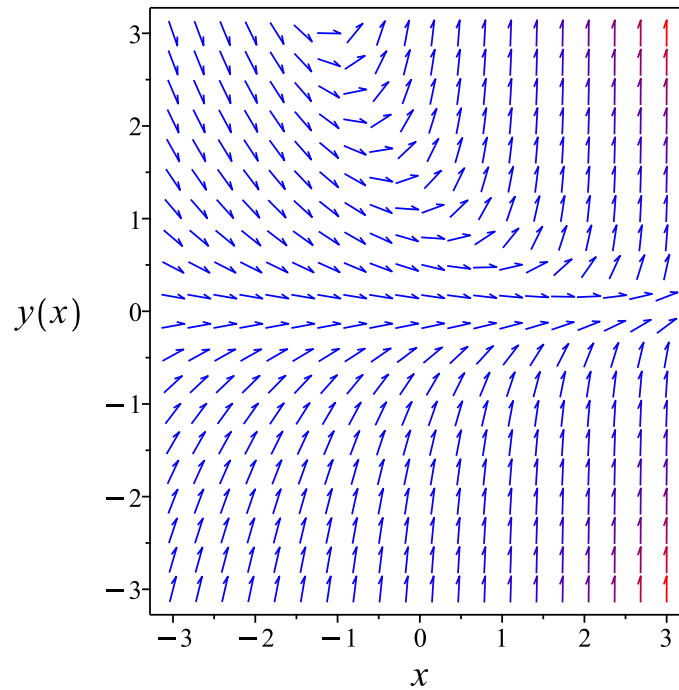


Figure 165: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{-x + c_1}$$

Verified OK.

4.5.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + y^2 e^x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y + y^2 e^x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -1$ and $f_2(x) = e^x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^x u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= e^x \\ f_1 f_2 &= -e^x \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^x u''(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 e^{-x}}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{c_3 e^{-x}}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3 e^{-x}}{c_3 x + 1} \tag{1}$$

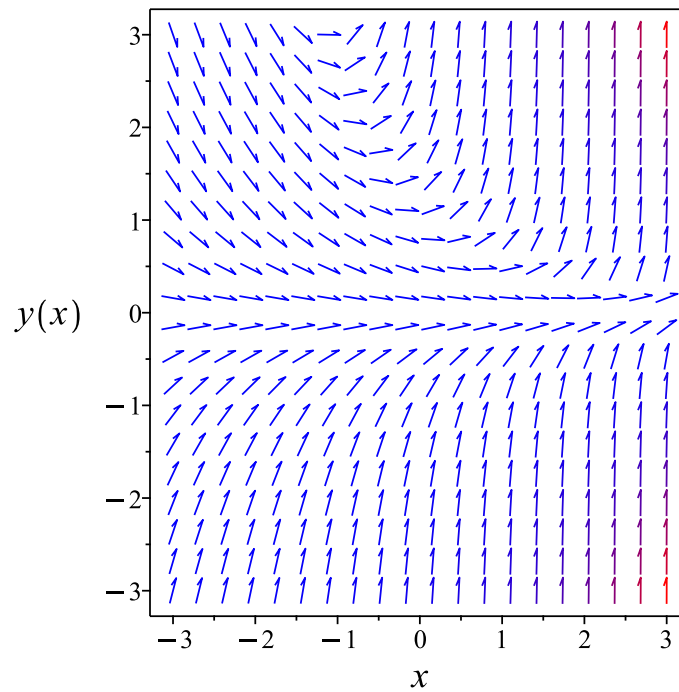


Figure 166: Slope field plot

Verification of solutions

$$y = -\frac{c_3 e^{-x}}{c_3 x + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)=y(x)^2*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}}{-x + c_1}$$

✓ Solution by Mathematica

Time used: 0.204 (sec). Leaf size: 25

```
DSolve[y'[x]+y[x]==y[x]^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{e^{-x}}{x - c_1}$$
$$y(x) \rightarrow 0$$

4.6 problem 19 (g)

4.6.1 Solving as exact ode 775

Internal problem ID [5303]

Internal file name [OUTPUT/4794_Friday_February_02_2024_05_11_57_AM_30991145/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (g).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [_Abel, `2nd type`, `class B`]]
```

$$y + (xy + x - 3y)y' = 0$$

4.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned}(xy + x - 3y) dy &= (-y) dx \\ (y) dx + (xy + x - 3y) dy &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y \\ N(x, y) &= xy + x - 3y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy + x - 3y) \\ &= 1 + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{(x-3)y+x} ((1) - (1+y)) \\ &= -\frac{y}{(x-3)y+x} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y} ((1+y) - (1)) \\ &= 1 \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int 1 \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^y \\ &= e^y \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= e^y(y) \\ &= e^y y \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= e^y(xy + x - 3y) \\ &= (xy + x - 3y) e^y \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ (e^y y) + ((xy + x - 3y) e^y) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int e^y y dx \\ \phi &= e^y xy + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= e^y xy + x e^y + f'(y) \\ &= x e^y (1 + y) + f'(y)\end{aligned} \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (xy + x - 3y) e^y$. Therefore equation (4) becomes

$$(xy + x - 3y) e^y = x e^y (1 + y) + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3 e^y y$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int (-3 e^y y) dy \\ f(y) &= -3(y - 1) e^y + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^y xy - 3(y - 1) e^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^y xy - 3(y - 1) e^y$$

The solution becomes

$$y = \frac{\text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) x - 3 \text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) - 3}{x - 3}$$

Summary

The solution(s) found are the following

$$y = \frac{\text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) x - 3 \text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) - 3}{x - 3} \quad (1)$$

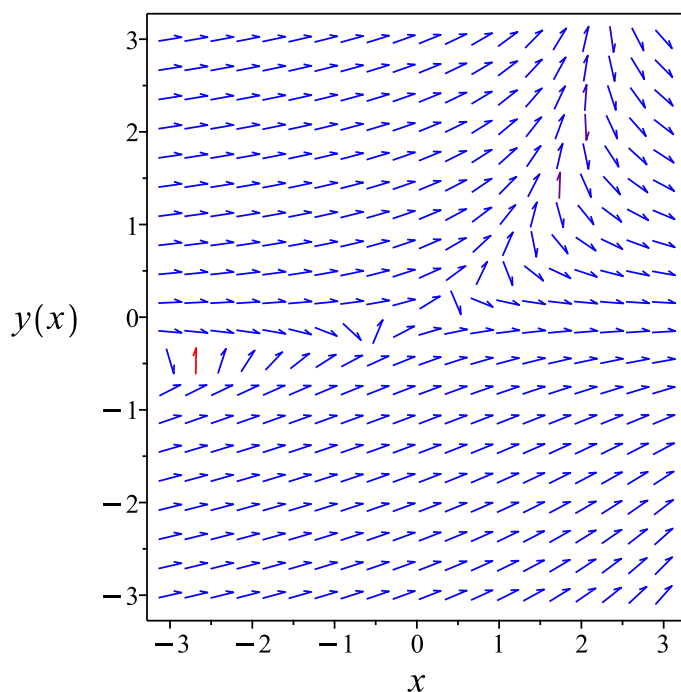


Figure 167: Slope field plot

Verification of solutions

$$y = \frac{\text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) x - 3 \text{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) - 3}{x - 3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
  -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
  <- Kovacics algorithm successful
  <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve(y(x)+(x*y(x)+x-3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3 + (x - 3) \text{LambertW}\left(\frac{e^{\frac{3}{x-3}}}{c_1(x-3)}\right)}{x - 3}$$

✓ Solution by Mathematica

Time used: 60.04 (sec). Leaf size: 31

```
DSolve[y[x]+(x*y[x]+x-3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{3}{x-3} + W\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right)$$

4.7 problem 19 (h)

4.7.1 Solving as exact ode	781
4.7.2 Maple step by step solution	785

Internal problem ID [5304]

Internal file name [OUTPUT/4795_Friday_February_02_2024_05_11_59_AM_65601271/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (h).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exact**"

Maple gives the following as the ode type

```
[_exact, [_1st_order, ` _with_symmetry_[F(x),G(x)]`], [_Abel, `2nd type`, `class A`]]
```

$$(2s - e^{2t}) s' - 2s e^{2t} = -2 \cos(2t)$$

4.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, s) dt + N(t, s) ds = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2s - e^{2t}) ds &= (2s e^{2t} - 2 \cos(2t)) dt \\ (-2s e^{2t} + 2 \cos(2t)) dt &+ (2s - e^{2t}) ds = 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t, s) &= -2s e^{2t} + 2 \cos(2t) \\ N(t, s) &= 2s - e^{2t}\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial s} &= \frac{\partial}{\partial s} (-2s e^{2t} + 2 \cos(2t)) \\ &= -2 e^{2t}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t} (2s - e^{2t}) \\ &= -2 e^{2t}\end{aligned}$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(t, s)$

$$\frac{\partial \phi}{\partial t} = M \quad (1)$$

$$\frac{\partial \phi}{\partial s} = N \quad (2)$$

Integrating (1) w.r.t. t gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial t} dt &= \int M dt \\ \int \frac{\partial \phi}{\partial t} dt &= \int -2s e^{2t} + 2 \cos(2t) dt \\ \phi &= -s e^{2t} + \sin(2t) + f(s) \end{aligned} \quad (3)$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both t and s . Taking derivative of equation (3) w.r.t s gives

$$\frac{\partial \phi}{\partial s} = -e^{2t} + f'(s) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = 2s - e^{2t}$. Therefore equation (4) becomes

$$2s - e^{2t} = -e^{2t} + f'(s) \quad (5)$$

Solving equation (5) for $f'(s)$ gives

$$f'(s) = 2s$$

Integrating the above w.r.t s gives

$$\begin{aligned} \int f'(s) ds &= \int (2s) ds \\ f(s) &= s^2 + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$\phi = -s e^{2t} + \sin(2t) + s^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -s e^{2t} + \sin(2t) + s^2$$

Summary

The solution(s) found are the following

$$-s e^{2t} + \sin(2t) + s^2 = c_1 \tag{1}$$

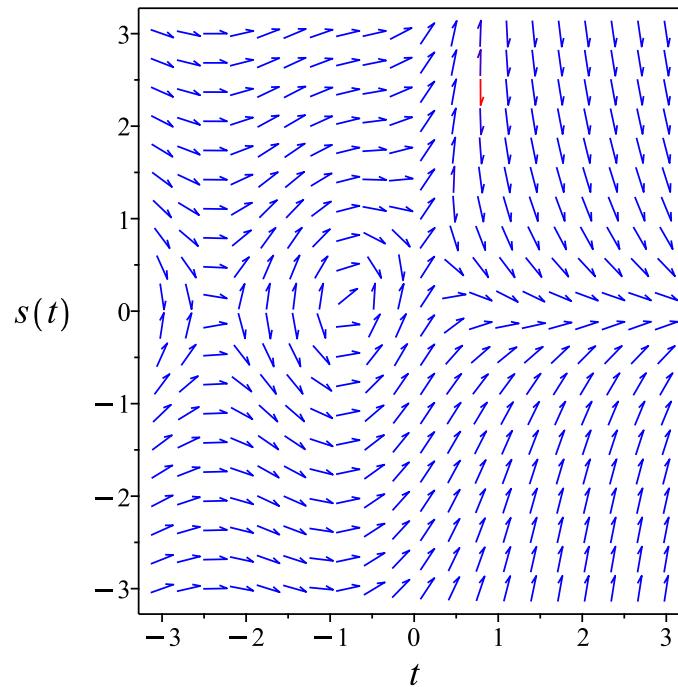


Figure 168: Slope field plot

Verification of solutions

$$-s e^{2t} + \sin(2t) + s^2 = c_1$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$$(2s - e^{2t}) s' - 2s e^{2t} = -2 \cos(2t)$$

- Highest derivative means the order of the ODE is 1
 s'
- Check if ODE is exact
 - ODE is exact if the lhs is the total derivative of a C^2 function
 $F'(t, s) = 0$
 - Compute derivative of lhs
 $F'(t, s) + \left(\frac{\partial}{\partial s} F(t, s)\right) s' = 0$
 - Evaluate derivatives
 $-2 e^{2t} = -2 e^{2t}$
 - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
 $[F(t, s) = c_1, M(t, s) = F'(t, s), N(t, s) = \frac{\partial}{\partial s} F(t, s)]$
- Solve for $F(t, s)$ by integrating $M(t, s)$ with respect to t
 $F(t, s) = \int (-2s e^{2t} + 2 \cos(2t)) dt + f_1(s)$
- Evaluate integral
 $F(t, s) = -s e^{2t} + \sin(2t) + f_1(s)$
- Take derivative of $F(t, s)$ with respect to s
 $N(t, s) = \frac{\partial}{\partial s} F(t, s)$
- Compute derivative
 $2s - e^{2t} = -e^{2t} + \frac{d}{ds} f_1(s)$
- Isolate for $\frac{d}{ds} f_1(s)$
 $\frac{d}{ds} f_1(s) = 2s$
- Solve for $f_1(s)$
 $f_1(s) = s^2$
- Substitute $f_1(s)$ into equation for $F(t, s)$
 $F(t, s) = -s e^{2t} + \sin(2t) + s^2$

- Substitute $F(t, s)$ into the solution of the ODE

$$-s e^{2t} + \sin(2t) + s^2 = c_1$$

- Solve for s

$$\left\{ s = \frac{e^{2t}}{2} - \frac{\sqrt{(e^{2t})^2 - 4 \sin(2t) + 4c_1}}{2}, s = \frac{e^{2t}}{2} + \frac{\sqrt{(e^{2t})^2 - 4 \sin(2t) + 4c_1}}{2} \right\}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve((2*s(t)-exp(2*t))*diff(s(t),t)=2*(s(t)*exp(2*t)-cos(2*t)),s(t), singsol=all)
```

$$s(t) = \frac{e^{2t}}{2} - \frac{\sqrt{e^{4t} - 4 \sin(2t) - 4c_1}}{2}$$

$$s(t) = \frac{e^{2t}}{2} + \frac{\sqrt{e^{4t} - 4 \sin(2t) - 4c_1}}{2}$$

✓ Solution by Mathematica

Time used: 15.59 (sec). Leaf size: 81

```
DSolve[(2*s[t]-Exp[2*t])*s'[t]==2*(s[t]*Exp[2*t]-Cos[2*t]),s[t],t,IncludeSingularSolutions
```

$$s(t) \rightarrow \frac{1}{2} \left(e^{2t} - i \sqrt{-e^{4t} + 4 \sin(2t) - 4c_1} \right)$$

$$s(t) \rightarrow \frac{1}{2} \left(e^{2t} + i \sqrt{-e^{4t} + 4 \sin(2t) - 4c_1} \right)$$

4.8 problem 19 (i)

4.8.1	Solving as first order ode lie symmetry lookup ode	788
4.8.2	Solving as bernoulli ode	792
4.8.3	Solving as exact ode	797

Internal problem ID [5305]

Internal file name [OUTPUT/4796_Friday_February_02_2024_05_12_05_AM_97857565/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (i).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$xy' + y - x^3y^6 = 0$$

4.8.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x^3y^5 - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type Bernoulli. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 99: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= y^6 x^5\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^6 x^5} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{5x^5 y^5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(x^3 y^5 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^6 y^5} \\ S_y &= \frac{1}{y^6 x^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^3} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \quad (4)$$

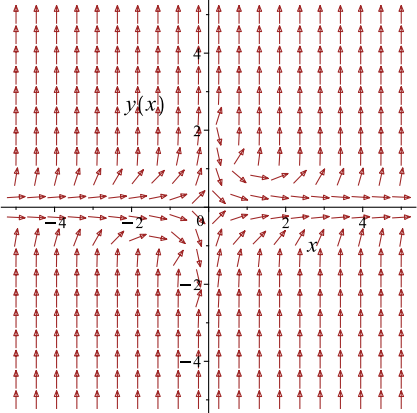
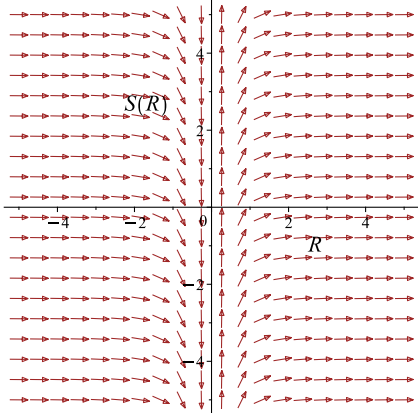
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

Which simplifies to

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^3y^5-1)}{x}$ 	$R = x$ $S = -\frac{1}{5x^5y^5}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1 \quad (1)$$

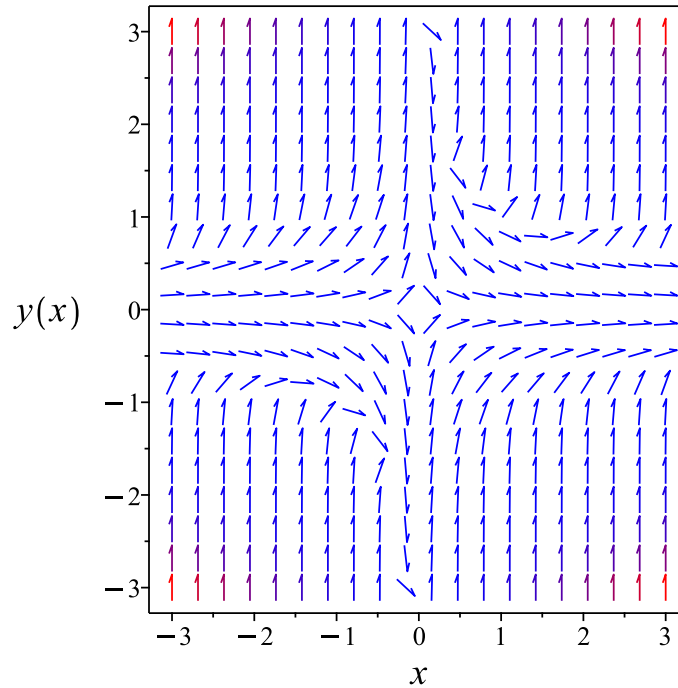


Figure 169: Slope field plot

Verification of solutions

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

Verified OK.

4.8.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x^3y^5 - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x^2y^6 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= x^2 \\ n &= 6 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^6$ gives

$$y' \frac{1}{y^6} = -\frac{1}{x y^5} + x^2 \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^5} \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{5}{y^6} y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{5} &= -\frac{w(x)}{x} + x^2 \\ w' &= \frac{5w}{x} - 5x^2 \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{5}{x} \\ q(x) &= -5x^2 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{5w(x)}{x} = -5x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{5}{x} dx} \\ &= \frac{1}{x^5}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-5x^2) \\ \frac{d}{dx}\left(\frac{w}{x^5}\right) &= \left(\frac{1}{x^5}\right)(-5x^2) \\ d\left(\frac{w}{x^5}\right) &= \left(-\frac{5}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^5} &= \int -\frac{5}{x^3} dx \\ \frac{w}{x^5} &= \frac{5}{2x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^5}$ results in

$$w(x) = \frac{5}{2}x^3 + c_1x^5$$

Replacing w in the above by $\frac{1}{y^5}$ using equation (5) gives the final solution.

$$\frac{1}{y^5} = \frac{5}{2}x^3 + c_1x^5$$

Solving for y gives

$$y(x) = \frac{2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{2c_1x^3 + 5x}$$

$$y(x) = \frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} + \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x}$$

$$y(x) = \frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} - \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x}$$

$$y(x) = -\frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} + \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x}$$

$$y(x) = -\frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} - \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{2c_1x^3 + 5x} \tag{1}$$

$$y = \frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} + \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x} \tag{2}$$

$$y = \frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} - \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x} \tag{3}$$

$$y = -\frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} + \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x} \tag{4}$$

$$y = -\frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} - \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2(2c_1x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1x^3 + 20x} \tag{5}$$

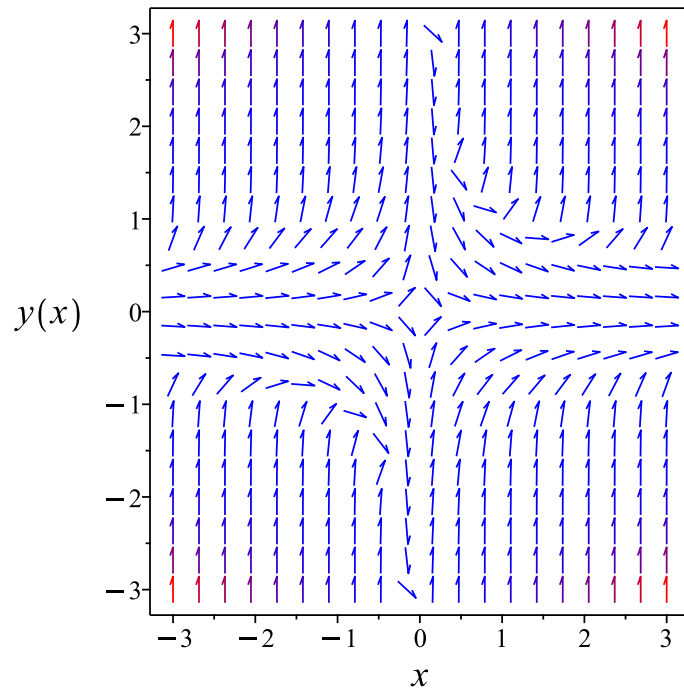


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{2c_1 x^3 + 5x}$$

Verified OK.

$$y = \frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} + \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

Verified OK.

$$y = \frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} - \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

Verified OK.

$$y = -\frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} + \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

Verified OK.

$$y = -\frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} - \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

Verified OK.

4.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) dy &= (x^3 y^6 - y) dx \\ (-x^3 y^6 + y) dx + (x) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^3 y^6 + y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3 y^6 + y) \\ &= -6x^3 y^5 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((-6x^3y^5 + 1) - (1)) \\ &= -6x^2y^5 \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{-x^3y^6 + y} ((1) - (-6x^3y^5 + 1)) \\ &= -\frac{6x^3y^4}{x^3y^5 - 1} \end{aligned}$$

Since B depends on x , it can not be used to obtain an integrating factor. We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1) - (-6x^3y^5 + 1)}{x(-x^3y^6 + y) - y(x)} \\ &= -\frac{6}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{6}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R dt} \\ &= e^{\int (-\frac{6}{t}) dt} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-6 \ln(t)} \\ &= \frac{1}{t^6}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^6 y^6}$$

Multiplying M and N by this integrating factor gives new \bar{M} and new \bar{N} which are called \bar{M} and \bar{N} so not to confuse them with the original M and N

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x^6 y^6} (-x^3 y^6 + y) \\ &= \frac{-x^3 y^5 + 1}{x^6 y^5}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x^6 y^6} (x) \\ &= \frac{1}{y^6 x^5}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-x^3 y^5 + 1}{x^6 y^5} \right) + \left(\frac{1}{y^6 x^5} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-x^3 y^5 + 1}{x^6 y^5} dx \\ \phi &= \frac{5x^3 y^5 - 2}{10y^5 x^5} + f(y)\end{aligned}\quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{5}{2y x^2} - \frac{5x^3 y^5 - 2}{2y^6 x^5} + f'(y) \\ &= \frac{1}{y^6 x^5} + f'(y)\end{aligned}\quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^6 x^5}$. Therefore equation (4) becomes

$$\frac{1}{y^6 x^5} = \frac{1}{y^6 x^5} + f'(y)\quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{5x^3 y^5 - 2}{10y^5 x^5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{5x^3 y^5 - 2}{10y^5 x^5}$$

Summary

The solution(s) found are the following

$$\frac{5x^3y^5 - 2}{10x^5y^5} = c_1 \quad (1)$$

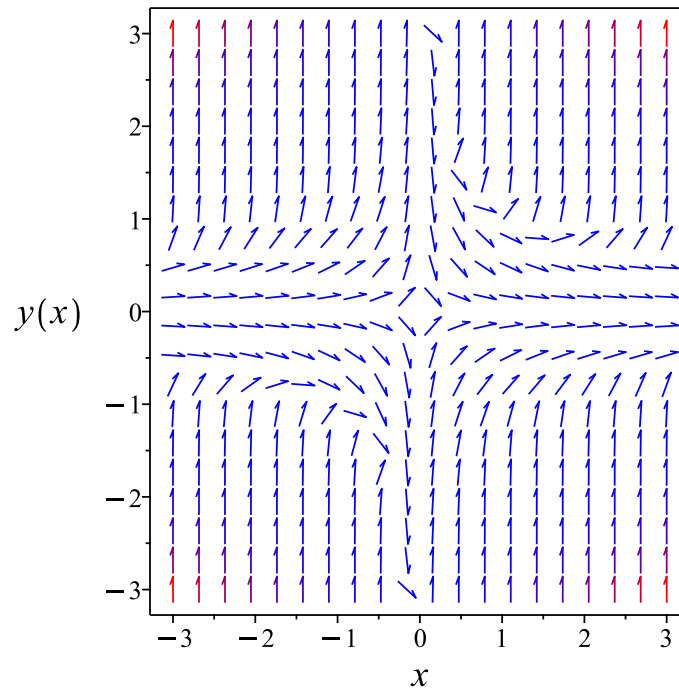


Figure 171: Slope field plot

Verification of solutions

$$\frac{5x^3y^5 - 2}{10x^5y^5} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 258

```
dsolve(x*diff(y(x),x)+(y(x)-x^3*y(x)^6)=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{2c_1 x^3 + 5x}$$

$$y(x) = -\frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} + \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = \frac{\left(i\sqrt{2} \sqrt{5 - \sqrt{5}} - \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = -\frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} - \sqrt{5} + 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = \frac{\left(i\sqrt{2} \sqrt{5 + \sqrt{5}} + \sqrt{5} - 1 \right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4 \right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 141

```
DSolve[x*y'[x]+(y[x]-x^3*y[x]^6)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt[5]{-2}}{\sqrt[5]{x^3(5+2c_1x^2)}}$$

$$y(x) \rightarrow \frac{1}{\sqrt[5]{\frac{5x^3}{2} + c_1x^5}}$$

$$y(x) \rightarrow \frac{(-1)^{2/5}}{\sqrt[5]{\frac{5x^3}{2} + c_1x^5}}$$

$$y(x) \rightarrow -\frac{(-1)^{3/5}}{\sqrt[5]{\frac{5x^3}{2} + c_1x^5}}$$

$$y(x) \rightarrow \frac{(-1)^{4/5}}{\sqrt[5]{\frac{5x^3}{2} + c_1x^5}}$$

$$y(x) \rightarrow 0$$

4.9 problem 19 (j)

4.9.1	Solving as linear ode	805
4.9.2	Solving as first order ode lie symmetry lookup ode	807
4.9.3	Solving as exact ode	811
4.9.4	Maple step by step solution	815

Internal problem ID [5306]

Internal file name [OUTPUT/4797_Friday_February_02_2024_05_12_07_AM_85124973/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (j).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$r' + 2r \cos(\theta) = -\sin(2\theta)$$

4.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = 2 \cos(\theta)$$

$$q(\theta) = -\sin(2\theta)$$

Hence the ode is

$$r' + 2r \cos(\theta) = -\sin(2\theta)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2 \cos(\theta) d\theta} \\ &= e^{2 \sin(\theta)}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{d\theta}(\mu r) &= (\mu) (-\sin(2\theta)) \\ \frac{d}{d\theta}(e^{2 \sin(\theta)} r) &= (e^{2 \sin(\theta)}) (-\sin(2\theta)) \\ d(e^{2 \sin(\theta)} r) &= (-\sin(2\theta) e^{2 \sin(\theta)}) d\theta\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2 \sin(\theta)} r &= \int -\sin(2\theta) e^{2 \sin(\theta)} d\theta \\ e^{2 \sin(\theta)} r &= -\sin(\theta) e^{2 \sin(\theta)} + \frac{e^{2 \sin(\theta)}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2 \sin(\theta)}$ results in

$$r = e^{-2 \sin(\theta)} \left(-\sin(\theta) e^{2 \sin(\theta)} + \frac{e^{2 \sin(\theta)}}{2} \right) + c_1 e^{-2 \sin(\theta)}$$

which simplifies to

$$r = -\sin(\theta) + \frac{1}{2} + c_1 e^{-2 \sin(\theta)}$$

Summary

The solution(s) found are the following

$$r = -\sin(\theta) + \frac{1}{2} + c_1 e^{-2 \sin(\theta)} \quad (1)$$

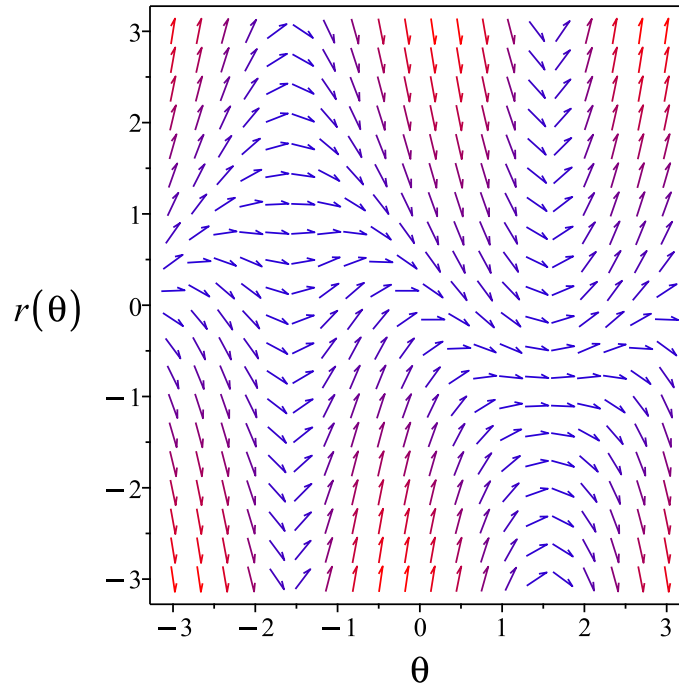


Figure 172: Slope field plot

Verification of solutions

$$r = -\sin(\theta) + \frac{1}{2} + c_1 e^{-2\sin(\theta)}$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} r' &= -2 \cos(\theta) r - \sin(2\theta) \\ r' &= \omega(\theta, r) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 101: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(\theta, r) &= 0 \\ \eta(\theta, r) &= e^{-2\sin(\theta)}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(\theta, r) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS\tag{1}$$

The above comes from the requirements that $(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r})S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2\sin(\theta)}} dy \end{aligned}$$

Which results in

$$S = e^{2\sin(\theta)} r$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r)S_r}{R_\theta + \omega(\theta, r)R_r} \quad (2)$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = -2 \cos(\theta) r - \sin(2\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= 2 \cos(\theta) e^{2\sin(\theta)} r \\ S_r &= e^{2\sin(\theta)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(2\theta) e^{2\sin(\theta)} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(2R) e^{2\sin(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 - \frac{e^{2\sin(R)}(-1 + 2\sin(R))}{2} \quad (4)$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$e^{2\sin(\theta)}r = c_1 - \frac{e^{2\sin(\theta)}(-1 + 2\sin(\theta))}{2}$$

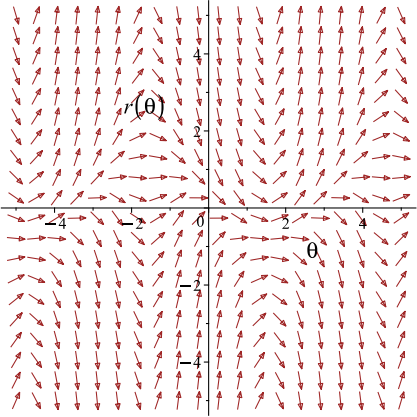
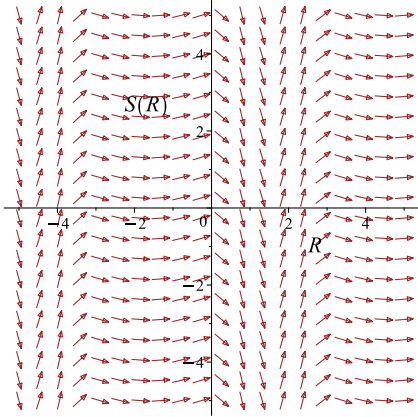
Which simplifies to

$$\frac{(2r + 2\sin(\theta) - 1)e^{2\sin(\theta)}}{2} - c_1 = 0$$

Which gives

$$r = -\frac{e^{-2\sin(\theta)}(2\sin(\theta)e^{2\sin(\theta)} - e^{2\sin(\theta)} - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = -2\cos(\theta)r - \sin(2\theta)$ 	$R = \theta$ $S = e^{2\sin(\theta)}r$	$\frac{dS}{dR} = -\sin(2R)e^{2\sin(R)}$ 

Summary

The solution(s) found are the following

$$r = -\frac{e^{-2\sin(\theta)}(2\sin(\theta)e^{2\sin(\theta)} - e^{2\sin(\theta)} - 2c_1)}{2} \quad (1)$$

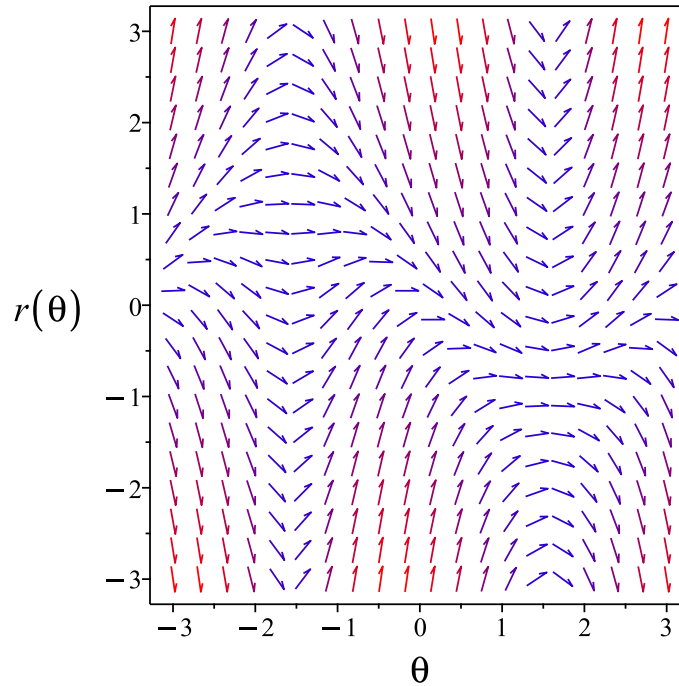


Figure 173: Slope field plot

Verification of solutions

$$r = -\frac{e^{-2\sin(\theta)}(2\sin(\theta)e^{2\sin(\theta)} - e^{2\sin(\theta)} - 2c_1)}{2}$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r) d\theta + N(\theta, r) dr = 0 \quad (1A)$$

Therefore

$$\begin{aligned}dr &= (-2 \cos(\theta) r - \sin(2\theta)) d\theta \\ (2 \cos(\theta) r + \sin(2\theta)) d\theta + dr &= 0\end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= 2 \cos(\theta) r + \sin(2\theta) \\ N(\theta, r) &= 1\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(2 \cos(\theta) r + \sin(2\theta)) \\ &= 2 \cos(\theta)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta} \right) \\ &= 1((2 \cos(\theta)) - (0)) \\ &= 2 \cos(\theta) \end{aligned}$$

Since A does not depend on r , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, d\theta} \\ &= e^{\int 2 \cos(\theta) \, d\theta} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{2 \sin(\theta)} \\ &= e^{2 \sin(\theta)} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= e^{2 \sin(\theta)} (2 \cos(\theta) r + \sin(2\theta)) \\ &= 2 \cos(\theta) (r + \sin(\theta)) e^{2 \sin(\theta)} \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= e^{2 \sin(\theta)} (1) \\ &= e^{2 \sin(\theta)} \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dr}{d\theta} &= 0 \\ (2 \cos(\theta) (r + \sin(\theta)) e^{2 \sin(\theta)} + (e^{2 \sin(\theta)}) \frac{dr}{d\theta} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial r} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} d\theta = \int \bar{M} d\theta$$

$$\int \frac{\partial \phi}{\partial \theta} d\theta = \int 2 \cos(\theta) (r + \sin(\theta)) e^{2 \sin(\theta)} d\theta$$

$$\phi = \frac{(2r + 2 \sin(\theta) - 1) e^{2 \sin(\theta)}}{2} + f(r) \quad (3)$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r . Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = e^{2 \sin(\theta)} + f'(r) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = e^{2 \sin(\theta)}$. Therefore equation (4) becomes

$$e^{2 \sin(\theta)} = e^{2 \sin(\theta)} + f'(r) \quad (5)$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \frac{(2r + 2 \sin(\theta) - 1) e^{2 \sin(\theta)}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2r + 2 \sin(\theta) - 1) e^{2 \sin(\theta)}}{2}$$

The solution becomes

$$r = - \frac{e^{-2 \sin(\theta)} (2 \sin(\theta) e^{2 \sin(\theta)} - e^{2 \sin(\theta)} - 2c_1)}{2}$$

Summary

The solution(s) found are the following

$$r = -\frac{e^{-2\sin(\theta)}(2\sin(\theta)e^{2\sin(\theta)} - e^{2\sin(\theta)} - 2c_1)}{2} \quad (1)$$

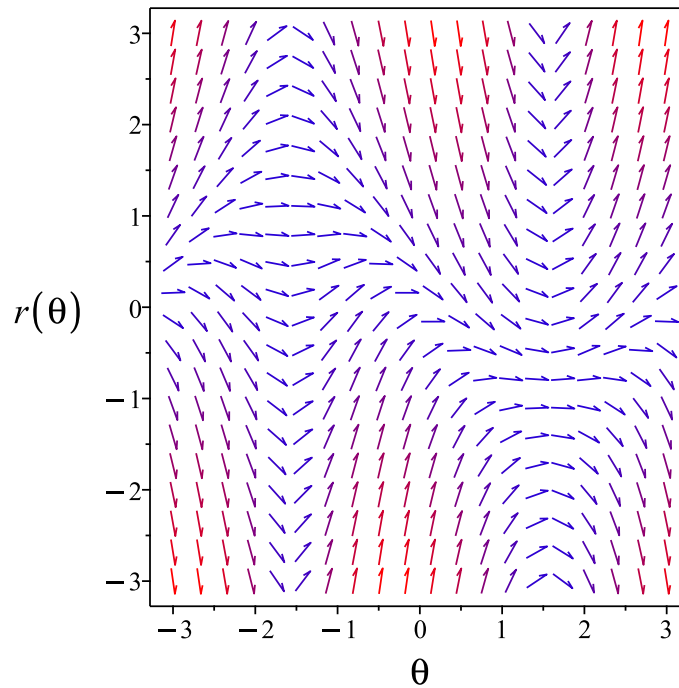


Figure 174: Slope field plot

Verification of solutions

$$r = -\frac{e^{-2\sin(\theta)}(2\sin(\theta)e^{2\sin(\theta)} - e^{2\sin(\theta)} - 2c_1)}{2}$$

Verified OK.

4.9.4 Maple step by step solution

Let's solve

$$r' + 2r \cos(\theta) = -\sin(2\theta)$$

- Highest derivative means the order of the ODE is 1

r'

- Isolate the derivative

$$r' = -2r \cos(\theta) - \sin(2\theta)$$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

$$r' + 2r \cos(\theta) = -\sin(2\theta)$$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

$$\mu(\theta) (r' + 2r \cos(\theta)) = -\mu(\theta) \sin(2\theta)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

$$\mu(\theta) (r' + 2r \cos(\theta)) = \mu'(\theta) r + \mu(\theta) r'$$

- Isolate $\mu'(\theta)$

$$\mu'(\theta) = 2\mu(\theta) \cos(\theta)$$

- Solve to find the integrating factor

$$\mu(\theta) = e^{2 \sin(\theta)}$$

- Integrate both sides with respect to θ

$$\int \left(\frac{d}{d\theta}(\mu(\theta) r) \right) d\theta = \int -\mu(\theta) \sin(2\theta) d\theta + c_1$$

- Evaluate the integral on the lhs

$$\mu(\theta) r = \int -\mu(\theta) \sin(2\theta) d\theta + c_1$$

- Solve for r

$$r = \frac{\int -\mu(\theta) \sin(2\theta) d\theta + c_1}{\mu(\theta)}$$

- Substitute $\mu(\theta) = e^{2 \sin(\theta)}$

$$r = \frac{\int -\sin(2\theta) e^{2 \sin(\theta)} d\theta + c_1}{e^{2 \sin(\theta)}}$$

- Evaluate the integrals on the rhs

$$r = \frac{-\sin(\theta) e^{2 \sin(\theta)} + \frac{e^{2 \sin(\theta)}}{2} + c_1}{e^{2 \sin(\theta)}}$$

- Simplify

$$r = -\sin(\theta) + \frac{1}{2} + c_1 e^{-2 \sin(\theta)}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(r(theta),theta)+(2*r(theta)*cos(theta)+sin(2*theta))=0,r(theta), singsol=all)
```

$$r(\theta) = -\sin(\theta) + \frac{1}{2} + e^{-2\sin(\theta)}c_1$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 22

```
DSolve[r'[t]+(2*r[t]*Cos[t]+Sin[2*t])==0,r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \rightarrow -\sin(t) + c_1 e^{-2\sin(t)} + \frac{1}{2}$$

4.10 problem 19 (k)

4.10.1 Solving as exact ode 818

Internal problem ID [5307]

Internal file name [OUTPUT/4798_Friday_February_02_2024_05_12_08_AM_80125195/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (k).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, ` _with_symmetry_ [F(x)*G(y),0] `]]
```

$$y(1 + y^2) - 2(1 - 2xy^2) y' = 0$$

4.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (4y^2x - 2) dy &= (-y(y^2 + 1)) dx \\ (y(y^2 + 1)) dx + (4y^2x - 2) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y(y^2 + 1) \\ N(x, y) &= 4y^2x - 2 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (y(y^2 + 1)) \\ &= 3y^2 + 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (4y^2x - 2) \\ &= 4y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{4y^2x - 2} ((3y^2 + 1) - (4y^2)) \\ &= \frac{-y^2 + 1}{4y^2x - 2} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y(y^2 + 1)} ((4y^2) - (3y^2 + 1)) \\ &= \frac{y^2 - 1}{y(y^2 + 1)} \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int \frac{y^2 - 1}{y(y^2 + 1)} \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(y^2 + 1) - \ln(y)} \\ &= \frac{y^2 + 1}{y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \frac{y^2 + 1}{y} (y(y^2 + 1)) \\ &= (y^2 + 1)^2 \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= \frac{y^2 + 1}{y} (4y^2 x - 2) \\ &= \frac{(4y^2 x - 2)(y^2 + 1)}{y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left((y^2 + 1)^2 \right) + \left(\frac{(4y^2 x - 2)(y^2 + 1)}{y} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int (y^2 + 1)^2 dx \\ \phi &= (y^2 + 1)^2 x + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4(y^2 + 1)xy + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(4y^2x-2)(y^2+1)}{y}$. Therefore equation (4) becomes

$$\frac{(4y^2x - 2)(y^2 + 1)}{y} = 4(y^2 + 1)xy + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2(y^2 + 1)}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) dy &= \int \left(\frac{-2y^2 - 2}{y} \right) dy \\ f(y) &= -y^2 - 2 \ln(y) + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = (y^2 + 1)^2 x - y^2 - 2 \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y^2 + 1)^2 x - y^2 - 2 \ln(y)$$

Summary

The solution(s) found are the following

$$(1 + y^2)^2 x - y^2 - 2 \ln(y) = c_1 \tag{1}$$

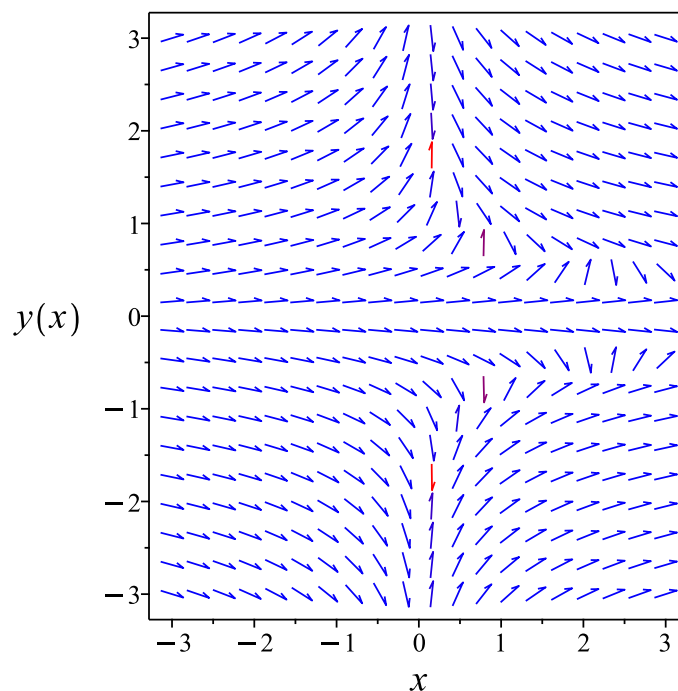


Figure 175: Slope field plot

Verification of solutions

$$(1 + y^2)^2 x - y^2 - 2 \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
<- 1st order linear successful  
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(y(x)*(1+y(x)^2)=2*(1-2*x*y(x)^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = e^{\text{RootOf}(-x e^{4-Z} - 2x e^{2-Z} + e^{2-Z} + c_1 + 2_Z - x)}$$

✓ Solution by Mathematica

Time used: 0.176 (sec). Leaf size: 36

```
DSolve[y[x]*(1+y[x]^2)==2*(1-2*x*y[x]^2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve} \left[x = \frac{y(x)^2 + 2 \log(y(x))}{(y(x)^2 + 1)^2} + \frac{c_1}{(y(x)^2 + 1)^2}, y(x) \right]$$

4.11 problem 19 (L)

4.11.1 Solving as separable ode	824
4.11.2 Solving as first order ode lie symmetry lookup ode	826
4.11.3 Solving as bernoulli ode	830
4.11.4 Solving as exact ode	833
4.11.5 Maple step by step solution	837

Internal problem ID [5308]

Internal file name [OUTPUT/4799_Friday_February_02_2024_05_12_09_AM_67397305/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (L).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$y'y - xy^2 = -x$$

4.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x(y^2 - 1)}{y}\end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{y^2-1}{y}$. Integrating both sides gives

$$\frac{1}{y^2-1} dy = x dx$$

$$\int \frac{1}{\frac{y^2-1}{y}} dy = \int x dx$$

$$\frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} = \frac{x^2}{2} + c_1$$

The above can be written as

$$\left(\frac{1}{2}\right) (\ln(y-1) + \ln(1+y)) = \frac{x^2}{2} + 2c_1$$

$$\ln(y-1) + \ln(1+y) = (2) \left(\frac{x^2}{2} + 2c_1\right)$$

$$= x^2 + 4c_1$$

Raising both side to exponential gives

$$e^{\ln(y-1)+\ln(1+y)} = e^{x^2+2c_1}$$

Which simplifies to

$$y^2 - 1 = 2c_1e^{x^2}$$

$$= c_2e^{x^2}$$

The solution is

$$y^2 - 1 = c_2e^{x^2}$$

Summary

The solution(s) found are the following

$$y^2 - 1 = c_2e^{x^2} \tag{1}$$

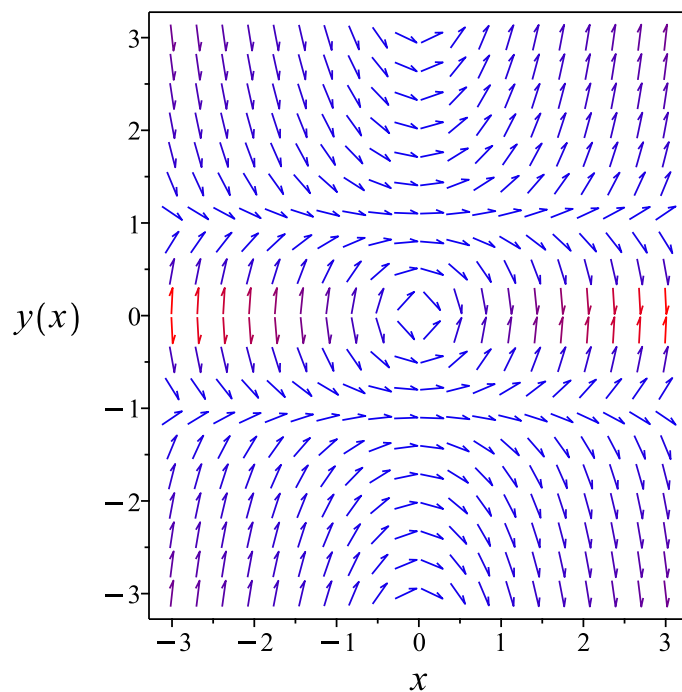


Figure 176: Slope field plot

Verification of solutions

$$y^2 - 1 = c_2 e^{x^2}$$

Verified OK.

4.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x(y^2 - 1)}{y}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **separable**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 104: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= \frac{1}{x} \\ \eta(x, y) &= 0\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x(y^2 - 1)}{y}$$

Evaluating all the partial derivatives gives

$$R_x = 0$$

$$R_y = 1$$

$$S_x = x$$

$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 - 1} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \quad (4)$$

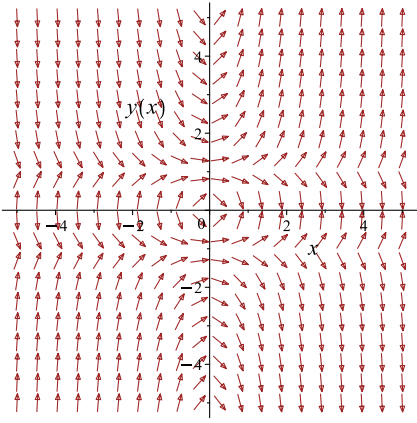
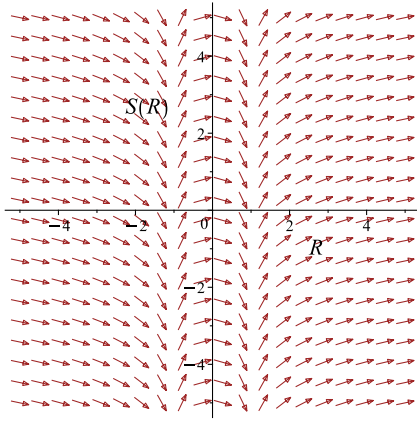
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(y^2-1)}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1 \quad (1)$$

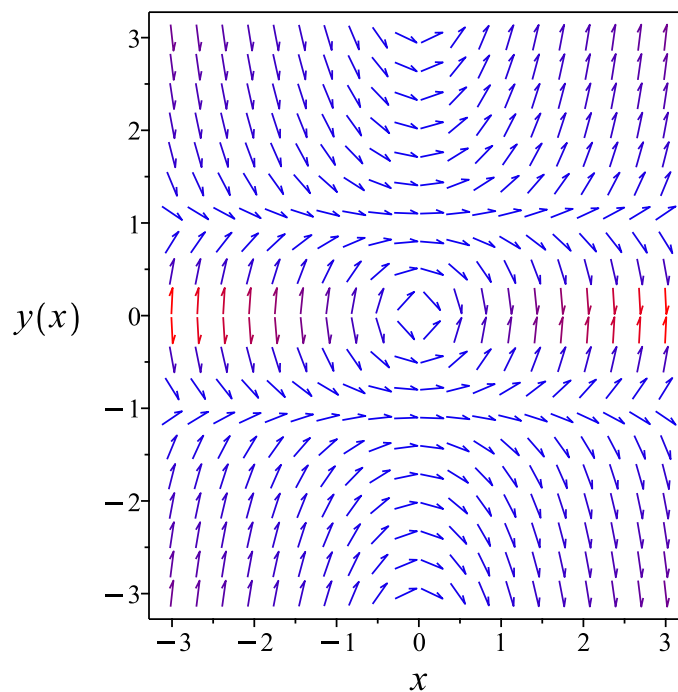


Figure 177: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

Verified OK.

4.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x(y^2 - 1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}f_0(x) &= x \\f_1(x) &= -x \\n &= -1\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = y^2x - x \tag{4}$$

Let

$$\begin{aligned}w &= y^{1-n} \\&= y^2\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}\frac{w'(x)}{2} &= w(x)x - x \\w' &= 2xw - 2x\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -2x \\q(x) &= -2x\end{aligned}$$

Hence the ode is

$$w'(x) - 2w(x)x = -2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2x dx} \\ &= e^{-x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\ \frac{d}{dx}(e^{-x^2} w) &= (e^{-x^2})(-2x) \\ d(e^{-x^2} w) &= (-2x e^{-x^2}) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x^2} w &= \int -2x e^{-x^2} dx \\ e^{-x^2} w &= e^{-x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$w(x) = e^{x^2} e^{-x^2} + c_1 e^{x^2}$$

which simplifies to

$$w(x) = 1 + c_1 e^{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = 1 + c_1 e^{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{1 + c_1 e^{x^2}} \\ y(x) &= -\sqrt{1 + c_1 e^{x^2}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{1 + c_1 e^{x^2}} \tag{1}$$

$$y = -\sqrt{1 + c_1 e^{x^2}} \tag{2}$$

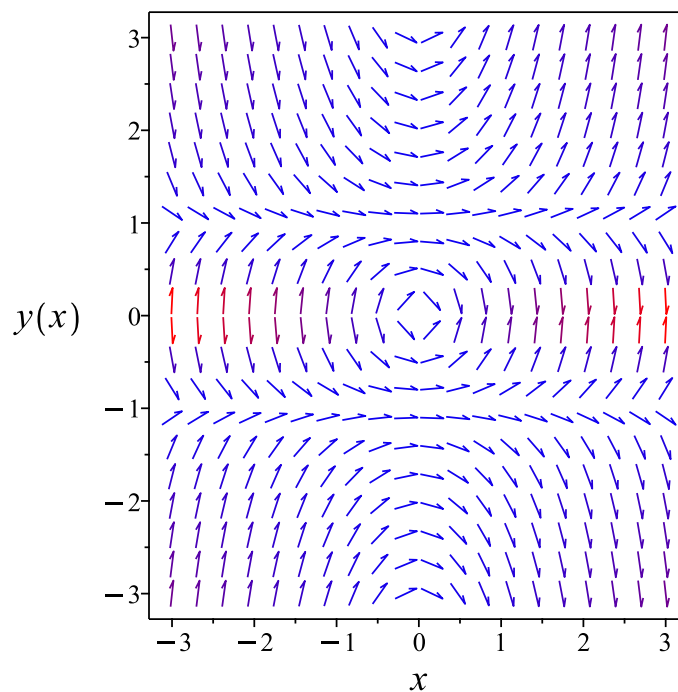


Figure 178: Slope field plot

Verification of solutions

$$y = \sqrt{1 + c_1 e^{x^2}}$$

Verified OK.

$$y = -\sqrt{1 + c_1 e^{x^2}}$$

Verified OK.

4.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} \left(\frac{y}{y^2 - 1} \right) dy &= (x) dx \\ (-x) dx + \left(\frac{y}{y^2 - 1} \right) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x \\ N(x, y) &= \frac{y}{y^2 - 1} \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{y}{y^2 - 1} \right) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact. The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \quad (1)$$

$$\frac{\partial \phi}{\partial y} = N \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int M dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int -x dx \\ \phi &= -\frac{x^2}{2} + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2 - 1}$. Therefore equation (4) becomes

$$\frac{y}{y^2 - 1} = 0 + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2 - 1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) dy &= \int \left(\frac{y}{y^2 - 1} \right) dy \\ f(y) &= \frac{\ln(y - 1)}{2} + \frac{\ln(1 + y)}{2} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} = c_1 \quad (1)$$

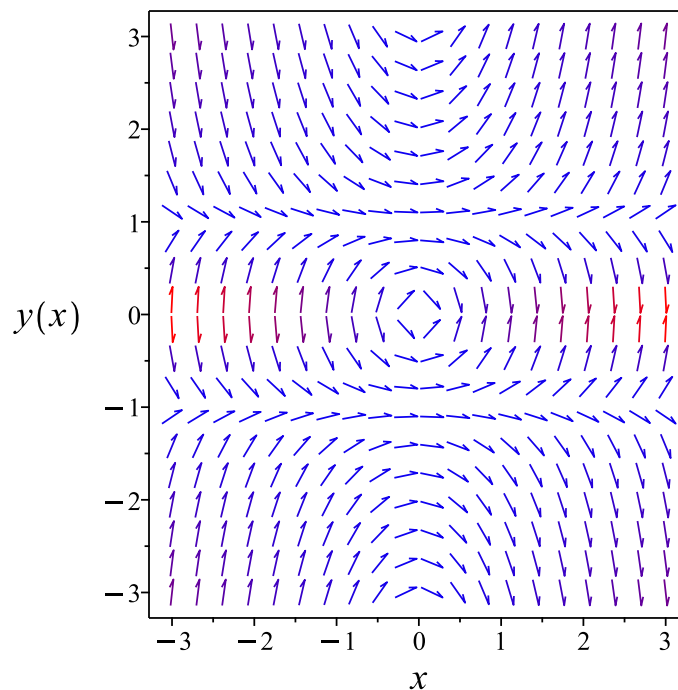


Figure 179: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} = c_1$$

Verified OK.

4.11.5 Maple step by step solution

Let's solve

$$y'y - xy^2 = -x$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'y}{(y-1)(1+y)} = x$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{(y-1)(1+y)} dx = \int x dx + c_1$$

- Evaluate integral

$$\frac{\ln((y-1)(1+y))}{2} = \frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{ y = \sqrt{1 + e^{x^2+2c_1}}, y = -\sqrt{1 + e^{x^2+2c_1}} \right\}$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(y(x)*diff(y(x),x)-x*y(x)^2+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{x^2} c_1 + 1}$$
$$y(x) = -\sqrt{e^{x^2} c_1 + 1}$$

✓ Solution by Mathematica

Time used: 1.859 (sec). Leaf size: 53

```
DSolve[y[x]*y'[x]-x*y[x]^2+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{1 + e^{x^2+2c_1}}$$

$$y(x) \rightarrow \sqrt{1 + e^{x^2+2c_1}}$$

$$y(x) \rightarrow -1$$

$$y(x) \rightarrow 1$$

4.12 problem 19 (m)

Internal problem ID [5309]

Internal file name [OUTPUT/4800_Friday_February_02_2024_05_12_10_AM_19531194/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (m).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\left(x - x\sqrt{x^2 - y^2}\right) y' - y = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)/((x^2-y^2)^(1/2)-1)]
```


✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 27

```
dsolve((x-x*sqrt(x^2-y(x)^2))*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) - \arctan\left(\frac{y(x)}{\sqrt{x^2 - y(x)^2}}\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.518 (sec). Leaf size: 29

```
DSolve[(x-x*Sqrt[x^2-y[x]^2])*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{\sqrt{x^2 - y(x)^2}}{y(x)}\right) + y(x) = c_1, y(x)\right]$$

4.13 problem 19 (o)

4.13.1 Solving as first order ode lie symmetry lookup ode	841
4.13.2 Solving as bernoulli ode	845
4.13.3 Solving as exact ode	849

Internal problem ID [5310]

Internal file name [OUTPUT/4801_Friday_February_02_2024_05_12_12_AM_22931473/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (o).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_Bernoulli]

$$2x' - \frac{x}{y} + x^3 \cos(y) = 0$$

4.13.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{x(x^2 \cos(y) y - 1)}{2y}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type Bernoulli. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(y, x) &= 0 \\ \eta(y, x) &= \frac{x^3}{y}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(y, x) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dx}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial x}\right)S(y, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y}} dy \end{aligned}$$

Which results in

$$S = -\frac{y}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x)S_x}{R_y + \omega(y, x)R_x} \quad (2)$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = -\frac{x(x^2 \cos(y) y - 1)}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= -\frac{1}{2x^2} \\ S_x &= \frac{y}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y \cos(y)}{2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R \cos(R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{\cos(R)}{2} - \frac{R \sin(R)}{2} + c_1 \quad (4)$$

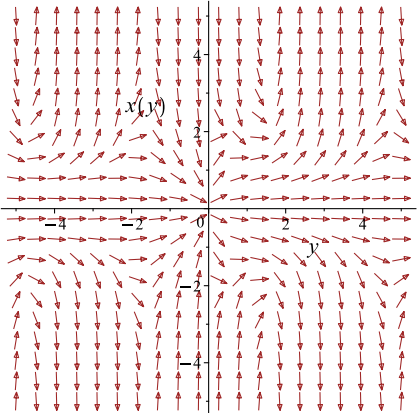
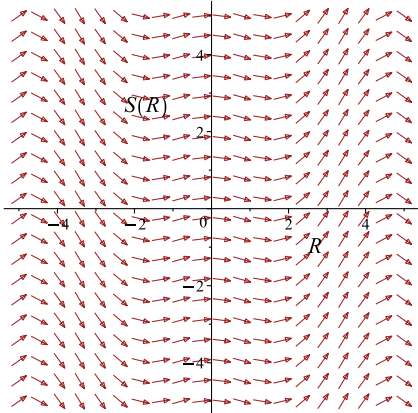
To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y \sin(y)}{2} + c_1$$

Which simplifies to

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y \sin(y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = -\frac{x(x^2 \cos(y)y-1)}{2y}$ 	$R = y$ $S = -\frac{y}{2x^2}$	$\frac{dS}{dR} = -\frac{R \cos(R)}{2}$ 

Summary

The solution(s) found are the following

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y \sin(y)}{2} + c_1 \quad (1)$$

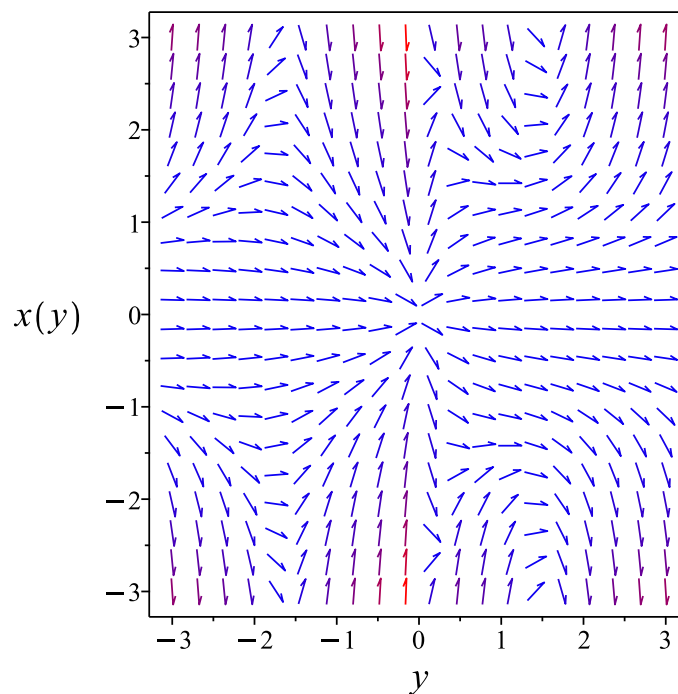


Figure 180: Slope field plot

Verification of solutions

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y \sin(y)}{2} + c_1$$

Verified OK.

4.13.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(y, x) \\ &= -\frac{x(x^2 \cos(y) y - 1)}{2y} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = \frac{1}{2y}x - \frac{\cos(y)}{2}x^3 \quad (1)$$

The standard Bernoulli ODE has the form

$$x' = f_0(y)x + f_1(y)x^n \quad (2)$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(y)x^{1-n} + f_1(y) \quad (3)$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(y)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(y)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(y) &= \frac{1}{2y} \\ f_1(y) &= -\frac{\cos(y)}{2} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $x^n = x^3$ gives

$$x' \frac{1}{x^3} = \frac{1}{2y x^2} - \frac{\cos(y)}{2} \quad (4)$$

Let

$$\begin{aligned} w &= x^{1-n} \\ &= \frac{1}{x^2} \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t y gives

$$w' = -\frac{2}{x^3} x' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(y)}{2} &= \frac{w(y)}{2y} - \frac{\cos(y)}{2} \\ w' &= -\frac{w}{y} + \cos(y) \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(y)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(y) + p(y)w(y) = q(y)$$

Where here

$$\begin{aligned} p(y) &= \frac{1}{y} \\ q(y) &= \cos(y) \end{aligned}$$

Hence the ode is

$$w'(y) + \frac{w(y)}{y} = \cos(y)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{y} dy} \\ &= y\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dy}(\mu w) &= (\mu) (\cos(y)) \\ \frac{d}{dy}(yw) &= (y) (\cos(y)) \\ d(yw) &= (y \cos(y)) dy\end{aligned}$$

Integrating gives

$$\begin{aligned}yw &= \int y \cos(y) dy \\ yw &= y \sin(y) + \cos(y) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = y$ results in

$$w(y) = \frac{y \sin(y) + \cos(y)}{y} + \frac{c_1}{y}$$

which simplifies to

$$w(y) = \frac{y \sin(y) + \cos(y) + c_1}{y}$$

Replacing w in the above by $\frac{1}{x^2}$ using equation (5) gives the final solution.

$$\frac{1}{x^2} = \frac{y \sin(y) + \cos(y) + c_1}{y}$$

Solving for x gives

$$\begin{aligned}x(y) &= \frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1} \\ x(y) &= -\frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1}\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1} \quad (1)$$

$$x = -\frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1} \quad (2)$$

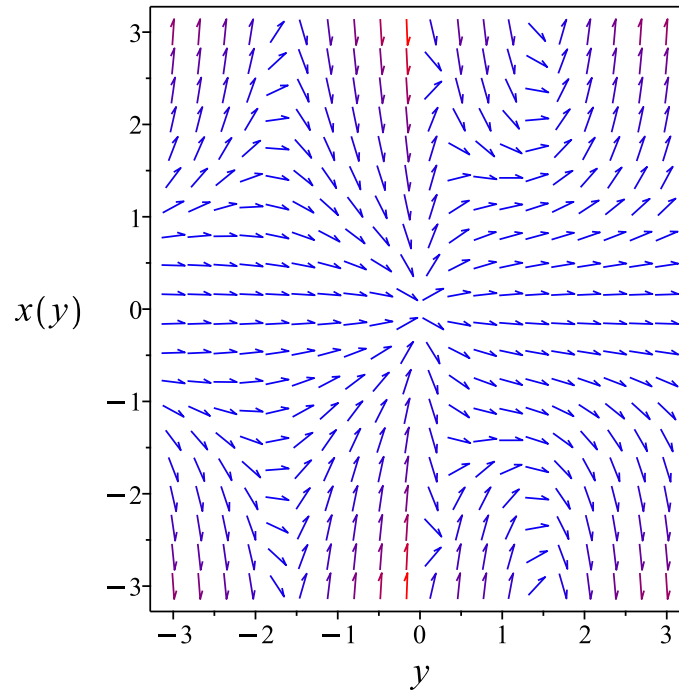


Figure 181: Slope field plot

Verification of solutions

$$x = \frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1}$$

Verified OK.

$$x = -\frac{\sqrt{(y \sin(y) + \cos(y) + c_1) y}}{y \sin(y) + \cos(y) + c_1}$$

Verified OK.

4.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y, x) dy + N(y, x) dx = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (2y) dx &= -(x^2 \cos(y) y - 1) x) dy \\ ((x^2 \cos(y) y - 1) x) dy + (2y) dx &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(y, x) &= (x^2 \cos(y) y - 1) x \\ N(y, x) &= 2y \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x} ((x^2 \cos(y) y - 1) x) \\ &= 3x^2 \cos(y) y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y} (2y) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) \\ &= \frac{1}{2y} ((3x^2 \cos(y) y - 1) - (2)) \\ &= \frac{3x^2 \cos(y) y - 3}{2y}\end{aligned}$$

Since A depends on x , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M} \left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x} \right) \\ &= \frac{1}{x(x^2 \cos(y) y - 1)} ((2) - (3x^2 \cos(y) y - 1)) \\ &= -\frac{3}{x}\end{aligned}$$

Since B does not depend on y , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned}\mu &= e^{\int B dx} \\ &= e^{\int -\frac{3}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^3} ((x^2 \cos(y) y - 1) x) \\ &= \frac{x^2 \cos(y) y - 1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^3} (2y) \\ &= \frac{2y}{x^3}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dx}{dy} &= 0 \\ \left(\frac{x^2 \cos(y) y - 1}{x^2} \right) + \left(\frac{2y}{x^3} \right) \frac{dx}{dy} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$\frac{\partial \phi}{\partial y} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial x} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y} dy &= \int \overline{M} dy \\ \int \frac{\partial \phi}{\partial y} dy &= \int \frac{x^2 \cos(y) y - 1}{x^2} dy \\ \phi &= \frac{x^2 \sin(y) y + \cos(y) x^2 - y}{x^2} + f(x)\end{aligned} \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x . Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= -\frac{2(x^2 \sin(y) y + \cos(y) x^2 - y)}{x^3} + \frac{2 \sin(y) xy + 2 \cos(y) x}{x^2} + f'(x) \quad (4) \\ &= \frac{2y}{x^3} + f'(x)\end{aligned}$$

But equation (2) says that $\frac{\partial\phi}{\partial x} = \frac{2y}{x^3}$. Therefore equation (4) becomes

$$\frac{2y}{x^3} = \frac{2y}{x^3} + f'(x) \quad (5)$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 \sin(y) y + \cos(y) x^2 - y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 \sin(y) y + \cos(y) x^2 - y}{x^2}$$

Summary

The solution(s) found are the following

$$\frac{x^2 y \sin(y) + \cos(y) x^2 - y}{x^2} = c_1 \quad (1)$$

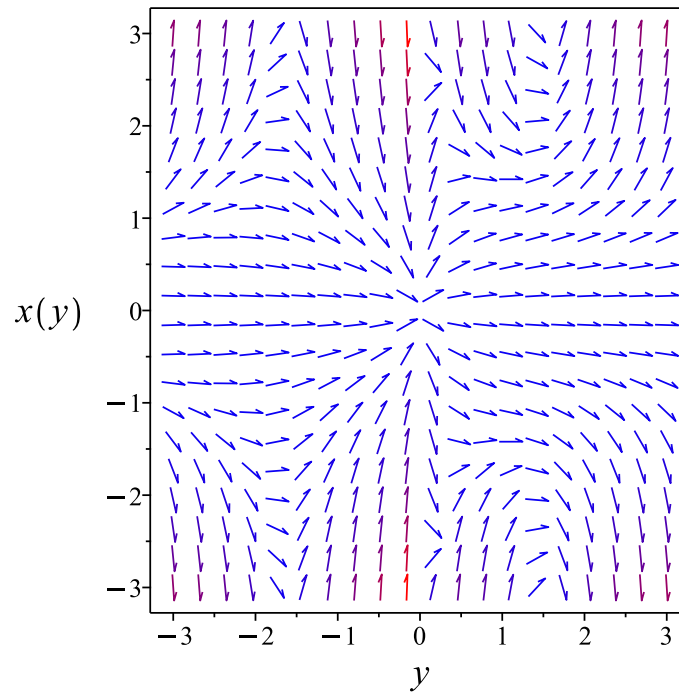


Figure 182: Slope field plot

Verification of solutions

$$\frac{x^2 y \sin(y) + \cos(y) x^2 - y}{x^2} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve(2*diff(x(y),y)-x(y)/y+x(y)^3*cos(y)=0,x(y), singsol=all)
```

$$x(y) = \frac{\sqrt{(\cos(y) + y \sin(y) + c_1) y}}{\cos(y) + y \sin(y) + c_1}$$
$$x(y) = -\frac{\sqrt{(\cos(y) + y \sin(y) + c_1) y}}{\cos(y) + y \sin(y) + c_1}$$

✓ Solution by Mathematica

Time used: 0.261 (sec). Leaf size: 53

```
DSolve[2*x'[y]-x[y]/y+x[y]^3*Cos[y]==0,x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \rightarrow -\frac{\sqrt{y}}{\sqrt{y \sin(y) + \cos(y) + c_1}}$$
$$x(y) \rightarrow \frac{\sqrt{y}}{\sqrt{y \sin(y) + \cos(y) + c_1}}$$
$$x(y) \rightarrow 0$$

4.14 problem 19 (p)

4.14.1 Solving as linear ode	855
4.14.2 Solving as first order ode lie symmetry lookup ode	857
4.14.3 Solving as exact ode	861
4.14.4 Maple step by step solution	866

Internal problem ID [5311]

Internal file name [OUTPUT/4802_Friday_February_02_2024_05_12_14_AM_12081669/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (p).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$xy' - y(1 - x \tan(x)) = \cos(x) x^2$$

4.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1 - x \tan(x)}{x}$$

$$q(x) = \cos(x) x$$

Hence the ode is

$$y' - \frac{(1 - x \tan(x)) y}{x} = \cos(x) x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1-x \tan(x)}{x} dx} \\ &= e^{-\ln(\cos(x))-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{1}{\cos(x)x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu) (\cos(x)x) \\ \frac{d}{dx} \left(\frac{y}{\cos(x)x} \right) &= \left(\frac{1}{\cos(x)x} \right) (\cos(x)x) \\ d \left(\frac{y}{\cos(x)x} \right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{\cos(x)x} &= \int dx \\ \frac{y}{\cos(x)x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\cos(x)x}$ results in

$$y = \cos(x)x^2 + \cos(x)c_1x$$

which simplifies to

$$y = \cos(x)x(x + c_1)$$

Summary

The solution(s) found are the following

$$y = \cos(x)x(x + c_1) \tag{1}$$

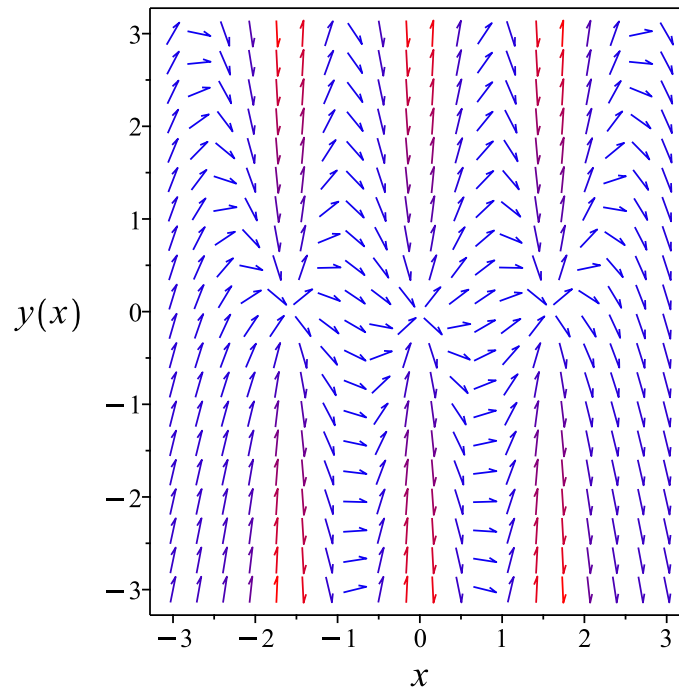


Figure 183: Slope field plot

Verification of solutions

$$y = \cos(x) x(x + c_1)$$

Verified OK.

4.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y \tan(x) x - \cos(x) x^2 - y}{x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 109: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= e^{\ln(\cos(x))+\ln(x)}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\ln(\cos(x)) + \ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x) x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y \tan(x) x - \cos(x) x^2 - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y \sec(x) (x \tan(x) - 1)}{x^2} \\ S_y &= \frac{\sec(x)}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\cos(x)x} = x + c_1$$

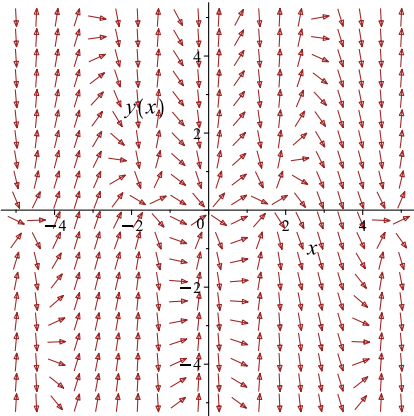
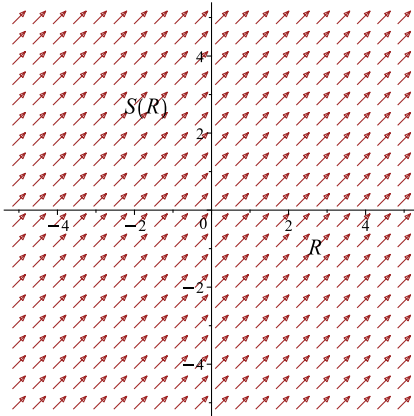
Which simplifies to

$$\frac{y}{\cos(x)x} = x + c_1$$

Which gives

$$y = \cos(x)x^2 + \cos(x)c_1x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y \tan(x)x - \cos(x)x^2 - y}{x}$ 	$R = x$ $S = \frac{y}{\cos(x)x}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \cos(x)x^2 + \cos(x)c_1x \tag{1}$$

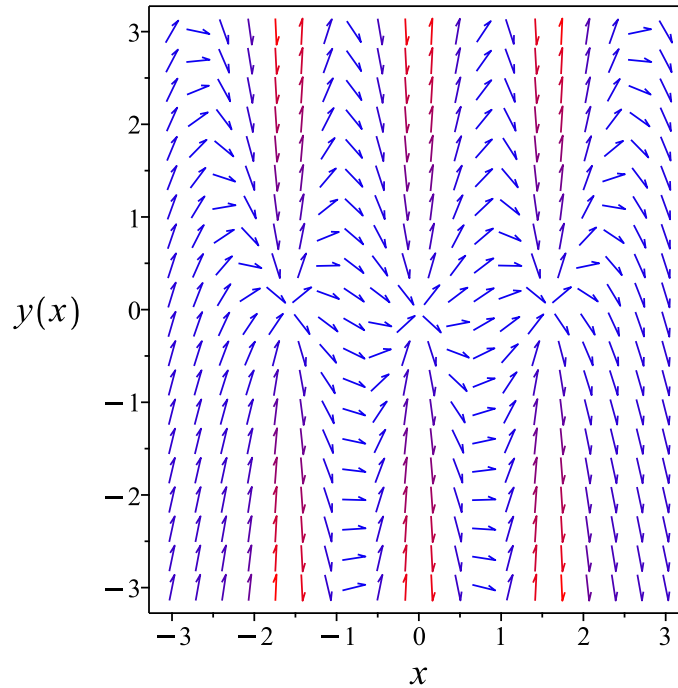


Figure 184: Slope field plot

Verification of solutions

$$y = \cos(x) x^2 + \cos(x) c_1 x$$

Verified OK.

4.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) dy &= (y(1 - x \tan(x)) + \cos(x) x^2) dx \\ (-y(1 - x \tan(x)) - \cos(x) x^2) dx &+ (x) dy = 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y(1 - x \tan(x)) - \cos(x) x^2 \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-y(1 - x \tan(x)) - \cos(x) x^2) \\ &= x \tan(x) - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((x \tan(x) - 1) - (1)) \\ &= \frac{x \tan(x) - 2}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A dx} \\ &= e^{\int \frac{x \tan(x) - 2}{x} dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(x)) - 2 \ln(x)} \\ &= \frac{1}{\cos(x) x^2} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \frac{1}{\cos(x) x^2} (-y(1 - x \tan(x)) - \cos(x) x^2) \\ &= \frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= \frac{1}{\cos(x) x^2} (x) \\ &= \frac{\sec(x)}{x} \end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} \right) + \left(\frac{\sec(x)}{x} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} dx \\ \phi &= \int^x \frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} dx + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2 e^{ix}}{x (e^{2ix} + 1)} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sec(x)}{x}$. Therefore equation (4) becomes

$$\frac{\sec(x)}{x} = \frac{2 e^{ix}}{x (e^{2ix} + 1)} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x \frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} dx + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x \frac{y \sec(a) (a \tan(a) - 1) - a^2}{a^2} da$$

Summary

The solution(s) found are the following

$$\int^x \frac{y \sec(a) (a \tan(a) - 1) - a^2}{a^2} da = c_1 \quad (1)$$

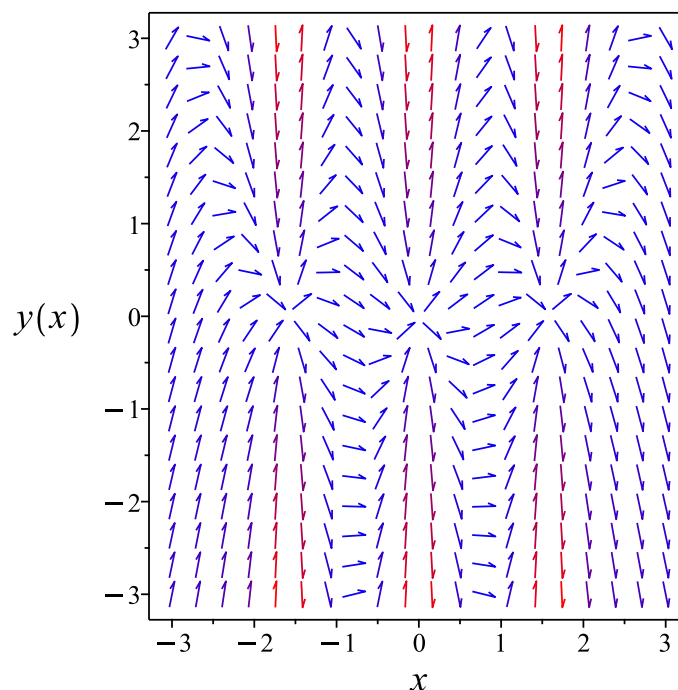


Figure 185: Slope field plot

Verification of solutions

$$\int^x \frac{y \sec(a) (a \tan(a) - 1) - a^2}{a^2} da = c_1$$

Verified OK.

4.14.4 Maple step by step solution

Let's solve

$$xy' - y(1 - x \tan(x)) = \cos(x) x^2$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = -\frac{(x \tan(x)-1)y}{x} + \cos(x) x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' + \frac{(x \tan(x)-1)y}{x} = \cos(x) x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' + \frac{(x \tan(x)-1)y}{x} \right) = \mu(x) \cos(x) x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) \left(y' + \frac{(x \tan(x)-1)y}{x} \right) = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)(x \tan(x)-1)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{\cos(x)x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int \mu(x) \cos(x) x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int \mu(x) \cos(x) x dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \cos(x) x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{\cos(x)x}$

$$y = \cos(x) x \left(\int 1 dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = \cos(x) x(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)=y(x)*(1-x*tan(x))+x^2*cos(x),y(x), singsol=all)
```

$$y(x) = (x + c_1) \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 13

```
DSolve[x*y'[x]==y[x]*(1-x*Tan[x])+x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x(x + c_1) \cos(x)$$

4.15 problem 19 (q)

4.15.1 Solving as first order ode lie symmetry calculated ode 868

4.15.2 Solving as exact ode 875

Internal problem ID [5312]

Internal file name [OUTPUT/4803_Friday_February_02_2024_05_12_22_AM_50983171/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (q).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, ` _with_symmetry_[F(x)*G(y),0]`]]
```

$$y^2 - (xy + 2y + y^3) y' = -2$$

4.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2 + 2}{y(y^2 + x + 2)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + x y a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + x y b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & 2xb_4 + yb_5 + b_2 + \frac{(y^2 + 2)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{y(y^2 + x + 2)} \\ & - \frac{(y^2 + 2)^2(xa_5 + 2ya_6 + a_3)}{y^2(y^2 + x + 2)^2} \\ & + \frac{(y^2 + 2)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{y(y^2 + x + 2)^2} - \left(\frac{2}{y^2 + x + 2} \right. \\ & \left. - \frac{y^2 + 2}{y^2(y^2 + x + 2)} - \frac{2(y^2 + 2)}{(y^2 + x + 2)^2} \right) (x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\begin{aligned} & \underline{2x y^6 b_4 + y^7 b_5 + 5x^2 y^4 b_4 - 2x y^5 a_4 + 4x y^5 b_5 - y^6 a_5 + y^6 b_2 + 3y^6 b_6 + x^3 y^2 b_4 - x^2 y^3 a_4 + x^2 y^3 b_5 - x y^4 a_5} \\ & = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & 2x y^6 b_4 + y^7 b_5 + 5x^2 y^4 b_4 - 2x y^5 a_4 + 4x y^5 b_5 - y^6 a_5 + y^6 b_2 + 3y^6 b_6 \\ & + x^3 y^2 b_4 - x^2 y^3 a_4 + x^2 y^3 b_5 - x y^4 a_5 + 3x y^4 b_2 + 8x y^4 b_4 + x y^4 b_6 - y^5 a_2 \\ & - y^5 a_6 + 2y^5 b_3 + 4y^5 b_5 + 12x^2 y^2 b_4 - 8x y^3 a_4 + 12x y^3 b_5 - 4y^4 a_5 + y^4 b_1 \\ & + 4y^4 b_2 + 12y^4 b_6 + 2x^3 b_4 - 2x^2 y a_4 + 4x^2 y b_5 - 4x y^2 a_5 - x y^2 b_1 + 8x y^2 b_2 \\ & + 8x b_4 y^2 + 6x y^2 b_6 + y^3 a_1 - 4y^3 a_2 - 6y^3 a_6 + 8y^3 b_3 + 4y^3 b_5 + 2x^2 b_2 + 4x^2 b_4 \\ & - 8x y a_4 + 4x y b_3 + 8x y b_5 - 2y^2 a_3 - 4y^2 a_5 + 4y^2 b_1 + 4b_2 y^2 + 12y^2 b_6 \\ & - 4x a_5 + 2x b_1 + 4x b_2 + 2y a_1 - 4y a_2 - 8y a_6 + 8y b_3 - 4a_3 + 4b_1 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 2b_4v_1v_2^6 + b_5v_2^7 - 2a_4v_1v_2^5 - a_5v_2^6 + b_2v_2^6 + 5b_4v_1^2v_2^4 + 4b_5v_1v_2^5 + 3b_6v_2^6 \\
& - a_2v_2^5 - a_4v_1^2v_2^3 - a_5v_1v_2^4 - a_6v_2^5 + 3b_2v_1v_2^4 + 2b_3v_2^5 + b_4v_1^3v_2^2 \\
& + 8b_4v_1v_2^4 + b_5v_1^2v_2^3 + 4b_5v_2^5 + b_6v_1v_2^4 - 8a_4v_1v_2^3 - 4a_5v_2^4 + b_1v_2^4 \\
& + 4b_2v_2^4 + 12b_4v_1^2v_2^2 + 12b_5v_1v_2^3 + 12b_6v_2^4 + a_1v_2^3 - 4a_2v_2^3 - 2a_4v_1^2v_2 \\
& - 4a_5v_1v_2^2 - 6a_6v_2^3 - b_1v_1v_2^2 + 8b_2v_1v_2^2 + 8b_3v_2^3 + 2b_4v_1^3 + 8b_4v_1v_2^2 \\
& + 4b_5v_1^2v_2 + 4b_5v_2^3 + 6b_6v_1v_2^2 - 2a_3v_2^2 - 8a_4v_1v_2 - 4a_5v_2^2 + 4b_1v_2^2 \\
& + 2b_2v_1^2 + 4b_2v_2^2 + 4b_3v_1v_2 + 4b_4v_1^2 + 8b_5v_1v_2 + 12b_6v_2^2 + 2a_1v_2 \\
& - 4a_2v_2 - 4a_5v_1 - 8a_6v_2 + 2b_1v_1 + 4b_2v_1 + 8b_3v_2 - 4a_3 + 4b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
& b_4v_1^3v_2^2 + 2b_4v_1^3 + 5b_4v_1^2v_2^4 + (-a_4 + b_5)v_1^2v_2^3 + 12b_4v_1^2v_2^2 \\
& + (-2a_4 + 4b_5)v_1^2v_2 + (2b_2 + 4b_4)v_1^2 + 2b_4v_1v_2^6 + (-2a_4 + 4b_5)v_1v_2^5 \\
& + (-a_5 + 3b_2 + 8b_4 + b_6)v_1v_2^4 + (-8a_4 + 12b_5)v_1v_2^3 \\
& + (-4a_5 - b_1 + 8b_2 + 8b_4 + 6b_6)v_1v_2^2 + (-8a_4 + 4b_3 + 8b_5)v_1v_2 \\
& + (-4a_5 + 2b_1 + 4b_2)v_1 + b_5v_2^7 + (-a_5 + b_2 + 3b_6)v_2^6 \\
& + (-a_2 - a_6 + 2b_3 + 4b_5)v_2^5 + (-4a_5 + b_1 + 4b_2 + 12b_6)v_2^4 \\
& + (a_1 - 4a_2 - 6a_6 + 8b_3 + 4b_5)v_2^3 + (-2a_3 - 4a_5 + 4b_1 + 4b_2 + 12b_6)v_2^2 \\
& + (2a_1 - 4a_2 - 8a_6 + 8b_3)v_2 - 4a_3 + 4b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$b_4 = 0$$

$$b_5 = 0$$

$$2b_4 = 0$$

$$5b_4 = 0$$

$$12b_4 = 0$$

$$-4a_3 + 4b_1 = 0$$

$$-8a_4 + 12b_5 = 0$$

$$-2a_4 + 4b_5 = 0$$

$$-a_4 + b_5 = 0$$

$$2b_2 + 4b_4 = 0$$

$$-8a_4 + 4b_3 + 8b_5 = 0$$

$$-4a_5 + 2b_1 + 4b_2 = 0$$

$$-a_5 + b_2 + 3b_6 = 0$$

$$2a_1 - 4a_2 - 8a_6 + 8b_3 = 0$$

$$-a_2 - a_6 + 2b_3 + 4b_5 = 0$$

$$-4a_5 + b_1 + 4b_2 + 12b_6 = 0$$

$$-a_5 + 3b_2 + 8b_4 + b_6 = 0$$

$$a_1 - 4a_2 - 6a_6 + 8b_3 + 4b_5 = 0$$

$$-2a_3 - 4a_5 + 4b_1 + 4b_2 + 12b_6 = 0$$

$$-4a_5 - b_1 + 8b_2 + 8b_4 + 6b_6 = 0$$

Solving the above equations for the unknowns gives

$$a_1 = 2a_6$$

$$a_2 = -a_6$$

$$a_3 = 0$$

$$a_4 = 0$$

$$a_5 = 0$$

$$a_6 = a_6$$

$$b_1 = 0$$

$$b_2 = 0$$

$$b_3 = 0$$

$$b_4 = 0$$

$$b_5 = 0$$

$$b_6 = 0$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y^2 - x + 2$$

$$\eta = 0$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y) \xi \\ &= 0 - \left(\frac{y^2 + 2}{y(y^2 + x + 2)} \right) (y^2 - x + 2) \\ &= \frac{-y^4 + y^2x - 4y^2 + 2x - 4}{y^3 + xy + 2y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y^4 + y^2 x - 4y^2 + 2x - 4}{y^3 + xy + 2y}} dy \end{aligned}$$

Which results in

$$S = -\ln(y^2 - x + 2) + \frac{\ln(y^2 + 2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 + 2}{y(y^2 + x + 2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y^2 - x + 2} \\ S_y &= -\frac{2y}{y^2 - x + 2} + \frac{y}{y^2 + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

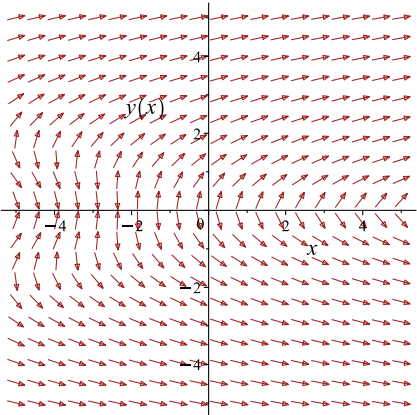
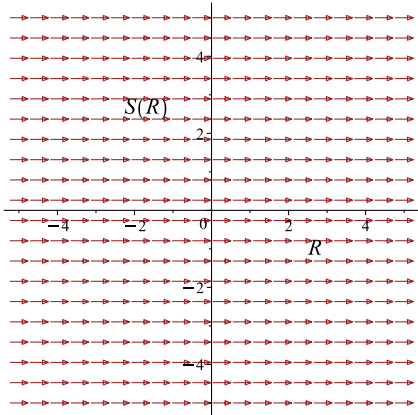
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y^2 - x + 2) + \frac{\ln(y^2 + 2)}{2} = c_1$$

Which simplifies to

$$-\ln(y^2 - x + 2) + \frac{\ln(y^2 + 2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+2}{y(y^2+x+2)}$ 	$R = x$ $S = -\ln(y^2 - x + 2) + \frac{1}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\ln(y^2 - x + 2) + \frac{\ln(y^2 + 2)}{2} = c_1 \quad (1)$$

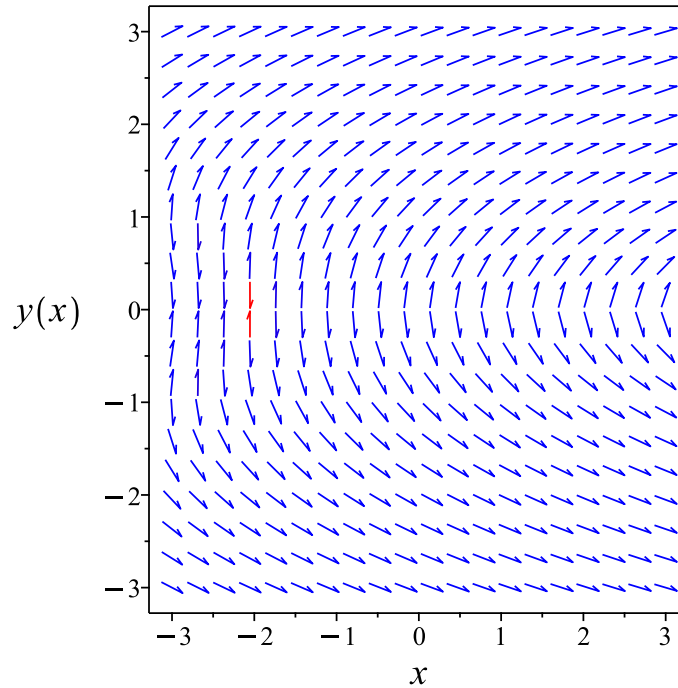


Figure 186: Slope field plot

Verification of solutions

$$-\ln(y^2 - x + 2) + \frac{\ln(y^2 + 2)}{2} = c_1$$

Verified OK.

4.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y^3 - xy - 2y) dy &= (-y^2 - 2) dx \\ (y^2 + 2) dx + (-y^3 - xy - 2y) dy &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y^2 + 2 \\ N(x, y) &= -y^3 - xy - 2y\end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2 + 2) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y^3 - xy - 2y) \\ &= -y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= -\frac{1}{(y^2 + x + 2)y} ((2y) - (-y)) \\ &= -\frac{3}{y^2 + x + 2} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y^2 + 2} ((-y) - (2y)) \\ &= -\frac{3y}{y^2 + 2} \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int -\frac{3y}{y^2+2} \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{3 \ln(y^2+2)}{2}} \\ &= \frac{1}{(y^2 + 2)^{\frac{3}{2}}} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \frac{1}{(y^2 + 2)^{\frac{3}{2}}} (y^2 + 2) \\ &= \frac{1}{\sqrt{y^2 + 2}} \end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{(y^2 + 2)^{\frac{3}{2}}} (-y^3 - xy - 2y) \\ &= \frac{(-y^2 - x - 2)y}{(y^2 + 2)^{\frac{3}{2}}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{1}{\sqrt{y^2 + 2}} \right) + \left(\frac{(-y^2 - x - 2)y}{(y^2 + 2)^{\frac{3}{2}}} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{1}{\sqrt{y^2 + 2}} dx \\ \phi &= \frac{x}{\sqrt{y^2 + 2}} + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{xy}{(y^2 + 2)^{\frac{3}{2}}} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(-y^2 - x - 2)y}{(y^2 + 2)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$\frac{(-y^2 - x - 2)y}{(y^2 + 2)^{\frac{3}{2}}} = -\frac{xy}{(y^2 + 2)^{\frac{3}{2}}} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{\sqrt{y^2 + 2}}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{y}{\sqrt{y^2 + 2}} \right) dy$$

$$f(y) = -\sqrt{y^2 + 2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{\sqrt{y^2 + 2}} - \sqrt{y^2 + 2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{\sqrt{y^2 + 2}} - \sqrt{y^2 + 2}$$

Summary

The solution(s) found are the following

$$\frac{x}{\sqrt{y^2 + 2}} - \sqrt{y^2 + 2} = c_1 \quad (1)$$

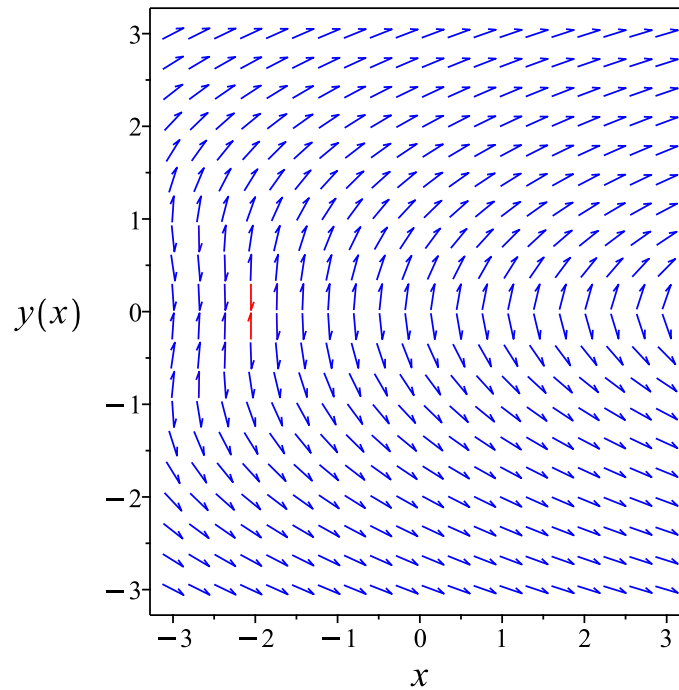


Figure 187: Slope field plot

Verification of solutions

$$\frac{x}{\sqrt{y^2 + 2}} - \sqrt{y^2 + 2} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve((2+y(x)^2)-(x*y(x)+2*y(x)+y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x - y(x)^2 - 2 - \sqrt{y(x)^2 + 2c_1} = 0$$

✓ Solution by Mathematica

Time used: 5.808 (sec). Leaf size: 189

```
DSolve[(2+y[x]^2)-(x*y[x]+2*y[x]+y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{2x - \sqrt{4c_1^2x + c_1^4} - 4 + c_1^2}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{2x - \sqrt{4c_1^2x + c_1^4} - 4 + c_1^2}}{\sqrt{2}}$$

$$y(x) \rightarrow -\frac{\sqrt{2x + \sqrt{4c_1^2x + c_1^4} - 4 + c_1^2}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{2x + \sqrt{4c_1^2x + c_1^4} - 4 + c_1^2}}{\sqrt{2}}$$

$$y(x) \rightarrow -i\sqrt{2}$$

$$y(x) \rightarrow i\sqrt{2}$$

4.16 problem 19 (r)

4.16.1 Solving as first order ode lie symmetry calculated ode 882

4.16.2 Solving as exact ode 889

Internal problem ID [5313]

Internal file name [OUTPUT/4804_Friday_February_02_2024_05_12_26_AM_45050703/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (r).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order , `_with_symmetry_[F(x),G(y)]`]]
```

$$y^2 - (\arctan(y) - x) y' = -1$$

4.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2 + 1}{\arctan(y) - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + x y a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + x y b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & 2xb_4 + yb_5 + b_2 + \frac{(y^2 + 1)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{\arctan(y) - x} \\ & - \frac{(y^2 + 1)^2(xa_5 + 2ya_6 + a_3)}{(\arctan(y) - x)^2} \\ & - \frac{(y^2 + 1)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{(\arctan(y) - x)^2} - \left(\frac{2y}{\arctan(y) - x} \right. \\ & \left. - \frac{1}{(\arctan(y) - x)^2} \right) (x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\begin{aligned} & \frac{-x y^4 a_5 + 2x^3 y b_4 + x^2 y^2 a_4 + x^2 y^2 b_5 + x^2 y b_5 - 2x y^2 a_5 - 2x y b_6 - \arctan(y) y^3 a_5 - 2 \arctan(y) x y b_2 - a_1 - a_3 + b_1}{\arctan(y) - x} \\ & - \frac{2y}{\arctan(y) - x} (x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1) \\ & + \frac{1}{(\arctan(y) - x)^2} (x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1) = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -x y^4 a_5 + 2x^3 y b_4 + x^2 y^2 a_4 + x^2 y^2 b_5 + x^2 y b_5 - 2x y^2 a_5 \\ & - 2x y b_6 - \arctan(y) y^3 a_5 - 2 \arctan(y) x y b_2 - a_1 - a_3 + b_1 \\ & - 2 \arctan(y) x^2 y b_4 - 2 \arctan(y) x y^2 a_4 - \arctan(y) x y^2 b_5 \\ & - 2 \arctan(y) x y b_5 + 2 \arctan(y)^2 x b_4 + \arctan(y)^2 y b_5 \\ & - 4 \arctan(y) x^2 b_4 - 2 \arctan(y) x a_4 + \arctan(y) x b_5 \\ & - \arctan(y) y a_5 + 2 \arctan(y) y b_6 + 2x y b_1 - \arctan(y) y^2 a_2 \\ & - \arctan(y) y^2 b_3 - 2 \arctan(y) x b_2 - 2 \arctan(y) y b_1 - 4y^3 a_6 - x^2 b_5 \\ & - x a_5 - 2y a_6 - 2y^5 a_6 - y^4 a_6 + 2x^3 b_4 + 2x^2 y b_2 + x y^2 b_3 + x^2 a_4 - y^2 a_6 \\ & + x^2 b_4 + y^2 b_6 - y^4 a_3 - y^3 a_3 + x^2 b_2 - y^2 a_1 - 2y^2 a_3 - x b_3 + \arctan(y)^2 b_2 \\ & - \arctan(y) a_2 + \arctan(y) b_3 + x y b_5 - y a_3 + x b_2 + y b_3 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \arctan(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \arctan(y) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -v_1v_2^4a_5 - 2v_2^5a_6 - v_2^4a_3 + v_1^2v_2^2a_4 - 2v_3v_1v_2^2a_4 - v_3v_2^3a_5 - v_2^4a_6 \\ & + 2v_1^3v_2b_4 - 2v_3v_1^2v_2b_4 + v_1^2v_2^2b_5 - v_3v_1v_2^2b_5 - v_3v_2^2a_2 - v_2^3a_3 - 2v_1v_2^2a_5 \\ & - 4v_2^3a_6 + 2v_1^2v_2b_2 - 2v_3v_1v_2b_2 + v_1v_2^2b_3 - v_3v_2^2b_3 + 2v_1^3b_4 - 4v_3v_1^2b_4 \\ & + 2v_3^2v_1b_4 + v_1^2v_2b_5 - 2v_3v_1v_2b_5 + v_3^2v_2b_5 - v_2^2a_1 - 2v_2^2a_3 + v_1^2a_4 \\ & - 2v_3v_1a_4 - v_3v_2a_5 - v_2^2a_6 + 2v_1v_2b_1 - 2v_3v_2b_1 + v_1^2b_2 - 2v_3v_1b_2 + v_3^2b_2 \\ & + v_1^2b_4 - v_1^2b_5 + v_1v_2b_5 + v_3v_1b_5 - 2v_1v_2b_6 + v_2^2b_6 + 2v_3v_2b_6 - v_3a_2 \\ & - v_2a_3 - v_1a_5 - 2v_2a_6 + v_1b_2 - v_1b_3 + v_2b_3 + v_3b_3 - a_1 - a_3 + b_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & -2v_3v_1^2v_2b_4 + (a_4 + b_5)v_1^2v_2^2 + (2b_2 + b_5)v_1^2v_2 + (-2a_5 + b_3)v_1v_2^2 \\ & + (2b_1 + b_5 - 2b_6)v_1v_2 + (-2a_4 - 2b_2 + b_5)v_1v_3 + (-a_2 - b_3)v_2^2v_3 \\ & + (-a_5 - 2b_1 + 2b_6)v_2v_3 - a_1 - a_3 + b_1 + (-2a_4 - b_5)v_1v_2^2v_3 \\ & + (-2b_2 - 2b_5)v_1v_2v_3 + (b_3 - a_2)v_3 + (-a_5 + b_2 - b_3)v_1 \\ & + (-a_3 - a_6)v_2^4 + (-a_3 - 4a_6)v_2^3 + (-a_1 - 2a_3 - a_6 + b_6)v_2^2 \\ & + (-a_3 - 2a_6 + b_3)v_2 + (a_4 + b_2 + b_4 - b_5)v_1^2 - 2v_2^5a_6 + 2v_1^3b_4 + v_3^2b_2 \\ & - v_1v_2^4a_5 + 2v_1^3v_2b_4 - v_3v_2^3a_5 + 2v_3^2v_1b_4 + v_3^2v_2b_5 - 4v_3v_1^2b_4 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ b_5 &= 0 \\ -a_5 &= 0 \\ -2a_6 &= 0 \\ -4b_4 &= 0 \\ -2b_4 &= 0 \\ 2b_4 &= 0 \\ -a_2 - b_3 &= 0 \\ -a_3 - 4a_6 &= 0 \\ -a_3 - a_6 &= 0 \\ -2a_4 - b_5 &= 0 \\ a_4 + b_5 &= 0 \\ -2a_5 + b_3 &= 0 \\ -2b_2 - 2b_5 &= 0 \\ 2b_2 + b_5 &= 0 \\ b_3 - a_2 &= 0 \\ -a_1 - a_3 + b_1 &= 0 \\ -a_3 - 2a_6 + b_3 &= 0 \\ -2a_4 - 2b_2 + b_5 &= 0 \\ -a_5 - 2b_1 + 2b_6 &= 0 \\ -a_5 + b_2 - b_3 &= 0 \\ 2b_1 + b_5 - 2b_6 &= 0 \\ -a_1 - 2a_3 - a_6 + b_6 &= 0 \\ a_4 + b_2 + b_4 - b_5 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= b_6 \\
 a_2 &= 0 \\
 a_3 &= 0 \\
 a_4 &= 0 \\
 a_5 &= 0 \\
 a_6 &= 0 \\
 b_1 &= b_6 \\
 b_2 &= 0 \\
 b_3 &= 0 \\
 b_4 &= 0 \\
 b_5 &= 0 \\
 b_6 &= b_6
 \end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= 1 \\
 \eta &= y^2 + 1
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= y^2 + 1 - \left(\frac{y^2 + 1}{\arctan(y) - x} \right) (1) \\
 &= \frac{y^2 \arctan(y) - y^2 x - y^2 + \arctan(y) - x - 1}{\arctan(y) - x} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2 \arctan(y) - y^2 x - y^2 + \arctan(y) - x - 1}{\arctan(y) - x}} dy \end{aligned}$$

Which results in

$$S = \arctan(y) + \ln(-1 + \arctan(y) - x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 + 1}{\arctan(y) - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{1 - \arctan(y) + x} \\ S_y &= \frac{1 + \frac{1}{-1 + \arctan(y) - x}}{y^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(y) + \ln(-1 + \arctan(y) - x) = c_1$$

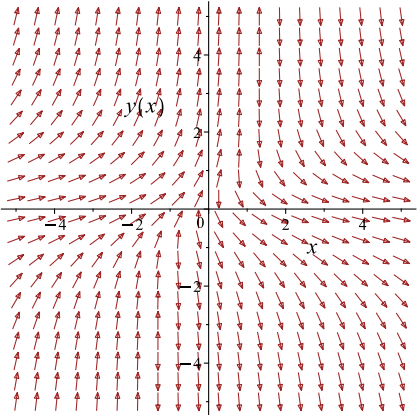
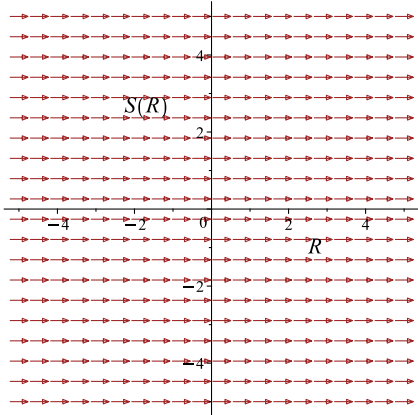
Which simplifies to

$$\arctan(y) + \ln(-1 + \arctan(y) - x) = c_1$$

Which gives

$$y = \tan(\text{LambertW}(e^{-1+c_1-x}) + 1 + x)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{\arctan(y)-x}$ 	$R = x$ $S = \arctan(y) + \ln(-1 + \arctan(y) - x)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \tan(\text{LambertW}(e^{-1+c_1-x}) + 1 + x) \quad (1)$$

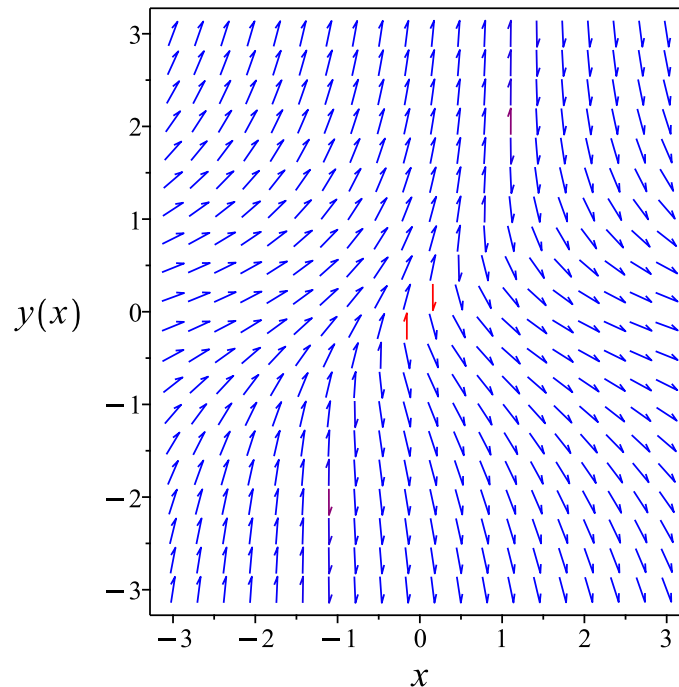


Figure 188: Slope field plot

Verification of solutions

$$y = \tan(\text{LambertW}(e^{-1+c_1-x}) + 1 + x)$$

Verified OK.

4.16.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (-\arctan(y) + x) dy &= (-y^2 - 1) dx \\ (y^2 + 1) dx + (-\arctan(y) + x) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y^2 + 1 \\ N(x, y) &= -\arctan(y) + x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2 + 1) \\ &= 2y \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\arctan(y) + x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{-\arctan(y) + x} ((2y) - (1)) \\ &= \frac{-2y + 1}{\arctan(y) - x}\end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y^2 + 1} ((1) - (2y)) \\ &= \frac{-2y + 1}{y^2 + 1}\end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned}\mu &= e^{\int B \, dy} \\ &= e^{\int \frac{-2y+1}{y^2+1} \, dy}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(y^2+1)+\arctan(y)} \\ &= \frac{e^{\arctan(y)}}{y^2 + 1}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{e^{\arctan(y)}}{y^2 + 1} (y^2 + 1) \\ &= e^{\arctan(y)}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{e^{\arctan(y)}}{y^2 + 1} (-\arctan(y) + x) \\ &= \frac{(-\arctan(y) + x) e^{\arctan(y)}}{y^2 + 1}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (e^{\arctan(y)}) + \left(\frac{(-\arctan(y) + x) e^{\arctan(y)}}{y^2 + 1} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int e^{\arctan(y)} dx \\ \phi &= e^{\arctan(y)} x + f(y)\end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{e^{\arctan(y)} x}{y^2 + 1} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(-\arctan(y)+x)e^{\arctan(y)}}{y^2+1}$. Therefore equation (4) becomes

$$\frac{(-\arctan(y) + x) e^{\arctan(y)}}{y^2 + 1} = \frac{e^{\arctan(y)} x}{y^2 + 1} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{e^{\arctan(y)} \arctan(y)}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int \left(-\frac{e^{\arctan(y)} \arctan(y)}{y^2 + 1} \right) dy$$

$$f(y) = -e^{\arctan(y)} \arctan(y) + e^{\arctan(y)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{\arctan(y)} x - e^{\arctan(y)} \arctan(y) + e^{\arctan(y)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{\arctan(y)} x - e^{\arctan(y)} \arctan(y) + e^{\arctan(y)}$$

The solution becomes

$$y = \tan(\text{LambertW}(-c_1 e^{-1-x}) + 1 + x)$$

Summary

The solution(s) found are the following

$$y = \tan(\text{LambertW}(-c_1 e^{-1-x}) + 1 + x) \quad (1)$$

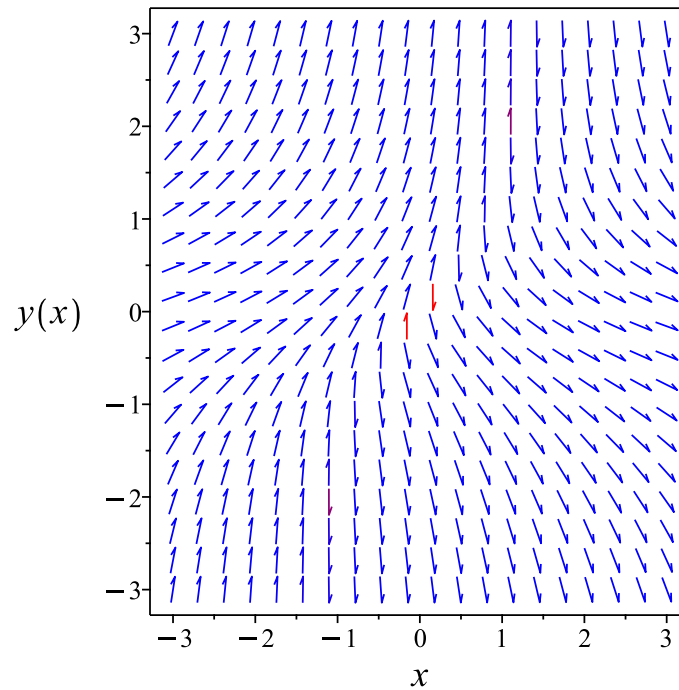


Figure 189: Slope field plot

Verification of solutions

$$y = \tan(\text{LambertW}(-c_1 e^{-1-x}) + 1 + x)$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve((1+y(x)^2)=(arctan(y(x))-x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan(\text{LambertW}(-c_1 e^{-x-1}) + x + 1)$$

✓ Solution by Mathematica

Time used: 60.157 (sec). Leaf size: 21

```
DSolve[(1+y[x]^2)==(ArcTan[y[x]]-x)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \tan(W(c_1(-e^{-x-1})) + x + 1)$$

4.17 problem 19 (s)

4.17.1 Solving as first order ode lie symmetry lookup ode	896
4.17.2 Solving as bernoulli ode	900

Internal problem ID [5314]

Internal file name [OUTPUT/4805_Friday_February_02_2024_05_12_29_AM_93928993/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (s).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**bernoulli**", "**first_order_ode_lie_symmetry_lookup**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$2y^5x - y + 2xy' = 0$$

4.17.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(2y^4x - 1)}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{y^5}{x^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^5}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{4y^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(2y^4x - 1)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{2y^4} \\ S_y &= \frac{x^2}{y^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^2 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{R^3}{3} + c_1 \quad (4)$$

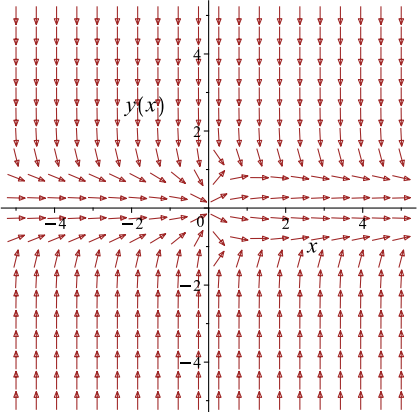
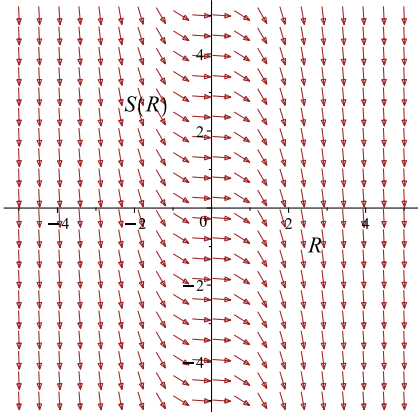
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

Which simplifies to

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2y^4x-1)}{2x}$ 	$R = x$ $S = -\frac{x^2}{4y^4}$	$\frac{dS}{dR} = -R^2$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1 \quad (1)$$

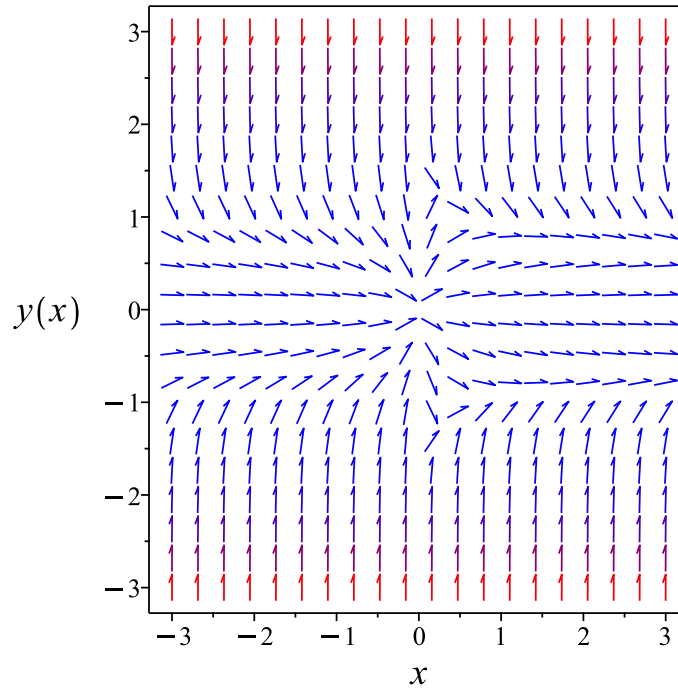


Figure 190: Slope field plot

Verification of solutions

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

Verified OK.

4.17.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(2y^4x - 1)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - y^5 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{2x} \\ f_1(x) &= -1 \\ n &= 5 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^5$ gives

$$y' \frac{1}{y^5} = \frac{1}{2x y^4} - 1 \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^4} \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{4}{y^5} y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{4} &= \frac{w(x)}{2x} - 1 \\ w' &= -\frac{2w}{x} + 4 \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= 4 \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 4$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu) (4) \\ \frac{d}{dx}(x^2 w) &= (x^2) (4) \\ d(x^2 w) &= (4x^2) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int 4x^2 dx \\ x^2 w &= \frac{4x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{4x}{3} + \frac{c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y^4}$ using equation (5) gives the final solution.

$$\frac{1}{y^4} = \frac{4x}{3} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \\ y(x) &= -\frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \\ y(x) &= \frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \\ y(x) &= -\frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{3^{\frac{1}{4}} \sqrt{x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \quad (1)$$

$$y = -\frac{3^{\frac{1}{4}} \sqrt{x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \quad (2)$$

$$y = \frac{3^{\frac{1}{4}} \sqrt{-x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \quad (3)$$

$$y = -\frac{3^{\frac{1}{4}} \sqrt{-x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \quad (4)$$

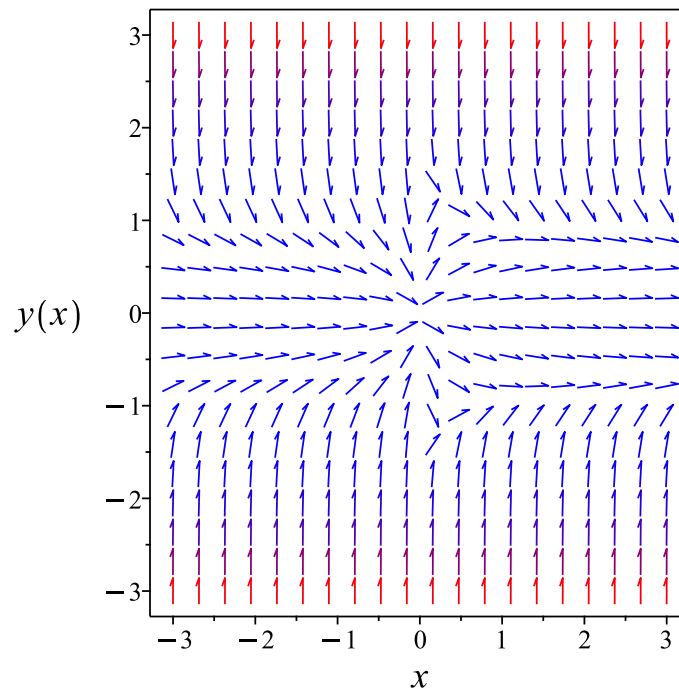


Figure 191: Slope field plot

Verification of solutions

$$y = \frac{3^{\frac{1}{4}} \sqrt{x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = -\frac{3^{\frac{1}{4}} \sqrt{x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = \frac{3^{\frac{1}{4}} \sqrt{-x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = -\frac{3^{\frac{1}{4}} \sqrt{-x \sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 137

```
dsolve((2*x*y(x)^5-y(x))+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{1}{4}} \sqrt{-\sqrt{4x^3 + 3c_1}} x}{\sqrt{4x^3 + 3c_1}}$$
$$y(x) = \frac{3^{\frac{1}{4}} \sqrt{\sqrt{4x^3 + 3c_1}} x}{\sqrt{4x^3 + 3c_1}}$$
$$y(x) = -\frac{3^{\frac{1}{4}} \sqrt{-\sqrt{4x^3 + 3c_1}} x}{\sqrt{4x^3 + 3c_1}}$$
$$y(x) = -\frac{3^{\frac{1}{4}} \sqrt{\sqrt{4x^3 + 3c_1}} x}{\sqrt{4x^3 + 3c_1}}$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 109

```
DSolve[(2*x*y[x]^5-y[x])+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3} + c_1}}$$
$$y(x) \rightarrow -\frac{i\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3} + c_1}}$$
$$y(x) \rightarrow \frac{i\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3} + c_1}}$$
$$y(x) \rightarrow \frac{\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3} + c_1}}$$
$$y(x) \rightarrow 0$$

4.18 problem 19 (t)

4.18.1 Solving as exact ode 906

Internal problem ID [5315]

Internal file name [OUTPUT/4806_Friday_February_02_2024_05_12_30_AM_81058194/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 19 (t).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

```
[[_1st_order , ` _with_symmetry_[F(x)*G(y),0] `]]
```

$$\sin(y) - (2y \cos(y) - x(\sec(y) + \tan(y)))y' = -1$$

4.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-2y \cos(y) + x(\sec(y) + \tan(y))) dy &= (-1 - \sin(y)) dx \\ (1 + \sin(y)) dx + (-2y \cos(y) + x(\sec(y) + \tan(y))) dy &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 1 + \sin(y) \\ N(x, y) &= -2y \cos(y) + x(\sec(y) + \tan(y)) \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 + \sin(y)) \\ &= \cos(y) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2y \cos(y) + x(\sec(y) + \tan(y))) \\ &= \sec(y) + \tan(y) \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{\cos(y)}{-2y \cos(y)^2 + \sin(y)x + x} ((\cos(y)) - (\sec(y) + \tan(y))) \\ &= -\frac{\sin(y)}{2y \sin(y) + x - 2y} \end{aligned}$$

Since A depends on y , it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{1 + \sin(y)} ((\sec(y) + \tan(y)) - (\cos(y))) \\ &= \tan(y) \end{aligned}$$

Since B does not depend on x , it can be used to obtain an integrating factor. Let the integrating factor be μ . Then

$$\begin{aligned} \mu &= e^{\int B \, dy} \\ &= e^{\int \tan(y) \, dy} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(\cos(y))} \\ &= \sec(y) \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} so not to confuse them with the original M and N .

$$\begin{aligned} \bar{M} &= \mu M \\ &= \sec(y) (1 + \sin(y)) \\ &= \sec(y) + \tan(y) \end{aligned}$$

And

$$\begin{aligned} \bar{N} &= \mu N \\ &= \sec(y) (-2y \cos(y) + x(\sec(y) + \tan(y))) \\ &= \frac{-2y \sin(y) - x + 2y}{\sin(y) - 1} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ (\sec(y) + \tan(y)) + \left(\frac{-2y \sin(y) - x + 2y}{\sin(y) - 1} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \quad (1)$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \quad (2)$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \sec(y) + \tan(y) dx \\ \phi &= x(\sec(y) + \tan(y)) + f(y) \end{aligned} \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= x(\sec(y) \tan(y) + 1 + \tan(y)^2) + f'(y) \\ &= -\frac{x}{\sin(y) - 1} + f'(y) \end{aligned} \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-2y \sin(y) - x + 2y}{\sin(y) - 1}$. Therefore equation (4) becomes

$$\frac{-2y \sin(y) - x + 2y}{\sin(y) - 1} = -\frac{x}{\sin(y) - 1} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) dy = \int (-2y) dy$$

$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(\sec(y) + \tan(y)) - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(\sec(y) + \tan(y)) - y^2$$

Summary

The solution(s) found are the following

$$x(\sec(y) + \tan(y)) - y^2 = c_1 \tag{1}$$

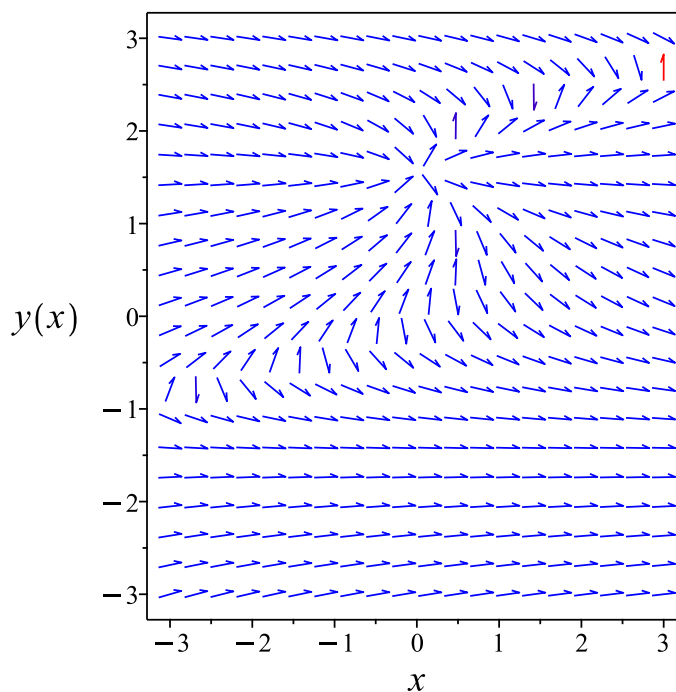


Figure 192: Slope field plot

Verification of solutions

$$x(\sec(y) + \tan(y)) - y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
<- 1st order linear successful  
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 25

```
dsolve((1+sin(y(x)))=(2*y(x)*cos(y(x))-x*(sec(y(x))+tan(y(x))))*diff(y(x),x),y(x), singsol=
```

$$x + \frac{-y(x)^2 - c_1}{\sec(y(x)) + \tan(y(x))} = 0$$

✓ Solution by Mathematica

Time used: 1.489 (sec). Leaf size: 66

```
DSolve[(1+Sin[y[x]])==(2*y[x]*Cos[y[x]]-x*(Sec[y[x]]+Tan[y[x]]))*y'[x],y[x],x,IncludeSingular
```

$$y(x) \rightarrow -\frac{\pi}{2}$$

$$y(x) \rightarrow \frac{3\pi}{2}$$

$$\text{Solve}\left[x = y(x)^2 e^{-2\arctanh\left(\tan\left(\frac{y(x)}{2}\right)\right)} + c_1 e^{-2\arctanh\left(\tan\left(\frac{y(x)}{2}\right)\right)}, y(x)\right]$$

$$y(x) \rightarrow -\frac{\pi}{2}$$

4.19 problem 22 (a)

4.19.1 Existence and uniqueness analysis	912
4.19.2 Solving as linear ode	913
4.19.3 Solving as first order ode lie symmetry lookup ode	915
4.19.4 Solving as exact ode	919
4.19.5 Maple step by step solution	924

Internal problem ID [5316]

Internal file name [OUTPUT/4807_Friday_February_02_2024_05_13_49_AM_31079458/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 22 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

[_linear]

$$xy' - 2y = x^3e^x$$

With initial conditions

$$[y(1) = 0]$$

4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = x^2e^x$$

Hence the ode is

$$y' - \frac{2y}{x} = x^2 e^x$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2 e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(x^2 e^x) \\ \frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)(x^2 e^x) \\ d\left(\frac{y}{x^2}\right) &= e^x dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int e^x dx \\ \frac{y}{x^2} &= e^x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 e^x + c_1 x^2$$

which simplifies to

$$y = x^2(e^x + c_1)$$

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e + c_1$$

$$c_1 = -e$$

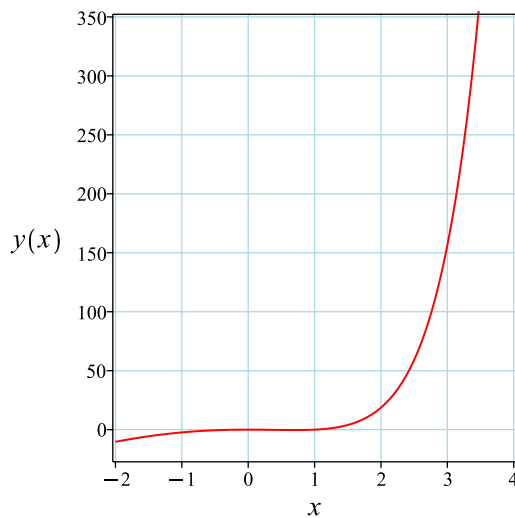
Substituting c_1 found above in the general solution gives

$$y = x^2(e^x - e)$$

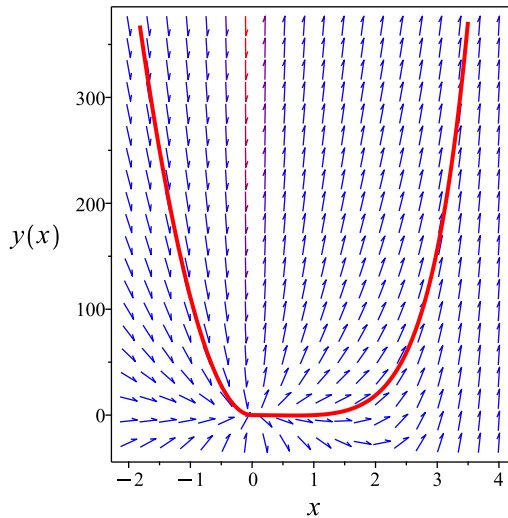
Summary

The solution(s) found are the following

$$y = x^2(e^x - e) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(e^x - e)$$

Verified OK.

4.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y + x^3 e^x}{x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 114: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x^2\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y}\tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y + x^3 e^x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\R_y &= 0 \\S_x &= -\frac{2y}{x^3} \\S_y &= \frac{1}{x^2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = e^x + c_1$$

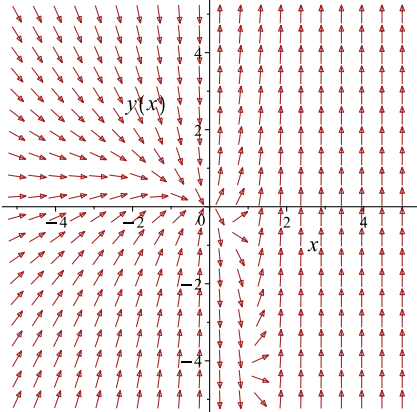
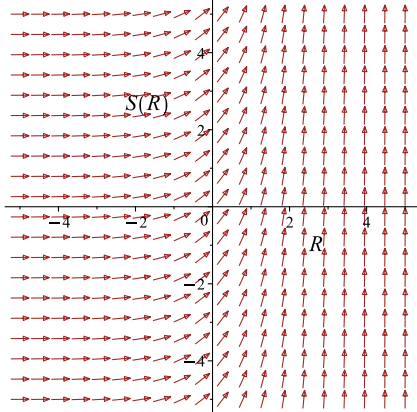
Which simplifies to

$$\frac{y}{x^2} = e^x + c_1$$

Which gives

$$y = x^2(e^x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+x^3e^x}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = e^R$ 

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e + c_1$$

$$c_1 = -e$$

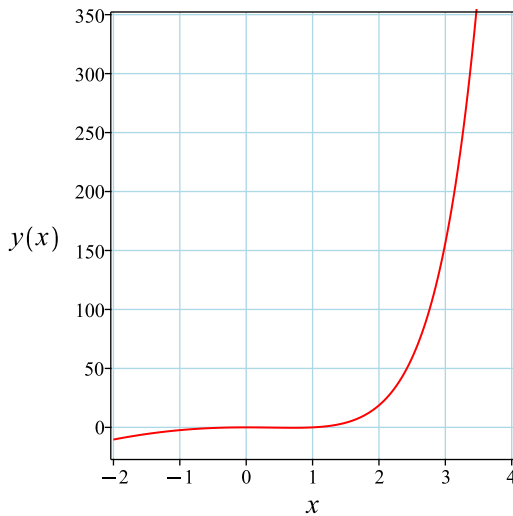
Substituting c_1 found above in the general solution gives

$$y = x^2(e^x - e)$$

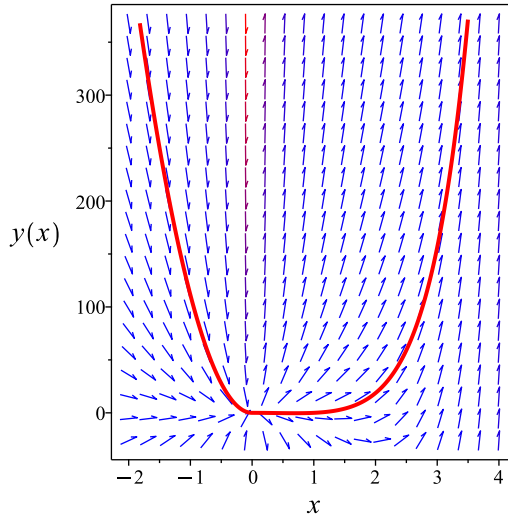
Summary

The solution(s) found are the following

$$y = x^2(e^x - e) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(e^x - e)$$

Verified OK.

4.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (x) dy &= (2y + x^3 e^x) dx \\ (-2y - x^3 e^x) dx + (x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -2y - x^3 e^x \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-2y - x^3 e^x) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((-2) - (1)) \\ &= -\frac{3}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int -\frac{3}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^3}(-2y - x^3 e^x) \\ &= \frac{-2y - x^3 e^x}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-2y - x^3 e^x}{x^3} \right) + \left(\frac{1}{x^2} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-2y - x^3 e^x}{x^3} dx \\ \phi &= \frac{-x^2 e^x + y}{x^2} + f(y)\end{aligned}\quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y)\quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y)\quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2 e^x + y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2 e^x + y}{x^2}$$

The solution becomes

$$y = x^2(e^x + c_1)$$

Initial conditions are used to solve for c_1 . Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e + c_1$$

$$c_1 = -e$$

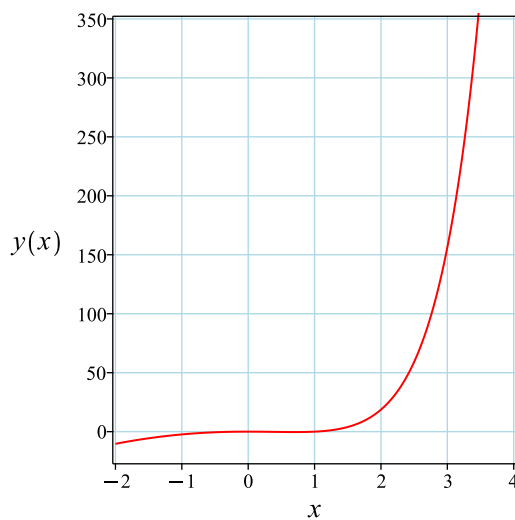
Substituting c_1 found above in the general solution gives

$$y = x^2(e^x - e)$$

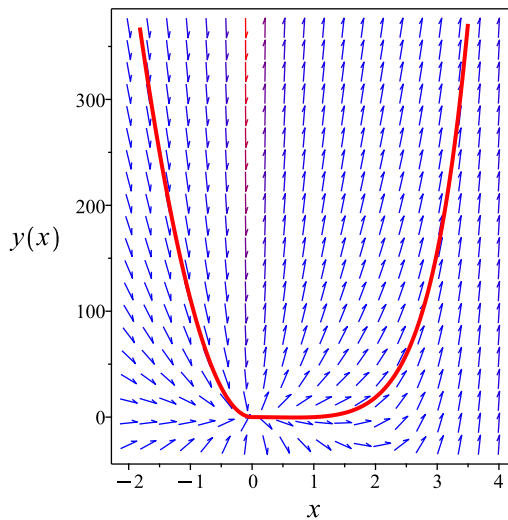
Summary

The solution(s) found are the following

$$y = x^2(e^x - e) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(e^x - e)$$

Verified OK.

4.19.5 Maple step by step solution

Let's solve

$$[xy' - 2y = x^3e^x, y(1) = 0]$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{2y}{x} + x^2e^x$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{2y}{x} = x^2e^x$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = \mu(x) x^2e^x$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x^2}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int \mu(x) x^2e^x dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int \mu(x) x^2e^x dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x)x^2e^x dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^2}$

$$y = x^2 \left(\int e^x dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x^2(e^x + c_1)$$

- Use initial condition $y(1) = 0$

$$0 = e + c_1$$

- Solve for c_1
 $c_1 = -e$
- Substitute $c_1 = -e$ into general solution and simplify
 $y = x^2(e^x - e)$
- Solution to the IVP
 $y = x^2(e^x - e)$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([x*diff(y(x),x)=2*y(x)+x^3*exp(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = -(-e^x + e) x^2$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 16

```
DSolve[{x*y'[x]==2*y[x]+x^3*Exp[x],{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (e^x - e) x^2$$

4.20 problem 22 (b)

4.20.1 Existence and uniqueness analysis	926
4.20.2 Solving as linear ode	927
4.20.3 Solving as first order ode lie symmetry lookup ode	928
4.20.4 Solving as exact ode	932
4.20.5 Maple step by step solution	936

Internal problem ID [5317]

Internal file name [OUTPUT/4808_Friday_February_02_2024_05_13_50_AM_63251945/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 22 (b).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$Li' + Ri = E \sin(2t)$$

With initial conditions

$$[i(0) = 0]$$

4.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$i' + p(t)i = q(t)$$

Where here

$$p(t) = \frac{R}{L}$$
$$q(t) = \frac{E \sin(2t)}{L}$$

Hence the ode is

$$i' + \frac{Ri}{L} = \frac{E \sin(2t)}{L}$$

The domain of $p(t) = \frac{R}{L}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{E \sin(2t)}{L}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{R}{L} dt} \\ &= e^{\frac{Rt}{L}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu i) &= (\mu) \left(\frac{E \sin(2t)}{L} \right) \\ \frac{d}{dt} \left(e^{\frac{Rt}{L}} i \right) &= \left(e^{\frac{Rt}{L}} \right) \left(\frac{E \sin(2t)}{L} \right) \\ d \left(e^{\frac{Rt}{L}} i \right) &= \left(\frac{E \sin(2t) e^{\frac{Rt}{L}}}{L} \right) dt \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{\frac{Rt}{L}} i &= \int \frac{E \sin(2t) e^{\frac{Rt}{L}}}{L} dt \\ e^{\frac{Rt}{L}} i &= \frac{E \left(-\frac{2 e^{\frac{Rt}{L}} \cos(2t)}{\frac{R^2}{L^2} + 4} + \frac{R e^{\frac{Rt}{L}} \sin(2t)}{L \left(\frac{R^2}{L^2} + 4 \right)} \right)}{L} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{Rt}{L}}$ results in

$$i = \frac{e^{-\frac{Rt}{L}} E \left(-\frac{2 e^{\frac{Rt}{L}} \cos(2t)}{\frac{R^2}{L^2} + 4} + \frac{R e^{\frac{Rt}{L}} \sin(2t)}{L \left(\frac{R^2}{L^2} + 4 \right)} \right)}{L} + c_1 e^{-\frac{Rt}{L}}$$

which simplifies to

$$i = \frac{c_1(4L^2 + R^2)e^{-\frac{Rt}{L}} - 2EL \cos(2t) + ER \sin(2t)}{4L^2 + R^2}$$

Initial conditions are used to solve for c_1 . Substituting $t = 0$ and $i = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4L^2c_1 + R^2c_1 - 2EL}{4L^2 + R^2}$$

$$c_1 = \frac{2EL}{4L^2 + R^2}$$

Substituting c_1 found above in the general solution gives

$$i = \frac{-2EL \cos(2t) + 2EL e^{-\frac{Rt}{L}} + ER \sin(2t)}{4L^2 + R^2}$$

Summary

The solution(s) found are the following

$$i = \frac{-2EL \cos(2t) + 2EL e^{-\frac{Rt}{L}} + ER \sin(2t)}{4L^2 + R^2} \quad (1)$$

Verification of solutions

$$i = \frac{-2EL \cos(2t) + 2EL e^{-\frac{Rt}{L}} + ER \sin(2t)}{4L^2 + R^2}$$

Verified OK.

4.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$i' = \frac{-Ri + E \sin(2t)}{L}$$

$$i' = \omega(t, i)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_i - \xi_t) - \omega^2 \xi_i - \omega_t \xi - \omega_i \eta = 0 \quad (A)$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(t, i) &= 0 \\ \eta(t, i) &= e^{-\frac{Rt}{L}}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(t, i) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{di}{\eta} = dS\tag{1}$$

The above comes from the requirements that $(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial i}) S(t, i) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{Rt}{L}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{Rt}{L}} i$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, i)S_i}{R_t + \omega(t, i)R_i} \quad (2)$$

Where in the above R_t, R_i, S_t, S_i are all partial derivatives and $\omega(t, i)$ is the right hand side of the original ode given by

$$\omega(t, i) = \frac{-Ri + E \sin(2t)}{L}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_i &= 0 \\ S_t &= \frac{R e^{\frac{Rt}{L}} i}{L} \\ S_i &= e^{\frac{Rt}{L}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{E \sin(2t) e^{\frac{Rt}{L}}}{L} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for t, i in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{E \sin(2R) e^{\frac{RR}{L}}}{L}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{Rt}{L}} (2L \cos(2R) - \sin(2R) R)}{4L^2 + R^2} \quad (4)$$

To complete the solution, we just need to transform (4) back to t, i coordinates. This results in

$$e^{\frac{Rt}{L}} i = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{Rt}{L}} (2L \cos(2t) - \sin(2t) R)}{4L^2 + R^2}$$

Which simplifies to

$$e^{\frac{Rt}{L}} i = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{Rt}{L}} (2L \cos(2t) - \sin(2t) R)}{4L^2 + R^2}$$

Which gives

$$i = -\frac{e^{-\frac{Rt}{L}} \left(2E \cos(2t) e^{\frac{Rt}{L}} L - E \sin(2t) R e^{\frac{Rt}{L}} - 4L^2 c_1 - R^2 c_1 \right)}{4L^2 + R^2}$$

Initial conditions are used to solve for c_1 . Substituting $t = 0$ and $i = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4L^2 c_1 + R^2 c_1 - 2EL}{4L^2 + R^2}$$

$$c_1 = \frac{2EL}{4L^2 + R^2}$$

Substituting c_1 found above in the general solution gives

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2}$$

Summary

The solution(s) found are the following

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2} \quad (1)$$

Verification of solutions

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2}$$

Verified OK.

4.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t, i) dt + N(t, i) di = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (L) di &= (-Ri + E \sin(2t)) dt \\ (Ri - E \sin(2t)) dt + (L) di &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(t, i) &= Ri - E \sin(2t) \\ N(t, i) &= L \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial i} &= \frac{\partial}{\partial i}(Ri - E \sin(2t)) \\ &= R\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(L) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial i} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial i} - \frac{\partial N}{\partial t} \right) \\ &= \frac{1}{L} ((R) - (0)) \\ &= \frac{R}{L}\end{aligned}$$

Since A does not depend on i , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dt} \\ &= e^{\int \frac{R}{L} dt}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{Rt}{L}} \\ &= e^{\frac{Rt}{L}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= e^{\frac{Rt}{L}} (Ri - E \sin(2t)) \\ &= -e^{\frac{Rt}{L}} (-Ri + E \sin(2t))\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= e^{\frac{Rt}{L}}(L) \\ &= L e^{\frac{Rt}{L}}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{di}{dt} &= 0 \\ \left(-e^{\frac{Rt}{L}}(-Ri + E \sin(2t))\right) + \left(L e^{\frac{Rt}{L}}\right) \frac{di}{dt} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t, i)$

$$\frac{\partial \phi}{\partial t} = \bar{M} \quad (1)$$

$$\frac{\partial \phi}{\partial i} = \bar{N} \quad (2)$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t} dt &= \int \bar{M} dt \\ \int \frac{\partial \phi}{\partial t} dt &= \int -e^{\frac{Rt}{L}}(-Ri + E \sin(2t)) dt \\ \phi &= \frac{e^{\frac{Rt}{L}} L(2EL \cos(2t) - ER \sin(2t) + 4iL^2 + iR^2)}{4L^2 + R^2} + f(i)\end{aligned} \quad (3)$$

Where $f(i)$ is used for the constant of integration since ϕ is a function of both t and i . Taking derivative of equation (3) w.r.t i gives

$$\frac{\partial \phi}{\partial i} = L e^{\frac{Rt}{L}} + f'(i) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial i} = L e^{\frac{Rt}{L}}$. Therefore equation (4) becomes

$$L e^{\frac{Rt}{L}} = L e^{\frac{Rt}{L}} + f'(i) \quad (5)$$

Solving equation (5) for $f'(i)$ gives

$$f'(i) = 0$$

Therefore

$$f(i) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(i)$ into equation (3) gives ϕ

$$\phi = \frac{e^{\frac{Rt}{L}} L(2EL \cos(2t) - ER \sin(2t) + 4iL^2 + iR^2)}{4L^2 + R^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{e^{\frac{Rt}{L}} L(2EL \cos(2t) - ER \sin(2t) + 4iL^2 + iR^2)}{4L^2 + R^2}$$

The solution becomes

$$i = - \frac{\left(2EL^2 \cos(2t) e^{\frac{Rt}{L}} - EL \sin(2t) e^{\frac{Rt}{L}} R - 4L^2 c_1 - R^2 c_1\right) e^{-\frac{Rt}{L}}}{(4L^2 + R^2) L}$$

Initial conditions are used to solve for c_1 . Substituting $t = 0$ and $i = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-2EL^2 + 4L^2 c_1 + R^2 c_1}{4L^3 + LR^2}$$

$$c_1 = \frac{2EL^2}{4L^2 + R^2}$$

Substituting c_1 found above in the general solution gives

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2}$$

Summary

The solution(s) found are the following

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2} \quad (1)$$

Verification of solutions

$$i = \frac{-2EL \cos(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} + E \sin(2t) e^{\frac{Rt}{L}} e^{-\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2}$$

Verified OK.

4.20.5 Maple step by step solution

Let's solve

$$[Li' + Ri = E \sin(2t), i(0) = 0]$$

- Highest derivative means the order of the ODE is 1

$$i'$$

- Isolate the derivative

$$i' = -\frac{Ri}{L} + \frac{E \sin(2t)}{L}$$

- Group terms with i on the lhs of the ODE and the rest on the rhs of the ODE

$$i' + \frac{Ri}{L} = \frac{E \sin(2t)}{L}$$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

$$\mu(t) \left(i' + \frac{Ri}{L} \right) = \frac{\mu(t)E \sin(2t)}{L}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t) i)$

$$\mu(t) \left(i' + \frac{Ri}{L} \right) = \mu'(t) i + \mu(t) i'$$

- Isolate $\mu'(t)$

$$\mu'(t) = \frac{\mu(t)R}{L}$$

- Solve to find the integrating factor

$$\mu(t) = e^{\frac{Rt}{L}}$$

- Integrate both sides with respect to t

$$\int \left(\frac{d}{dt}(\mu(t) i) \right) dt = \int \frac{\mu(t)E \sin(2t)}{L} dt + c_1$$

- Evaluate the integral on the lhs

$$\mu(t) i = \int \frac{\mu(t)E \sin(2t)}{L} dt + c_1$$

- Solve for i

$$i = \frac{\int \frac{\mu(t)E \sin(2t)}{L} dt + c_1}{\mu(t)}$$

- Substitute $\mu(t) = e^{\frac{Rt}{L}}$

$$i = \frac{\int \frac{E \sin(2t)e^{\frac{Rt}{L}}}{L} dt + c_1}{e^{\frac{Rt}{L}}}$$

- Evaluate the integrals on the rhs

$$i = \frac{E \left(-\frac{2e^{-\frac{Rt}{L}} \cos(2t)}{\frac{R^2}{L^2} + 4} + \frac{R e^{-\frac{Rt}{L}} \sin(2t)}{L \left(\frac{R^2}{L^2} + 4 \right)} \right) + c_1}{e^{-\frac{Rt}{L}}}$$

- Simplify

$$i = \frac{c_1(4L^2 + R^2)e^{-\frac{Rt}{L}} - 2EL \cos(2t) + ER \sin(2t)}{4L^2 + R^2}$$

- Use initial condition $i(0) = 0$

$$0 = \frac{c_1(4L^2 + R^2) - 2EL}{4L^2 + R^2}$$

- Solve for c_1

$$c_1 = \frac{2EL}{4L^2 + R^2}$$

- Substitute $c_1 = \frac{2EL}{4L^2 + R^2}$ into general solution and simplify

$$i = -\frac{2E \left(L \cos(2t) - L e^{-\frac{Rt}{L}} - \frac{\sin(2t)R}{2} \right)}{4L^2 + R^2}$$

- Solution to the IVP

$$i = -\frac{2E \left(L \cos(2t) - L e^{-\frac{Rt}{L}} - \frac{\sin(2t)R}{2} \right)}{4L^2 + R^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([L*diff(i(t),t)+R*i(t)=E*sin(2*t),i(0) = 0],i(t), singsol=all)
```

$$i(t) = \frac{E \left(2L e^{-\frac{Rt}{L}} - 2L \cos(2t) + \sin(2t) R \right)}{4L^2 + R^2}$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 49

```
DSolve[{L*i'[t]+R*i[t]==e*Sin[2*t],{i[0]==0}},i[t],t,IncludeSingularSolutions -> True]
```

$$i(t) \rightarrow \frac{2e \left(L \left(e^{-\frac{Rt}{L}} + \sin^2(t) \right) - L \cos^2(t) + R \sin(t) \cos(t) \right)}{4L^2 + R^2}$$

4.21 problem 23 (a)

4.21.1 Solving as exact ode 939

Internal problem ID [5318]

Internal file name [OUTPUT/4809_Friday_February_02_2024_05_13_52_AM_30399592/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 23 (a).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**exactWithIntegrationFactor**"

Maple gives the following as the ode type

[`y=_G(x,y)´]

$$x^2 \cos(y) y' - 2x \sin(y) = -1$$

4.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (\cos(y) x^2) dy &= (2 \sin(y) x - 1) dx \\ (-2 \sin(y) x + 1) dx + (\cos(y) x^2) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -2 \sin(y) x + 1 \\ N(x, y) &= \cos(y) x^2 \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (-2 \sin(y) x + 1) \\ &= -2 \cos(y) x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (\cos(y) x^2) \\ &= 2 \cos(y) x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{\sec(y)}{x^2} ((-2 \cos(y) x) - (2 \cos(y) x)) \\ &= -\frac{4}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int -\frac{4}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4 \ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x^4}(-2 \sin(y) x + 1) \\ &= \frac{-2 \sin(y) x + 1}{x^4}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x^4}(\cos(y) x^2) \\ &= \frac{\cos(y)}{x^2}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-2 \sin(y) x + 1}{x^4} \right) + \left(\frac{\cos(y)}{x^2} \right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \bar{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int \frac{-2 \sin(y) x + 1}{x^4} dx$$

$$\phi = \frac{3 \sin(y) x - 1}{3x^3} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2}$. Therefore equation (4) becomes

$$\frac{\cos(y)}{x^2} = \frac{\cos(y)}{x^2} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{3 \sin(y) x - 1}{3x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{3 \sin(y) x - 1}{3x^3}$$

Summary

The solution(s) found are the following

$$\frac{3x \sin(y) - 1}{3x^3} = c_1 \quad (1)$$

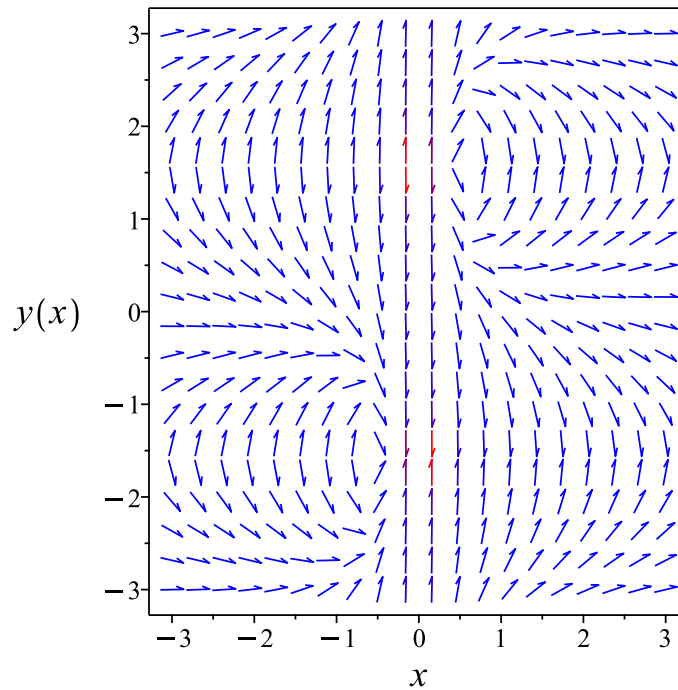


Figure 196: Slope field plot

Verification of solutions

$$\frac{3x \sin(y) - 1}{3x^3} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```


✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*cos(y(x))*diff(y(x),x)=2*x*sin(y(x))-1,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\frac{3c_1x^3 - 1}{3x}\right)$$

✓ Solution by Mathematica

Time used: 10.185 (sec). Leaf size: 21

```
DSolve[x^2*Cos[y[x]]*y'[x]==2*x*Sin[y[x]]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \arcsin\left(\frac{1}{3x} + 2c_1x^2\right)$$

4.22 problem 23 (b)

Internal problem ID [5319]

Internal file name [OUTPUT/4810_Friday_February_02_2024_05_13_54_AM_49569275/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 23 (b).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$4yx^2y' - 3x(3y^2 + 2) - 2(3y^2 + 2)^3 = 0$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2` [0, (2*x*y^4+x*y^6+8/27*x+8/81*x^2+4/3*x*y^2+4/27
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 175

```
dsolve(4*x^2*y(x)*diff(y(x),x)=3*x*(3*y(x)^2+2)+2*(3*y(x)^2+2)^3,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6} \sqrt{\frac{-3c_1x^8 + \sqrt{-3(c_1x^8 - \frac{1}{3})c_1x^9 + 1}}{3c_1x^8 - 1}}}{3}$$

$$y(x) = \frac{\sqrt{6} \sqrt{\frac{-3c_1x^8 + \sqrt{-3(c_1x^8 - \frac{1}{3})c_1x^9 + 1}}{3c_1x^8 - 1}}}{3}$$

$$y(x) = -\frac{\sqrt{\frac{-18c_1x^8 - 6\sqrt{-3(c_1x^8 - \frac{1}{3})c_1x^9 + 6}}{3c_1x^8 - 1}}}{3}$$

$$y(x) = \frac{\sqrt{\frac{-18c_1x^8 - 6\sqrt{-3(c_1x^8 - \frac{1}{3})c_1x^9 + 6}}{3c_1x^8 - 1}}}{3}$$

✓ Solution by Mathematica

Time used: 19.518 (sec). Leaf size: 277

```
DSolve[4*x^2*y[x]*y'[x]==3*x*(3*y[x]^2+2)+2*(3*y[x]^2+2)^3,y[x],x,IncludeSingularSolutions
```

$$y(x) \rightarrow -\frac{1}{3}\sqrt{2}\sqrt{-\frac{3x^8 + \sqrt{3}\sqrt{-x^9(x^8 + 72c_1)} + 216c_1}{x^8 + 72c_1}}$$

$$y(x) \rightarrow \frac{1}{3}\sqrt{2}\sqrt{-\frac{3x^8 + \sqrt{3}\sqrt{-x^9(x^8 + 72c_1)} + 216c_1}{x^8 + 72c_1}}$$

$$y(x) \rightarrow -\frac{1}{3}\sqrt{2}\sqrt{\frac{-3x^8 + \sqrt{3}\sqrt{-x^9(x^8 + 72c_1)} - 216c_1}{x^8 + 72c_1}}$$

$$y(x) \rightarrow \frac{1}{3}\sqrt{2}\sqrt{\frac{-3x^8 + \sqrt{3}\sqrt{-x^9(x^8 + 72c_1)} - 216c_1}{x^8 + 72c_1}}$$

$$y(x) \rightarrow -i\sqrt{\frac{2}{3}}$$

$$y(x) \rightarrow i\sqrt{\frac{2}{3}}$$

4.23 problem 23 (c)

4.23.1 Solving as first order ode lie symmetry lookup ode	948
4.23.2 Solving as bernoulli ode	952
4.23.3 Solving as exact ode	956

Internal problem ID [5320]

Internal file name [OUTPUT/4811_Friday_February_02_2024_05_13_56_AM_14351785/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 23 (c).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_Bernoulli]

$$y^3x - y^3 + 3y'y^2x = x^2e^x$$

4.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-xy^3 + y^3 + x^2e^x}{3y^2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type Bernoulli. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{e^{-x+\ln(x)}}{y^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-x+\ln(x)}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 e^x}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x y^3 + y^3 + x^2 e^x}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3 e^x (x - 1)}{3x^2} \\ S_y &= \frac{y^2 e^x}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{2x}}{3} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{2R}}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{e^{2R}}{6} + c_1 \quad (4)$$

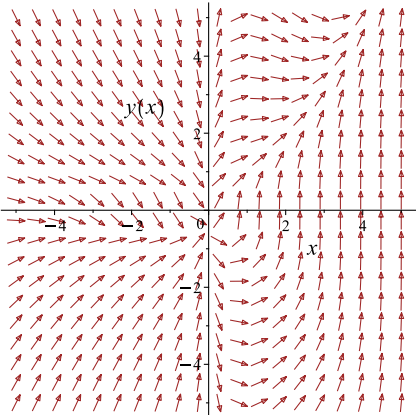
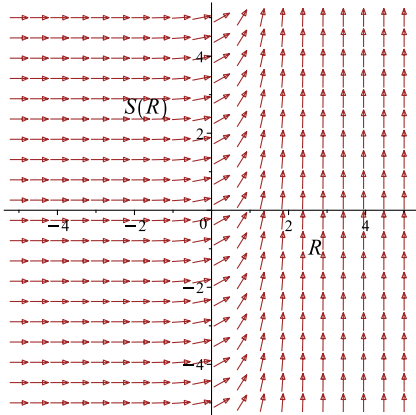
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

Which simplifies to

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x y^3 + y^3 + x^2 e^x}{3y^2 x}$ 	$R = x$ $S = \frac{y^3 e^x}{3x}$	$\frac{dS}{dR} = \frac{e^{2R}}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1 \quad (1)$$

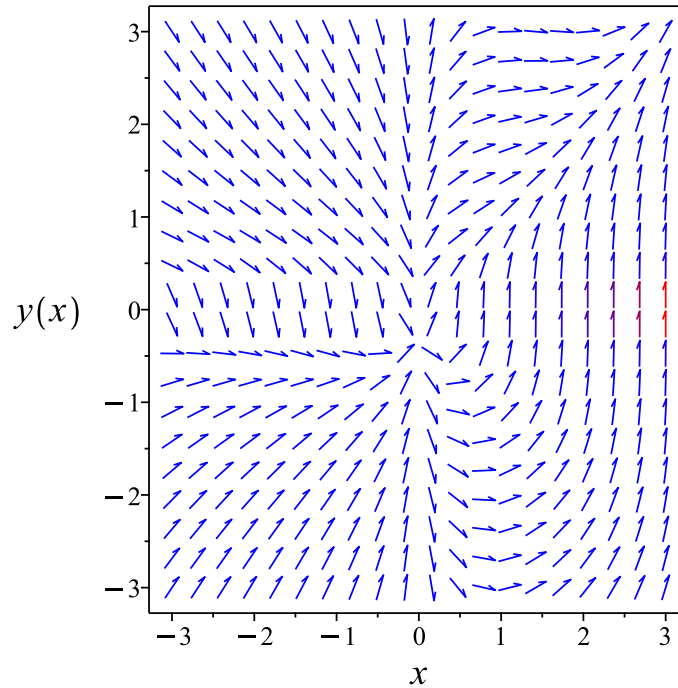


Figure 197: Slope field plot

Verification of solutions

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

Verified OK.

4.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{-x y^3 + y^3 + x^2 e^x}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1-x}{3x} y + \frac{x e^x}{3} \frac{1}{y^2} \quad (1)$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \quad (2)$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \quad (3)$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1-x}{3x} \\ f_1(x) &= \frac{x e^x}{3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y' y^2 = \frac{(1-x)y^3}{3x} + \frac{x e^x}{3} \quad (4)$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \quad (5)$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2 y' \quad (6)$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{(1-x)w(x)}{3x} + \frac{x e^x}{3} \\ w' &= \frac{(1-x)w}{x} + x e^x \end{aligned} \quad (7)$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1-x}{x} \\ q(x) &= x e^x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{(1-x)w(x)}{x} = x e^x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1-x}{x} dx} \\ &= e^{x-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{e^x}{x}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x e^x) \\ \frac{d}{dx}\left(\frac{e^x w}{x}\right) &= \left(\frac{e^x}{x}\right)(x e^x) \\ d\left(\frac{e^x w}{x}\right) &= e^{2x} dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{e^x w}{x} &= \int e^{2x} dx \\ \frac{e^x w}{x} &= \frac{e^{2x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{e^x}{x}$ results in

$$w(x) = \frac{x e^{-x} e^{2x}}{2} + c_1 x e^{-x}$$

which simplifies to

$$w(x) = \frac{x(2c_1 e^{-x} + e^x)}{2}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = \frac{x(2c_1 e^{-x} + e^x)}{2}$$

Solving for y gives

$$y(x) = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}e^{-x}}{2}$$

$$y(x) = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(i\sqrt{3} - 1)e^{-x}}{4}$$

$$y(x) = -\frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(1 + i\sqrt{3})e^{-x}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}e^{-x}}{2} \quad (1)$$

$$y = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(i\sqrt{3} - 1)e^{-x}}{4} \quad (2)$$

$$y = -\frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(1 + i\sqrt{3})e^{-x}}{4} \quad (3)$$

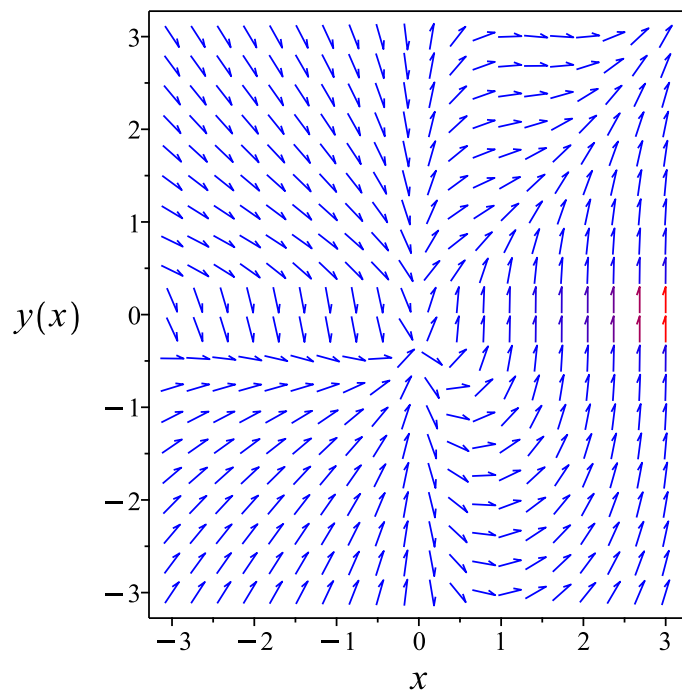


Figure 198: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}e^{-x}}{2}$$

Verified OK.

$$y = \frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(i\sqrt{3} - 1)e^{-x}}{4}$$

Verified OK.

$$y = -\frac{2^{\frac{2}{3}}(x(e^{2x} + 2c_1)e^{2x})^{\frac{1}{3}}(1 + i\sqrt{3})e^{-x}}{4}$$

Verified OK.

4.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (3y^2x) dy &= (-xy^3 + y^3 + x^2e^x) dx \\ (xy^3 - y^3 - x^2e^x) dx + (3y^2x) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= xy^3 - y^3 - x^2e^x \\ N(x, y) &= 3y^2x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(xy^3 - y^3 - x^2e^x) \\ &= 3y^2(x - 1) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2x) \\ &= 3y^2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{3y^2x} ((3y^2x - 3y^2) - (3y^2)) \\ &= \frac{-2 + x}{x} \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int \frac{-2+x}{x} dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x-2\ln(x)} \\ &= \frac{e^x}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \overline{M} and \overline{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{e^x}{x^2}(xy^3 - y^3 - x^2e^x) \\ &= -\frac{e^x(x^2e^x - y^3(x-1))}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{e^x}{x^2}(3y^2x) \\ &= \frac{3y^2e^x}{x}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(-\frac{e^x(x^2e^x - y^3(x-1))}{x^2}\right) + \left(\frac{3y^2e^x}{x}\right) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} dx = \int \overline{M} dx$$

$$\int \frac{\partial \phi}{\partial x} dx = \int -\frac{e^x(x^2 e^x - y^3(x-1))}{x^2} dx$$

$$\phi = -\frac{e^x(-2y^3 + x e^x)}{2x} + f(y) \quad (3)$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2 e^x}{x} + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2 e^x}{x}$. Therefore equation (4) becomes

$$\frac{3y^2 e^x}{x} = \frac{3y^2 e^x}{x} + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{e^x(-2y^3 + x e^x)}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{e^x(-2y^3 + x e^x)}{2x}$$

Summary

The solution(s) found are the following

$$-\frac{e^x(-2y^3 + x e^x)}{2x} = c_1 \quad (1)$$

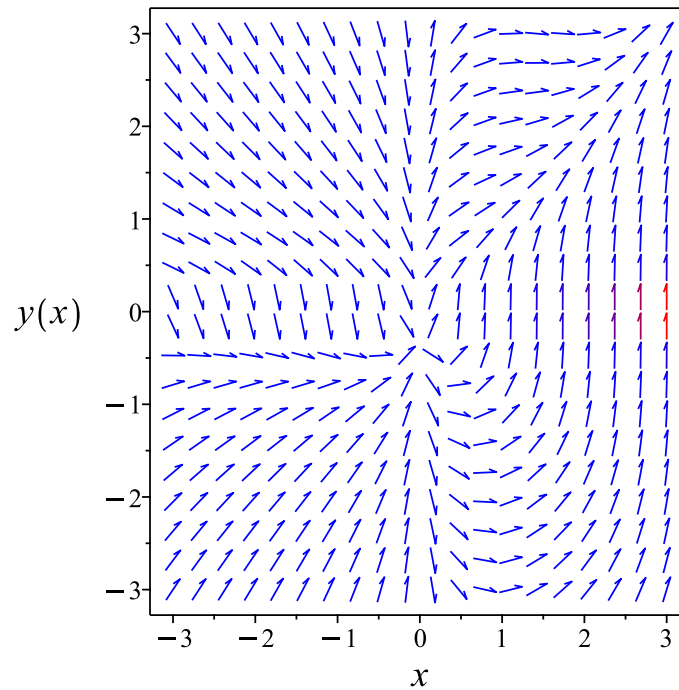


Figure 199: Slope field plot

Verification of solutions

$$-\frac{e^x(-2y^3 + x e^x)}{2x} = c_1$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
dsolve((x*y(x)^3-y(x)^3-x^2*exp(x))+(3*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{2}{3}}((e^{2x} + 2c_1) x e^{2x})^{\frac{1}{3}} e^{-x}}{2}$$
$$y(x) = -\frac{2^{\frac{2}{3}}((e^{2x} + 2c_1) x e^{2x})^{\frac{1}{3}} (1 + i\sqrt{3}) e^{-x}}{4}$$
$$y(x) = \frac{2^{\frac{2}{3}}((e^{2x} + 2c_1) x e^{2x})^{\frac{1}{3}} (i\sqrt{3} - 1) e^{-x}}{4}$$

✓ Solution by Mathematica

Time used: 0.854 (sec). Leaf size: 117

```
DSolve[(x*y[x]^3-y[x]^3-x^2*Exp[x])+(3*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow -\sqrt[3]{-\frac{1}{2}e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x} + 2c_1}}$$
$$y(x) \rightarrow \frac{e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x} + 2c_1}}{\sqrt[3]{2}}$$
$$y(x) \rightarrow \frac{(-1)^{2/3}e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x} + 2c_1}}{\sqrt[3]{2}}$$

4.24 problem 23 (d)

4.24.1 Solving as `abelFirstKind` ode 962

Internal problem ID [5321]

Internal file name [OUTPUT/4812_Friday_February_02_2024_05_13_57_AM_51034746/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 23 (d).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**abelFirstKind**"

Maple gives the following as the ode type

`[_Abel]`

$$y' + x(x + y) - x^3(x + y)^3 = -1$$

4.24.1 Solving as `abelFirstKind` ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = x^3y^3 + 3y^2x^4 + (3x^5 - x)y + x^6 - x^2 - 1 \tag{1}$$

Therefore

$$f_0(x) = x^6 - x^2 - 1$$

$$f_1(x) = 3x^5 - x$$

$$f_2(x) = 3x^4$$

$$f_3(x) = x^3$$

Since $f_2(x) = 3x^4$ is not zero, then the first step is to apply the following transformation to remove f_2 . Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} y &= u(x) - \left(\frac{3x^4}{3x^3} \right) \\ &= u(x) - x \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = x^3 u(x)^3 - xu(x) \quad (2)$$

The above ODE (2) can now be solved as separable.

Writing the ode as

$$\begin{aligned} u'(x) &= x^3 u^3 - xu \\ u'(x) &= \omega(x, u) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_u - \xi_x) - \omega^2 \xi_u - \omega_x \xi - \omega_u \eta = 0 \quad (A)$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 122: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, u) &= 0 \\ \eta(x, u) &= u^3 e^{x^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, u) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{du}{\eta} = dS\tag{1}$$

The above comes from the requirements that $(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}) S(x, u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{u^3 e^{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{-x^2}}{2u^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, u)S_u}{R_x + \omega(x, u)R_u} \quad (2)$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x, u)$ is the right hand side of the original ode given by

$$\omega(x, u) = x^3 u^3 - xu$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_u &= 0 \\ S_x &= \frac{x e^{-x^2}}{u^2} \\ S_u &= \frac{e^{-x^2}}{u^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 e^{-x^2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 e^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{(R^2 + 1)e^{-R^2}}{2} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, u coordinates. This results in

$$-\frac{e^{-x^2}}{2u(x)^2} = -\frac{(x^2 + 1)e^{-x^2}}{2} + c_1$$

Which simplifies to

$$-\frac{e^{-x^2}}{2u(x)^2} = -\frac{(x^2 + 1)e^{-x^2}}{2} + c_1$$

Substituting $u = -x + y$ in the above solution gives

$$-\frac{e^{-x^2}}{2(-x + y)^2} = -\frac{(x^2 + 1)e^{-x^2}}{2} + c_1$$

Summary

The solution(s) found are the following

$$-\frac{e^{-x^2}}{2(-x + y)^2} = -\frac{(x^2 + 1)e^{-x^2}}{2} + c_1 \quad (1)$$

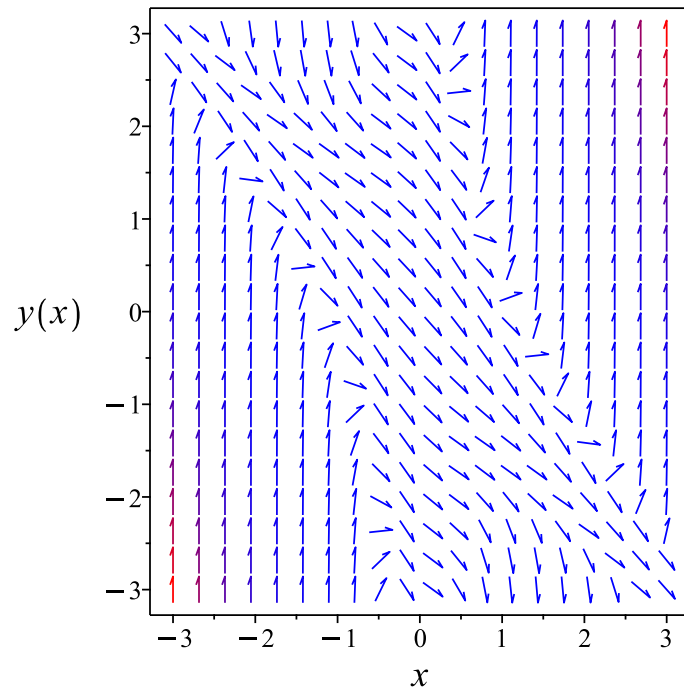


Figure 200: Slope field plot

Verification of solutions

$$-\frac{e^{-x^2}}{2(-x+y)^2} = -\frac{(x^2+1)e^{-x^2}}{2} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
trying Bernoulli  
trying separable  
trying inverse linear  
trying homogeneous types:  
trying Chini  
differential order: 1; looking for linear symmetries  
trying exact  
trying Abel  
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 109

```
dsolve(diff(y(x),x)+x*(x+y(x))=x^3*(x+y(x))^3-1,y(x), singsol=all)
```

$$y(x) = -\frac{x\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1} + e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1}}$$
$$y(x) = \frac{-x\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1} + e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1}}$$

✓ Solution by Mathematica

Time used: 10.062 (sec). Leaf size: 85

```
DSolve[y'[x]+x*(x+y[x])==x^3*(x+y[x])^3-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -x - \frac{e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}(x^2 + 1) + c_1}}$$
$$y(x) \rightarrow -x + \frac{e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}(x^2 + 1) + c_1}}$$
$$y(x) \rightarrow -x$$

4.25 problem 23 (e)

4.25.1 Solving as exact ode 969

Internal problem ID [5322]

Internal file name [OUTPUT/4813_Friday_February_02_2024_05_13_59_AM_76254145/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplementary problems. Page 39

Problem number: 23 (e).

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

[`y=_G(x,y)´]

$$y + e^y + (1 + e^y) y' = e^{-x}$$

4.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (1A)$$

Therefore

$$\begin{aligned} (1 + e^y) dy &= (-y - e^y + e^{-x}) dx \\ (y + e^y - e^{-x}) dx + (1 + e^y) dy &= 0 \end{aligned} \quad (2A)$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y + e^y - e^{-x} \\ N(x, y) &= 1 + e^y \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} (y + e^y - e^{-x}) \\ &= 1 + e^y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (1 + e^y) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{1 + e^y} ((1 + e^y) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A dx} \\ &= e^{\int 1 dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^x \\ &= e^x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= e^x (y + e^y - e^{-x}) \\ &= e^{x+y} + y e^x - 1\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= e^x (1 + e^y) \\ &= (1 + e^y) e^x\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\bar{M} + \bar{N} \frac{dy}{dx} &= 0 \\ (e^{x+y} + y e^x - 1) + ((1 + e^y) e^x) \frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \bar{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \bar{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} dx &= \int \bar{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int e^{x+y} + y e^x - 1 dx \\ \phi &= -x + y e^x + e^{x+y} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{x+y} + e^x + f'(y) \quad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (1 + e^y) e^x$. Therefore equation (4) becomes

$$(1 + e^y) e^x = e^{x+y} + e^x + f'(y) \quad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + y e^x + e^{x+y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + y e^x + e^{x+y}$$

The solution becomes

$$y = -\left(\text{LambertW}\left(e^{-x+(x e^x+c_1+x)e^{-x}}\right) e^x - c_1 - x\right) e^{-x}$$

Summary

The solution(s) found are the following

$$y = -\left(\text{LambertW}\left(e^{-x+(x e^x+c_1+x)e^{-x}}\right) e^x - c_1 - x\right) e^{-x} \quad (1)$$

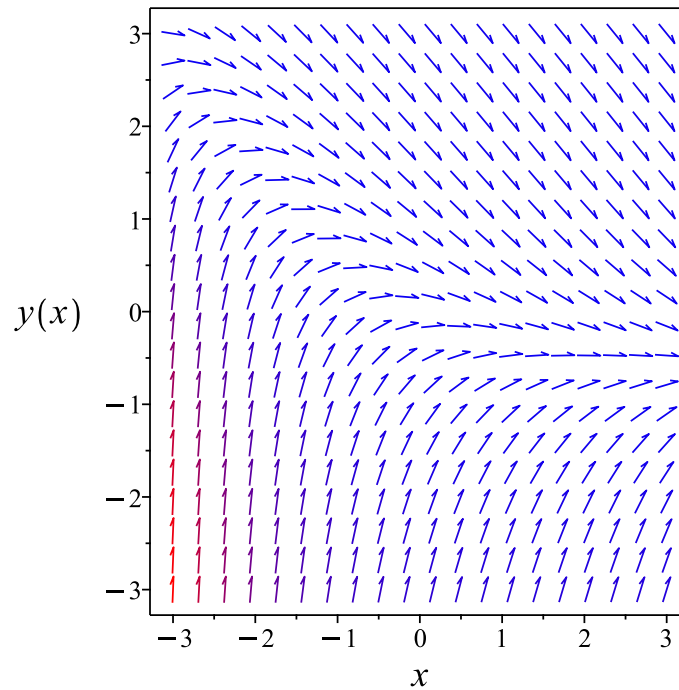


Figure 201: Slope field plot

Verification of solutions

$$y = -\left(\text{LambertW}\left(e^{-x+(x e^x+c_1+x)e^{-x}}\right) e^x - c_1 - x\right) e^{-x}$$

Verified OK.

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`

```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 29

```
dsolve((y(x)+exp(y(x))-exp(-x))*(1+exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\text{LambertW}\left(e^{e^{-x}(x-c_1)}\right) + e^{-x}(x - c_1)$$

✓ Solution by Mathematica

Time used: 6.265 (sec). Leaf size: 33

```
DSolve[(y[x]+Exp[y[x]]-Exp[-x])*(1+Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-x}\left(-e^x W\left(e^{e^{-x}(x+c_1)}\right) + x + c_1\right)$$

5 Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

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5.1 problem 17

5.1.1 Maple step by step solution 978

Internal problem ID [5323]

Internal file name [OUTPUT/4814_Friday_February_02_2024_05_14_01_AM_61480249/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 17.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_separable]

$$x^2y'^2 + xyy' - 6y^2 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -\frac{3y}{x} \tag{1}$$

$$y' = \frac{2y}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{3y}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= -\frac{3}{x} dx \\ \int \frac{1}{y} dy &= \int -\frac{3}{x} dx \\ \ln(y) &= -3 \ln(x) + c_1 \\ y &= e^{-3 \ln(x) + c_1} \\ &= \frac{c_1}{x^3}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{2y}{x}\end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= \frac{2}{x} dx \\ \int \frac{1}{y} dy &= \int \frac{2}{x} dx \\ \ln(y) &= 2 \ln(x) + c_2 \\ y &= e^{2 \ln(x) + c_2} \\ &= c_2 x^2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^2$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$$x^2 y'^2 + x y y' - 6 y^2 = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'}{y} = \frac{2}{x}$$

- Integrate both sides with respect to x

$$\int \frac{y'}{y} dx = \int \frac{2}{x} dx + c_1$$

- Evaluate integral

$$\ln(y) = 2 \ln(x) + c_1$$

- Solve for y

$$y = x^2 e^{c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)^2+x*y(x)*diff(y(x),x)-6*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2$$
$$y(x) = \frac{c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 26

```
DSolve[x^2*(y'[x])^2+x*y[x]*y'[x]-6*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_1}{x^3}$$
$$y(x) \rightarrow c_1 x^2$$
$$y(x) \rightarrow 0$$

5.2 problem 18

5.2.1 Maple step by step solution 982

Internal problem ID [5324]

Internal file name [OUTPUT/4815_Friday_February_02_2024_05_14_02_AM_34816384/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 18.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "exact", "linear", "quadrature", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"

Maple gives the following as the ode type

[_quadrature]

$$xy'^2 + (y - 1 - x^2)y' - (y - 1)x = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = x \tag{1}$$

$$y' = -\frac{y-1}{x} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\begin{aligned} y &= \int x \, dx \\ &= \frac{x^2}{2} + c_1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{2} + c_1 \quad (1)$$

Verification of solutions

$$y = \frac{x^2}{2} + c_1$$

Verified OK.

Solving equation (2)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{1-y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = 1 - y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1-y} dy &= \frac{1}{x} dx \\ \int \frac{1}{1-y} dy &= \int \frac{1}{x} dx \\ -\ln(y-1) &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-1} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{y-1} = c_3 x$$

Which simplifies to

$$y = \frac{(c_3 x e^{c_2} + 1) e^{-c_2}}{c_3 x}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3 x e^{c_2} + 1) e^{-c_2}}{c_3 x} \quad (1)$$

Verification of solutions

$$y = \frac{(c_3 x e^{c_2} + 1) e^{-c_2}}{c_3 x}$$

Verified OK.

5.2.1 Maple step by step solution

Let's solve

$$xy'^2 + (y - 1 - x^2)y' - (y - 1)x = 0$$

- Highest derivative means the order of the ODE is 1

y'

- Integrate both sides with respect to x

$$\int (xy'^2 + (y - 1 - x^2)y' - (y - 1)x) dx = \int 0 dx + c_1$$

- Cannot compute integral

$$\int (xy'^2 + (y - 1 - x^2)y' - (y - 1)x) dx = c_1$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
<- quadrature successful  
Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)^2+(y(x)-1-x^2)*diff(y(x),x)-x*(y(x)-1)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + c_1$$

$$y(x) = \frac{x + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 32

```
DSolve[x*(y'[x])^2+(y[x]-1-x^2)*y'[x]-x*(y[x]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^2}{2} + c_1$$

$$y(x) \rightarrow \frac{x + c_1}{x}$$

$$y(x) \rightarrow 1$$

5.3 problem 19

5.3.1 Solving as dAlembert ode 984

Internal problem ID [5325]

Internal file name [OUTPUT/4816_Friday_February_02_2024_05_14_02_AM_44066862/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 19.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$xy'^2 - 2y'y = -4x$$

5.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp^2 - 2py = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2 + 4}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2 + 4}{2p} = x \left(1 - \frac{p^2 + 4}{2p^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2 + 4}{2p} = 0$$

Solving for p from the above gives

$$p = 2$$
$$p = -2$$

Substituting these in (1A) gives

$$y = -2x$$
$$y = 2x$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2 + 4}{2p(x)}}{x \left(1 - \frac{p(x)^2 + 4}{2p(x)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx} \mu p &= 0 \\ \frac{d}{dx} \left(\frac{p}{x} \right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituting the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$

$$y = 2x \tag{2}$$

$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
*** Sublevel 2 ***  
Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 43

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -2x \cosh(-\log(x) + c_1)$$

$$y(x) \rightarrow -2x \cosh(\log(x) + c_1)$$

$$y(x) \rightarrow -2x$$

$$y(x) \rightarrow 2x$$

5.4 problem 20

Internal problem ID [5326]

Internal file name [OUTPUT/4817_Friday_February_02_2024_05_14_02_AM_30867913/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 20.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "first_order_ode_lie_symmetry_calculated"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$3x^4y'^2 - xy' - y = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 12yx^2}}{6x^3} \quad (1)$$

$$y' = -\frac{-1 + \sqrt{1 + 12yx^2}}{6x^3} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{1 + \sqrt{12yx^2 + 1}}{6x^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(1 + \sqrt{12yx^2 + 1})(b_3 - a_2)}{6x^3} - \frac{(1 + \sqrt{12yx^2 + 1})^2 a_3}{36x^6} \\ - \left(-\frac{1 + \sqrt{12yx^2 + 1}}{2x^4} + \frac{2y}{x^2\sqrt{12yx^2 + 1}} \right) (xa_2 + ya_3 + a_1) \\ - \frac{xb_2 + yb_3 + b_1}{x\sqrt{12yx^2 + 1}} = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\begin{aligned} \frac{-36b_2x^6\sqrt{12yx^2 + 1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 + 36x^5b_1 - 144x^4ya_1 - 12\sqrt{12yx^2 + 1}}{x^6} \\ = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} 36b_2x^6\sqrt{12yx^2 + 1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 \\ - 36x^5b_1 + 144x^4ya_1 + 12\sqrt{12yx^2 + 1}x^3a_2 + 6\sqrt{12yx^2 + 1}x^3b_3 \\ + 18\sqrt{12yx^2 + 1}x^2ya_3 - (12yx^2 + 1)^{\frac{3}{2}}a_3 + 18\sqrt{12yx^2 + 1}x^2a_1 \\ + 12x^3a_2 + 6x^3b_3 - 6x^2ya_3 + 18x^2a_1 - a_3\sqrt{12yx^2 + 1} - 2a_3 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} 36b_2x^6\sqrt{12yx^2 + 1} - 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 72x^4y^2a_3 \\ + 12(12yx^2 + 1)x^3a_2 + 6(12yx^2 + 1)x^3b_3 + 18(12yx^2 + 1)x^2ya_3 \\ - 36x^5b_1 - 72x^4ya_1 + 18(12yx^2 + 1)x^2a_1 + 12\sqrt{12yx^2 + 1}x^3a_2 \\ + 6\sqrt{12yx^2 + 1}x^3b_3 + 18\sqrt{12yx^2 + 1}x^2ya_3 - (12yx^2 + 1)^{\frac{3}{2}}a_3 \\ + 18\sqrt{12yx^2 + 1}x^2a_1 - 2(12yx^2 + 1)a_3 - a_3\sqrt{12yx^2 + 1} = 0 \end{aligned} \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 36b_2x^6\sqrt{12yx^2+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 - 36x^5b_1 + 144x^4ya_1 \\
& + 12\sqrt{12yx^2+1}x^3a_2 + 6\sqrt{12yx^2+1}x^3b_3 + 6\sqrt{12yx^2+1}x^2ya_3 + 12x^3a_2 \\
& + 6x^3b_3 + 18\sqrt{12yx^2+1}x^2a_1 - 6x^2ya_3 + 18x^2a_1 - 2a_3\sqrt{12yx^2+1} - 2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{12yx^2+1}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{12yx^2+1} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 36b_2v_1^6v_3 + 72v_1^5v_2a_2 + 144v_1^4v_2^2a_3 - 36v_1^6b_2 + 36v_1^5v_2b_3 + 144v_1^4v_2a_1 \\
& - 36v_1^5b_1 + 12v_3v_1^3a_2 + 6v_3v_1^2v_2a_3 + 6v_3v_1^3b_3 + 18v_3v_1^2a_1 \\
& + 12v_1^3a_2 - 6v_1^2v_2a_3 + 6v_1^3b_3 + 18v_1^2a_1 - 2a_3v_3 - 2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 36b_2v_1^6v_3 - 36v_1^6b_2 + (72a_2 + 36b_3)v_1^5v_2 - 36v_1^5b_1 + 144v_1^4v_2^2a_3 \\
& + 144v_1^4v_2a_1 + (12a_2 + 6b_3)v_1^3v_3 + (12a_2 + 6b_3)v_1^3 + 6v_3v_1^2v_2a_3 \\
& - 6v_1^2v_2a_3 + 18v_3v_1^2a_1 + 18v_1^2a_1 - 2a_3v_3 - 2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 18a_1 &= 0 \\
 144a_1 &= 0 \\
 -6a_3 &= 0 \\
 -2a_3 &= 0 \\
 6a_3 &= 0 \\
 144a_3 &= 0 \\
 -36b_1 &= 0 \\
 -36b_2 &= 0 \\
 36b_2 &= 0 \\
 12a_2 + 6b_3 &= 0 \\
 72a_2 + 36b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= a_2 \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= -2a_2
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= x \\
 \eta &= -2y
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= -2y - \left(\frac{1 + \sqrt{12y x^2 + 1}}{6x^3} \right) (x) \\
 &= \frac{-12y x^2 - \sqrt{12y x^2 + 1} - 1}{6x^2} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12yx^2 - \sqrt{12yx^2 + 1} - 1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{12yx^2 + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{12yx^2 + 1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{12yx^2 + 1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{12yx^2 + 1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1 \quad (1)$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12yx^2}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1 + \sqrt{12yx^2 + 1}}{6x^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(-1 + \sqrt{12yx^2 + 1})(b_3 - a_2)}{6x^3} - \frac{(-1 + \sqrt{12yx^2 + 1})^2 a_3}{36x^6} \\ - \left(-\frac{2y}{x^2 \sqrt{12yx^2 + 1}} + \frac{-1 + \sqrt{12yx^2 + 1}}{2x^4} \right) (xa_2 + ya_3 + a_1) \\ + \frac{xb_2 + yb_3 + b_1}{x \sqrt{12yx^2 + 1}} = 0 \end{aligned} \quad (\text{5E})$$

Putting the above in normal form gives

$$\begin{aligned} & \frac{-36b_2x^6\sqrt{12yx^2+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 - 36x^5b_1 + 144x^4ya_1 - 12\sqrt{12yx^2+1}}{36x^6} \\ & = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & 36b_2x^6\sqrt{12yx^2+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 \\ & + 36x^5b_1 - 144x^4ya_1 + 12\sqrt{12yx^2+1}x^3a_2 + 6\sqrt{12yx^2+1}x^3b_3 \\ & + 18\sqrt{12yx^2+1}x^2ya_3 - (12yx^2+1)^{\frac{3}{2}}a_3 + 18\sqrt{12yx^2+1}x^2a_1 \\ & - 12x^3a_2 - 6x^3b_3 + 6x^2ya_3 - 18x^2a_1 - a_3\sqrt{12yx^2+1} + 2a_3 = 0 \end{aligned} \quad (\text{6E})$$

Simplifying the above gives

$$\begin{aligned}
& 36b_2x^6\sqrt{12yx^2+1} + 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 72x^4y^2a_3 \\
& - 12(12yx^2+1)x^3a_2 - 6(12yx^2+1)x^3b_3 - 18(12yx^2+1)x^2ya_3 \\
& + 36x^5b_1 + 72x^4ya_1 - 18(12yx^2+1)x^2a_1 + 12\sqrt{12yx^2+1}x^3a_2 \\
& + 6\sqrt{12yx^2+1}x^3b_3 + 18\sqrt{12yx^2+1}x^2ya_3 - (12yx^2+1)^{\frac{3}{2}}a_3 \\
& + 18\sqrt{12yx^2+1}x^2a_1 + 2(12yx^2+1)a_3 - a_3\sqrt{12yx^2+1} = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 36b_2x^6\sqrt{12yx^2+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 + 36x^5b_1 - 144x^4ya_1 \\
& + 12\sqrt{12yx^2+1}x^3a_2 + 6\sqrt{12yx^2+1}x^3b_3 + 6\sqrt{12yx^2+1}x^2ya_3 - 12x^3a_2 \\
& - 6x^3b_3 + 18\sqrt{12yx^2+1}x^2a_1 + 6x^2ya_3 - 18x^2a_1 - 2a_3\sqrt{12yx^2+1} + 2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{12yx^2+1}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{12yx^2+1} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 36b_2v_1^6v_3 - 72v_1^5v_2a_2 - 144v_1^4v_2^2a_3 + 36v_1^6b_2 - 36v_1^5v_2b_3 - 144v_1^4v_2a_1 \\
& + 36v_1^5b_1 + 12v_3v_1^3a_2 + 6v_3v_1^2v_2a_3 + 6v_3v_1^3b_3 + 18v_3v_1^2a_1 \\
& - 12v_1^3a_2 + 6v_1^2v_2a_3 - 6v_1^3b_3 - 18v_1^2a_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 36b_2v_1^6v_3 + 36v_1^6b_2 + (-72a_2 - 36b_3)v_1^5v_2 + 36v_1^5b_1 - 144v_1^4v_2^2a_3 \\
& - 144v_1^4v_2a_1 + (12a_2 + 6b_3)v_1^3v_3 + (-12a_2 - 6b_3)v_1^3 \\
& + 6v_3v_1^2v_2a_3 + 6v_1^2v_2a_3 + 18v_3v_1^2a_1 - 18v_1^2a_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 -144a_1 &= 0 \\
 -18a_1 &= 0 \\
 18a_1 &= 0 \\
 -144a_3 &= 0 \\
 -2a_3 &= 0 \\
 2a_3 &= 0 \\
 6a_3 &= 0 \\
 36b_1 &= 0 \\
 36b_2 &= 0 \\
 -72a_2 - 36b_3 &= 0 \\
 -12a_2 - 6b_3 &= 0 \\
 12a_2 + 6b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= a_2 \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= -2a_2
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= x \\
 \eta &= -2y
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= -2y - \left(-\frac{-1 + \sqrt{12y x^2 + 1}}{6x^3} \right) (x) \\
 &= \frac{-12y x^2 + \sqrt{12y x^2 + 1} - 1}{6x^2} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12yx^2 + \sqrt{12yx^2 + 1} - 1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{12yx^2 + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{12yx^2 + 1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{12yx^2 + 1}} \\ S_y &= \frac{-\frac{1}{\sqrt{12yx^2 + 1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1 \quad (1)$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12yx^2}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying dAlembert
  trying simple symmetries for implicit equations
  Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
    *** Sublevel 3 ***
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying homogeneous types:
    trying homogeneous G
    <- homogeneous successful
    -----
  * Tackling next ODE.
    *** Sublevel 3 ***
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying homogeneous types:
    trying homogeneous G
    <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 97

```
dsolve(3*x^4*diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{12x^2}$$
$$y(x) = \frac{-i\sqrt{3}c_1 - 3x}{3x c_1^2}$$
$$y(x) = \frac{i\sqrt{3}c_1 - 3x}{3x c_1^2}$$
$$y(x) = \frac{i\sqrt{3}c_1 - 3x}{3x c_1^2}$$
$$y(x) = \frac{-i\sqrt{3}c_1 - 3x}{3x c_1^2}$$

✓ Solution by Mathematica

Time used: 0.512 (sec). Leaf size: 123

```
DSolve[3*x^4*y'[x]^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve} \left[\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x) \right]$$
$$\text{Solve} \left[\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x) \right]$$
$$y(x) \rightarrow 0$$

5.5 problem 21

5.5.1 Solving as dAlembert ode 1002

Internal problem ID [5327]

Internal file name [OUTPUT/4818_Friday_February_02_2024_05_14_06_AM_65941968/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 21.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$8yy'^2 - 2xy' + y = 0$$

5.5.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$8yp^2 - 2xp + y = 0$$

Solving for y from the above results in

$$y = \frac{2xp}{8p^2 + 1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{2p}{8p^2 + 1}$$

$$g = 0$$

Hence (2) becomes

$$p - \frac{2p}{8p^2 + 1} = x \left(\frac{2}{8p^2 + 1} - \frac{32p^2}{(8p^2 + 1)^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{2p}{8p^2 + 1} = 0$$

Solving for p from the above gives

$$p = 0$$

$$p = \frac{\sqrt{2}}{4}$$

$$p = -\frac{\sqrt{2}}{4}$$

Substituting these in (1A) gives

$$y = 0$$

$$y = -\frac{x\sqrt{2}}{4}$$

$$y = \frac{x\sqrt{2}}{4}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{2p(x)}{8p(x)^2 + 1}}{x \left(\frac{2}{8p(x)^2 + 1} - \frac{32p(x)^2}{(8p(x)^2 + 1)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{x(p) \left(\frac{2}{8p^2 + 1} - \frac{32p^2}{(8p^2 + 1)^2} \right)}{p - \frac{2p}{8p^2 + 1}} \quad (4)$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{2}{8p^3 + p}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{8p^3 + p} = 0$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{8p^3 + p} dp}$$
$$= e^{2\ln(p) - \ln(8p^2 + 1)}$$

Which simplifies to

$$\mu = \frac{p^2}{8p^2 + 1}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$
$$\frac{d}{dp}\left(\frac{p^2 x}{8p^2 + 1}\right) = 0$$

Integrating gives

$$\frac{p^2 x}{8p^2 + 1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{8p^2 + 1}$ results in

$$x(p) = \frac{c_3(8p^2 + 1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p . This results in

$$p = \frac{x + \sqrt{x^2 - 8y^2}}{8y}$$

$$p = -\frac{-x + \sqrt{x^2 - 8y^2}}{8y}$$

Substituting the above in the solution for x found above gives

$$x = \frac{16c_3x}{x + \sqrt{x^2 - 8y^2}}$$

$$x = -\frac{16c_3x}{-x + \sqrt{x^2 - 8y^2}}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = -\frac{x\sqrt{2}}{4} \tag{2}$$

$$y = \frac{x\sqrt{2}}{4} \tag{3}$$

$$x = \frac{16c_3x}{x + \sqrt{x^2 - 8y^2}} \tag{4}$$

$$x = -\frac{16c_3x}{-x + \sqrt{x^2 - 8y^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -\frac{x\sqrt{2}}{4}$$

Verified OK.

$$y = \frac{x\sqrt{2}}{4}$$

Verified OK.

$$x = \frac{16c_3x}{x + \sqrt{x^2 - 8y^2}}$$

Verified OK.

$$x = -\frac{16c_3x}{-x + \sqrt{x^2 - 8y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
*** Sublevel 2 ***  
Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying simple symmetries for implicit equations  
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 95

```
dsolve(8*y(x)*diff(y(x),x)^2-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}x}{4}$$

$$y(x) = \frac{\sqrt{2}x}{4}$$

$$y(x) = 0$$

$$\ln(x) + \operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{8y(x)^2-x^2}{x^2}}}\right) + \ln\left(\frac{y(x)}{x}\right) - c_1 = 0$$

$$\ln(x) - \operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{8y(x)^2-x^2}{x^2}}}\right) + \ln\left(\frac{y(x)}{x}\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 174

```
DSolve[8*y[x]*y'[x]^2-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{e^{4c_1}\sqrt{e^{8c_1}-2ix}}{2\sqrt{2}}$$

$$y(x) \rightarrow \frac{e^{4c_1}\sqrt{e^{8c_1}-2ix}}{2\sqrt{2}}$$

$$y(x) \rightarrow -\frac{e^{4c_1}\sqrt{2ix+e^{8c_1}}}{2\sqrt{2}}$$

$$y(x) \rightarrow \frac{e^{4c_1}\sqrt{2ix+e^{8c_1}}}{2\sqrt{2}}$$

$$y(x) \rightarrow 0$$

$$y(x) \rightarrow -\frac{x}{2\sqrt{2}}$$

$$y(x) \rightarrow \frac{x}{2\sqrt{2}}$$

5.6 problem 22

Internal problem ID [5328]

Internal file name [OUTPUT/4819_Friday_February_02_2024_05_14_06_AM_46240253/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 22.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _rational]
```

$$y^2 y' + 3xy' - y = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-3x + \sqrt{9x^2 + 4y^3}}{2y^2} \quad (1)$$

$$y' = -\frac{3x + \sqrt{9x^2 + 4y^3}}{2y^2} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(-3x + \sqrt{4y^3 + 9x^2})(b_3 - a_2)}{2y^2} - \frac{(-3x + \sqrt{4y^3 + 9x^2})^2 a_3}{4y^4} \\ - \frac{\left(-3 + \frac{9x}{\sqrt{4y^3 + 9x^2}}\right)(xa_2 + ya_3 + a_1)}{2y^2} \\ - \left(-\frac{-3x + \sqrt{4y^3 + 9x^2}}{y^3} + \frac{3}{\sqrt{4y^3 + 9x^2}}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{-4b_2y^4\sqrt{4y^3 + 9x^2} - 4xy^4b_2 + 8y^5a_2 - 12y^5b_3 + 12\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}xy^2a_2}{y^6} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} 4b_2y^4\sqrt{4y^3 + 9x^2} + 4xy^4b_2 - 8y^5a_2 + 12y^5b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\ + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\ + 36x^3yb_2 - 36x^2y^2a_2 + 54x^2y^2b_3 + 6xy^3a_3 + 4y^4b_1 \\ - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 - 9\sqrt{4y^3 + 9x^2}x^2a_3 - 12\sqrt{4y^3 + 9x^2}xyb_1 \\ + 6\sqrt{4y^3 + 9x^2}y^2a_1 + 54x^3a_3 + 36x^2yb_1 - 18xy^2a_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3+9x^2} - 12xy^4b_2 - 12y^5b_3 + 4(4y^3+9x^2)xyb_2 \\
& - 2(4y^3+9x^2)y^2a_2 + 6(4y^3+9x^2)y^2b_3 - 12\sqrt{4y^3+9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3+9x^2}xy^2a_2 - 18\sqrt{4y^3+9x^2}xy^2b_3 + 6\sqrt{4y^3+9x^2}y^3a_3 \quad (6E) \\
& - 18x^2y^2a_2 - 18xy^3a_3 - 12y^4b_1 - (4y^3+9x^2)^{\frac{3}{2}}a_3 \\
& + 6(4y^3+9x^2)xa_3 + 4(4y^3+9x^2)yb_1 - 9\sqrt{4y^3+9x^2}x^2a_3 \\
& - 12\sqrt{4y^3+9x^2}xyb_1 + 6\sqrt{4y^3+9x^2}y^2a_1 - 18xy^2a_1 = 0
\end{aligned}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 4xy^4b_2 + 4b_2y^4\sqrt{4y^3+9x^2} - 8y^5a_2 + 12y^5b_3 + 36x^3yb_2 - 12\sqrt{4y^3+9x^2}x^2yb_2 \\
& - 36x^2y^2a_2 + 54x^2y^2b_3 + 12\sqrt{4y^3+9x^2}xy^2a_2 - 18\sqrt{4y^3+9x^2}xy^2b_3 \\
& + 6xy^3a_3 + 2\sqrt{4y^3+9x^2}y^3a_3 + 4y^4b_1 + 54x^3a_3 - 18\sqrt{4y^3+9x^2}x^2a_3 \\
& + 36x^2yb_1 - 12\sqrt{4y^3+9x^2}xyb_1 - 18xy^2a_1 + 6\sqrt{4y^3+9x^2}y^2a_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4y^3+9x^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4y^3+9x^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& -8v_2^5a_2 + 4v_1v_2^4b_2 + 4b_2v_2^4v_3 + 12v_2^5b_3 - 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 + 6v_1v_2^3a_3 \quad (7E) \\
& + 2v_3v_2^3a_3 + 4v_2^4b_1 + 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 + 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\
& - 18v_1v_2^2a_1 + 6v_3v_2^2a_1 + 54v_1^3a_3 - 18v_3v_1^2a_3 + 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0
\end{aligned}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
 & 36v_1^3v_2b_2 + 54v_1^3a_3 + (-36a_2 + 54b_3)v_1^2v_2^2 - 12v_3v_1^2v_2b_2 + 36v_1^2v_2b_1 \\
 & - 18v_3v_1^2a_3 + 4v_1v_2^4b_2 + 6v_1v_2^3a_3 + (12a_2 - 18b_3)v_1v_2^2v_3 - 18v_1v_2^2a_1 \\
 & - 12v_3v_1v_2b_1 + (-8a_2 + 12b_3)v_2^5 + 4b_2v_2^4v_3 + 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0
 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 -18a_1 &= 0 \\
 6a_1 &= 0 \\
 -18a_3 &= 0 \\
 2a_3 &= 0 \\
 6a_3 &= 0 \\
 54a_3 &= 0 \\
 -12b_1 &= 0 \\
 4b_1 &= 0 \\
 36b_1 &= 0 \\
 -12b_2 &= 0 \\
 4b_2 &= 0 \\
 36b_2 &= 0 \\
 -36a_2 + 54b_3 &= 0 \\
 -8a_2 + 12b_3 &= 0 \\
 12a_2 - 18b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= \frac{3b_3}{2} \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= b_3
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for

any unknown in the RHS) gives

$$\xi = \frac{3x}{2}$$

$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2} \right) \left(\frac{3x}{2} \right) \\ &= \frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2} x}{4y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2} x}{4y^2}} dy\end{aligned}$$

Which results in

$$S = \ln(y) - \frac{2x \operatorname{arctanh} \left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}} \right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 + \frac{3x}{\sqrt{4y^3 + 9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \quad (1)$$

Verification of solutions

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2 - \frac{(\sqrt{4y^3 + 9x^2} + 3x)(b_3 - a_2)}{2y^2} - \frac{(\sqrt{4y^3 + 9x^2} + 3x)^2 a_3}{4y^4} \\
& + \frac{\left(\frac{9x}{\sqrt{4y^3 + 9x^2}} + 3\right)(xa_2 + ya_3 + a_1)}{2y^2} \\
& - \left(-\frac{3}{\sqrt{4y^3 + 9x^2}} + \frac{\sqrt{4y^3 + 9x^2} + 3x}{y^3}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
& \frac{-4b_2y^4\sqrt{4y^3 + 9x^2} + 4xy^4b_2 - 8y^5a_2 + 12y^5b_3 + 12\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}x^2yb_2}{-} \\
& = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3 + 9x^2} - 4xy^4b_2 + 8y^5a_2 - 12y^5b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\
& - 36x^3yb_2 + 36x^2y^2a_2 - 54x^2y^2b_3 - 6xy^3a_3 - 4y^4b_1 \\
& - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 - 9\sqrt{4y^3 + 9x^2}x^2a_3 - 12\sqrt{4y^3 + 9x^2}xyb_1 \\
& + 6\sqrt{4y^3 + 9x^2}y^2a_1 - 54x^3a_3 - 36x^2yb_1 + 18xy^2a_1 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3 + 9x^2} + 12xy^4b_2 + 12y^5b_3 - 4(4y^3 + 9x^2)xyb_2 \\
& + 2(4y^3 + 9x^2)y^2a_2 - 6(4y^3 + 9x^2)y^2b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\
& + 18x^2y^2a_2 + 18xy^3a_3 + 12y^4b_1 - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 \\
& - 6(4y^3 + 9x^2)xa_3 - 4(4y^3 + 9x^2)yb_1 - 9\sqrt{4y^3 + 9x^2}x^2a_3 \\
& - 12\sqrt{4y^3 + 9x^2}xyb_1 + 6\sqrt{4y^3 + 9x^2}y^2a_1 + 18xy^2a_1 = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& -4xy^4b_2 + 4b_2y^4\sqrt{4y^3 + 9x^2} + 8y^5a_2 - 12y^5b_3 - 36x^3yb_2 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 36x^2y^2a_2 - 54x^2y^2b_3 + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 \\
& - 6xy^3a_3 + 2\sqrt{4y^3 + 9x^2}y^3a_3 - 4y^4b_1 - 54x^3a_3 - 18\sqrt{4y^3 + 9x^2}x^2a_3 \\
& - 36x^2yb_1 - 12\sqrt{4y^3 + 9x^2}xyb_1 + 18xy^2a_1 + 6\sqrt{4y^3 + 9x^2}y^2a_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4y^3 + 9x^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4y^3 + 9x^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8v_2^5a_2 - 4v_1v_2^4b_2 + 4b_2v_2^4v_3 - 12v_2^5b_3 + 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 - 6v_1v_2^3a_3 \\ &+ 2v_3v_2^3a_3 - 4v_2^4b_1 - 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 - 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\ &+ 18v_1v_2^2a_1 + 6v_3v_2^2a_1 - 54v_1^3a_3 - 18v_3v_1^2a_3 - 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-36v_1^3v_2b_2 - 54v_1^3a_3 + (36a_2 - 54b_3)v_1^2v_2^2 - 12v_3v_1^2v_2b_2 - 36v_1^2v_2b_1 \\ &- 18v_3v_1^2a_3 - 4v_1v_2^4b_2 - 6v_1v_2^3a_3 + (12a_2 - 18b_3)v_1v_2^2v_3 + 18v_1v_2^2a_1 \\ &- 12v_3v_1v_2b_1 + (8a_2 - 12b_3)v_2^5 + 4b_2v_2^4v_3 - 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_1 &= 0 \\18a_1 &= 0 \\-54a_3 &= 0 \\-18a_3 &= 0 \\-6a_3 &= 0 \\2a_3 &= 0 \\-36b_1 &= 0 \\-12b_1 &= 0 \\-4b_1 &= 0 \\-36b_2 &= 0 \\-12b_2 &= 0 \\-4b_2 &= 0 \\4b_2 &= 0 \\8a_2 - 12b_3 &= 0 \\12a_2 - 18b_3 &= 0 \\36a_2 - 54b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= \frac{3b_3}{2} \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= \frac{3x}{2} \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(-\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2} \right) \left(\frac{3x}{2} \right) \\ &= \frac{4y^3 + 3\sqrt{4y^3 + 9x^2}x + 9x^2}{4y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^3 + 3\sqrt{4y^3 + 9x^2}x + 9x^2}{4y^2}} dy\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 - \frac{3x}{\sqrt{4y^3 + 9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \quad (1)$$

Verification of solutions

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
*** Sublevel 3 ***
Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3`[3/2*x, y], [2/3*y^3+3*x^2, y*x]
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 126

```
dsolve(y(x)^2*diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{18^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}}{2}$$

$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}} \left(3i3^{\frac{1}{6}} + 3^{\frac{2}{3}} \right)}{4}$$

$$y(x) = \frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}} \left(-3^{\frac{2}{3}} + 3i3^{\frac{1}{6}} \right)}{4}$$

$$y(x) = 0$$

$$y(x) = \text{RootOf} \left(-2 \ln(x) + 3 \left(\int^{-z} -\frac{4a^3 - 3\sqrt{4a^3 + 9} + 9}{-a(4a^3 + 9)} da \right) + 2c_1 \right) x^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.574 (sec). Leaf size: 239

```
DSolve[y[x]^2*y'[x]^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow -\sqrt[3]{-1} e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow (-1)^{2/3} e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow -\sqrt[3]{-1} e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow (-1)^{2/3} e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow 0$$

$$y(x) \rightarrow -\left(-\frac{3}{2}\right)^{2/3} x^{2/3}$$

$$y(x) \rightarrow -\left(\frac{3}{2}\right)^{2/3} x^{2/3}$$

$$y(x) \rightarrow \frac{\sqrt[3]{-13}^{2/3} x^{2/3}}{2^{2/3}}$$

5.7 problem 23

5.7.1 Solving as clairaut ode 1022

Internal problem ID [5329]

Internal file name [OUTPUT/4820_Friday_February_02_2024_05_14_11_AM_82456155/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 23.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**clairaut**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries], _Clairaut]
```

$$y'^2 - xy' + y = 0$$

5.7.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^2 + xp \\ &= -p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x . Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x \frac{dp}{dx} \right) + \left(g' \frac{dp}{dx} \right) \\ p &= p + (x + g') \frac{dp}{dx} \\ 0 &= (x + g') \frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p . The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -c_1^2 + c_1x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 2p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = -c_1^2 + c_1x \quad (1)$$

$$y = \frac{x^2}{4} \quad (2)$$

Verification of solutions

$$y = -c_1^2 + c_1x$$

Verified OK.

$$y = \frac{x^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
*** Sublevel 2 ***  
Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4}$$
$$y(x) = c_1(x - c_1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[y'[x]^2-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1(x - c_1)$$
$$y(x) \rightarrow \frac{x^2}{4}$$

5.8 problem 24

Internal problem ID [5330]

Internal file name [OUTPUT/4821_Friday_February_02_2024_05_14_11_AM_7693094/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 24.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "first_order_ode_lie_symmetry_calculated"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries], _rational]
```

$$16y^3y'^2 - 4xy' + y = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x + \sqrt{x^2 - 4y^4}}{8y^3} \quad (1)$$

$$y' = -\frac{-x + \sqrt{x^2 - 4y^4}}{8y^3} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{x + \sqrt{-4y^4 + x^2}}{8y^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(x + \sqrt{-4y^4 + x^2})(b_3 - a_2)}{8y^3} - \frac{(x + \sqrt{-4y^4 + x^2})^2 a_3}{64y^6} \\ - \frac{\left(1 + \frac{x}{\sqrt{-4y^4 + x^2}}\right)(xa_2 + ya_3 + a_1)}{8y^3} \\ - \left(-\frac{3(x + \sqrt{-4y^4 + x^2})}{8y^4} - \frac{1}{\sqrt{-4y^4 + x^2}}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{-64b_2y^6\sqrt{-4y^4 + x^2} + 32xy^6b_2 - 32y^7a_2 + 64y^7b_3 + 32y^6b_1 - 24\sqrt{-4y^4 + x^2}x^2y^2b_2 + 16\sqrt{-4y^4 + x^2}x^2y^2a_3}{8y^6} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} 64b_2y^6\sqrt{-4y^4 + x^2} - 32xy^6b_2 + 32y^7a_2 - 64y^7b_3 - 32y^6b_1 \\ + 24\sqrt{-4y^4 + x^2}x^2y^2b_2 - 16\sqrt{-4y^4 + x^2}xy^3a_2 + 32\sqrt{-4y^4 + x^2}xy^3b_3 \\ - 8\sqrt{-4y^4 + x^2}y^4a_3 + 24x^3y^2b_2 - 16x^2y^3a_2 + 32x^2y^3b_3 \\ + 24\sqrt{-4y^4 + x^2}xy^2b_1 - 8\sqrt{-4y^4 + x^2}y^3a_1 + 24x^2y^2b_1 \\ - 8xy^3a_1 - (-4y^4 + x^2)^{\frac{3}{2}}a_3 - \sqrt{-4y^4 + x^2}x^2a_3 - 2x^3a_3 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned}
& 64b_2y^6\sqrt{-4y^4+x^2} + 64xy^6b_2 + 64y^7b_3 + 64y^6b_1 \\
& + 24(-4y^4+x^2)xy^2b_2 - 8(-4y^4+x^2)y^3a_2 + 32(-4y^4+x^2)y^3b_3 \\
& + 24\sqrt{-4y^4+x^2}x^2y^2b_2 - 16\sqrt{-4y^4+x^2}xy^3a_2 + 32\sqrt{-4y^4+x^2}xy^3b_3 \quad (6E) \\
& - 8\sqrt{-4y^4+x^2}y^4a_3 - 8x^2y^3a_2 - 8xy^4a_3 + 24(-4y^4+x^2)y^2b_1 \\
& + 24\sqrt{-4y^4+x^2}xy^2b_1 - 8\sqrt{-4y^4+x^2}y^3a_1 - 8xy^3a_1 \\
& - (-4y^4+x^2)^{\frac{3}{2}}a_3 - 2(-4y^4+x^2)xa_3 - \sqrt{-4y^4+x^2}x^2a_3 = 0
\end{aligned}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& -32xy^6b_2 + 64b_2y^6\sqrt{-4y^4+x^2} + 32y^7a_2 - 64y^7b_3 - 32y^6b_1 + 24x^3y^2b_2 \\
& + 24\sqrt{-4y^4+x^2}x^2y^2b_2 - 16x^2y^3a_2 + 32x^2y^3b_3 - 16\sqrt{-4y^4+x^2}xy^3a_2 \\
& + 32\sqrt{-4y^4+x^2}xy^3b_3 - 4\sqrt{-4y^4+x^2}y^4a_3 + 24x^2y^2b_1 + 24\sqrt{-4y^4+x^2}xy^2b_1 \\
& - 8xy^3a_1 - 8\sqrt{-4y^4+x^2}y^3a_1 - 2x^3a_3 - 2\sqrt{-4y^4+x^2}x^2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{-4y^4+x^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{-4y^4+x^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 32v_2^7a_2 - 32v_1v_2^6b_2 + 64b_2v_2^6v_3 - 64v_2^7b_3 - 32v_2^6b_1 - 16v_1^2v_2^3a_2 - 16v_3v_1v_2^3a_2 \quad (7E) \\
& - 4v_3v_2^4a_3 + 24v_1^3v_2^2b_2 + 24v_3v_1^2v_2^2b_2 + 32v_1^2v_2^3b_3 + 32v_3v_1v_2^3b_3 \\
& - 8v_1v_2^3a_1 - 8v_3v_2^3a_1 + 24v_1^2v_2^2b_1 + 24v_3v_1v_2^2b_1 - 2v_1^3a_3 - 2v_3v_1^2a_3 = 0
\end{aligned}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 24v_1^3v_2^2b_2 - 2v_1^3a_3 + (-16a_2 + 32b_3)v_1^2v_2^3 + 24v_3v_1^2v_2^2b_2 + 24v_1^2v_2^2b_1 \\
& - 2v_3v_1^2a_3 - 32v_1v_2^6b_2 + (-16a_2 + 32b_3)v_1v_2^3v_3 - 8v_1v_2^3a_1 + 24v_3v_1v_2^2b_1 \\
& + (32a_2 - 64b_3)v_2^7 + 64b_2v_2^6v_3 - 32v_2^6b_1 - 4v_3v_2^4a_3 - 8v_3v_2^3a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-8a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-32b_1 &= 0 \\
24b_1 &= 0 \\
-32b_2 &= 0 \\
24b_2 &= 0 \\
64b_2 &= 0 \\
-16a_2 + 32b_3 &= 0 \\
32a_2 - 64b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{x + \sqrt{-4y^4 + x^2}}{8y^3} \right) (2x) \\ &= \frac{4y^4 - \sqrt{-4y^4 + x^2} x - x^2}{4y^3} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^4 - \sqrt{-4y^4 + x^2} x - x^2}{4y^3}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(4y^4 - x^2)}{4} - \frac{\ln(2y^2 + x)}{4} + \ln(y) - \frac{\ln(2y^2 - x)}{4} + \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2} \sqrt{-4y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + \sqrt{-4y^4 + x^2}}{8y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + \sqrt{-4y^4 + x^2}}{2\sqrt{-4y^4 + x^2} x} \\ S_y &= -\frac{4y^3}{\sqrt{-4y^4 + x^2} (x + \sqrt{-4y^4 + x^2})} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \frac{\ln(R)}{2} + c_1 \quad (4)$$

Summary

The solution(s) found are the following

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln(x + \sqrt{x^2 - 4y^4})}{2} = \frac{\ln(x)}{2} + c_1 \quad (1)$$

Verification of solutions

$$\frac{\ln(2)}{2} + \frac{\ln(x)}{2} + \frac{\ln(x + \sqrt{x^2 - 4y^4})}{2} = \frac{\ln(x)}{2} + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\sqrt{-4y^4 + x^2} - x}{8y^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(\sqrt{-4y^4 + x^2} - x)(b_3 - a_2)}{8y^3} - \frac{(\sqrt{-4y^4 + x^2} - x)^2 a_3}{64y^6}$$

$$+ \frac{\left(-1 + \frac{x}{\sqrt{-4y^4 + x^2}}\right)(xa_2 + ya_3 + a_1)}{8y^3} \quad (\text{5E})$$

$$- \left(\frac{1}{\sqrt{-4y^4 + x^2}} + \frac{\frac{3\sqrt{-4y^4 + x^2}}{8} - \frac{3x}{8}}{y^4}\right)(xb_2 + yb_3 + b_1) = 0$$

Putting the above in normal form gives

$$\frac{-64b_2y^6\sqrt{-4y^4 + x^2} - 32xy^6b_2 + 32y^7a_2 - 64y^7b_3 - 32y^6b_1 - 24\sqrt{-4y^4 + x^2}x^2y^2b_2 + 16\sqrt{-4y^4 + x^2}}{64y^6}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
& 64b_2y^6\sqrt{-4y^4+x^2} + 32xy^6b_2 - 32y^7a_2 + 64y^7b_3 + 32y^6b_1 \\
& + 24\sqrt{-4y^4+x^2}x^2y^2b_2 - 16\sqrt{-4y^4+x^2}xy^3a_2 + 32\sqrt{-4y^4+x^2}xy^3b_3 \\
& - 8\sqrt{-4y^4+x^2}y^4a_3 - 24x^3y^2b_2 + 16x^2y^3a_2 - 32x^2y^3b_3 \\
& + 24\sqrt{-4y^4+x^2}xy^2b_1 - 8\sqrt{-4y^4+x^2}y^3a_1 - 24x^2y^2b_1 \\
& + 8xy^3a_1 - (-4y^4+x^2)^{\frac{3}{2}}a_3 - \sqrt{-4y^4+x^2}x^2a_3 + 2x^3a_3 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
& 64b_2y^6\sqrt{-4y^4+x^2} - 64xy^6b_2 - 64y^7b_3 - 64y^6b_1 \\
& - 24(-4y^4+x^2)xy^2b_2 + 8(-4y^4+x^2)y^3a_2 - 32(-4y^4+x^2)y^3b_3 \\
& + 24\sqrt{-4y^4+x^2}x^2y^2b_2 - 16\sqrt{-4y^4+x^2}xy^3a_2 + 32\sqrt{-4y^4+x^2}xy^3b_3 \\
& - 8\sqrt{-4y^4+x^2}y^4a_3 + 8x^2y^3a_2 + 8xy^4a_3 - 24(-4y^4+x^2)y^2b_1 \\
& + 24\sqrt{-4y^4+x^2}xy^2b_1 - 8\sqrt{-4y^4+x^2}y^3a_1 + 8xy^3a_1 \\
& - (-4y^4+x^2)^{\frac{3}{2}}a_3 + 2(-4y^4+x^2)xa_3 - \sqrt{-4y^4+x^2}x^2a_3 = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 32xy^6b_2 + 64b_2y^6\sqrt{-4y^4+x^2} - 32y^7a_2 + 64y^7b_3 + 32y^6b_1 - 24x^3y^2b_2 \\
& + 24\sqrt{-4y^4+x^2}x^2y^2b_2 + 16x^2y^3a_2 - 32x^2y^3b_3 - 16\sqrt{-4y^4+x^2}xy^3a_2 \\
& + 32\sqrt{-4y^4+x^2}xy^3b_3 - 4\sqrt{-4y^4+x^2}y^4a_3 - 24x^2y^2b_1 + 24\sqrt{-4y^4+x^2}xy^2b_1 \\
& + 8xy^3a_1 - 8\sqrt{-4y^4+x^2}y^3a_1 + 2x^3a_3 - 2\sqrt{-4y^4+x^2}x^2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \sqrt{-4y^4+x^2} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \sqrt{-4y^4+x^2} = v_3 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& -32v_2^7a_2 + 32v_1v_2^6b_2 + 64b_2v_2^6v_3 + 64v_2^7b_3 + 32v_2^6b_1 + 16v_1^2v_2^3a_2 \\
& - 16v_3v_1v_2^3a_2 - 4v_3v_2^4a_3 - 24v_1^3v_2^2b_2 + 24v_3v_1^2v_2^2b_2 - 32v_1^2v_2^3b_3 + 32v_3v_1v_2^3b_3 \\
& + 8v_1v_2^3a_1 - 8v_3v_2^3a_1 - 24v_1^2v_2^2b_1 + 24v_3v_1v_2^2b_1 + 2v_1^3a_3 - 2v_3v_1^2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} & -24v_1^3v_2^2b_2 + 2v_1^3a_3 + (16a_2 - 32b_3)v_1^2v_2^3 + 24v_3v_1^2v_2^2b_2 - 24v_1^2v_2^2b_1 \\ & - 2v_3v_1^2a_3 + 32v_1v_2^6b_2 + (-16a_2 + 32b_3)v_1v_2^3v_3 + 8v_1v_2^3a_1 + 24v_3v_1v_2^2b_1 \\ & + (-32a_2 + 64b_3)v_2^7 + 64b_2v_2^6v_3 + 32v_2^6b_1 - 4v_3v_2^4a_3 - 8v_3v_2^3a_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_1 &= 0 \\ 8a_1 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ 2a_3 &= 0 \\ -24b_1 &= 0 \\ 24b_1 &= 0 \\ 32b_1 &= 0 \\ -24b_2 &= 0 \\ 24b_2 &= 0 \\ 32b_2 &= 0 \\ 64b_2 &= 0 \\ -32a_2 + 64b_3 &= 0 \\ -16a_2 + 32b_3 &= 0 \\ 16a_2 - 32b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 2x$$

$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(-\frac{\sqrt{-4y^4 + x^2} - x}{8y^3} \right) (2x) \\ &= \frac{4y^4 + \sqrt{-4y^4 + x^2} x - x^2}{4y^3}\end{aligned}$$

$$\xi = 0$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^4 + \sqrt{-4y^4 + x^2} x - x^2}{4y^3}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(4y^4 - x^2)}{4} - \frac{\ln(2y^2 + x)}{4} + \ln(y) - \frac{\ln(2y^2 - x)}{4} - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2} \sqrt{-4y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{-4y^4 + x^2} - x}{8y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x + \sqrt{-4y^4 + x^2}}{2\sqrt{-4y^4 + x^2}x} \\ S_y &= \frac{-4y^4 + 2x^2 + 2\sqrt{-4y^4 + x^2}x}{y\sqrt{-4y^4 + x^2}(x + \sqrt{-4y^4 + x^2})} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{-4y^4 + \sqrt{-4y^4 + x^2}x + x^2}{2\sqrt{-4y^4 + x^2}x(x + \sqrt{-4y^4 + x^2})} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \quad (4)$$

Which gives

$$y = e^{\frac{\ln(2)}{4} + \frac{\ln(-8e^{4c_1}e^{-2c_1} + 2e^{2c_1}e^{-2c_1}x)}{4} + \frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(2)}{4} + \frac{\ln(-8e^{4c_1}e^{-2c_1} + 2e^{2c_1}e^{-2c_1x})}{4}} + \frac{c_1}{2} \quad (1)$$

Verification of solutions

$$y = e^{\frac{\ln(2)}{4} + \frac{\ln(-8e^{4c_1}e^{-2c_1} + 2e^{2c_1}e^{-2c_1x})}{4}} + \frac{c_1}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
*** Sublevel 3 ***
Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3`[2*x, y]
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 105

```
dsolve(16*y(x)^3*diff(y(x),x)^2-4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\sqrt{-x}}{2}$$

$$y(x) = \frac{\sqrt{2}\sqrt{-x}}{2}$$

$$y(x) = -\frac{\sqrt{x}\sqrt{2}}{2}$$

$$y(x) = \frac{\sqrt{x}\sqrt{2}}{2}$$

$$y(x) = 0$$

$$y(x) = \text{RootOf}\left(-\ln(x) + 2\left(\int^{-z} -\frac{4a^4 - \sqrt{-4a^4 + 1} - 1}{-a(4a^4 - 1)} da\right) + c_1\right) \sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.563 (sec). Leaf size: 303

```
DSolve[16*y[x]^3*y'[x]^2-4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -e^{\frac{c_1}{4}} \sqrt[4]{e^{c_1} - ix}$$

$$y(x) \rightarrow -ie^{\frac{c_1}{4}} \sqrt[4]{e^{c_1} - ix}$$

$$y(x) \rightarrow ie^{\frac{c_1}{4}} \sqrt[4]{e^{c_1} - ix}$$

$$y(x) \rightarrow e^{\frac{c_1}{4}} \sqrt[4]{e^{c_1} - ix}$$

$$y(x) \rightarrow -e^{\frac{c_1}{4}} \sqrt[4]{ix + e^{c_1}}$$

$$y(x) \rightarrow -ie^{\frac{c_1}{4}} \sqrt[4]{ix + e^{c_1}}$$

$$y(x) \rightarrow ie^{\frac{c_1}{4}} \sqrt[4]{ix + e^{c_1}}$$

$$y(x) \rightarrow e^{\frac{c_1}{4}} \sqrt[4]{ix + e^{c_1}}$$

$$y(x) \rightarrow 0$$

$$y(x) \rightarrow -\frac{\sqrt{x}}{\sqrt{2}}$$

$$y(x) \rightarrow -\frac{i\sqrt{x}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{i\sqrt{x}}{\sqrt{2}}$$

$$y(x) \rightarrow \frac{\sqrt{x}}{\sqrt{2}}$$

5.9 problem 25

5.9.1 Solving as clairaut ode 1040

Internal problem ID [5331]

Internal file name [OUTPUT/4822_Friday_February_02_2024_05_14_16_AM_58506166/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 25.

ODE order: 1.

ODE degree: 5.

The type(s) of ODE detected by this program : "**clairaut**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Clairaut]
```

$$xy'^5 - yy'^4 + (x^2 + 1)y'^3 - 2xyy'^2 + (x + y^2)y' - y = 0$$

5.9.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$xp^5 - yp^4 + (x^2 + 1)p^3 - 2xyp^2 + (y^2 + x)p - y = 0$$

Solving for y from the above results in

$$y = \frac{xp^2 + 1}{p} \tag{1A}$$

$$y = p^3 + px \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved. Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{1}{p} \\ &= px + \frac{1}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x . Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x \frac{dp}{dx} \right) + \left(g' \frac{dp}{dx} \right) \\ p &= p + (x + g') \frac{dp}{dx} \\ 0 &= (x + g') \frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p . The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1x + \frac{1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{1}{p^2} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{1}{\sqrt{x}}$$
$$p_2 = -\frac{1}{\sqrt{x}}$$

Substituting the above back in (1) results in

$$y_1 = 2\sqrt{x}$$
$$y_2 = -2\sqrt{x}$$

Solving ode 2A We start by replacing y' by p which gives

$$y = p^3 + px$$
$$= p^3 + px$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x . Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = p^3$$

Taking derivative of (1) w.r.t. x gives

$$p = \frac{d}{dx}(xp + g)$$
$$p = \left(p + x \frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right)$$
$$p = p + (x + g') \frac{dp}{dx}$$
$$0 = (x + g') \frac{dp}{dx}$$

Where g' is derivative of $g(p)$ w.r.t. p . The general solution is given by

$$\frac{dp}{dx} = 0$$
$$p = c_1$$

Substituting this in (1) gives the general solution as

$$y = c_2^3 + c_2x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^3$, then the above equation becomes

$$\begin{aligned}x + g'(p) &= 3p^2 + x \\ &= 0\end{aligned}$$

Solving the above for p results in

$$\begin{aligned}p_1 &= \frac{\sqrt{-3x}}{3} \\ p_2 &= -\frac{\sqrt{-3x}}{3}\end{aligned}$$

Substituting the above back in (1) results in

$$\begin{aligned}y_1 &= -\frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9} \\ y_2 &= \frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + \frac{1}{c_1} \tag{1}$$

$$y = 2\sqrt{x} \tag{2}$$

$$y = -2\sqrt{x} \tag{3}$$

$$y = c_2^3 + c_2x \tag{4}$$

$$y = -\frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9} \tag{5}$$

$$y = \frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9} \tag{6}$$

Verification of solutions

$$y = c_1x + \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{x}$$

Verified OK.

$$y = -2\sqrt{x}$$

Verified OK.

$$y = c_2^3 + c_2x$$

Verified OK.

$$y = -\frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

$$y = \frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying dAlembert
  <- dAlembert successful
Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying dAlembert
  <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 63

```
dsolve(x*diff(y(x),x)^5-y(x)*diff(y(x),x)^4+(1+x^2)*diff(y(x),x)^3-2*x*y(x)*diff(y(x),x)^2+
```

$$y(x) = \frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9}$$

$$y(x) = -\frac{2\sqrt{3}(-x)^{\frac{3}{2}}}{9}$$

$$y(x) = c_1(c_1^2 + x)$$

$$y(x) = -2\sqrt{x}$$

$$y(x) = 2\sqrt{x}$$

$$y(x) = c_1x + \frac{1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 142

```
DSolve[x*y'[x]^5-y[x]*y'[x]^4+(1+x^2)*y'[x]^3-2*x*y[x]*y'[x]^2+(x+y[x]^2)*y'[x]-y[x]==0,y[x]
```

$$y(x) \rightarrow c_1 x + \frac{1}{c_1}$$

$$y(x) \rightarrow c_1(x + c_1^2)$$

$$y(x) \rightarrow \text{Indeterminate}$$

$$y(x) \rightarrow -x - 1$$

$$y(x) \rightarrow -2\sqrt{x}$$

$$y(x) \rightarrow 2\sqrt{x}$$

$$y(x) \rightarrow -\frac{2ix^{3/2}}{3\sqrt{3}}$$

$$y(x) \rightarrow \frac{2ix^{3/2}}{3\sqrt{3}}$$

$$y(x) \rightarrow x + 1$$

$$y(x) \rightarrow -\sqrt{-(x-1)^2}$$

$$y(x) \rightarrow \sqrt{-(x-1)^2}$$

5.10 problem 26

5.10.1 Solving as dAlembert ode 1047

Internal problem ID [5332]

Internal file name [OUTPUT/4823_Friday_February_02_2024_05_14_16_AM_5154403/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 26.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$xy'^2 - y'y - y = 0$$

5.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp^2 - py - y = 0$$

Solving for y from the above results in

$$y = \frac{xp^2}{p+1} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2}{p+1}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2}{p+1} = x \left(\frac{2p}{p+1} - \frac{p^2}{(p+1)^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2}{p+1} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2}{p(x)+1}}{x \left(\frac{2p(x)}{p(x)+1} - \frac{p(x)^2}{(p(x)+1)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{x(p) \left(\frac{2p}{p+1} - \frac{p^2}{(p+1)^2} \right)}{p - \frac{p^2}{p+1}} \quad (4)$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp} x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{p+2}{p+1}$$
$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{x(p)(p+2)}{p+1} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{p+2}{p+1} dp}$$
$$= e^{-p - \ln(p+1)}$$

Which simplifies to

$$\mu = \frac{e^{-p}}{p+1}$$

The ode becomes

$$\frac{d}{dp}\mu x = 0$$
$$\frac{d}{dp}\left(\frac{e^{-p}x}{p+1}\right) = 0$$

Integrating gives

$$\frac{e^{-p}x}{p+1} = c_2$$

Dividing both sides by the integrating factor $\mu = \frac{e^{-p}}{p+1}$ results in

$$x(p) = c_2(p+1)e^p$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p . This results in

$$p = \frac{y + \sqrt{y^2 + 4xy}}{2x}$$
$$p = -\frac{-y + \sqrt{y^2 + 4xy}}{2x}$$

Substituting the above in the solution for x found above gives

$$x = \frac{c_2 \left(y + \sqrt{y(y+4x)} + 2x \right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x}$$

$$x = -\frac{c_2 \left(-y + \sqrt{y(y+4x)} - 2x \right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{c_2 \left(y + \sqrt{y(y+4x)} + 2x \right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x} \tag{2}$$

$$x = -\frac{c_2 \left(-y + \sqrt{y(y+4x)} - 2x \right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{c_2 \left(y + \sqrt{y(y+4x)} + 2x \right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x}$$

Verified OK.

$$x = -\frac{c_2 \left(-y + \sqrt{y(y+4x)} - 2x \right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-y(x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{\left(\text{LambertW}\left(\frac{x e}{c_1}\right) - 1\right)^2 x}{\text{LambertW}\left(\frac{x e}{c_1}\right)}$$

✓ Solution by Mathematica

Time used: 2.255 (sec). Leaf size: 158

```
DSolve[x*y'[x]^2-y[x]*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{y(x)}{4x} + \frac{1}{4}\sqrt{\frac{y(x)}{x}}\sqrt{\frac{y(x)}{x} + 4} - \log\left(\sqrt{\frac{y(x)}{x} + 4} - \sqrt{\frac{y(x)}{x}}\right) = -\frac{\log(x)}{2} + c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{1}{4}\left(\frac{y(x)}{x} + \sqrt{\frac{y(x)}{x}}\sqrt{\frac{y(x)}{x} + 4} - 4\log\left(\sqrt{\frac{y(x)}{x} + 4} - \sqrt{\frac{y(x)}{x}}\right)\right) = \frac{\log(x)}{2} + c_1, y(x)\right]$$
$$y(x) \rightarrow 0$$

5.11 problem 27

Internal problem ID [5333]

Internal file name [OUTPUT/4824_Friday_February_02_2024_05_14_17_AM_71139267/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 27.

ODE order: 1.

ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries]]
```

$$y - 2xy' - y^2y'^3 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{6y} - \frac{4x}{y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \quad (1)$$

$$y' = -\frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{12y} + \frac{2x}{y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{6y}\right)}{\dots} \quad (2)$$

$$y' = -\frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{12y} + \frac{2x}{y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{6y}\right)}{\dots} \quad (3)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x}{6y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2 + \frac{\left((108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) (b_3 - a_2)}{6y (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \\
& - \frac{\left((108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right)^2 a_3}{36y^2 (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}}} \\
& - \left(\frac{\frac{384\sqrt{3}x^2}{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}\sqrt{27y^4 + 32x^3}} - 24}{6y (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
& \left. - \frac{32 \left((108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) \sqrt{3} x^2}{y (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{4}{3}} \sqrt{27y^4 + 32x^3}} \right) (xa_2 + ya_3 + a_1) \quad (5E) \\
& - \left(\frac{216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}}}{9y (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}}} \right. \\
& - \frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x}{6y^2 (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \\
& \left. - \frac{\left((108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{18y (108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{4}{3}}} \right) (xb_2 \\
& + yb_3 + b_1) = 0
\end{aligned}$$

Putting the above in normal form gives

$$\begin{aligned}
& \frac{15552\sqrt{3} x y^5 a_2 - 18432\sqrt{3} x^4 y b_3 + 5184\sqrt{27y^4 + 32x^3} x y^3 a_2 - 192(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} \sqrt{3} x}{=} \\
& = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
& 15552\sqrt{3}xy^5a_2 - 18432\sqrt{3}x^4yb_3 \\
& + 5184\sqrt{27y^4 + 32x^3}xy^3a_2 \\
& - 192\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}x^3ya_2 \\
& - 192\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}x^2y^2a_3 \\
& - 72\left(108y^2\right. \\
& \left.+ 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{27y^4 + 32x^3}xy^2b_2 \\
& - 192\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}x^2ya_1 \\
& - 216\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}y^5b_3 \\
& + \left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{5}{3}}\sqrt{27y^4 + 32x^3}xb_2 \\
& - \left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{5}{3}}\sqrt{27y^4 + 32x^3}ya_2 \\
& + 2\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{5}{3}}\sqrt{27y^4 + 32x^3}yb_3 \\
& + 6b_2y^2\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4 + 32x^3} \\
& - 216\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}y^4b_1 \\
& - 12960\sqrt{3}x^2y^4b_2 - 20736\sqrt{3}xy^5b_3 \\
& + 8\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{4}{3}}\sqrt{27y^4 + 32x^3}xa_3 \\
& - 72\left(108y^2\right. \\
& \left.+ 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{27y^4 + 32x^3}y^3b_3 \\
& + 13824\sqrt{3}x^4ya_2 - 9216\sqrt{3}x^3y^2a_3 \\
& - 12960\sqrt{3}xy^4b_1 - 96\left(108y^2\right. \\
& \left.+ 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{27y^4 + 32x^3}x^2a_3 \\
& - 72\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{27y^4 + 32x^3}y^2b_1 \\
& - 4320\sqrt{27y^4 + 32x^3}x^2y^2b_2 - 6912\sqrt{27y^4 + 32x^3}xy^3b_3 \\
& + 4608\sqrt{3}x^3ya_1 - 4320\sqrt{27y^4 + 32x^3}xy^2b_1 \\
& - 216\left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}\sqrt{3}xy^4b_2 \\
& + \left(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3}\right)^{\frac{5}{3}}\sqrt{27y^4 + 32x^3}b_1 \\
& + 648\sqrt{27y^4 + 32x^3}y^4a_3 + 2592\sqrt{27y^4 + 32x^3}y^3a_1 \\
& - 3888\sqrt{3}y^6a_3 - 9216\sqrt{3}x^5b_2 + 7776\sqrt{3}y^5a_1
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4 + 32x^3} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ \begin{aligned} x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 \right. \\ \left. + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4 + 32x^3} = v_5 \end{aligned} \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -110592\sqrt{3} v_1^4 v_2 b_3 + 31104 v_5 v_1 v_2^3 a_2 - 648 v_4 v_5 v_2^3 a_2 \\ & + 2592 v_4 \sqrt{3} v_2^5 b_3 + 648 v_4 \sqrt{3} v_2^4 b_1 - 77760 \sqrt{3} v_1^2 v_2^4 b_2 \\ & - 124416 \sqrt{3} v_1 v_2^5 b_3 + 864 v_4 v_5 v_2^3 b_3 + 82944 \sqrt{3} v_1^4 v_2 a_2 \\ & - 55296 \sqrt{3} v_1^3 v_2^2 a_3 - 77760 \sqrt{3} v_1 v_2^4 b_1 - 576 v_4 v_5 v_1^2 a_3 \\ & + 216 v_4 v_5 v_2^2 b_1 - 25920 v_5 v_1^2 v_2^2 b_2 - 41472 v_5 v_1 v_2^3 b_3 \\ & + 27648 \sqrt{3} v_1^3 v_2 a_1 - 25920 v_5 v_1 v_2^2 b_1 + 3888 v_3 v_5 v_2^4 b_2 \\ & - 1944 \sqrt{3} v_4 v_2^5 a_2 + 11664 \sqrt{3} v_3 v_2^6 b_2 + 2304 \sqrt{3} v_1^4 v_4 b_2 \\ & + 18432 \sqrt{3} v_1^4 v_3 a_3 + 2304 \sqrt{3} v_1^3 v_4 b_1 + 5184 v_1 v_3 v_5 v_2^2 a_3 \\ & - 3456 v_4 \sqrt{3} v_1^3 v_2 a_2 - 1152 v_4 \sqrt{3} v_1^2 v_2^2 a_3 + 216 v_4 v_5 v_1 v_2^2 b_2 \\ & - 1152 v_4 \sqrt{3} v_1^2 v_2 a_1 + 648 v_4 \sqrt{3} v_1 v_2^4 b_2 + 4608 \sqrt{3} v_1^3 v_4 v_2 b_3 \\ & + 13824 \sqrt{3} v_1^3 v_3 v_2^2 b_2 + 15552 \sqrt{3} v_1 v_3 v_2^4 a_3 + 93312 \sqrt{3} v_1 v_2^5 a_2 \\ & - 13824 v_1^3 v_5 a_3 - 7776 v_5 v_2^4 a_3 + 15552 v_5 v_2^3 a_1 - 23328 \sqrt{3} v_2^6 a_3 \\ & - 55296 \sqrt{3} v_1^5 b_2 + 46656 \sqrt{3} v_2^5 a_1 - 55296 \sqrt{3} v_1^4 b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 648v_4\sqrt{3}v_2^4b_1 - 77760\sqrt{3}v_1^2v_2^4b_2 - 55296\sqrt{3}v_1^3v_2^2a_3 \\
& - 77760\sqrt{3}v_1v_2^4b_1 - 576v_4v_5v_1^2a_3 + 216v_4v_5v_2^2b_1 - 25920v_5v_1^2v_2^2b_2 \\
& + 27648\sqrt{3}v_1^3v_2a_1 - 25920v_5v_1v_2^2b_1 + 3888v_3v_5v_2^4b_2 \\
& + 11664\sqrt{3}v_3v_2^6b_2 + 2304\sqrt{3}v_1^4v_4b_2 + 18432\sqrt{3}v_1^4v_3a_3 \\
& + 2304\sqrt{3}v_1^3v_4b_1 + \left(93312\sqrt{3}a_2 - 124416\sqrt{3}b_3\right)v_1v_2^5 \\
& + \left(-1944\sqrt{3}a_2 + 2592\sqrt{3}b_3\right)v_2^5v_4 \\
& + \left(82944\sqrt{3}a_2 - 110592\sqrt{3}b_3\right)v_1^4v_2 + (-648a_2 + 864b_3)v_2^3v_4v_5 \\
& + 5184v_1v_3v_5v_2^2a_3 - 1152v_4\sqrt{3}v_1^2v_2^2a_3 + 216v_4v_5v_1v_2^2b_2 \\
& - 1152v_4\sqrt{3}v_1^2v_2a_1 + 648v_4\sqrt{3}v_1v_2^4b_2 + 13824\sqrt{3}v_1^3v_3v_2^2b_2 \\
& + 15552\sqrt{3}v_1v_3v_2^4a_3 + \left(-3456\sqrt{3}a_2 + 4608\sqrt{3}b_3\right)v_1^3v_2v_4 \\
& + (31104a_2 - 41472b_3)v_1v_2^3v_5 - 13824v_1^3v_5a_3 \\
& - 7776v_5v_2^4a_3 + 15552v_5v_2^3a_1 - 23328\sqrt{3}v_2^6a_3 \\
& - 55296\sqrt{3}v_1^5b_2 + 46656\sqrt{3}v_2^5a_1 - 55296\sqrt{3}v_1^4b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 15552a_1 &= 0 \\
 -13824a_3 &= 0 \\
 -7776a_3 &= 0 \\
 -576a_3 &= 0 \\
 5184a_3 &= 0 \\
 -25920b_1 &= 0 \\
 216b_1 &= 0 \\
 -25920b_2 &= 0 \\
 216b_2 &= 0 \\
 3888b_2 &= 0 \\
 -1152\sqrt{3}a_1 &= 0 \\
 27648\sqrt{3}a_1 &= 0 \\
 46656\sqrt{3}a_1 &= 0 \\
 -55296\sqrt{3}a_3 &= 0 \\
 -23328\sqrt{3}a_3 &= 0 \\
 -1152\sqrt{3}a_3 &= 0 \\
 15552\sqrt{3}a_3 &= 0 \\
 18432\sqrt{3}a_3 &= 0 \\
 -77760\sqrt{3}b_1 &= 0 \\
 -55296\sqrt{3}b_1 &= 0 \\
 648\sqrt{3}b_1 &= 0 \\
 2304\sqrt{3}b_1 &= 0 \\
 -77760\sqrt{3}b_2 &= 0 \\
 -55296\sqrt{3}b_2 &= 0 \\
 648\sqrt{3}b_2 &= 0 \\
 2304\sqrt{3}b_2 &= 0 \\
 11664\sqrt{3}b_2 &= 0 \\
 13824\sqrt{3}b_2 &= 0 \\
 -648a_2 + 864b_3 &= 0 \\
 31104a_2 - 41472b_3 &= 0 \\
 -3456\sqrt{3}a_2 + 4608\sqrt{3}b_3 &= 0 \\
 -1944\sqrt{3}a_2 + 2592\sqrt{3}b_3 &= 0 \\
 82944\sqrt{3}a_2 - 110592\sqrt{3}b_3 &= 0 \\
 93312\sqrt{3}a_2 - 124416\sqrt{3}b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= \frac{4b_3}{3} \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= \frac{4x}{3} \\ \eta &= y\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{y}{\frac{4x}{3}} \\ &= \frac{3y}{4x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x^{\frac{3}{4}}$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^{\frac{3}{4}}}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{4x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{3 \ln(x)}{4} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x}{6y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{4x^{\frac{7}{4}}} \\ R_y &= \frac{1}{x^{\frac{3}{4}}} \\ S_x &= \frac{3}{4x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^{\frac{3}{4}}y(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}{2(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{2}{3}}x - 9y^2(108y^2 + 12\sqrt{3}\sqrt{27y^4 + 32x^3})^{\frac{1}{3}} - 48x^2} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9R12^{\frac{1}{3}}(\sqrt{3}\sqrt{27R^4+32}+9R^2)^{\frac{1}{3}}}{212^{\frac{2}{3}}(\sqrt{3}\sqrt{27R^4+32}+9R^2)^{\frac{2}{3}}-912^{\frac{1}{3}}(\sqrt{3}\sqrt{27R^4+32}+9R^2)^{\frac{1}{3}}R^2-48}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \int \frac{9R(12\sqrt{81R^4+96}+108R^2)^{\frac{1}{3}}}{418^{\frac{1}{3}}\left((\sqrt{81R^4+96}+9R^2)^2\right)^{\frac{1}{3}}-9R^2(12\sqrt{81R^4+96}+108R^2)^{\frac{1}{3}}-48}dR + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x)}{4} = \int_{x^{\frac{3}{4}}}^{\frac{y}{x^{\frac{3}{4}}}} \frac{9_a(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}}{418^{\frac{1}{3}}\left((\sqrt{81_a^4+96}+9_a^2)^2\right)^{\frac{1}{3}}-9_a^2(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}-48}d_a + c_1$$

Which simplifies to

$$\frac{3\ln(x)}{4} = \int_{x^{\frac{3}{4}}}^{\frac{y}{x^{\frac{3}{4}}}} \frac{9_a(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}}{418^{\frac{1}{3}}\left((\sqrt{81_a^4+96}+9_a^2)^2\right)^{\frac{1}{3}}-9_a^2(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}-48}d_a + c_1$$

Summary

The solution(s) found are the following

$$\begin{aligned} & \frac{3\ln(x)}{4} \quad (1) \\ & = \int_{x^{\frac{3}{4}}}^{\frac{y}{x^{\frac{3}{4}}}} \frac{9_a(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}}{418^{\frac{1}{3}}\left((\sqrt{81_a^4+96}+9_a^2)^2\right)^{\frac{1}{3}}-9_a^2(12\sqrt{81_a^4+96}+108_a^2)^{\frac{1}{3}}-48}d_a \\ & + c_1 \end{aligned}$$

Verification of solutions

$$\frac{3 \ln(x)}{4} = \int \frac{y^{\frac{3}{4}}}{x^{\frac{3}{4}}} \frac{9_a(12\sqrt{81_a^4 + 96} + 108_a a^2)^{\frac{1}{3}}}{4 \cdot 18^{\frac{1}{3}} \left((\sqrt{81_a^4 + 96} + 9_a a^2)^{\frac{1}{3}} - 9_a a^2 (12\sqrt{81_a^4 + 96} + 108_a a^2)^{\frac{1}{3}} - 48 \right)} d_a a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
 & b_2 \tag{5E} \\
 & + \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x \right) (b_3 - a_2)}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \\
 & - \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x \right)^2 a_3}{144y^2 (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}}} \\
 & - \left(\frac{\frac{1152ix^2}{(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}} \sqrt{27y^4 + 32x^3}} + 24i\sqrt{3} - \frac{384\sqrt{3}x^2}{(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}} \sqrt{27y^4 + 32x^3}} + 24}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. - \frac{16 \left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x \right) \sqrt{3} x^2}{y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{4}{3}} \sqrt{27y^4 + 32x^3}} \right) \\
 & + ya_3 + a_1) - \left(\frac{\frac{2i\sqrt{3} \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{3(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} - \frac{2 \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{3(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. - \frac{i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x}{12y^2 (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. - \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x - (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24x \right) \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{36y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{4}{3}}} \right) \\
 & + yb_3 + b_1) = 0
 \end{aligned}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4 + 32x^3} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4 + 32x^3} = v_5 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 139968iv_2^6a_3 + 331776iv_1^5b_2 - 279936iv_2^5a_1 \\
& + 331776iv_1^4b_1 + 27648v_1^3v_5a_3 + 46656\sqrt{3}v_2^6a_3 \\
& + 110592\sqrt{3}v_1^5b_2 - 93312\sqrt{3}v_2^5a_1 + 110592\sqrt{3}v_1^4b_1 \\
& + 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1 + 432iv_4v_5\sqrt{3}v_1v_2^2b_2 \\
& - 5184v_4\sqrt{3}v_2^5b_3 - 1296v_4\sqrt{3}v_2^4b_1 + 155520\sqrt{3}v_1^2v_2^4b_2 \\
& + 248832\sqrt{3}v_1v_2^5b_3 - 1728v_4v_5v_2^3b_3 - 165888\sqrt{3}v_1^4v_2a_2 \\
& + 110592\sqrt{3}v_1^3v_2^2a_3 + 155520\sqrt{3}v_1v_2^4b_1 - 432v_4v_5v_2^2b_1 \\
& + 51840v_5v_1^2v_2^2b_2 + 82944v_5v_1v_2^3b_3 - 55296\sqrt{3}v_1^3v_2a_1 \\
& + 51840v_5v_1v_2^2b_1 - 165888iv_1^3v_2a_1 + 466560iv_1^2v_2^4b_2 \\
& - 559872iv_1v_2^5a_2 + 746496iv_1v_2^5b_3 - 497664iv_1^4v_2a_2 \\
& + 663552iv_1^4v_2b_3 + 331776iv_1^3v_2^2a_3 + 466560iv_1v_2^4b_1 \\
& - 11664iv_4v_2^5a_2 + 15552iv_4v_2^5b_3 + 13824iv_1^4v_4b_2 \\
& + 3888iv_4v_2^4b_1 + 13824iv_1^3v_4b_1 + 15552v_5v_3v_2^4b_2 \\
& - 62208v_5v_1v_2^3a_2 + 3888\sqrt{3}v_4v_2^5a_2 + 46656\sqrt{3}v_3v_2^6b_2 \\
& - 4608\sqrt{3}v_1^4v_4b_2 + 73728\sqrt{3}v_1^4v_3a_3 - 4608\sqrt{3}v_1^3v_4b_1 \\
& - 186624\sqrt{3}v_1v_2^5a_2 + 221184\sqrt{3}v_1^4v_2b_3 + 1152v_4v_5v_1^2a_3 \\
& + 1296v_5v_4v_2^3a_2 - 1152iv_4v_5\sqrt{3}v_1^2a_3 - 1296i\sqrt{3}v_5v_4v_2^3a_2 \\
& + 1728iv_4v_5\sqrt{3}v_2^3b_3 + 432iv_4v_5\sqrt{3}v_2^2b_1 \\
& + 51840iv_5\sqrt{3}v_1^2v_2^2b_2 - 62208iv_5\sqrt{3}v_1v_2^3a_2 \\
& + 82944iv_5\sqrt{3}v_1v_2^3b_3 + 51840iv_5\sqrt{3}v_1v_2^2b_1 \\
& - 6912iv_4v_1^2v_2^2a_3 - 6912iv_4v_1^2v_2a_1 - 31104iv_5\sqrt{3}v_2^3a_1 \\
& + 15552iv_5\sqrt{3}v_2^4a_3 + 27648i\sqrt{3}v_1^3v_5a_3 + 3888iv_4v_1v_2^4b_2 \\
& - 20736iv_4v_1^3v_2a_2 + 27648iv_1^3v_4v_2b_3 + 20736v_1v_5v_3v_2^2a_3 \\
& - 9216\sqrt{3}v_1^3v_4v_2b_3 + 55296\sqrt{3}v_1^3v_3v_2^2b_2 \\
& + 62208\sqrt{3}v_1v_3v_2^4a_3 - 1296v_4\sqrt{3}v_1v_2^4b_2 \\
& + 6912v_4\sqrt{3}v_1^3v_2a_2 + 2304v_4\sqrt{3}v_1^2v_2^2a_3 \\
& - 432v_4v_5v_1v_2^2b_2 + 2304v_4\sqrt{3}v_1^2v_2a_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& \left(-31104i\sqrt{3}a_1 - 31104a_1\right)v_2^3v_5 \\
& + \left(331776ib_2 + 110592\sqrt{3}b_2\right)v_1^5 \\
& + \left(331776ib_1 + 110592\sqrt{3}b_1\right)v_1^4 \\
& + \left(139968ia_3 + 46656\sqrt{3}a_3\right)v_2^6 \\
& + \left(-279936ia_1 - 93312\sqrt{3}a_1\right)v_2^5 + 15552v_5v_3v_2^4b_2 \\
& + 46656\sqrt{3}v_3v_2^6b_2 + 73728\sqrt{3}v_1^4v_3a_3 + \left(-20736ia_2\right. \\
& + 27648ib_3 + 6912\sqrt{3}a_2 - 9216\sqrt{3}b_3\left.)v_1^3v_2v_4\right. \\
& + \left(-6912ia_3 + 2304\sqrt{3}a_3\right)v_1^2v_2^2v_4 \\
& + \left(51840i\sqrt{3}b_2 + 51840b_2\right)v_1^2v_2^2v_5 \\
& + \left(-6912ia_1 + 2304\sqrt{3}a_1\right)v_1^2v_2v_4 \\
& + \left(-1152i\sqrt{3}a_3 + 1152a_3\right)v_1^2v_4v_5 \\
& + \left(3888ib_2 - 1296\sqrt{3}b_2\right)v_1v_2^4v_4 \\
& + \left(-62208i\sqrt{3}a_2 + 82944i\sqrt{3}b_3 - 62208a_2\right. \\
& + 82944b_3\left.)v_1v_2^3v_5 + \left(51840i\sqrt{3}b_1 + 51840b_1\right)v_1v_2^2v_5\right. \\
& + \left(-1296i\sqrt{3}a_2 + 1728i\sqrt{3}b_3 + 1296a_2 - 1728b_3\right)v_2^3v_4v_5 \\
& + \left(432i\sqrt{3}b_1 - 432b_1\right)v_2^2v_4v_5 + \left(-497664ia_2\right. \\
& + 663552ib_3 - 165888\sqrt{3}a_2 + 221184\sqrt{3}b_3\left.)v_1^4v_2\right. \\
& + \left(13824ib_2 - 4608\sqrt{3}b_2\right)v_1^4v_4 \\
& + \left(331776ia_3 + 110592\sqrt{3}a_3\right)v_1^3v_2^2 \\
& + \left(-165888ia_1 - 55296\sqrt{3}a_1\right)v_1^3v_2 \\
& + \left(13824ib_1 - 4608\sqrt{3}b_1\right)v_1^3v_4 \\
& + \left(27648i\sqrt{3}a_3 + 27648a_3\right)v_1^3v_5 \\
& + \left(466560ib_2 + 155520\sqrt{3}b_2\right)v_1^2v_2^4 \\
& + \left(-559872ia_2 + 746496ib_3 - 186624\sqrt{3}a_2\right. \\
& + 248832\sqrt{3}b_3\left.)v_1v_2^5 + \left(466560ib_1 + 155520\sqrt{3}b_1\right)v_1v_2^4\right. \\
& + \left(-11664ia_2 + 15552ib_3 + \frac{3888\sqrt{3}a_2 - 5184\sqrt{3}b_3}{1066}\right)v_2^5v_4 \\
& + \left(3888ib_1 - 1296\sqrt{3}b_1\right)v_2^4v_4 \\
& + \left(15552i\sqrt{3}a_2 + 15552a_2\right)v_2^4v_4
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
20736a_3 &= 0 \\
15552b_2 &= 0 \\
62208\sqrt{3}a_3 &= 0 \\
73728\sqrt{3}a_3 &= 0 \\
46656\sqrt{3}b_2 &= 0 \\
55296\sqrt{3}b_2 &= 0 \\
-279936ia_1 - 93312\sqrt{3}a_1 &= 0 \\
-165888ia_1 - 55296\sqrt{3}a_1 &= 0 \\
-6912ia_1 + 2304\sqrt{3}a_1 &= 0 \\
-6912ia_3 + 2304\sqrt{3}a_3 &= 0 \\
3888ib_1 - 1296\sqrt{3}b_1 &= 0 \\
3888ib_2 - 1296\sqrt{3}b_2 &= 0 \\
13824ib_1 - 4608\sqrt{3}b_1 &= 0 \\
13824ib_2 - 4608\sqrt{3}b_2 &= 0 \\
139968ia_3 + 46656\sqrt{3}a_3 &= 0 \\
331776ia_3 + 110592\sqrt{3}a_3 &= 0 \\
331776ib_1 + 110592\sqrt{3}b_1 &= 0 \\
331776ib_2 + 110592\sqrt{3}b_2 &= 0 \\
466560ib_1 + 155520\sqrt{3}b_1 &= 0 \\
466560ib_2 + 155520\sqrt{3}b_2 &= 0 \\
-31104i\sqrt{3}a_1 - 31104a_1 &= 0 \\
-1152i\sqrt{3}a_3 + 1152a_3 &= 0 \\
432i\sqrt{3}b_1 - 432b_1 &= 0 \\
432i\sqrt{3}b_2 - 432b_2 &= 0 \\
15552i\sqrt{3}a_3 + 15552a_3 &= 0 \\
27648i\sqrt{3}a_3 + 27648a_3 &= 0 \\
51840i\sqrt{3}b_1 + 51840b_1 &= 0 \\
51840i\sqrt{3}b_2 + 51840b_2 &= 0 \\
-559872ia_2 + 746496ib_3 - 186624\sqrt{3}a_2 + 248832\sqrt{3}b_3 &= 0 \\
-497664ia_2 + 663552ib_3 - 165888\sqrt{3}a_2 + 221184\sqrt{3}b_3 &= 0 \\
-20736ia_2 + 27648ib_3 + 6912\sqrt{3}a_2 - 9216\sqrt{3}b_3 &= 0 \\
-11664ia_2 + 15552ib_3 + 3888\sqrt{3}a_2 - 5184\sqrt{3}b_3 &= 0 \\
-62208i\sqrt{3}a_2 + 82944i\sqrt{3}b_3 - 62208a_2 + 82944b_3 &= 0 \\
-1296i\sqrt{3}a_2 + 1728i\sqrt{3}b_3 + 1296a_2 - 1728b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{4b_3}{3} \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= \frac{4x}{3} \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R . Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$y' = -\frac{i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
 & b_2 \tag{5E} \\
 & \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) (b_3 - a_2)}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \\
 & - \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right)^2 a_3}{144y^2 (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}}} \\
 & - \left(\frac{\frac{1152ix^2}{(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}} \sqrt{27y^4 + 32x^3}} + 24i\sqrt{3} + \frac{384\sqrt{3}x^2}{(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}} \sqrt{27y^4 + 32x^3}} - 24}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. + \frac{16 \left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) \sqrt{3} x^2}{y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{4}{3}} \sqrt{27y^4 + 32x^3}} \right) \\
 & + ya_3 + a_1 - \left(\frac{\frac{2i\sqrt{3} \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{3(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} + \frac{144y + \frac{432\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}}}{(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}}}{12y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. + \frac{i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x}{12y^2 (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{1}{3}}} \right. \\
 & \left. + \frac{\left(i\sqrt{3} (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} + 24i\sqrt{3}x + (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{2}{3}} - 24x \right) \left(216y + \frac{648\sqrt{3}y^3}{\sqrt{27y^4 + 32x^3}} \right)}{36y (108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3})^{\frac{4}{3}}} \right) \\
 & + yb_3 + b_1 = 0
 \end{aligned}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}}, \sqrt{27y^4 + 32x^3} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} = v_3, \left(108y^2 + 12\sqrt{3} \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4 + 32x^3} = v_5 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& -51840i\sqrt{3}v_5v_1^2v_2^2b_2 + 62208i\sqrt{3}v_5v_1v_2^3a_2 \\
& - 82944i\sqrt{3}v_5v_1v_2^3b_3 - 51840i\sqrt{3}v_5v_1v_2^2b_1 \\
& + 3888\sqrt{3}v_4v_2^5a_2 + 46656\sqrt{3}v_3v_2^6b_2 - 4608\sqrt{3}v_1^4v_4b_2 \\
& + 73728\sqrt{3}v_1^4v_3a_3 - 4608\sqrt{3}v_1^3v_4b_1 - 62208v_5v_1v_2^3a_2 \\
& - 186624\sqrt{3}v_1v_2^5a_2 + 221184\sqrt{3}v_1^4v_2b_3 + 1152v_4v_5v_1^2a_3 \\
& - 5184\sqrt{3}v_4v_2^5b_3 - 1296\sqrt{3}v_4v_2^4b_1 + 155520\sqrt{3}v_1^2v_2^4b_2 \\
& + 248832\sqrt{3}v_1v_2^5b_3 - 165888\sqrt{3}v_1^4v_2a_2 \\
& + 110592\sqrt{3}v_1^3v_2^2a_3 + 155520\sqrt{3}v_1v_2^4b_1 \\
& - 1728v_4v_5v_2^3b_3 - 55296\sqrt{3}v_1^3v_2a_1 - 432v_4v_5v_2^2b_1 \\
& + 51840v_5v_1^2v_2^2b_2 + 82944v_5v_1v_2^3b_3 + 51840v_5v_1v_2^2b_1 \\
& + 1296v_5v_4v_2^3a_2 + 15552v_5v_3v_2^4b_2 + 559872iv_1v_2^5a_2 \\
& - 746496iv_1v_2^5b_3 + 497664iv_1^4v_2a_2 - 663552iv_1^4v_2b_3 \\
& - 331776iv_1^3v_2^2a_3 - 466560iv_1v_2^4b_1 + 11664iv_4v_2^5a_2 \\
& - 15552iv_4v_2^5b_3 - 13824iv_1^4v_4b_2 - 3888iv_4v_2^4b_1 \\
& - 13824iv_1^3v_4b_1 + 165888iv_1^3v_2a_1 - 466560iv_1^2v_2^4b_2 \\
& + 6912iv_4v_1^2v_2^2a_3 + 6912iv_4v_1^2v_2a_1 + 31104i\sqrt{3}v_5v_2^3a_1 \\
& - 3888iv_4v_1v_2^4b_2 + 20736iv_4v_1^3v_2a_2 - 27648iv_1^3v_4v_2b_3 \\
& - 15552i\sqrt{3}v_5v_2^4a_3 - 27648i\sqrt{3}v_1^3v_5a_3 \\
& - 1296\sqrt{3}v_4v_1v_2^4b_2 + 6912\sqrt{3}v_4v_1^3v_2a_2 \\
& + 2304\sqrt{3}v_4v_1^2v_2^2a_3 + 2304\sqrt{3}v_4v_1^2v_2a_1 - 432v_4v_5v_1v_2^2b_2 \\
& - 9216\sqrt{3}v_1^3v_4v_2b_3 + 55296\sqrt{3}v_1^3v_3v_2^2b_2 \\
& + 62208\sqrt{3}v_1v_3v_2^4a_3 + 20736v_1v_5v_3v_2^2a_3 \\
& + 1296i\sqrt{3}v_5v_4v_2^3a_2 - 1728i\sqrt{3}v_4v_5v_2^3b_3 \\
& - 432i\sqrt{3}v_4v_5v_2^2b_1 + 1152i\sqrt{3}v_4v_5v_1^2a_3 \\
& - 432i\sqrt{3}v_4v_5v_1v_2^2b_2 - 139968iv_2^6a_3 - 331776iv_1^5b_2 \\
& + 279936iv_2^5a_1 - 331776iv_1^4b_1 + 27648v_1^3v_5a_3 \\
& + 46656\sqrt{3}v_2^6a_3 + 110592\sqrt{3}v_1^5b_2 - 93312\sqrt{3}v_2^5a_1 \\
& + 110592\sqrt{3}v_1^4b_1 + 15552v_5v_2^4a_3 - 31104v_5v_2^3a_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 46656\sqrt{3}v_3v_2^6b_2 + 73728\sqrt{3}v_1^4v_3a_3 \\
& + \left(6912\sqrt{3}a_2 - 9216\sqrt{3}b_3 + 20736ia_2 - 27648ib_3\right)v_1^3v_2v_4 \\
& + \left(2304\sqrt{3}a_3 + 6912ia_3\right)v_1^2v_2^2v_4 \\
& + \left(-51840i\sqrt{3}b_2 + 51840b_2\right)v_1^2v_2^2v_5 \\
& + \left(2304\sqrt{3}a_1 + 6912ia_1\right)v_1^2v_2v_4 \\
& + \left(1152i\sqrt{3}a_3 + 1152a_3\right)v_1^2v_4v_5 \\
& + \left(-1296\sqrt{3}b_2 - 3888ib_2\right)v_1v_2^4v_4 + \left(62208i\sqrt{3}a_2 \right. \\
& \quad \left. - 82944i\sqrt{3}b_3 - 62208a_2 + 82944b_3\right)v_1v_2^3v_5 \\
& + \left(-51840i\sqrt{3}b_1 + 51840b_1\right)v_1v_2^2v_5 \\
& + \left(1296i\sqrt{3}a_2 - 1728i\sqrt{3}b_3 + 1296a_2 - 1728b_3\right)v_2^3v_4v_5 \\
& + \left(-432i\sqrt{3}b_1 - 432b_1\right)v_2^2v_4v_5 + 15552v_5v_3v_2^4b_2 \\
& + 55296\sqrt{3}v_1^3v_3v_2^2b_2 + 62208\sqrt{3}v_1v_3v_2^4a_3 \\
& + 20736v_1v_5v_3v_2^2a_3 + \left(-432i\sqrt{3}b_2 - 432b_2\right)v_1v_2^2v_4v_5 \\
& + \left(46656\sqrt{3}a_3 - 139968ia_3\right)v_2^6 \\
& + \left(-93312\sqrt{3}a_1 + 279936ia_1\right)v_2^5 \\
& + \left(3888\sqrt{3}a_2 - 5184\sqrt{3}b_3 + 11664ia_2 - 15552ib_3\right)v_2^5v_4 \\
& + \left(-1296\sqrt{3}b_1 - 3888ib_1\right)v_2^4v_4 \\
& + \left(-15552i\sqrt{3}a_3 + 15552a_3\right)v_2^4v_5 \\
& + \left(31104i\sqrt{3}a_1 - 31104a_1\right)v_2^3v_5 + \left(-165888\sqrt{3}a_2 \right. \\
& \quad \left. + 221184\sqrt{3}b_3 + 497664ia_2 - 663552ib_3\right)v_1^4v_2 \\
& + \left(-4608\sqrt{3}b_2 - 13824ib_2\right)v_1^4v_4 \\
& + \left(110592\sqrt{3}a_3 - 331776ia_3\right)v_1^3v_2^2 \\
& + \left(-55296\sqrt{3}a_1 + 165888ia_1\right)v_1^3v_2 \\
& + \left(-4608\sqrt{3}b_1 - 13824ib_1\right)v_1^3v_4 \\
& + \left(-27648i\sqrt{3}a_3 + 27648a_3\right)v_1^3v_5 \\
& + \left(155520\sqrt{3}b_2 - 466560ib_2\right)v_1^2v_2^4 \\
& + \left(-186624\sqrt{3}a_2 + 248832\sqrt{3}b_3 + 559872ia_2 \right. \\
& \quad \left. - 746496ib_3\right)v_1v_2^5 + \left(155520\sqrt{3}b_1 - 466560ib_1\right)v_1v_2^4
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
20736a_3 &= 0 \\
15552b_2 &= 0 \\
62208\sqrt{3}a_3 &= 0 \\
73728\sqrt{3}a_3 &= 0 \\
46656\sqrt{3}b_2 &= 0 \\
55296\sqrt{3}b_2 &= 0 \\
-93312\sqrt{3}a_1 + 279936ia_1 &= 0 \\
-55296\sqrt{3}a_1 + 165888ia_1 &= 0 \\
2304\sqrt{3}a_1 + 6912ia_1 &= 0 \\
2304\sqrt{3}a_3 + 6912ia_3 &= 0 \\
46656\sqrt{3}a_3 - 139968ia_3 &= 0 \\
110592\sqrt{3}a_3 - 331776ia_3 &= 0 \\
-4608\sqrt{3}b_1 - 13824ib_1 &= 0 \\
-1296\sqrt{3}b_1 - 3888ib_1 &= 0 \\
110592\sqrt{3}b_1 - 331776ib_1 &= 0 \\
155520\sqrt{3}b_1 - 466560ib_1 &= 0 \\
-4608\sqrt{3}b_2 - 13824ib_2 &= 0 \\
-1296\sqrt{3}b_2 - 3888ib_2 &= 0 \\
110592\sqrt{3}b_2 - 331776ib_2 &= 0 \\
155520\sqrt{3}b_2 - 466560ib_2 &= 0 \\
-51840i\sqrt{3}b_1 + 51840b_1 &= 0 \\
-51840i\sqrt{3}b_2 + 51840b_2 &= 0 \\
-27648i\sqrt{3}a_3 + 27648a_3 &= 0 \\
-15552i\sqrt{3}a_3 + 15552a_3 &= 0 \\
-432i\sqrt{3}b_1 - 432b_1 &= 0 \\
-432i\sqrt{3}b_2 - 432b_2 &= 0 \\
1152i\sqrt{3}a_3 + 1152a_3 &= 0 \\
31104i\sqrt{3}a_1 - 31104a_1 &= 0 \\
-186624\sqrt{3}a_2 + 248832\sqrt{3}b_3 + 559872ia_2 - 746496ib_3 &= 0 \\
-165888\sqrt{3}a_2 + 221184\sqrt{3}b_3 + 497664ia_2 - 663552ib_3 &= 0 \\
3888\sqrt{3}a_2 - 5184\sqrt{3}b_3 + 11664ia_2 - 15552ib_3 &= 0 \\
6912\sqrt{3}a_2 - 9216\sqrt{3}b_3 + 20736ia_2 - 27648ib_3 &= 0 \\
1296i\sqrt{3}a_2 - 1728i\sqrt{3}b_3 + 1296a_2 - 1728b_3 &= 0 \\
62208i\sqrt{3}a_2 - 82944i\sqrt{3}b_3 - 62208a_2 + 82944b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= \frac{4b_3}{3} \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= \frac{4x}{3} \\ \eta &= y\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R . Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying simple symmetries for implicit equations
  Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
    *** Sublevel 3 ***
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying homogeneous types:
    trying exact
    Looking for potential symmetries
    trying an equivalence to an Abel ODE
    trying 1st order ODE linearizable_by_differentiation
  -> Solving 1st order ODE of high degree, Lie methods, 1st trial
  `, `-> Computing symmetries using: way = 2
  `, `-> Computing symmetries using: way = 2
  -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
  trying dAlembert
  -> Calling odsolve with the ODE`,  $\text{diff}(y(x), x) = \frac{-2*y(x)^2*x^3-y(x)}{2*y(x)*x^4+x}$ , y(
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying a quadrature
    trying 1st order linear
    <- 1st order linear successful
  <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 97

```
dsolve(y(x)=2*x*diff(y(x),x)+y(x)^2*diff(y(x),x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$

$$y(x) = \frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$

$$y(x) = -\frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$

$$y(x) = \frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$

$$y(x) = 0$$

$$y(x) = \sqrt{c_1(c_1^2 + 2x)}$$

$$y(x) = -\sqrt{c_1(c_1^2 + 2x)}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 119

```
DSolve[y[x]==2*x*y'[x]+y[x]^2*y'[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{2c_1x + c_1^3}$$

$$y(x) \rightarrow \sqrt{2c_1x + c_1^3}$$

$$y(x) \rightarrow (-1 - i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

$$y(x) \rightarrow (1 - i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

$$y(x) \rightarrow (-1 + i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

$$y(x) \rightarrow (1 + i) \left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

5.12 problem 28

5.12.1 Solving as dAlembert ode 1078

Internal problem ID [5334]

Internal file name [OUTPUT/4825_Friday_February_02_2024_05_14_23_AM_52910488/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 28.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _dAlembert]
```

$$y'^2 - xy' - y = 0$$

5.12.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - xp - y = 0$$

Solving for y from the above results in

$$y = p^2 - xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned}f &= -p \\g &= p^2\end{aligned}$$

Hence (2) becomes

$$2p = (-x + 2p) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$2p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{2p(x)}{-x + 2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-x(p) + 2p}{2p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned}p(p) &= \frac{1}{2p} \\q(p) &= 1\end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{2p} = 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2p} dp} \\ &= \sqrt{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= \mu \\ \frac{d}{dp}(\sqrt{p} x) &= \sqrt{p} \\ d(\sqrt{p} x) &= \sqrt{p} dp\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{p} x &= \int \sqrt{p} dp \\ \sqrt{p} x &= \frac{2p^{\frac{3}{2}}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{p}$ results in

$$x(p) = \frac{2p}{3} + \frac{c_1}{\sqrt{p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p . This results in

$$\begin{aligned}p &= \frac{x}{2} + \frac{\sqrt{x^2 + 4y}}{2} \\ p &= \frac{x}{2} - \frac{\sqrt{x^2 + 4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}} \\ x &= \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}} \tag{2}$$

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}}$$

Verified OK.

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 77

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y(x)}}} + \frac{2x}{3} + \frac{\sqrt{x^2 + 4y(x)}}{3} = 0$$

$$\frac{c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y(x)}}} + \frac{2x}{3} - \frac{\sqrt{x^2 + 4y(x)}}{3} = 0$$

✓ Solution by Mathematica

Time used: 60.129 (sec). Leaf size: 1003

`DSolve[y'[x]^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$y(x) \rightarrow \frac{\left(x^2 + \sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}}\right)^2 + 8e^{3c_1}x}{4\sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}}}$$

$$y(x) \rightarrow \frac{1}{8} \left(4x^2 - \frac{i(\sqrt{3} - i)x(x^3 + 8e^{3c_1})}{\sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}}} + i(\sqrt{3} + i)\sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}} \right)$$

$$y(x) \rightarrow \frac{1}{8} \left(4x^2 + \frac{i(\sqrt{3} + i)x(x^3 + 8e^{3c_1})}{\sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}}} - (1 + i\sqrt{3})\sqrt[3]{-x^6 + 20e^{3c_1}x^3 + 8\sqrt{e^{3c_1}(-x^3 + e^{3c_1})^3 + 8e^{6c_1}}} \right)$$

$$y(x) \rightarrow \frac{2\sqrt[3]{2}x^4 + 2^{2/3}\left(-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}\right)^{2/3} + 4x^2\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}}}{8\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}}}$$

$$y(x) \rightarrow \frac{1}{16} \left(8x^2 + \frac{2\sqrt[3]{2}(1 + i\sqrt{3})x(-x^3 + 2e^{3c_1})}{\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}}} + i2^{2/3}(\sqrt{3} + i)\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}} \right)$$

$$y(x) \rightarrow \frac{1}{16} \left(8x^2 + \frac{2i\sqrt[3]{2}(\sqrt{3} + i)x(x^3 - 2e^{3c_1})}{\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}}} - 2^{2/3}(1 + i\sqrt{3})\sqrt[3]{-2x^6 - 10e^{3c_1}x^3 + \sqrt{e^{3c_1}(4x^3 + e^{3c_1})^3 + e^{6c_1}}} \right)$$

5.13 problem 29

5.13.1 Solving as dAlembert ode 1084

Internal problem ID [5335]

Internal file name [OUTPUT/4826_Friday_February_02_2024_05_14_23_AM_73749985/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 29.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _dAlembert]
```

$$y - (1 + y')x - y'^2 = 0$$

5.13.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - (1 + p)x - p^2 = 0$$

Solving for y from the above results in

$$y = (1 + p)x + p^2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned}f &= 1 + p \\g &= p^2\end{aligned}$$

Hence (2) becomes

$$-1 = (x + 2p)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-1 = 0$$

No singular solution are found

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{1}{x + 2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -x(p) - 2p \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned}p(p) &= 1 \\q(p) &= -2p\end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + x(p) = -2p$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1 dp} \\&= e^p\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= (\mu) (-2p) \\ \frac{d}{dp}(e^p x) &= (e^p) (-2p) \\ d(e^p x) &= (-2p e^p) dp\end{aligned}$$

Integrating gives

$$\begin{aligned}e^p x &= \int -2p e^p dp \\ e^p x &= -2(p-1)e^p + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^p$ results in

$$x(p) = -2e^{-p}(p-1)e^p + c_1e^{-p}$$

which simplifies to

$$x(p) = -2p + 2 + c_1e^{-p}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p . This results in

$$\begin{aligned}p &= -\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2} \\ p &= -\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= x - \sqrt{x^2 - 4x + 4y} + 2 + c_1e^{\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}} \\ x &= x + \sqrt{x^2 - 4x + 4y} + 2 + c_1e^{\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2}}\end{aligned}$$

Summary

The solution(s) found are the following

$$x = x - \sqrt{x^2 - 4x + 4y} + 2 + c_1e^{\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}} \quad (1)$$

$$x = x + \sqrt{x^2 - 4x + 4y} + 2 + c_1e^{\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2}} \quad (2)$$

Verification of solutions

$$x = x - \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}}$$

Verified OK.

$$x = x + \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 36

```
dsolve(y(x)=(1+diff(y(x),x))*x+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = x - \frac{x^2}{4} + \text{LambertW}\left(\frac{c_1 e^{-1 + \frac{x}{2}}}{2}\right)^2 + 2 \text{LambertW}\left(\frac{c_1 e^{-1 + \frac{x}{2}}}{2}\right) + 1$$

✓ Solution by Mathematica

Time used: 1.048 (sec). Leaf size: 177

```
DSolve[y[x]==(1+y'[x])*x+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\sqrt{x^2+4y(x)-4x}+2\log\left(\sqrt{x^2+4y(x)-4x}-x+2\right)\right. \\ \left.-2\log\left(-x\sqrt{x^2+4y(x)-4x}+x^2+4y(x)-2x-4\right)+x=c_1,y(x)\right]$$
$$\text{Solve}\left[-4\text{arctanh}\left(\frac{(x-5)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+7x-6}{(x-3)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+5x-2}\right)\right. \\ \left.+\sqrt{x^2+4y(x)-4x}+x=c_1,y(x)\right]$$

5.14 problem 30

5.14.1 Maple step by step solution 1092

Internal problem ID [5336]

Internal file name [OUTPUT/4827_Friday_February_02_2024_05_14_23_AM_68613889/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 30.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$y - 2y' - \sqrt{1 + y'^2} = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2y}{3} + \frac{\sqrt{y^2 + 3}}{3} \tag{1}$$

$$y' = \frac{2y}{3} - \frac{\sqrt{y^2 + 3}}{3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} dy = \int dx$$

$$\begin{aligned}
& -\frac{\sqrt{(y-1)^2+2y+2}}{2} - \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\
& + \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2+2y+2}}\right) + \frac{\sqrt{(1+y)^2-2y+2}}{2} \\
& - \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2-2y+2}}\right) + \ln(y-1) + \ln(1+y) = x + c_1
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
& -\frac{\sqrt{(y-1)^2+2y+2}}{2} - \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\
& + \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2+2y+2}}\right) + \frac{\sqrt{(1+y)^2-2y+2}}{2} \\
& - \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2-2y+2}}\right) + \ln(y-1) + \ln(1+y) = x + c_1
\end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
& -\frac{\sqrt{(y-1)^2+2y+2}}{2} - \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\
& + \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2+2y+2}}\right) + \frac{\sqrt{(1+y)^2-2y+2}}{2} \\
& - \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2-2y+2}}\right) + \ln(y-1) + \ln(1+y) = x + c_1
\end{aligned}$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{\frac{2y}{3} - \frac{\sqrt{y^2+3}}{3}} dy = \int dx$$

$$\begin{aligned} & \frac{\sqrt{(y-1)^2 + 2y + 2}}{2} + \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\ & - \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2 + 2y + 2}}\right) - \frac{\sqrt{(1+y)^2 - 2y + 2}}{2} \\ & + \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2 - 2y + 2}}\right) + \ln(y-1) + \ln(1+y) = c_2 + x \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} & \frac{\sqrt{(y-1)^2 + 2y + 2}}{2} + \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\ & - \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2 + 2y + 2}}\right) - \frac{\sqrt{(1+y)^2 - 2y + 2}}{2} \\ & + \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2 - 2y + 2}}\right) + \ln(y-1) + \ln(1+y) = c_2 + x \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} & \frac{\sqrt{(y-1)^2 + 2y + 2}}{2} + \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) \\ & - \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2 + 2y + 2}}\right) - \frac{\sqrt{(1+y)^2 - 2y + 2}}{2} \\ & + \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2 - 2y + 2}}\right) + \ln(y-1) + \ln(1+y) = c_2 + x \end{aligned}$$

Verified OK.

5.14.1 Maple step by step solution

Let's solve

$$y - 2y' - \sqrt{1 + y'^2} = 0$$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$$\frac{y'}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{y'}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} dx = \int 1 dx + c_1$$

- Evaluate integral

$$-\frac{\sqrt{(y-1)^2+2y+2}}{2} - \operatorname{arcsinh}\left(\frac{y\sqrt{3}}{3}\right) + \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(y-1)^2+2y+2}}\right) + \frac{\sqrt{(1+y)^2-2y+2}}{2} - \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(1+y)^2-2y+2}}\right)$$

Maple trace

```
`Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
<- differential order: 1; missing x successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve(y(x)=2*diff(y(x),x)+sqrt(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$\begin{aligned} & x - \operatorname{arcsinh}\left(\frac{\sqrt{3}y(x)}{3}\right) + \operatorname{arctanh}\left(\frac{3+y(x)}{2\sqrt{y(x)^2+3}}\right) \\ & + \operatorname{arctanh}\left(\frac{-3+y(x)}{2\sqrt{y(x)^2+3}}\right) - \ln(y(x)-1) - \ln(y(x)+1) - c_1 = 0 \\ & x + \operatorname{arcsinh}\left(\frac{\sqrt{3}y(x)}{3}\right) - \operatorname{arctanh}\left(\frac{3+y(x)}{2\sqrt{y(x)^2+3}}\right) \\ & - \operatorname{arctanh}\left(\frac{-3+y(x)}{2\sqrt{y(x)^2+3}}\right) - \ln(y(x)-1) - \ln(y(x)+1) - c_1 = 0 \end{aligned}$$

✓ Solution by Mathematica

Time used: 60.301 (sec). Leaf size: 4821

```
DSolve[y[x]==2*y'[x]+Sqrt[1+y'[x]^2],y[x],x,IncludeSingularSolutions->True]
```

Too large to display

5.15 problem 31

5.15.1 Solving as dAlembert ode 1094

Internal problem ID [5337]

Internal file name [OUTPUT/4828_Friday_February_02_2024_05_14_27_AM_37782308/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 9. Equations of first order and higher degree. Supplementary problems. Page 65

Problem number: 31.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**dAlembert**"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$yy'^2 - xy' + 3y = 0$$

5.15.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$yp^2 - xp + 3y = 0$$

Solving for y from the above results in

$$y = \frac{xp}{p^2 + 3} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{p^2 + 3}$$

$$g = 0$$

Hence (2) becomes

$$p - \frac{p}{p^2 + 3} = x \left(\frac{1}{p^2 + 3} - \frac{2p^2}{(p^2 + 3)^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p}{p^2 + 3} = 0$$

Solving for p from the above gives

$$p = 0$$

$$p = i\sqrt{2}$$

$$p = -i\sqrt{2}$$

Substituting these in (1A) gives

$$y = 0$$

$$y = -i\sqrt{2}x$$

$$y = i\sqrt{2}x$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)}{p(x)^2 + 3}}{x \left(\frac{1}{p(x)^2 + 3} - \frac{2p(x)^2}{(p(x)^2 + 3)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$. In canonical form the ODE is

$$p' = F(x, p)$$

$$= f(x)g(p)$$

$$= -\frac{(p^2 + 2)p(p^2 + 3)}{x(p^2 - 3)}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = \frac{(p^2+2)p(p^2+3)}{p^2-3}$. Integrating both sides gives

$$\frac{1}{\frac{(p^2+2)p(p^2+3)}{p^2-3}} dp = -\frac{1}{x} dx$$

$$\int \frac{1}{\frac{(p^2+2)p(p^2+3)}{p^2-3}} dp = \int -\frac{1}{x} dx$$

$$-\ln(p^2+3) - \frac{\ln(p)}{2} + \frac{5\ln(p^2+2)}{4} = -\ln(x) + c_1$$

Raising both side to exponential gives

$$e^{-\ln(p^2+3) - \frac{\ln(p)}{2} + \frac{5\ln(p^2+2)}{4}} = e^{-\ln(x) + c_1}$$

Which simplifies to

$$\frac{(p^2+2)^{\frac{5}{4}}}{(p^2+3)\sqrt{p}} = \frac{c_2}{x}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x^3 \text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^2 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2 \left(\frac{\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^4 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2} \right)}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = -i\sqrt{2}x \tag{2}$$

$$y = i\sqrt{2}x \tag{3}$$

$$y = \frac{x^3 \text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^2 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2 \left(\frac{\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^4 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2} \right)}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -i\sqrt{2}x$$

Verified OK.

$$y = i\sqrt{2}x$$

Verified OK.

y

$$= \frac{x^3 \text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^2 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2 \left(\frac{\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4)}{c_2^4 \left(\text{RootOf}((c_2^4 - x^4)Z^{20} + 2c_2^4 Z^{16} - 2c_2^4 Z^{12} - 8c_2^4 Z^8 - 7c_2^4 Z^4 - 2c_2^4) + 1 \right)^2} \right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 5.141 (sec). Leaf size: 159

`dsolve(y(x)*diff(y(x),x)^2-x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)`

$$y(x) = 0$$

$$\ln(x) - \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-x^2+12y(x)^2}}\right)}{4} + \frac{5 \operatorname{arctanh}\left(\frac{\sqrt{\frac{x^2-12y(x)^2}{x^2}}}{5}\right)}{4}$$

$$+ \frac{5 \ln\left(\frac{2x^2+y(x)^2}{x^2}\right)}{8} - \frac{\ln\left(\frac{y(x)}{x}\right)}{4} - c_1 = 0$$

$$\ln(x) + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-x^2+12y(x)^2}}\right)}{4} - \frac{5 \operatorname{arctanh}\left(\frac{\sqrt{\frac{x^2-12y(x)^2}{x^2}}}{5}\right)}{4}$$

$$+ \frac{5 \ln\left(\frac{2x^2+y(x)^2}{x^2}\right)}{8} - \frac{\ln\left(\frac{y(x)}{x}\right)}{4} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.281 (sec). Leaf size: 1131

`DSolve[y[x]*y'[x]^2-x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$\begin{aligned} & y(x) \rightarrow -\sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow \sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow -\sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow \sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow -\sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow \sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow -\sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow \sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow -\sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \\ & y(x) \rightarrow \sqrt{\text{Root}[62208\#1^5 + 622080\#1^4x^2 + \#1^3(2488320x^4 - 864e^{8c_1}) + \#1^2(4976640x^6 + 16416e^{8c_1}x^2) + \dots]} \end{aligned}$$

**6 Chapter 10. Singular solutions, Extraneous loci.
Supplementary problems. Page 74**

6.1	problem 10	1101
6.2	problem 11	1105
6.3	problem 12	1119
6.4	problem 13	1124
6.5	problem 14	1129
6.6	problem 15	1134
6.7	problem 16	1147
6.8	problem 17	1153
6.9	problem 18	1156
6.10	problem 19	1184

6.1 problem 10

6.1.1 Solving as clairaut ode 1101

Internal problem ID [5338]

Internal file name [OUTPUT/4829_Sunday_February_04_2024_12_46_05_AM_16279652/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 10.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**clairaut**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _Clairaut]
```

$$y - xy' + 2y'^2 = 0$$

6.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$2p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -2p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -2p^2 + xp \\ &= -2p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x . Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -2p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x \frac{dp}{dx} \right) + \left(g' \frac{dp}{dx} \right) \\ p &= p + (x + g') \frac{dp}{dx} \\ 0 &= (x + g') \frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p . The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -2c_1^2 + c_1x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -2p^2$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - 4p \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{4}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{8}$$

Summary

The solution(s) found are the following

$$y = -2c_1^2 + c_1x \quad (1)$$

$$y = \frac{x^2}{8} \quad (2)$$

Verification of solutions

$$y = -2c_1^2 + c_1x$$

Verified OK.

$$y = \frac{x^2}{8}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```


✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(y(x)=diff(y(x),x)*x-2*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{8}$$
$$y(x) = c_1(x - 2c_1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[y[x]==y'[x]*x-2*y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1(x - 2c_1)$$
$$y(x) \rightarrow \frac{x^2}{8}$$

6.2 problem 11

Internal problem ID [5339]

Internal file name [OUTPUT/4830_Sunday_February_04_2024_12_46_05_AM_48988675/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 11.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _rational]
```

$$y^2 y' + 3xy' - y = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-3x + \sqrt{9x^2 + 4y^3}}{2y^2} \quad (1)$$

$$y' = -\frac{3x + \sqrt{9x^2 + 4y^3}}{2y^2} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(-3x + \sqrt{4y^3 + 9x^2})(b_3 - a_2)}{2y^2} - \frac{(-3x + \sqrt{4y^3 + 9x^2})^2 a_3}{4y^4} \\ - \frac{\left(-3 + \frac{9x}{\sqrt{4y^3 + 9x^2}}\right)(xa_2 + ya_3 + a_1)}{2y^2} \\ - \left(-\frac{-3x + \sqrt{4y^3 + 9x^2}}{y^3} + \frac{3}{\sqrt{4y^3 + 9x^2}}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{-4b_2y^4\sqrt{4y^3 + 9x^2} - 4xy^4b_2 + 8y^5a_2 - 12y^5b_3 + 12\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}xy^2a_2}{y^6} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} 4b_2y^4\sqrt{4y^3 + 9x^2} + 4xy^4b_2 - 8y^5a_2 + 12y^5b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\ + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\ + 36x^3yb_2 - 36x^2y^2a_2 + 54x^2y^2b_3 + 6xy^3a_3 + 4y^4b_1 \\ - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 - 9\sqrt{4y^3 + 9x^2}x^2a_3 - 12\sqrt{4y^3 + 9x^2}xyb_1 \\ + 6\sqrt{4y^3 + 9x^2}y^2a_1 + 54x^3a_3 + 36x^2yb_1 - 18xy^2a_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3+9x^2} - 12xy^4b_2 - 12y^5b_3 + 4(4y^3+9x^2)xyb_2 \\
& - 2(4y^3+9x^2)y^2a_2 + 6(4y^3+9x^2)y^2b_3 - 12\sqrt{4y^3+9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3+9x^2}xy^2a_2 - 18\sqrt{4y^3+9x^2}xy^2b_3 + 6\sqrt{4y^3+9x^2}y^3a_3 \quad (6E) \\
& - 18x^2y^2a_2 - 18xy^3a_3 - 12y^4b_1 - (4y^3+9x^2)^{\frac{3}{2}}a_3 \\
& + 6(4y^3+9x^2)xa_3 + 4(4y^3+9x^2)yb_1 - 9\sqrt{4y^3+9x^2}x^2a_3 \\
& - 12\sqrt{4y^3+9x^2}xyb_1 + 6\sqrt{4y^3+9x^2}y^2a_1 - 18xy^2a_1 = 0
\end{aligned}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 4xy^4b_2 + 4b_2y^4\sqrt{4y^3+9x^2} - 8y^5a_2 + 12y^5b_3 + 36x^3yb_2 - 12\sqrt{4y^3+9x^2}x^2yb_2 \\
& - 36x^2y^2a_2 + 54x^2y^2b_3 + 12\sqrt{4y^3+9x^2}xy^2a_2 - 18\sqrt{4y^3+9x^2}xy^2b_3 \\
& + 6xy^3a_3 + 2\sqrt{4y^3+9x^2}y^3a_3 + 4y^4b_1 + 54x^3a_3 - 18\sqrt{4y^3+9x^2}x^2a_3 \\
& + 36x^2yb_1 - 12\sqrt{4y^3+9x^2}xyb_1 - 18xy^2a_1 + 6\sqrt{4y^3+9x^2}y^2a_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4y^3+9x^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4y^3+9x^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& -8v_2^5a_2 + 4v_1v_2^4b_2 + 4b_2v_2^4v_3 + 12v_2^5b_3 - 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 + 6v_1v_2^3a_3 \quad (7E) \\
& + 2v_3v_2^3a_3 + 4v_2^4b_1 + 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 + 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\
& - 18v_1v_2^2a_1 + 6v_3v_2^2a_1 + 54v_1^3a_3 - 18v_3v_1^2a_3 + 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0
\end{aligned}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
 & 36v_1^3v_2b_2 + 54v_1^3a_3 + (-36a_2 + 54b_3)v_1^2v_2^2 - 12v_3v_1^2v_2b_2 + 36v_1^2v_2b_1 \\
 & - 18v_3v_1^2a_3 + 4v_1v_2^4b_2 + 6v_1v_2^3a_3 + (12a_2 - 18b_3)v_1v_2^2v_3 - 18v_1v_2^2a_1 \\
 & - 12v_3v_1v_2b_1 + (-8a_2 + 12b_3)v_2^5 + 4b_2v_2^4v_3 + 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0
 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 -18a_1 &= 0 \\
 6a_1 &= 0 \\
 -18a_3 &= 0 \\
 2a_3 &= 0 \\
 6a_3 &= 0 \\
 54a_3 &= 0 \\
 -12b_1 &= 0 \\
 4b_1 &= 0 \\
 36b_1 &= 0 \\
 -12b_2 &= 0 \\
 4b_2 &= 0 \\
 36b_2 &= 0 \\
 -36a_2 + 54b_3 &= 0 \\
 -8a_2 + 12b_3 &= 0 \\
 12a_2 - 18b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= \frac{3b_3}{2} \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= b_3
 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for

any unknown in the RHS) gives

$$\xi = \frac{3x}{2}$$

$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(\frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2} \right) \left(\frac{3x}{2} \right) \\ &= \frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2} x}{4y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2} x}{4y^2}} dy\end{aligned}$$

Which results in

$$S = \ln(y) - \frac{2x \operatorname{arctanh} \left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}} \right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 + \frac{3x}{\sqrt{4y^3 + 9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \quad (1)$$

Verification of solutions

$$\ln(y) - \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2 - \frac{(\sqrt{4y^3 + 9x^2} + 3x)(b_3 - a_2)}{2y^2} - \frac{(\sqrt{4y^3 + 9x^2} + 3x)^2 a_3}{4y^4} \\
& + \frac{\left(\frac{9x}{\sqrt{4y^3 + 9x^2}} + 3\right)(xa_2 + ya_3 + a_1)}{2y^2} \\
& - \left(-\frac{3}{\sqrt{4y^3 + 9x^2}} + \frac{\sqrt{4y^3 + 9x^2} + 3x}{y^3}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
& \frac{-4b_2y^4\sqrt{4y^3 + 9x^2} + 4xy^4b_2 - 8y^5a_2 + 12y^5b_3 + 12\sqrt{4y^3 + 9x^2}x^2yb_2 - 12\sqrt{4y^3 + 9x^2}xy^2a_2 + 18\sqrt{4y^3 + 9x^2}x^2yb_2}{-} \\
& = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3 + 9x^2} - 4xy^4b_2 + 8y^5a_2 - 12y^5b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\
& - 36x^3yb_2 + 36x^2y^2a_2 - 54x^2y^2b_3 - 6xy^3a_3 - 4y^4b_1 \\
& - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 - 9\sqrt{4y^3 + 9x^2}x^2a_3 - 12\sqrt{4y^3 + 9x^2}xyb_1 \\
& + 6\sqrt{4y^3 + 9x^2}y^2a_1 - 54x^3a_3 - 36x^2yb_1 + 18xy^2a_1 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
& 4b_2y^4\sqrt{4y^3 + 9x^2} + 12xy^4b_2 + 12y^5b_3 - 4(4y^3 + 9x^2)xyb_2 \\
& + 2(4y^3 + 9x^2)y^2a_2 - 6(4y^3 + 9x^2)y^2b_3 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 + 6\sqrt{4y^3 + 9x^2}y^3a_3 \\
& + 18x^2y^2a_2 + 18xy^3a_3 + 12y^4b_1 - (4y^3 + 9x^2)^{\frac{3}{2}}a_3 \\
& - 6(4y^3 + 9x^2)xa_3 - 4(4y^3 + 9x^2)yb_1 - 9\sqrt{4y^3 + 9x^2}x^2a_3 \\
& - 12\sqrt{4y^3 + 9x^2}xyb_1 + 6\sqrt{4y^3 + 9x^2}y^2a_1 + 18xy^2a_1 = 0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& -4xy^4b_2 + 4b_2y^4\sqrt{4y^3 + 9x^2} + 8y^5a_2 - 12y^5b_3 - 36x^3yb_2 - 12\sqrt{4y^3 + 9x^2}x^2yb_2 \\
& + 36x^2y^2a_2 - 54x^2y^2b_3 + 12\sqrt{4y^3 + 9x^2}xy^2a_2 - 18\sqrt{4y^3 + 9x^2}xy^2b_3 \\
& - 6xy^3a_3 + 2\sqrt{4y^3 + 9x^2}y^3a_3 - 4y^4b_1 - 54x^3a_3 - 18\sqrt{4y^3 + 9x^2}x^2a_3 \\
& - 36x^2yb_1 - 12\sqrt{4y^3 + 9x^2}xyb_1 + 18xy^2a_1 + 6\sqrt{4y^3 + 9x^2}y^2a_1 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4y^3 + 9x^2}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4y^3 + 9x^2} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8v_2^5a_2 - 4v_1v_2^4b_2 + 4b_2v_2^4v_3 - 12v_2^5b_3 + 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 - 6v_1v_2^3a_3 \\ &+ 2v_3v_2^3a_3 - 4v_2^4b_1 - 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 - 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\ &+ 18v_1v_2^2a_1 + 6v_3v_2^2a_1 - 54v_1^3a_3 - 18v_3v_1^2a_3 - 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-36v_1^3v_2b_2 - 54v_1^3a_3 + (36a_2 - 54b_3)v_1^2v_2^2 - 12v_3v_1^2v_2b_2 - 36v_1^2v_2b_1 \\ &- 18v_3v_1^2a_3 - 4v_1v_2^4b_2 - 6v_1v_2^3a_3 + (12a_2 - 18b_3)v_1v_2^2v_3 + 18v_1v_2^2a_1 \\ &- 12v_3v_1v_2b_1 + (8a_2 - 12b_3)v_2^5 + 4b_2v_2^4v_3 - 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}6a_1 &= 0 \\18a_1 &= 0 \\-54a_3 &= 0 \\-18a_3 &= 0 \\-6a_3 &= 0 \\2a_3 &= 0 \\-36b_1 &= 0 \\-12b_1 &= 0 \\-4b_1 &= 0 \\-36b_2 &= 0 \\-12b_2 &= 0 \\-4b_2 &= 0 \\4b_2 &= 0 \\8a_2 - 12b_3 &= 0 \\12a_2 - 18b_3 &= 0 \\36a_2 - 54b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= \frac{3b_3}{2} \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= \frac{3x}{2} \\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \left(-\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2} \right) \left(\frac{3x}{2} \right) \\ &= \frac{4y^3 + 3\sqrt{4y^3 + 9x^2}x + 9x^2}{4y^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4y^3 + 3\sqrt{4y^3 + 9x^2}x + 9x^2}{4y^2}} dy\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 - \frac{3x}{\sqrt{4y^3 + 9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2 + 4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \quad (1)$$

Verification of solutions

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
*** Sublevel 3 ***
Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3`[3/2*x, y], [2/3*y^3+3*x^2, y*x]
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 123

`dsolve(y(x)^2*diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)`

$$y(x) = \frac{18^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}}{2}$$

$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}} \left(3i3^{\frac{1}{6}} + 3^{\frac{2}{3}} \right)}{4}$$

$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}} \left(3^{\frac{2}{3}} - 3i3^{\frac{1}{6}} \right)}{4}$$

$$y(x) = 0$$

$$y(x) = \text{RootOf} \left(-2 \ln(x) - 3 \left(\int^{-Z} \frac{4_a^3 + 3\sqrt{4_a^3 + 9} + 9}{_a(4_a^3 + 9)} d_a \right) + 2c_1 \right) x^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.597 (sec). Leaf size: 239

`DSolve[y[x]^2*y'[x]^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]`

$$y(x) \rightarrow e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow -\sqrt[3]{-1} e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow (-1)^{2/3} e^{\frac{c_1}{3}} \sqrt[3]{-3x + e^{c_1}}$$

$$y(x) \rightarrow e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow -\sqrt[3]{-1} e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow (-1)^{2/3} e^{\frac{c_1}{3}} \sqrt[3]{3x + e^{c_1}}$$

$$y(x) \rightarrow 0$$

$$y(x) \rightarrow -\left(-\frac{3}{2}\right)^{2/3} x^{2/3}$$

$$y(x) \rightarrow -\left(\frac{3}{2}\right)^{2/3} x^{2/3}$$

$$y(x) \rightarrow \frac{\sqrt[3]{-13}^{2/3} x^{2/3}}{2^{2/3}}$$

6.3 problem 12

6.3.1 Solving as dAlembert ode 1119

Internal problem ID [5340]

Internal file name [OUTPUT/4831_Sunday_February_04_2024_12_46_09_AM_99128311/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 12.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "dAlembert"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$xy'^2 - 2y'y = -4x$$

6.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp^2 - 2py = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2 + 4}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2 + 4}{2p} = x \left(1 - \frac{p^2 + 4}{2p^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2 + 4}{2p} = 0$$

Solving for p from the above gives

$$p = 2$$
$$p = -2$$

Substituting these in (1A) gives

$$y = -2x$$
$$y = 2x$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2 + 4}{2p(x)}}{x \left(1 - \frac{p(x)^2 + 4}{2p(x)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx} \mu p &= 0 \\ \frac{d}{dx} \left(\frac{p}{x} \right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituting the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$

$$y = 2x \tag{2}$$

$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 43

```
DSolve[x*y'[x]^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -2x \cosh(-\log(x) + c_1)$$

$$y(x) \rightarrow -2x \cosh(\log(x) + c_1)$$

$$y(x) \rightarrow -2x$$

$$y(x) \rightarrow 2x$$

6.4 problem 13

6.4.1 Solving as dAlembert ode 1124

Internal problem ID [5341]

Internal file name [OUTPUT/4832_Sunday_February_04_2024_12_46_10_AM_46687559/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 13.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "dAlembert"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$xy'^2 - 2y'y + 2y = -x$$

6.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$xp^2 - 2py + 2y = -x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 1)}{2p - 2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p^2 + 1}{2p - 2}$$

$$g = 0$$

Hence (2) becomes

$$p - \frac{p^2 + 1}{2p - 2} = x \left(\frac{2p}{2p - 2} - \frac{2(p^2 + 1)}{(2p - 2)^2} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2 + 1}{2p - 2} = 0$$

Solving for p from the above gives

$$p = 1 + \sqrt{2}$$

$$p = 1 - \sqrt{2}$$

Substituting these in (1A) gives

$$y = x - x\sqrt{2}$$

$$y = x + x\sqrt{2}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2 + 1}{2p(x) - 2}}{x \left(\frac{2p(x)}{2p(x) - 2} - \frac{2(p(x)^2 + 1)}{(2p(x) - 2)^2} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$

$$q(x) = -\frac{1}{x}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = -\frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu) \left(-\frac{1}{x}\right) \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right) \left(-\frac{1}{x}\right) \\ d\left(\frac{p}{x}\right) &= \left(-\frac{1}{x^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int -\frac{1}{x^2} dx \\ \frac{p}{x} &= \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x + 1$$

Substituting the above solution for p in (2A) gives

$$y = \frac{(c_1 x + 1)^2 + 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = x - x\sqrt{2} \tag{1}$$

$$y = x + x\sqrt{2} \tag{2}$$

$$y = \frac{(c_1 x + 1)^2 + 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = x - x\sqrt{2}$$

Verified OK.

$$y = x + x\sqrt{2}$$

Verified OK.

$$y = \frac{(c_1x + 1)^2 + 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+x+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = (1 - \sqrt{2})x$$

$$y(x) = (1 + \sqrt{2})x$$

$$y(x) = \frac{2c_1^2 + 2c_1x + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 78

```
DSolve[x*y'[x]^2-2*y[x]*y'[x]+x+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2}e^{-c_1}x^2 + x - e^{c_1}$$

$$y(x) \rightarrow -e^{c_1}x^2 + x - \frac{e^{-c_1}}{2}$$

$$y(x) \rightarrow x - \sqrt{2}x$$

$$y(x) \rightarrow (1 + \sqrt{2})x$$

6.5 problem 14

6.5.1 Maple step by step solution 1130

Internal problem ID [5342]

Internal file name [OUTPUT/4833_Sunday_February_04_2024_12_46_10_AM_26499802/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 14.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

[_quadrature]

$$(-1 + 3y)^2 y'^2 - 4y = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2\sqrt{y}}{-1 + 3y} \quad (1)$$

$$y' = -\frac{2\sqrt{y}}{-1 + 3y} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{-1 + 3y}{2\sqrt{y}} dy = \int dx$$
$$\sqrt{y}(y - 1) = x + c_1$$

Summary

The solution(s) found are the following

$$\sqrt{y}(y - 1) = x + c_1 \quad (1)$$

Verification of solutions

$$\sqrt{y}(y-1) = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{-1+3y}{2\sqrt{y}} dy = \int dx$$
$$-\sqrt{y}(y-1) = c_2 + x$$

Summary

The solution(s) found are the following

$$-\sqrt{y}(y-1) = c_2 + x \tag{1}$$

Verification of solutions

$$-\sqrt{y}(y-1) = c_2 + x$$

Verified OK.

6.5.1 Maple step by step solution

Let's solve

$$(-1+3y)^2 y'^2 - 4y = 0$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Separate variables

$$\frac{y'(-1+3y)}{\sqrt{y}} = 2$$

- Integrate both sides with respect to x

$$\int \frac{y'(-1+3y)}{\sqrt{y}} dx = \int 2 dx + c_1$$

- Evaluate integral

$$2y^{\frac{3}{2}} - 2\sqrt{y} = c_1 + 2x$$

- Solve for y

$$y = \left(\frac{\left(54c_1 + 108x + 6\sqrt{81c_1^2 + 324c_1x + 324x^2 - 48}\right)^{\frac{1}{3}}}{6} + \frac{2}{\left(54c_1 + 108x + 6\sqrt{81c_1^2 + 324c_1x + 324x^2 - 48}\right)^{\frac{1}{3}}} \right)^2$$

Maple trace

```

`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing x successful`

```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 519

`dsolve((3*y(x)-1)^2*diff(y(x),x)^2=4*y(x),y(x), singsol=all)`

$$y(x) = 0$$

$$y(x) = \frac{\left(\left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} + 12 \right)^2}{36 \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

$$y(x)$$

$$= \frac{\left(i\sqrt{3} \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} - 12i\sqrt{3} + \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} \right)^2}{144 \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

$$y(x)$$

$$= \frac{\left((i\sqrt{3} - 1) \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} - 12i\sqrt{3} - 12 \right)^2}{144 \left(-108x + 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(\left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} + 12 \right)^2}{36 \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

$$y(x)$$

$$= \frac{\left(i\sqrt{3} \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} - 12i\sqrt{3} + \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} \right)^2}{144 \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

$$y(x)$$

$$= \frac{\left((i\sqrt{3} - 1) \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}} - 12i\sqrt{3} - 12 \right)^2}{144 \left(108x - 108c_1 + 12\sqrt{81c_1^2 - 162c_1x + 81x^2 - 12} \right)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 4.472 (sec). Leaf size: 892

`DSolve[(3*y[x]-1)^2*y'[x]^2==4*y[x],y[x],x,IncludeSingularSolutions -> True]`

$$y(x) \rightarrow \frac{\left(2 + \sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2}\right)^2}{6\sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2}}$$

$$y(x) \rightarrow \frac{1}{24} \left(2i(\sqrt{3} + i) \sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2} - 8 - 8i\sqrt{3} \right. \\ \left. + \frac{-8 - 8i\sqrt{3}}{\sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2}} \right) + 16$$

$$y(x) \rightarrow \frac{1}{24} \left(-2(1 + i\sqrt{3}) \sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2} - 8 + 8i\sqrt{3} \right. \\ \left. + \frac{-8 + 8i\sqrt{3}}{\sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(-2x + c_1)^2(108x^2 - 108c_1x - 16 + 27c_1^2)} - 108c_1x - 8 + 27c_1^2}} \right) + 16$$

$$y(x) \rightarrow \frac{\left(2 + \sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(2x + c_1)^2(108x^2 + 108c_1x - 16 + 27c_1^2)} + 108c_1x - 8 + 27c_1^2}\right)^2}{6\sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(2x + c_1)^2(108x^2 + 108c_1x - 16 + 27c_1^2)} + 108c_1x - 8 + 27c_1^2}}$$

$$y(x) \rightarrow \frac{1}{24} \left(2i(\sqrt{3} + i) \sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(2x + c_1)^2(108x^2 + 108c_1x - 16 + 27c_1^2)} + 108c_1x - 8 + 27c_1^2} - 8(1 + i\sqrt{3}) \right. \\ \left. - \frac{8(1 + i\sqrt{3})}{\sqrt[3]{108x^2 + 3\sqrt{3}\sqrt{(2x + c_1)^2(108x^2 + 108c_1x - 16 + 27c_1^2)} + 108c_1x - 8 + 27c_1^2}} \right) + 16$$

6.6 problem 15

Internal problem ID [5343]

Internal file name [OUTPUT/4834_Sunday_February_04_2024_12_46_11_AM_27770854/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 15.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "**first_order_ode_lie_symmetry_calculated**"

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$y + xy' - x^4y'^2 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 4yx^2}}{2x^3} \quad (1)$$

$$y' = -\frac{-1 + \sqrt{1 + 4yx^2}}{2x^3} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{1 + \sqrt{4yx^2 + 1}}{2x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \frac{(1 + \sqrt{4yx^2 + 1})(b_3 - a_2)}{2x^3} - \frac{(1 + \sqrt{4yx^2 + 1})^2 a_3}{4x^6} \\ - \left(-\frac{3(1 + \sqrt{4yx^2 + 1})}{2x^4} + \frac{2y}{x^2\sqrt{4yx^2 + 1}} \right) (xa_2 + ya_3 + a_1) \\ - \frac{xb_2 + yb_3 + b_1}{x\sqrt{4yx^2 + 1}} = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\begin{aligned} & \frac{-4b_2x^6\sqrt{4yx^2 + 1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 - 4\sqrt{4yx^2 + 1}x^3a_2 - 2\sqrt{4yx^2 + 1}x^3a_3}{x^6\sqrt{4yx^2 + 1}} \\ & = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & 4b_2x^6\sqrt{4yx^2 + 1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 \\ & - 4x^5b_1 + 16x^4ya_1 + 4\sqrt{4yx^2 + 1}x^3a_2 + 2\sqrt{4yx^2 + 1}x^3b_3 \\ & + 6\sqrt{4yx^2 + 1}x^2ya_3 - (4yx^2 + 1)^{\frac{3}{2}}a_3 + 6\sqrt{4yx^2 + 1}x^2a_1 \\ & + 4x^3a_2 + 2x^3b_3 - 2x^2ya_3 + 6x^2a_1 - a_3\sqrt{4yx^2 + 1} - 2a_3 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} & 4b_2x^6\sqrt{4yx^2 + 1} - 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 8x^4y^2a_3 + 4(4yx^2 + 1)x^3a_2 \\ & + 2(4yx^2 + 1)x^3b_3 + 6(4yx^2 + 1)x^2ya_3 - 4x^5b_1 - 8x^4ya_1 + 6(4yx^2 + 1)x^2a_1 \\ & + 4\sqrt{4yx^2 + 1}x^3a_2 + 2\sqrt{4yx^2 + 1}x^3b_3 + 6\sqrt{4yx^2 + 1}x^2ya_3 \\ & - (4yx^2 + 1)^{\frac{3}{2}}a_3 + 6\sqrt{4yx^2 + 1}x^2a_1 - 2(4yx^2 + 1)a_3 - a_3\sqrt{4yx^2 + 1} = 0 \end{aligned} \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 4b_2x^6\sqrt{4yx^2+1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 \\
& + 4\sqrt{4yx^2+1}x^3a_2 + 2\sqrt{4yx^2+1}x^3b_3 + 2\sqrt{4yx^2+1}x^2ya_3 + 4x^3a_2 \\
& + 2x^3b_3 + 6\sqrt{4yx^2+1}x^2a_1 - 2x^2ya_3 + 6x^2a_1 - 2a_3\sqrt{4yx^2+1} - 2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4yx^2+1}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4yx^2+1} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 4b_2v_1^6v_3 + 8v_1^5v_2a_2 + 16v_1^4v_2^2a_3 - 4v_1^6b_2 + 4v_1^5v_2b_3 + 16v_1^4v_2a_1 \\
& - 4v_1^5b_1 + 4v_3v_1^3a_2 + 2v_3v_1^2v_2a_3 + 2v_3v_1^3b_3 + 6v_3v_1^2a_1 \\
& + 4v_1^3a_2 - 2v_1^2v_2a_3 + 2v_1^3b_3 + 6v_1^2a_1 - 2a_3v_3 - 2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 4b_2v_1^6v_3 - 4v_1^6b_2 + (8a_2 + 4b_3)v_1^5v_2 - 4v_1^5b_1 + 16v_1^4v_2^2a_3 \\
& + 16v_1^4v_2a_1 + (4a_2 + 2b_3)v_1^3v_3 + (4a_2 + 2b_3)v_1^3 + 2v_3v_1^2v_2a_3 \\
& - 2v_1^2v_2a_3 + 6v_3v_1^2a_1 + 6v_1^2a_1 - 2a_3v_3 - 2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 6a_1 &= 0 \\
 16a_1 &= 0 \\
 -2a_3 &= 0 \\
 2a_3 &= 0 \\
 16a_3 &= 0 \\
 -4b_1 &= 0 \\
 -4b_2 &= 0 \\
 4b_2 &= 0 \\
 4a_2 + 2b_3 &= 0 \\
 8a_2 + 4b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= a_2 \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= -2a_2
 \end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= x \\
 \eta &= -2y
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= -2y - \left(\frac{1 + \sqrt{4y x^2 + 1}}{2x^3} \right) (x) \\
 &= \frac{-4y x^2 - \sqrt{4y x^2 + 1} - 1}{2x^2} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4yx^2 - \sqrt{4yx^2 + 1} - 1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{4yx^2 + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{4yx^2 + 1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{4yx^2 + 1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{4yx^2 + 1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1 \quad (1)$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1 + \sqrt{4yx^2 + 1}}{2x^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(-1 + \sqrt{4yx^2 + 1})(b_3 - a_2)}{2x^3} - \frac{(-1 + \sqrt{4yx^2 + 1})^2 a_3}{4x^6} - \left(-\frac{2y}{x^2 \sqrt{4yx^2 + 1}} + \frac{-\frac{3}{2} + \frac{3\sqrt{4yx^2 + 1}}{2}}{x^4} \right) (xa_2 + ya_3 + a_1) + \frac{xb_2 + yb_3 + b_1}{x\sqrt{4yx^2 + 1}} = 0 \quad (5E)$$

Putting the above in normal form gives

$$\frac{-4b_2x^6\sqrt{4yx^2 + 1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 - 4\sqrt{4yx^2 + 1}x^3a_2 - 2\sqrt{4yx^2 + 1}x^3a_3}{x^6} = 0$$

Setting the numerator to zero gives

$$4b_2x^6\sqrt{4yx^2 + 1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 + 4\sqrt{4yx^2 + 1}x^3a_2 + 2\sqrt{4yx^2 + 1}x^3b_3 + 6\sqrt{4yx^2 + 1}x^2ya_3 - (4yx^2 + 1)^{\frac{3}{2}}a_3 + 6\sqrt{4yx^2 + 1}x^2a_1 - 4x^3a_2 - 2x^3b_3 + 2x^2ya_3 - 6x^2a_1 - a_3\sqrt{4yx^2 + 1} + 2a_3 = 0 \quad (6E)$$

Simplifying the above gives

$$4b_2x^6\sqrt{4yx^2 + 1} + 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 8x^4y^2a_3 - 4(4yx^2 + 1)x^3a_2 - 2(4yx^2 + 1)x^3b_3 - 6(4yx^2 + 1)x^2ya_3 + 4x^5b_1 + 8x^4ya_1 - 6(4yx^2 + 1)x^2a_1 + 4\sqrt{4yx^2 + 1}x^3a_2 + 2\sqrt{4yx^2 + 1}x^3b_3 + 6\sqrt{4yx^2 + 1}x^2ya_3 - (4yx^2 + 1)^{\frac{3}{2}}a_3 + 6\sqrt{4yx^2 + 1}x^2a_1 + 2(4yx^2 + 1)a_3 - a_3\sqrt{4yx^2 + 1} = 0 \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 4b_2x^6\sqrt{4yx^2+1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 \\
& + 4\sqrt{4yx^2+1}x^3a_2 + 2\sqrt{4yx^2+1}x^3b_3 + 2\sqrt{4yx^2+1}x^2ya_3 - 4x^3a_2 \\
& - 2x^3b_3 + 6\sqrt{4yx^2+1}x^2a_1 + 2x^2ya_3 - 6x^2a_1 - 2a_3\sqrt{4yx^2+1} + 2a_3 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{4yx^2+1}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{4yx^2+1} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 4b_2v_1^6v_3 - 8v_1^5v_2a_2 - 16v_1^4v_2^2a_3 + 4v_1^6b_2 - 4v_1^5v_2b_3 - 16v_1^4v_2a_1 \\
& + 4v_1^5b_1 + 4v_3v_1^3a_2 + 2v_3v_1^2v_2a_3 + 2v_3v_1^3b_3 + 6v_3v_1^2a_1 \\
& - 4v_1^3a_2 + 2v_1^2v_2a_3 - 2v_1^3b_3 - 6v_1^2a_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 4b_2v_1^6v_3 + 4v_1^6b_2 + (-8a_2 - 4b_3)v_1^5v_2 + 4v_1^5b_1 - 16v_1^4v_2^2a_3 \\
& - 16v_1^4v_2a_1 + (4a_2 + 2b_3)v_1^3v_3 + (-4a_2 - 2b_3)v_1^3 \\
& + 2v_3v_1^2v_2a_3 + 2v_1^2v_2a_3 + 6v_3v_1^2a_1 - 6v_1^2a_1 - 2a_3v_3 + 2a_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
 -16a_1 &= 0 \\
 -6a_1 &= 0 \\
 6a_1 &= 0 \\
 -16a_3 &= 0 \\
 -2a_3 &= 0 \\
 2a_3 &= 0 \\
 4b_1 &= 0 \\
 4b_2 &= 0 \\
 -8a_2 - 4b_3 &= 0 \\
 -4a_2 - 2b_3 &= 0 \\
 4a_2 + 2b_3 &= 0
 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
 a_1 &= 0 \\
 a_2 &= a_2 \\
 a_3 &= 0 \\
 b_1 &= 0 \\
 b_2 &= 0 \\
 b_3 &= -2a_2
 \end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
 \xi &= x \\
 \eta &= -2y
 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
 \eta &= \eta - \omega(x, y) \xi \\
 &= -2y - \left(-\frac{-1 + \sqrt{4y x^2 + 1}}{2x^3} \right) (x) \\
 &= \frac{-4y x^2 + \sqrt{4y x^2 + 1} - 1}{2x^2} \\
 \xi &= 0
 \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4yx^2 + \sqrt{4yx^2 + 1} - 1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{4yx^2 + 1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{4yx^2 + 1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{4yx^2 + 1}} \\ S_y &= \frac{-\frac{1}{\sqrt{4yx^2 + 1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1 \quad (1)$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4yx^2}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying dAlembert
  trying simple symmetries for implicit equations
  Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
    *** Sublevel 3 ***
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying homogeneous types:
    trying homogeneous G
    <- homogeneous successful
    -----
  * Tackling next ODE.
    *** Sublevel 3 ***
    Methods for first order ODEs:
    --- Trying classification methods ---
    trying homogeneous types:
    trying homogeneous G
    <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 81

```
dsolve(y(x)=-x*diff(y(x),x)+x^4*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{1}{4x^2}$$
$$y(x) = \frac{-c_1 i - x}{x c_1^2}$$
$$y(x) = \frac{c_1 i - x}{x c_1^2}$$
$$y(x) = \frac{c_1 i - x}{x c_1^2}$$
$$y(x) = \frac{-c_1 i - x}{x c_1^2}$$

✓ Solution by Mathematica

Time used: 0.498 (sec). Leaf size: 123

```
DSolve[y[x]==-x*y'[x]+x^4*y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve} \left[\frac{x \sqrt{4x^2 y(x) + 1} \operatorname{arctanh}(\sqrt{4x^2 y(x) + 1})}{\sqrt{4x^4 y(x) + x^2}} - \frac{1}{2} \log(y(x)) = c_1, y(x) \right]$$
$$\text{Solve} \left[\frac{x \sqrt{4x^2 y(x) + 1} \operatorname{arctanh}(\sqrt{4x^2 y(x) + 1})}{\sqrt{4x^4 y(x) + x^2}} - \frac{1}{2} \log(y(x)) = c_1, y(x) \right]$$
$$y(x) \rightarrow 0$$

6.7 problem 16

6.7.1 Solving as dAlembert ode 1147

Internal problem ID [5344]

Internal file name [OUTPUT/4835_Sunday_February_04_2024_12_46_13_AM_25059485/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 16.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "dAlembert"

Maple gives the following as the ode type

```
[[_1st_order , _with_linear_symmetries] , _dAlembert]
```

$$2y - y'^2 - 4xy' = 0$$

6.7.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2 - 4xp + 2y = 0$$

Solving for y from the above results in

$$y = \frac{1}{2}p^2 + 2xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned}f &= 2p \\g &= \frac{p^2}{2}\end{aligned}$$

Hence (2) becomes

$$-p = (2x + p)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x + p(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) + p}{p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned}p(p) &= \frac{2}{p} \\q(p) &= -1\end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p} dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dp}(\mu x) &= (\mu)(-1) \\ \frac{d}{dp}(p^2 x) &= (p^2)(-1) \\ d(p^2 x) &= (-p^2) dp\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -p^2 dp \\ p^2 x &= -\frac{p^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{p}{3} + \frac{c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p . This results in

$$\begin{aligned}p &= -2x + \sqrt{4x^2 + 2y} \\ p &= -2x - \sqrt{4x^2 + 2y}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{(-16x^2 - 2y)\sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x - \sqrt{4x^2 + 2y})^2} \\ x &= \frac{2(8x^2 + y)\sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x + \sqrt{4x^2 + 2y})^2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{(-16x^2 - 2y) \sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x - \sqrt{4x^2 + 2y})^2} \tag{2}$$

$$x = \frac{2(8x^2 + y) \sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x + \sqrt{4x^2 + 2y})^2} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{(-16x^2 - 2y) \sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x - \sqrt{4x^2 + 2y})^2}$$

Verified OK.

$$x = \frac{2(8x^2 + y) \sqrt{4x^2 + 2y} + 32x^3 + 12xy + 3c_1}{3(2x + \sqrt{4x^2 + 2y})^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:  
-> Solving 1st order ODE of high degree, 1st attempt  
trying 1st order WeierstrassP solution for high degree ODE  
trying 1st order WeierstrassPPrime solution for high degree ODE  
trying 1st order JacobiSN solution for high degree ODE  
trying 1st order ODE linearizable_by_differentiation  
trying differential order: 1; missing variables  
trying dAlembert  
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 700

```
dsolve(2*y(x)=diff(y(x),x)^2+4*x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 2x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} + 4x^2 \right) \left(\left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 2x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} + 4x^2 \right)}{8 \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 2x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} + 4x^2}$$

$$y(x) = \frac{\left(i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 4i\sqrt{3}x^2 + \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x \right) \left(i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 4i\sqrt{3}x^2 + \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x \right)}{\left(i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} - 4i\sqrt{3}x^2 + \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x \right)^2}$$

$$y(x) = \frac{\left(4i\sqrt{3}x^2 - i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x^2 + 4x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} \right) \left(4i\sqrt{3}x^2 - i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x^2 + 4x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} \right)}{\left(4i\sqrt{3}x^2 - i\sqrt{3} \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{2}{3}} + 4x^2 + 4x \left(12c_1 - 8x^3 + 4\sqrt{3} \sqrt{-4c_1x^3 + 3c_1^2} \right)^{\frac{1}{3}} \right)^2}$$

✓ Solution by Mathematica

Time used: 60.241 (sec). Leaf size: 1344

`DSolve[2*y[x]==y'[x]^2+4*x*y'[x],y[x],x,IncludeSingularSolutions -> True]`

$y(x)$

$$\rightarrow \frac{1}{2} \left(-x^2 + \frac{x(x^3 + 2\sqrt{2}e^{3c_1})}{\sqrt[3]{-x^6 + 5\sqrt{2}e^{3c_1}x^3 + \sqrt{e^{3c_1}(-16\sqrt{2}x^9 + 24e^{3c_1}x^6 - 6\sqrt{2}e^{6c_1}x^3 + e^{9c_1})} + e^{6c_1}}} + \sqrt[3]{-x^6 + 5\sqrt{2}e^{3c_1}x^3 + \sqrt{e^{3c_1}(-16\sqrt{2}x^9 + 24e^{3c_1}x^6 - 6\sqrt{2}e^{6c_1}x^3 + e^{9c_1})} + e^{6c_1}} \right)$$

$y(x) \rightarrow \frac{1}{4} \left(-2x^2 \right.$

$$\left. - \frac{(1 + i\sqrt{3})x(x^3 + 2\sqrt{2}e^{3c_1})}{\sqrt[3]{-x^6 + 5\sqrt{2}e^{3c_1}x^3 + \sqrt{e^{3c_1}(-16\sqrt{2}x^9 + 24e^{3c_1}x^6 - 6\sqrt{2}e^{6c_1}x^3 + e^{9c_1})} + e^{6c_1}}} + i(\sqrt{3} + i) \sqrt[3]{-x^6 + 5\sqrt{2}e^{3c_1}x^3 + \sqrt{e^{3c_1}(-16\sqrt{2}x^9 + 24e^{3c_1}x^6 - 6\sqrt{2}e^{6c_1}x^3 + e^{9c_1})} + e^{6c_1}} \right)$$

$y(x) \rightarrow \frac{1}{4} \left(-2x^2 \right.$

$$\left. + \frac{i(\sqrt{3} + i)x(x^3 + 2\sqrt{2}e^{3c_1})}{\sqrt[3]{-x^6 + 5\sqrt{2}e^{3c_1}x^3 + \sqrt{e^{3c_1}(-16\sqrt{2}x^9 + 24e^{3c_1}x^6 - 6\sqrt{2}e^{6c_1}x^3 + e^{9c_1})} + e^{6c_1}}} \right)$$

6.8 problem 17

6.8.1 Maple step by step solution 1154

Internal problem ID [5345]

Internal file name [OUTPUT/4836_Sunday_February_04_2024_12_46_13_AM_76649283/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 17.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

[_quadrature]

$$y(3 - 4y)^2 y'^2 + 4y = 4$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2\sqrt{-y(y-1)}}{y(4y-3)} \quad (1)$$

$$y' = -\frac{2\sqrt{-y(y-1)}}{y(4y-3)} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y(4y-3)}{2\sqrt{-y(y-1)}} dy = \int dx$$
$$\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = x + c_1$$

Summary

The solution(s) found are the following

$$\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = x + c_1 \quad (1)$$

Verification of solutions

$$\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y(4y-3)}{2\sqrt{-y(y-1)}} dy = \int dx$$
$$-\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = c_2 + x$$

Summary

The solution(s) found are the following

$$-\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = c_2 + x \quad (1)$$

Verification of solutions

$$-\frac{y^2(y-1)}{\sqrt{-y(y-1)}} = c_2 + x$$

Verified OK.

6.8.1 Maple step by step solution

Let's solve

$$y(3-4y)^2 y'^2 + 4y = 4$$

- Highest derivative means the order of the ODE is 1
 y'
- Separate variables

$$\frac{y'y(4y-3)}{\sqrt{-y(y-1)}} = 2$$

- Integrate both sides with respect to x

$$\int \frac{y'y(4y-3)}{\sqrt{-y(y-1)}} dx = \int 2dx + c_1$$

- Evaluate integral

$$-2y\sqrt{-y^2 + y} = c_1 + 2x$$

Maple trace

```

Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing x successful

```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 58

```
dsolve(y(x)*(3-4*y(x))^2*diff(y(x),x)^2=4*(1-y(x)),y(x), singsol=all)
```

$$y(x) = 1$$

$$x + \frac{y(x)^2 (y(x) - 1)}{\sqrt{-y(x)(y(x) - 1)}} - c_1 = 0$$

$$x - \frac{y(x)^2 (y(x) - 1)}{\sqrt{-y(x)(y(x) - 1)}} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.264 (sec). Leaf size: 3751

```
DSolve[y[x]*(3-4*y[x])^2*y'[x]^2==4*(1-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

6.9 problem 18

Internal problem ID [5346]

Internal file name [OUTPUT/4837_Sunday_February_04_2024_12_46_14_AM_38484777/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 18.

ODE order: 1.

ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order , _with_linear_symmetries]]`

$$y'^3 - 4x^4y' + 8yx^3 = 0$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2 \left(\frac{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{6} + \frac{2x^2}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right) x \quad (1)$$

$$y' = 2 \left(-\frac{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{12} - \frac{x^2}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} + \frac{i\sqrt{3} \left(\frac{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{6} \right)}{\dots} \right) \quad (2)$$

$$y' = 2 \left(-\frac{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{12} - \frac{x^2}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} - \frac{i\sqrt{3} \left(\frac{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{6} \right)}{\dots} \right) \quad (3)$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{x \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)}{3 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2 + \frac{x \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right) (b_3 - a_2)}{3 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}}} \\
& - \frac{x^2 \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)^2 a_3}{9 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}}} \\
& - \left(\frac{\left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} + 12x^2}{3 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}}} \right. \\
& + \frac{x \left(-\frac{288x^5}{\left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2}} + 24x \right)}{3 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}}} \\
& \left. + \frac{48x^6 \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)}{\left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2}} \right) (xa_2 + ya_3 + a_1) \\
& - \left(\frac{2x \left(-108 + \frac{972y}{\sqrt{-12x^6 + 81y^2}} \right)}{9 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}}} \right. \\
& \left. - \frac{x \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right) \left(-108 + \frac{972y}{\sqrt{-12x^6 + 81y^2}} \right)}{9 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{4}{3}}} \right) (xb_2 + yb_3 + b_1) \\
& = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}
& \frac{-3888\sqrt{-12x^6 + 81y^2} x^2 a_1 y + 6912x^8 y a_3 - 3888x^4 y b_2 + 8 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2}}{\dots} \\
& = 0
\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}
& 3888\sqrt{-12x^6 + 81y^2} x^2 a_1 y - 6912x^8 y a_3 + 3888x^4 y b_2 \\
& - 8\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2} x^4 a_3 \\
& - 48\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^6 a_3 \\
& + 144\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^6 y a_3 \\
& - 2\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} x a_2 \\
& + \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} x b_3 \\
& - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} y a_3 \\
& + 36\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^2 b_2 \\
& - 324\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^2 y b_2 \\
& - 324\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x y^2 b_3 \\
& - 1728\sqrt{-12x^6 + 81y^2} x^3 y b_3 \\
& + 36\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x b_1 \\
& - 324\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x y b_1 \\
& - 1728x^9 b_3 + 5184x^9 a_2 + 3456x^8 a_1 + 15552x^3 y^2 b_3 \\
& + 3888x^3 y b_1 + 144\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^7 a_2 \\
& + 144\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^6 a_1 \\
& - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} a_1 \\
& + 3b_2\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2} \\
& - 432\sqrt{-12x^6 + 81y^2} x^4 b_2 \\
& - 432\sqrt{-12x^6 + 81y^2} x^3 b_1 - 48(-12x^6 + 81y^2)^{\frac{3}{2}} x^2 a_3 \\
& + 36\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x y b_3 \\
& - 46656x^3 y^2 a_2 + 34992x^2 y^3 a_3 - 34992x^2 y^2 a_1 \\
& + 5184\sqrt{-12x^6 + 81y^2} x^3 a_2 y = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
& -5184x^8ya_3 + 11664x^4yb_2 - \left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^2 \sqrt{-12x^6 + 81y^2} x^2a_3 \\
& \quad - 24\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2} x^4a_3 \\
& \quad - 144\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^6a_3 \\
& \quad + 432\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^6ya_3 - 6\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} xa_2 \\
& \quad + 3\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} xb_3 \\
& \quad - 3\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{5}{3}} \sqrt{-12x^6 + 81y^2} ya_3 \\
& \quad - 144\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right) \sqrt{-12x^6 + 81y^2} x^3a_2 \\
& \quad + 36\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right) \sqrt{-12x^6 + 81y^2} x^3b_3 \\
& \quad - 108\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right) \sqrt{-12x^6 + 81y^2} x^2a_1 \\
& \quad + 108\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^2b_2 \\
& \quad - 972\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x^2yb_2 \\
& \quad - 972\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} x y^2b_3 \\
& \quad - 1296\sqrt{-12x^6 + 81y^2} x^3yb_3 + 108\left(-108y\right. \\
& \quad \left.+ 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} xb_1 \\
& \quad - 972\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} xyb_1 \\
& \quad - 5184x^9a_2 - 5184x^8a_1 + 11664x^3y^2b_3 + 11664x^3yb_1
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
& 11664\sqrt{-12x^6 + 81y^2} x^2 a_1 y + 1728x^8 \sqrt{-12x^6 + 81y^2} a_3 - 20736x^8 y a_3 \\
& + 11664x^4 y b_2 - 2916 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} y^3 a_3 \\
& - 2916 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} y^2 a_1 + 8748 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} y^2 b_2 \\
& - 144 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^6 a_3 \\
& + 864 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x^6 y a_3 \\
& + 108 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x^2 b_2 \\
& - 972 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x^2 y b_2 + 1944 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x y^2 b_3 \\
& - 5184 \sqrt{-12x^6 + 81y^2} x^3 y b_3 + 108 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x b_1 \\
& - 972 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x y b_1 \\
& + 2592 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2} x^4 a_3 y \\
& + 648 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x a_2 y \\
& - 432x^7 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} b_3 - 1296x^6 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} b_2 \\
& - 5184x^9 b_3 + 15552x^9 a_2 + 10368x^8 a_1 + 46656x^3 y^2 b_3 + 11664x^3 y b_1 \\
& + 1296 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x^7 a_2 + 864 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} x^6 a_1 \\
& - 1296 \sqrt{-12x^6 + 81y^2} x^4 b_2 - 1296 \sqrt{-12x^6 + 81y^2} x^3 b_1 \\
& - 216 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} x y b_3 \\
& + 3456x^{10} \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} a_3 - 139968x^3 y^2 a_2 + 104976x^2 y^3 a_3 \\
& - 104976x^2 y^2 a_1 + 15552 \sqrt{-12x^6 + 81y^2} x^3 a_2 y - 11664x^2 \sqrt{-12x^6 + 81y^2} y^2 a_3 \\
& + 324 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} y^2 a_3 \\
& + 324 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} \sqrt{-12x^6 + 81y^2} a_1 y \\
& - 972b_2 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2} y \\
& - 23328x^4 \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{1}{3}} y^2 a_3 \\
& - 5832x \left(-108y + 12\sqrt{-12x^6 + 81y^2} \right)^{\frac{2}{3}} y^2 a_2 = 0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6 + 81y^2} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6 + 81y^2} = v_5 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & 3456v_1^{10}v_3a_3 + 15552v_1^9a_2 - 20736v_1^8v_2a_3 + 1728v_1^8v_5a_3 \\ & - 5184v_1^9b_3 + 10368v_1^8a_1 + 1296v_4v_1^7a_2 + 864v_4v_1^6v_2a_3 \\ & - 144v_4v_5v_1^6a_3 - 432v_1^7v_4b_3 + 864v_4v_1^6a_1 - 23328v_1^4v_3v_2^2a_3 \\ & + 2592v_3v_5v_1^4a_3v_2 - 1296v_1^6v_3b_2 - 139968v_1^3v_2^2a_2 + 15552v_5v_1^3a_2v_2 \\ & + 104976v_1^2v_2^3a_3 - 11664v_1^2v_5v_2^2a_3 + 11664v_1^4v_2b_2 - 1296v_5v_1^4b_2 \\ & + 46656v_1^3v_2^2b_3 - 5184v_5v_1^3v_2b_3 - 104976v_1^2v_2^2a_1 + 11664v_5v_1^2a_1v_2 \\ & - 5832v_1v_4v_2^2a_2 + 648v_4v_5v_1a_2v_2 - 2916v_4v_2^3a_3 + 324v_4v_5v_2^2a_3 \\ & + 11664v_1^3v_2b_1 - 1296v_5v_1^3b_1 - 972v_4v_1^2v_2b_2 + 108v_4v_5v_1^2b_2 \\ & + 1944v_4v_1v_2^2b_3 - 216v_4v_5v_1v_2b_3 - 2916v_4v_2^2a_1 + 324v_4v_5a_1v_2 \\ & - 972v_4v_1v_2b_1 + 108v_4v_5v_1b_1 + 8748v_3v_2^2b_2 - 972b_2v_3v_5v_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& 2592v_3v_5v_1^4a_3v_2 - 1296v_5v_1^3b_1 + 3456v_1^{10}v_3a_3 + 104976v_1^2v_2^3a_3 \\
& - 104976v_1^2v_2^2a_1 + 10368v_1^8a_1 + 1728v_1^8v_5a_3 - 20736v_1^8v_2a_3 \\
& - 2916v_4v_2^3a_3 + 11664v_1^4v_2b_2 - 2916v_4v_2^2a_1 + 8748v_3v_2^2b_2 \\
& - 1296v_1^6v_3b_2 + 11664v_1^3v_2b_1 + 864v_4v_1^6a_1 - 1296v_5v_1^4b_2 \\
& + (15552a_2 - 5184b_3)v_1^3v_2v_5 + (648a_2 - 216b_3)v_1v_2v_4v_5 \\
& + (-5832a_2 + 1944b_3)v_1v_2^2v_4 + (15552a_2 - 5184b_3)v_1^9 - 144v_4v_5v_1^6a_3 \\
& + 864v_4v_1^6v_2a_3 + 108v_4v_5v_1^2b_2 - 972v_4v_1^2v_2b_2 + 108v_4v_5v_1b_1 \\
& - 972v_4v_1v_2b_1 - 11664v_1^2v_5v_2^2a_3 + 324v_4v_5v_2^2a_3 + 324v_4v_5a_1v_2 \\
& - 972b_2v_3v_5v_2 - 23328v_1^4v_3v_2^2a_3 + (1296a_2 - 432b_3)v_1^7v_4 \\
& + (-139968a_2 + 46656b_3)v_1^3v_2^2 + 11664v_5v_1^2a_1v_2 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$-104976a_1 = 0$$

$$-2916a_1 = 0$$

$$324a_1 = 0$$

$$864a_1 = 0$$

$$10368a_1 = 0$$

$$11664a_1 = 0$$

$$-23328a_3 = 0$$

$$-20736a_3 = 0$$

$$-11664a_3 = 0$$

$$-2916a_3 = 0$$

$$-144a_3 = 0$$

$$324a_3 = 0$$

$$864a_3 = 0$$

$$1728a_3 = 0$$

$$2592a_3 = 0$$

$$3456a_3 = 0$$

$$104976a_3 = 0$$

$$-1296b_1 = 0$$

$$-972b_1 = 0$$

$$108b_1 = 0$$

$$11664b_1 = 0$$

$$-1296b_2 = 0$$

$$-972b_2 = 0$$

$$108b_2 = 0$$

$$8748b_2 = 0$$

$$11664b_2 = 0$$

$$-139968a_2 + 46656b_3 = 0$$

$$-5832a_2 + 1944b_3 = 0$$

$$648a_2 - 216b_3 = 0$$

$$1296a_2 - 432b_3 = 0$$

$$15552a_2 - 5184b_3 = 0$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= a_2 \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= 3a_2\end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= 3y\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{3y}{x} \\ &= \frac{3y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x^3$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^3}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x \left((-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)}{3(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{x^4} \\ R_y &= \frac{1}{x^3} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3x^3(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}{x^2(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^4 - 9(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}y} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3 \cdot 12^{\frac{1}{3}} (\sqrt{3} \sqrt{27R^2 - 4} - 9R)^{\frac{1}{3}}}{12^{\frac{2}{3}} (\sqrt{3} \sqrt{27R^2 - 4} - 9R)^{\frac{2}{3}} - 9 \cdot 12^{\frac{1}{3}} (\sqrt{3} \sqrt{27R^2 - 4} - 9R)^{\frac{1}{3}} R + 12}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = \int \frac{3(12\sqrt{81R^2 - 12} - 108R)^{\frac{1}{3}}}{218^{\frac{1}{3}} \left((\sqrt{81R^2 - 12} - 9R)^2 \right)^{\frac{1}{3}} - 9R (12\sqrt{81R^2 - 12} - 108R)^{\frac{1}{3}} + 12} dR + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int \frac{y}{x^3} \frac{3(12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}}}{218^{\frac{1}{3}} \left((\sqrt{81_a a^2 - 12} - 9_a a)^2 \right)^{\frac{1}{3}} - 9_a a (12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}} + 12} d_a a + c_1$$

Which simplifies to

$$\ln(x) = \int \frac{y}{x^3} \frac{3(12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}}}{218^{\frac{1}{3}} \left((\sqrt{81_a a^2 - 12} - 9_a a)^2 \right)^{\frac{1}{3}} - 9_a a (12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}} + 12} d_a a + c_1$$

Summary

The solution(s) found are the following

$$\begin{aligned} \ln(x) &= \int \frac{y}{x^3} \frac{3(12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}}}{218^{\frac{1}{3}} \left((\sqrt{81_a a^2 - 12} - 9_a a)^2 \right)^{\frac{1}{3}} - 9_a a (12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}} + 12} d_a a \\ &+ c_1 \end{aligned} \quad (1)$$

Verification of solutions

$$\begin{aligned} \ln(x) &= \int \frac{y}{x^3} \frac{3(12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}}}{218^{\frac{1}{3}} \left((\sqrt{81_a a^2 - 12} - 9_a a)^2 \right)^{\frac{1}{3}} - 9_a a (12\sqrt{81_a a^2 - 12} - 108_a a)^{\frac{1}{3}} + 12} d_a a \\ &+ c_1 \end{aligned}$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3} x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2 \right)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
 & b_2 \tag{5E} \\
 & + \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2 \right) (b_3 - a_2)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \\
 & - \frac{x^2 \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2 \right)^2 a_3}{36 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}}} \\
 & - \left(\frac{i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right. \\
 & + \frac{x \left(-\frac{288i\sqrt{3}x^5}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2}} - 24i\sqrt{3}x + \frac{288x^5}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2}} - 24x \right)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \\
 & \left. + \frac{24x^6 \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2 \right)}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2}} \right) (xa_2) \\
 & + ya_3 + a_1) - \left(\frac{x \left(\frac{2i\sqrt{3} \left(-108 + \frac{972y}{\sqrt{-12x^6 + 81y^2}} \right)}{3(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} - \frac{2 \left(-108 + \frac{972y}{\sqrt{-12x^6 + 81y^2}} \right)}{3(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right) \\
 & - \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 - (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12x^2 \right) (-108 + 12\sqrt{-12x^6 + 81y^2})}{18 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{4}{3}}} \\
 & + yb_3 + b_1) = 0
 \end{aligned}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6 + 81y^2} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6 + 81y^2} = v_5 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& -1728v_4v_1^6v_2a_3 - 31104v_5v_1^3a_2v_2 - 23328v_5v_1^2a_1v_2 - 93312v_1^4v_3v_2^2a_3 + 11664v_1v_4v_2^2a_2 \\
& - 216v_4v_5v_1^2b_2 + 1944v_4v_1^2v_2b_2 - 3888v_4v_1v_2^2b_3 + 10368v_5v_1^3v_2b_3 - 216v_4v_5v_1b_1 \\
& + 1944v_4v_1v_2b_1 - 648v_4v_5a_1v_2 - 3888b_2v_3v_5v_2 - 648v_4v_5v_2^2a_3 + 23328v_1^2v_5v_2^2a_3 \\
& - 93312i\sqrt{3}v_1^3v_2^2b_3 - 209952i\sqrt{3}v_1^2v_2^3a_3 - 31104i\sqrt{3}v_1^9a_2 + 10368i\sqrt{3}v_1^9b_3 \\
& - 20736i\sqrt{3}v_1^8a_1 - 1296v_4v_5v_1a_2v_2 + 10368v_3v_5v_1^4a_3v_2 - 3456v_1^8v_5a_3 + 279936v_1^3v_2^2a_2 \\
& + 209952v_1^2v_2^2a_1 - 209952v_1^2v_2^3a_3 + 13824v_1^{10}v_3a_3 + 864v_1^7v_4b_3 - 5184v_1^6v_3b_2 + 5832v_4v_2^3a_3 \\
& + 432v_4v_5v_1v_2b_3 + 209952i\sqrt{3}v_1^2v_2^2a_1 + 41472i\sqrt{3}v_1^8v_2a_3 + 279936i\sqrt{3}v_1^3v_2^2a_2 \\
& + 2592iv_4\sqrt{3}v_1^7a_2 - 864i\sqrt{3}v_1^7v_4b_3 + 1728iv_4\sqrt{3}v_1^6a_1 - 5832i\sqrt{3}v_4v_2^3a_3 \\
& - 5832i\sqrt{3}v_4v_2^2a_1 - 3456i\sqrt{3}v_1^8v_5a_3 + 2592iv_5\sqrt{3}v_1^4b_2 - 23328i\sqrt{3}v_1^4v_2b_2 \\
& + 2592iv_5\sqrt{3}v_1^3b_1 - 23328i\sqrt{3}v_1^3v_2b_1 + 288v_4v_5v_1^6a_3 + 5832v_4v_2^2a_1 + 34992v_3v_2^2b_2 \\
& + 41472v_1^8v_2a_3 - 23328v_1^4v_2b_2 - 93312v_1^3v_2^2b_3 - 23328v_1^3v_2b_1 - 2592v_4v_1^7a_2 \\
& - 1728v_4v_1^6a_1 + 2592v_5v_1^4b_2 + 2592v_5v_1^3b_1 - 31104v_1^9a_2 - 20736v_1^8a_1 + 10368v_1^9b_3 \\
& + 10368iv_5\sqrt{3}v_1^3v_2b_3 - 432iv_4v_5\sqrt{3}v_1v_2b_3 + 1296iv_4v_5\sqrt{3}v_1a_2v_2 + 648iv_4v_5\sqrt{3}v_2^2a_3 \\
& + 23328i\sqrt{3}v_1^2v_5v_2^2a_3 + 216iv_4v_5\sqrt{3}v_1^2b_2 - 1944iv_4\sqrt{3}v_1^2v_2b_2 - 288iv_4v_5\sqrt{3}v_1^6a_3 \\
& + 216iv_4v_5\sqrt{3}v_1b_1 - 1944iv_4\sqrt{3}v_1v_2b_1 - 31104iv_5\sqrt{3}v_1^3a_2v_2 - 23328iv_5\sqrt{3}v_1^2a_1v_2 \\
& + 3888iv_4\sqrt{3}v_1v_2^2b_3 + 1728iv_4\sqrt{3}v_1^6v_2a_3 - 11664i\sqrt{3}v_1v_4v_2^2a_2 + 648iv_4v_5\sqrt{3}a_1v_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& \left(1728i\sqrt{3}a_3 - 1728a_3\right) v_1^6 v_2 v_4 \\
& + \left(-288i\sqrt{3}a_3 + 288a_3\right) v_1^6 v_4 v_5 + \left(-31104i\sqrt{3}a_2 \right. \\
& + \left.10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3\right) v_1^3 v_2 v_5 \\
& + \left(23328i\sqrt{3}a_3 + 23328a_3\right) v_1^2 v_2^2 v_5 \\
& + \left(-1944i\sqrt{3}b_2 + 1944b_2\right) v_1^2 v_2 v_4 - 93312v_1^4 v_3 v_2^2 a_3 \\
& - 3888b_2 v_3 v_5 v_2 + \left(-23328i\sqrt{3}b_2 - 23328b_2\right) v_1^4 v_2 \\
& + \left(2592i\sqrt{3}b_2 + 2592b_2\right) v_1^4 v_5 + \left(279936i\sqrt{3}a_2 \right. \\
& - \left.93312i\sqrt{3}b_3 + 279936a_2 - 93312b_3\right) v_1^3 v_2^2 \\
& + \left(-23328i\sqrt{3}b_1 - 23328b_1\right) v_1^3 v_2 \\
& + \left(2592i\sqrt{3}b_1 + 2592b_1\right) v_1^3 v_5 \\
& + \left(-209952i\sqrt{3}a_3 - 209952a_3\right) v_1^2 v_2^3 \\
& + \left(209952i\sqrt{3}a_1 + 209952a_1\right) v_1^2 v_2^2 \\
& + \left(-5832i\sqrt{3}a_3 + 5832a_3\right) v_2^3 v_4 \\
& + \left(-5832i\sqrt{3}a_1 + 5832a_1\right) v_2^2 v_4 + 10368v_3 v_5 v_1^4 a_3 v_2 \\
& + \left(41472i\sqrt{3}a_3 + 41472a_3\right) v_1^8 v_2 \\
& + \left(-3456i\sqrt{3}a_3 - 3456a_3\right) v_1^8 v_5 \\
& + \left(2592i\sqrt{3}a_2 - 864i\sqrt{3}b_3 - 2592a_2 + 864b_3\right) v_1^7 v_4 \\
& + \left(1728i\sqrt{3}a_1 - 1728a_1\right) v_1^6 v_4 \\
& + 13824v_1^{10} v_3 a_3 - 5184v_1^6 v_3 b_2 \\
& + \left(-31104i\sqrt{3}a_2 + 10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3\right) v_1^9 \\
& + \left(-20736i\sqrt{3}a_1 - 20736a_1\right) v_1^8 \\
& + \left(1296i\sqrt{3}a_2 - 432i\sqrt{3}b_3 - 1296a_2 + 432b_3\right) v_1 v_2 v_4 v_5 \\
& + 34992v_3 v_2^2 b_2 + \left(-23328i\sqrt{3}a_1 - 23328a_1\right) v_1^2 v_2 v_5 \\
& + \left(216i\sqrt{3}b_2 - 216b_2\right) v_1^2 v_4 v_5 + \left(-11664i\sqrt{3}a_2 \right. \\
& + \left.3888i\sqrt{3}b_3 + 11664a_2 - 3888b_3\right) v_1 v_2^2 v_4 \\
& + \left(-1944i\sqrt{3}b_1 + 1944b_1\right) v_1 v_2 v_4 \\
& + \left(216i\sqrt{3}b_1 - 216b_1\right) v_1 v_4 v_5 \\
& + \left(648i\sqrt{3}a_3 - 648a_3\right) v_2^2 v_4 v_5
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-93312a_3 &= 0 \\
10368a_3 &= 0 \\
13824a_3 &= 0 \\
-5184b_2 &= 0 \\
-3888b_2 &= 0 \\
34992b_2 &= 0 \\
-209952i\sqrt{3}a_3 - 209952a_3 &= 0 \\
-23328i\sqrt{3}a_1 - 23328a_1 &= 0 \\
-23328i\sqrt{3}b_1 - 23328b_1 &= 0 \\
-23328i\sqrt{3}b_2 - 23328b_2 &= 0 \\
-20736i\sqrt{3}a_1 - 20736a_1 &= 0 \\
-5832i\sqrt{3}a_1 + 5832a_1 &= 0 \\
-5832i\sqrt{3}a_3 + 5832a_3 &= 0 \\
-3456i\sqrt{3}a_3 - 3456a_3 &= 0 \\
-1944i\sqrt{3}b_1 + 1944b_1 &= 0 \\
-1944i\sqrt{3}b_2 + 1944b_2 &= 0 \\
-288i\sqrt{3}a_3 + 288a_3 &= 0 \\
216i\sqrt{3}b_1 - 216b_1 &= 0 \\
216i\sqrt{3}b_2 - 216b_2 &= 0 \\
648i\sqrt{3}a_1 - 648a_1 &= 0 \\
648i\sqrt{3}a_3 - 648a_3 &= 0 \\
1728i\sqrt{3}a_1 - 1728a_1 &= 0 \\
1728i\sqrt{3}a_3 - 1728a_3 &= 0 \\
2592i\sqrt{3}b_1 + 2592b_1 &= 0 \\
2592i\sqrt{3}b_2 + 2592b_2 &= 0 \\
23328i\sqrt{3}a_3 + 23328a_3 &= 0 \\
41472i\sqrt{3}a_3 + 41472a_3 &= 0 \\
209952i\sqrt{3}a_1 + 209952a_1 &= 0 \\
-31104i\sqrt{3}a_2 + 10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3 &= 0 \\
-11664i\sqrt{3}a_2 + 3888i\sqrt{3}b_3 + 11664a_2 - 3888b_3 &= 0 \\
1296i\sqrt{3}a_2 - 432i\sqrt{3}b_3 - 1296a_2 + 432b_3 &= 0 \\
2592i\sqrt{3}a_2 - 864i\sqrt{3}b_3 - 2592a_2 + 864b_3 &= 0 \\
279936i\sqrt{3}a_2 - 93312i\sqrt{3}b_3 - 279936a_2 + 93312b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 3a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= 3y \end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R . Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$y' = - \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
 & b_2 \tag{5E} \\
 & \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right) (b_3 - a_2)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \\
 & - \frac{x^2 \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)^2 a_3}{36 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}}} \\
 & - \left(\frac{i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right. \\
 & \left. x \left(-\frac{288i\sqrt{3}x^5}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2}} - 24i\sqrt{3}x - \frac{288x^5}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}} \sqrt{-12x^6 + 81y^2}} + 24x \right) \right. \\
 & \left. - \frac{24x^6 \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right)}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{4}{3}} \sqrt{-12x^6 + 81y^2}} \right) (xa) \\
 & + ya_3 + a_1) - \left(\frac{x \left(\frac{2i\sqrt{3} \left(-108 + \frac{972y}{\sqrt{-12x^6 + 81y^2}} \right)}{3 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} + \frac{-72 + \frac{648y}{\sqrt{-12x^6 + 81y^2}}}{(-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right)}{6 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{1}{3}}} \right) \\
 & + \frac{x \left(i\sqrt{3} (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} - 12i\sqrt{3}x^2 + (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{2}{3}} + 12x^2 \right) (-108 +}{18 (-108y + 12\sqrt{-12x^6 + 81y^2})^{\frac{4}{3}}} \\
 & + yb_3 + b_1) = 0
 \end{aligned}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{ x, y, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6 + 81y^2} \right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{ x = v_1, y = v_2, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6 + 81y^2} = v_5 \right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 10368v_5v_3v_1^4a_3v_2 - 1296v_5v_4v_1a_2v_2 - 31104v_1^9a_2 \\
& - 20736v_1^8a_1 + 10368v_1^9b_3 - 2592i\sqrt{3}v_5v_1^3b_1 \\
& + 23328i\sqrt{3}v_1^3v_2b_1 - 2592i\sqrt{3}v_5v_1^4b_2 \\
& + 23328i\sqrt{3}v_1^4v_2b_2 + 5832i\sqrt{3}v_4v_2^2a_1 - 2592i\sqrt{3}v_4v_1^7a_2 \\
& + 864i\sqrt{3}v_1^7v_4b_3 - 1728i\sqrt{3}v_4v_1^6a_1 + 5832i\sqrt{3}v_4v_2^3a_3 \\
& + 432v_5v_4v_1v_2b_3 + 3456i\sqrt{3}v_1^8v_5a_3 - 41472i\sqrt{3}v_1^8v_2a_3 \\
& - 279936i\sqrt{3}v_1^3v_2^2a_2 + 93312i\sqrt{3}v_1^3v_2^2b_3 \\
& + 209952i\sqrt{3}v_1^2v_2^3a_3 - 209952i\sqrt{3}v_1^2v_2^2a_1 \\
& + 31104i\sqrt{3}v_5v_1^3a_2v_2 - 10368i\sqrt{3}v_5v_1^3v_2b_3 \\
& + 23328i\sqrt{3}v_5v_1^2a_1v_2 - 648i\sqrt{3}v_5v_4v_2^2a_3 \\
& - 648i\sqrt{3}v_5v_4a_1v_2 + 288i\sqrt{3}v_5v_4v_1^6a_3 \\
& - 216i\sqrt{3}v_5v_4v_1^2b_2 + 1944i\sqrt{3}v_4v_1^2v_2b_2 \\
& - 216i\sqrt{3}v_5v_4v_1b_1 + 1944i\sqrt{3}v_4v_1v_2b_1 \\
& - 23328i\sqrt{3}v_1^2v_5v_2^2a_3 - 3888i\sqrt{3}v_4v_1v_2^2b_3 \\
& - 1728i\sqrt{3}v_4v_1^6v_2a_3 + 11664i\sqrt{3}v_1v_4v_2^2a_2 \\
& - 23328v_5v_1^2a_1v_2 + 288v_5v_4v_1^6a_3 - 1728v_4v_1^6v_2a_3 \\
& - 216v_5v_4v_1^2b_2 + 10368v_5v_1^3v_2b_3 + 1944v_4v_1^2v_2b_2 \\
& - 3888v_4v_1v_2^2b_3 - 216v_5v_4v_1b_1 + 1944v_4v_1v_2b_1 \\
& + 11664v_1v_4v_2^2a_2 - 93312v_1^4v_3v_2^2a_3 - 3888b_2v_3v_5v_2 \\
& - 648v_5v_4v_2^2a_3 + 23328v_1^2v_5v_2^2a_3 - 648v_5v_4a_1v_2 \\
& + 31104i\sqrt{3}v_1^9a_2 - 10368i\sqrt{3}v_1^9b_3 + 20736i\sqrt{3}v_1^8a_1 \\
& - 31104v_5v_1^3a_2v_2 + 41472v_1^8v_2a_3 - 23328v_1^4v_2b_2 \\
& - 23328v_1^3v_2b_1 - 93312v_1^3v_2^2b_3 - 2592v_4v_1^7a_2 \\
& - 1728v_4v_1^6a_1 + 2592v_5v_1^4b_2 + 2592v_5v_1^3b_1 - 3456v_1^8v_5a_3 \\
& + 13824v_1^{10}v_3a_3 + 864v_1^7v_4b_3 - 5184v_1^6v_3b_2 \\
& + 5832v_4v_2^3a_3 + 5832v_4v_2^2a_1 + 34992v_3v_2^2b_2 \\
& + 279936v_1^3v_2^2a_2 - 209952v_1^2v_2^3a_3 + 209952v_1^2v_2^2a_1 \\
& + 432i\sqrt{3}v_5v_4v_1v_2b_3 - 1296i\sqrt{3}v_5v_4v_1a_2v_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$\begin{aligned}
& \left(31104i\sqrt{3}a_2 - 10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3 \right) v_1^9 \\
& + \left(20736i\sqrt{3}a_1 - 20736a_1 \right) v_1^8 + 10368v_5v_3v_1^4a_3v_2 \\
& + \left(-1296i\sqrt{3}a_2 + 432i\sqrt{3}b_3 - 1296a_2 + 432b_3 \right) v_1v_2v_4v_5 \\
& - 93312v_1^4v_3v_2^2a_3 - 3888b_2v_3v_5v_2 \\
& + \left(-1728i\sqrt{3}a_3 - 1728a_3 \right) v_1^6v_2v_4 \\
& + \left(288i\sqrt{3}a_3 + 288a_3 \right) v_1^6v_4v_5 + \left(31104i\sqrt{3}a_2 \right. \\
& \quad \left. - 10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3 \right) v_1^3v_2v_5 \\
& + \left(-23328i\sqrt{3}a_3 + 23328a_3 \right) v_1^2v_2^2v_5 \\
& + \left(1944i\sqrt{3}b_2 + 1944b_2 \right) v_1^2v_2v_4 \\
& + \left(23328i\sqrt{3}a_1 - 23328a_1 \right) v_1^2v_2v_5 \\
& + \left(-216i\sqrt{3}b_2 - 216b_2 \right) v_1^2v_4v_5 \\
& + \left(11664i\sqrt{3}a_2 - 3888i\sqrt{3}b_3 + 11664a_2 - 3888b_3 \right) v_1v_2^2v_4 \\
& + \left(1944i\sqrt{3}b_1 + 1944b_1 \right) v_1v_2v_4 \\
& + \left(-216i\sqrt{3}b_1 - 216b_1 \right) v_1v_4v_5 \\
& + \left(-648i\sqrt{3}a_3 - 648a_3 \right) v_2^2v_4v_5 \\
& + \left(-648i\sqrt{3}a_1 - 648a_1 \right) v_2v_4v_5 \\
& + 13824v_1^{10}v_3a_3 - 5184v_1^6v_3b_2 + 34992v_3v_2^2b_2 \\
& + \left(23328i\sqrt{3}b_2 - 23328b_2 \right) v_1^4v_2 \\
& + \left(-2592i\sqrt{3}b_2 + 2592b_2 \right) v_1^4v_5 + \left(-279936i\sqrt{3}a_2 \right. \\
& \quad \left. + 93312i\sqrt{3}b_3 + 279936a_2 - 93312b_3 \right) v_1^3v_2^2 \\
& + \left(23328i\sqrt{3}b_1 - 23328b_1 \right) v_1^3v_2 \\
& + \left(-2592i\sqrt{3}b_1 + 2592b_1 \right) v_1^3v_5 \\
& + \left(209952i\sqrt{3}a_3 - 209952a_3 \right) v_1^2v_2^3 \\
& + \left(-209952i\sqrt{3}a_1 + 209952a_1 \right) v_1^2v_2^2 \\
& + \left(5832i\sqrt{3}a_3 + 5832a_3 \right) v_2^3v_4 \\
& + \left(5832i\sqrt{3}a_1 + 5832a_1 \right) v_2^2v_4 \\
& + \left(-41472i\sqrt{3}a_3 + 41472a_3 \right) v_1^8v_2 \\
& + \left(3456i\sqrt{3}a_3 - 3456a_3 \right) v_1^8v_5
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
& -93312a_3 = 0 \\
& 10368a_3 = 0 \\
& 13824a_3 = 0 \\
& -5184b_2 = 0 \\
& -3888b_2 = 0 \\
& 34992b_2 = 0 \\
& -209952i\sqrt{3}a_1 + 209952a_1 = 0 \\
& -41472i\sqrt{3}a_3 + 41472a_3 = 0 \\
& -23328i\sqrt{3}a_3 + 23328a_3 = 0 \\
& -2592i\sqrt{3}b_1 + 2592b_1 = 0 \\
& -2592i\sqrt{3}b_2 + 2592b_2 = 0 \\
& -1728i\sqrt{3}a_1 - 1728a_1 = 0 \\
& -1728i\sqrt{3}a_3 - 1728a_3 = 0 \\
& -648i\sqrt{3}a_1 - 648a_1 = 0 \\
& -648i\sqrt{3}a_3 - 648a_3 = 0 \\
& -216i\sqrt{3}b_1 - 216b_1 = 0 \\
& -216i\sqrt{3}b_2 - 216b_2 = 0 \\
& 288i\sqrt{3}a_3 + 288a_3 = 0 \\
& 1944i\sqrt{3}b_1 + 1944b_1 = 0 \\
& 1944i\sqrt{3}b_2 + 1944b_2 = 0 \\
& 3456i\sqrt{3}a_3 - 3456a_3 = 0 \\
& 5832i\sqrt{3}a_1 + 5832a_1 = 0 \\
& 5832i\sqrt{3}a_3 + 5832a_3 = 0 \\
& 20736i\sqrt{3}a_1 - 20736a_1 = 0 \\
& 23328i\sqrt{3}a_1 - 23328a_1 = 0 \\
& 23328i\sqrt{3}b_1 - 23328b_1 = 0 \\
& 23328i\sqrt{3}b_2 - 23328b_2 = 0 \\
& 209952i\sqrt{3}a_3 - 209952a_3 = 0 \\
& -279936i\sqrt{3}a_2 + 93312i\sqrt{3}b_3 + 279936a_2 - 93312b_3 = 0 \\
& -2592i\sqrt{3}a_2 + 864i\sqrt{3}b_3 - 2592a_2 + 864b_3 = 0 \\
& -1296i\sqrt{3}a_2 + 432i\sqrt{3}b_3 - 1296a_2 + 432b_3 = 0 \\
& 11664i\sqrt{3}a_2 - 3888i\sqrt{3}b_3 + 11664a_2 - 3888b_3 = 0 \\
& 31104i\sqrt{3}a_2 - 10368i\sqrt{3}b_3 - 31104a_2 + 10368b_3 = 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0 \\a_2 &= a_2 \\a_3 &= 0 \\b_1 &= 0 \\b_2 &= 0 \\b_3 &= 3a_2\end{aligned}$$

Substituting the above solution in the ansatz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x \\ \eta &= 3y\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R . Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
  *** Sublevel 2 ***
  Methods for first order ODEs:
  -> Solving 1st order ODE of high degree, 1st attempt
  trying 1st order WeierstrassP solution for high degree ODE
  trying 1st order WeierstrassPPrime solution for high degree ODE
  trying 1st order JacobiSN solution for high degree ODE
  trying 1st order ODE linearizable_by_differentiation
  trying differential order: 1; missing variables
  trying dAlembert
  trying simple symmetries for implicit equations
  Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
  *** Sublevel 3 ***
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying homogeneous types:
  trying exact
  Looking for potential symmetries
  trying an equivalence to an Abel ODE
  trying 1st order ODE linearizable_by_differentiation
  -> Solving 1st order ODE of high degree, Lie methods, 1st trial
  `, `-> Computing symmetries using: way = 2
  `, `-> Computing symmetries using: way = 2
  -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
  -> Calling odsolve with the ODE`, diff(y(x), x) = (-4*y(x)^5+3*y(x)*x^2)/(-4*y(x)^4*x+3*x
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying 1st order linear
  <- 1st order linear successful
  <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 40

```
dsolve(diff(y(x),x)^3-4*x^4*diff(y(x),x)+8*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\sqrt{3}x^3}{9}$$
$$y(x) = \frac{2\sqrt{3}x^3}{9}$$
$$y(x) = \frac{x^2}{2c_1} - \frac{1}{8c_1^3}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^3-4*x^4*y'[x]+8*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

6.10 problem 19

6.10.1 Solving as dAlembert ode 1184

Internal problem ID [5347]

Internal file name [OUTPUT/4838_Sunday_February_04_2024_12_46_20_AM_38250850/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 10. Singular solutions, Extraneous loci. Supplementary problems. Page 74

Problem number: 19.

ODE order: 1.

ODE degree: 2.

The type(s) of ODE detected by this program : "dAlembert"

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$(1 + y'^2)(x - y)^2 - (x + y'y)^2 = 0$$

6.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$(p^2 + 1)(x - y)^2 - (py + x)^2 = 0$$

Solving for y from the above results in

$$y = (p^2 + p + 1 + \sqrt{p^4 + 2p^3 + 2p^2 + 2p + 1})x \quad (1A)$$

$$y = (p^2 + p + 1 - \sqrt{p^4 + 2p^3 + 2p^2 + 2p + 1})x \quad (2A)$$

This has the form

$$y = xf(p) + g(p) \quad (*)$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \quad (2)$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= p^2 + p + 1 + \sqrt{(p^2 + 1)(p + 1)^2} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$-p^2 - 1 - \sqrt{(p^2 + 1)(p + 1)^2} = x \left(2p + 1 + \frac{2p(p + 1)^2 + 2(p^2 + 1)(p + 1)}{2\sqrt{(p^2 + 1)(p + 1)^2}} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 - 1 - \sqrt{(p^2 + 1)(p + 1)^2} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= i \\ p &= -i \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= -ix \\ y &= ix \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 - 1 - \sqrt{(p(x)^2 + 1)(p(x) + 1)^2}}{x \left(2p(x) + 1 + \frac{2p(x)(p(x)+1)^2 + 2(p(x)^2+1)(p(x)+1)}{2\sqrt{(p(x)^2+1)(p(x)+1)^2}} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{x(p) \left(2p + 1 + \frac{2p(p+1)^2 + 2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}} \right)}{-p^2 - 1 - \sqrt{(p^2 + 1)(p + 1)^2}} \quad (4)$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left((-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1 \right) x(p)}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp}$$

The ode becomes

$$\frac{d}{dp} \left(e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp} x \right) = 0$$

Integrating gives

$$e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp} x = c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp}$ results in

$$x(p) = c_2 e^{-\left(\int \frac{(2p+1)\sqrt{(p^2+1)(p+1)^2} + 2p^3 + 3p^2 + 2p + 1}{\sqrt{(p^2+1)(p+1)^2} \left(p^2+1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp \right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g') \frac{dp}{dx} \\ p - f &= (xf' + g') \frac{dp}{dx} \end{aligned} \quad (2)$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= p^2 + p + 1 - \sqrt{(p^2 + 1)(p + 1)^2} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$-p^2 - 1 + \sqrt{(p^2 + 1)(p + 1)^2} = x \left(2p + 1 - \frac{2p(p + 1)^2 + 2(p^2 + 1)(p + 1)}{2\sqrt{(p^2 + 1)(p + 1)^2}} \right) p'(x) \quad (2A)$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 - 1 + \sqrt{(p^2 + 1)(p + 1)^2} = 0$$

Solving for p from the above gives

$$\begin{aligned} p &= 0 \\ p &= i \\ p &= -i \end{aligned}$$

Substituting these in (1A) gives

$$\begin{aligned} y &= 0 \\ y &= -ix \\ y &= ix \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 - 1 + \sqrt{(p(x)^2 + 1)(p(x) + 1)^2}}{x \left(2p(x) + 1 - \frac{2p(x)(p(x)+1)^2 + 2(p(x)^2+1)(p(x)+1)}{2\sqrt{(p(x)^2+1)(p(x)+1)^2}} \right)} \quad (3)$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp} x(p) = \frac{x(p) \left(2p + 1 - \frac{2p(p+1)^2 + 2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}} \right)}{-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2}} \quad (4)$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp} x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-2p^3 + 2p\sqrt{(p^2+1)(p+1)^2} - 3p^2 + \sqrt{(p^2+1)(p+1)^2} - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} \right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp} x(p) - \frac{x(p) \left(-2p^3 + 2p\sqrt{(p^2+1)(p+1)^2} - 3p^2 + \sqrt{(p^2+1)(p+1)^2} - 2p - 1 \right)}{\sqrt{(p^2+1)(p+1)^2} \left(-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} \right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2p^3 + 2p\sqrt{(p^2+1)(p+1)^2} - 3p^2 + \sqrt{(p^2+1)(p+1)^2} - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp}$$

The ode becomes

$$\frac{d}{dp} \left(e^{\int -\frac{-2p^3 + 2p\sqrt{(p^2+1)(p+1)^2} - 3p^2 + \sqrt{(p^2+1)(p+1)^2} - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp} x \right) = 0$$

Integrating gives

$$e^{\int -\frac{-2p^3 + 2p\sqrt{(p^2+1)(p+1)^2} - 3p^2 + \sqrt{(p^2+1)(p+1)^2} - 2p - 1}{\sqrt{(p^2+1)(p+1)^2} \left(-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} \right)} dp} x = c_4$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2-3p^2+\sqrt{(p^2+1)(p+1)^2-2p-1}}{\sqrt{(p^2+1)(p+1)^2(-p^2-1+\sqrt{(p^2+1)(p+1)^2})}} dp}$ results in

$$x(p) = c_4 e^{\int \frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2-3p^2+\sqrt{(p^2+1)(p+1)^2-2p-1}}{\sqrt{(p^2+1)(p+1)^2(-p^2-1+\sqrt{(p^2+1)(p+1)^2})}} dp}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = 0 \tag{3}$$

$$y = -ix \tag{4}$$

$$y = ix \tag{5}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
*** Sublevel 2 ***
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 103

```
dsolve((diff(y(x),x)^2+1)*(x-y(x))^2=(x+y(x)*diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \text{RootOf} \left(-2 \ln(x) - \left(\int^{-z} \frac{2_a^2 + \sqrt{2} \sqrt{-a} (_a - 1)^2}{_a (_a^2 + 1)} d_a \right) + 2c_1 \right) x$$

$$y(x) = \text{RootOf} \left(-2 \ln(x) + \int^{-z} \frac{\sqrt{2} \sqrt{-a} (_a - 1)^2 - 2_a^2}{_a (_a^2 + 1)} d_a + 2c_1 \right) x$$

✓ Solution by Mathematica

Time used: 4.36 (sec). Leaf size: 167

```
DSolve[(y'[x]^2+1)*(x-y[x])^2==(x+y[x]*y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)}-e^{\frac{c_1}{2}}$$

$$y(x) \rightarrow \sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)}-e^{\frac{c_1}{2}}$$

$$y(x) \rightarrow e^{\frac{c_1}{2}}-\sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)}$$

$$y(x) \rightarrow \sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)}+e^{\frac{c_1}{2}}$$

$$y(x) \rightarrow -\sqrt{-x^2}$$

$$y(x) \rightarrow \sqrt{-x^2}$$

7 Chapter 12. Linear equations of order n .

Supplementary problems. Page 81

7.1	problem 10	1193
7.2	problem 11	1201
7.3	problem 12	1203
7.4	problem 13	1214
7.5	problem 14	1225
7.6	problem 15	1241
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7.8	problem 17	1264
7.9	problem 18	1269
7.10	problem 19	1279

7.1 problem 10

7.1.1 Solving as second order linear constant coeff ode	1193
7.1.2 Solving using Kovacic algorithm	1195
7.1.3 Maple step by step solution	1199

Internal problem ID [5348]

Internal file name [OUTPUT/4839_Sunday_February_04_2024_12_46_22_AM_33866243/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 10.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' + y' - 6y = 0$$

7.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2 - (4)(1)(-6)} \\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{5}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{5}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-3x} \tag{1}$$

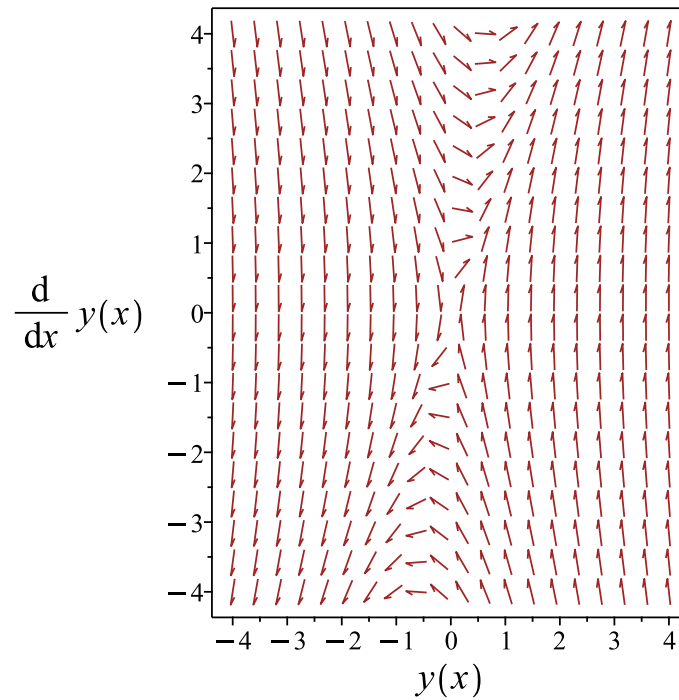


Figure 202: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Verified OK.

7.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1}{1} dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(e^{-\frac{x}{2}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{5x}}{5} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-3x}) + c_2 \left(e^{-3x} \left(\frac{e^{5x}}{5} \right) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} \quad (1)$$

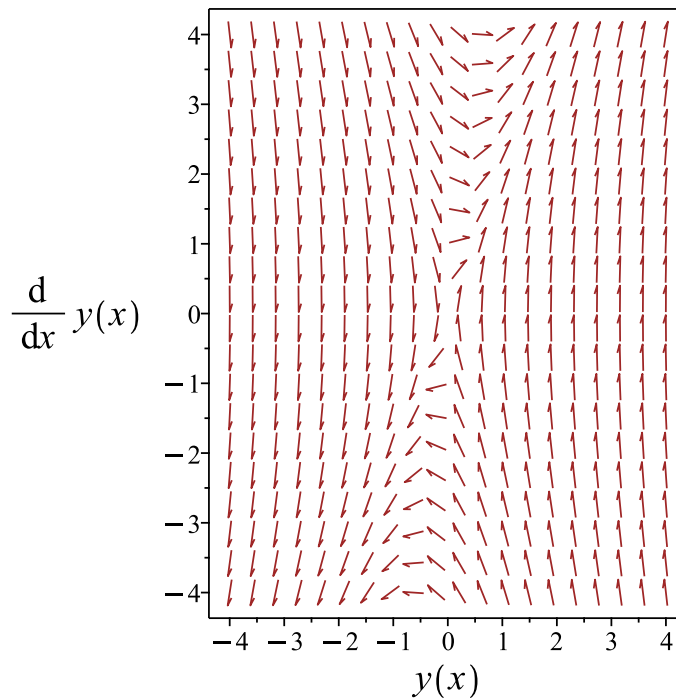


Figure 203: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5}$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$$y'' + y' - 6y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 + r - 6 = 0$$

- Factor the characteristic polynomial

$$(r + 3)(r - 2) = 0$$

- Roots of the characteristic polynomial

$$r = (-3, 2)$$

- 1st solution of the ODE

$$y_1(x) = e^{-3x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-3x} + c_2 e^{2x}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```


✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_1 e^{5x} + c_2) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-3x}(c_2 e^{5x} + c_1)$$

7.2 problem 11

Internal problem ID [5349]

Internal file name [OUTPUT/4840_Sunday_February_04_2024_12_46_23_AM_24262917/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 11.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_x]]
```

$$y''' - 6y'' + 12y' - 8y = 0$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$

$$\lambda_2 = 2$$

$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + x e^{2x} c_2 + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$

$$y_2 = e^{2x} x$$

$$y_3 = x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + x e^{2x} c_2 + x^2 e^{2x} c_3 \quad (1)$$

Verification of solutions

$$y = c_1 e^{2x} + x e^{2x} c_2 + x^2 e^{2x} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_3 x^2 + c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 23

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{2x}(x(c_3 x + c_2) + c_1)$$

7.3 problem 12

7.3.1 Solving as second order linear constant coeff ode	1203
7.3.2 Solving using Kovacic algorithm	1206
7.3.3 Maple step by step solution	1211

Internal problem ID [5350]

Internal file name [OUTPUT/4841_Sunday_February_04_2024_12_46_23_AM_84963062/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 12.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - 3y' + 2y = e^{5x}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = e^{5x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 2 \\ \lambda_2 &= 1 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(1)x} \end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^x$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^{5x} = e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{5x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^x) + \left(\frac{e^{5x}}{12} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + \frac{e^{5x}}{12} \quad (1)$$

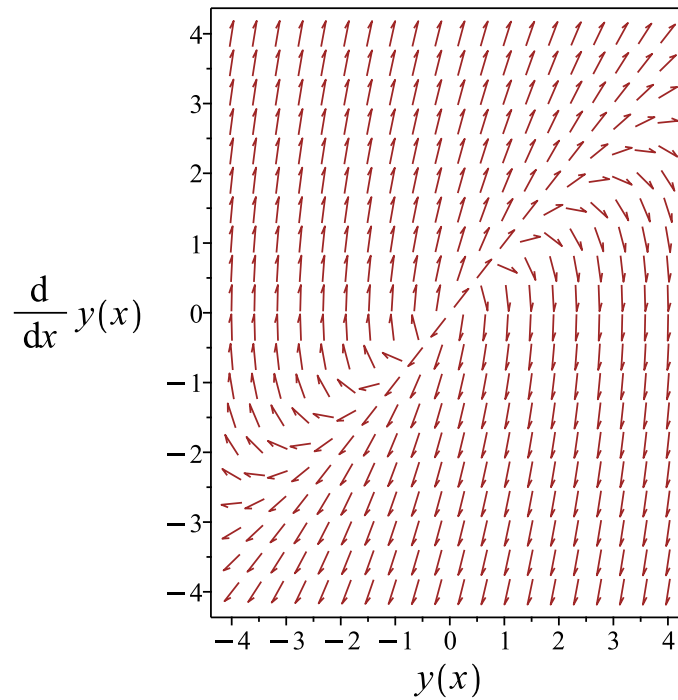


Figure 204: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^x + \frac{e^{5x}}{12}$$

Verified OK.

7.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(e^{\frac{3x}{2}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2} dx \\ &= y_1(e^x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1(e^x) + c_2(e^x(e^x))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1e^x + c_2e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1e^{5x} = e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{5x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{2x}) + \left(\frac{e^{5x}}{12} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} + \frac{e^{5x}}{12} \tag{1}$$

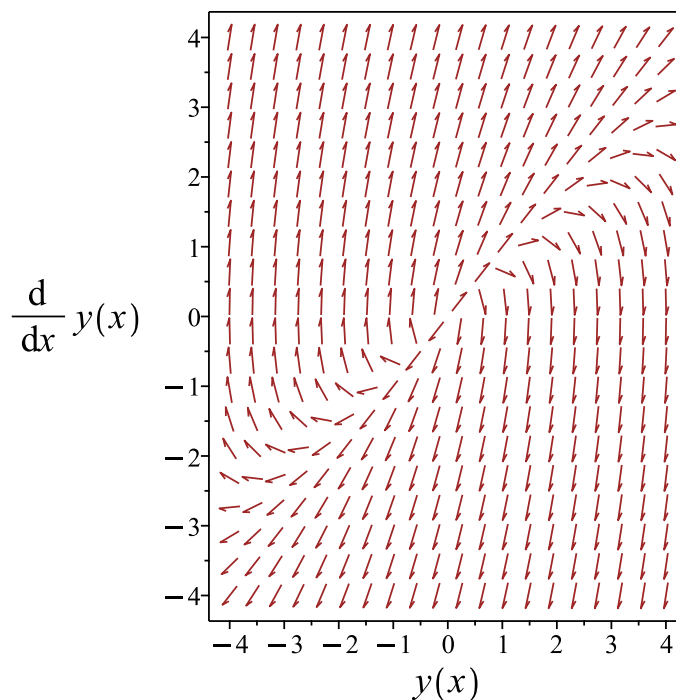


Figure 205: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x} + \frac{e^{5x}}{12}$$

Verified OK.

7.3.3 Maple step by step solution

Let's solve

$$y'' - 3y' + 2y = e^{5x}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 3r + 2 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 2) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 2)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^x$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{2x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^{5x} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{3x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^x \left(\int e^{4x} dx \right) + e^{2x} \left(\int e^{3x} dx \right)$$

- Compute integrals

$$y_p(x) = \frac{e^{5x}}{12}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{2x} + \frac{e^{5x}}{12}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x), x$2) - 3*diff(y(x), x) + 2*y(x) = exp(5*x), y(x), singsol=all)
```

$$y(x) = \frac{(e^{4x} + 12 e^x c_1 + 12 c_2) e^x}{12}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Exp[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^{5x}}{12} + c_1 e^x + c_2 e^{2x}$$

7.4 problem 13

7.4.1	Solving as second order linear constant coeff ode	1214
7.4.2	Solving using Kovacic algorithm	1217
7.4.3	Maple step by step solution	1222

Internal problem ID [5351]

Internal file name [OUTPUT/4842_Sunday_February_04_2024_12_46_23_AM_16762942/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 13.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 9y = \cos(x) x$$

7.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \cos(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i \end{aligned}$$

Hence

$$\lambda_1 = +3i$$

$$\lambda_2 = -3i$$

Which simplifies to

$$\lambda_1 = 3i$$

$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)x, \sin(x)x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)x + A_2 \sin(x)x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 8A_1 \cos(x)x - 2A_1 \sin(x) + 8A_2 \sin(x)x + 2A_2 \cos(x) + 8A_3 \cos(x) + 8A_4 \sin(x) \\ = \cos(x)x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = 0, A_4 = \frac{1}{32} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{\cos(x)x}{8} + \frac{\sin(x)}{32} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x)x}{8} + \frac{\sin(x)}{32} \quad (1)$$

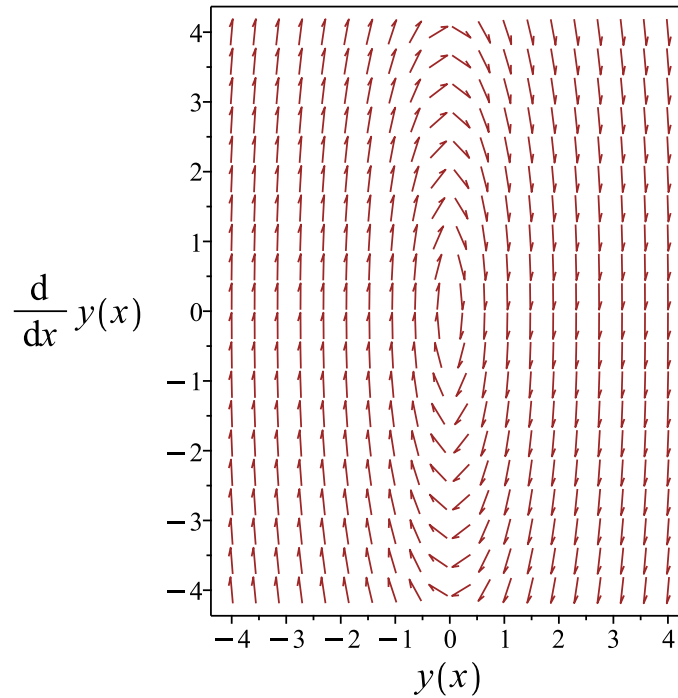


Figure 206: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= 9\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -9 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= \cos(3x)
 \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2} dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\cos(3x)) + c_2 \left(\cos(3x) \left(\frac{\tan(3x)}{3} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)x, \sin(x)x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sin(3x)}{3}, \cos(3x) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)x + A_2 \sin(x)x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 8A_1 \cos(x)x - 2A_1 \sin(x) + 8A_2 \sin(x)x + 2A_2 \cos(x) + 8A_3 \cos(x) + 8A_4 \sin(x) \\ = \cos(x)x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = 0, A_4 = \frac{1}{32} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} \right) + \left(\frac{\cos(x)x}{8} + \frac{\sin(x)}{32} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\cos(x)x}{8} + \frac{\sin(x)}{32} \quad (1)$$

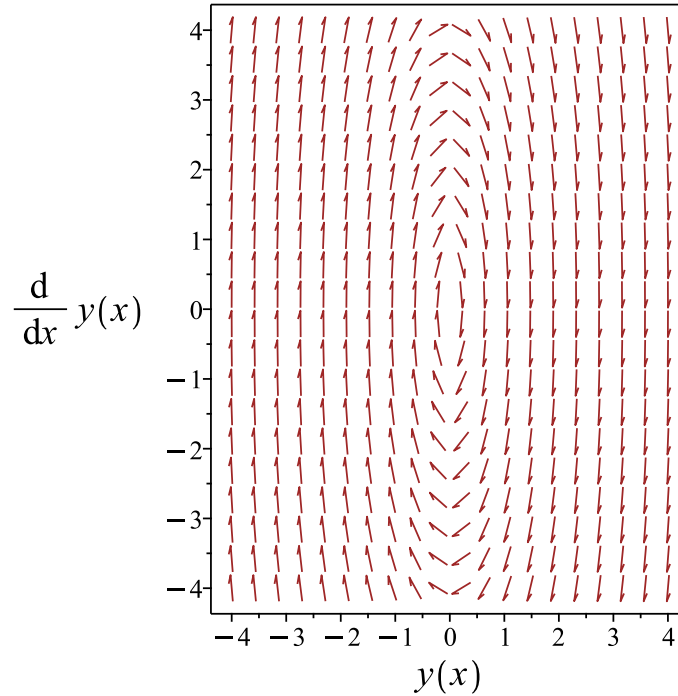


Figure 207: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$$y'' + 9y = \cos(x)x$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 + 9 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-36})}{2}$$

- Roots of the characteristic polynomial

$$r = (-3I, 3I)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(3x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(3x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \cos(x)x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 3$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)(\int \sin(3x) \cos(x)x dx)}{3} + \frac{\sin(3x)(\int \cos(3x) \cos(x)x dx)}{3}$$

- Compute integrals

$$y_p(x) = \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x)x}{8} + \frac{\sin(x)}{32}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+9*y(x)=x*cos(x),y(x), singsol=all)
```

$$y(x) = \sin(3x) c_2 + \cos(3x) c_1 + \frac{\sin(x)}{32} + \frac{\cos(x) x}{8}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 32

```
DSolve[y''[x]+9*y[x]==x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{32}(\sin(x) + 4x \cos(x)) + c_1 \cos(3x) + c_2 \sin(3x)$$

7.5 problem 14

7.5.1	Solving as second order euler ode ode	1225
7.5.2	Solving as second order change of variable on x method 2 ode .	1226
7.5.3	Solving as second order change of variable on x method 1 ode .	1229
7.5.4	Solving as second order change of variable on y method 2 ode .	1231
7.5.5	Solving using Kovacic algorithm	1233
7.5.6	Maple step by step solution	1238

Internal problem ID [5352]

Internal file name [OUTPUT/4843_Sunday_February_04_2024_12_46_24_AM_71040505/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 14.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$x^2y'' - 3xy' + 4y = 0$$

7.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3rx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)x^r - 3rx^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r - 1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \quad (1)$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$

$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \quad (1)$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

7.5.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)} dx \\ &= \int e^{-(\int -\frac{3}{x}dx)} dx \\ &= \int e^{3\ln(x)} dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4}{x^6} \\ &= \frac{4}{x^8} \end{aligned} \quad (7)$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\tau^r + 0\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau} \ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2} \quad (1)$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2}$$

Verified OK.

7.5.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$

$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}x^3} \end{aligned} \quad (6)$$

Substituting the above into (4) results in

$$\begin{aligned}
 p_1(\tau) &= \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \\
 &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}x^3} - \frac{3}{x} \frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
 &= -2c
 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
 y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
 \frac{d^2}{d\tau^2} y(\tau) - 2c \left(\frac{d}{d\tau} y(\tau) \right) + c^2 y(\tau) &= 0 \tag{7}
 \end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}
 \tau &= \int \frac{1}{c} \sqrt{q} dx \\
 &= \frac{\int 2\sqrt{\frac{1}{x^2}} dx}{c} \\
 &= \frac{2\sqrt{\frac{1}{x^2}} x \ln(x)}{c}
 \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

7.5.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 2 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \frac{v'(x)}{x} = 0$$
$$v''(x) + \frac{v'(x)}{x} = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{1}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{1}{x} dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x) x^n \\ &= (c_1 \ln(x) + c_2) x^2 \\ &= (c_1 \ln(x) + c_2) x^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2) x^2 \quad (1)$$

Verification of solutions

$$y = (c_1 \ln(x) + c_2) x^2$$

Verified OK.

7.5.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 4y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ , which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω .

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} dx} \\&= z_1 e^{\frac{3 \ln(x)}{2}} \\&= z_1 \left(x^{\frac{3}{2}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{3 \ln(x)}}{(y_1)^2} dx \\&= y_1 (\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 (x^2) + c_2 (x^2 (\ln(x)))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

7.5.6 Maple step by step solution

Let's solve

$$x^2y'' - 3xy' + 4y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{3y'}{x} - \frac{4y}{x^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$$

- Multiply by denominators of the ODE

$$x^2y'' - 3xy' + 4y = 0$$

- Make a change of variables

$$t = \ln(x)$$

- Substitute the change of variables back into the ODE

- Calculate the 1st derivative of y with respect to x , using the chain rule

$$y' = \left(\frac{d}{dt}y(t)\right) t'(x)$$

- Compute derivative

$$y' = \frac{\frac{d}{dt}y(t)}{x}$$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

$$y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$$

- Compute derivative

$$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$$

Substitute the change of variables back into the ODE

$$x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2} \right) - 3 \frac{d}{dt}y(t) + 4y(t) = 0$$

- Simplify

$$\frac{d^2}{dt^2}y(t) - 4 \frac{d}{dt}y(t) + 4y(t) = 0$$

- Characteristic polynomial of ODE

$$r^2 - 4r + 4 = 0$$

- Factor the characteristic polynomial
 $(r - 2)^2 = 0$
- Root of the characteristic polynomial
 $r = 2$
- 1st solution of the ODE
 $y_1(t) = e^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence
 $y_2(t) = t e^{2t}$
- General solution of the ODE
 $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions
 $y(t) = c_1 e^{2t} + c_2 t e^{2t}$
- Change variables back using $t = \ln(x)$
 $y = c_1 x^2 + c_2 x^2 \ln(x)$
- Simplify
 $y = x^2(c_1 + c_2 \ln(x))$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y'[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x^2(2c_2 \log(x) + c_1)$$

7.6 problem 15

7.6.1 Maple step by step solution 1246

Internal problem ID [5353]

Internal file name [OUTPUT/4844_Sunday_February_04_2024_12_46_25_AM_48686744/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 15.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_ODE_non_constant_coefficients_of_type_Euler**"

Maple gives the following as the ode type

```
[[_3rd_order , _with_linear_symmetries]]
```

$$x^3 y''' + xy' - y = 3x^4$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3 y''' + xy' - y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$y' = \lambda x^{\lambda-1}$$

$$y'' = \lambda(\lambda - 1) x^{\lambda-2}$$

$$y''' = \lambda(\lambda - 1)(\lambda - 2) x^{\lambda-3}$$

Substituting these back into

$$x^3 y''' + xy' - y = 3x^4$$

gives

$$x\lambda x^{\lambda-1} + x^3\lambda(\lambda-1)(\lambda-2)x^{\lambda-3} - x^\lambda = 0$$

Which simplifies to

$$\lambda x^\lambda + \lambda(\lambda-1)(\lambda-2)x^\lambda - x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ , the above becomes

$$\lambda + \lambda(\lambda-1)(\lambda-2) - 1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-1)^3 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicity two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicity three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1x + c_2 \ln(x)x + c_3 \ln(x)^2 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$

$$y_2 = \ln(x)x$$

$$y_3 = \ln(x)^2 x$$

Now the particular solution to the given ODE is found

$$x^3 y''' + xy' - y = 3x^4$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i -th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & \ln(x)x & \ln(x)^2 x \\ 1 & 1 + \ln(x) & \ln(x)(\ln(x) + 2) \\ 0 & \frac{1}{x} & \frac{2+2\ln(x)}{x} \end{bmatrix}$$

$$|W| = 2$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i .

$$W_1(x) = \det \begin{bmatrix} \ln(x)x & \ln(x)^2 x \\ 1 + \ln(x) & \ln(x)(\ln(x) + 2) \end{bmatrix}$$

$$= \ln(x)^2 x$$

$$\begin{aligned}
W_2(x) &= \det \begin{bmatrix} x & \ln(x)^2 x \\ 1 & \ln(x)(\ln(x) + 2) \end{bmatrix} \\
&= 2 \ln(x) x
\end{aligned}$$

$$\begin{aligned}
W_3(x) &= \det \begin{bmatrix} x & \ln(x) x \\ 1 & 1 + \ln(x) \end{bmatrix} \\
&= x
\end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)} dx \\
&= (-1)^2 \int \frac{(3x^4)(\ln(x)^2 x)}{(x^3)(2)} dx \\
&= \int \frac{3x^5 \ln(x)^2}{2x^3} dx \\
&= \int \left(\frac{3x^2 \ln(x)^2}{2} \right) dx \\
&= \frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9}
\end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)} dx \\
&= (-1)^1 \int \frac{(3x^4)(2 \ln(x) x)}{(x^3)(2)} dx \\
&= - \int \frac{6x^5 \ln(x)}{2x^3} dx \\
&= - \int (3 \ln(x) x^2) dx \\
&= -x^3 \ln(x) + \frac{x^3}{3}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)} dx \\
&= (-1)^0 \int \frac{(3x^4)(x)}{(x^3)(2)} dx \\
&= \int \frac{3x^5}{2x^3} dx \\
&= \int \left(\frac{3x^2}{2} \right) dx \\
&= \frac{x^3}{2}
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Hence

$$\begin{aligned}
y_p &= \left(\frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9} \right) (x) \\
&\quad + \left(-x^3 \ln(x) + \frac{x^3}{3} \right) (\ln(x) x) \\
&\quad + \left(\frac{x^3}{2} \right) (\ln(x)^2 x)
\end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{x^4}{9}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1x + c_2 \ln(x) x + c_3 \ln(x)^2 x) + \left(\frac{x^4}{9} \right)
\end{aligned}$$

Which simplifies to

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9}$$

Summary

The solution(s) found are the following

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9} \quad (1)$$

Verification of solutions

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9}$$

Verified OK.

7.6.1 Maple step by step solution

Let's solve

$$x^3 y''' + xy' - y = 3x^4$$

- Highest derivative means the order of the ODE is 3
 y'''

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]  
trying high order linear exact nonhomogeneous  
trying differential order: 3; missing the dependent variable  
checking if the LODE is of Euler type  
<- LODE of Euler type successful  
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)+x*diff(y(x),x)-y(x)=3*x^4,y(x), singsol=all)
```

$$y(x) = \frac{x(9c_2 \ln(x)^2 + x^3 + 9c_3 \ln(x) + 9c_1)}{9}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 31

```
DSolve[x^3*y''[x]+x*y'[x]-y[x]==3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^4}{9} + c_1x + c_3x \log^2(x) + c_2x \log(x)$$

7.7 problem 16

7.7.1	Solving as second order change of variable on x method 2 ode .	1248
7.7.2	Solving as second order change of variable on x method 1 ode .	1251
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7.7.5	Maple step by step solution	1260

Internal problem ID [5354]

Internal file name [OUTPUT/4845_Sunday_February_04_2024_12_46_25_AM_91415232/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$xy'' - y' + 4yx^3 = 0$$

7.7.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)} dx \\ &= \int e^{-(\int -\frac{1}{x}dx)} dx \\ &= \int e^{\ln(x)} dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4x^2}{x^2} \\ &= 4 \end{aligned} \quad (7)$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4 e^{\lambda\tau} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i \end{aligned}$$

Hence

$$\lambda_1 = +2i$$

$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$

$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau} (c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0 (c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) \quad (1)$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

7.7.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{x^2}}\end{aligned}\tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0\end{aligned}\tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q} dx \\ &= \frac{\int 2\sqrt{x^2} dx}{c} \\ &= \frac{x\sqrt{x^2}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)\tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

7.7.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - xy' + 4yx^4 = 0 \quad (1)$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 1$$

$$\beta = 1$$

$$n = \frac{1}{2}$$

$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}}$$

Verified OK.

7.7.4 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' + 4yx^3 = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 4x^3 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for \sqrt{r} at ∞ . Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^v a_i x^i \\ &= \sum_{i=0}^1 a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of \sqrt{r} at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^1 a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b , where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r . How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_{\infty} &= 2ix \\ \alpha_{\infty}^{+} &= \frac{1}{2} \left(\frac{b}{a} - v \right) = \frac{1}{2} \left(\frac{0}{2i} - 1 \right) = -\frac{1}{2} \\ \alpha_{\infty}^{-} &= \frac{1}{2} \left(-\frac{b}{a} - v \right) = \frac{1}{2} \left(-\frac{0}{2i} - 1 \right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^{\pm} have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated α_{∞}^{\pm} have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2 \left(-\frac{1}{2x} - 2ix \right) (0) + \left(\left(\frac{1}{2x^2} - 2i \right) + \left(-\frac{1}{2x} - 2ix \right)^2 - \left(\frac{-16x^4 + 3}{4x^2} \right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix \right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} dx} \\&= z_1 e^{\frac{\ln(x)}{2}} \\&= z_1 (\sqrt{x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{-ix^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} dx \\&= y_1 \left(-\frac{ie^{2ix^2}}{4} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 (e^{-ix^2}) + c_2 \left(e^{-ix^2} \left(-\frac{ie^{2ix^2}}{4} \right) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-ix^2} - \frac{ic_2 e^{ix^2}}{4} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-ix^2} - \frac{ic_2 e^{ix^2}}{4}$$

Verified OK.

7.7.5 Maple step by step solution

Let's solve

$$y''x - y' + 4yx^3 = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{y'}{x} - 4yx^2$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{y'}{x} + 4yx^2 = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x - y' + 4yx^3 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

$$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k \rightarrow k - 3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) x^{-1+r} + a_1 (1+r)(-1+r) x^r + a_2 (2+r)r x^{1+r} + a_3 (3+r)(1+r) x^{2+r} + \left(\sum_{k=3}^{\infty} a_k \right)$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$
- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$
- The coefficients of each power of x must be 0

$$[a_1(1+r)(-1+r) = 0, a_2(2+r)r = 0, a_3(3+r)(1+r) = 0]$$
- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$
- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r-1) + 4a_{k-3} = 0$$

- Shift index using $k \rightarrow k + 3$

$$a_{k+4}(k+4+r)(k+2+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0 \right]$$

- Recursion relation for $r = 2$

$$a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2} \right), a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+6)(k+4)} \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x^2) + c_2 \cos(x^2)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(x^2) + c_2 \sin(x^2)$$

7.8 problem 17

7.8.1	Solving as second order ode missing y ode	1264
7.8.2	Solving as second order ode missing x ode	1265
7.8.3	Maple step by step solution	1267

Internal problem ID [5355]

Internal file name [OUTPUT/4846_Sunday_February_04_2024_12_46_26_AM_79790319/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x", "second_order_ode_missing_y"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x], [_2nd_order , _reducible , _mu_xy]]
```

$$y'' + y'^2 = -1$$

7.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y . Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2 - 1} dp = x + c_1$$
$$- \arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(x + c_1) \, dx \\ &= -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \quad (1)$$

Verification of solutions

$$y = -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2$$

Verified OK.

7.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y) \left(\frac{d}{dy} p(y) \right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2 + 1} dp = \int dy$$

$$-\frac{\ln(p^2 + 1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p^2 + 1}} = e^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p^2 + 1}} = c_2 e^y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \text{RootOf}(-Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1)$$

Integrating both sides gives

$$\int \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1)} dy = \int dx$$

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3 \quad (1)$$

Verification of solutions

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3$$

Verified OK.

7.8.3 Maple step by step solution

Let's solve

$$y'' + y'^2 = -1$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Make substitution $u = y'$ to reduce order of ODE

$$u'(x) + u(x)^2 = -1$$

- Separate variables

$$\frac{u'(x)}{-u(x)^2-1} = 1$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{-u(x)^2-1} dx = \int 1 dx + c_1$$

- Evaluate integral

$$-\arctan(u(x)) = x + c_1$$

- Solve for $u(x)$

$$u(x) = -\tan(x + c_1)$$

- Solve 1st ODE for $u(x)$

$$u(x) = -\tan(x + c_1)$$

- Make substitution $u = y'$

$$y' = -\tan(x + c_1)$$

- Integrate both sides to solve for y

$$\int y' dx = \int -\tan(x + c_1) dx + c_2$$

- Compute integrals

$$y = -\frac{\ln(1+\tan(x+c_1)^2)}{2} + c_2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying 2nd order Liouville  
trying 2nd order WeierstrassP  
trying 2nd order JacobiSN  
differential order: 2; trying a linearization to 3rd order  
trying 2nd order ODE linearizable_by_differentiation  
trying 2nd order, 2 integrating factors of the form mu(x,y)  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful  
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \ln(\cos(x) c_2 - c_1 \sin(x))$$

✓ Solution by Mathematica

Time used: 1.861 (sec). Leaf size: 16

```
DSolve[y''[x]+y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \log(\cos(x - c_1)) + c_2$$

7.9 problem 18

7.9.1	Solving as second order integrable as is ode	1269
7.9.2	Solving as second order ode missing x ode	1270
7.9.3	Solving as type second_order_integrable_as_is (not using ABC version)	1273
7.9.4	Solving as exact nonlinear second order ode ode	1274
7.9.5	Maple step by step solution	1275

Internal problem ID [5356]

Internal file name [OUTPUT/4847_Sunday_February_04_2024_12_46_27_AM_79210250/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$yy'' + y'^2 = 2$$

7.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + y'^2) dx = \int 2dx$$
$$y'y = 2x + c_1$$

Which is now solved for y . In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{c_1 + 2x}{y}\end{aligned}$$

Where $f(x) = c_1 + 2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}} dy &= c_1 + 2x dx \\ \int \frac{1}{\frac{1}{y}} dy &= \int c_1 + 2x dx \\ \frac{y^2}{2} &= c_1x + x^2 + c_2\end{aligned}$$

The solution is

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0$$

Verified OK.

7.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned}y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) + p(y)^2 = 2$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y, p) \\ &= f(y)g(p) \\ &= -\frac{p^2 - 2}{yp}\end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = \frac{p^2-2}{p}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{p^2-2}{p}} dp &= -\frac{1}{y} dy \\ \int \frac{1}{\frac{p^2-2}{p}} dp &= \int -\frac{1}{y} dy \\ \frac{\ln(p^2 - 2)}{2} &= -\ln(y) + c_1\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{p^2 - 2} = e^{-\ln(y)+c_1}$$

Which simplifies to

$$\sqrt{p^2 - 2} = \frac{c_2}{y}$$

Which simplifies to

$$\sqrt{p(y)^2 - 2} = \frac{c_2 e^{c_1}}{y}$$

The solution is

$$\sqrt{p(y)^2 - 2} = \frac{c_2 e^{c_1}}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{y'^2 - 2} = \frac{c_2 e^{c_1}}{y}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{y} \quad (1)$$

$$y' = -\frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{y} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{e^{2c_1} c_2^2 + 2y^2}} dy = \int dx$$
$$\frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{2} = x + c_3$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{e^{2c_1} c_2^2 + 2y^2}} dy = \int dx$$
$$-\frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{2} = x + c_4$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{2} = x + c_3 \quad (1)$$

$$-\frac{\sqrt{e^{2c_1} c_2^2 + 2y^2}}{2} = x + c_4 \quad (2)$$

Verification of solutions

$$\frac{\sqrt{e^{2c_1}c_2^2 + 2y^2}}{2} = x + c_3$$

Verified OK.

$$-\frac{\sqrt{e^{2c_1}c_2^2 + 2y^2}}{2} = x + c_4$$

Verified OK.

7.9.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 2$$

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + y'^2) dx = \int 2dx$$
$$y'y = 2x + c_1$$

Which is now solved for y . In canonical form the ODE is

$$y' = F(x, y)$$
$$= f(x)g(y)$$
$$= \frac{c_1 + 2x}{y}$$

Where $f(x) = c_1 + 2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}} dy = c_1 + 2x dx$$
$$\int \frac{1}{\frac{1}{y}} dy = \int c_1 + 2x dx$$
$$\frac{y^2}{2} = c_1x + x^2 + c_2$$

The solution is

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0 \quad (1)$$

Verification of solutions

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0$$

Verified OK.

7.9.4 Solving as exact nonlinear second order ode

An exact non-linear second order ode has the form

$$a_2(x, y, y') y'' + a_1(x, y, y') y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned} \frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y} \end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned} a_2 &= y \\ a_1 &= y' \\ a_0 &= -2 \end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode.

Therefore it can be reduced to first order ode given by

$$\begin{aligned} \int a_2 dy' + \int a_1 dy + \int a_0 dx &= c_1 \\ \int y dy' + \int y' dy + \int -2 dx &= c_1 \end{aligned}$$

Which results in

$$2y'y - 2x = c_1$$

Which is now solved In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x + \frac{c_1}{2}}{y}\end{aligned}$$

Where $f(x) = x + \frac{c_1}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= x + \frac{c_1}{2} dx \\ \int \frac{1}{y} dy &= \int x + \frac{c_1}{2} dx \\ \frac{y^2}{2} &= \frac{1}{2}x^2 + \frac{1}{2}c_1x + c_2\end{aligned}$$

The solution is

$$\frac{y^2}{2} - \frac{x^2}{2} - \frac{c_1x}{2} - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - \frac{x^2}{2} - \frac{c_1x}{2} - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - \frac{x^2}{2} - \frac{c_1x}{2} - c_2 = 0$$

Warning, solution could not be verified

7.9.5 Maple step by step solution

Let's solve

$$yy'' + y'^2 = 2$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Define new dependent variable u
 $u(x) = y'$
- Compute y''
 $u'(x) = y''$
- Use chain rule on the lhs
 $y' \left(\frac{d}{dy} u(y) \right) = y''$
- Substitute in the definition of u
 $u(y) \left(\frac{d}{dy} u(y) \right) = y''$
- Make substitutions $y' = u(y)$, $y'' = u(y) \left(\frac{d}{dy} u(y) \right)$ to reduce order of ODE
 $yu(y) \left(\frac{d}{dy} u(y) \right) + u(y)^2 = 2$
- Separate variables
 $\frac{\left(\frac{d}{dy} u(y) \right) u(y)}{-u(y)^2 + 2} = \frac{1}{y}$
- Integrate both sides with respect to y
 $\int \frac{\left(\frac{d}{dy} u(y) \right) u(y)}{-u(y)^2 + 2} dy = \int \frac{1}{y} dy + c_1$
- Evaluate integral
 $-\frac{\ln(u(y)^2 - 2)}{2} = \ln(y) + c_1$
- Solve for $u(y)$
 $\left\{ u(y) = \frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y}, u(y) = -\frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y} \right\}$
- Solve 1st ODE for $u(y)$
 $u(y) = \frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y}$
- Revert to original variables with substitution $u(y) = y'$, $y = y$
 $y' = \frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y}$
- Separate variables
 $\frac{y' y}{\sqrt{2(e^{c_1})^2 y^2 + 1}} = \frac{1}{e^{c_1}}$
- Integrate both sides with respect to x

$$\int \frac{y'y}{\sqrt{2(e^{c_1})^2 y^2 + 1}} dx = \int \frac{1}{e^{c_1}} dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{2(e^{c_1})^2} = \frac{x}{e^{c_1}} + c_2$$

- Solve for y

$$\left\{ y = -\frac{\sqrt{-2+8(e^{c_1})^4 c_2^2 + 16(e^{c_1})^3 c_2 x + 8(e^{c_1})^2 x^2}}{2 e^{c_1}}, y = \frac{\sqrt{-2+8(e^{c_1})^4 c_2^2 + 16(e^{c_1})^3 c_2 x + 8(e^{c_1})^2 x^2}}{2 e^{c_1}} \right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{e^{c_1} y}$$

- Separate variables

$$\frac{y'y}{\sqrt{2(e^{c_1})^2 y^2 + 1}} = -\frac{1}{e^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y'y}{\sqrt{2(e^{c_1})^2 y^2 + 1}} dx = \int -\frac{1}{e^{c_1}} dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2(e^{c_1})^2 y^2 + 1}}{2(e^{c_1})^2} = -\frac{x}{e^{c_1}} + c_2$$

- Solve for y

$$\left\{ y = -\frac{\sqrt{-2+8(e^{c_1})^4 c_2^2 - 16(e^{c_1})^3 c_2 x + 8(e^{c_1})^2 x^2}}{2 e^{c_1}}, y = \frac{\sqrt{-2+8(e^{c_1})^4 c_2^2 - 16(e^{c_1})^3 c_2 x + 8(e^{c_1})^2 x^2}}{2 e^{c_1}} \right\}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying 2nd order Liouville  
trying 2nd order WeierstrassP  
trying 2nd order JacobiSN  
differential order: 2; trying a linearization to 3rd order  
trying 2nd order ODE linearizable_by_differentiation  
trying 2nd order, 2 integrating factors of the form mu(x,y)  
trying a quadrature  
<- quadrature successful  
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=2,y(x), singsol=all)
```

$$y(x) = \sqrt{-2c_1x + 2x^2 + 2c_2}$$
$$y(x) = -\sqrt{-2c_1x + 2x^2 + 2c_2}$$

✓ Solution by Mathematica

Time used: 6.295 (sec). Leaf size: 101

```
DSolve[y[x]*y'[x]+y'[x]^2==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{4(x+c_2)^2 - e^{2c_1}}}{\sqrt{2}}$$

$$y(x) \rightarrow \sqrt{2(x+c_2)^2 - \frac{e^{2c_1}}{2}}$$

$$y(x) \rightarrow -\sqrt{2}\sqrt{(x+c_2)^2}$$

$$y(x) \rightarrow \sqrt{2}\sqrt{(x+c_2)^2}$$

7.10 problem 19

7.10.1 Solving as second order ode missing x ode 1279

7.10.2 Maple step by step solution 1281

Internal problem ID [5357]

Internal file name [OUTPUT/4848_Sunday_February_04_2024_12_46_29_AM_29877559/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 12. Linear equations of order n. Supplementary problems. Page 81

Problem number: 19.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],  
 [_2nd_order, _reducible, _mu_y_y1]]
```

$$yy'' + y'^3 = 0$$

7.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y)g(p) \\ &= -\frac{p^2}{y} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p^2} dp &= -\frac{1}{y} dy \\ \int \frac{1}{p^2} dp &= \int -\frac{1}{y} dy \\ -\frac{1}{p} &= -\ln(y) + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{p(y)} + \ln(y) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} + \ln(y) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} \int (\ln(y) - c_1) dy &= c_2 + x \\ -c_1 y + \ln(y) y - y &= c_2 + x \end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = e^{\text{LambertW}((c_2+x)e^{-1-c_1})+1+c_1} \quad (1)$$

Verification of solutions

$$y = e^{\text{LambertW}((c_2+x)e^{-1-c_1})+1+c_1}$$

Verified OK.

7.10.2 Maple step by step solution

Let's solve

$$yy'' + y'^3 = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Define new dependent variable u

$$u(x) = y'$$

- Compute y''

$$u'(x) = y''$$

- Use chain rule on the lhs

$$y' \left(\frac{d}{dy} u(y) \right) = y''$$

- Substitute in the definition of u

$$u(y) \left(\frac{d}{dy} u(y) \right) = y''$$

- Make substitutions $y' = u(y)$, $y'' = u(y) \left(\frac{d}{dy} u(y) \right)$ to reduce order of ODE

$$yu(y) \left(\frac{d}{dy} u(y) \right) + u(y)^3 = 0$$

- Separate variables

$$\frac{\frac{d}{dy} u(y)}{u(y)^2} = -\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy} u(y)}{u(y)^2} dy = \int -\frac{1}{y} dy + c_1$$

- Evaluate integral

$$-\frac{1}{u(y)} = -\ln(y) + c_1$$

- Solve for $u(y)$

$$u(y) = \frac{1}{\ln(y) - c_1}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{1}{\ln(y) - c_1}$$

- Revert to original variables with substitution $u(y) = y'$, $y = y$

$$y' = \frac{1}{\ln(y) - c_1}$$

- Separate variables

$$y'(\ln(y) - c_1) = 1$$
- Integrate both sides with respect to x

$$\int y'(\ln(y) - c_1) dx = \int 1 dx + c_2$$
- Evaluate integral

$$-yc_1 + y \ln(y) - y = c_2 + x$$
- Solve for y

$$y = e^{LambertW((c_2+x)e^{-1-c_1})+1+c_1}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^3/_a = 0, _b(_a)` *** S
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying 1st order linear
  trying Bernoulli
  <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`

```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = c_1$$

$$y(x) = \frac{x + c_2}{\text{LambertW}((x + c_2) e^{c_1 - 1})}$$

✓ Solution by Mathematica

Time used: 60.091 (sec). Leaf size: 26

```
DSolve[y[x]*y'[x]+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x + c_2}{W(e^{-1-c_1}(x + c_2))}$$

8 Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems.

Page 86

8.1	problem 16	1285
8.2	problem 17	1293
8.3	problem 18	1298
8.4	problem 19	1307
8.5	problem 20	1309
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8.7	problem 22	1327
8.8	problem 23	1332
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8.1 problem 16

- 8.1.1 Solving as second order linear constant coeff ode 1285
- 8.1.2 Solving using Kovacic algorithm 1287
- 8.1.3 Maple step by step solution 1291

Internal problem ID [5358]

Internal file name [OUTPUT/4849_Sunday_February_04_2024_12_46_29_AM_58753252/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' + 2y' - 15y = 0$$

8.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -15$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} - 15e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 15 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -15$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{2^2 - (4)(1)(-15)} \\ &= -1 \pm 4\end{aligned}$$

Hence

$$\lambda_1 = -1 + 4$$

$$\lambda_2 = -1 - 4$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = -5$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(3)x} + c_2 e^{(-5)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^{-5x}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{-5x} \tag{1}$$

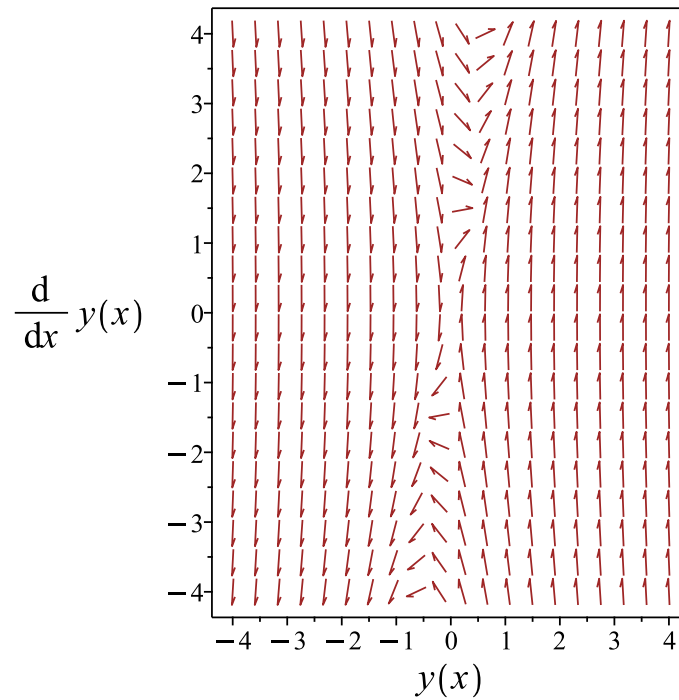


Figure 208: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^{-5x}$$

Verified OK.

8.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 15y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 2 \tag{3}$$

$$C = -15$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 16$$

$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2}{1} dx} \\ &= z_1 e^{-x} \\ &= z_1 (e^{-x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{8x}}{8} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-5x}) + c_2 \left(e^{-5x} \left(\frac{e^{8x}}{8} \right) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-5x} + \frac{c_2 e^{3x}}{8} \tag{1}$$

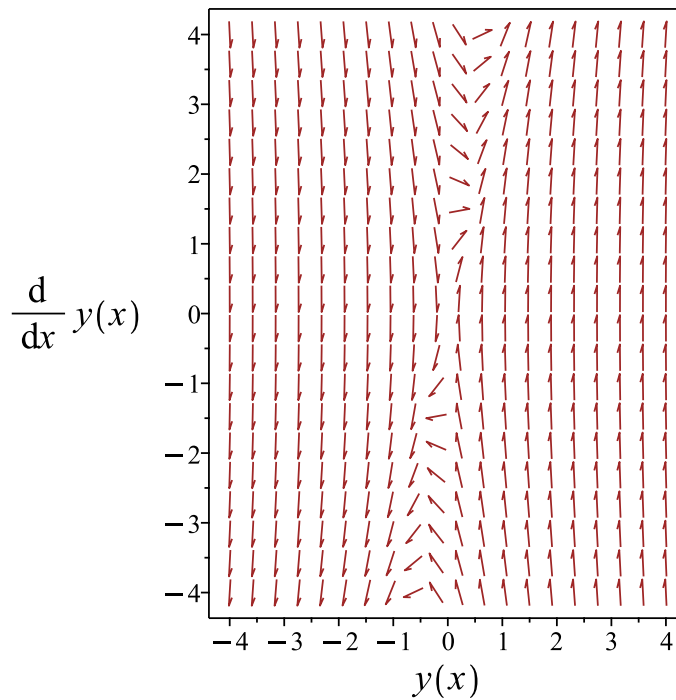


Figure 209: Slope field plot

Verification of solutions

$$y = c_1 e^{-5x} + \frac{c_2 e^{3x}}{8}$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$$y'' + 2y' - 15y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 + 2r - 15 = 0$$

- Factor the characteristic polynomial

$$(r + 5)(r - 3) = 0$$

- Roots of the characteristic polynomial

$$r = (-5, 3)$$

- 1st solution of the ODE

$$y_1(x) = e^{-5x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{3x}$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x)$$

- Substitute in solutions

$$y = c_1e^{-5x} + c_2e^{3x}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-15*y(x)=0,y(x), singsol=all)
```

$$y(x) = (e^{8x}c_2 + c_1) e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]-15*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-5x}(c_2 e^{8x} + c_1)$$

8.2 problem 17

8.2.1 Maple step by step solution 1294

Internal problem ID [5359]

Internal file name [OUTPUT/4850_Sunday_February_04_2024_12_46_30_AM_68380213/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 17.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_x]]
```

$$y''' + y'' - 2y' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 1$$

$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2e^{-2x} + c_3e^x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = e^{-2x}$$

$$y_3 = e^x$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2e^{-2x} + c_3e^x \quad (1)$$

Verification of solutions

$$y = c_1 + c_2e^{-2x} + c_3e^x$$

Verified OK.

8.2.1 Maple step by step solution

Let's solve

$$y''' + y'' - 2y' = 0$$

- Highest derivative means the order of the ODE is 3
 y'''

□ Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = -y_3(x) + 2y_2(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + 2y_2(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 e^{3x} + 4c_2 e^{2x} + c_1) e^{-2x}}{4}$$

Maple trace

```

`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_3 e^{3x} + e^{2x} c_1 + c_2) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[y'''[x]+y''[x]-2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2}c_1 e^{-2x} + c_2 e^x + c_3$$

8.3 problem 18

8.3.1	Solving as second order linear constant coeff ode	1298
8.3.2	Solving as linear second order ode solved by an integrating factor ode	1300
8.3.3	Solving using Kovacic algorithm	1301
8.3.4	Maple step by step solution	1305

Internal problem ID [5360]

Internal file name [OUTPUT/4851_Sunday_February_04_2024_12_46_30_AM_10925479/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' + 6y' + 9y = 0$$

8.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 9e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(6)^2 - (4)(1)(9)} \\ &= -3 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \quad (1)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + x e^{-3x} c_2 \quad (1)$$

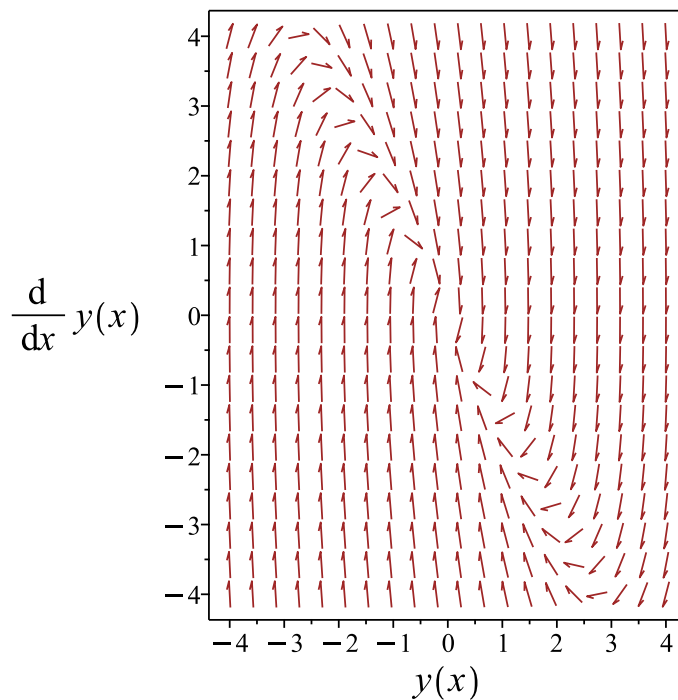


Figure 210: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + x e^{-3x} c_2$$

Verified OK.

8.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x)^2 + p'(x))y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2} \int p dx} \\ &= e^{\int 6 dx} \\ &= e^{3x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 0 \\ (e^{3x}y)'' &= 0\end{aligned}$$

Integrating once gives

$$(e^{3x}y)' = c_1$$

Integrating again gives

$$(e^{3x}y) = c_1x + c_2$$

Hence the solution is

$$y = \frac{c_1x + c_2}{e^{3x}}$$

Or

$$y = c_1x e^{-3x} + c_2e^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1x e^{-3x} + c_2e^{-3x} \tag{1}$$

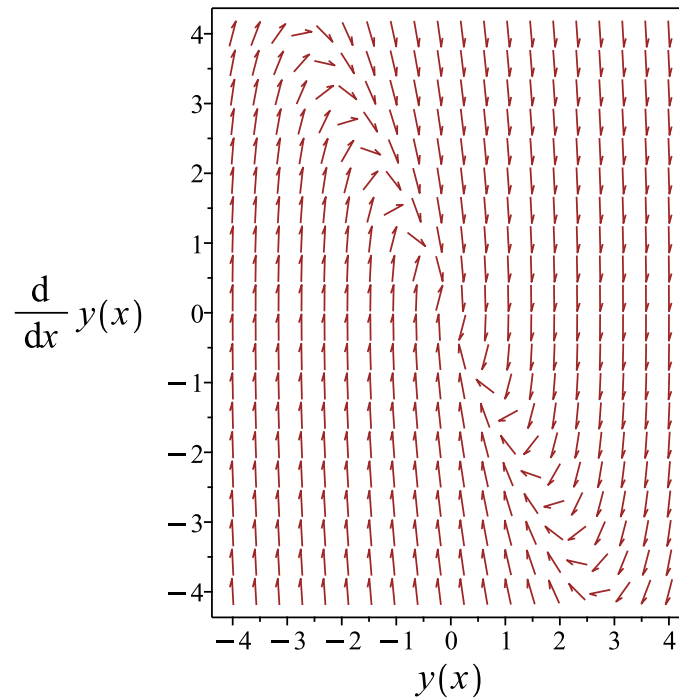


Figure 211: Slope field plot

Verification of solutions

$$y = c_1 x e^{-3x} + c_2 e^{-3x}$$

Verified OK.

8.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0 \\ t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 146: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6}{1} dx} \\ &= z_1 e^{-3x} \\ &= z_1 (e^{-3x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2} dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-3x}) + c_2 (e^{-3x}(x))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + x e^{-3x} c_2 \tag{1}$$

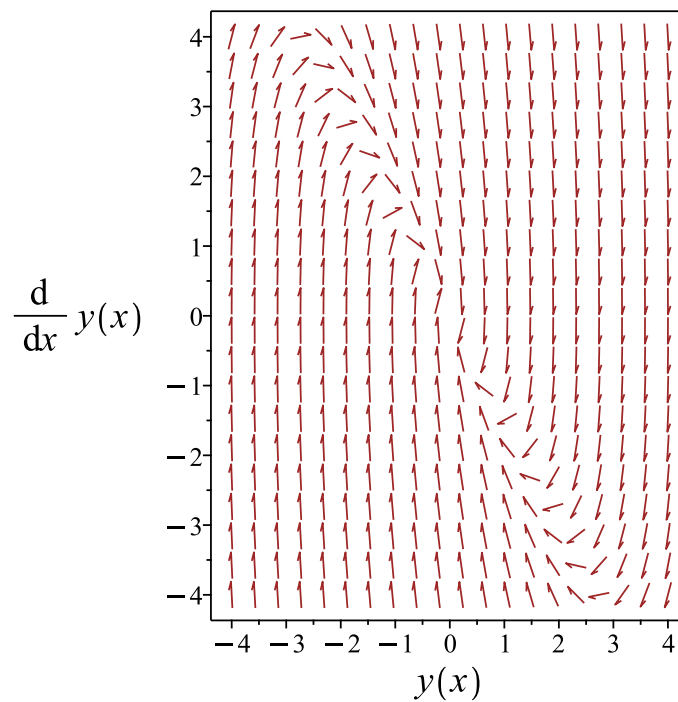


Figure 212: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + x e^{-3x} c_2$$

Verified OK.

8.3.4 Maple step by step solution

Let's solve

$$y'' + 6y' + 9y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 + 6r + 9 = 0$$

- Factor the characteristic polynomial

$$(r + 3)^2 = 0$$

- Root of the characteristic polynomial

$$r = -3$$

- 1st solution of the ODE

$$y_1(x) = e^{-3x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = x e^{-3x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-3x} + x e^{-3x} c_2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-3x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[y''[x]+6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-3x}(c_2x + c_1)$$

8.4 problem 19

Internal problem ID [5361]

Internal file name [OUTPUT/4852_Sunday_February_04_2024_12_46_30_AM_84496447/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 19.

ODE order: 4.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order , _missing_x]]
```

$$y'''' - 6y'''' + 12y'' - 8y' = 0$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 12\lambda^2 - 8\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 2$$

$$\lambda_3 = 2$$

$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{2x} + x e^{2x} c_3 + x^2 e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = e^{2x}$$

$$y_3 = e^{2x} x$$

$$y_4 = x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{2x} + x e^{2x} c_3 + x^2 e^{2x} c_4 \quad (1)$$

Verification of solutions

$$y = c_1 + c_2 e^{2x} + x e^{2x} c_3 + x^2 e^{2x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+12*diff(y(x),x$2)-8*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (c_4 x^2 + c_3 x + c_2) e^{2x} + c_1$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 43

```
DSolve[y''''[x]-6*y'''[x]+12*y''[x]-8*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4} e^{2x} (c_3 (2x^2 - 2x + 1) + c_2 (2x - 1) + 2c_1) + c_4$$

8.5 problem 20

8.5.1	Solving as second order linear constant coeff ode	1309
8.5.2	Solving using Kovacic algorithm	1311
8.5.3	Maple step by step solution	1315

Internal problem ID [5362]

Internal file name [OUTPUT/4853_Sunday_February_04_2024_12_46_30_AM_31817868/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' - 4y' + 13y = 0$$

8.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 13e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(13)} \\ &= 2 \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = 2 + 3i$$

$$\lambda_2 = 2 - 3i$$

Which simplifies to

$$\lambda_1 = 2 + 3i$$

$$\lambda_2 = 2 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x)) \quad (1)$$

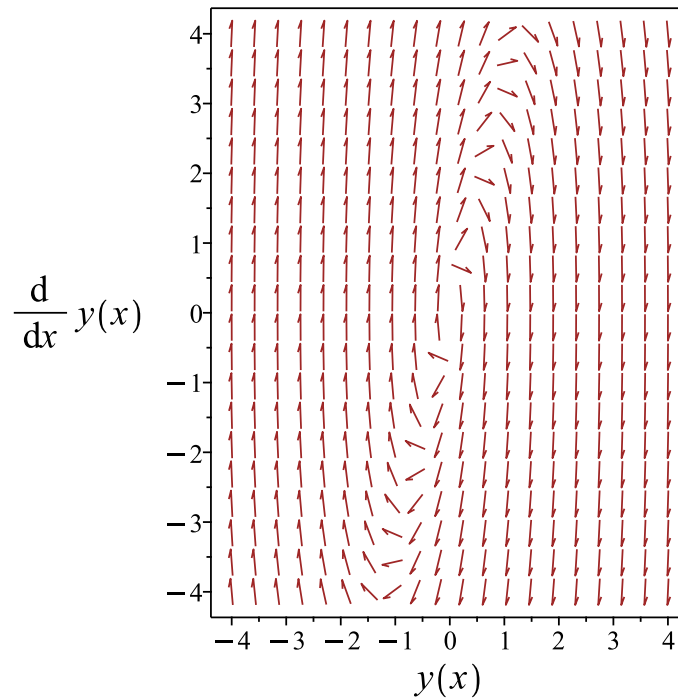


Figure 213: Slope field plot

Verification of solutions

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Verified OK.

8.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = -4 \tag{3}$$

$$C = 13$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 148: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x}) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x} \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{\tan(3x)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{2x} \cos(3x)) + c_2 \left(e^{2x} \cos(3x) \left(\frac{\tan(3x)}{3} \right) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3} \quad (1)$$

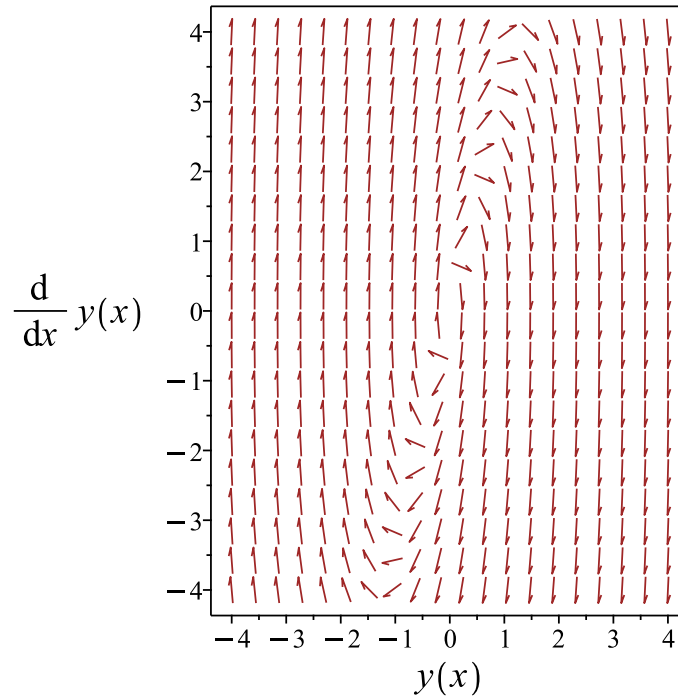


Figure 214: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3}$$

Verified OK.

8.5.3 Maple step by step solution

Let's solve

$$y'' - 4y' + 13y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 - 4r + 13 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{4 \pm (\sqrt{-36})}{2}$$

- Roots of the characteristic polynomial

$$r = (2 - 3I, 2 + 3I)$$

- 1st solution of the ODE

$$y_1(x) = e^{2x} \cos(3x)$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x} \sin(3x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{2x} \cos(3x) + c_2 e^{2x} \sin(3x)$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[y''[x]-4*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{2x}(c_2 \cos(3x) + c_1 \sin(3x))$$

8.6 problem 21

8.6.1	Solving as second order linear constant coeff ode	1317
8.6.2	Solving as second order ode can be made integrable ode	1319
8.6.3	Solving using Kovacic algorithm	1321
8.6.4	Maple step by step solution	1325

Internal problem ID [5363]

Internal file name [OUTPUT/4854_Sunday_February_04_2024_12_46_31_AM_33768958/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 21.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$y'' + 25y = 0$$

8.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 25 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(25)} \\ &= \pm 5i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +5i \\ \lambda_2 &= -5i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 5i \\ \lambda_2 &= -5i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (c_1 \cos(5x) + c_2 \sin(5x))$$

Or

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(5x) + c_2 \sin(5x) \tag{1}$$

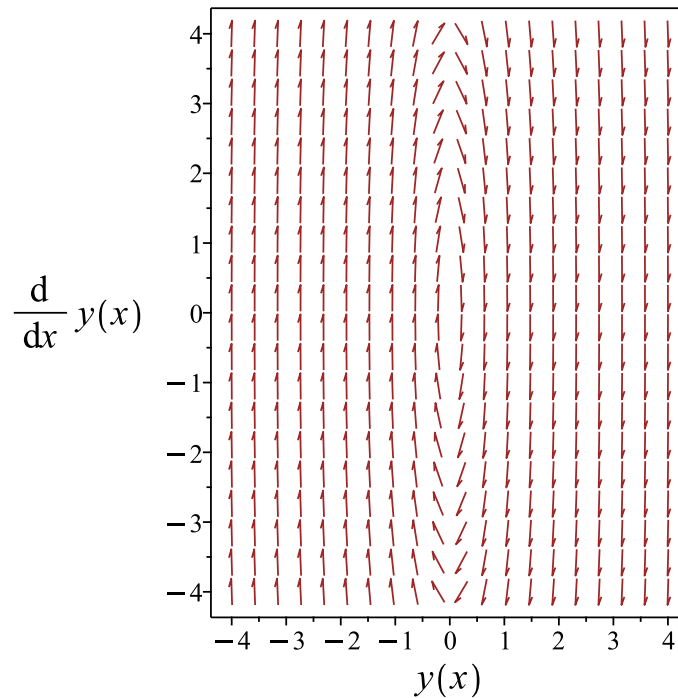


Figure 215: Slope field plot

Verification of solutions

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Verified OK.

8.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 25y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 25y'y) dx = 0$$

$$\frac{y'^2}{2} + \frac{25y^2}{2} = c_2$$

Which is now solved for y . Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-25y^2 + 2c_1} \quad (1)$$

$$y' = -\sqrt{-25y^2 + 2c_1} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-25y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2 + 2c_1}}\right)}{5} = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-25y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2 + 2c_1}}\right)}{5} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2 + 2c_1}}\right)}{5} = c_2 + x \quad (1)$$

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2 + 2c_1}}\right)}{5} = x + c_3 \quad (2)$$

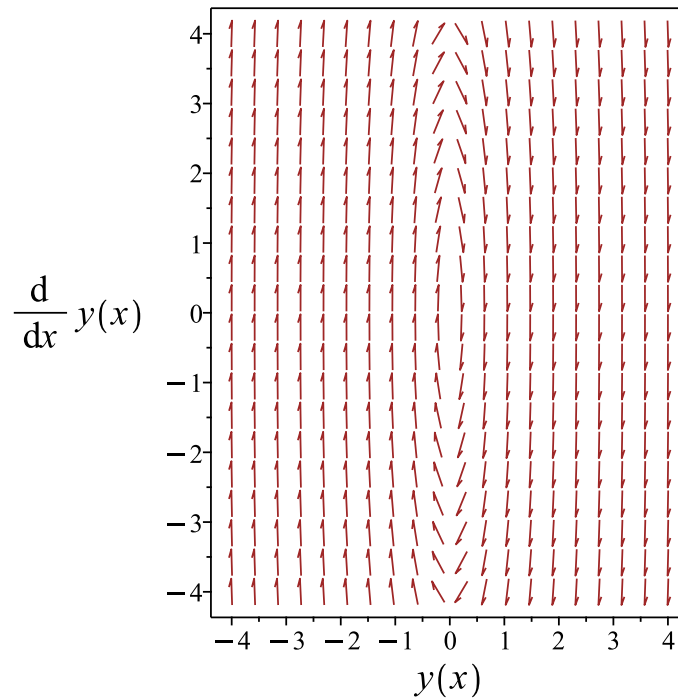


Figure 216: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_2 + x$$

Verified OK.

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_3$$

Verified OK.

8.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 25y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= 25\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -25 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(5x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= \cos(5x)
 \end{aligned}$$

Which simplifies to

$$y_1 = \cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(5x) \int \frac{1}{\cos(5x)^2} dx \\ &= \cos(5x) \left(\frac{\tan(5x)}{5} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(5x)) + c_2 \left(\cos(5x) \left(\frac{\tan(5x)}{5} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} \tag{1}$$

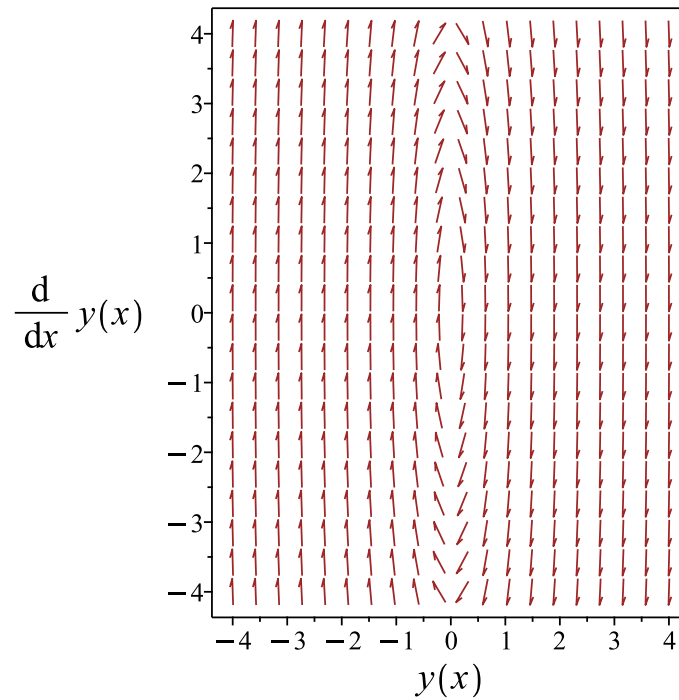


Figure 217: Slope field plot

Verification of solutions

$$y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5}$$

Verified OK.

8.6.4 Maple step by step solution

Let's solve

$$y'' + 25y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of ODE

$$r^2 + 25 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-100})}{2}$$

- Roots of the characteristic polynomial

$$r = (-5I, 5I)$$

- 1st solution of the ODE
 $y_1(x) = \cos(5x)$
- 2nd solution of the ODE
 $y_2(x) = \sin(5x)$
- General solution of the ODE
 $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions
 $y = c_1 \cos(5x) + c_2 \sin(5x)$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(5x) + c_2 \cos(5x)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[y''[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(5x) + c_2 \sin(5x)$$

8.7 problem 22

8.7.1 Maple step by step solution 1328

Internal problem ID [5364]

Internal file name [OUTPUT/4855_Sunday_February_04_2024_12_46_31_AM_84621573/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 22.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_x]]
```

$$y''' - y'' + 9y' - 9y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 9\lambda - 9 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$

$$\lambda_2 = 3i$$

$$\lambda_3 = -3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = e^{-3ix}$$

$$y_3 = e^{3ix}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3 \quad (1)$$

Verification of solutions

$$y = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3$$

Verified OK.

8.7.1 Maple step by step solution

Let's solve

$$y''' - y'' + 9y' - 9y = 0$$

- Highest derivative means the order of the ODE is 3

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = y_3(x) - 9y_2(x) + 9y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - 9y_2(x) + 9y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -9 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -9 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right], \left[-3I, \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right], \left[3I, \begin{bmatrix} -\frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3I, \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-3Ix} \cdot \begin{bmatrix} -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(3x) - I \sin(3x)) \cdot \begin{bmatrix} -\frac{1}{9} \\ \frac{I}{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\cos(3x)}{9} + \frac{I \sin(3x)}{9} \\ \frac{I}{3}(\cos(3x) - I \sin(3x)) \\ \cos(3x) - I \sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(3x)}{9} \\ \frac{\sin(3x)}{3} \\ \cos(3x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \frac{\sin(3x)}{9} \\ \frac{\cos(3x)}{3} \\ -\sin(3x) \end{bmatrix} \right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_2 \cos(3x)}{9} + \frac{c_3 \sin(3x)}{9} \\ \frac{c_2 \sin(3x)}{3} + \frac{c_3 \cos(3x)}{3} \\ c_2 \cos(3x) - c_3 \sin(3x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = c_1 e^x + \frac{c_3 \sin(3x)}{9} - \frac{c_2 \cos(3x)}{9}$$

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+9*diff(y(x),x)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + \sin(3x) c_2 + c_3 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]-y''[x]+9*y'[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_3 e^x + c_1 \cos(3x) + c_2 \sin(3x)$$

8.8 problem 23

8.8.1 Maple step by step solution 1333

Internal problem ID [5365]

Internal file name [OUTPUT/4856_Sunday_February_04_2024_12_46_32_AM_25985800/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 23.

ODE order: 4.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order , _missing_x]]
```

$$y'''' + 4y'' = 0$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 2i$$

$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{-2ix}c_3 + e^{2ix}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = x$$

$$y_3 = e^{-2ix}$$

$$y_4 = e^{2ix}$$

Summary

The solution(s) found are the following

$$y = c_2x + c_1 + e^{-2ix}c_3 + e^{2ix}c_4 \quad (1)$$

Verification of solutions

$$y = c_2x + c_1 + e^{-2ix}c_3 + e^{2ix}c_4$$

Verified OK.

8.8.1 Maple step by step solution

Let's solve

$$y'''' + 4y'' = 0$$

- Highest derivative means the order of the ODE is 4

$$y''''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Define new variable $y_4(x)$

$$y_4(x) = y'''$$

- Isolate for $y_4'(x)$ using original ODE

$$y_4'(x) = -4y_3(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_3(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right], \left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right], \left[-2I, \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right], \left[2I, \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2I, \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-2Ix} \cdot \begin{bmatrix} -\frac{1}{8} \\ -\frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - I \sin(2x)) \cdot \begin{bmatrix} -\frac{I}{8} \\ -\frac{1}{4} \\ \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{I}{8}(\cos(2x) - I \sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{I \sin(2x)}{4} \\ \frac{I}{2}(\cos(2x) - I \sin(2x)) \\ \cos(2x) - I \sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -\frac{c_4 \cos(2x)}{8} - \frac{c_3 \sin(2x)}{8} + c_1 \\ \frac{c_4 \sin(2x)}{4} - \frac{c_3 \cos(2x)}{4} \\ \frac{c_4 \cos(2x)}{2} + \frac{c_3 \sin(2x)}{2} \\ -c_4 \sin(2x) + c_3 \cos(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4 \cos(2x)}{8} - \frac{c_3 \sin(2x)}{8} + c_1$$

Maple trace

```
`Methods for high order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2x + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 32

```
DSolve[y''''[x]+4*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_4x - \frac{1}{4}c_1 \cos(2x) - \frac{1}{4}c_2 \sin(2x) + c_3$$

8.9 problem 24

8.9.1 Maple step by step solution 1339

Internal problem ID [5366]

Internal file name [OUTPUT/4857_Sunday_February_04_2024_12_46_32_AM_35836327/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 24.

ODE order: 4.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$y'''' - 6y''' + 13y'' - 12y' + 4y = 0$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

$$\lambda_3 = 2$$

$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x e^x + e^{2x} c_3 + x e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$

$$y_2 = x e^x$$

$$y_3 = e^{2x}$$

$$y_4 = e^{2x} x$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x e^x + e^{2x} c_3 + x e^{2x} c_4 \quad (1)$$

Verification of solutions

$$y = c_1 e^x + c_2 x e^x + e^{2x} c_3 + x e^{2x} c_4$$

Verified OK.

8.9.1 Maple step by step solution

Let's solve

$$y'''' - 6y''' + 13y'' - 12y' + 4y = 0$$

- Highest derivative means the order of the ODE is 4

$$y''''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Define new variable $y_4(x)$

$$y_4(x) = y'''$$

- Isolate for $y_4'(x)$ using original ODE

$$y_4'(x) = 6y_4(x) - 13y_3(x) + 12y_2(x) - 4y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 6y_4(x) - 13y_3(x) + 12y_2(x) - 4y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right], \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right], \left[\begin{bmatrix} 2 \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right], \left[\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right] \right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where \vec{p} is to be solved for, $\lambda = 1$ is the eigenvalue, and

$$\vec{y}_2(x) = e^{\lambda x} (x\vec{v} + \vec{p})$$

- Note that the x multiplying \vec{v} makes this solution linearly independent to the 1st solution obtained

- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda e^{\lambda x} (x\vec{v} + \vec{p}) + e^{\lambda x} \vec{v} = (e^{\lambda x} A) \cdot (x\vec{v} + \vec{p})$$

- Use the fact that \vec{v} is an eigenvector of A

$$\lambda e^{\lambda x} (x\vec{v} + \vec{p}) + e^{\lambda x} \vec{v} = e^{\lambda x} (\lambda x\vec{v} + A \cdot \vec{p})$$

- Simplify equation

$$\lambda \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition \vec{p} must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose \vec{p} to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of \vec{p}

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- First solution from eigenvalue 2

$$\vec{y}_3(x) = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where \vec{p} is to be solved for, $\lambda = 2$ is the eigenvalue, and

$$\vec{y}_4(x) = e^{\lambda x} (x\vec{v} + \vec{p})$$

- Note that the x multiplying \vec{v} makes this solution linearly independent to the 1st solution obtained

- Substitute $\vec{y}_4(x)$ into the homogeneous system

$$\lambda e^{\lambda x} (x\vec{v} + \vec{p}) + e^{\lambda x} \vec{v} = (e^{\lambda x} A) \cdot (x\vec{v} + \vec{p})$$

- Use the fact that \vec{v} is an eigenvector of A

$$\lambda e^{\lambda x} (x\vec{v} + \vec{p}) + e^{\lambda x} \vec{v} = e^{\lambda x} (\lambda x\vec{v} + A \cdot \vec{p})$$

- Simplify equation

$$\lambda \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition \vec{p} must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose \vec{p} to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \cdot \vec{p} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Choice of \vec{p}

$$\vec{p} = \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2

$$\vec{y}_4(x) = e^{2x} \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^x \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + e^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + e^{2x} c_4 \cdot \left(x \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

- First component of the vector is the solution to the ODE

$$y = \frac{((2x-1)c_4+2c_3)e^{2x}}{16} + ((x-1)c_2 + c_1)e^x$$

Maple trace

```

`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+13*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=0,y(x),sing
```

$$y(x) = (c_2x + c_1)e^{2x} + e^x(c_4x + c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[y''''[x]-6*y'''[x]+13*y''[x]-12*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \rightarrow e^x(c_3e^x + x(c_4e^x + c_2) + c_1)$$

8.10 problem 25

Internal problem ID [5367]

Internal file name [OUTPUT/4858_Sunday_February_04_2024_12_46_32_AM_13284602/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplementary problems. Page 86

Problem number: 25.

ODE order: 6.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order , _missing_x]]
```

$$y^{(6)} + 9y'''' + 24y'' + 16y = 0$$

The characteristic equation is

$$\lambda^6 + 9\lambda^4 + 24\lambda^2 + 16 = 0$$

The roots of the above equation are

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

$$\lambda_3 = 2i$$

$$\lambda_4 = -2i$$

$$\lambda_5 = 2i$$

$$\lambda_6 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-ix} + e^{-2ix} c_2 + x e^{-2ix} c_3 + e^{2ix} c_4 + x e^{2ix} c_5 + e^{ix} c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
 y_1 &= e^{-ix} \\
 y_2 &= e^{-2ix} \\
 y_3 &= x e^{-2ix} \\
 y_4 &= e^{2ix} \\
 y_5 &= x e^{2ix} \\
 y_6 &= e^{ix}
 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-ix} + e^{-2ix} c_2 + x e^{-2ix} c_3 + e^{2ix} c_4 + x e^{2ix} c_5 + e^{ix} c_6 \quad (1)$$

Verification of solutions

$$y = c_1 e^{-ix} + e^{-2ix} c_2 + x e^{-2ix} c_3 + e^{2ix} c_4 + x e^{2ix} c_5 + e^{ix} c_6$$

Verified OK.

Maple trace

```

`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$6)+9*diff(y(x),x$4)+24*diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_6 x + c_4) \cos(2x) + (c_5 x + c_3) \sin(2x) + c_1 \sin(x) + \cos(x) c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
DSolve[y''''''[x]+9*y''''[x]+24*y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (c_2 x + c_1) \cos(2x) + c_6 \sin(x) + \cos(x)(2(c_4 x + c_3) \sin(x) + c_5)$$

9 Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

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9.1 problem 11

- 9.1.1 Solving as second order linear constant coeff ode 1348
- 9.1.2 Solving using Kovacic algorithm 1351
- 9.1.3 Maple step by step solution 1356

Internal problem ID [5368]

Internal file name [OUTPUT/4859_Sunday_February_04_2024_12_46_32_AM_63793941/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 11.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x]]
```

$$y'' - 4y' + 3y = 1$$

9.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 3 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(3)} \\ &= 2 \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$

$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^x$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^x$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{3x}c_1 + c_2e^x) + \left(\frac{1}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2e^x + \frac{1}{3} \quad (1)$$

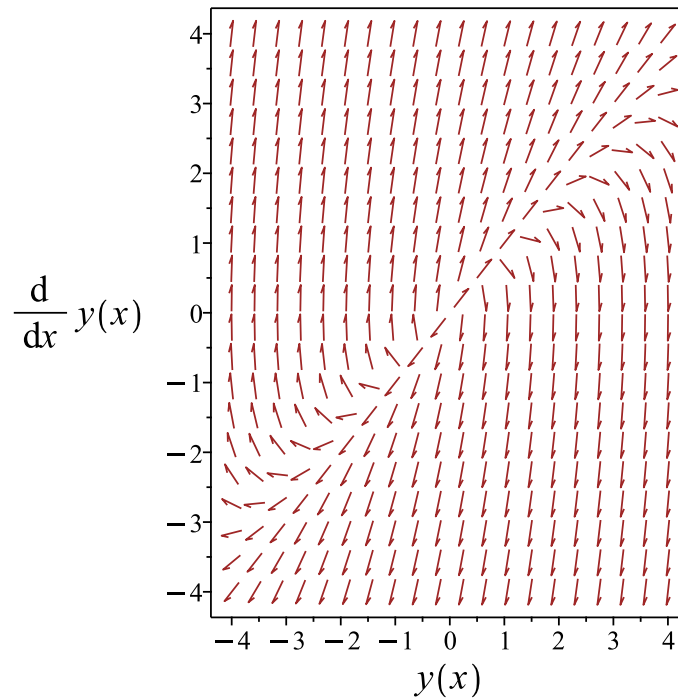


Figure 218: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^x + \frac{1}{3}$$

Verified OK.

9.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = -4 \tag{3}$$

$$C = 3$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1 \\ t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x}) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^x) + c_2\left(e^x\left(\frac{e^{2x}}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + \frac{c_2 e^{3x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{2}, e^x\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^{3x}}{2} \right) + \left(\frac{1}{3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{1}{3} \tag{1}$$

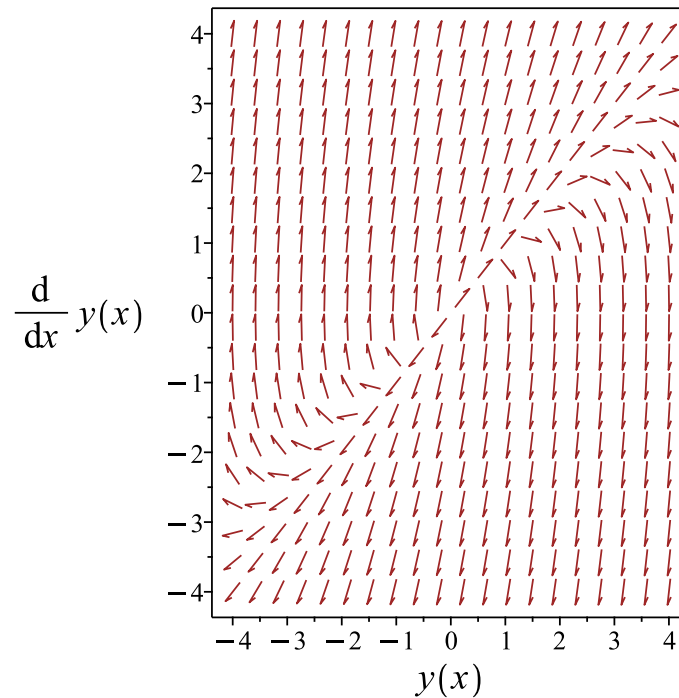


Figure 219: Slope field plot

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{1}{3}$$

Verified OK.

9.1.3 Maple step by step solution

Let's solve

$$y'' - 4y' + 3y = 1$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r + 3 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 3) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 3)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^x$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{3x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{3x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right), f(x) = 1 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^x \left(\int e^{-x} dx \right)}{2} + \frac{e^{3x} \left(\int e^{-3x} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{1}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{3x} + \frac{1}{3}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=1,y(x), singsol=all)
```

$$y(x) = e^x c_2 + e^{3x} c_1 + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+3*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x + c_2 e^{3x} + \frac{1}{3}$$

9.2 problem 12

9.2.1	Solving as second order linear constant coeff ode	1359
9.2.2	Solving as second order integrable as is ode	1363
9.2.3	Solving as second order ode missing y ode	1365
9.2.4	Solving as type second_order_integrable_as_is (not using ABC version)	1366
9.2.5	Solving using Kovacic algorithm	1368
9.2.6	Solving as exact linear second order ode ode	1373
9.2.7	Maple step by step solution	1376

Internal problem ID [5369]

Internal file name [OUTPUT/4860_Sunday_February_04_2024_12_46_33_AM_16089114/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems.

Page 92

Problem number: 12.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$y'' - 4y' = 5$$

9.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 0, f(x) = 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(0)} \\ &= 2 \pm 2 \end{aligned}$$

Hence

$$\lambda_1 = 2 + 2$$

$$\lambda_2 = 2 - 2$$

Which simplifies to

$$\lambda_1 = 4$$

$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(4)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 e^{4x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{4x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{5x}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 e^{4x} + c_2) + \left(-\frac{5x}{4}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 - \frac{5x}{4} \tag{1}$$

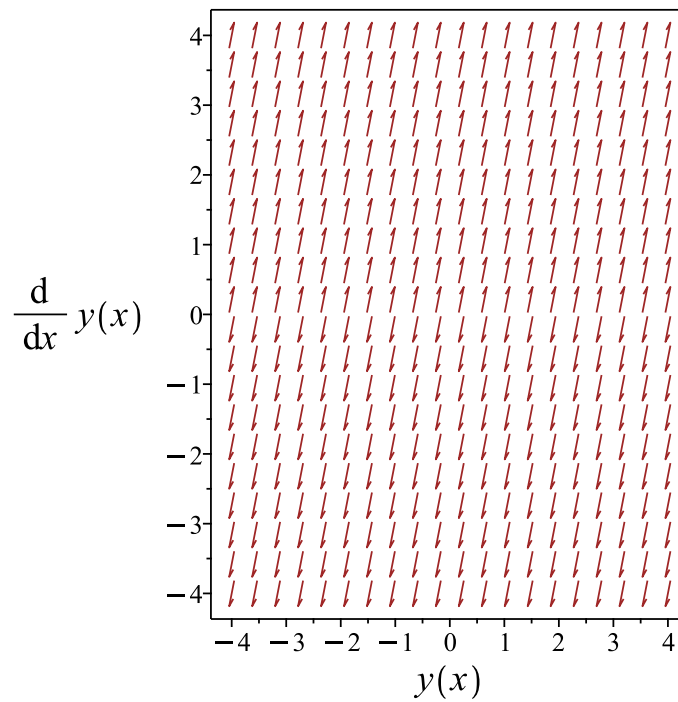


Figure 220: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 - \frac{5x}{4}$$

Verified OK.

9.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y') dx = \int 5dx$$
$$-4y + y' = 5x + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = 5x + c_1$$

Hence the ode is

$$-4y + y' = 5x + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-4)dx}$$
$$= e^{-4x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(5x + c_1)$$
$$\frac{d}{dx}(e^{-4x}y) = (e^{-4x})(5x + c_1)$$
$$d(e^{-4x}y) = ((5x + c_1)e^{-4x}) dx$$

Integrating gives

$$e^{-4x}y = \int (5x + c_1)e^{-4x} dx$$
$$e^{-4x}y = -\frac{(20x + 4c_1 + 5)e^{-4x}}{16} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 4c_1 + 5)e^{-4x}}{16} + c_2e^{4x}$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x} \tag{1}$$

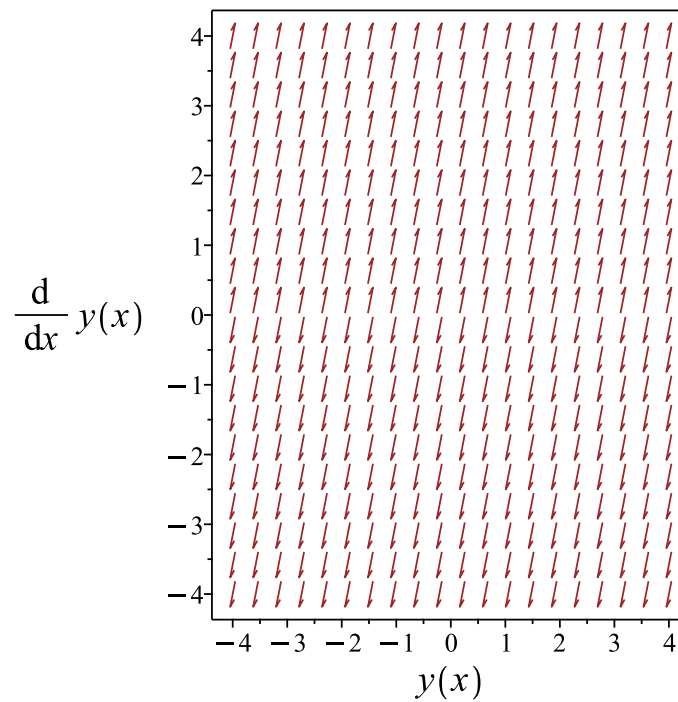


Figure 221: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y . Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4p(x) - 5 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{4p+5} dp = \int dx$$
$$\frac{\ln(4p+5)}{4} = x + c_1$$

Raising both side to exponential gives

$$(4p+5)^{\frac{1}{4}} = e^{x+c_1}$$

Which simplifies to

$$(4p+5)^{\frac{1}{4}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_2^4 e^{4x}}{4} - \frac{5}{4}$$

Integrating both sides gives

$$y = \int \frac{c_2^4 e^{4x}}{4} - \frac{5}{4} dx$$
$$= -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3 \quad (1)$$

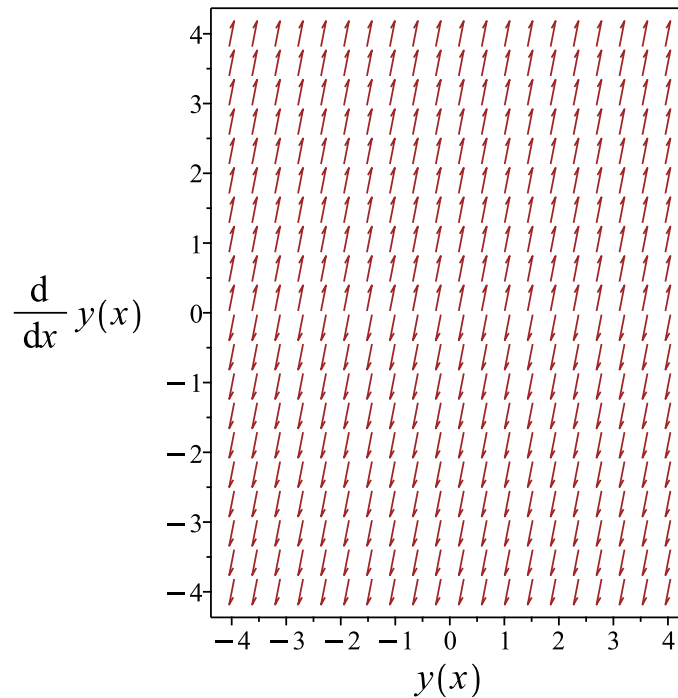


Figure 222: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3$$

Verified OK.

9.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = 5$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y') dx = \int 5 dx$$

$$-4y + y' = 5x + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -4 \\q(x) &= 5x + c_1\end{aligned}$$

Hence the ode is

$$-4y + y' = 5x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int(-4)dx} \\ &= e^{-4x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(5x + c_1) \\ \frac{d}{dx}(e^{-4x}y) &= (e^{-4x})(5x + c_1) \\ d(e^{-4x}y) &= ((5x + c_1)e^{-4x}) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-4x}y &= \int (5x + c_1)e^{-4x} dx \\ e^{-4x}y &= -\frac{(20x + 4c_1 + 5)e^{-4x}}{16} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 4c_1 + 5)e^{-4x}}{16} + c_2e^{4x}$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2e^{4x} \tag{1}$$

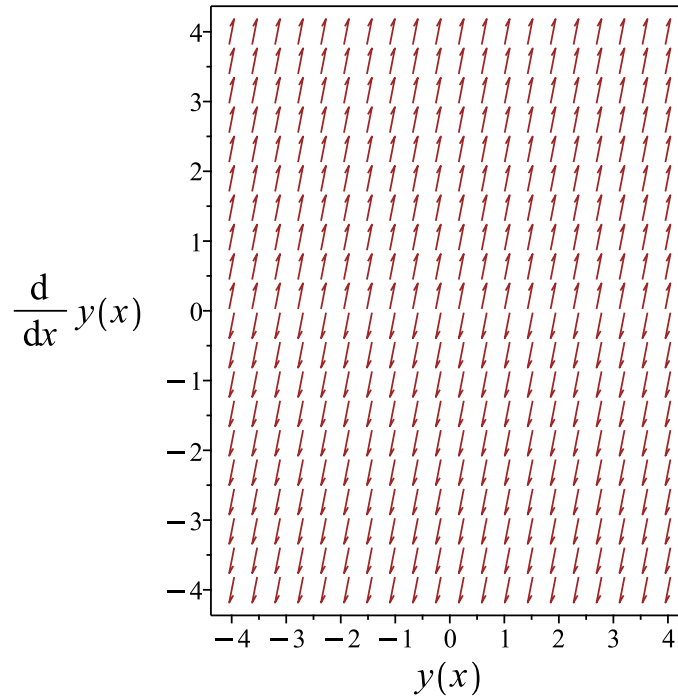


Figure 223: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 4$$

$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x}) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{4x}}{4} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2 \left(1 \left(\frac{e^{4x}}{4} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 e^{4x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ 1, \frac{e^{4x}}{4} \right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{5x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{4x}}{4} \right) + \left(-\frac{5x}{4} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{4} \tag{1}$$

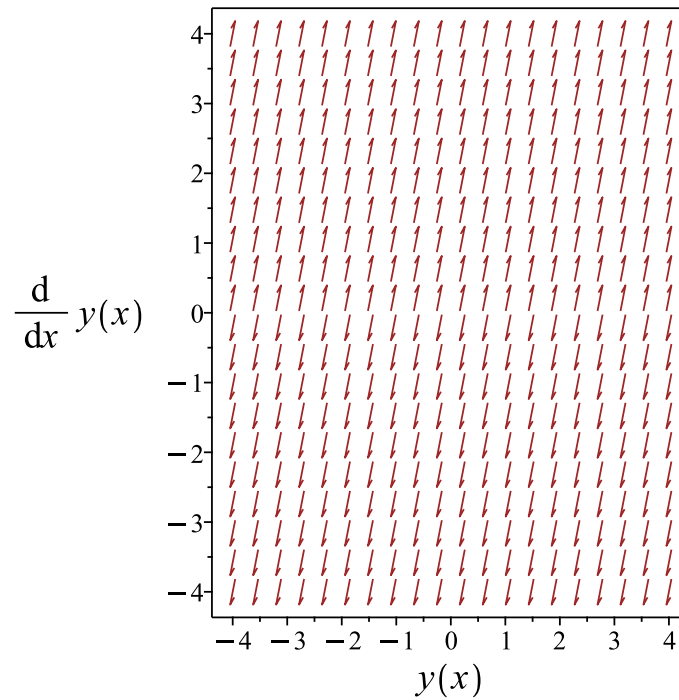


Figure 224: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{4}$$

Verified OK.

9.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= 5 \end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0 \\q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)y' + (q(x) - p'(x))y)' = s(x)$$

Integrating gives

$$p(x)y' + (q(x) - p'(x))y = \int s(x) dx$$

Substituting the above values for p, q, r, s gives

$$-4y + y' = \int 5 dx$$

We now have a first order ode to solve which is

$$-4y + y' = 5x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -4 \\q(x) &= 5x + c_1\end{aligned}$$

Hence the ode is

$$-4y + y' = 5x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-4)dx} \\&= e^{-4x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(5x + c_1) \\ \frac{d}{dx}(e^{-4x}y) &= (e^{-4x})(5x + c_1) \\ d(e^{-4x}y) &= ((5x + c_1)e^{-4x}) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-4x}y &= \int (5x + c_1)e^{-4x} dx \\ e^{-4x}y &= -\frac{(20x + 4c_1 + 5)e^{-4x}}{16} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 4c_1 + 5)e^{-4x}}{16} + c_2e^{4x}$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2e^{4x} \tag{1}$$

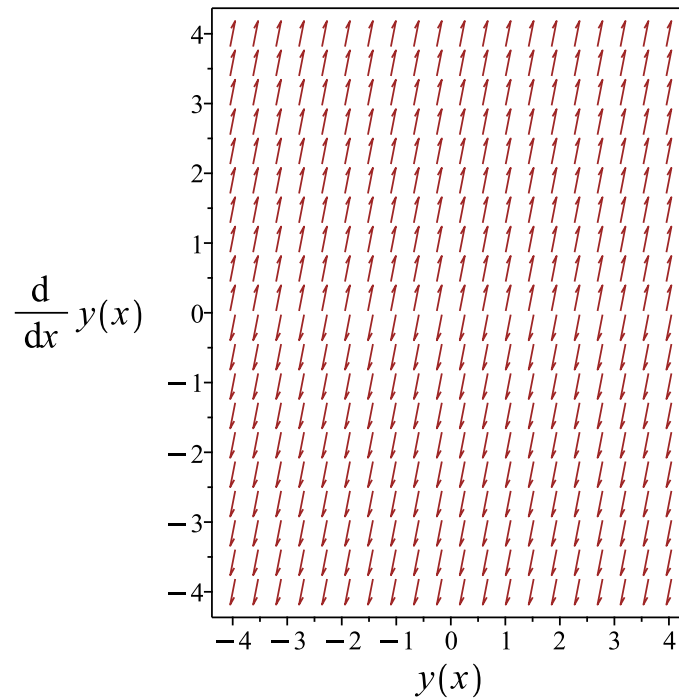


Figure 225: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.7 Maple step by step solution

Let's solve

$$y'' - 4y' = 5$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r = 0$$

- Factor the characteristic polynomial

$$r(r - 4) = 0$$

- Roots of the characteristic polynomial

$$r = (0, 4)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = 1$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{4x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 + c_2 e^{4x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right), f(x) = 5 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{4x} \\ 0 & 4e^{4x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 4e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{5(\int 1 dx)}{4} + \frac{5e^{4x}(\int e^{-4x} dx)}{4}$$

- Compute integrals

$$y_p(x) = -\frac{5x}{4} - \frac{5}{16}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 + c_2 e^{4x} - \frac{5x}{4} - \frac{5}{16}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*_b(_a)+5, _b(_a)` *** Sublevel 2 **  
  Methods for first order ODEs:  
    --- Trying classification methods ---  
    trying a quadrature  
    trying 1st order linear  
    <- 1st order linear successful  
  <- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)=5,y(x), singsol=all)
```

$$y(x) = \frac{e^{4x}c_1}{4} - \frac{5x}{4} + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 24

```
DSolve[y''[x]-4*y'[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{5x}{4} + \frac{1}{4}c_1e^{4x} + c_2$$

9.3 problem 13

Internal problem ID [5370]

Internal file name [OUTPUT/4861_Sunday_February_04_2024_12_46_34_AM_82222025/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 13.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_x]]
```

$$y''' - 4y'' = 5$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2x + c_1 + e^{4x}c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}y_1 &= 1 \\y_2 &= x \\y_3 &= e^{4x}\end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - 4y'' = 5$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{8} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{5x^2}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2x + c_1 + e^{4x}c_3) + \left(-\frac{5x^2}{8} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2x + c_1 + e^{4x}c_3 - \frac{5x^2}{8} \quad (1)$$

Verification of solutions

$$y = c_2x + c_1 + e^{4x}c_3 - \frac{5x^2}{8}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)=5,y(x), singsol=all)
```

$$y(x) = \frac{e^{4x}c_1}{16} - \frac{5x^2}{8} + c_2x + c_3$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 30

```
DSolve[y'''[x]-4*y''[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{5x^2}{8} + c_3x + \frac{1}{16}c_1e^{4x} + c_2$$

9.4 problem 14

Internal problem ID [5371]

Internal file name [OUTPUT/4862_Sunday_February_04_2024_12_46_34_AM_81323681/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 14.

ODE order: 5.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order , _missing_x]]
```

$$y^{(5)} - 4y''' = 5$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(5)} - 4y''' = 0$$

The characteristic equation is

$$\lambda^5 - 4\lambda^3 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

$$\lambda_4 = 2$$

$$\lambda_5 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{-2x}c_4 + e^{2x}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = x$$

$$y_3 = x^2$$

$$y_4 = e^{-2x}$$

$$y_5 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y^{(5)} - 4y''' = 5$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^{-2x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^2\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-24A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{24} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{5x^3}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_3 x^2 + c_2 x + c_1 + e^{-2x} c_4 + e^{2x} c_5) + \left(-\frac{5x^3}{24} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3 x^2 + c_2 x + c_1 + e^{-2x} c_4 + e^{2x} c_5 - \frac{5x^3}{24} \quad (1)$$

Verification of solutions

$$y = c_3 x^2 + c_2 x + c_1 + e^{-2x} c_4 + e^{2x} c_5 - \frac{5x^3}{24}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 5; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 4*_b(_a)+5, _b(_a)` *** Sub
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
checking if the LODE has constant coefficients
<- constant coefficients successful
<- solving first the homogeneous part of the ODE successful
<- differential order: 5; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$5)-4*diff(y(x),x$3)=5,y(x), singsol=all)
```

$$y(x) = -\frac{5x^3}{24} + \frac{c_2 e^{2x}}{8} + \frac{c_3 x^2}{2} - \frac{e^{-2x} c_1}{8} + c_4 x + c_5$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 47

```
DSolve[y'''''[x]-4*y''''[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{5x^3}{24} + c_5 x^2 + c_4 x + \frac{1}{8} c_1 e^{2x} - \frac{1}{8} c_2 e^{-2x} + c_3$$

9.5 problem 15

9.5.1 Maple step by step solution 1389

Internal problem ID [5372]

Internal file name [OUTPUT/4863_Sunday_February_04_2024_12_46_34_AM_22823233/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 15.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_y]]
```

$$y''' - 4y' = x$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = 2$$

$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = e^{-2x}$$

$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y' = x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8xA_2 - 4A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{8} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{x^2}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 e^{-2x} + e^{2x} c_3) + \left(-\frac{x^2}{8} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + e^{2x} c_3 - \frac{x^2}{8} \quad (1)$$

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + e^{2x} c_3 - \frac{x^2}{8}$$

Verified OK.

9.5.1 Maple step by step solution

Let's solve

$$y''' - 4y' = x$$

- Highest derivative means the order of the ODE is 3
 y'''
- Convert linear ODE into a system of first order ODEs
 - Define new variable $y_1(x)$
 $y_1(x) = y$
 - Define new variable $y_2(x)$
 $y_2(x) = y'$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = x + 4y_2(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x + 4y_2(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \\ 0 & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} \\ 0 & -e^{-2x} + e^{2x} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x) \cdot \vec{v}(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} -\frac{1}{16} + \frac{e^{2x}}{32} + \frac{e^{-2x}}{32} - \frac{x^2}{8} \\ -\frac{x}{4} - \frac{e^{-2x}}{16} + \frac{e^{2x}}{16} \\ \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} -\frac{1}{16} + \frac{e^{2x}}{32} + \frac{e^{-2x}}{32} - \frac{x^2}{8} \\ -\frac{x}{4} - \frac{e^{-2x}}{16} + \frac{e^{2x}}{16} \\ \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(8c_1+1)e^{-2x}}{32} + \frac{(8c_3+1)e^{2x}}{32} - \frac{x^2}{8} + c_2 - \frac{1}{16}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 4*_b(_a)+_a, _b(_a)` *** Su
  Methods for second order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying high order exact linear fully integrable
  trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
  trying a double symmetry of the form [xi=0, eta=F(x)]
  -> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
    <- solving first the homogeneous part of the ODE successful
  <- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{8} + \frac{c_2 e^{2x}}{2} - \frac{e^{-2x} c_1}{2} + c_3$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[y'''[x]-4*y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x^2}{8} + \frac{1}{2}c_1 e^{2x} - \frac{1}{2}c_2 e^{-2x} + c_3$$

9.6 problem 16

9.6.1	Solving as second order linear constant coeff ode	1395
9.6.2	Solving as linear second order ode solved by an integrating factor ode	1398
9.6.3	Solving using Kovacic algorithm	1400
9.6.4	Maple step by step solution	1405

Internal problem ID [5373]

Internal file name [OUTPUT/4864_Sunday_February_04_2024_12_46_35_AM_19377599/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems.
Page 92

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - 6y' + 9y = e^{2x}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} \quad (1)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 x e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{3x} c_1 + c_2 x e^{3x}) + (e^{2x}) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + e^{2x} \tag{1}$$

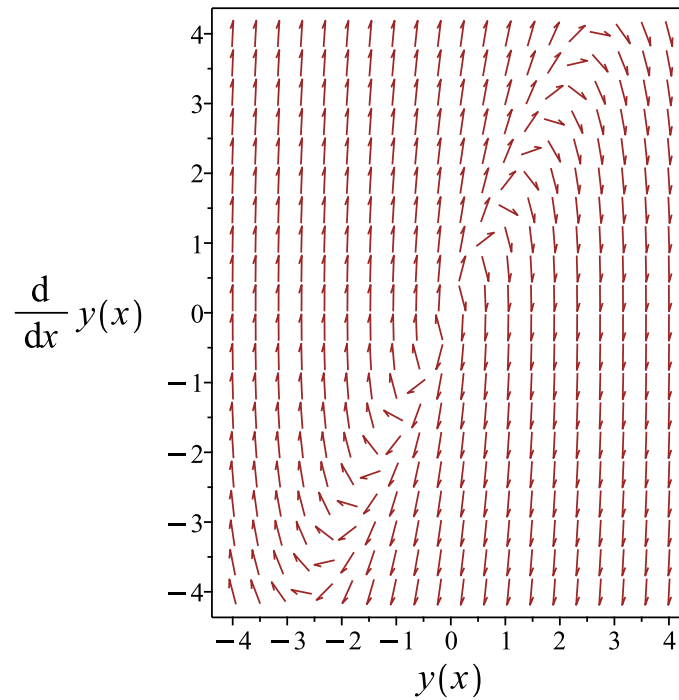


Figure 226: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + e^{2x}$$

Verified OK.

9.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x))^2 + p'(x)}{2}y = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int -6 \, dx} \\ &= e^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{-3x}e^{2x}$$

$$(e^{-3x}y)'' = e^{-3x}e^{2x}$$

Integrating once gives

$$(e^{-3x}y)' = -e^{-x} + c_1$$

Integrating again gives

$$(e^{-3x}y) = c_1x + e^{-x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + e^{-x} + c_2}{e^{-3x}}$$

Or

$$y = c_1x e^{3x} + c_2 e^{3x} + e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1x e^{3x} + c_2 e^{3x} + e^{2x} \tag{1}$$

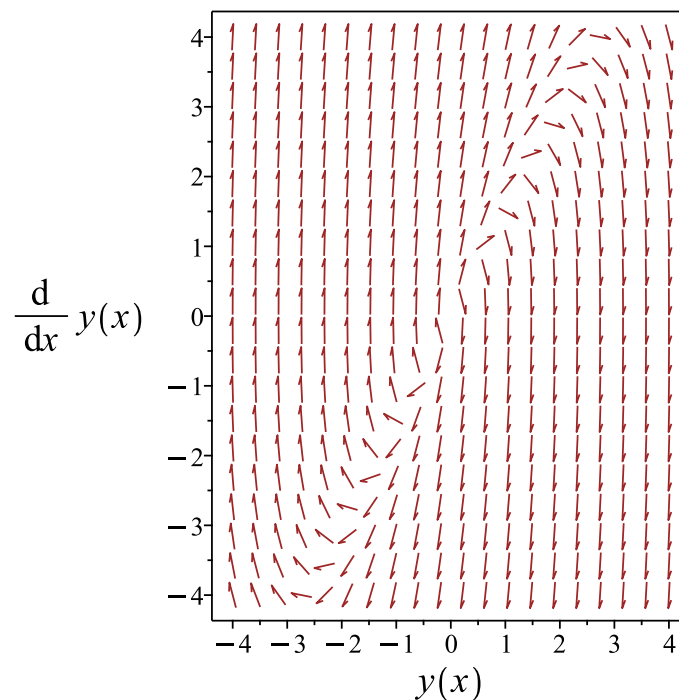


Figure 227: Slope field plot

Verification of solutions

$$y = c_1 x e^{3x} + c_2 e^{3x} + e^{2x}$$

Verified OK.

9.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-6}{1} dx} \\ &= z_1 e^{3x} \\ &= z_1 (e^{3x}) \end{aligned}$$

Which simplifies to

$$y_1 = e^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2} dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{3x}) + c_2 (e^{3x}(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{3x} c_1 + c_2 x e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{3x} c_1 + c_2 x e^{3x}) + (e^{2x}) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2x + c_1) + e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2x + c_1) + e^{2x} \tag{1}$$

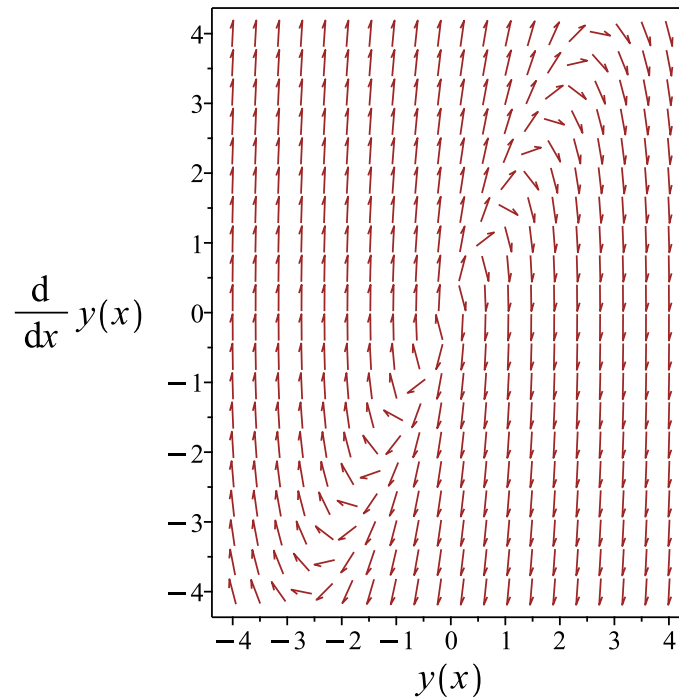


Figure 228: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + e^{2x}$$

Verified OK.

9.6.4 Maple step by step solution

Let's solve

$$y'' - 6y' + 9y = e^{2x}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 6r + 9 = 0$$

- Factor the characteristic polynomial

$$(r - 3)^2 = 0$$

- Root of the characteristic polynomial

$$r = 3$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{3x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = e^{3x}x$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = e^{3x}c_1 + c_2x e^{3x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^{2x} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & e^{3x}x \\ 3e^{3x} & 3e^{3x}x + e^{3x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{6x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = e^{3x} \left(- \left(\int x e^{-x} dx \right) + \left(\int e^{-x} dx \right) x \right)$$

- Compute integrals

$$y_p(x) = e^{2x}$$

- Substitute particular solution into general solution to ODE

$$y = c_2 x e^{3x} + e^{3x} c_1 + e^{2x}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2) e^{3x} + e^{2x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 24

```
DSolve[y''[x]-6*y'[x]+9*y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{2x}(1 + e^x(c_2 x + c_1))$$

9.7 problem 17

9.7.1 Solving as second order linear constant coeff ode	1407
9.7.2 Solving using Kovacic algorithm	1410
9.7.3 Maple step by step solution	1415

Internal problem ID [5374]

Internal file name [OUTPUT/4865_Sunday_February_04_2024_12_46_35_AM_92023248/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' + y' - 2y = -2x^2 + 2x + 2$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = -2x^2 + 2x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2 - (4)(1)(-2)} \\ &= -\frac{1}{2} \pm \frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= -2 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x} \end{aligned}$$

Or

$$y = c_1 e^x + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = -2x^2 + 2x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^x + c_2e^{-2x}) + (x^2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{-2x} + x^2 \tag{1}$$

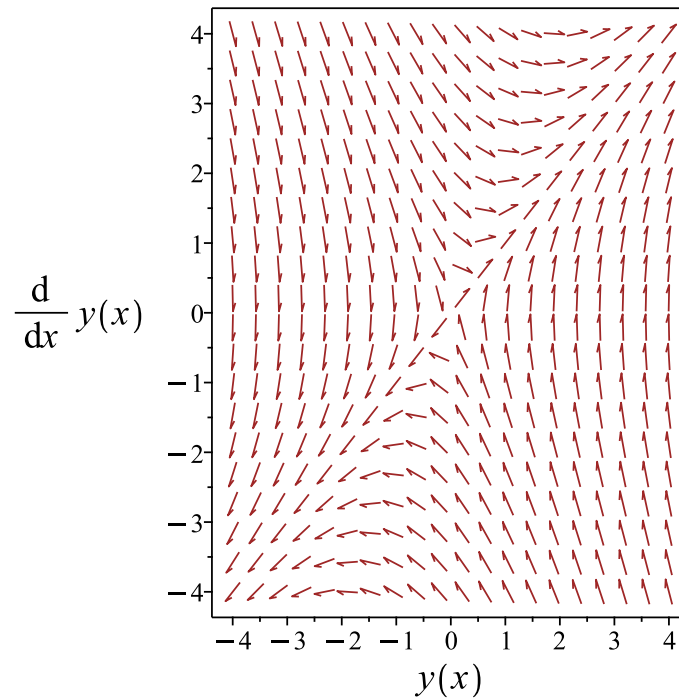


Figure 229: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-2x} + x^2$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 162: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{1}{1} dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(e^{-\frac{x}{2}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{3x}}{3} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-2x}) + c_2 \left(e^{-2x} \left(\frac{e^{3x}}{3} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^x}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{3}, e^{-2x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = -2x^2 + 2x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-2x} + \frac{c_2e^x}{3} \right) + (x^2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-2x} + \frac{c_2e^x}{3} + x^2 \tag{1}$$

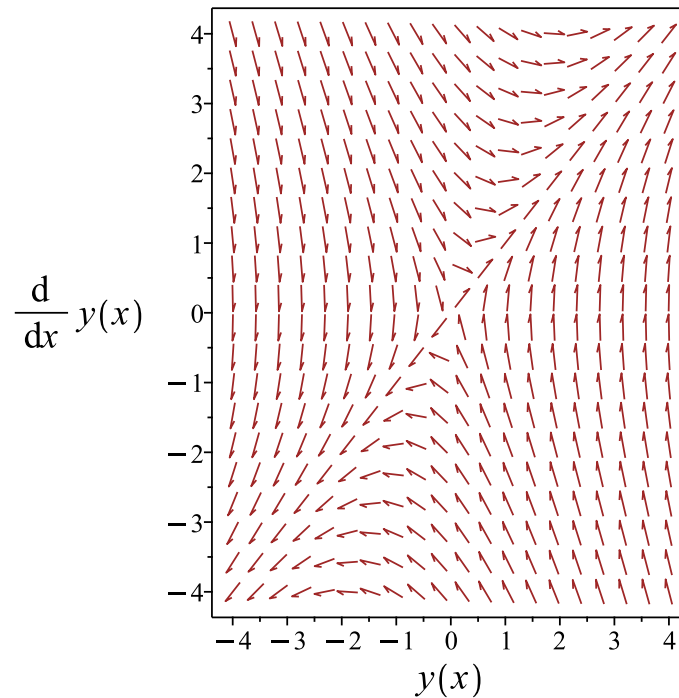


Figure 230: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^x}{3} + x^2$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$$y'' + y' - 2y = -2x^2 + 2x + 2$$

- Highest derivative means the order of the ODE is 2
- $$y''$$
- Characteristic polynomial of homogeneous ODE
- $$r^2 + r - 2 = 0$$
- Factor the characteristic polynomial
- $$(r + 2)(r - 1) = 0$$
- Roots of the characteristic polynomial

$$r = (-2, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-2x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-2x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = -2x^2 + 2x + 2 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^x \\ -2e^{-2x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 3e^{-x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2(e^{3x}(\int e^{-x}(x^2-x-1)dx) - (\int e^{2x}(x^2-x-1)dx)e^{-2x}}{3}$$

- Compute integrals

$$y_p(x) = x^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-2x} + c_2 e^x + x^2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=2*(1+x-x^2),y(x), singsol=all)
```

$$y(x) = (e^{3x}c_1 + e^{2x}x^2 + c_2) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[y''[x]+y'[x]-2*y[x]==2*(1+x-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x^2 + c_1 e^{-2x} + c_2 e^x$$

9.8 problem 18

- 9.8.1 Solving as second order linear constant coeff ode 1418
- 9.8.2 Solving using Kovacic algorithm 1421
- 9.8.3 Maple step by step solution 1427

Internal problem ID [5375]

Internal file name [OUTPUT/4866_Sunday_February_04_2024_12_46_36_AM_94845678/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = 4x e^x$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 4x e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x e^x, x^2 e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x + A_2 x^2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x + 2A_2 e^x + 4A_2 x e^x = 4x e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -x e^x + x^2 e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + (-x e^x + x^2 e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - x e^x + x^2 e^x \quad (1)$$

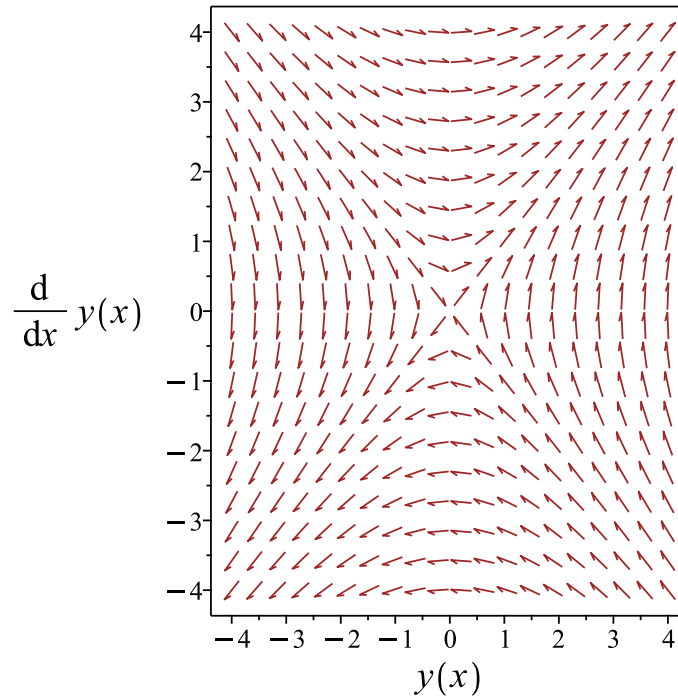


Figure 231: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - x e^x + x^2 e^x$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 0 \quad (3)$$

$$C = -1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$

$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x}) \left(\frac{e^x}{2}\right) - \left(\frac{e^x}{2}\right) (-e^{-x})$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{2 e^{2x} x}{1} dx$$

Which simplifies to

$$u_1 = - \int 2 e^{2x} x dx$$

Hence

$$u_1 = - \frac{(2x - 1) e^{2x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 e^{-x} e^x x}{1} dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \frac{(2x - 1) e^{2x} e^{-x}}{2} + x^2 e^x$$

Which simplifies to

$$y_p(x) = e^x \left(\frac{1}{2} + x^2 - x \right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(e^x \left(\frac{1}{2} + x^2 - x \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + e^x \left(\frac{1}{2} + x^2 - x \right) \quad (1)$$

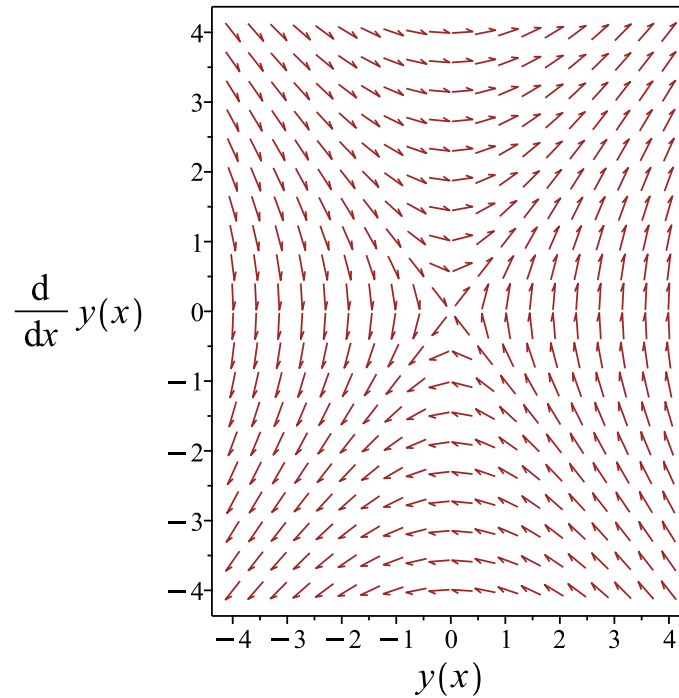


Figure 232: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + e^x \left(\frac{1}{2} + x^2 - x \right)$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$$y'' - y = 4x e^x$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = 4x e^x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -2 e^{-x} \left(\int e^{2x} x dx \right) + 2 e^x \left(\int x dx \right)$$

- Compute integrals

$$y_p(x) = e^x \left(\frac{1}{2} + x^2 - x \right)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + e^x \left(\frac{1}{2} + x^2 - x \right)$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=4*x*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x (x^2 + c_1 - x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 30

```
DSolve[y''[x]-y[x]==4*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x \left(x^2 - x + \frac{1}{2} + c_1 \right) + c_2 e^{-x}$$

9.9 problem 19

9.9.1 Solving as second order linear constant coeff ode	1430
9.9.2 Solving using Kovacic algorithm	1433
9.9.3 Maple step by step solution	1438

Internal problem ID [5376]

Internal file name [OUTPUT/4867_Sunday_February_04_2024_12_46_36_AM_65151594/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 19.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = \sin(x)^2$$

9.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10} \quad (1)$$

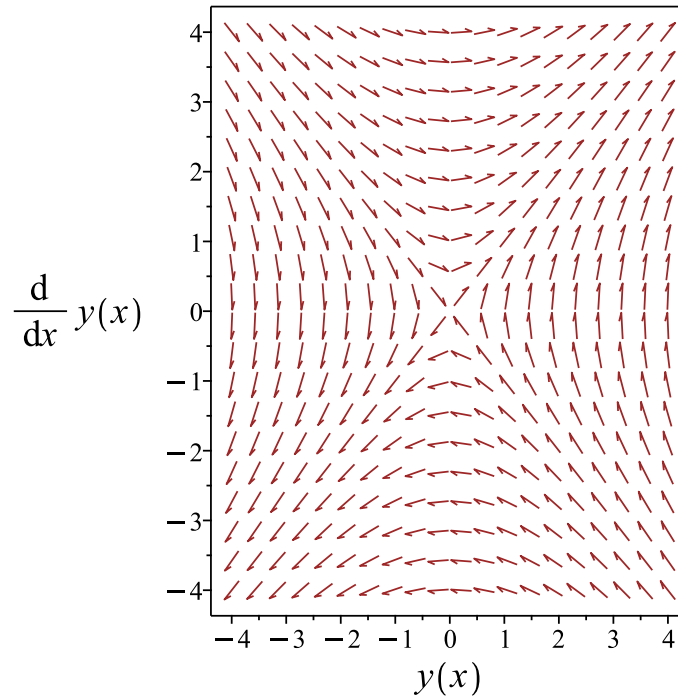


Figure 233: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= -1\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 1 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{2}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{1}{2} + \frac{\cos(2x)}{10} \quad (1)$$

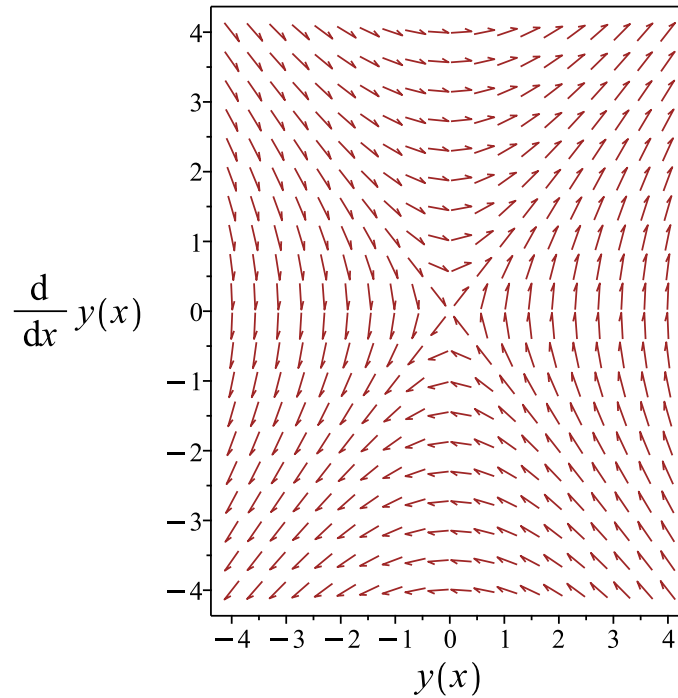


Figure 234: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$$y'' - y = \sin(x)^2$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right), f(x) = \sin(x)^2 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x} \left(\int e^x \sin(x)^2 dx \right)}{2} + \frac{e^x \left(\int \sin(x)^2 e^{-x} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 + \frac{\cos(x)^2}{5} - \frac{3}{5}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 30

```
DSolve[y''[x]-y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{10}(\cos(2x) - 5) + c_1 e^x + c_2 e^{-x}$$

9.10 problem 20

9.10.1 Solving as second order linear constant coeff ode	1441
9.10.2 Solving using Kovacic algorithm	1446
9.10.3 Maple step by step solution	1452

Internal problem ID [5377]

Internal file name [OUTPUT/4868_Sunday_February_04_2024_12_46_37_AM_60385736/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = \frac{1}{(1 + e^{-x})^2}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \frac{1}{(1+e^{-x})^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)(-e^{-x}) - (e^{-x})(e^x)$$

Which simplifies to

$$W = -2e^{-x}e^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{-2} dx$$

Which simplifies to

$$u_1 = - \int -\frac{e^{-x}}{2(1+e^{-x})^2} dx$$

Hence

$$u_1 = \frac{1}{2e^{-x} + 2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^x}{(1+e^{-x})^2}}{-2} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{3x}}{2(1+e^x)^2} dx$$

Hence

$$u_2 = -\frac{e^x}{2} + \ln(1+e^x) + \frac{1}{2e^x + 2}$$

Which simplifies to

$$u_1 = \frac{1}{2e^{-x} + 2}$$
$$u_2 = \frac{(2e^x + 2) \ln(1+e^x) - e^x - e^{2x} + 1}{2e^x + 2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2e^{-x} + 2} + \frac{((2e^x + 2) \ln(1+e^x) - e^x - e^{2x} + 1) e^{-x}}{2e^x + 2}$$

Which simplifies to

$$y_p(x) = \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(\frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \quad (1)$$

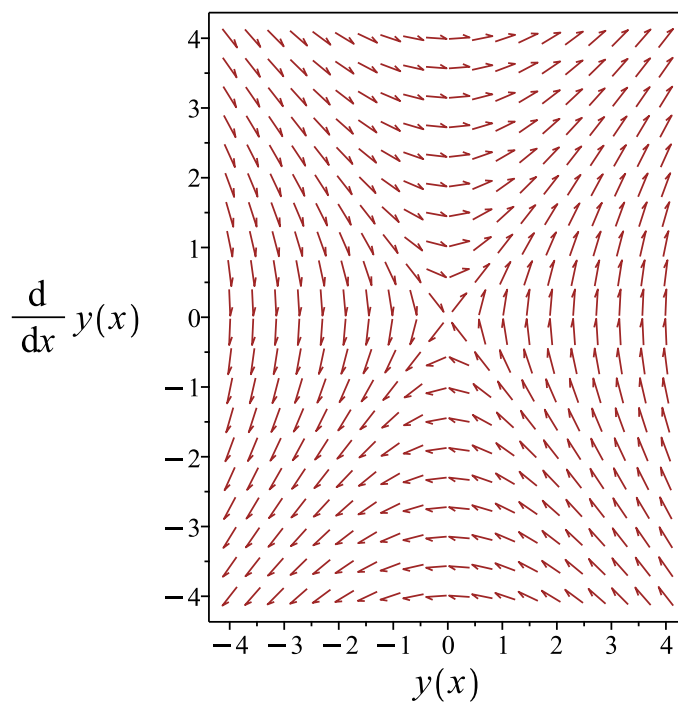


Figure 235: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 168: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= e^{-x}\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$

$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x}) \left(\frac{e^x}{2} \right) - \left(\frac{e^x}{2} \right) (-e^{-x})$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^x}{2(1+e^{-x})^2}}{1} dx$$

Which simplifies to

$$u_1 = - \int \frac{e^{3x}}{2(1+e^x)^2} dx$$

Hence

$$u_1 = -\frac{e^x}{2} + \ln(1+e^x) + \frac{1}{2e^x+2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}}{(1+e^{-x})^2} dx$$

Hence

$$u_2 = \frac{1}{1+e^{-x}}$$

Which simplifies to

$$u_1 = \frac{(2e^x+2)\ln(1+e^x) - e^x - e^{2x} + 1}{2e^x+2}$$

$$u_2 = \frac{1}{1+e^{-x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2e^{-x} + 2} + \frac{((2e^x + 2) \ln(1 + e^x) - e^x - e^{2x} + 1)e^{-x}}{2e^x + 2}$$

Which simplifies to

$$y_p(x) = \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(\frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \quad (1)$$

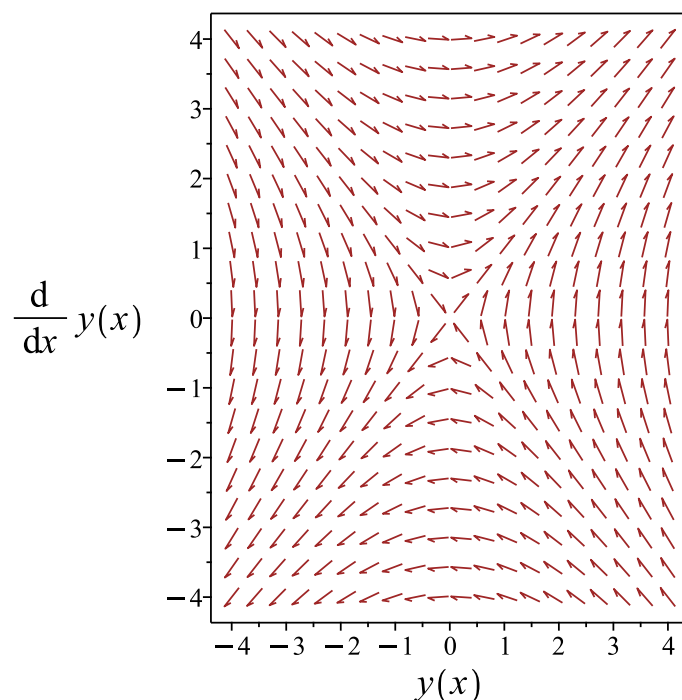


Figure 236: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$$y'' - y = \frac{1}{(1+e^{-x})^2}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \frac{1}{(1+e^{-x})^2} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x} \left(\int \frac{e^{3x}}{(1+e^x)^2} dx \right)}{2} + \frac{e^x \left(\int \frac{e^{-x}}{(1+e^{-x})^2} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{(2e^{-x}+2) \ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{(2e^{-x}+2) \ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(y(x), x$2) - y(x) = (1+exp(-x))^-2, y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 + \frac{e^x}{2} - 1 + e^{-x} \ln(e^x + 1) + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 42

```
DSolve[y''[x]-y[x]==(1+Exp[-x])^(-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}e^{-x}(-2e^x + 2\log(e^x + 1) + 2c_1e^{2x} + 1 + 2c_2)$$

9.11 problem 21

- 9.11.1 Solving as second order linear constant coeff ode 1455
- 9.11.2 Solving using Kovacic algorithm 1459
- 9.11.3 Maple step by step solution 1465

Internal problem ID [5378]

Internal file name [OUTPUT/4869_Sunday_February_04_2024_12_46_38_AM_25923686/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 21.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + y = \csc(x)$$

9.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$

$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_1 = - \int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2\sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x) \quad (1)$$

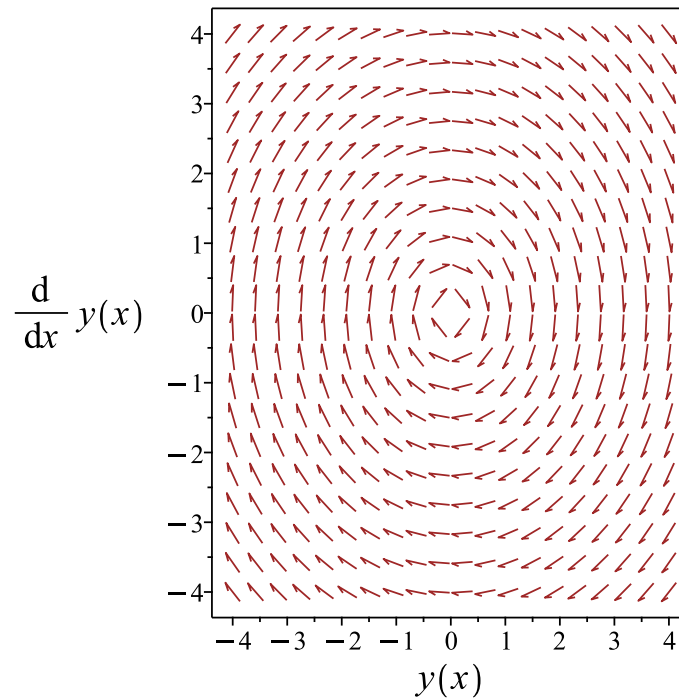


Figure 237: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 0 \quad (3)$$

$$C = 1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= \cos(x)
 \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_1 = - \int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2 \sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)c_1 + c_2 \sin(x) - \cos(x)x + \ln(\sin(x))\sin(x) \quad (1)$$

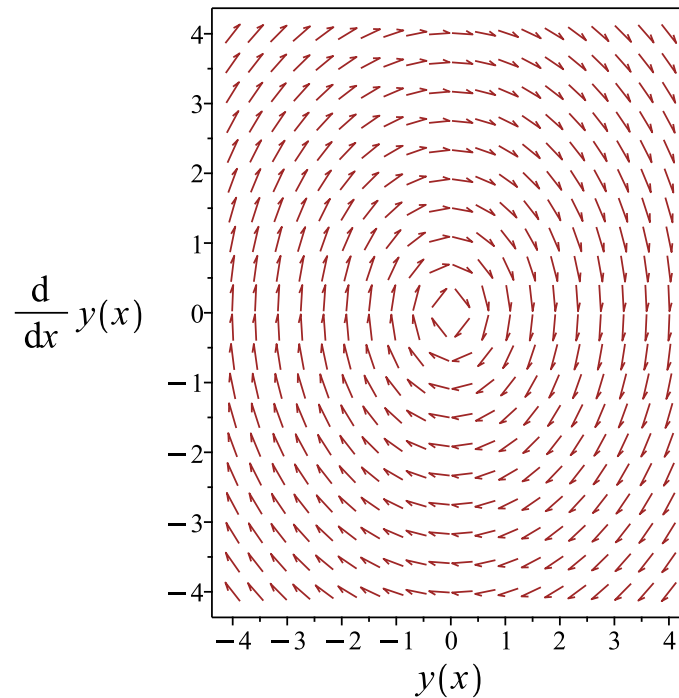


Figure 238: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.11.3 Maple step by step solution

Let's solve

$$y'' + y = \csc(x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-4})}{2}$$

- Roots of the characteristic polynomial

$$r = (-I, I)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \csc(x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 1$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x) \left(\int 1 dx \right) + \sin(x) \left(\int \cot(x) dx \right)$$

- Compute integrals

$$y_p(x) = -\cos(x) x + \ln(\sin(x)) \sin(x)$$

- Substitute particular solution into general solution to ODE

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=csc(x),y(x), singsol=all)
```

$$y(x) = -\ln(\csc(x)) \sin(x) + (-x + c_1) \cos(x) + \sin(x) c_2$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 86

```
DSolve[y''[x]-y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \left(\frac{1}{2} + \frac{i}{2}\right) e^{ix} \left(\text{Hypergeometric2F1} \left(\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, e^{2ix} \right) \right. \\ \left. + i \text{Hypergeometric2F1} \left(\frac{1}{2} + \frac{i}{2}, 1, \frac{3}{2} + \frac{i}{2}, e^{2ix} \right) \right) + c_1 e^x + c_2 e^{-x}$$

9.12 problem 22

9.12.1 Solving as second order linear constant coeff ode	1468
9.12.2 Solving using Kovacic algorithm	1473
9.12.3 Maple step by step solution	1479

Internal problem ID [5379]

Internal file name [OUTPUT/4870_Sunday_February_04_2024_12_46_38_AM_28670253/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 14. Linear equations with constant coefficients. Supplementary problems. Page 92

Problem number: 22.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 3y' + 2y = \sin(e^{-x})$$

9.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = \sin(e^{-x})$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} + 2e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$

$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^x$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$

$$y_2 = e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^x \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^x \\ 2e^{2x} & e^x \end{vmatrix}$$

Therefore

$$W = (e^{2x})(e^x) - (e^x)(2e^{2x})$$

Which simplifies to

$$W = -e^{2x} e^x$$

Which simplifies to

$$W = -e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^x \sin(e^{-x})}{-e^{3x}} dx$$

Which simplifies to

$$u_1 = - \int -\sin(e^{-x}) e^{-2x} dx$$

Hence

$$u_1 = - \frac{e^{-x} \tan\left(\frac{e^{-x}}{2}\right)^2 - e^{-x} + 2 \tan\left(\frac{e^{-x}}{2}\right)}{1 + \tan\left(\frac{e^{-x}}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \sin(e^{-x})}{-e^{3x}} dx$$

Which simplifies to

$$u_2 = \int -e^{-x} \sin(e^{-x}) dx$$

Hence

$$u_2 = -\cos(e^{-x})$$

Which simplifies to

$$u_1 = \cos(e^{-x}) e^{-x} - \sin(e^{-x})$$

$$u_2 = -\cos(e^{-x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\cos(e^{-x}) e^{-x} - \sin(e^{-x})) e^{2x} - \cos(e^{-x}) e^x$$

Which simplifies to

$$y_p(x) = -e^{2x} \sin(e^{-x})$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^x) + (-e^{2x} \sin(e^{-x})) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x - e^{2x} \sin(e^{-x}) \quad (1)$$

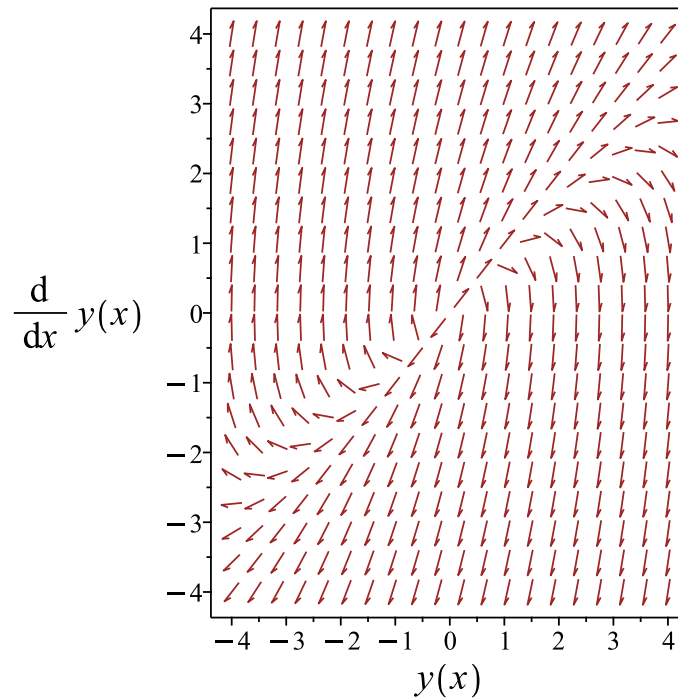


Figure 239: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^x - e^{2x} \sin(e^{-x})$$

Verified OK.

9.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = -3 \quad (3)$$

$$C = 2$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} \quad (5)$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \quad (6)$$

Comparing the above to (5) shows that

$$s = 1$$

$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} dx} \end{aligned}$$

$$\begin{aligned}
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(e^{\frac{3x}{2}} \right)
\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2} dx \\
&= y_1(e^x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2(e^x(e^x))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + c_2 e^{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$

$$y_2 = e^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{2x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^x)(2e^{2x}) - (e^{2x})(e^x)$$

Which simplifies to

$$W = e^{2x}e^x$$

Which simplifies to

$$W = e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{2x} \sin(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_1 = - \int e^{-x} \sin(e^{-x}) dx$$

Hence

$$u_1 = - \cos(e^{-x})$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x \sin(e^{-x})}{e^{3x}} dx$$

Which simplifies to

$$u_2 = \int \sin(e^{-x}) e^{-2x} dx$$

Hence

$$u_2 = \frac{-e^{-x} \tan\left(\frac{e^{-x}}{2}\right)^2 + e^{-x} - 2 \tan\left(\frac{e^{-x}}{2}\right)}{1 + \tan\left(\frac{e^{-x}}{2}\right)^2}$$

Which simplifies to

$$\begin{aligned} u_1 &= - \cos(e^{-x}) \\ u_2 &= \cos(e^{-x}) e^{-x} - \sin(e^{-x}) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (\cos(e^{-x}) e^{-x} - \sin(e^{-x})) e^{2x} - \cos(e^{-x}) e^x$$

Which simplifies to

$$y_p(x) = -e^{2x} \sin(e^{-x})$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{2x}) + (-e^{2x} \sin(e^{-x}))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin(e^{-x}) \quad (1)$$

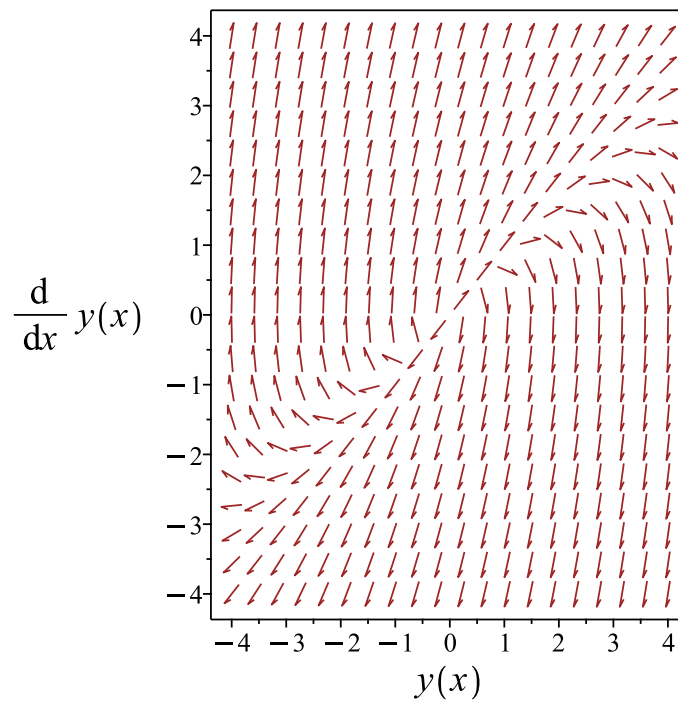


Figure 240: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin(e^{-x})$$

Verified OK.

9.12.3 Maple step by step solution

Let's solve

$$y'' - 3y' + 2y = \sin(e^{-x})$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 3r + 2 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 2) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 2)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^x$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{2x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \sin(e^{-x}) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{3x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^x \left(\int e^{-x} \sin(e^{-x}) dx \right) + e^{2x} \left(\int \sin(e^{-x}) e^{-2x} dx \right)$$

- Compute integrals

$$y_p(x) = -e^{2x} \sin(e^{-x})$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin(e^{-x})$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=sin(exp(-x)),y(x), singsol=all)
```

$$y(x) = (e^x c_1 - e^x \sin(e^{-x}) + c_2) e^x$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Sin[Exp[-x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x (-e^x \sin(e^{-x}) + c_2 e^x + c_1)$$

10 Chapter 15. Linear equations with constant coefficients (Variation of parameters).

Supplementary problems. Page 98

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10.1 problem 10

10.1.1 Solving as second order linear constant coeff ode	1482
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10.1.3 Maple step by step solution	1492

Internal problem ID [5380]

Internal file name [OUTPUT/4871_Sunday_February_04_2024_12_46_39_AM_21402703/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 10.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + y = \csc(x)$$

10.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$

$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_1 = - \int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2\sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x) \quad (1)$$

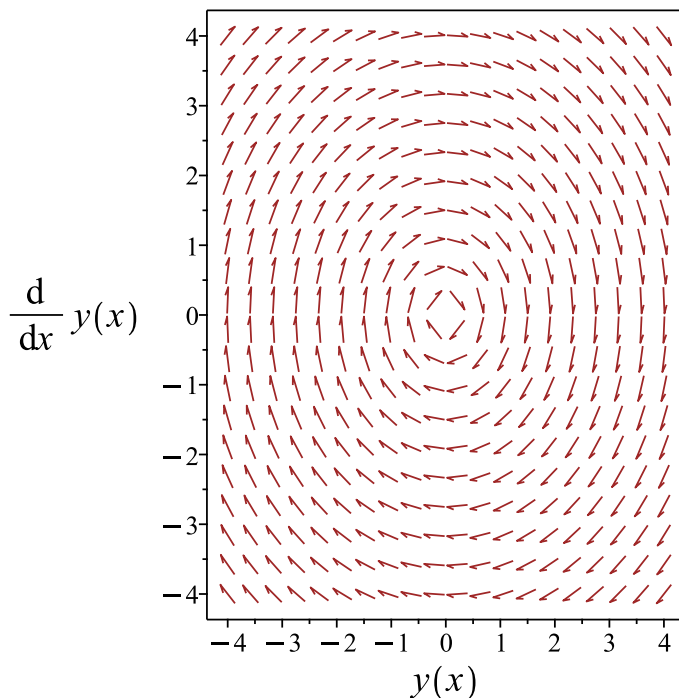


Figure 241: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 174: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= \cos(x)
 \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$

$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_1 = - \int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x) \csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)c_1 + c_2 \sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)c_1 + c_2 \sin(x) - \cos(x)x + \ln(\sin(x))\sin(x) \quad (1)$$

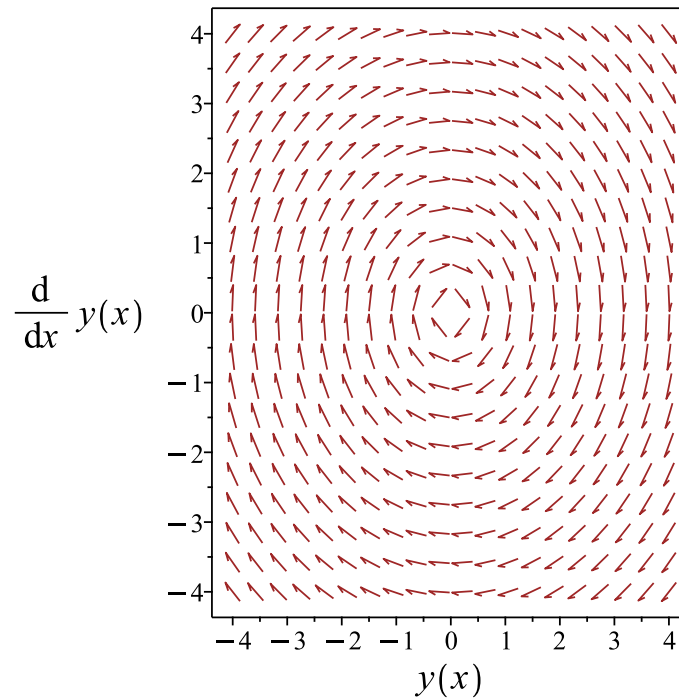


Figure 242: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$$y'' + y = \csc(x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-4})}{2}$$

- Roots of the characteristic polynomial

$$r = (-I, I)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \csc(x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 1$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x) \left(\int 1 dx \right) + \sin(x) \left(\int \cot(x) dx \right)$$

- Compute integrals

$$y_p(x) = -\cos(x) x + \ln(\sin(x)) \sin(x)$$

- Substitute particular solution into general solution to ODE

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=csc(x),y(x), singsol=all)
```

$$y(x) = -\ln(\csc(x)) \sin(x) + (-x + c_1) \cos(x) + \sin(x) c_2$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (-x + c_1) \cos(x) + \sin(x)(\log(\sin(x)) + c_2)$$

10.2 problem 11

10.2.1 Solving as second order linear constant coeff ode	1495
10.2.2 Solving using Kovacic algorithm	1500
10.2.3 Maple step by step solution	1506

Internal problem ID [5381]

Internal file name [OUTPUT/4872_Sunday_February_04_2024_12_46_40_AM_16041237/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 11.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 4y = 4 \sec(x)^2$$

10.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 4 \sec(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +2i \\ \lambda_2 &= -2i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 2i \\ \lambda_2 &= -2i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$

$$y_2 = \sin(2x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2 \sin(2x) & 2 \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(2 \cos(2x)) - (\sin(2x))(-2 \sin(2x))$$

Which simplifies to

$$W = 2 \cos(2x)^2 + 2 \sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{4 \sin(2x) \sec(x)^2}{2} dx$$

Which simplifies to

$$u_1 = - \int 4 \tan(x) dx$$

Hence

$$u_1 = 4 \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 \cos(2x) \sec(x)^2}{2} dx$$

Which simplifies to

$$u_2 = \int 2 \cos(2x) \sec(x)^2 dx$$

Hence

$$u_2 = 4x - 2 \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 4 \ln(\cos(x)) \cos(2x) + (4x - 2 \tan(x)) \sin(2x)$$

Which simplifies to

$$y_p(x) = (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= (c_1 \cos(2x) + c_2 \sin(2x)) + ((8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2 \quad (1)$$

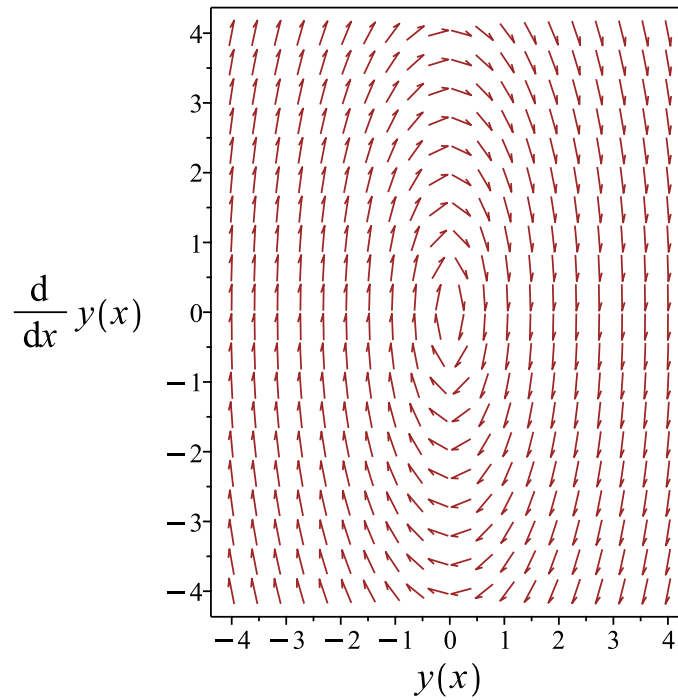


Figure 243: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

Verified OK.

10.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2} dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2 \left(\cos(2x) \left(\frac{\tan(2x)}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$

$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2 \sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{2 \sin(2x) \sec(x)^2}{1} dx$$

Which simplifies to

$$u_1 = - \int 4 \tan(x) dx$$

Hence

$$u_1 = 4 \ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 \cos(2x) \sec(x)^2}{1} dx$$

Which simplifies to

$$u_2 = \int 4 \cos(2x) \sec(x)^2 dx$$

Hence

$$u_2 = 8x - 4 \tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 4 \ln(\cos(x)) \cos(2x) + \frac{(8x - 4 \tan(x)) \sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \right) + ((8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2 \quad (1)$$

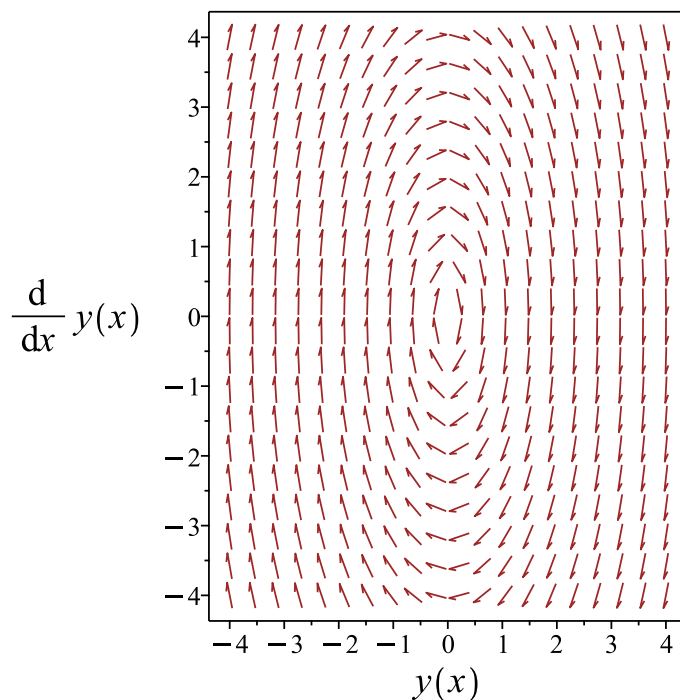


Figure 244: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

Verified OK.

10.2.3 Maple step by step solution

Let's solve

$$y'' + 4y = 4 \sec(x)^2$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 4 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-16})}{2}$$

- Roots of the characteristic polynomial

$$r = (-2i, 2i)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(2x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = 4 \sec(x)^2 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2 \sin(2x) & 2 \cos(2x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -4 \cos(2x) \left(\int \tan(x) dx \right) + 2 \sin(2x) \left(\int \cos(2x) \sec(x)^2 dx \right)$$

- Compute integrals

$$y_p(x) = (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + (8 \cos(x)^2 - 4) \ln(\cos(x)) + 8 \cos(x) \sin(x) x - 4 \sin(x)^2$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$2)+4*y(x)=4*sec(x)^2,y(x), singsol=all)
```

$$y(x) = (-8 \cos(x)^2 + 4) \ln(\sec(x)) + 2c_1 \cos(x)^2 + 8 \left(x + \frac{c_2}{4} \right) \sin(x) \cos(x) - 4 \sin(x)^2 - c_1$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 44

```
DSolve[y''[x]+4*y[x]==4*Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 2 \sin(2x) \arctan(\tan(x)) + 2x \sin(2x) + c_2 \sin(2x) + \cos(2x)(4 \log(\cos(x)) + 2 + c_1) - 2$$

10.3 problem 12

10.3.1 Solving as second order linear constant coeff ode	1508
10.3.2 Solving using Kovacic algorithm	1512
10.3.3 Maple step by step solution	1518

Internal problem ID [5382]

Internal file name [OUTPUT/4873_Sunday_February_04_2024_12_46_40_AM_96331104/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 12.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 4y' + 3y = \frac{1}{1 + e^{-x}}$$

10.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = \frac{1}{1+e^{-x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 3e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(3)} \\ &= 2 \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$

$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^x$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{3x}$$

$$y_2 = e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & e^x \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}(e^x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & e^x \\ 3e^{3x} & e^x \end{vmatrix}$$

Therefore

$$W = (e^{3x})(e^x) - (e^x)(3e^{3x})$$

Which simplifies to

$$W = -2e^{3x}e^x$$

Which simplifies to

$$W = -2 e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^x}{1+e^{-x}}}{-2 e^{4x}} dx$$

Which simplifies to

$$u_1 = - \int -\frac{e^{-3x}}{2 e^{-x} + 2} dx$$

Hence

$$u_1 = -\frac{\ln(1 + e^x)}{2} - \frac{e^{-2x}}{4} + \frac{\ln(e^x)}{2} + \frac{e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{3x}}{1+e^{-x}}}{-2 e^{4x}} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{-x}}{2 e^{-x} + 2} dx$$

Hence

$$u_2 = \frac{\ln(1 + e^{-x})}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(1 + e^x)}{2} - \frac{e^{-2x}}{4} + \frac{\ln(e^x)}{2} + \frac{e^{-x}}{2} \right) e^{3x} + \frac{e^x \ln(1 + e^{-x})}{2}$$

Which simplifies to

$$y_p(x) = -\frac{e^x(2 \ln(1 + e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1 + e^{-x}) + 1)}{4}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= (e^{3x}c_1 + c_2e^x) + \left(-\frac{e^x(2\ln(1+e^x)e^{2x} - 2\ln(e^x)e^{2x} - 2e^x - 2\ln(1+e^{-x}) + 1)}{4} \right)$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2e^x - \frac{e^x(2\ln(1+e^x)e^{2x} - 2\ln(e^x)e^{2x} - 2e^x - 2\ln(1+e^{-x}) + 1)}{4} \quad (1)$$

Verification of solutions

$$y = e^{3x}c_1 + c_2e^x - \frac{e^x(2\ln(1+e^x)e^{2x} - 2\ln(e^x)e^{2x} - 2e^x - 2\ln(1+e^{-x}) + 1)}{4}$$

Verified OK.

10.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = -4$$

$$C = 3 \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} \quad (5)$$

$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x}) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + \frac{c_2 e^{3x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= e^x \\
y_2 &= \frac{e^{3x}}{2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{e^{3x}}{2} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{e^{3x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{e^{3x}}{2} \\ e^x & \frac{3e^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = (e^x) \left(\frac{3e^{3x}}{2} \right) - \left(\frac{e^{3x}}{2} \right) (e^x)$$

Which simplifies to

$$W = e^{3x} e^x$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^{3x}}{2e^{-x}+2}}{e^{4x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{e^{-x}}{2e^{-x} + 2} dx$$

Hence

$$u_1 = \frac{\ln(1 + e^{-x})}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^x}{1+e^{-x}}}{e^{4x}} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-3x}}{1 + e^{-x}} dx$$

Hence

$$u_2 = -\ln(1 + e^x) - \frac{e^{-2x}}{2} + \ln(e^x) + e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x \ln(1 + e^{-x})}{2} + \frac{\left(-\ln(1 + e^x) - \frac{e^{-2x}}{2} + \ln(e^x) + e^{-x}\right) e^{3x}}{2}$$

Which simplifies to

$$y_p(x) = -\frac{e^x(2 \ln(1 + e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1 + e^{-x}) + 1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^{3x}}{2}\right) + \left(-\frac{e^x(2 \ln(1 + e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1 + e^{-x}) + 1)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} - \frac{e^x(2 \ln(1 + e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1 + e^{-x}) + 1)}{4} \quad (1)$$

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} - \frac{e^x(2 \ln(1 + e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1 + e^{-x}) + 1)}{4}$$

Verified OK.

10.3.3 Maple step by step solution

Let's solve

$$y'' - 4y' + 3y = \frac{1}{1+e^{-x}}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r + 3 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r - 3) = 0$$

- Roots of the characteristic polynomial

$$r = (1, 3)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^x$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{3x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^x + c_2 e^{3x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \frac{1}{1+e^{-x}} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^x \left(\int \frac{e^{-x}}{1+e^{-x}} dx \right)}{2} + \frac{e^{3x} \left(\int \frac{e^{-3x}}{1+e^{-x}} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{e^x (2 \ln(1+e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1+e^{-x}) + 1)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{3x} - \frac{e^x (2 \ln(1+e^x) e^{2x} - 2 \ln(e^x) e^{2x} - 2 e^x - 2 \ln(1+e^{-x}) + 1)}{4}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=1/(1+exp(-x)),y(x), singsol=all)
```

$$y(x) = \frac{\ln(1+e^{-x})e^x}{2} - \frac{e^{3x} \ln(e^x+1)}{2} + \frac{(4c_1+2 \ln(e^x))e^{3x}}{4} + \frac{e^{2x}}{2} + \frac{(4c_2-1)e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 49

```
DSolve[y''[x]-4*y'[x]+3*y[x]==1/(1+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{4} e^x (-4(e^{2x}-1) \operatorname{arctanh}(2e^x+1) + 2e^x + 4c_2 e^{2x} - 1 + 4c_1)$$

10.4 problem 13

10.4.1 Solving as second order linear constant coeff ode	1520
10.4.2 Solving using Kovacic algorithm	1525
10.4.3 Maple step by step solution	1531

Internal problem ID [5383]

Internal file name [OUTPUT/4874_Sunday_February_04_2024_12_46_41_AM_90539611/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 13.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$$

10.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)(-e^{-x}) - (e^{-x})(e^x)$$

Which simplifies to

$$W = -2e^{-x}e^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{-x}(e^{-x} \sin(e^{-x}) + \cos(e^{-x}))}{-2} dx$$

Which simplifies to

$$u_1 = - \int - \frac{e^{-x}(e^{-x} \sin(e^{-x}) + \cos(e^{-x}))}{2} dx$$

Hence

$$u_1 = - \sin(e^{-x}) + \frac{\cos(e^{-x}) e^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x(e^{-x} \sin(e^{-x}) + \cos(e^{-x}))}{-2} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{\cos(e^{-x}) e^x}{2} - \frac{\sin(e^{-x})}{2} \right) dx$$

Hence

$$u_2 = -\frac{\cos(e^{-x}) e^x}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin(e^{-x}) + \frac{\cos(e^{-x}) e^{-x}}{2} \right) e^x - \frac{\cos(e^{-x}) e^x e^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -e^x \sin(e^{-x})$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + (-e^x \sin(e^{-x}))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x}) \quad (1)$$

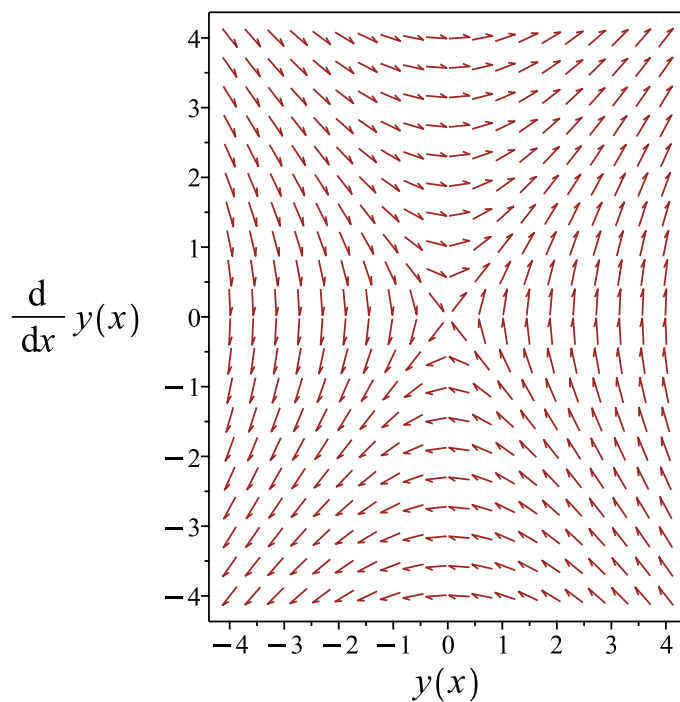


Figure 245: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x})$$

Verified OK.

10.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= e^{-x}\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$
$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x}) \left(\frac{e^x}{2} \right) - \left(\frac{e^x}{2} \right) (-e^{-x})$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^x (e^{-x} \sin(e^{-x}) + \cos(e^{-x}))}{\frac{2}{1}} dx$$

Which simplifies to

$$u_1 = - \int \left(\frac{\cos(e^{-x}) e^x}{2} + \frac{\sin(e^{-x})}{2} \right) dx$$

Hence

$$u_1 = - \frac{\cos(e^{-x}) e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} (e^{-x} \sin(e^{-x}) + \cos(e^{-x}))}{1} dx$$

Which simplifies to

$$u_2 = \int e^{-x} (e^{-x} \sin(e^{-x}) + \cos(e^{-x})) dx$$

Hence

$$u_2 = \cos(e^{-x}) e^{-x} - 2 \sin(e^{-x})$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \frac{\cos(e^{-x}) e^x e^{-x}}{2} + \frac{(\cos(e^{-x}) e^{-x} - 2 \sin(e^{-x})) e^x}{2}$$

Which simplifies to

$$y_p(x) = -e^x \sin(e^{-x})$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + (-e^x \sin(e^{-x})) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - e^x \sin(e^{-x}) \quad (1)$$

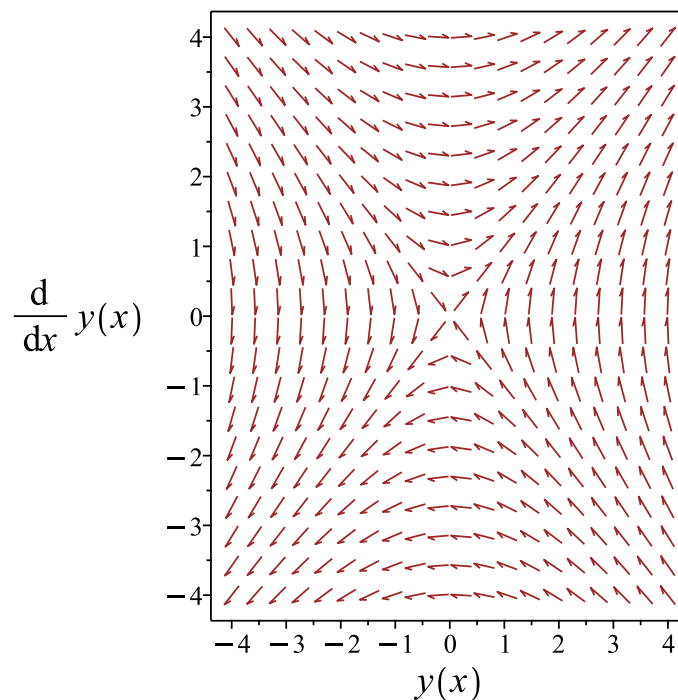


Figure 246: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - e^x \sin(e^{-x})$$

Verified OK.

10.4.3 Maple step by step solution

Let's solve

$$y'' - y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right) \right], f(x) = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}(\int(\cos(e^{-x})e^x + \sin(e^{-x}))dx)}{2} + \frac{e^x(\int e^{-x}(e^{-x}\sin(e^{-x}) + \cos(e^{-x}))dx)}{2}$$

- Compute integrals

$$y_p(x) = -e^x \sin(e^{-x})$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x - e^x \sin(e^{-x})$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-y(x)=exp(-x)*sin(exp(-x))+cos(exp(-x)),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 - e^x \sin(e^{-x})$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 31

```
DSolve[y''[x]-y[x]==Exp[-x]*Sin[Exp[-x]]+Cos[Exp[-x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -e^x \sin(e^{-x}) + c_1 e^x + c_2 e^{-x}$$

10.5 problem 14

10.5.1 Solving as second order linear constant coeff ode	1533
10.5.2 Solving using Kovacic algorithm	1538
10.5.3 Maple step by step solution	1544

Internal problem ID [5384]

Internal file name [OUTPUT/4875_Sunday_February_04_2024_12_46_42_AM_81101878/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 14.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = \frac{1}{(1 + e^{-x})^2}$$

10.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \frac{1}{(1+e^{-x})^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$

$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)(-e^{-x}) - (e^{-x})(e^x)$$

Which simplifies to

$$W = -2e^{-x}e^x$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{-2} dx$$

Which simplifies to

$$u_1 = - \int -\frac{e^{-x}}{2(1+e^{-x})^2} dx$$

Hence

$$u_1 = \frac{1}{2e^{-x} + 2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^x}{(1+e^{-x})^2}}{-2} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{3x}}{2(1+e^x)^2} dx$$

Hence

$$u_2 = -\frac{e^x}{2} + \ln(1+e^x) + \frac{1}{2e^x + 2}$$

Which simplifies to

$$u_1 = \frac{1}{2e^{-x} + 2}$$
$$u_2 = \frac{(2e^x + 2) \ln(1+e^x) - e^x - e^{2x} + 1}{2e^x + 2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2e^{-x} + 2} + \frac{((2e^x + 2) \ln(1+e^x) - e^x - e^{2x} + 1) e^{-x}}{2e^x + 2}$$

Which simplifies to

$$y_p(x) = \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(\frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \quad (1)$$

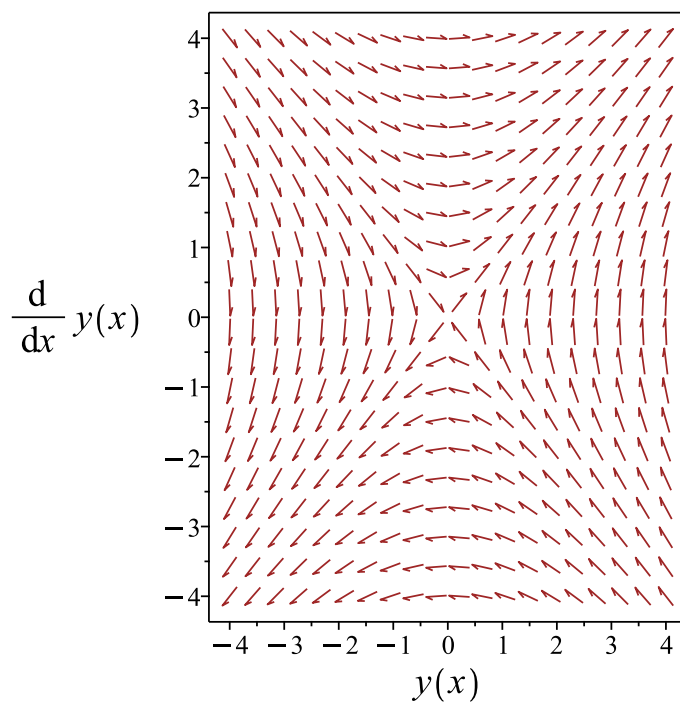


Figure 247: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Verified OK.

10.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= e^{-x}\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$

$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x}) \left(\frac{e^x}{2} \right) - \left(\frac{e^x}{2} \right) (-e^{-x})$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{e^x}{2(1+e^{-x})^2}}{1} dx$$

Which simplifies to

$$u_1 = - \int \frac{e^{3x}}{2(1+e^x)^2} dx$$

Hence

$$u_1 = -\frac{e^x}{2} + \ln(1+e^x) + \frac{1}{2e^x+2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-x}}{(1+e^{-x})^2} dx$$

Hence

$$u_2 = \frac{1}{1+e^{-x}}$$

Which simplifies to

$$u_1 = \frac{(2e^x+2)\ln(1+e^x) - e^x - e^{2x} + 1}{2e^x+2}$$

$$u_2 = \frac{1}{1+e^{-x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2e^{-x} + 2} + \frac{((2e^x + 2) \ln(1 + e^x) - e^x - e^{2x} + 1)e^{-x}}{2e^x + 2}$$

Which simplifies to

$$y_p(x) = \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(\frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2} \quad (1)$$

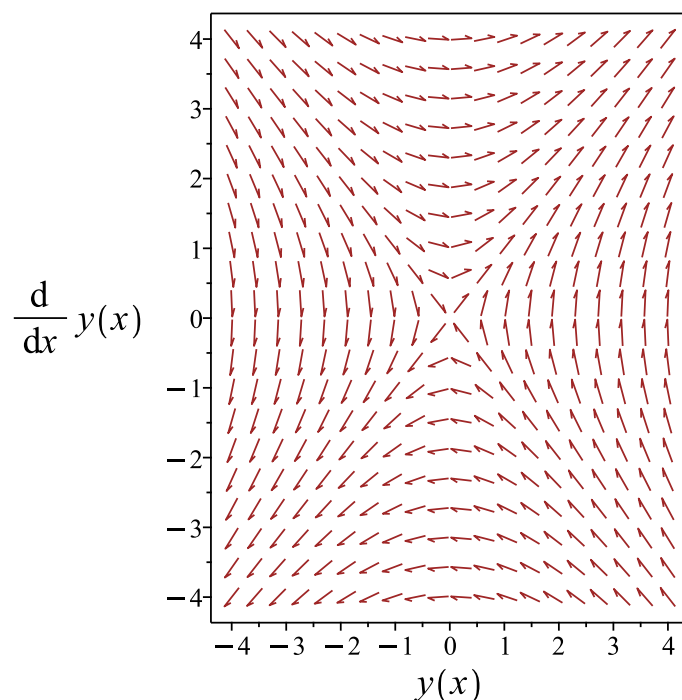


Figure 248: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2e^{-x} + 2) \ln(1 + e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Verified OK.

10.5.3 Maple step by step solution

Let's solve

$$y'' - y = \frac{1}{(1+e^{-x})^2}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \frac{1}{(1+e^{-x})^2} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x} \left(\int \frac{e^{3x}}{(1+e^x)^2} dx \right)}{2} + \frac{e^x \left(\int \frac{e^{-x}}{(1+e^{-x})^2} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{(2e^{-x}+2) \ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{(2e^{-x}+2) \ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2e^x + 2}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-y(x)=1/(1+exp(-x))^2,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 + \frac{e^x}{2} - 1 + e^{-x} \ln(e^x + 1) + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 42

```
DSolve[y''[x]-y[x]==1/(1+Exp[-x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}e^{-x}(-2e^x + 2\log(e^x + 1) + 2c_1e^{2x} + 1 + 2c_2)$$

10.6 problem 15

10.6.1 Solving as second order linear constant coeff ode	1547
10.6.2 Solving using Kovacic algorithm	1550
10.6.3 Maple step by step solution	1555

Internal problem ID [5385]

Internal file name [OUTPUT/4876_Sunday_February_04_2024_12_46_42_AM_79064302/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 15.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' + 2y = e^x + 2$$

10.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = e^x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}))$$

Or

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + 2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x\sqrt{2}), \sin(x\sqrt{2})\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2 e^x + 2A_1 = e^x + 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{3} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = 1 + \frac{e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) \right) + \left(1 + \frac{e^x}{3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + 1 + \frac{e^x}{3} \quad (1)$$

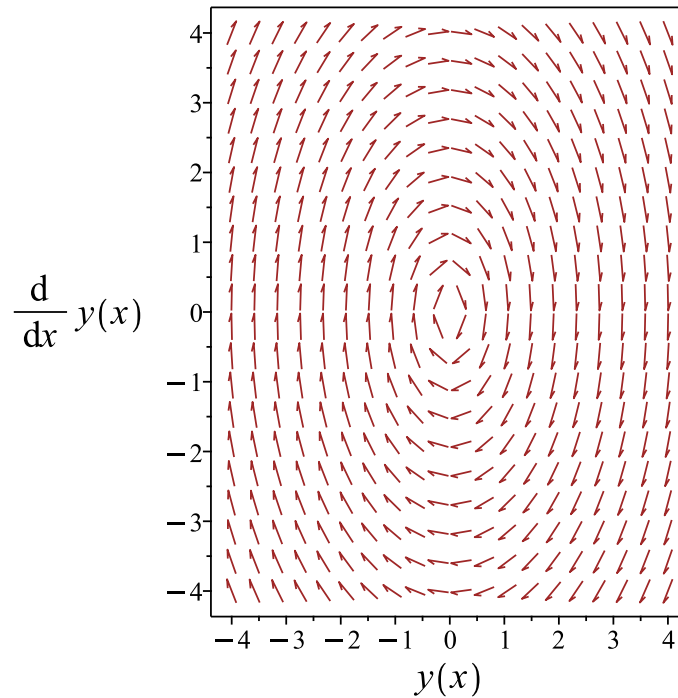


Figure 249: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + 1 + \frac{e^x}{3}$$

Verified OK.

10.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= 2\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -2 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x\sqrt{2})$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= \cos(x\sqrt{2})
 \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x\sqrt{2})$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x\sqrt{2}) \int \frac{1}{\cos^2(x\sqrt{2})} dx \\ &= \cos(x\sqrt{2}) \left(\frac{\sqrt{2} \tan(x\sqrt{2})}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\cos(x\sqrt{2}) \right) + c_2 \left(\cos(x\sqrt{2}) \left(\frac{\sqrt{2} \tan(x\sqrt{2})}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x\sqrt{2}) + \frac{c_2 \sqrt{2} \sin(x\sqrt{2})}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + 2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sqrt{2} \sin(x\sqrt{2})}{2}, \cos(x\sqrt{2}) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2 e^x + 2A_1 = e^x + 2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{3} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = 1 + \frac{e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(x\sqrt{2}) + \frac{c_2 \sqrt{2} \sin(x\sqrt{2})}{2} \right) + \left(1 + \frac{e^x}{3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2} + 1 + \frac{e^x}{3} \quad (1)$$

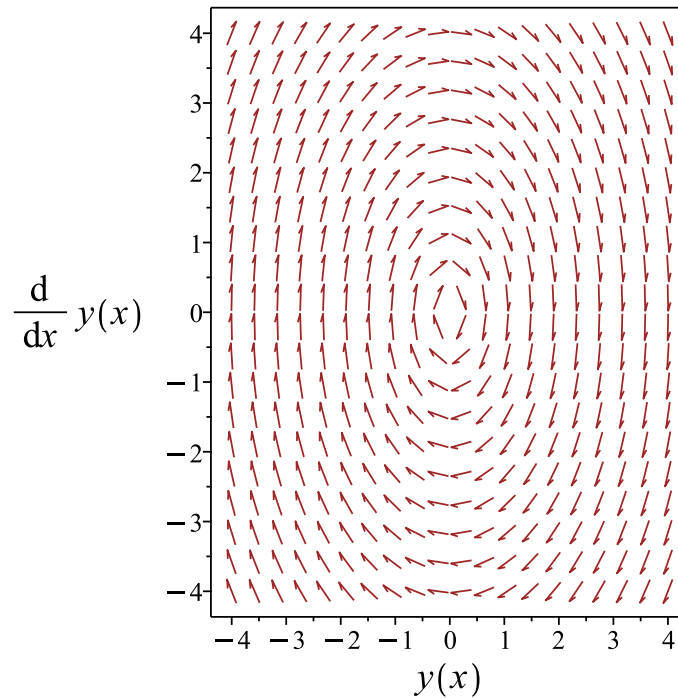


Figure 250: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2} + 1 + \frac{e^x}{3}$$

Verified OK.

10.6.3 Maple step by step solution

Let's solve

$$y'' + 2y = e^x + 2$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 + 2 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-8})}{2}$$

- Roots of the characteristic polynomial

$$r = (-I\sqrt{2}, I\sqrt{2})$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x\sqrt{2})$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x\sqrt{2})$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right), f(x) = e^x + 2 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x\sqrt{2}) & \sin(x\sqrt{2}) \\ -\sqrt{2} \sin(x\sqrt{2}) & \sqrt{2} \cos(x\sqrt{2}) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \sqrt{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\sqrt{2} (\cos(x\sqrt{2}) (\int (e^x + 2) \sin(x\sqrt{2}) dx) - \sin(x\sqrt{2}) (\int (e^x + 2) \cos(x\sqrt{2}) dx))}{2}$$

- Compute integrals

$$y_p(x) = 1 + \frac{e^x}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + 1 + \frac{e^x}{3}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+2*y(x)=2+exp(x),y(x), singsol=all)
```

$$y(x) = \sin(\sqrt{2}x) c_2 + \cos(\sqrt{2}x) c_1 + 1 + \frac{e^x}{3}$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 36

```
DSolve[y''[x]+2*y[x]==2+Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{e^x}{3} + c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x) + 1$$

10.7 problem 16

10.7.1 Solving as second order linear constant coeff ode	1558
10.7.2 Solving using Kovacic algorithm	1561
10.7.3 Maple step by step solution	1566

Internal problem ID [5386]

Internal file name [OUTPUT/4877_Sunday_February_04_2024_12_46_43_AM_3374812/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = e^x \sin(2x)$$

10.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = e^x \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^x \cos(2x) - 4A_1 e^x \sin(2x) - 4A_2 e^x \sin(2x) + 4A_2 e^x \cos(2x) = e^x \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = -\frac{1}{8} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(-\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - \frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8} \quad (1)$$

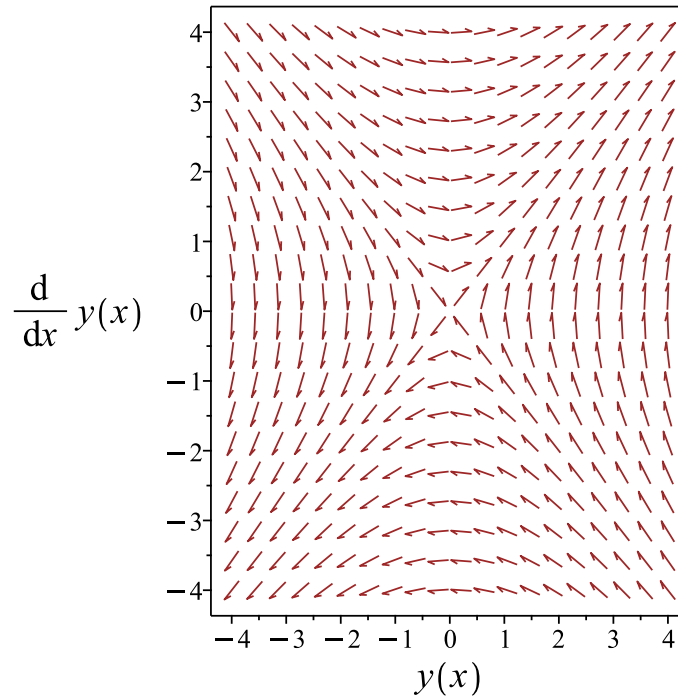


Figure 251: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - \frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}$$

Verified OK.

10.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= -1\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 1 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{2}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^x \cos(2x) - 4A_1 e^x \sin(2x) - 4A_2 e^x \sin(2x) + 4A_2 e^x \cos(2x) = e^x \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = -\frac{1}{8} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(-\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8} \quad (1)$$

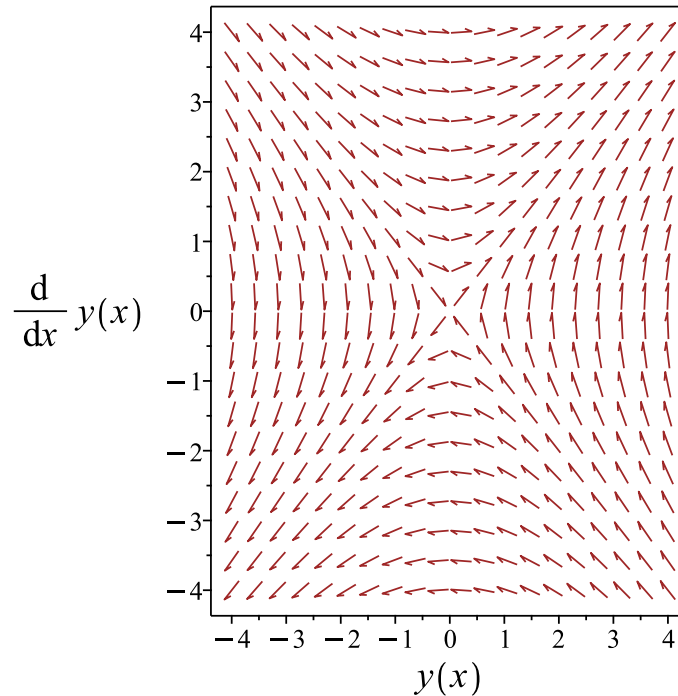


Figure 252: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - \frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}$$

Verified OK.

10.7.3 Maple step by step solution

Let's solve

$$y'' - y = e^x \sin(2x)$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^x \sin(2x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x} \left(\int e^{2x} \sin(2x) dx \right)}{2} + \frac{e^x \left(\int \sin(2x) dx \right)}{2}$$

- Compute integrals

$$y_p(x) = -\frac{e^x (\sin(2x) + \cos(2x))}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x - \frac{e^x (\sin(2x) + \cos(2x))}{8}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-y(x)=exp(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^x \cos(2x)}{8} + c_2 e^{-x} + e^x \left(c_1 - \frac{\sin(2x)}{8} \right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 37

```
DSolve[y''[x]-y[x]==Exp[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x + c_2 e^{-x} - \frac{1}{8} e^x (\sin(2x) + \cos(2x) + 2)$$

10.8 problem 17

10.8.1 Solving as second order linear constant coeff ode	1569
10.8.2 Solving using Kovacic algorithm	1572
10.8.3 Maple step by step solution	1577

Internal problem ID [5387]

Internal file name [OUTPUT/4878_Sunday_February_04_2024_12_46_43_AM_88184614/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 2y' + 2y = x^2 + \sin(x)$$

10.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = x^2 + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{2^2 - (4)(1)(2)} \\ &= -1 \pm i \end{aligned}$$

Hence

$$\lambda_1 = -1 + i$$

$$\lambda_2 = -1 - i$$

Which simplifies to

$$\lambda_1 = -1 + i$$

$$\lambda_2 = -1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x} (\cos(x) c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x} (\cos(x) c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(x), e^{-x} \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} A_1 \cos(x) + A_2 \sin(x) + 2A_5 - 2A_1 \sin(x) + 2A_2 \cos(x) \\ + 2A_4 + 4A_5 x + 2A_3 + 2A_4 x + 2A_5 x^2 = x^2 + \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}, A_3 = \frac{1}{2}, A_4 = -1, A_5 = \frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{-x}(\cos(x) c_1 + c_2 \sin(x))) + \left(-\frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \quad (1)$$

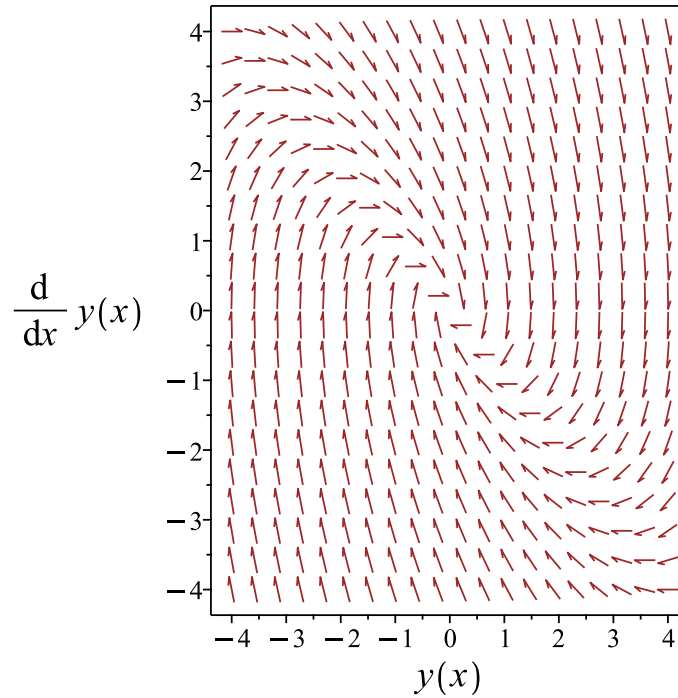


Figure 253: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Verified OK.

10.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 2 \\C &= 2\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -1 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 O(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
 y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
 &= z_1 e^{-\int \frac{1}{2} \frac{2}{1} dx} \\
 &= z_1 e^{-x} \\
 &= z_1 (e^{-x})
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2} dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^{-x} \cos(x)) + c_2(e^{-x} \cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(x), e^{-x} \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} A_1 \cos(x) + A_2 \sin(x) + 2A_5 - 2A_1 \sin(x) + 2A_2 \cos(x) \\ + 2A_4 + 4A_5 x + 2A_3 + 2A_4 x + 2A_5 x^2 = x^2 + \sin(x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}, A_3 = \frac{1}{2}, A_4 = -1, A_5 = \frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x)) + \left(-\frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \right) \end{aligned}$$

Which simplifies to

$$y = e^{-x} (\cos(x) c_1 + c_2 \sin(x)) - \frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \quad (1)$$

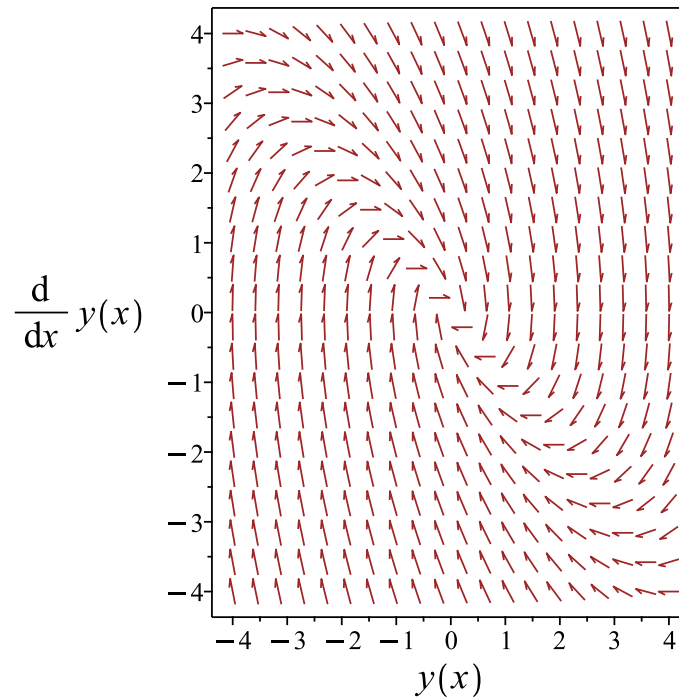


Figure 254: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Verified OK.

10.8.3 Maple step by step solution

Let's solve

$$y'' + 2y' + 2y = x^2 + \sin(x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 2r + 2 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{(-2) \pm (\sqrt{-4})}{2}$$

- Roots of the characteristic polynomial

$$r = (-1 - I, -1 + I)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x} \cos(x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{-x} \sin(x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = x^2 + \sin(x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} \cos(x) & e^{-x} \sin(x) \\ -e^{-x} \cos(x) - e^{-x} \sin(x) & -e^{-x} \sin(x) + e^{-x} \cos(x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{-2x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = e^{-x} (-\cos(x) \left(\int \sin(x) e^x (x^2 + \sin(x)) dx \right) + \sin(x) \left(\int \cos(x) e^x (x^2 + \sin(x)) dx \right))$$

- Compute integrals

$$y_p(x) = -\frac{2 \cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_2 e^{-x} \sin(x) + c_1 e^{-x} \cos(x) + \frac{x^2}{2} + \frac{\sin(x)}{5} - \frac{2 \cos(x)}{5} - x + \frac{1}{2}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=x^2+sin(x),y(x), singsol=all)
```

$$y(x) = e^{-x} \sin(x) c_2 + \cos(x) e^{-x} c_1 + \frac{x^2}{2} - x + \frac{1}{2} + \frac{\sin(x)}{5} - \frac{2 \cos(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 50

```
DSolve[y''[x]+2*y'[x]+2*y[x]==x^2+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{10} e^{-x} (5e^x (x-1)^2 + (-4e^x + 10c_2) \cos(x) + 2(e^x + 5c_1) \sin(x))$$

10.9 problem 18

10.9.1 Solving as second order linear constant coeff ode	1580
10.9.2 Solving using Kovacic algorithm	1583
10.9.3 Maple step by step solution	1588

Internal problem ID [5388]

Internal file name [OUTPUT/4879_Sunday_February_04_2024_12_46_44_AM_76522474/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 9y = x + e^{2x} - \sin(2x)$$

10.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -9, f(x) = x + e^{2x} - \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 9 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-9)} \\ &= \pm 3 \end{aligned}$$

Hence

$$\lambda_1 = +3$$

$$\lambda_2 = -3$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(3)x} + c_2 e^{(-3)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x} - \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{2x} - 13A_4 \cos(2x) - 13A_5 \sin(2x) - 9A_2 - 9A_3 x = x + e^{2x} - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0, A_3 = -\frac{1}{9}, A_4 = 0, A_5 = \frac{1}{13} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{3x} c_1 + c_2 e^{-3x}) + \left(-\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2e^{-3x} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \quad (1)$$

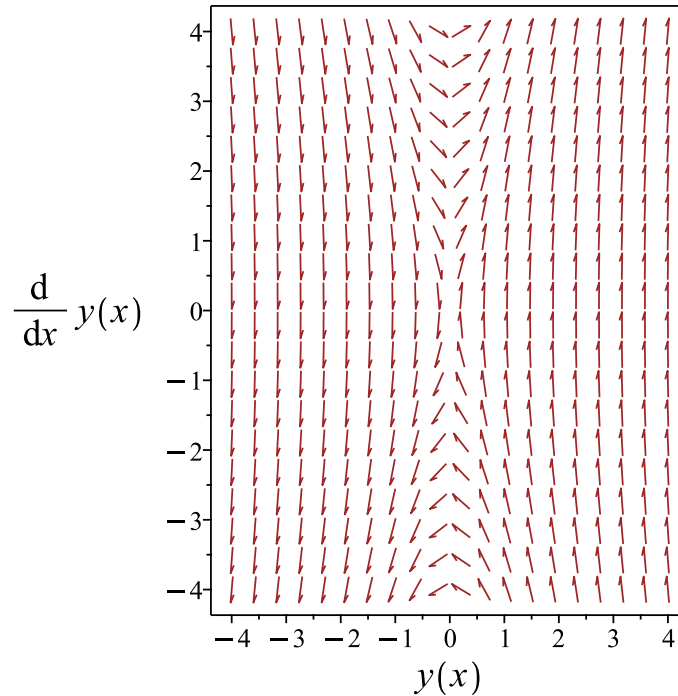


Figure 255: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^{-3x} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Verified OK.

10.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= -9\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 9 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-3x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-3x} \int \frac{1}{e^{-6x}} dx \\ &= e^{-3x} \left(\frac{e^{6x}}{6} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-3x}) + c_2 \left(e^{-3x} \left(\frac{e^{6x}}{6} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' - 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x} - \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^{3x}}{6}, e^{-3x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{2x} - 13A_4 \cos(2x) - 13A_5 \sin(2x) - 9A_2 - 9A_3 x = x + e^{2x} - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0, A_3 = -\frac{1}{9}, A_4 = 0, A_5 = \frac{1}{13} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} \right) + \left(-\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \quad (1)$$

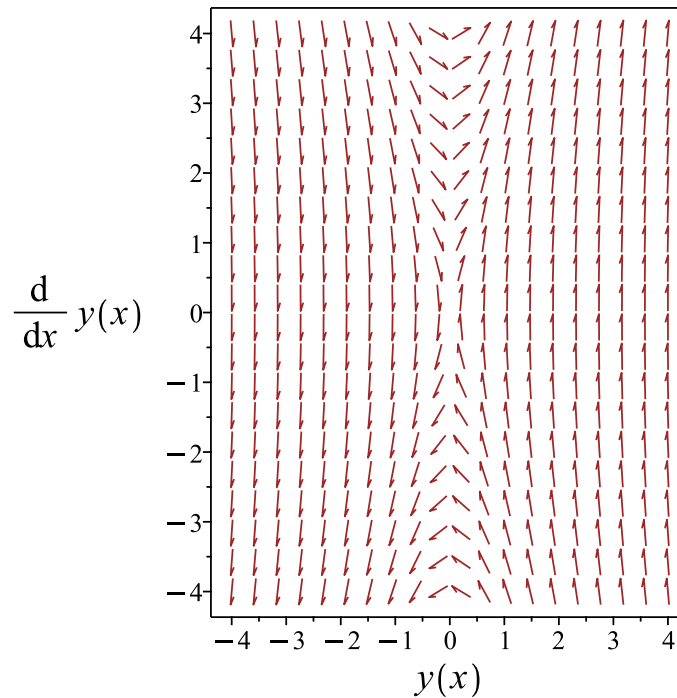


Figure 256: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Verified OK.

10.9.3 Maple step by step solution

Let's solve

$$y'' - 9y = x + e^{2x} - \sin(2x)$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 - 9 = 0$$

- Factor the characteristic polynomial

$$(r - 3)(r + 3) = 0$$

- Roots of the characteristic polynomial

$$r = (-3, 3)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-3x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{3x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-3x} + c_2 e^{3x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right), f(x) = x + e^{2x} - \sin(2x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{3x} \\ -3e^{-3x} & 3e^{3x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 6$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-3x} \left(\int e^{3x} (x + e^{2x} - \sin(2x)) dx \right)}{6} + \frac{e^{3x} \left(\int e^{-3x} (x + e^{2x} - \sin(2x)) dx \right)}{6}$$

- Compute integrals

$$y_p(x) = -\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-3x} + c_2 e^{3x} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)-9*y(x)=x+exp(2*x)-sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{-3x} \left(\left(x - \frac{9 \sin(2x)}{13} \right) e^{3x} - 9 e^{6x} c_1 - 9 c_2 + \frac{9 e^{5x}}{5} \right)}{9}$$

✓ Solution by Mathematica

Time used: 0.846 (sec). Leaf size: 44

```
DSolve[y''[x]-9*y[x]==x+Exp[2*x]-Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{x}{9} - \frac{e^{2x}}{5} + \frac{1}{13} \sin(2x) + c_1 e^{3x} + c_2 e^{-3x}$$

10.10 problem 19

10.10.1 Maple step by step solution 1593

Internal problem ID [5389]

Internal file name [OUTPUT/4880_Sunday_February_04_2024_12_46_45_AM_62370958/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 19.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_y]]
```

$$y''' + 3y'' + 2y' = x^2 + 4x + 8$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 2y' = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$

$$\lambda_2 = -1$$

$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + e^{-2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$

$$y_2 = 1$$

$$y_3 = e^{-2x}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 2y' = x^2 + 4x + 8$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6x^2 A_3 + 4x A_2 + 18x A_3 + 2A_1 + 6A_2 + 6A_3 = x^2 + 4x + 8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{11}{4}, A_2 = \frac{1}{4}, A_3 = \frac{1}{6} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-x} + c_2 + e^{-2x}c_3) + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + c_2 + e^{-2x}c_3 + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} \quad (1)$$

Verification of solutions

$$y = c_1e^{-x} + c_2 + e^{-2x}c_3 + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4}$$

Verified OK.

10.10.1 Maple step by step solution

Let's solve

$$y''' + 3y'' + 2y' = x^2 + 4x + 8$$

- Highest derivative means the order of the ODE is 3

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = x^2 - 3y_3(x) - 2y_2(x) + 4x + 8$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 - 3y_3(x) - 2y_2(x) + 4x + 8]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 + 4x + 8 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 + 4x + 8 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & \frac{e^{-2x}}{2} - 2e^{-x} + \frac{3}{2} & \frac{e^{-2x}}{2} - e^{-x} + \frac{1}{2} \\ 0 & -e^{-2x} + 2e^{-x} & -e^{-2x} + e^{-x} \\ 0 & 2e^{-2x} - 2e^{-x} & 2e^{-2x} - e^{-x} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{11x}{4} + \frac{x^2}{4} + \frac{x^3}{6} + 6e^{-x} - \frac{13e^{-2x}}{8} - \frac{35}{8} \\ -6e^{-x} + \frac{13e^{-2x}}{4} + \frac{11}{4} + \frac{x^2}{2} + \frac{x}{2} \\ 6e^{-x} - \frac{13e^{-2x}}{2} + \frac{1}{2} + x \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} \frac{11x}{4} + \frac{x^2}{4} + \frac{x^3}{6} + 6e^{-x} - \frac{13e^{-2x}}{8} - \frac{35}{8} \\ -6e^{-x} + \frac{13e^{-2x}}{4} + \frac{11}{4} + \frac{x^2}{2} + \frac{x}{2} \\ 6e^{-x} - \frac{13e^{-2x}}{2} + \frac{1}{2} + x \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{35}{8} + \frac{(-13+2c_1)e^{-2x}}{8} + (c_2 + 6)e^{-x} + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} + c_3$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2-3*(diff(_b(_a), _a))-2*_
  Methods for second order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying high order exact linear fully integrable
  trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
  trying a double symmetry of the form [xi=0, eta=F(x)]
  <- double symmetry of the form [xi=0, eta=F(x)] successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+2*diff(y(x),x)=x^2+4*x+8,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4} + \frac{x^3}{6} - c_2 e^{-x} + \frac{e^{-2x} c_1}{2} + \frac{11x}{4} + c_3$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 47

```
DSolve[y'''[x]+3*y''[x]+2*y'[x]==x^2+4*x+8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} - \frac{1}{2} c_1 e^{-2x} - c_2 e^{-x} + c_3$$

10.11 problem 20

10.11.1 Solving as second order linear constant coeff ode	1599
10.11.2 Solving using Kovacic algorithm	1603
10.11.3 Maple step by step solution	1607

Internal problem ID [5390]

Internal file name [OUTPUT/4881_Sunday_February_04_2024_12_46_45_AM_57829874/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + y = -2 \sin(x) + 4 \cos(x) x$$

10.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = -2 \sin(x) + 4 \cos(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$

$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2 \sin(x) + 4 \cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ = -2 \sin(x) + 4 \cos(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 0, A_4 = 0]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2 \sin(x) + 2 \cos(x) x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (x^2 \sin(x) + 2 \cos(x) x)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2 \cos(x) x \quad (1)$$

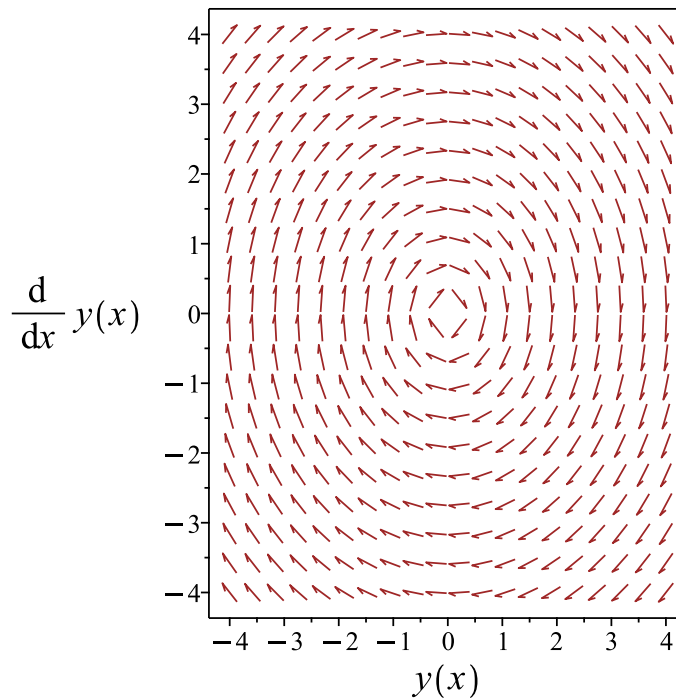


Figure 257: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2 \cos(x) x$$

Verified OK.

10.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (\cos(x)) + c_2 (\cos(x) (\tan(x)))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2 \sin(x) + 4 \cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ = -2 \sin(x) + 4 \cos(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 0, A_4 = 0]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = x^2 \sin(x) + 2 \cos(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (x^2 \sin(x) + 2 \cos(x) x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2 \cos(x) x \quad (1)$$

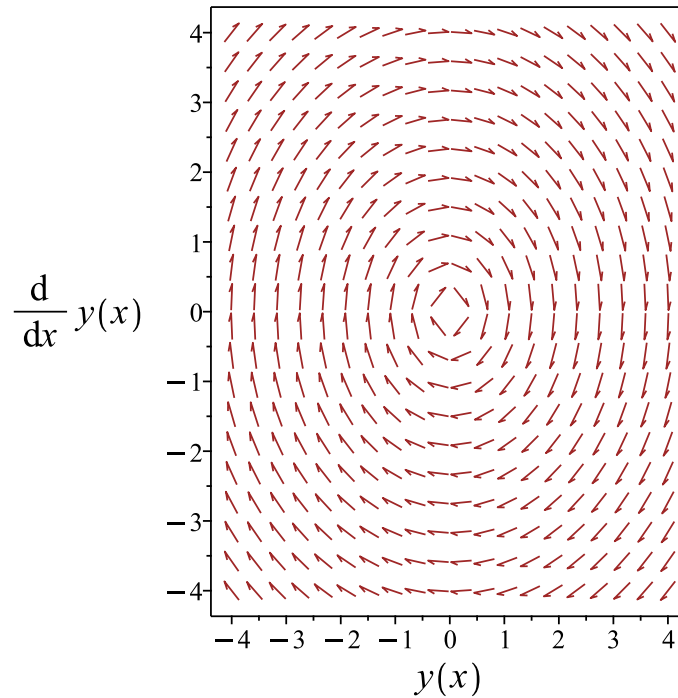


Figure 258: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2 \cos(x) x$$

Verified OK.

10.11.3 Maple step by step solution

Let's solve

$$y'' + y = -2 \sin(x) + 4 \cos(x) x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-4})}{2}$$
- Roots of the characteristic polynomial

$$r = (-I, I)$$
- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x)$$
- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x)$$
- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$
- Substitute in solutions of the homogeneous ODE

$$y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$$
- Find a particular solution $y_p(x)$ of the ODE
 - Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right) \right], f(x) = -2 \sin(x) + 4 \cos(x)$$
 - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$
 - Compute Wronskian

$$W(y_1(x), y_2(x)) = 1$$
 - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x) \left(\int (2x \sin(2x) - 1 + \cos(2x)) dx \right) + \sin(x) \left(\int (2x \cos(2x) + 2x - \sin(2x)) dx \right)$$
 - Compute integrals

$$y_p(x) = x^2 \sin(x) + 2 \cos(x) x - \sin(x)$$
- Substitute particular solution into general solution to ODE

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2 \cos(x) x - \sin(x)$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=-2*sin(x)+4*x*cos(x),y(x), singsol=all)
```

$$y(x) = (x^2 + c_2 - 1) \sin(x) + 2 \cos(x) \left(x + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 32

```
DSolve[y''[x]+y[x]==-2*Sin[x]+4*x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}(2x^2 - 1 + 2c_2) \sin(x) + (2x + c_1) \cos(x)$$

10.12 problem 21

10.12.1 Maple step by step solution 1612

Internal problem ID [5391]

Internal file name [OUTPUT/4882_Sunday_February_04_2024_12_46_46_AM_39250907/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters).
Supplementary problems. Page 98

Problem number: 21.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

[[_3rd_order , _linear , _nonhomogeneous]]

$$y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5e^{2x}x + e^{2x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - 4y' + 4y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$

$$\lambda_2 = 2$$

$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-2x}$$

$$y_2 = e^x$$

$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5e^{2x} x + e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x^2 - 4x - 1 + 2x^2 e^{2x} + 5e^{2x} x + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}, \{x^2 e^{2x}, e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{1, x, x^2\}, \{x^2 e^{2x}, e^{2x} x, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1 + A_4 x^2 e^{2x} + A_5 e^{2x} x + A_6 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 12A_6 e^{2x} x^2 + 30A_6 e^{2x} x + 8A_4 x e^{2x} + 4A_1 - 4A_2 - 2A_3 + 4A_3 x^2 - 8A_3 x + 10A_4 e^{2x} + 6A_6 e^{2x} + 4A_5 e^{2x} = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5e^{2x} x + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{2}, A_4 = 0, A_5 = 0, A_6 = \frac{1}{6} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{x^2}{2} + \frac{e^{2x}x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-2x} + c_2e^x + e^{2x}c_3) + \left(\frac{x^2}{2} + \frac{e^{2x}x^3}{6} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-2x} + c_2e^x + e^{2x}c_3 + \frac{x^2}{2} + \frac{e^{2x}x^3}{6} \quad (1)$$

Verification of solutions

$$y = c_1e^{-2x} + c_2e^x + e^{2x}c_3 + \frac{x^2}{2} + \frac{e^{2x}x^3}{6}$$

Verified OK.

10.12.1 Maple step by step solution

Let's solve

$$y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2e^{2x} + 5e^{2x}x + e^{2x}$$

- Highest derivative means the order of the ODE is 3

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = 2x^2e^{2x} + 5e^{2x}x + 2x^2 + y_3(x) + 4y_2(x) - 4y_1(x) + e^{2x} - 4x - 1$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = 2x^2e^{2x} + 5e^{2x}x + 2x^2 + y_3(x) + 4y_2(x) - 4y_1(x) + e^{2x} -$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2x^2 - 4x - 1 + 2x^2e^{2x} + 5e^{2x}x + e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2x^2 - 4x - 1 + 2x^2e^{2x} + 5e^{2x}x + e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = e^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \vec{y}_p(x)$$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^x & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^x & \frac{e^{2x}}{2} \\ e^{-2x} & e^x & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^x & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^x & \frac{e^{2x}}{2} \\ e^{-2x} & e^x & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{(3e^{4x} - 8e^{3x} - 1)e^{-2x}}{6} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{(3e^{4x} - 4e^{3x} + 1)e^{-2x}}{12} \\ -\frac{(3e^{4x} - 4e^{3x} + 1)e^{-2x}}{3} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & \frac{(3e^{4x} - 2e^{3x} - 1)e^{-2x}}{6} \\ -\frac{2(3e^{4x} - 2e^{3x} - 1)e^{-2x}}{3} & -e^{-2x} + e^{2x} & -\frac{(-3e^{4x} + e^{3x} - 1)e^{-2x}}{3} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x) \cdot \vec{v}(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(2e^{4x}x^3 - 3e^{4x} + 6x^2e^{2x} + 4e^{3x} - 1)e^{-2x}}{12} \\ \frac{(2e^{4x}x^3 + 3e^{4x}x^2 - 3e^{4x} + 2e^{3x} + 6e^{2x}x + 1)e^{-2x}}{6} \\ \frac{e^{-2x}((2x^3 + 6x^2 + 3x - 3)e^{4x} + 3e^{2x} + e^{3x} - 1)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 + \begin{bmatrix} \frac{(2e^{4x}x^3 - 3e^{4x} + 6x^2e^{2x} + 4e^{3x} - 1)e^{-2x}}{12} \\ \frac{(2e^{4x}x^3 + 3e^{4x}x^2 - 3e^{4x} + 2e^{3x} + 6e^{2x}x + 1)e^{-2x}}{6} \\ \frac{e^{-2x}((2x^3 + 6x^2 + 3x - 3)e^{4x} + 3e^{2x} + e^{3x} - 1)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{e^{-2x}((2x^3 + 3c_3 - 3)e^{4x} + 6x^2e^{2x} + 12c_2e^{3x} + 3c_1 + 4e^{3x} - 1)}{12}$$

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]  
trying high order linear exact nonhomogeneous  
trying differential order: 3; missing the dependent variable  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=2*x^2-4*x-1+2*x^2*exp(2*x)+5*x*exp(2*x),y(x))
```

$$y(x) = \frac{((x^3 + 6c_3)e^{4x} + 3e^{2x}x^2 + 6e^{3x}c_1 + 6c_2)e^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 0.519 (sec). Leaf size: 44

```
DSolve[y'''[x]-y''[x]-4*y'[x]+4*y[x]==2*x^2-4*x-1+2*x^2*Exp[2*x]+5*x*Exp[2*x]+Exp[2*x],y[x],x]
```

$$y(x) \rightarrow \frac{1}{6}(e^{2x}x + 3)x^2 + c_1e^{-2x} + c_2e^x + c_3e^{2x}$$

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11.1 problem 26

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Internal problem ID [5392]

Internal file name [OUTPUT/4883_Sunday_February_04_2024_12_46_46_AM_17206234/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

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Problem number: 26.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + y' + y = e^{3x} + 6e^x - 3e^{-2x} + 5$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = (e^{5x} + 6e^{3x} + 5e^{2x} - 3)e^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}x}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}x}{2} \right) \right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}x}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}x}{2} \right) \right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right)$$

$$y_2 = e^{-\frac{x}{2}} \sin \left(\frac{\sqrt{3}x}{2} \right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right) & e^{-\frac{x}{2}} \sin \left(\frac{\sqrt{3}x}{2} \right) \\ \frac{d}{dx} \left(e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right) \right) & \frac{d}{dx} \left(e^{-\frac{x}{2}} \sin \left(\frac{\sqrt{3}x}{2} \right) \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \right) \left(-\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3}}{2} \right) - \left(e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \right) \left(-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \right)$$

Which simplifies to

$$W = \frac{e^{-x} \cos\left(\frac{\sqrt{3}x}{2}\right)^2 \sqrt{3}}{2} + \frac{e^{-x} \sin\left(\frac{\sqrt{3}x}{2}\right)^2 \sqrt{3}}{2}$$

Which simplifies to

$$W = \frac{\sqrt{3} e^{-x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) e^{-2x}}{\frac{\sqrt{3} e^{-x}}{2}} dx$$

Which simplifies to

$$u_1 = - \int \frac{2\sqrt{3} e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3)}{3} dx$$

Hence

$$u_1 = \frac{2\sqrt{3}}{3} \left(\frac{\sqrt{3} e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{3e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{\sqrt{3} e^{\frac{7x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{26} + \frac{7 \sin\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{7x}{2}}}{26} - \sqrt{3} e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + 3 \sin\left(\frac{\sqrt{3}x}{2}\right) \right)$$

3

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) e^{-2x}}{\frac{\sqrt{3}e^{-x}}{2}} dx$$

Which simplifies to

$$u_2 = \int \frac{2\sqrt{3}e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3)}{3} dx$$

Hence

$$u_2 = \frac{2\sqrt{3} \left(\frac{3e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{\sqrt{3}e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{7 \cos\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{7x}{2}}}{26} + \frac{\sqrt{3}e^{\frac{7x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{26} + 3e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + \sqrt{3}e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \right)}{3}$$

Which simplifies to

$$u_1 = \frac{5 \left(\left(-3e^{2x} - \frac{6e^{3x}}{5} - \frac{3e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(e^{2x} + \frac{6e^{3x}}{5} + \frac{7e^{5x}}{65} + \frac{3}{5} \right) \right) e^{-\frac{3x}{2}}}{3}$$

$$u_2 = \frac{5 \left(\sqrt{3} \left(e^{2x} + \frac{6e^{3x}}{5} + \frac{7e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + 3 \left(-\frac{1}{5} + e^{2x} + \frac{2e^{3x}}{5} + \frac{e^{5x}}{65} \right) \sin\left(\frac{\sqrt{3}x}{2}\right) \right) e^{-\frac{3x}{2}}}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5 \left(\left(-3e^{2x} - \frac{6e^{3x}}{5} - \frac{3e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(e^{2x} + \frac{6e^{3x}}{5} + \frac{7e^{5x}}{65} + \frac{3}{5} \right) \right) e^{-\frac{3x}{2}} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

$$+ \frac{5 \left(\sqrt{3} \left(e^{2x} + \frac{6e^{3x}}{5} + \frac{7e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + 3 \left(-\frac{1}{5} + e^{2x} + \frac{2e^{3x}}{5} + \frac{e^{5x}}{65} \right) \sin\left(\frac{\sqrt{3}x}{2}\right) \right) e^{-\frac{3x}{2}} e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

Which simplifies to

$$y_p(x) = \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}x}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}x}{2} \right) \right) \right) + \left(\frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13} \right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}x}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}x}{2} \right) \right) + \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13} \quad (1)$$

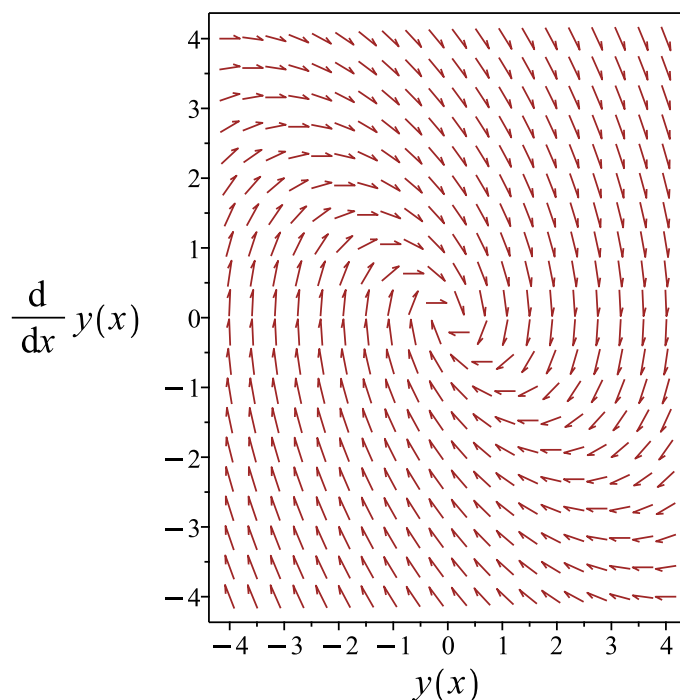


Figure 259: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos \left(\frac{\sqrt{3}x}{2} \right) + c_2 \sin \left(\frac{\sqrt{3}x}{2} \right) \right) + \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13}$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{1}{1} dx} \\&= z_1 e^{-\frac{x}{2}} \\&= z_1 \left(e^{-\frac{x}{2}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-x}}{(y_1)^2} dx \\&= y_1 \left(\frac{2\sqrt{3} \tan \left(\frac{\sqrt{3}x}{2} \right)}{3} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 \left(e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right) \right) + c_2 \left(e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}x}{2} \right) \left(\frac{2\sqrt{3} \tan \left(\frac{\sqrt{3}x}{2} \right)}{3} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)$$

$$y_2 = \frac{2 e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & \frac{2e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} \\ \frac{d}{dx}\left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)\right) & \frac{d}{dx}\left(\frac{2e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & \frac{2e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} & -\frac{e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)\right) \left(-\frac{e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)\right) - \left(\frac{2e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}\right) \left(-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2}\right)$$

Which simplifies to

$$W = e^{-x} \cos\left(\frac{\sqrt{3}x}{2}\right)^2 + e^{-x} \sin\left(\frac{\sqrt{3}x}{2}\right)^2$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{2e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3)e^{-2x}\sqrt{3}}{3e^{-x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{2\sqrt{3}e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3)}{3} dx$$

Hence

$$u_1 = \frac{2\sqrt{3} \left(\frac{\sqrt{3} e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{3 e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{\sqrt{3} e^{\frac{7x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{26} + \frac{7 \sin\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{7x}{2}}}{26} - \sqrt{3} e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + 3 \sin\left(\frac{\sqrt{3}x}{2}\right) \right)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) e^{-2x}}{e^{-x}} dx$$

Which simplifies to

$$u_2 = \int e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) dx$$

Hence

$$u_2 = \frac{3 e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{\sqrt{3} e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{7 \cos\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{7x}{2}}}{26} + \frac{\sqrt{3} e^{\frac{7x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{26} + 3 e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) + \sqrt{3} e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) + \frac{5 e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{5 e^{\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2}$$

Which simplifies to

$$u_1 = \frac{5 \left(\left(-3 e^{2x} - \frac{6 e^{3x}}{5} - \frac{3 e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(e^{2x} + \frac{6 e^{3x}}{5} + \frac{7 e^{5x}}{65} + \frac{3}{5} \right) \right) e^{-\frac{3x}{2}}}{3}$$

$$u_2 = \frac{5 \left(\left(e^{2x} + \frac{6 e^{3x}}{5} + \frac{7 e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(-\frac{1}{5} + e^{2x} + \frac{2 e^{3x}}{5} + \frac{e^{5x}}{65} \right) \right) e^{-\frac{3x}{2}}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{5 \left(\left(-3 e^{2x} - \frac{6 e^{3x}}{5} - \frac{3 e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(e^{2x} + \frac{6 e^{3x}}{5} + \frac{7 e^{5x}}{65} + \frac{3}{5} \right) \right) e^{-\frac{3x}{2}} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

$$+ \frac{5 \left(\left(e^{2x} + \frac{6 e^{3x}}{5} + \frac{7 e^{5x}}{65} + \frac{3}{5} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) + \sin\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3} \left(-\frac{1}{5} + e^{2x} + \frac{2 e^{3x}}{5} + \frac{e^{5x}}{65} \right) \right) e^{-\frac{3x}{2}} e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3}$$

Which simplifies to

$$y_p(x) = \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} \right) + \left(\frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13} \right)$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} + \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13} \quad (1)$$

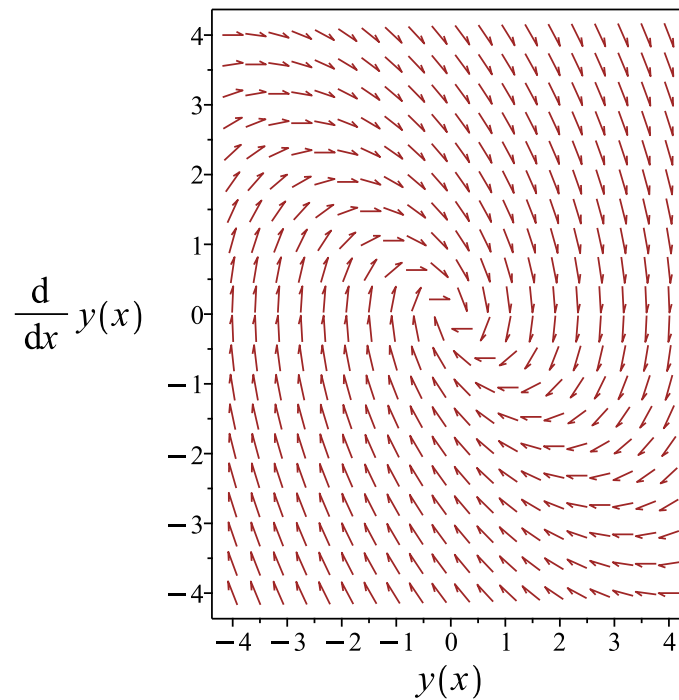


Figure 260: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{3} + \frac{e^{-2x}(65e^{2x} + e^{5x} + 26e^{3x} - 13)}{13}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$$y'' + y' + y = (e^{5x} + 6e^{3x} + 5e^{2x} - 3)e^{-2x}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + r + 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{(-1) \pm (\sqrt{-3})}{2}$$

- Roots of the characteristic polynomial

$$r = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) c_2 + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x), y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x), y_2(x))} dx \right) \right], f(x) = (e^{5x} + 6e^{3x} + 5e^{2x} - 3)e^{-2x}$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) \sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \frac{\sqrt{3} e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2 e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{\sqrt{3}x}{2}\right) \left(\int e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) dx \right) - \sin\left(\frac{\sqrt{3}x}{2}\right) \left(\int e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) (e^{5x} + 6e^{3x} + 5e^{2x} - 3) dx \right) \right)}{3}$$

- Compute integrals

$$y_p(x) = \frac{e^{-2x} (65 e^{2x} + e^{5x} + 26 e^{3x} - 13)}{13}$$

- Substitute particular solution into general solution to ODE

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) c_2 + \frac{e^{-2x} (65 e^{2x} + e^{5x} + 26 e^{3x} - 13)}{13}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(x), x$2)+diff(y(x), x)+y(x)=exp(3*x)+6*exp(x)-3*exp(-2*x)+5, y(x), singsol=all)
```

$$y(x) = \left(\frac{e^{5x}}{13} + 2e^{3x} + \sin\left(\frac{\sqrt{3}x}{2}\right) c_2 e^{\frac{3x}{2}} + \cos\left(\frac{\sqrt{3}x}{2}\right) c_1 e^{\frac{3x}{2}} + 5e^{2x} - 1 \right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 6.996 (sec). Leaf size: 70

```
DSolve[y''[x]+y'[x]+y[x]==Exp[3*x]+6*Exp[x]-3*Exp[-2*x]+5,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow -e^{-2x} + 2e^x + \frac{e^{3x}}{13} + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + 5$$

11.2 problem 27

11.2.1 Solving as second order linear constant coeff ode	1635
11.2.2 Solving using Kovacic algorithm	1638
11.2.3 Maple step by step solution	1644

Internal problem ID [5393]

Internal file name [OUTPUT/4884_Sunday_February_04_2024_12_46_47_AM_72162890/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 27.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - y = e^x$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{x e^x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(\frac{x e^x}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \frac{x e^x}{2} \quad (1)$$

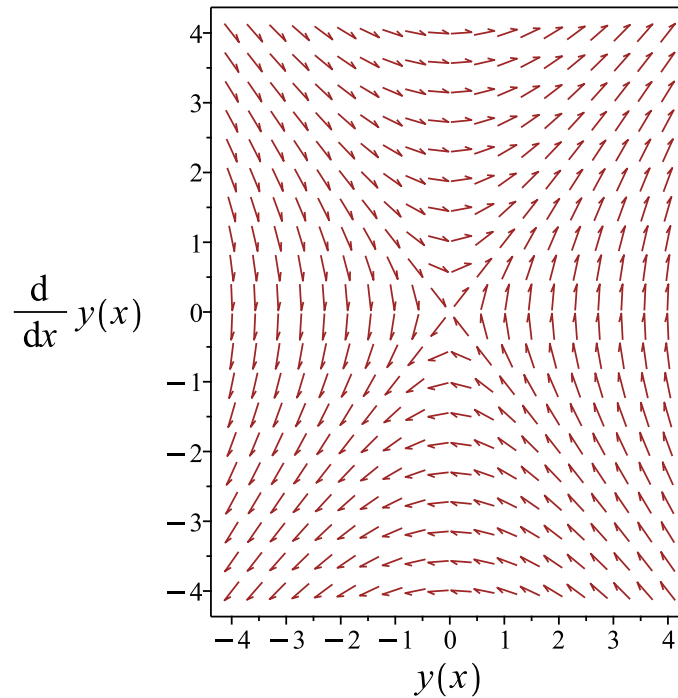


Figure 261: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{x e^x}{2}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1 \\B &= 0 \\C &= -1\end{aligned}\tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x)\tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned}\tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1}\tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 1 \\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x)\tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
 y_1 &= z_1 \\
 &= e^{-x}
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$

$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}(e^{-x}) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^{-x}) \left(\frac{e^x}{2}\right) - \left(\frac{e^x}{2}\right) (-e^{-x})$$

Which simplifies to

$$W = e^{-x} e^x$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{2x}}{1} dx$$

Which simplifies to

$$u_1 = - \int \frac{e^{2x}}{2} dx$$

Hence

$$u_1 = - \frac{e^{2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x}e^x}{1} dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-x}e^{2x}}{4} + \frac{x e^x}{2}$$

Which simplifies to

$$y_p(x) = \frac{(2x - 1) e^x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2} \right) + \left(\frac{(2x - 1) e^x}{4} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2x - 1) e^x}{4} \quad (1)$$

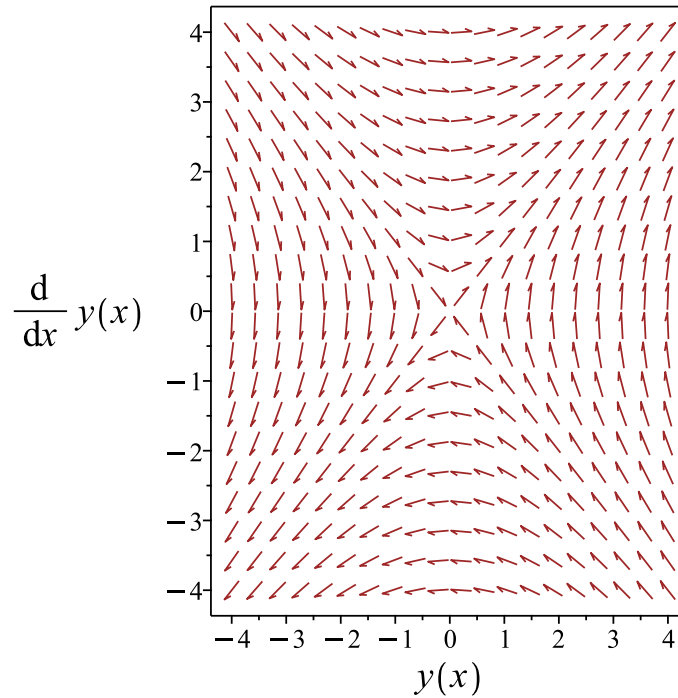


Figure 262: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2x - 1) e^x}{4}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$$y'' - y = e^x$$

- Highest derivative means the order of the ODE is 2
- y''
- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}(\int e^{2x} dx)}{2} + \frac{e^x(\int 1 dx)}{2}$$

- Compute integrals

$$y_p(x) = \frac{(2x-1)e^x}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{(2x-1)e^x}{4}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + \frac{e^x(x + 2c_1)}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 29

```
DSolve[y''[x]-y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^x \left(\frac{x}{2} - \frac{1}{4} + c_1 \right) + c_2 e^{-x}$$

11.3 problem 28

11.3.1 Solving as second order linear constant coeff ode	1647
11.3.2 Solving as linear second order ode solved by an integrating factor ode	1650
11.3.3 Solving using Kovacic algorithm	1652
11.3.4 Maple step by step solution	1657

Internal problem ID [5394]

Internal file name [OUTPUT/4885_Sunday_February_04_2024_12_46_48_AM_44622789/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 28.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 4y' + 4y = e^x + e^{2x}x$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = e^x + e^{2x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 4e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-4)^2 - (4)(1)(4)} \\ &= 2 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \quad (1)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + x e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + e^{2x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{e^x\}, \{x^2e^{2x}, e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{e^x\}, \{x^2e^{2x}, e^{2x}x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1e^x + A_2x^2e^{2x} + A_3e^{2x}x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1e^x + 2A_2e^{2x} + 6A_3e^{2x}x = e^x + e^{2x}x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{6} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = e^x + \frac{e^{2x}x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{2x} + xe^{2x}c_2) + \left(e^x + \frac{e^{2x}x^3}{6} \right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2x + c_1) + e^x + \frac{e^{2x}x^3}{6}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2x + c_1) + e^x + \frac{e^{2x}x^3}{6} \quad (1)$$

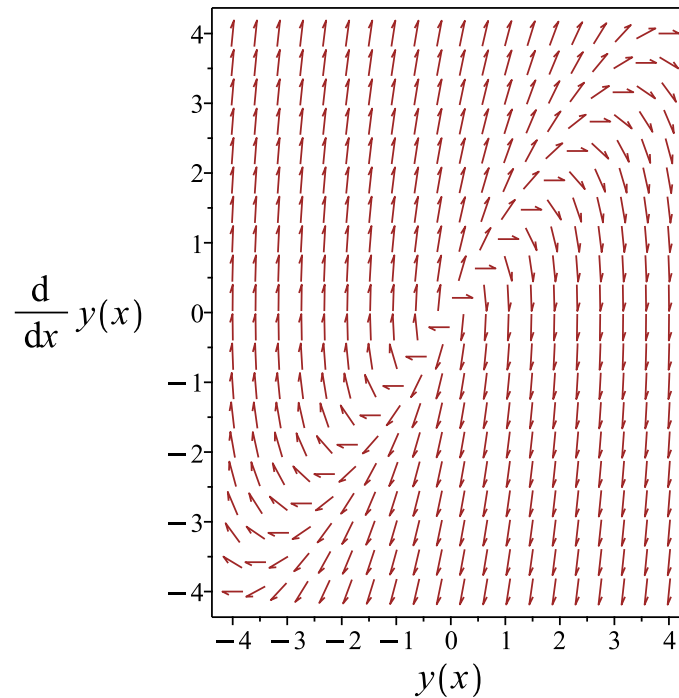


Figure 263: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^x + \frac{e^{2x}x^3}{6}$$

Verified OK.

11.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x)^2 + p'(x))y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int -4 \, dx} \\ &= e^{-2x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= e^{-2x}(e^x + e^{2x}x) \\ (ye^{-2x})'' &= e^{-2x}(e^x + e^{2x}x)\end{aligned}$$

Integrating once gives

$$(ye^{-2x})' = \frac{x^2}{2} - e^{-x} + c_1$$

Integrating again gives

$$(ye^{-2x}) = \frac{x^3}{6} + c_1x + e^{-x} + c_2$$

Hence the solution is

$$y = \frac{\frac{x^3}{6} + c_1x + e^{-x} + c_2}{e^{-2x}}$$

Or

$$y = \frac{e^{2x}x^3}{6} + c_1x e^{2x} + c_2e^{2x} + e^x$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2x}x^3}{6} + c_1x e^{2x} + c_2e^{2x} + e^x \tag{1}$$

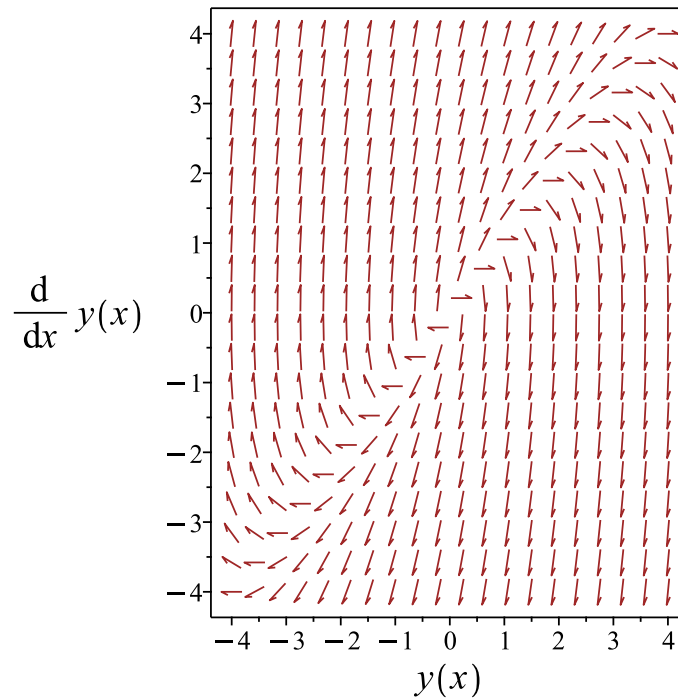


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{e^{2x} x^3}{6} + c_1 x e^{2x} + c_2 e^{2x} + e^x$$

Verified OK.

11.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$

$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{2x}) + c_2 (e^{2x}(x))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + x e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + e^{2x}x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{2x}x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{e^x\}, \{x^2 e^{2x}, e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{e^x\}, \{x^2 e^{2x}, e^{2x}x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x^2 e^{2x} + A_3 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^x + 2A_2 e^{2x} + 6A_3 e^{2x} x = e^x + e^{2x} x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{6} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = e^x + \frac{e^{2x} x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + x e^{2x} c_2) + \left(e^x + \frac{e^{2x} x^3}{6} \right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^x + \frac{e^{2x} x^3}{6}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^x + \frac{e^{2x} x^3}{6} \quad (1)$$

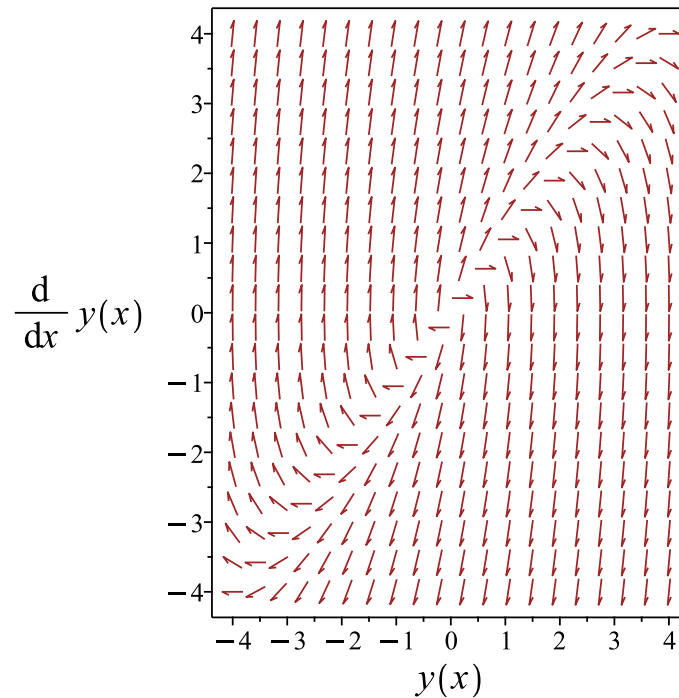


Figure 265: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^x + \frac{e^{2x}x^3}{6}$$

Verified OK.

11.3.4 Maple step by step solution

Let's solve

$$y'' - 4y' + 4y = e^x + e^{2x}x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r + 4 = 0$$

- Factor the characteristic polynomial

$$(r - 2)^2 = 0$$

- Root of the characteristic polynomial

$$r = 2$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{2x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = e^{2x}x$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1e^{2x} + xe^{2x}c_2 + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^x + e^{2x}x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x}x \\ 2e^{2x} & 2e^{2x}x + e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = e^{2x} \left(- \left(\int x(x + e^{-x}) dx \right) + \left(\int (x + e^{-x}) dx \right) x \right)$$

- Compute integrals

$$y_p(x) = e^x + \frac{e^{2x}x^3}{6}$$

- Substitute particular solution into general solution to ODE

$$y = xe^{2x}c_2 + c_1e^{2x} + e^x + \frac{e^{2x}x^3}{6}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(x)+x*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(x^3 + 6c_1x + 6c_2)e^{2x}}{6} + e^x$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 31

```
DSolve[y''[x]-4*y'[x]+4*y[x]==Exp[x]+x*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6}e^x(6 + e^x(x^3 + 6c_2x + 6c_1))$$

11.4 problem 29

11.4.1 Maple step by step solution 1662

Internal problem ID [5395]

Internal file name [OUTPUT/4886_Sunday_February_04_2024_12_46_49_AM_13873342/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 29.

ODE order: 4.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_high_order , _linear , _nonhomogeneous]]
```

$$y'''' - y = \sin(2x)$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

$$\lambda_3 = i$$

$$\lambda_4 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 e^x + e^{-ix} c_3 + e^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$

$$y_2 = e^x$$

$$y_3 = e^{-ix}$$

$$y_4 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y'''' - y = \sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{ix}, e^{-x}, e^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1 \cos(2x) + 15A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{15} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\sin(2x)}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 e^x + e^{-ix} c_3 + e^{ix} c_4) + \left(\frac{\sin(2x)}{15} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 e^x + e^{-ix} c_3 + e^{ix} c_4 + \frac{\sin(2x)}{15} \quad (1)$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 e^x + e^{-ix} c_3 + e^{ix} c_4 + \frac{\sin(2x)}{15}$$

Verified OK.

11.4.1 Maple step by step solution

Let's solve

$$y''' - y = \sin(2x)$$

- Highest derivative means the order of the ODE is 4

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Define new variable $y_4(x)$

$$y_4(x) = y''''$$

- Isolate for $y_4'(x)$ using original ODE

$$y_4'(x) = \sin(2x) + y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \sin(2x) + y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right], \left[-I, \begin{bmatrix} -I \\ -1 \\ I \\ 1 \end{bmatrix} \right], \left[I, \begin{bmatrix} I \\ -1 \\ -I \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = e^x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathbf{I}, \begin{bmatrix} -\mathbf{I} \\ -1 \\ \mathbf{I} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-\mathbf{I}x} \cdot \begin{bmatrix} -\mathbf{I} \\ -1 \\ \mathbf{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathbf{I} \sin(x)) \cdot \begin{bmatrix} -\mathbf{I} \\ -1 \\ \mathbf{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathbf{I}(\cos(x) - \mathbf{I} \sin(x)) \\ -\cos(x) + \mathbf{I} \sin(x) \\ \mathbf{I}(\cos(x) - \mathbf{I} \sin(x)) \\ \cos(x) - \mathbf{I} \sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix} \right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \vec{y}_p(x)$$

- Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^x & -\sin(x) & -\cos(x) \\ e^{-x} & e^x & -\cos(x) & \sin(x) \\ -e^{-x} & e^x & \sin(x) & \cos(x) \\ e^{-x} & e^x & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix
 $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^x & -\sin(x) & -\cos(x) \\ e^{-x} & e^x & -\cos(x) & \sin(x) \\ -e^{-x} & e^x & \sin(x) & \cos(x) \\ e^{-x} & e^x & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^x}{4} & -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^x}{4} & -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^x}{4} & -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^x}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^x}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^x}{4} \end{bmatrix}$$

□

Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\sin(2x)}{15} - \frac{\sin(x)}{3} - \frac{e^{-x}}{10} + \frac{e^x}{10} \\ \frac{4 \cos(x)^2}{15} - \frac{2}{15} + \frac{e^{-x}}{10} + \frac{e^x}{10} - \frac{\cos(x)}{3} \\ -\frac{4 \sin(2x)}{15} + \frac{\sin(x)}{3} - \frac{e^{-x}}{10} + \frac{e^x}{10} \\ -\frac{16 \cos(x)^2}{15} + \frac{8}{15} + \frac{e^{-x}}{10} + \frac{e^x}{10} + \frac{\cos(x)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x) + \begin{bmatrix} \frac{\sin(2x)}{15} - \frac{\sin(x)}{3} - \frac{e^{-x}}{10} + \frac{e^x}{10} \\ \frac{4 \cos(x)^2}{15} - \frac{2}{15} + \frac{e^{-x}}{10} + \frac{e^x}{10} - \frac{\cos(x)}{3} \\ -\frac{4 \sin(2x)}{15} + \frac{\sin(x)}{3} - \frac{e^{-x}}{10} + \frac{e^x}{10} \\ -\frac{16 \cos(x)^2}{15} + \frac{8}{15} + \frac{e^{-x}}{10} + \frac{e^x}{10} + \frac{\cos(x)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-1-10c_1)e^{-x}}{10} + \frac{(-5-15c_3+2 \cos(x)) \sin(x)}{15} - c_4 \cos(x) + \frac{e^x(10c_2+1)}{10}$$

Maple trace

```
`Methods for high order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]  
trying high order linear exact nonhomogeneous  
trying differential order: 4; missing the dependent variable  
checking if the LODE has constant coefficients  
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)-y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)}{15} + \cos(x) c_1 + e^x c_2 + c_3 \sin(x) + c_4 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[y''''[x]-y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x + c_3 e^{-x} + c_4 \sin(x) + \cos(x) \left(\frac{2 \sin(x)}{15} + c_2 \right)$$

11.5 problem 30

11.5.1 Maple step by step solution 1671

Internal problem ID [5396]

Internal file name [OUTPUT/4887_Sunday_February_04_2024_12_46_49_AM_92657304/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 30.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _linear , _nonhomogeneous]]
```

$$y''' + y = \cos(x)$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y = 0$$

The characteristic equation is

$$\lambda^3 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$

$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

$$\lambda_3 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-x} \\ y_2 &= e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} \\ y_3 &= e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y = \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x}, e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \sin(x) - A_2 \cos(x) + A_1 \cos(x) + A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_3 \right) + \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \quad (1)$$

Verification of solutions

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

11.5.1 Maple step by step solution

Let's solve

$$y''' + y = \cos(x)$$

- Highest derivative means the order of the ODE is 3

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = \cos(x) - y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \cos(x) - y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \cos(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \cos(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right], \left[\frac{1}{2} - \frac{i\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix} \right], \left[\frac{1}{2} + \frac{i\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} + \frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{i\sqrt{3}}{2}, \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} \cdot \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}x}{2}\right) - i \sin\left(\frac{\sqrt{3}x}{2}\right) \right) \cdot \begin{bmatrix} \frac{1}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}x}{2}\right) - i \sin\left(\frac{\sqrt{3}x}{2}\right)}{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}x}{2}\right) - i \sin\left(\frac{\sqrt{3}x}{2}\right)}{\frac{1}{2} - \frac{i\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}x}{2}\right) - i \sin\left(\frac{\sqrt{3}x}{2}\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\vec{y}_2(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}x}{2}\right) \end{bmatrix}, \vec{y}_3(x) = e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}x}{2}\right) \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$
 $\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

□ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} e^{-x} & e^{\frac{x}{2}} \left(-\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \right) & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \right) \\ -e^{-x} & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \right) & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \right) \\ e^{-x} & e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & -e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix
 $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & e^{\frac{x}{2}} \left(-\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \right) & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \right) \\ -e^{-x} & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} \right) & e^{\frac{x}{2}} \left(\frac{\cos\left(\frac{\sqrt{3}x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}x}{2}\right)}{2} \right) \\ e^{-x} & e^{\frac{x}{2}} \cos\left(\frac{\sqrt{3}x}{2}\right) & -e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right) \end{bmatrix} \cdot \begin{bmatrix} 1 & & \\ 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{2\cos\left(\frac{\sqrt{3}x}{2}\right)e^{-x}e^{\frac{3x}{2}}}{3} + \frac{e^{-x}}{3} & \frac{(\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} & -\frac{(-\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} \\ \frac{(-\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} & \frac{2\cos\left(\frac{\sqrt{3}x}{2}\right)e^{-x}e^{\frac{3x}{2}}}{3} + \frac{e^{-x}}{3} & \frac{(\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} \\ -\frac{(\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} & \frac{(-\sqrt{3}e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}x}{2}\right) + e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}x}{2}\right) - 1)e^{-x}}{3} & \frac{2\cos\left(\frac{\sqrt{3}x}{2}\right)e^{-x}e^{\frac{3x}{2}}}{3} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$$

- Take the derivative of the particular solution

$$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(-2\sqrt{3}e^{\frac{3x}{2}} \sin(\frac{\sqrt{3}x}{2}) + 2e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2}) - 3\cos(x)e^x + 3\sin(x)e^x + 1)e^{-x}}{6} \\ \frac{(2\sqrt{3}e^{\frac{3x}{2}} \sin(\frac{\sqrt{3}x}{2}) + 2e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2}) - 3\cos(x)e^x - 3\sin(x)e^x + 1)e^{-x}}{6} \\ e^{-x} \left(-\frac{4e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2})}{3} + \frac{1}{3} + (-\sin(x) + \cos(x))e^x \right) \\ \frac{\phantom{e^{-x} \left(-\frac{4e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2})}{3} + \frac{1}{3} + (-\sin(x) + \cos(x))e^x \right)}}{2} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} \frac{(-2\sqrt{3}e^{\frac{3x}{2}} \sin(\frac{\sqrt{3}x}{2}) + 2e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2}) - 3\cos(x)e^x + 3\sin(x)e^x + 1)e^{-x}}{6} \\ \frac{(2\sqrt{3}e^{\frac{3x}{2}} \sin(\frac{\sqrt{3}x}{2}) + 2e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2}) - 3\cos(x)e^x - 3\sin(x)e^x + 1)e^{-x}}{6} \\ e^{-x} \left(-\frac{4e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2})}{3} + \frac{1}{3} + (-\sin(x) + \cos(x))e^x \right) \\ \frac{\phantom{e^{-x} \left(-\frac{4e^{\frac{3x}{2}} \cos(\frac{\sqrt{3}x}{2})}{3} + \frac{1}{3} + (-\sin(x) + \cos(x))e^x \right)}}{2} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{-x} \left(e^{\frac{3x}{2}} \left(-\sqrt{3} c_3 + c_2 + \frac{2}{3} \right) \cos\left(\frac{\sqrt{3}x}{2}\right) - e^{\frac{3x}{2}} \left((c_2 + \frac{2}{3}) \sqrt{3} + c_3 \right) \sin\left(\frac{\sqrt{3}x}{2}\right) + (\sin(x) - \cos(x)) e^x - 2c_1 + \frac{1}{3} \right)}{2}$$

Maple trace

```

`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$3)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \left(\frac{(\cos(x) - \sin(x)) e^x}{2} + c_3 \sin\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{3x}{2}} + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right) e^{\frac{3x}{2}} + c_1 \right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.515 (sec). Leaf size: 68

```
DSolve[y'''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x} + c_3 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

11.6 problem 31

11.6.1 Solving as second order linear constant coeff ode	1677
11.6.2 Solving using Kovacic algorithm	1681
11.6.3 Maple step by step solution	1686

Internal problem ID [5397]

Internal file name [OUTPUT/4888_Sunday_February_04_2024_12_46_49_AM_1617631/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 31.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 4y = \sin(2x)$$

11.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i \end{aligned}$$

Hence

$$\lambda_1 = +2i$$

$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$

$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 (c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x \cos(2x), x \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{x \cos(2x)}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(-\frac{x \cos(2x)}{4}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{x \cos(2x)}{4} \quad (1)$$

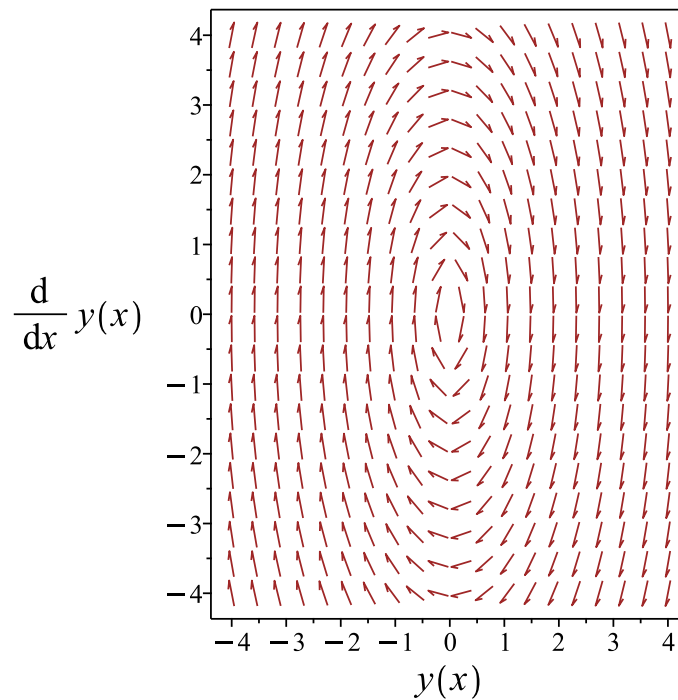


Figure 266: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{x \cos(2x)}{4}$$

Verified OK.

11.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2} dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2 \left(\cos(2x) \left(\frac{\tan(2x)}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sin(2x)}{2}, \cos(2x) \right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x \cos(2x), x \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{x \cos(2x)}{4}$$

Therefore the general solution is

$$y = y_h + y_p \\ = \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \right) + \left(-\frac{x \cos(2x)}{4} \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{x \cos(2x)}{4} \quad (1)$$

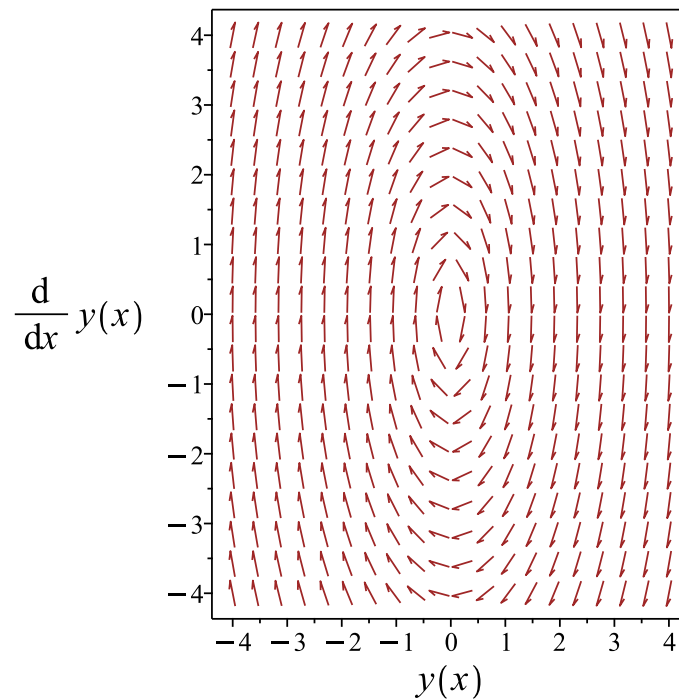


Figure 267: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{x \cos(2x)}{4}$$

Verified OK.

11.6.3 Maple step by step solution

Let's solve

$$y'' + 4y = \sin(2x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 4 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-16})}{2}$$

- Roots of the characteristic polynomial

$$r = (-2i, 2i)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(2x)$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \sin(2x) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \sin(4x) dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{\sin(2x)}{16} - \frac{x \cos(2x)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)}{16} - \frac{x \cos(2x)}{4}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-x + 4c_1) \cos(2x)}{4} + \sin(2x) c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \left(-\frac{x}{4} + c_1\right) \cos(2x) + \frac{1}{8}(1 + 16c_2) \sin(x) \cos(x)$$

11.7 problem 32

11.7.1 Solving as second order linear constant coeff ode	1688
11.7.2 Solving using Kovacic algorithm	1692
11.7.3 Maple step by step solution	1697

Internal problem ID [5398]

Internal file name [OUTPUT/4889_Sunday_February_04_2024_12_46_50_AM_80123366/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 32.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 5y = \cos(x\sqrt{5})$$

11.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 5, f(x) = \cos(x\sqrt{5})$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 5$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(5)} \\ &= \pm i\sqrt{5} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +i\sqrt{5} \\ \lambda_2 &= -i\sqrt{5} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= i\sqrt{5} \\ \lambda_2 &= -i\sqrt{5} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 \left(c_1 \cos \left(x\sqrt{5} \right) + c_2 \sin \left(x\sqrt{5} \right) \right)$$

Or

$$y = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5})$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5})$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x\sqrt{5})$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{ \cos(x\sqrt{5}), \sin(x\sqrt{5}) \right\} \right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \cos(x\sqrt{5}), \sin(x\sqrt{5}) \right\}$$

Since $\cos(x\sqrt{5})$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$\left[\left\{ x \cos(x\sqrt{5}), x \sin(x\sqrt{5}) \right\} \right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x\sqrt{5}) + A_2 x \sin(x\sqrt{5})$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{5} \sin(x\sqrt{5}) + 2A_2\sqrt{5} \cos(x\sqrt{5}) = \cos(x\sqrt{5})$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{5}}{10} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\sqrt{5} x \sin(x\sqrt{5})}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5}) \right) + \left(\frac{\sqrt{5} x \sin(x\sqrt{5})}{10} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5}) + \frac{\sqrt{5} x \sin(x\sqrt{5})}{10} \quad (1)$$

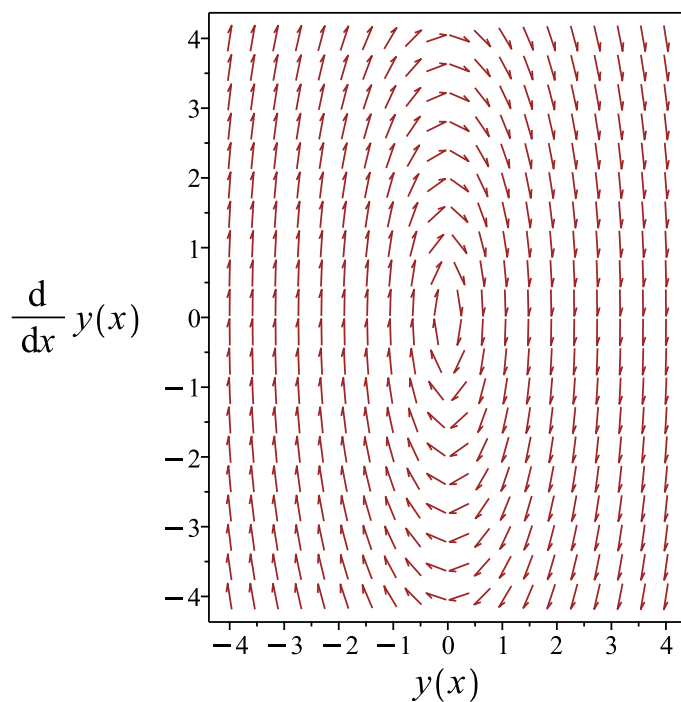


Figure 268: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5}) + \frac{\sqrt{5} x \sin(x\sqrt{5})}{10}$$

Verified OK.

11.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 5 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -5z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -5$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x\sqrt{5})$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x\sqrt{5})\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x\sqrt{5})$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x\sqrt{5}) \int \frac{1}{\cos^2(x\sqrt{5})} dx \\ &= \cos(x\sqrt{5}) \left(\frac{\sqrt{5} \tan(x\sqrt{5})}{5} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\cos(x\sqrt{5}) \right) + c_2 \left(\cos(x\sqrt{5}) \left(\frac{\sqrt{5} \tan(x\sqrt{5})}{5} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x\sqrt{5}) + \frac{c_2\sqrt{5} \sin(x\sqrt{5})}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x\sqrt{5})$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{ \cos(x\sqrt{5}), \sin(x\sqrt{5}) \right\} \right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sqrt{5} \sin(x\sqrt{5})}{5}, \cos(x\sqrt{5}) \right\}$$

Since $\cos(x\sqrt{5})$ is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$\left[\left\{ x \cos(x\sqrt{5}), x \sin(x\sqrt{5}) \right\} \right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(x\sqrt{5}) + A_2 x \sin(x\sqrt{5})$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{5} \sin(x\sqrt{5}) + 2A_2\sqrt{5} \cos(x\sqrt{5}) = \cos(x\sqrt{5})$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{5}}{10} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{\sqrt{5} x \sin(x\sqrt{5})}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(x\sqrt{5}) + \frac{c_2 \sqrt{5} \sin(x\sqrt{5})}{5} \right) + \left(\frac{\sqrt{5} x \sin(x\sqrt{5})}{10} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{5}) + \frac{c_2 \sqrt{5} \sin(x\sqrt{5})}{5} + \frac{\sqrt{5} x \sin(x\sqrt{5})}{10} \quad (1)$$

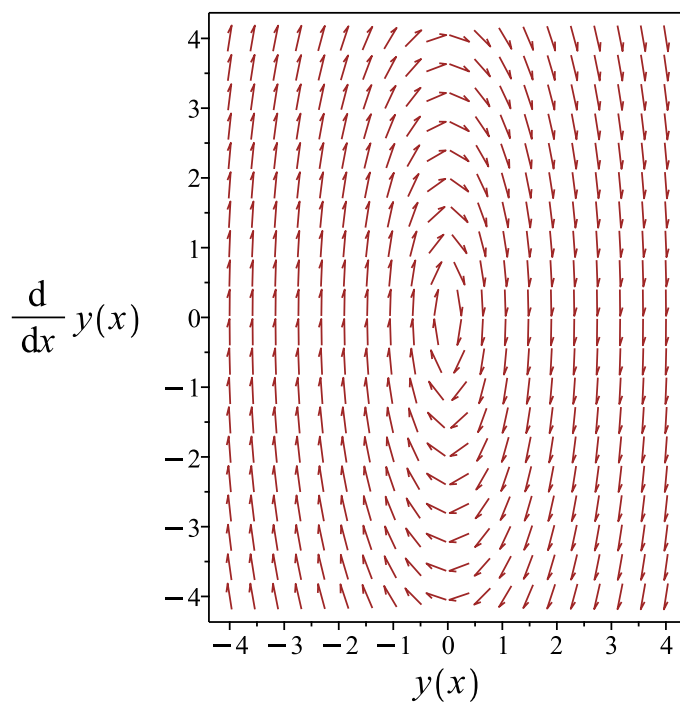


Figure 269: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{5}) + \frac{c_2 \sqrt{5} \sin(x\sqrt{5})}{5} + \frac{\sqrt{5} x \sin(x\sqrt{5})}{10}$$

Verified OK.

11.7.3 Maple step by step solution

Let's solve

$$y'' + 5y = \cos(x\sqrt{5})$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 5 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-20})}{2}$$

- Roots of the characteristic polynomial

$$r = (-i\sqrt{5}, i\sqrt{5})$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x\sqrt{5})$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x\sqrt{5})$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5}) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \cos(x\sqrt{5}) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x\sqrt{5}) & \sin(x\sqrt{5}) \\ -\sqrt{5} \sin(x\sqrt{5}) & \sqrt{5} \cos(x\sqrt{5}) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \sqrt{5}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\sqrt{5} \left(\cos(x\sqrt{5}) \left(\int \sin(2x\sqrt{5}) dx \right) - 2 \sin(x\sqrt{5}) \left(\int \cos(x\sqrt{5})^2 dx \right) \right)}{10}$$

- Compute integrals

$$y_p(x) = \frac{\sqrt{5} x \sin(x\sqrt{5})}{10} + \frac{\cos(x\sqrt{5})}{20}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x\sqrt{5}) + c_2 \sin(x\sqrt{5}) + \frac{\sqrt{5} x \sin(x\sqrt{5})}{10} + \frac{\cos(x\sqrt{5})}{20}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x), x$2)+5*y(x)=cos(sqrt(5)*x), y(x), singsol=all)
```

$$y(x) = \frac{(10c_1 + 1) \cos(\sqrt{5}x)}{10} + \frac{\sin(\sqrt{5}x) (\sqrt{5}x + 10c_2)}{10}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 45

```
DSolve[y''[x]+5*y[x]==Cos[Sqrt[5]*x], y[x], x, IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \left(\frac{1}{20} + c_1 \right) \cos(\sqrt{5}x) + \frac{1}{10} (\sqrt{5}x + 10c_2) \sin(\sqrt{5}x)$$

11.8 problem 33

11.8.1 Maple step by step solution 1703

Internal problem ID [5399]

Internal file name [OUTPUT/4890_Sunday_February_04_2024_12_46_50_AM_73408246/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 33.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

[[_3rd_order , _linear , _nonhomogeneous]]

$$y''' + y'' + y' + y = e^x + e^{-x} + \sin(x)$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' + y' + y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$

$$\lambda_2 = i$$

$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{-ix} c_2 + e^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$

$$y_2 = e^{-ix}$$

$$y_3 = e^{ix}$$

Now the particular solution to the given ODE is found

$$y''' + y'' + y' + y = e^x + e^{-x} + \sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i -th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} e^{-x} & e^{-ix} & e^{ix} \\ -e^{-x} & -ie^{-ix} & ie^{ix} \\ e^{-x} & -e^{-ix} & -e^{ix} \end{bmatrix}$$

$$|W| = 4ie^{-x} e^{-ix} e^{ix}$$

The determinant simplifies to

$$|W| = 4ie^{-x}$$

Now we determine W_i for each U_i .

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} e^{-ix} & e^{ix} \\ -ie^{-ix} & ie^{ix} \end{bmatrix} \\ &= 2i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} e^{-x} & e^{ix} \\ -e^{-x} & ie^{ix} \end{bmatrix} \\ &= (1+i)e^{(-1+i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} e^{-x} & e^{-ix} \\ -e^{-x} & -ie^{-ix} \end{bmatrix} \\ &= (1-i)e^{(-1-i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)} dx \\ &= (-1)^2 \int \frac{(e^x + e^{-x} + \sin(x))(2i)}{(1)(4ie^{-x})} dx \\ &= \int \frac{2i(e^x + e^{-x} + \sin(x))}{4ie^{-x}} dx \\ &= \int \left(\frac{e^{2x}}{2} + \frac{1}{2} + \frac{\sin(x)e^x}{2} \right) dx \\ &= \frac{x}{2} - \frac{\cos(x)e^x}{4} + \frac{\sin(x)e^x}{4} + \frac{e^{2x}}{4} \\ &= \frac{x}{2} - \frac{\cos(x)e^x}{4} + \frac{\sin(x)e^x}{4} + \frac{e^{2x}}{4} \end{aligned}$$

$$\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)} dx \\
&= (-1)^1 \int \frac{(e^x + e^{-x} + \sin(x)) ((1+i)e^{(-1+i)x})}{(1)(4ie^{-x})} dx \\
&= - \int \frac{(1+i)(e^x + e^{-x} + \sin(x))e^{(-1+i)x}}{4ie^{-x}} dx \\
&= - \int \left(\left(\frac{1}{4} - \frac{i}{4} \right) e^{(-1+i)x} (1 + e^{2x} + \sin(x)e^x) \right) dx \\
&= \frac{e^{(-1+i)x}}{4} + \frac{ie^{(1+i)x}}{4} + \frac{e^{2ix}}{16} - \frac{ie^{2ix}}{16} - \frac{x}{8} - \frac{ix}{8} \\
&= \frac{e^{(-1+i)x}}{4} + \frac{ie^{(1+i)x}}{4} + \frac{e^{2ix}}{16} - \frac{ie^{2ix}}{16} - \frac{x}{8} - \frac{ix}{8}
\end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)} dx \\
&= (-1)^0 \int \frac{(e^x + e^{-x} + \sin(x)) ((1-i)e^{(-1-i)x})}{(1)(4ie^{-x})} dx \\
&= \int \frac{(1-i)(e^x + e^{-x} + \sin(x))e^{(-1-i)x}}{4ie^{-x}} dx \\
&= \int \left(\left(-\frac{1}{4} - \frac{i}{4} \right) e^{(-1-i)x} (1 + e^{2x} + \sin(x)e^x) \right) dx \\
&= \frac{e^{(-1-i)x}}{4} - \frac{ie^{(1-i)x}}{4} - \frac{x}{8} + \frac{ix}{8} + \frac{e^{-2ix}}{16} + \frac{ie^{-2ix}}{16} \\
&= \frac{e^{(-1-i)x}}{4} - \frac{ie^{(1-i)x}}{4} - \frac{x}{8} + \frac{ix}{8} + \frac{e^{-2ix}}{16} + \frac{ie^{-2ix}}{16}
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Hence

$$\begin{aligned}
y_p &= \left(\frac{x}{2} - \frac{\cos(x)e^x}{4} + \frac{\sin(x)e^x}{4} + \frac{e^{2x}}{4} \right) (e^{-x}) \\
&+ \left(\frac{e^{(-1+i)x}}{4} + \frac{ie^{(1+i)x}}{4} + \frac{e^{2ix}}{16} - \frac{ie^{2ix}}{16} - \frac{x}{8} - \frac{ix}{8} \right) (e^{-ix}) \\
&+ \left(\frac{e^{(-1-i)x}}{4} - \frac{ie^{(1-i)x}}{4} - \frac{x}{8} + \frac{ix}{8} + \frac{e^{-2ix}}{16} + \frac{ie^{-2ix}}{16} \right) (e^{ix})
\end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{(4x+4)e^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{e^x}{4}$$

Therefore the general solution is

$$y = y_h + y_p \\ = (c_1 e^{-x} + e^{-ix} c_2 + e^{ix} c_3) + \left(\frac{(4x+4)e^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{e^x}{4} \right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{-ix} c_2 + e^{ix} c_3 + \frac{(4x+4)e^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{e^x}{4} \quad (1)$$

Verification of solutions

$$y = c_1 e^{-x} + e^{-ix} c_2 + e^{ix} c_3 + \frac{(4x+4)e^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{e^x}{4}$$

Verified OK.

11.8.1 Maple step by step solution

Let's solve

$$y''' + y'' + y' + y = e^x + e^{-x} + \sin(x)$$

- Highest derivative means the order of the ODE is 3

$$y'''$$

- Convert linear ODE into a system of first order ODEs

- Define new variable $y_1(x)$

$$y_1(x) = y$$

- Define new variable $y_2(x)$

$$y_2(x) = y'$$

- Define new variable $y_3(x)$

$$y_3(x) = y''$$

- Isolate for $y_3'(x)$ using original ODE

$$y_3'(x) = e^x + e^{-x} + \sin(x) - y_3(x) - y_2(x) - y_1(x)$$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = e^x + e^{-x} + \sin(x) - y_3(x) - y_2(x) - y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ e^x + e^{-x} + \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ e^x + e^{-x} + \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right], \left[1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathbf{I}, \begin{bmatrix} -1 \\ \mathbf{I} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-Ix} \cdot \begin{bmatrix} -1 \\ \mathbf{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(x) - \mathbf{I} \sin(x)) \cdot \begin{bmatrix} -1 \\ \mathbf{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathbf{I} \sin(x) \\ \mathbf{I}(\cos(x) - \mathbf{I} \sin(x)) \\ \cos(x) - \mathbf{I} \sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix} \right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p(x)$

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \vec{y}_p(x)$$

- Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is the identity matrix. $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & \sin(x) & \frac{e^{-x}}{2} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{2} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2} & \cos(x) & -\frac{e^{-x}}{2} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ \frac{e^{-x}}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -\sin(x) & \frac{e^{-x}}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \end{bmatrix}$$

- Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$
 $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$$\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(x)$

$$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$$

- Cancel like terms

$$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$$

- Integrate to solve for $\vec{v}(x)$

$$\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(x)$ into the equation for the particular solution

$$\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{(2+2x)e^{-x}}{4} + \frac{(-x-3)\cos(x)}{4} - \frac{\sin(x)x}{4} + \frac{e^x}{4} \\ -\frac{x e^{-x}}{2} + \frac{(-1-x)\cos(x)}{4} + \frac{(x+2)\sin(x)}{4} + \frac{e^x}{4} \\ \frac{(2x-2)e^{-x}}{4} + \frac{(x+1)\cos(x)}{4} + \frac{(x+2)\sin(x)}{4} + \frac{e^x}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1 \vec{y}_1 + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + \begin{bmatrix} \frac{(2+2x)e^{-x}}{4} + \frac{(-x-3)\cos(x)}{4} - \frac{\sin(x)x}{4} + \frac{e^x}{4} \\ -\frac{x e^{-x}}{2} + \frac{(-1-x)\cos(x)}{4} + \frac{(x+2)\sin(x)}{4} + \frac{e^x}{4} \\ \frac{(2x-2)e^{-x}}{4} + \frac{(x+1)\cos(x)}{4} + \frac{(x+2)\sin(x)}{4} + \frac{e^x}{4} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(2x+4c_1+2)e^{-x}}{4} + \frac{(-x-4c_2-3)\cos(x)}{4} + \frac{(-x+4c_3)\sin(x)}{4} + \frac{e^x}{4}$$

Maple trace

```

`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)+diff(y(x),x)+y(x)=exp(x)+exp(-x)+sin(x),y(x), singsol=a
```

$$y(x) = \frac{(2x + 4c_3 + 2)e^{-x}}{4} + \frac{(-x + 4c_2 + 1)\sin(x)}{4} + \frac{(-x + 4c_1)\cos(x)}{4} + \frac{e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.265 (sec). Leaf size: 55

```
DSolve[y'''[x]+y''[x]+y'[x]+y[x]==Exp[x]+Exp[-x]+Sin[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \rightarrow \frac{1}{8}(2e^{-x}(2x + e^{2x} + 2 + 4c_3) + (-2x - 1 + 8c_1)\cos(x) + (-2x + 3 + 8c_2)\sin(x))$$

11.9 problem 34

11.9.1 Solving as second order linear constant coeff ode	1709
11.9.2 Solving using Kovacic algorithm	1712
11.9.3 Maple step by step solution	1717

Internal problem ID [5400]

Internal file name [OUTPUT/4891_Sunday_February_04_2024_12_46_51_AM_49659287/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 34.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - y = x^2$$

11.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^x + c_2e^{-x}) + (-x^2 - 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{-x} - x^2 - 2 \tag{1}$$

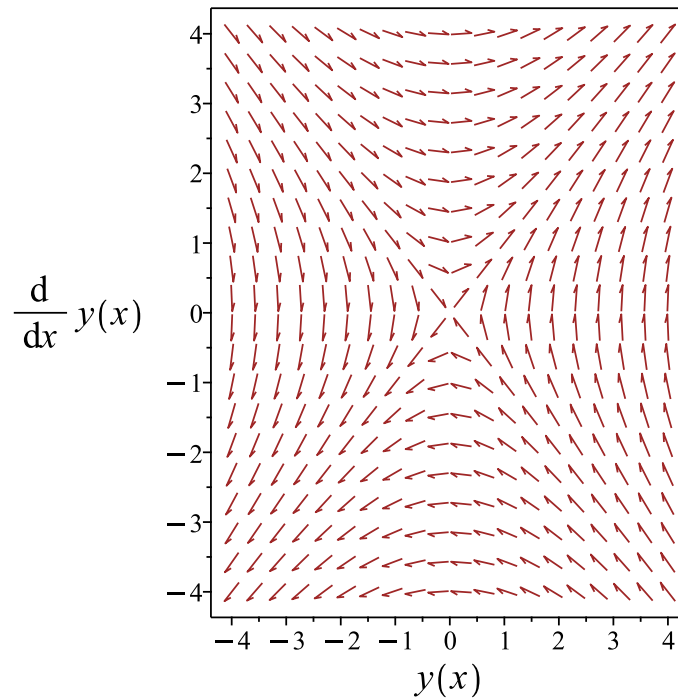


Figure 270: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - x^2 - 2$$

Verified OK.

11.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$

$$B = 0 \tag{3}$$

$$C = -1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{2}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-x} + \frac{c_2e^x}{2} \right) + (-x^2 - 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + \frac{c_2e^x}{2} - x^2 - 2 \quad (1)$$

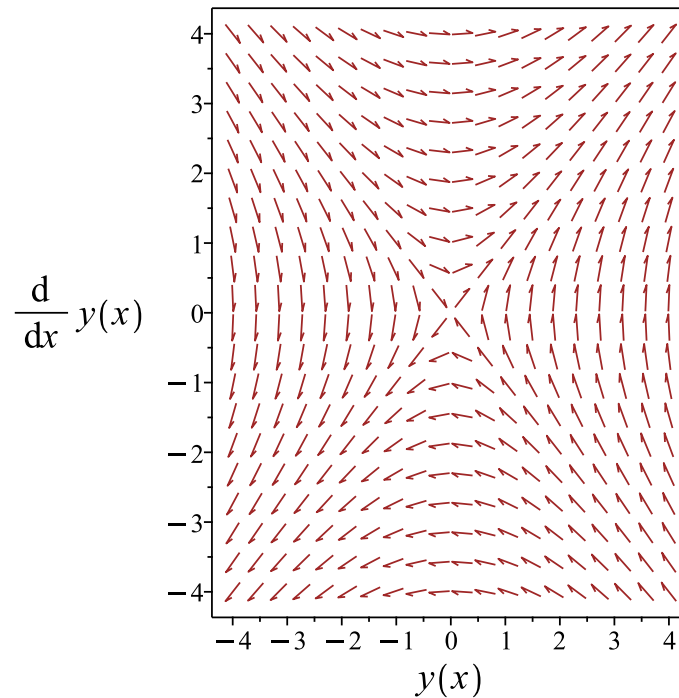


Figure 271: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - x^2 - 2$$

Verified OK.

11.9.3 Maple step by step solution

Let's solve

$$y'' - y = x^2$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = x^2 \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x} \left(\int x^2 e^x dx \right)}{2} + \frac{e^x \left(\int x^2 e^{-x} dx \right)}{2}$$

- Compute integrals

$$y_p(x) = -x^2 - 2$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x - x^2 - 2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-y(x)=x^2,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 - x^2 - 2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[y''[x]-y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -x^2 + c_1 e^x + c_2 e^{-x} - 2$$

11.10 problem 36

11.10.1 Solving as second order linear constant coeff ode	1720
11.10.2 Solving using Kovacic algorithm	1724
11.10.3 Maple step by step solution	1729

Internal problem ID [5401]

Internal file name [OUTPUT/4892_Sunday_February_04_2024_12_46_52_AM_3321707/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 36.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 2y = x^3 + x^2 + e^{-2x} + \cos(3x)$$

11.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = x^3 + x^2 + e^{-2x} + \cos(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2} \end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{2}$$

$$\lambda_2 = -i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = i\sqrt{2}$$

$$\lambda_2 = -i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0 \left(c_1 \cos \left(x\sqrt{2} \right) + c_2 \sin \left(x\sqrt{2} \right) \right)$$

Or

$$y = c_1 \cos \left(x\sqrt{2} \right) + c_2 \sin \left(x\sqrt{2} \right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + e^{-2x} + \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{\cos(3x), \sin(3x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x\sqrt{2}), \sin(x\sqrt{2})\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x} + A_2 \cos(3x) + A_3 \sin(3x) + A_4 + A_5 x + A_6 x^2 + A_7 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 6A_1 e^{-2x} - 7A_2 \cos(3x) - 7A_3 \sin(3x) + 2A_6 + 6A_7 x + 2A_4 + 2A_5 x + 2A_6 x^2 + 2A_7 x^3 \\ = x^3 + x^2 + e^{-2x} + \cos(3x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = -\frac{1}{7}, A_3 = 0, A_4 = -\frac{1}{2}, A_5 = -\frac{3}{2}, A_6 = \frac{1}{2}, A_7 = \frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) \right) + \left(\frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \quad (1)$$

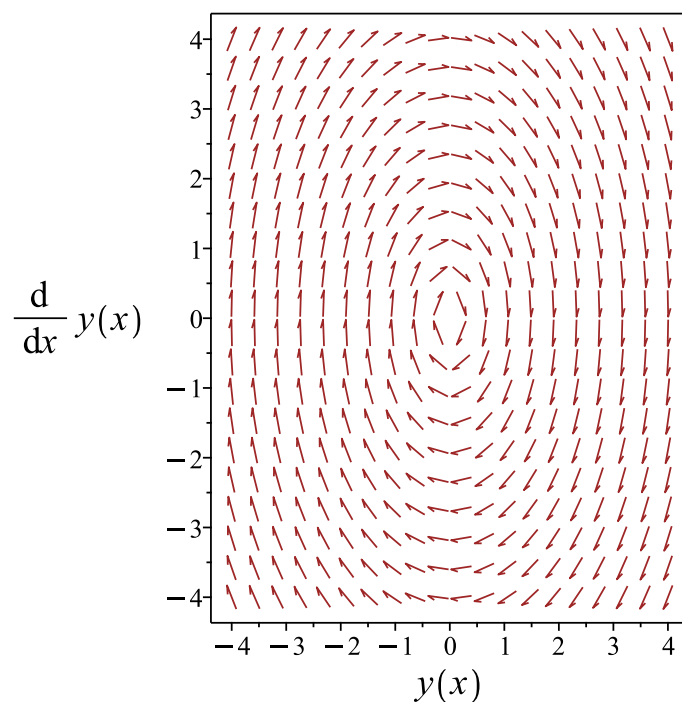


Figure 272: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Verified OK.

11.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x\sqrt{2})$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x\sqrt{2})\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x\sqrt{2})$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x\sqrt{2}) \int \frac{1}{\cos(x\sqrt{2})^2} dx \\ &= \cos(x\sqrt{2}) \left(\frac{\sqrt{2} \tan(x\sqrt{2})}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\cos(x\sqrt{2}) \right) + c_2 \left(\cos(x\sqrt{2}) \left(\frac{\sqrt{2} \tan(x\sqrt{2})}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + e^{-2x} + \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}, \{\cos(3x), \sin(3x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sqrt{2} \sin(x\sqrt{2})}{2}, \cos(x\sqrt{2}) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x} + A_2 \cos(3x) + A_3 \sin(3x) + A_4 + A_5 x + A_6 x^2 + A_7 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} 6A_1 e^{-2x} - 7A_2 \cos(3x) - 7A_3 \sin(3x) + 2A_6 + 6A_7 x + 2A_4 + 2A_5 x + 2A_6 x^2 + 2A_7 x^3 \\ = x^3 + x^2 + e^{-2x} + \cos(3x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = -\frac{1}{7}, A_3 = 0, A_4 = -\frac{1}{2}, A_5 = -\frac{3}{2}, A_6 = \frac{1}{2}, A_7 = \frac{1}{2} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2} \right) + \left(\frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2} + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \quad (1)$$

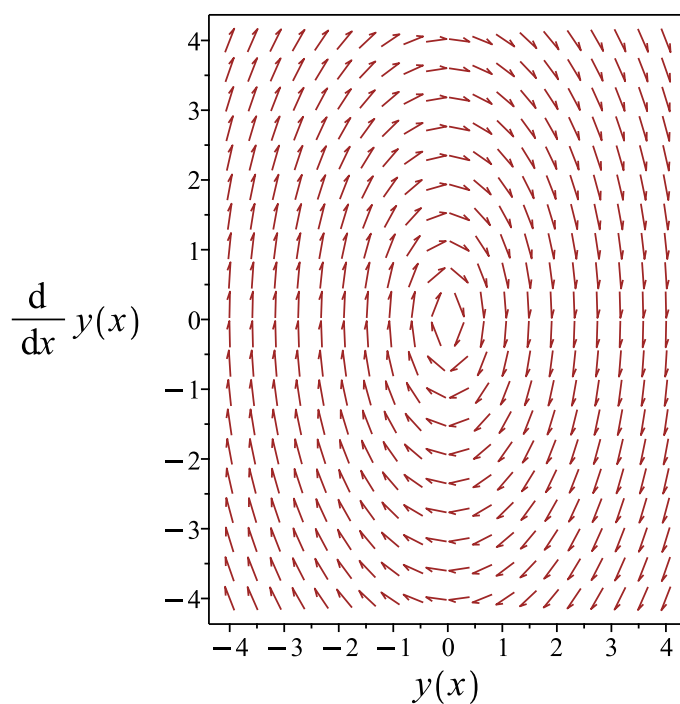


Figure 273: Slope field plot

Verification of solutions

$$y = c_1 \cos(x\sqrt{2}) + \frac{c_2\sqrt{2} \sin(x\sqrt{2})}{2} + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Verified OK.

11.10.3 Maple step by step solution

Let's solve

$$y'' + 2y = x^3 + x^2 + e^{-2x} + \cos(3x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 2 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{0 \pm (\sqrt{-8})}{2}$$

- Roots of the characteristic polynomial

$$r = (-I\sqrt{2}, I\sqrt{2})$$

- 1st solution of the homogeneous ODE

$$y_1(x) = \cos(x\sqrt{2})$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = \sin(x\sqrt{2})$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right) \right], f(x) = x^3 + x^2 + e^{-2x} + \cos(3x)$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x\sqrt{2}) & \sin(x\sqrt{2}) \\ -\sqrt{2} \sin(x\sqrt{2}) & \sqrt{2} \cos(x\sqrt{2}) \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \sqrt{2}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\sqrt{2} \left(\cos(x\sqrt{2}) \left(\int \sin(x\sqrt{2}) (x^3+x^2+e^{-2x}+\cos(3x)) dx \right) - \sin(x\sqrt{2}) \left(\int \cos(x\sqrt{2}) (x^3+x^2+e^{-2x}+\cos(3x)) dx \right) \right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)+2*y(x)=x^3+x^2+exp(-2*x)+cos(3*x),y(x), singsol=all)
```

$$y(x) = \sin(\sqrt{2}x) c_2 + \cos(\sqrt{2}x) c_1 + \frac{x^3}{2} + \frac{x^2}{2} - \frac{3x}{2} - \frac{\cos(3x)}{7} - \frac{1}{2} + \frac{e^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 4.775 (sec). Leaf size: 69

```
DSolve[y''[x]+2*y[x]==x^3+x^2+Exp[-2*x]+Cos[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{42} \left(21x^3 + 21x^2 - 63x + 7e^{-2x} + 9 \sin(x) \sin(2x) - 6 \cos^3(x) + 42c_1 \cos(\sqrt{2}x) + 42c_2 \sin(\sqrt{2}x) - 21 \right)$$

11.11 problem 37

11.11.1 Solving as second order linear constant coeff ode	1731
11.11.2 Solving using Kovacic algorithm	1735
11.11.3 Maple step by step solution	1740

Internal problem ID [5402]

Internal file name [OUTPUT/4893_Sunday_February_04_2024_12_46_52_AM_84754092/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 37.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 2y' - y = \cos(x) e^x$$

11.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -1, f(x) = \cos(x) e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-2^2 - (4)(1)(-1)} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

Hence

$$\lambda_1 = 1 + \sqrt{2}$$

$$\lambda_2 = 1 - \sqrt{2}$$

Which simplifies to

$$\lambda_1 = 1 + \sqrt{2}$$

$$\lambda_2 = 1 - \sqrt{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}$$

Or

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{x(1-\sqrt{2})}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{(1+\sqrt{2})x} + c_2 e^{x(1-\sqrt{2})}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x) e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^x, \sin(x) e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{x(1-\sqrt{2})}, e^{(1+\sqrt{2})x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^x + A_2 \sin(x) e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(x) e^x - 3A_2 \sin(x) e^x = \cos(x) e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{\cos(x) e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{(1+\sqrt{2})x} + c_2 e^{x(1-\sqrt{2})} \right) + \left(-\frac{\cos(x) e^x}{3} \right) \end{aligned}$$

Which simplifies to

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{-(\sqrt{2}-1)x} - \frac{\cos(x) e^x}{3}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{-(\sqrt{2}-1)x} - \frac{\cos(x) e^x}{3} \quad (1)$$

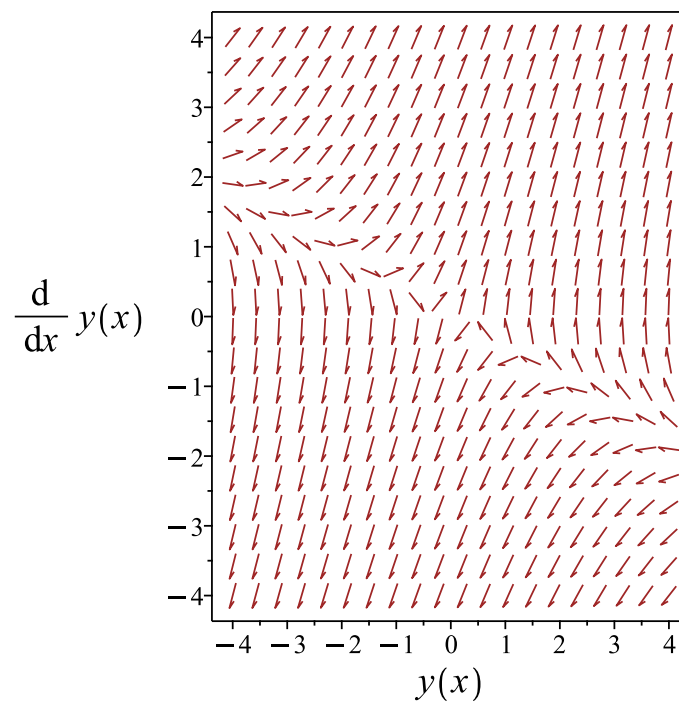


Figure 274: Slope field plot

Verification of solutions

$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{-(\sqrt{2}-1)x} - \frac{\cos(x) e^x}{3}$$

Verified OK.

11.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 2z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x\sqrt{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} dx} \end{aligned}$$

$$\begin{aligned}
&= z_1 e^x \\
&= z_1 (e^x)
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-(\sqrt{2}-1)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1} dx}}{(y_1)^2} dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2} dx \\
&= y_1 \left(\frac{\sqrt{2} e^{2x\sqrt{2}}}{4} \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-(\sqrt{2}-1)x} \right) + c_2 \left(e^{-(\sqrt{2}-1)x} \left(\frac{\sqrt{2} e^{2x\sqrt{2}}}{4} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-(\sqrt{2}-1)x} + \frac{c_2 \sqrt{2} e^{(1+\sqrt{2})x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x) e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) e^x, \sin(x) e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{\sqrt{2} e^{(1+\sqrt{2})x}}{4}, e^{-(\sqrt{2}-1)x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) e^x + A_2 \sin(x) e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(x) e^x - 3A_2 \sin(x) e^x = \cos(x) e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = 0 \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = -\frac{\cos(x) e^x}{3}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 e^{-(\sqrt{2}-1)x} + \frac{c_2 \sqrt{2} e^{(1+\sqrt{2})x}}{4} \right) + \left(-\frac{\cos(x) e^x}{3} \right)$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-(\sqrt{2}-1)x} + \frac{c_2 \sqrt{2} e^{(1+\sqrt{2})x}}{4} - \frac{\cos(x) e^x}{3} \quad (1)$$

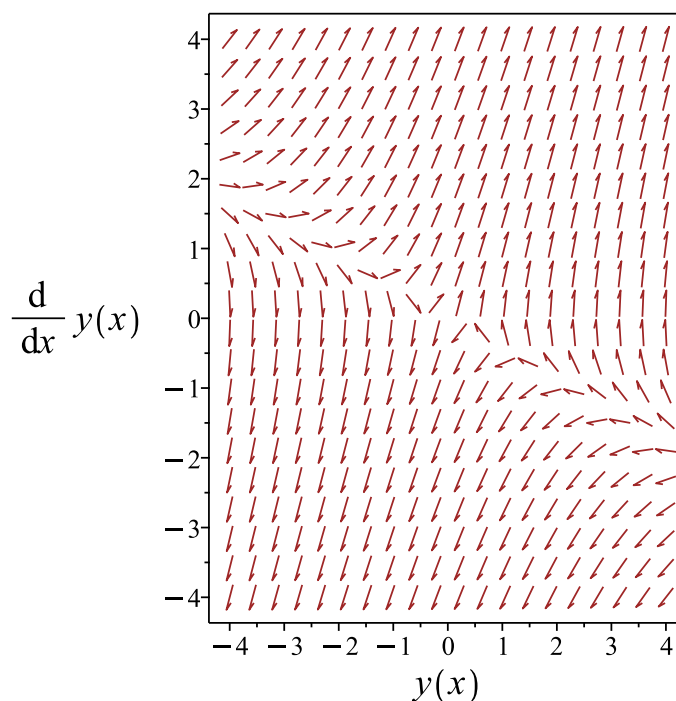


Figure 275: Slope field plot

Verification of solutions

$$y = c_1 e^{-(\sqrt{2}-1)x} + \frac{c_2 \sqrt{2} e^{(1+\sqrt{2})x}}{4} - \frac{\cos(x) e^x}{3}$$

Verified OK.

11.11.3 Maple step by step solution

Let's solve

$$y'' - 2y' - y = \cos(x) e^x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 2r - 1 = 0$$

- Use quadratic formula to solve for r

$$r = \frac{2 \pm (\sqrt{8})}{2}$$

- Roots of the characteristic polynomial

$$r = (1 - \sqrt{2}, 1 + \sqrt{2})$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{x(1-\sqrt{2})}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{(1+\sqrt{2})x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{x(1-\sqrt{2})} + c_2 e^{(1+\sqrt{2})x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \cos(x) e^x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{x(1-\sqrt{2})} & e^{(1+\sqrt{2})x} \\ (1-\sqrt{2})e^{x(1-\sqrt{2})} & (1+\sqrt{2})e^{(1+\sqrt{2})x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2\sqrt{2}e^{2x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\sqrt{2} \left(e^{(1+\sqrt{2})x} \left(\int \cos(x) e^{-x\sqrt{2}} dx \right) - e^{-(\sqrt{2}-1)x} \left(\int \cos(x) e^{x\sqrt{2}} dx \right) \right)}{4}$$

- Compute integrals

$$y_p(x) = -\frac{\cos(x)e^x}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{x(1-\sqrt{2})} + c_2 e^{(1+\sqrt{2})x} - \frac{\cos(x)e^x}{3}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-y(x)=exp(x)*cos(x),y(x), singsol=all)
```

$$y(x) = e^{(1+\sqrt{2})x} c_2 + e^{-(\sqrt{2}-1)x} c_1 - \frac{e^x \cos(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 56

```
DSolve[y''[x]-2*y'[x]-y[x]==Exp[x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{3} e^{-\sqrt{2}x} \left(-e^{(1+\sqrt{2})x} \cos(x) + 3e^x (c_2 e^{2\sqrt{2}x} + c_1) \right)$$

11.12 problem 38

11.12.1 Solving as second order linear constant coeff ode	1742
11.12.2 Solving as linear second order ode solved by an integrating factor ode	1746
11.12.3 Solving using Kovacic algorithm	1748
11.12.4 Maple step by step solution	1754

Internal problem ID [5403]

Internal file name [OUTPUT/4894_Sunday_February_04_2024_12_46_53_AM_6482310/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 38.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$$

11.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = \frac{e^{2x}}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 4e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{(-4)^2 - (4)(1)(4)} \\ &= 2 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \quad (1)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + x e^{2x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= e^{2x} \\ y_2 &= e^{2x} x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{2x}x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ 2e^{2x} & 2e^{2x}x + e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})(2e^{2x}x + e^{2x}) - (e^{2x}x)(2e^{2x})$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{4x}}{e^{4x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{1}{x} dx$$

Hence

$$u_1 = - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{4x}}{x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x) e^{2x} - e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + x e^{2x} c_2) + (e^{2x}(-1 - \ln(x))) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x))$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x)) \quad (1)$$

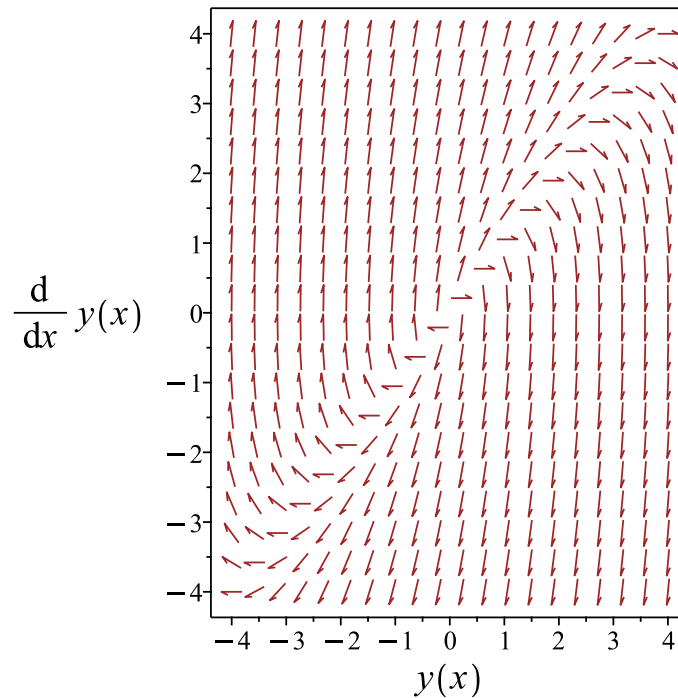


Figure 276: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^{2x}(-1 - \ln(x))$$

Verified OK.

11.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x))^2 + p'(x)}{2}y = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p dx} \\ &= e^{\int -4 dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \frac{e^{-2x}e^{2x}}{x^2}$$
$$(y e^{-2x})'' = \frac{e^{-2x}e^{2x}}{x^2}$$

Integrating once gives

$$(y e^{-2x})' = c_1 - \frac{1}{x}$$

Integrating again gives

$$(y e^{-2x}) = c_1 x - \ln(x) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \ln(x) + c_2}{e^{-2x}}$$

Or

$$y = c_1 x e^{2x} + c_2 e^{2x} - \ln(x) e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x e^{2x} + c_2 e^{2x} - \ln(x) e^{2x} \quad (1)$$

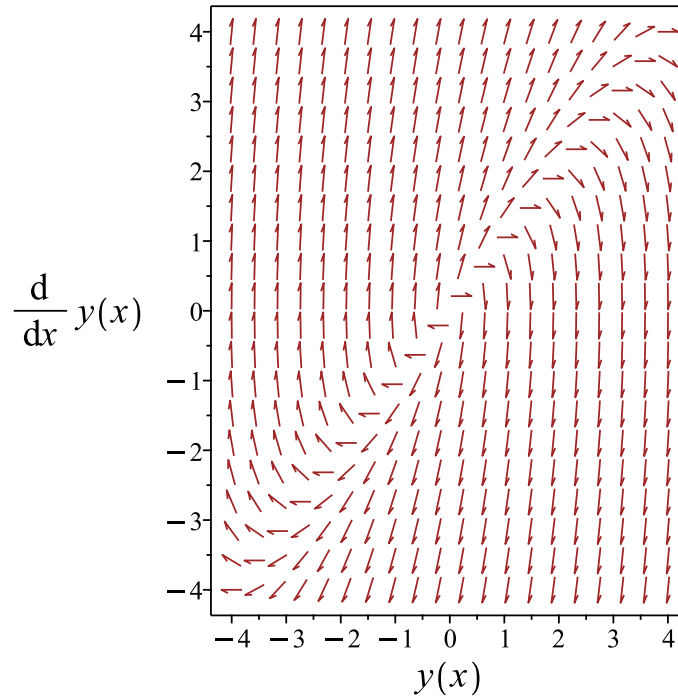


Figure 277: Slope field plot

Verification of solutions

$$y = c_1 x e^{2x} + c_2 e^{2x} - \ln(x) e^{2x}$$

Verified OK.

11.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0 \\ t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} dx} \\ &= z_1 e^{2x} \\ &= z_1 (e^{2x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2} dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{2x}) + c_2 (e^{2x}(x))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + x e^{2x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{2x} \\ y_2 &= e^{2x} x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{2x}x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ 2e^{2x} & 2e^{2x}x + e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})(2e^{2x}x + e^{2x}) - (e^{2x}x)(2e^{2x})$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{4x}}{e^{4x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{1}{x} dx$$

Hence

$$u_1 = - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{4x}}{e^{4x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x) e^{2x} - e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + x e^{2x} c_2) + (e^{2x}(-1 - \ln(x))) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x))$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x)) \quad (1)$$

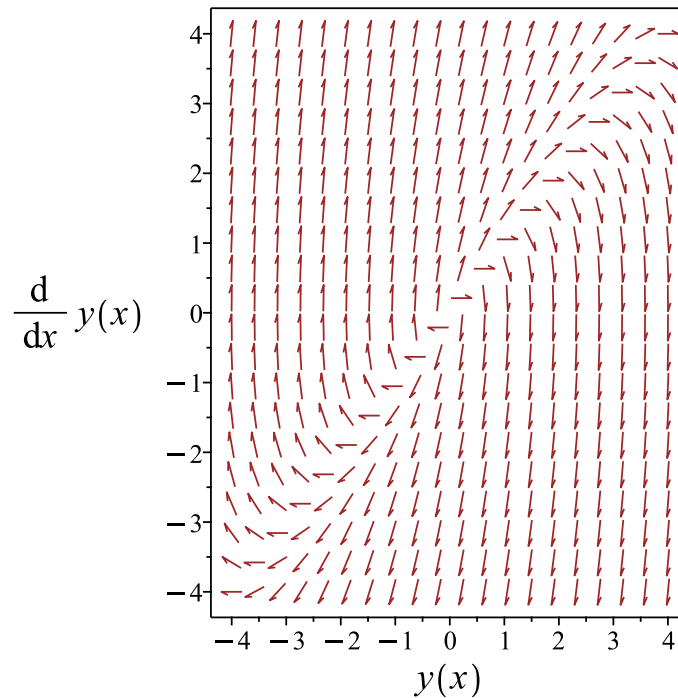


Figure 278: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^{2x}(-1 - \ln(x))$$

Verified OK.

11.12.4 Maple step by step solution

Let's solve

$$y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 4r + 4 = 0$$

- Factor the characteristic polynomial

$$(r - 2)^2 = 0$$

- Root of the characteristic polynomial

$$r = 2$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{2x}$$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

$$y_2(x) = e^{2x}x$$

- General solution of the ODE

$$y = c_1y_1(x) + c_2y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1e^{2x} + xe^{2x}c_2 + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = \frac{e^{2x}}{x^2} \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x}x \\ 2e^{2x} & 2e^{2x}x + e^{2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{4x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = e^{2x} \left(- \left(\int \frac{1}{x} dx \right) + \left(\int \frac{1}{x^2} dx \right) x \right)$$

- Compute integrals

$$y_p(x) = -e^{2x}(1 + \ln(x))$$

- Substitute particular solution into general solution to ODE

$$y = c_1e^{2x} + xe^{2x}c_2 - e^{2x}(1 + \ln(x))$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(2*x)/x^2,y(x), singsol=all)
```

$$y(x) = e^{2x}(-1 + c_1x - \ln(x) + c_2)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+4*y[x]==Exp[2*x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{2x}(-\log(x) + c_2x - 1 + c_1)$$

11.13 problem 39

11.13.1 Solving as second order linear constant coeff ode	1757
11.13.2 Solving using Kovacic algorithm	1760
11.13.3 Maple step by step solution	1765

Internal problem ID [5404]

Internal file name [OUTPUT/4895_Sunday_February_04_2024_12_46_54_AM_80399894/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 39.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_linear_constant_coeff"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' - y = e^{3x}x$$

11.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = e^{3x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1 \end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^x + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}x, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1e^{3x}x + A_2e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1e^{3x}x + 6A_1e^{3x} + 8A_2e^{3x} = e^{3x}x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = -\frac{3}{32} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{3x}x}{8} - \frac{3e^{3x}}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^x + c_2e^{-x}) + \left(\frac{e^{3x}x}{8} - \frac{3e^{3x}}{32} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{-x} + \frac{e^{3x}x}{8} - \frac{3e^{3x}}{32} \quad (1)$$

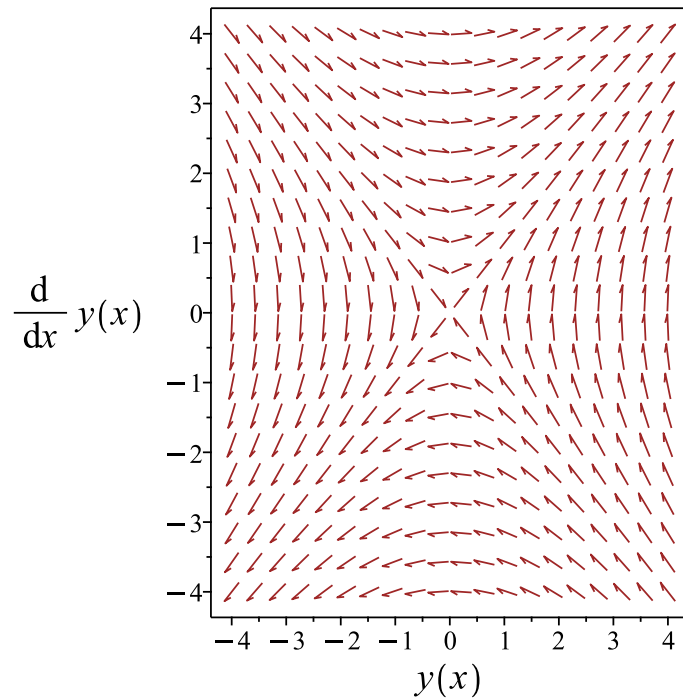


Figure 279: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{e^{3x} x}{8} - \frac{3 e^{3x}}{32}$$

Verified OK.

11.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1 \\ t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= e^{-x} \int \frac{1}{e^{-2x}} dx \\ &= e^{-x} \left(\frac{e^{2x}}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-x}) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x} x, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \frac{e^x}{2}, e^{-x} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x} x + A_2 e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1e^{3x}x + 6A_1e^{3x} + 8A_2e^{3x} = e^{3x}x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = -\frac{3}{32} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{e^{3x}x}{8} - \frac{3e^{3x}}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{-x} + \frac{c_2e^x}{2} \right) + \left(\frac{e^{3x}x}{8} - \frac{3e^{3x}}{32} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + \frac{c_2e^x}{2} + \frac{e^{3x}x}{8} - \frac{3e^{3x}}{32} \quad (1)$$

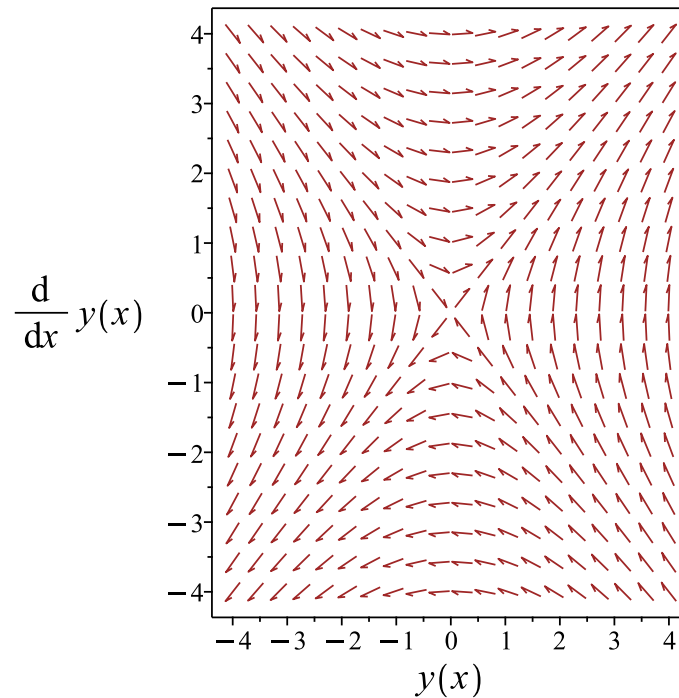


Figure 280: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{e^{3x} x}{8} - \frac{3 e^{3x}}{32}$$

Verified OK.

11.13.3 Maple step by step solution

Let's solve

$$y'' - y = e^{3x} x$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 - 1 = 0$$

- Factor the characteristic polynomial

$$(r - 1)(r + 1) = 0$$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^x$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-x} + c_2 e^x + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^{3x} x \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}(\int x e^{4x} dx)}{2} + \frac{e^x(\int e^{2x} x dx)}{2}$$

- Compute integrals

$$y_p(x) = \frac{(-3+4x)e^{3x}}{32}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^x + \frac{(-3+4x)e^{3x}}{32}$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    <- constant coefficients successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-y(x)=x*exp(3*x),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^x c_1 + \frac{(4x - 3) e^{3x}}{32}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 34

```
DSolve[y''[x]-y[x]==x*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{32} e^{3x} (4x - 3) + c_1 e^x + c_2 e^{-x}$$

11.14 problem 40

11.14.1 Solving as second order linear constant coeff ode	1768
11.14.2 Solving using Kovacic algorithm	1773
11.14.3 Maple step by step solution	1780

Internal problem ID [5405]

Internal file name [OUTPUT/4896_Sunday_February_04_2024_12_46_54_AM_96116373/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplementary problems. Page 107

Problem number: 40.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + 5y' + 6y = e^{-2x} \sec(x)^2 (1 + 2 \tan(x))$$

11.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 6, f(x) = e^{-2x} \sec(x)^2 (1 + 2 \tan(x))$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 6e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{5^2 - (4)(1)(6)} \\ &= -\frac{5}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{5}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= -2 \\ \lambda_2 &= -3 \end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned} y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-2)x} + c_2 e^{(-3)x} \end{aligned}$$

Or

$$y = c_1 e^{-2x} + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-2x} + c_2 e^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$

$$y_2 = e^{-3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ \frac{d}{dx}(e^{-2x}) & \frac{d}{dx}(e^{-3x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

Therefore

$$W = (e^{-2x})(-3e^{-3x}) - (e^{-3x})(-2e^{-2x})$$

Which simplifies to

$$W = -e^{-2x} e^{-3x}$$

Which simplifies to

$$W = -e^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{-3x} e^{-2x} \sec(x)^2 (1 + 2 \tan(x))}{-e^{-5x}} dx$$

Which simplifies to

$$u_1 = - \int (-1 - 2 \tan(x)) \sec(x)^2 dx$$

Hence

$$u_1 = \tan(x)^2 + \tan(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-4x} \sec(x)^2 (1 + 2 \tan(x))}{-e^{-5x}} dx$$

Which simplifies to

$$u_2 = \int (-1 - 2 \tan(x)) \sec(x)^2 e^x dx$$

Hence

$$u_2 = \frac{e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$

Which simplifies to

$$u_1 = \tan(x) (\tan(x) + 1)$$

$$u_2 = \frac{e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \tan(x) (\tan(x) + 1) e^{-2x} + \frac{(e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1) e^{-3x}}{(e^{2ix} + 1)^2}$$

Which simplifies to

$$y_p(x) = \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan(x) + 1)}{(e^{2ix} + 1)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-2x} + c_2 e^{-3x}) \\ &\quad + \left(\frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan(x) + 1)}{(e^{2ix} + 1)^2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 e^{-2x} + c_2 e^{-3x} \\ &\quad + \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan(x) + 1)}{(e^{2ix} + 1)^2} \end{aligned} \tag{1}$$

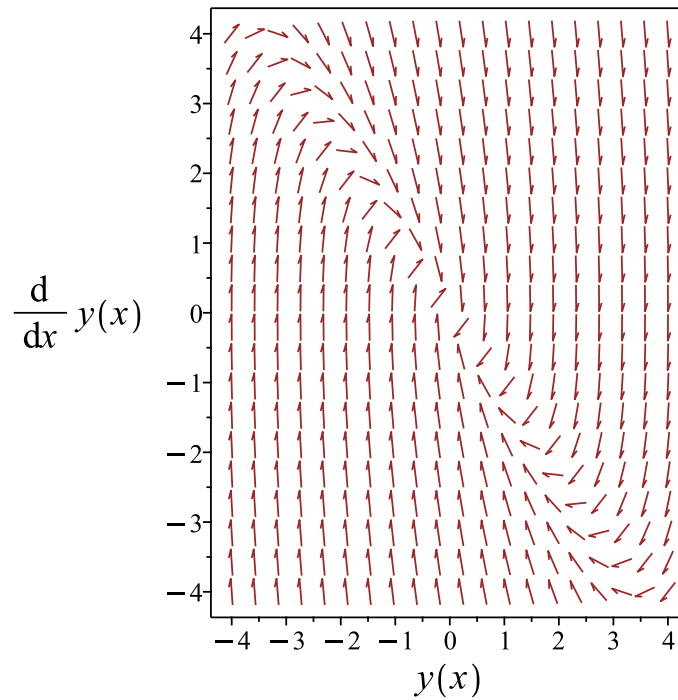


Figure 281: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2 e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan(x)^2 + \tan(x))}{(e^{2ix} + 1)^2}$$

Verified OK.

11.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}
 \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\
 &= 0 - 0 \\
 &= 0
 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
 y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
 &= z_1 e^{-\int \frac{1}{2} \frac{5}{1} dx} \\
 &= z_1 e^{-\frac{5x}{2}} \\
 &= z_1 \left(e^{-\frac{5x}{2}} \right)
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2} dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(e^{-3x}) + c_2(e^{-3x}(e^x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x}$$

$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} & e^{-2x} \\ \frac{d}{dx}(e^{-3x}) & \frac{d}{dx}(e^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{vmatrix}$$

Therefore

$$W = (e^{-3x})(-2e^{-2x}) - (e^{-2x})(-3e^{-3x})$$

Which simplifies to

$$W = e^{-2x}e^{-3x}$$

Which simplifies to

$$W = e^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{e^{-4x} \sec(x)^2 (1 + 2 \tan(x))}{e^{-5x}} dx$$

Which simplifies to

$$u_1 = - \int \sec(x)^2 e^x (1 + 2 \tan(x)) dx$$

Hence

$$u_1 = - \frac{-e^{4ix} - 2e^{2ix} + 4e^{(1+2i)x} - 1}{(e^{2ix} + 1)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x} e^{-2x} \sec(x)^2 (1 + 2 \tan(x))}{e^{-5x}} dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2 (1 + 2 \tan(x)) dx$$

Hence

$$u_2 = \tan(x)^2 + \tan(x)$$

Which simplifies to

$$u_1 = \frac{e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$

$$u_2 = \tan(x) (\tan(x) + 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \tan(x) (\tan(x) + 1) e^{-2x} + \frac{(e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1) e^{-3x}}{(e^{2ix} + 1)^2}$$

Which simplifies to

$$y_p(x) = \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan(x) + 1)}{(e^{2ix} + 1)^2}$$

Therefore the general solution is

$$\begin{aligned}
 y &= y_h + y_p \\
 &= (c_1 e^{-3x} + c_2 e^{-2x}) \\
 &\quad + \left(\frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2 e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x}}{(e^{2ix} + 1)^2} \right)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
 y &= c_1 e^{-3x} + c_2 e^{-2x} \\
 &\quad + \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2 e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x}}{(e^{2ix} + 1)^2} \tag{1}
 \end{aligned}$$

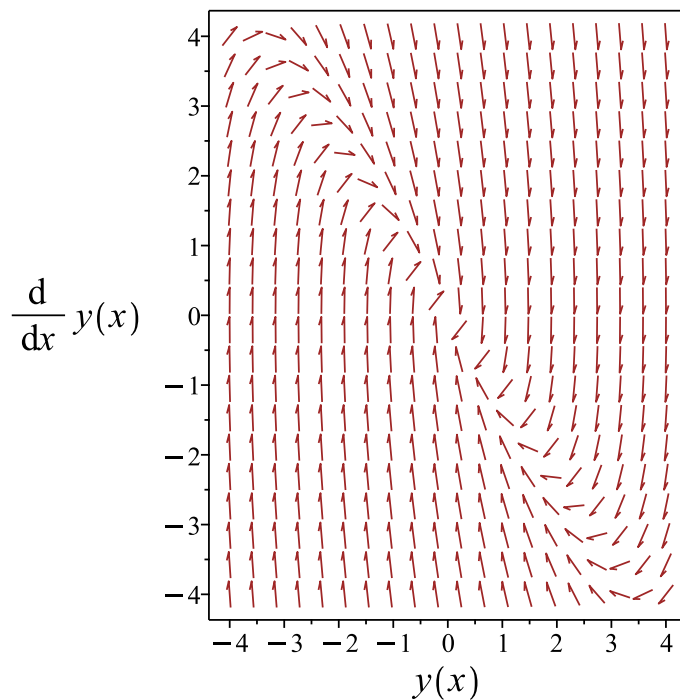


Figure 282: Slope field plot

Verification of solutions

$$\begin{aligned}
 y &= c_1 e^{-3x} + c_2 e^{-2x} \\
 &\quad + \frac{(2 \tan(x)^2 + 2 \tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2 e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x}}{(e^{2ix} + 1)^2}
 \end{aligned}$$

Verified OK.

11.14.3 Maple step by step solution

Let's solve

$$y'' + 5y' + 6y = e^{-2x} \sec(x)^2 (1 + 2 \tan(x))$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Characteristic polynomial of homogeneous ODE

$$r^2 + 5r + 6 = 0$$

- Factor the characteristic polynomial

$$(r + 3)(r + 2) = 0$$

- Roots of the characteristic polynomial

$$r = (-3, -2)$$

- 1st solution of the homogeneous ODE

$$y_1(x) = e^{-3x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x) = e^{-2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y = c_1 e^{-3x} + c_2 e^{-2x} + y_p(x)$$

- Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))} dx \right) + y_2(x) \left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))} dx \right), f(x) = e^{-2x} \sec(x)^2 (1 + 2 \tan(x)) \right]$$

- Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{-2x} \\ -3e^{-3x} & -2e^{-2x} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = e^{-5x}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -e^{-3x} \left(\int \sec(x)^2 e^x (1 + 2 \tan(x)) dx \right) + e^{-2x} \left(\int \sec(x)^2 (1 + 2 \tan(x)) dx \right)$$

- Compute integrals

$$y_p(x) = \frac{(2 \tan(x)^2 + 2 \tan(x) - 4)e^{(-2+2i)x} + \tan(x)(e^{-2x} + e^{(-2+4i)x})(\tan(x)+1)}{(e^{2ix} + 1)^2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{(2 \tan(x)^2 + 2 \tan(x) - 4)e^{(-2+2i)x} + \tan(x)(e^{-2x} + e^{(-2+4i)x})(\tan(x)+1)}{(e^{2ix} + 1)^2}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 95

```
dsolve(diff(y(x), x$2) + 5*diff(y(x), x) + 6*y(x) = exp(-2*x)*sec(x)^2*(1+2*tan(x)), y(x), singsol=all)
```

$$y(x) = \frac{(2 \sec(x)^2 + 2c_2 + 2 \tan(x) - 6) e^{(-2+2i)x} + (\tan(x)^2 + c_2 + \tan(x)) e^{(-2+4i)x} + 2c_1 e^{(-3+2i)x} + c_1 e^{(-3+4i)x}}{e^{4ix} + 2e^{2ix} + 1}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 26

```
DSolve[y''[x] + 5*y'[x] + 6*y[x] == Exp[-2*x]*Sec[x]^2*(1+2*Tan[x]), y[x], x, IncludeSingularSolution -> True]
```

$$y(x) \rightarrow e^{-3x}(e^x \tan(x) + c_2 e^x + c_1)$$

12 Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre).

Supplementary problems. Page 110

12.1 problem 6	1783
12.2 problem 7	1811
12.3 problem 8	1849
12.4 problem 9	1856
12.5 problem 10	1863
12.6 problem 11	1894

12.1 problem 6

12.1.1 Solving as second order euler ode ode	1783
12.1.2 Solving as second order change of variable on x method 2 ode .	1787
12.1.3 Solving as second order change of variable on x method 1 ode .	1792
12.1.4 Solving as second order change of variable on y method 2 ode .	1797
12.1.5 Solving using Kovacic algorithm	1802

Internal problem ID [5406]

Internal file name [OUTPUT/4897_Sunday_February_04_2024_12_46_55_AM_34394589/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre).
Supplementary problems. Page 110

Problem number: 6.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$x^2y'' - 3xy' + 4y = x + \ln(x)x^2$$

12.1.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -3x$, $C = 4$, $f(x) = x(\ln(x)x + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)x^r - 3rx^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$

$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 4y = x(\ln(x)x + 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$

$$y_2 = \ln(x) x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x) x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ 2x & x + 2 \ln(x) x \end{vmatrix}$$

Therefore

$$W = (x^2) (x + 2 \ln(x) x) - (\ln(x) x^2) (2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\ln(x) x^3 (\ln(x) x + 1)}{x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x)(\ln(x)x + 1)}{x^2} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3(\ln(x)x + 1)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)x + 1}{x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x} \right) x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x} \right) \ln(x) x^2$$

Which simplifies to

$$y_p(x) = \frac{x(\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{\ln(x)^3 x^2}{6} + c_2 x^2 \ln(x) + c_1 x^2 + x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^3 x^2}{6} + c_2 x^2 \ln(x) + c_1 x^2 + x \quad (1)$$

Verification of solutions

$$y = \frac{\ln(x)^3 x^2}{6} + c_2 x^2 \ln(x) + c_1 x^2 + x$$

Verified OK.

12.1.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x) \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x) dx)} dx \\ &= \int e^{-(\int -\frac{3}{x} dx)} dx \\ &= \int e^{3 \ln(x)} dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{4}{x^2}}{x^6} \\ &= \frac{4}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + \frac{4y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2} y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4 \left(\frac{d^2}{d\tau^2} y(\tau) \right) \tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\tau^r + 0\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$

$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau} \ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}(\sqrt{x^4}) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right) \left(-\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(-\ln(2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2}\right) x (\ln(x) x + 1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_1 = - \int - \frac{(\ln(2) - 2 \ln(x)) (\ln(x) x + 1)}{2x^2} dx$$

Hence

$$u_1 = - \frac{4 \ln(x)^3 x - 3 \ln(2) \ln(x)^2 x - 12 \ln(x) + 6 \ln(2) - 12}{12x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4} x (\ln(x) x + 1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x) x + 1}{2x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \frac{(4 \ln(x)^3 x - 3 \ln(2) \ln(x)^2 x - 12 \ln(x) + 6 \ln(2) - 12) \sqrt{x^4}}{12x} + \left(\frac{\ln(x)^2}{4} - \frac{1}{2x}\right) \left(-\ln(2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2}\right)$$

Which simplifies to

$$y_p(x) = \frac{x (\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2} \right) + \left(\frac{x(\ln(x)^3 x + 6)}{6} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2} + \frac{x(\ln(x)^3 x + 6)}{6} \quad (1)$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1) \sqrt{x^4}}{2} + \frac{x(\ln(x)^3 x + 6)}{6}$$

Verified OK. $\{0 < x\}$

12.1.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x(\ln(x)x + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$

$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}x^3} \end{aligned} \quad (6)$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2y(\tau) &= 0 \end{aligned} \quad (7)$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau}c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c} \sqrt{q} dx \\ &= \frac{\int 2\sqrt{\frac{1}{x^2}} dx}{c} \\ &= \frac{2\sqrt{\frac{1}{x^2}} x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 4y = x(\ln(x)x + 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}(\sqrt{x^4}) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x^4}) \left(-\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \right) \left(\frac{2x^3}{\sqrt{x^4}} \right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \right) x(\ln(x)x + 1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_1 = - \int -\frac{(\ln(2) - 2\ln(x))(\ln(x)x + 1)}{2x^2} dx$$

Hence

$$u_1 = - \frac{4\ln(x)^3 x - 3\ln(2)\ln(x)^2 x - 12\ln(x) + 6\ln(2) - 12}{12x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4} x (\ln(x) x + 1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\ln(x) x + 1}{2x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(4 \ln(x)^3 x - 3 \ln(2) \ln(x)^2 x - 12 \ln(x) + 6 \ln(2) - 12) \sqrt{x^4}}{12x} + \left(\frac{\ln(x)^2}{4} - \frac{1}{2x} \right) \left(-\ln(2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2} \right)$$

Which simplifies to

$$y_p(x) = \frac{x(\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + \left(\frac{x(\ln(x)^3 x + 6)}{6} \right) \\ &= \frac{x(\ln(x)^3 x + 6)}{6} + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{\ln(x)^3 x^2}{6} + c_1 x^2 + x$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^3 x^2}{6} + c_1 x^2 + x \quad (1)$$

Verification of solutions

$$y = \frac{\ln(x)^3 x^2}{6} + c_1 x^2 + x$$

Verified OK. $\{0 < x\}$

12.1.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x(\ln(x)x + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 2 \quad (6)$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{1}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{1}{x} dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x}\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \ln(x) + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x) x^n \\ &= (c_1 \ln(x) + c_2) x^2 \\ &= (c_1 \ln(x) + c_2) x^2\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 4y = x(\ln(x)x + 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x^2 \\ y_2 &= \ln(x) x^2\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x) x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ 2x & x + 2 \ln(x) x \end{vmatrix}$$

Therefore

$$W = (x^2) (x + 2 \ln(x) x) - (\ln(x) x^2) (2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\ln(x) x^3 (\ln(x) x + 1)}{x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x) (\ln(x) x + 1)}{x^2} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3(\ln(x)x + 1)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)x + 1}{x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}\right)x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x}\right)\ln(x)x^2$$

Which simplifies to

$$y_p(x) = \frac{x(\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= ((c_1 \ln(x) + c_2)x^2) + \left(\frac{x(\ln(x)^3 x + 6)}{6}\right) \\ &= \frac{x(\ln(x)^3 x + 6)}{6} + (c_1 \ln(x) + c_2)x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{\ln(x)^3 x^2}{6} + x^2 c_1 \ln(x) + c_2 x^2 + x$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^3 x^2}{6} + x^2 c_1 \ln(x) + c_2 x^2 + x \quad (1)$$

Verification of solutions

$$y = \frac{\ln(x)^3 x^2}{6} + x^2 c_1 \ln(x) + c_2 x^2 + x$$

Verified OK. $\{0 < x\}$

12.1.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3xy' + 4y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole

larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega dx}$$
$$= e^{\int \frac{1}{2x} dx}$$
$$= \sqrt{x}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$
$$= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} dx}$$
$$= z_1 e^{\frac{3 \ln(x)}{2}}$$
$$= z_1 \left(x^{\frac{3}{2}}\right)$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-3x}{x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{3 \ln(x)}}{(y_1)^2} dx$$
$$= y_1 (\ln(x))$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x^2) + c_2 (x^2 (\ln(x)))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' - 3xy' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^2 + c_2 x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x) x^2
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x) x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x) x^2 \\ 2x & x + 2 \ln(x) x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2 \ln(x) x) - (\ln(x) x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\ln(x) x^3 (\ln(x) x + 1)}{x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x) (\ln(x) x + 1)}{x^2} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3 (\ln(x) x + 1)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x) x + 1}{x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x} \right) x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x} \right) \ln(x) x^2$$

Which simplifies to

$$y_p(x) = \frac{x(\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2 + c_2 x^2 \ln(x)) + \left(\frac{x(\ln(x)^3 x + 6)}{6} \right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{x(\ln(x)^3 x + 6)}{6}$$

Summary

The solution(s) found are the following

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{x(\ln(x)^3 x + 6)}{6} \tag{1}$$

Verification of solutions

$$y = x^2(c_1 + c_2 \ln(x)) + \frac{x(\ln(x)^3 x + 6)}{6}$$

Verified OK. $\{0 < x\}$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    checking if the LODE is of Euler type  
    <- LODE of Euler type successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x+x^2*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)^3 x^2}{6} + \ln(x) x^2 c_1 + c_2 x^2 + x$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 30

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x+x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6}x(x \log^3(x) + 6c_1x + 12c_2x \log(x) + 6)$$

12.2 problem 7

12.2.1 Solving as second order euler ode	1812
12.2.2 Solving as linear second order ode solved by an integrating factor ode	1815
12.2.3 Solving as second order change of variable on x method 2 ode	1816
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12.2.7 Solving as second order ode non constant coeff transformation on B ode	1838
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Internal problem ID [5407]

Internal file name [OUTPUT/4898_Sunday_February_04_2024_12_46_57_AM_38265301/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre).
Supplementary problems. Page 110

Problem number: 7.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)$$

12.2.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -2x$, $C = 2$, $f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xr x^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)x^r - 2rx^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$

$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2 (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2 \ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2 \ln(x) - \ln(x^2)}{x}$$
$$u_2 = \frac{-2 \ln(x)^2 - 2 \ln(x) + 2 \ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x \quad (1)$$

Verification of solutions

$$y = \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x$$

Verified OK.

12.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)y' + \frac{(p(x))^2 + p'(x)}{2}y = f(x)$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2} \int p dx} \\ &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= \frac{\ln(x)^2 - \ln(x^2)}{x^3} \\ \left(\frac{y}{x}\right)'' &= \frac{\ln(x)^2 - \ln(x^2)}{x^3}\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x}\right)' = \frac{-2 \ln(x)^2 - 2 \ln(x) + 2 \ln(x^2) + 1}{4x^2} + c_1$$

Integrating again gives

$$\left(\frac{y}{x}\right) = \frac{4c_1x^2 + 2\ln(x)^2 + 6\ln(x) - 2\ln(x^2) + 1}{4x} + c_2$$

Hence the solution is

$$y = \frac{\frac{4c_1x^2 + 2\ln(x)^2 + 6\ln(x) - 2\ln(x^2) + 1}{4x} + c_2}{\frac{1}{x}}$$

Or

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \quad (1)$$

Verification of solutions

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)} dx \\ &= \int e^{-(\int -\frac{2}{x} dx)} dx \\ &= \int e^{2\ln(x)} dx \\ &= \int x^2 dx \\ &= \frac{x^3}{3} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{x^2}}{x^4} \\ &= \frac{2}{x^6} \end{aligned} \quad (7)$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} &= 0\end{aligned}$$

But in terms of τ

$$\frac{2}{x^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\tau^r + 0\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1 &= \frac{1}{3} \\ r_2 &= \frac{2}{3}\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^3)^{\frac{1}{3}}$$

$$y_2 = (x^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx} \left((x^3)^{\frac{1}{3}} \right) & \frac{d}{dx} \left((x^3)^{\frac{2}{3}} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}} \right) \left(\frac{2x^2}{(x^3)^{\frac{1}{3}}} \right) - \left((x^3)^{\frac{2}{3}} \right) \left(\frac{x^2}{(x^3)^{\frac{2}{3}}} \right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_1 = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = -(x^3)^{\frac{1}{3}} \left(- \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \right. \\ \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

Therefore the general solution is

$$y = y_h + y_p \\ = \left(\frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} \right) + \left(-(x^3)^{\frac{1}{3}} \left(- \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \right. \right. \\ \left. \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} - (x^3)^{\frac{1}{3}} \left(- \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \right. \\ \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \quad (1)$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} - (x^3)^{\frac{1}{3}} \left(- \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

Verified OK.

12.2.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2 y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c} \sqrt{q} \\ &= \frac{\sqrt{2} \sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c \sqrt{\frac{1}{x^2}} x^3} \end{aligned} \quad (6)$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{c \sqrt{\frac{1}{x^2}} x^3} - \frac{2}{x} \frac{\sqrt{2} \sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2} \sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2} \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) - \frac{3c\sqrt{2}}{2} \left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0 \end{aligned} \quad (7)$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{3\sqrt{2}c\tau}{4}} \left(c_1 \cosh \left(\frac{\sqrt{2}c\tau}{4} \right) + ic_2 \sinh \left(\frac{\sqrt{2}c\tau}{4} \right) \right)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c} \sqrt{q} dx \\ &= \frac{\int \sqrt{2} \sqrt{\frac{1}{x^2}} dx}{c} \\ &= \frac{\sqrt{2} \sqrt{\frac{1}{x^2}} x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = x^{\frac{3}{2}} \left(c_1 \cosh \left(\frac{\ln(x)}{2} \right) + ic_2 \sinh \left(\frac{\ln(x)}{2} \right) \right)$$

Now the particular solution to this ODE is found

$$x^2 y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^3)^{\frac{1}{3}}$$

$$y_2 = (x^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx} \left((x^3)^{\frac{1}{3}} \right) & \frac{d}{dx} \left((x^3)^{\frac{2}{3}} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}} \right) \left(\frac{2x^2}{(x^3)^{\frac{1}{3}}} \right) - \left((x^3)^{\frac{2}{3}} \right) \left(\frac{x^2}{(x^3)^{\frac{2}{3}}} \right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_1 = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}}$$

Therefore the general solution is

$$y = y_h + y_p \\ = \left(x^{\frac{3}{2}} \left(c_1 \cosh \left(\frac{\ln(x)}{2} \right) + ic_2 \sinh \left(\frac{\ln(x)}{2} \right) \right) \right) \\ + \left(- \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \right. \\ \left. + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}} \right) \\ = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}} \\ + x^{\frac{3}{2}} \left(c_1 \cosh \left(\frac{\ln(x)}{2} \right) + ic_2 \sinh \left(\frac{\ln(x)}{2} \right) \right)$$

Which simplifies to

$$y = i \sinh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_2 + \cosh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_1 \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}} \\ - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}}$$

Summary

The solution(s) found are the following

$$y = i \sinh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_2 + \cosh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_1 \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}} \quad (1) \\ - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}}$$

Verification of solutions

$$y = i \sinh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_2 + \cosh \left(\frac{\ln(x)}{2} \right) x^{\frac{3}{2}} c_1 \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}} \\ - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}}$$

Verified OK.

12.2.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' - 2xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x) z(x) \quad (3)$$

is used to change the original ode to a constant coefficients ode in v . In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= e^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-2}{2} dx} \\ &= x \end{aligned} \quad (5)$$

Hence (3) becomes

$$y = v(x) x \quad (4)$$

Applying this change of variable to the original ode results in

$$x^3 v''(x) = \ln(x)^2 - \ln(x^2)$$

Which is now solved for $v(x)$ Simplifying the ode gives

$$v''(x) = \frac{\ln(x)^2 - \ln(x^2)}{x^3}$$

Integrating once gives

$$v'(x) = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2} + c_1$$

Integrating again gives

$$v(x) = \frac{1}{4x} + \frac{\ln(x)^2}{2x} + \frac{3\ln(x)}{2x} - \frac{\ln(x^2)}{2x} + c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)z(x) \\ &= \left(c_1x + \frac{\ln(x)^2}{2x} + \frac{3\ln(x)}{2x} - \frac{\ln(x^2)}{2x} + \frac{1}{4x} + c_2 \right) (z(x)) \end{aligned} \quad (7)$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(c_1x + \frac{\ln(x)^2}{2x} + \frac{3\ln(x)}{2x} - \frac{\ln(x^2)}{2x} + \frac{1}{4x} + c_2 \right) x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1x + \frac{\ln(x)^2}{2x} + \frac{3\ln(x)}{2x} - \frac{\ln(x^2)}{2x} + \frac{1}{4x} + c_2 \right) x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^3)^{\frac{1}{3}}$$

$$y_2 = (x^3)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx} \left((x^3)^{\frac{1}{3}} \right) & \frac{d}{dx} \left((x^3)^{\frac{2}{3}} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}} \right) \left(\frac{2x^2}{(x^3)^{\frac{1}{3}}} \right) - \left((x^3)^{\frac{2}{3}} \right) \left(\frac{x^2}{(x^3)^{\frac{2}{3}}} \right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^3)^{\frac{2}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_1 = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} (\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} \\ + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) (x^3)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = -(x^3)^{\frac{1}{3}} \left(-(x^3)^{\frac{1}{3}} \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right. \\ \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

Therefore the general solution is

$$\begin{aligned}
 y &= y_h + y_p \\
 &= \left(\left(c_1 x + \frac{\ln(x)^2}{2x} + \frac{3\ln(x)}{2x} - \frac{\ln(x^2)}{2x} + \frac{1}{4x} + c_2 \right) x \right) \\
 &\quad + \left(-(x^3)^{\frac{1}{3}} \left(-(x^3)^{\frac{1}{3}} \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right. \right. \\
 &\qquad \qquad \qquad \left. \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right)
 \end{aligned}$$

Which simplifies to

$$\begin{aligned}
 y &= c_1 x^2 + c_2 x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\
 &\quad - (x^3)^{\frac{1}{3}} \left(-(x^3)^{\frac{1}{3}} \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right. \\
 &\qquad \qquad \qquad \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
 y &= c_1 x^2 + c_2 x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\
 &\quad - (x^3)^{\frac{1}{3}} \left(-(x^3)^{\frac{1}{3}} \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right. \\
 &\qquad \qquad \qquad \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \tag{1}
 \end{aligned}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x + \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\ - (x^3)^{\frac{1}{3}} \left(- (x^3)^{\frac{1}{3}} \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right) \right. \\ \left. + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} (\ln(\alpha)^2 - \ln(\alpha^2))}{\alpha^4} d\alpha \right)$$

Verified OK.

12.2.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -2x$, $C = 2$, $f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2 y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x} \\ q(x) = \frac{2}{x^2}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 2 \quad (6)$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{2}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{2}{x} dx \\ \ln(u) &= -2 \ln(x) + c_1 \\ u &= e^{-2 \ln(x) + c_1} \\ &= \frac{c_1}{x^2}\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= -\frac{c_1}{x} + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x) x^n \\ &= \left(-\frac{c_1}{x} + c_2\right) x^2 \\ &= (c_2 x - c_1) x\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= x^2\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2 \ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2 \ln(x) - \ln(x^2)}{x}$$
$$u_2 = \frac{-2 \ln(x)^2 - 2 \ln(x) + 2 \ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\left(-\frac{c_1}{x} + c_2 \right) x^2 \right) + \left(\frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \right)$$
$$= \frac{\ln(x)^2}{2} + \frac{3 \ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + \left(-\frac{c_1}{x} + c_2 \right) x^2$$

Which simplifies to

$$y = c_2x^2 + \frac{\ln(x)^2}{2} - c_1x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = c_2x^2 + \frac{\ln(x)^2}{2} - c_1x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \quad (1)$$

Verification of solutions

$$y = c_2x^2 + \frac{\ln(x)^2}{2} - c_1x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= x^2 \\B &= -2x \\C &= 2 \\F &= \ln(x)^2 - \ln(x^2)\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (x^2)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v'' + (0)v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3u'(x) = 0$$

Which is now solved for u . Integrating both sides gives

$$\begin{aligned}u(x) &= \int 0 \, dx \\ &= c_1\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int c_1 \, dx \\ &= c_1x + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-2x)(c_1x + c_2) \\ &= -2x(c_1x + c_2)\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= x^2\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2) \quad (1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2 \ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2 \ln(x) - \ln(x^2)}{x}$$
$$u_2 = \frac{-2 \ln(x)^2 - 2 \ln(x) + 2 \ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2x(c_1x + c_2)) + \left(\frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \right) \\ &= -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \quad (1)$$

Verification of solutions

$$y = -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2xy' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$

$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$

$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 222: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is *infinity* then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2} dx} \\ &= z_1 e^{\ln(x)} \\ &= z_1(x)\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2} dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1 + c_2y_2 \\ &= c_1(x) + c_2(x(x))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2x^2 + c_1x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_1 = - \int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2\ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(\ln(x)^2 - \ln(x^2))}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2\ln(x) - \ln(x^2)}{x}$$
$$u_2 = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_2x^2 + c_1x) + \left(\frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \right)$$

Which simplifies to

$$y = x(c_2x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = x(c_2x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \quad (1)$$

Verification of solutions

$$y = x(c_2x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
-> Try solving first the homogeneous part of the ODE  
    checking if the LODE has constant coefficients  
    checking if the LODE is of Euler type  
    <- LODE of Euler type successful  
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=(ln(x))^2-ln(x^2),y(x), singsol=all)
```

$$y(x) = c_2x + c_1x^2 + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 38

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*y[x]==(Log[x])^2-Log[x^2],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \rightarrow \frac{1}{4}(-2\log(x^2) + 2\log^2(x) + 6\log(x) + 1) + c_2x^2 + c_1x$$

12.3 problem 8

12.3.1 Maple step by step solution 1854

Internal problem ID [5408]

Internal file name [OUTPUT/4899_Sunday_February_04_2024_12_46_59_AM_3205007/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre). Supplementary problems. Page 110

Problem number: 8.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_ODE_non_constant_coefficients_of_type_Euler**"

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$x^3y''' + 2x^2y'' = x + \sin(\ln(x))$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + 2x^2y'' = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}y' &= \lambda x^{\lambda-1} \\y'' &= \lambda(\lambda-1)x^{\lambda-2} \\y''' &= \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}\end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' = x + \sin(\ln(x))$$

gives

$$2x^2\lambda(\lambda - 1)x^{\lambda-2} + x^3\lambda(\lambda - 1)(\lambda - 2)x^{\lambda-3} = 0$$

Which simplifies to

$$2\lambda(\lambda - 1)x^\lambda + \lambda(\lambda - 1)(\lambda - 2)x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ , the above becomes

$$2\lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) = 0$$

Simplifying gives the characteristic equation as

$$\lambda^2(\lambda - 1) = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

This table summarises the result

root	multiplicity	type of root
0	2	real root
1	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicity two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicity three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 + c_2 \ln(x) + c_3x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$

$$y_2 = \ln(x)$$

$$y_3 = x$$

Now the particular solution to the given ODE is found

$$x^3 y''' + 2x^2 y'' = x + \sin(\ln(x))$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i -th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \ln(x) & x \\ 0 & \frac{1}{x} & 1 \\ 0 & -\frac{1}{x^2} & 0 \end{bmatrix}$$

$$|W| = \frac{1}{x^2}$$

The determinant simplifies to

$$|W| = \frac{1}{x^2}$$

Now we determine W_i for each U_i .

$$W_1(x) = \det \begin{bmatrix} \ln(x) & x \\ \frac{1}{x} & 1 \end{bmatrix}$$

$$= \ln(x) - 1$$

$$\begin{aligned}
 W_2(x) &= \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 W_3(x) &= \det \begin{bmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{bmatrix} \\
 &= \frac{1}{x}
 \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
 U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)} dx \\
 &= (-1)^2 \int \frac{(x + \sin(\ln(x))) (\ln(x) - 1)}{(x^3) \left(\frac{1}{x^2}\right)} dx \\
 &= \int \frac{(x + \sin(\ln(x))) (\ln(x) - 1)}{x} dx \\
 &= \int \left(\frac{(x + \sin(\ln(x))) (\ln(x) - 1)}{x} \right) dx \\
 &= -2x + \cos(\ln(x)) + \ln(x)x + \sin(\ln(x)) - \cos(\ln(x)) \ln(x)
 \end{aligned}$$

$$\begin{aligned}
 U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)} dx \\
 &= (-1)^1 \int \frac{(x + \sin(\ln(x))) (1)}{(x^3) \left(\frac{1}{x^2}\right)} dx \\
 &= - \int \frac{x + \sin(\ln(x))}{x} dx \\
 &= - \int \left(\frac{x + \sin(\ln(x))}{x} \right) dx \\
 &= -x + \cos(\ln(x))
 \end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)} dx \\
&= (-1)^0 \int \frac{(x + \sin(\ln(x))) \left(\frac{1}{x}\right)}{(x^3) \left(\frac{1}{x^2}\right)} dx \\
&= \int \frac{\frac{x + \sin(\ln(x))}{x}}{x} dx \\
&= \int \left(\frac{x + \sin(\ln(x))}{x^2} \right) dx \\
&= \ln(x) + \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x}
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Hence

$$\begin{aligned}
y_p &= (-2x + \cos(\ln(x)) + \ln(x)x + \sin(\ln(x)) - \cos(\ln(x))\ln(x)) \\
&\quad + (-x + \cos(\ln(x)))(\ln(x)) \\
&\quad + \left(\ln(x) + \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1 + \tan\left(\frac{\ln(x)}{2}\right)^2\right) x} \right) (x)
\end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{\cos(\ln(x))}{2} - 2x + \ln(x)x + \frac{\sin(\ln(x))}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2 \ln(x) + c_3x) + \left(\frac{\cos(\ln(x))}{2} - 2x + \ln(x)x + \frac{\sin(\ln(x))}{2} \right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) + c_3x + \frac{\cos(\ln(x))}{2} - 2x + \ln(x)x + \frac{\sin(\ln(x))}{2} \quad (1)$$

Verification of solutions

$$y = c_1 + c_2 \ln(x) + c_3 x + \frac{\cos(\ln(x))}{2} - 2x + \ln(x)x + \frac{\sin(\ln(x))}{2}$$

Verified OK.

12.3.1 Maple step by step solution

Let's solve

$$x^3 y''' + 2x^2 y'' = x + \sin(\ln(x))$$

- Highest derivative means the order of the ODE is 3
 y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(2*_a^2*_b(_a)-sin(ln(_a))-_a)/_a^3,
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)=x+sin(ln(x)),y(x), singsol=all)
```

$$y(x) = -c_1 \ln(x) + \ln(x)x + c_2 x + c_3 - x + \frac{\tan\left(\frac{\ln(x)}{2}\right) + 1}{1 + \tan\left(\frac{\ln(x)}{2}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 36

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]==x+Sin[Log[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}(\sin(\log(x)) + \cos(\log(x)) + 2((-1 + c_3)x + (x - c_1)\log(x) + c_2))$$

12.4 problem 9

12.4.1 Maple step by step solution 1861

Internal problem ID [5409]

Internal file name [OUTPUT/4900_Sunday_February_04_2024_12_47_00_AM_45729767/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre). Supplementary problems. Page 110

Problem number: 9.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_ODE_non_constant_coefficients_of_type_Euler**"

Maple gives the following as the ode type

```
[[_3rd_order , _with_linear_symmetries]]
```

$$x^3y''' + xy' - y = 3x^4$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + xy' - y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}y' &= \lambda x^{\lambda-1} \\y'' &= \lambda(\lambda-1) x^{\lambda-2} \\y''' &= \lambda(\lambda-1)(\lambda-2) x^{\lambda-3}\end{aligned}$$

Substituting these back into

$$x^3y''' + xy' - y = 3x^4$$

gives

$$x\lambda x^{\lambda-1} + x^3\lambda(\lambda-1)(\lambda-2)x^{\lambda-3} - x^\lambda = 0$$

Which simplifies to

$$\lambda x^\lambda + \lambda(\lambda-1)(\lambda-2)x^\lambda - x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ , the above becomes

$$\lambda + \lambda(\lambda-1)(\lambda-2) - 1 = 0$$

Simplifying gives the characteristic equation as

$$(\lambda-1)^3 = 0$$

Solving the above gives the following roots

$$\lambda_1 = 1$$

$$\lambda_2 = 1$$

$$\lambda_3 = 1$$

This table summarises the result

root	multiplicity	type of root
1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicity two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicity three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1x + c_2x \ln(x) + c_3 \ln(x)^2 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x$$

$$y_2 = \ln(x)x$$

$$y_3 = \ln(x)^2 x$$

Now the particular solution to the given ODE is found

$$x^3 y''' + xy' - y = 3x^4$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)} dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i -th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & \ln(x)x & \ln(x)^2 x \\ 1 & 1 + \ln(x) & \ln(x)(\ln(x) + 2) \\ 0 & \frac{1}{x} & \frac{2+2\ln(x)}{x} \end{bmatrix}$$

$$|W| = 2$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i .

$$W_1(x) = \det \begin{bmatrix} \ln(x)x & \ln(x)^2 x \\ 1 + \ln(x) & \ln(x)(\ln(x) + 2) \end{bmatrix}$$

$$= \ln(x)^2 x$$

$$\begin{aligned}
 W_2(x) &= \det \begin{bmatrix} x & \ln(x)^2 x \\ 1 & \ln(x)(\ln(x) + 2) \end{bmatrix} \\
 &= 2 \ln(x) x
 \end{aligned}$$

$$\begin{aligned}
 W_3(x) &= \det \begin{bmatrix} x & \ln(x) x \\ 1 & 1 + \ln(x) \end{bmatrix} \\
 &= x
 \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned}
 U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)} dx \\
 &= (-1)^2 \int \frac{(3x^4)(\ln(x)^2 x)}{(x^3)(2)} dx \\
 &= \int \frac{3x^5 \ln(x)^2}{2x^3} dx \\
 &= \int \left(\frac{3x^2 \ln(x)^2}{2} \right) dx \\
 &= \frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9}
 \end{aligned}$$

$$\begin{aligned}
 U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)} dx \\
 &= (-1)^1 \int \frac{(3x^4)(2 \ln(x) x)}{(x^3)(2)} dx \\
 &= - \int \frac{6x^5 \ln(x)}{2x^3} dx \\
 &= - \int (3 \ln(x) x^2) dx \\
 &= -x^3 \ln(x) + \frac{x^3}{3}
 \end{aligned}$$

$$\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)} dx \\
&= (-1)^0 \int \frac{(3x^4)(x)}{(x^3)(2)} dx \\
&= \int \frac{3x^5}{2x^3} dx \\
&= \int \left(\frac{3x^2}{2} \right) dx \\
&= \frac{x^3}{2}
\end{aligned}$$

Now that all the U_i functions have been determined, the particular solution is found from

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Hence

$$\begin{aligned}
y_p &= \left(\frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9} \right) (x) \\
&\quad + \left(-x^3 \ln(x) + \frac{x^3}{3} \right) (\ln(x) x) \\
&\quad + \left(\frac{x^3}{2} \right) (\ln(x)^2 x)
\end{aligned}$$

Therefore the particular solution is

$$y_p = \frac{x^4}{9}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1x + c_2x \ln(x) + c_3 \ln(x)^2 x) + \left(\frac{x^4}{9} \right)
\end{aligned}$$

Which simplifies to

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9}$$

Summary

The solution(s) found are the following

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9} \tag{1}$$

Verification of solutions

$$y = x(c_1 + c_2 \ln(x) + c_3 \ln(x)^2) + \frac{x^4}{9}$$

Verified OK.

12.4.1 Maple step by step solution

Let's solve

$$x^3 y''' + xy' - y = 3x^4$$

- Highest derivative means the order of the ODE is 3
 y'''

Maple trace

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]  
trying high order linear exact nonhomogeneous  
trying differential order: 3; missing the dependent variable  
checking if the LODE is of Euler type  
<- LODE of Euler type successful  
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)+x*diff(y(x),x)-y(x)=3*x^4,y(x), singsol=all)
```

$$y(x) = \frac{x(9c_3 \ln(x)^2 + x^3 + 9c_2 \ln(x) + 9c_1)}{9}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 31

```
DSolve[x^3*y''[x]+x*y'[x]-y[x]==3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^4}{9} + c_1x + c_3x \log^2(x) + c_2x \log(x)$$

12.5 problem 10

12.5.1 Solving as second order change of variable on x method 2 ode .	1864
12.5.2 Solving as second order change of variable on x method 1 ode .	1869
12.5.3 Solving as second order integrable as is ode	1875
12.5.4 Solving as second order ode non constant coeff transformation on B ode	1876
12.5.5 Solving as type second_order_integrable_as_is (not using ABC version)	1881
12.5.6 Solving using Kovacic algorithm	1883
12.5.7 Solving as exact linear second order ode ode	1891

Internal problem ID [5410]

Internal file name [OUTPUT/4901_Sunday_February_04_2024_12_47_00_AM_5832621/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre).
Supplementary problems. Page 110

Problem number: 10.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _exact , _linear , _nonhomogeneous]]
```

$$(x + 1)^2 y'' + (x + 1) y' - y = \ln(x + 1)^2 + x - 1$$

12.5.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x + 1)^2 y'' + (x + 1) y' - y = 0$$

In normal form the ode

$$(x + 1)^2 y'' + (x + 1) y' - y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = \frac{1}{x + 1}$$
$$q(x) = -\frac{1}{(x + 1)^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x) \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
 \tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
 &= \int e^{-\left(\int \frac{1}{x+1} dx\right)} dx \\
 &= \int e^{-\ln(x+1)} dx \\
 &= \int \frac{1}{x+1} dx \\
 &= \ln(x+1)
 \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
 q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
 &= \frac{1}{(x+1)^2} \\
 &= \frac{1}{(x+1)^2} \\
 &= -1
 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
 \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\
 \frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
 \end{aligned}$$

The above ode is now solved for $y(\tau)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$

$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$

$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1(x+1)^2 + c_2}{x+1}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1(x+1)^2 + c_2}{x+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$

$$y_2 = \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ \frac{d}{dx} \left(\frac{1}{x+1} \right) & \frac{d}{dx} \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1} \right) \left(-\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \right) - \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right) \left(-\frac{1}{(x+1)^2} \right)$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right) (\ln(x+1)^2 + x - 1)}{2 + 2x} dx$$

Which simplifies to

$$u_1 = - \int \left(\frac{\ln(x+1)^2}{2} + \frac{x}{2} - \frac{1}{2} \right) dx$$

Hence

$$u_1 = -\frac{x}{2} - \frac{x^2}{4} - \frac{\ln(x+1)^2(x+1)}{2} + \ln(x+1)(x+1) - 1$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2 + x - 1}{x+1}}{2 + 2x} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2 + x - 1}{2(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} + \frac{\ln(x+1)}{2}$$

Which simplifies to

$$u_1 = -1 + \frac{(-1-x)\ln(x+1)^2}{2} + \ln(x+1)(x+1) - \frac{x^2}{4} - \frac{x}{2}$$

$$u_2 = \frac{\ln(x+1)(-\ln(x+1) - 1 + x)}{2 + 2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-1 + \frac{(-1-x)\ln(x+1)^2}{2} + \ln(x+1)(x+1) - \frac{x^2}{4} - \frac{x}{2}}{x+1} + \frac{\ln(x+1)(-\ln(x+1) - 1 + x) \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right)}{2+2x}$$

Which simplifies to

$$y_p(x) = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}$$

Therefore the general solution is

$$y = y_h + y_p = \left(\frac{c_1(x+1)^2 + c_2}{x+1} \right) + \left(\frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x+1)^2 + c_2}{x+1} + \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4} \quad (1)$$

Verification of solutions

$$y = \frac{c_1(x+1)^2 + c_2}{x+1} + \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}$$

Verified OK.

12.5.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = (x+1)^2$, $B = x+1$, $C = -1$, $f(x) = \ln(x+1)^2 + x - 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$(x + 1)^2 y'' + (x + 1) y' - y = 0$$

In normal form the ode

$$(x + 1)^2 y'' + (x + 1) y' - y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = \frac{1}{x + 1}$$

$$q(x) = -\frac{1}{(x + 1)^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\tau' = \frac{1}{c} \sqrt{q}$$

$$= \frac{\sqrt{-\frac{1}{(x+1)^2}}}{c} \quad (6)$$

$$\tau'' = \frac{1}{c \sqrt{-\frac{1}{(x+1)^2}} (x + 1)^3}$$

Substituting the above into (4) results in

$$\begin{aligned}
 p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\
 &= \frac{\frac{1}{c\sqrt{-\frac{1}{(x+1)^2}}(x+1)^3} + \frac{1}{x+1} \frac{\sqrt{-\frac{1}{(x+1)^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{(x+1)^2}}}{c}\right)^2} \\
 &= 0
 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
 y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\
 \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0 \tag{7}
 \end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
 \tau &= \int \frac{1}{c} \sqrt{q} dx \\
 &= \frac{\int \sqrt{-\frac{1}{(x+1)^2}} dx}{c} \\
 &= \frac{\sqrt{-\frac{1}{(x+1)^2}} (x+1) \ln(x+1)}{c}
 \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(c_2i + c_1)x^2 + (2c_2i + 2c_1)x + 2c_1}{2 + 2x}$$

Now the particular solution to this ODE is found

$$(x+1)^2 y'' + (x+1)y' - y = \ln(x+1)^2 + x - 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x+1}$$

$$y_2 = \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ \frac{d}{dx} \left(\frac{1}{x+1} \right) & \frac{d}{dx} \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1} \right) \left(-\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \right) - \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right) \left(-\frac{1}{(x+1)^2} \right)$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right) (\ln(x+1)^2 + x - 1)}{2 + 2x} dx$$

Which simplifies to

$$u_1 = - \int \left(\frac{\ln(x+1)^2}{2} + \frac{x}{2} - \frac{1}{2} \right) dx$$

Hence

$$u_1 = -\frac{x}{2} - \frac{x^2}{4} - \frac{\ln(x+1)^2(x+1)}{2} + \ln(x+1)(x+1) - 1$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2 + x - 1}{x+1}}{2 + 2x} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2 + x - 1}{2(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} + \frac{\ln(x+1)}{2}$$

Which simplifies to

$$u_1 = -1 + \frac{(-1-x)\ln(x+1)^2}{2} + \ln(x+1)(x+1) - \frac{x^2}{4} - \frac{x}{2}$$

$$u_2 = \frac{\ln(x+1)(-\ln(x+1) - 1 + x)}{2 + 2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-1 + \frac{(-1-x)\ln(x+1)^2}{2} + \ln(x+1)(x+1) - \frac{x^2}{4} - \frac{x}{2}}{x+1} + \frac{\ln(x+1)(-\ln(x+1) - 1 + x) \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \right)}{2+2x}$$

Which simplifies to

$$y_p(x) = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_2i + c_1)x^2 + (2c_2i + 2c_1)x + 2c_1}{2+2x} \right) \\ &\quad + \left(\frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4} \right) \\ &= \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4} \\ &\quad + \frac{(c_2i + c_1)x^2 + (2c_2i + 2c_1)x + 2c_1}{2+2x} \end{aligned}$$

Which simplifies to

$$y = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (2c_2i + 2c_1 - 1)x^2 + (4c_2i + 4c_1 - 2)x + 4c_1 - 4}{4x+4}$$

Summary

The solution(s) found are the following

$$y = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (2c_2i + 2c_1 - 1)x^2 + (4c_2i + 4c_1 - 2)x + 4c_1 - 4}{4x+4} \quad (1)$$

Verification of solutions

$$y = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (2c_2i + 2c_1 - 1)x^2 + (4c_2i + 4c_1 - 2)x + 4c_1 - 4}{4x+4}$$

Verified OK.

12.5.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((x+1)^2 y'' + (x+1)y' - y) dx = \int (\ln(x+1) +$$

$$(-1-x)y + (x^2 + 2x + 1)y' = x + \frac{x^2}{2} + \ln(x+1)^2(x+1) - 2\ln(x+1)(x+1) + 2 + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x+1}$$

$$q(x) = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x+1} dx} \\ &= \frac{1}{x+1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right) \\ \frac{d}{dx} \left(\frac{y}{x+1} \right) &= \left(\frac{1}{x+1} \right) \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right) \\ d \left(\frac{y}{x+1} \right) &= \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} \right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x+1} &= \int \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} dx \\ \frac{y}{x+1} &= -\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = (x + 1) \left(-\frac{\ln(x + 1)^2}{x + 1} + \frac{\ln(x + 1)}{2} - \frac{c_1}{2(x + 1)^2} - \frac{3}{4(x + 1)^2} \right) + c_2(x + 1)$$

Summary

The solution(s) found are the following

$$y = (x + 1) \left(-\frac{\ln(x + 1)^2}{x + 1} + \frac{\ln(x + 1)}{2} - \frac{c_1}{2(x + 1)^2} - \frac{3}{4(x + 1)^2} \right) + c_2(x + 1) \quad (1)$$

Verification of solutions

$$y = (x + 1) \left(-\frac{\ln(x + 1)^2}{x + 1} + \frac{\ln(x + 1)}{2} - \frac{c_1}{2(x + 1)^2} - \frac{3}{4(x + 1)^2} \right) + c_2(x + 1)$$

Verified OK.

12.5.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2) u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= (x + 1)^2 \\ B &= x + 1 \\ C &= -1 \\ F &= \ln(x + 1)^2 + x - 1 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= ((x + 1)^2) (0) + (x + 1) (1) + (-1) (x + 1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$(x + 1)^3 v'' + (3(x + 1)^2) v' = 0$$

Now by applying $v' = u$ the above becomes

$$(x + 1)^2 ((x + 1) u'(x) + 3u(x)) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x + 1} \end{aligned}$$

Where $f(x) = -\frac{3}{x+1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{3}{x + 1} dx \\ \int \frac{1}{u} du &= \int -\frac{3}{x + 1} dx \\ \ln(u) &= -3 \ln(x + 1) + c_1 \\ u &= e^{-3 \ln(x+1) + c_1} \\ &= \frac{c_1}{(x + 1)^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= \frac{c_1}{(x+1)^3}\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{c_1}{(x+1)^3} dx \\ &= -\frac{c_1}{2(x+1)^2} + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (x+1) \left(-\frac{c_1}{2(x+1)^2} + c_2 \right) \\ &= \frac{2c_2(x+1)^2 - c_1}{2+2x}\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \frac{1}{2+2x} \\ y_2 &= \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{2+2x} & \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \\ \frac{d}{dx} \left(\frac{1}{2+2x} \right) & \frac{d}{dx} \left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{2+2x} & \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \\ -\frac{2}{(2+2x)^2} & -\frac{4x^2}{(2+2x)^2} + \frac{4x}{2+2x} - \frac{8x}{(2+2x)^2} + \frac{4}{2+2x} - \frac{4}{(2+2x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{2+2x} \right) \left(-\frac{4x^2}{(2+2x)^2} + \frac{4x}{2+2x} - \frac{8x}{(2+2x)^2} + \frac{4}{2+2x} - \frac{4}{(2+2x)^2} \right) - \left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \right) \left(-\frac{2}{(2+2x)^2} \right)$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \right) (\ln(x+1)^2 + x - 1)}{x+1} dx$$

Which simplifies to

$$u_1 = - \int (\ln(x+1)^2 + x - 1) dx$$

Hence

$$u_1 = -x - \frac{x^2}{2} - \ln(x+1)^2(x+1) + 2 \ln(x+1)(x+1) - 2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2+x-1}{2+2x}}{x+1} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2+x-1}{2(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} + \frac{\ln(x+1)}{2}$$

Which simplifies to

$$u_1 = -2 + (-1-x)\ln(x+1)^2 + 2\ln(x+1)(x+1) - \frac{x^2}{2} - x$$

$$u_2 = \frac{\ln(x+1)(-\ln(x+1)-1+x)}{2+2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-2 + (-1-x)\ln(x+1)^2 + 2\ln(x+1)(x+1) - \frac{x^2}{2} - x}{2+2x}$$

$$+ \frac{\ln(x+1)(-\ln(x+1)-1+x)\left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x}\right)}{2+2x}$$

Which simplifies to

$$y_p(x) = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}$$

Hence the complete solution is

$$y(x) = y_h + y_p$$

$$= \left(\frac{2c_2(x+1)^2 - c_1}{2+2x}\right) + \left(\frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}\right)$$

$$= \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (4c_2-1)x^2 + (8c_2-2)x - 2c_1 + 4c_2 - 4}{4x+4}$$

Summary

The solution(s) found are the following

$$y = \frac{(-4x - 4) \ln(x + 1)^2 + 2(x + 1)^2 \ln(x + 1) + (4c_2 - 1)x^2 + (8c_2 - 2)x - 2c_1 + 4c_2 - 4}{4x + 4} \quad (1)$$

Verification of solutions

$$y = \frac{(-4x - 4) \ln(x + 1)^2 + 2(x + 1)^2 \ln(x + 1) + (4c_2 - 1)x^2 + (8c_2 - 2)x - 2c_1 + 4c_2 - 4}{4x + 4}$$

Verified OK.

12.5.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x + 1)^2 y'' + (x + 1) y' - y = \ln(x + 1)^2 + x - 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int ((x + 1)^2 y'' + (x + 1) y' - y) dx = \int (\ln(x + 1)^2 + x - 1) dx$$
$$(-1 - x)y + (x^2 + 2x + 1)y' = x + \frac{x^2}{2} + \ln(x + 1)^2(x + 1) - 2\ln(x + 1)(x + 1) + 2 + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x + 1}$$
$$q(x) = \frac{(2 + 2x) \ln(x + 1)^2 + (-4x - 4) \ln(x + 1) + x^2 + 2x + 2c_1 + 4}{2(x + 1)^2}$$

Hence the ode is

$$y' - \frac{y}{x + 1} = \frac{(2 + 2x) \ln(x + 1)^2 + (-4x - 4) \ln(x + 1) + x^2 + 2x + 2c_1 + 4}{2(x + 1)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x+1} dx} \\ &= \frac{1}{x+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{(2+2x) \ln(x+1)^2 + (-4x-4) \ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right) \\ \frac{d}{dx} \left(\frac{y}{x+1} \right) &= \left(\frac{1}{x+1} \right) \left(\frac{(2+2x) \ln(x+1)^2 + (-4x-4) \ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right) \\ d \left(\frac{y}{x+1} \right) &= \left(\frac{(2+2x) \ln(x+1)^2 + (-4x-4) \ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x+1} &= \int \frac{(2+2x) \ln(x+1)^2 + (-4x-4) \ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} dx \\ \frac{y}{x+1} &= -\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = (x+1) \left(-\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} \right) + c_2(x+1)$$

Summary

The solution(s) found are the following

$$y = (x+1) \left(-\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} \right) + c_2(x+1) \quad (1)$$

Verification of solutions

$$y = (x+1) \left(-\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} \right) + c_2(x+1)$$

Verified OK.

12.5.6 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)^2 y'' + (x+1)y' - y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= x+1 \\ C &= -1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4(x+1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4(x+1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x + 1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x + 1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ , which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4(x+1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω .

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2(x+1)} + (-)(0) \\ &= -\frac{1}{2(x+1)} \\ &= -\frac{1}{2(x+1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x+1)^2}\right) + \left(-\frac{1}{2(x+1)}\right)^2 - \left(\frac{3}{4(x+1)^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int -\frac{1}{2(x+1)} dx} \\ &= \frac{1}{\sqrt{x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{x+1}{(x+1)^2} dx} \\&= z_1 e^{-\frac{\ln(x+1)}{2}} \\&= z_1 \left(\frac{1}{\sqrt{x+1}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x+1}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{(x+1)^2} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-\ln(x+1)}}{(y_1)^2} dx \\&= y_1 \left(\frac{1}{2} x^2 + x \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 \left(\frac{1}{x+1} \right) + c_2 \left(\frac{1}{x+1} \left(\frac{1}{2} x^2 + x \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x + 1)^2 y'' + (x + 1) y' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x + 1} + \frac{c_2 x(x + 2)}{2 + 2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x + 1}$$

$$y_2 = \frac{x(x + 2)}{2 + 2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x(x+2)}{2+2x} \\ \frac{d}{dx} \left(\frac{1}{x+1} \right) & \frac{d}{dx} \left(\frac{x(x+2)}{2+2x} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x(x+2)}{2+2x} \\ -\frac{1}{(x+1)^2} & \frac{x+2}{2+2x} + \frac{x}{2+2x} - \frac{2x(x+2)}{(2+2x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1} \right) \left(\frac{x+2}{2+2x} + \frac{x}{2+2x} - \frac{2x(x+2)}{(2+2x)^2} \right) - \left(\frac{x(x+2)}{2+2x} \right) \left(-\frac{1}{(x+1)^2} \right)$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{x(x+2)(\ln(x+1)^2+x-1)}{2+2x}}{x+1} dx$$

Which simplifies to

$$u_1 = - \int \frac{x(x+2)(\ln(x+1)^2+x-1)}{2(x+1)^2} dx$$

Hence

$$u_1 = - \frac{\ln(x+1)^2(x+1)}{2} + \ln(x+1)(x+1) - \frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} - \frac{(x+1)^2}{4} + \frac{\ln(x+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2+x-1}{x+1}}{x+1} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2 + x - 1}{(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{x+1} - \frac{2\ln(x+1)}{x+1} + \ln(x+1)$$

Which simplifies to

$$u_1 = \frac{(-2x^2 - 4x - 4)\ln(x+1)^2 + (4x^2 + 10x + 2)\ln(x+1) - (x+1)^3}{4x+4}$$

$$u_2 = \frac{\ln(x+1)(-\ln(x+1) - 1 + x)}{x+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(-2x^2 - 4x - 4)\ln(x+1)^2 + (4x^2 + 10x + 2)\ln(x+1) - (x+1)^3}{(4x+4)(x+1)} + \frac{\ln(x+1)(-\ln(x+1) - 1 + x)x(x+2)}{(x+1)(2+2x)}$$

Which simplifies to

$$y_p(x) = -\ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{x}{4} - \frac{1}{4}$$

Therefore the general solution is

$$y = y_h + y_p = \left(\frac{c_1}{x+1} + \frac{c_2x(x+2)}{2+2x} \right) + \left(-\ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{x}{4} - \frac{1}{4} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x+1} + \frac{c_2x(x+2)}{2+2x} - \ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{x}{4} - \frac{1}{4} \quad (1)$$

Verification of solutions

$$y = \frac{c_1}{x+1} + \frac{c_2x(x+2)}{2+2x} - \ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{x}{4} - \frac{1}{4}$$

Verified OK.

12.5.7 Solving as exact linear second order ode

An ode of the form

$$p(x)y'' + q(x)y' + r(x)y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \quad (1)$$

For the given ode we have

$$\begin{aligned} p(x) &= (x+1)^2 \\ q(x) &= x+1 \\ r(x) &= -1 \\ s(x) &= \ln(x+1)^2 + x - 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2 - (1) + (-1) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)y' + (q(x) - p'(x))y)' = s(x)$$

Integrating gives

$$p(x)y' + (q(x) - p'(x))y = \int s(x) dx$$

Substituting the above values for p, q, r, s gives

$$(x+1)^2 y' + (-1-x)y = \int \ln(x+1)^2 + x - 1 dx$$

We now have a first order ode to solve which is

$$(x+1)^2 y' + (-1-x)y = x + \frac{x^2}{2} + \ln(x+1)^2(x+1) - 2\ln(x+1)(x+1) + 2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x+1}$$

$$q(x) = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x+1} dx}$$

$$= \frac{1}{x+1}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right)$$

$$\frac{d}{dx} \left(\frac{y}{x+1} \right) = \left(\frac{1}{x+1} \right) \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2} \right)$$

$$d \left(\frac{y}{x+1} \right) = \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} \right) dx$$

Integrating gives

$$\frac{y}{x+1} = \int \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3} dx$$

$$\frac{y}{x+1} = -\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = (x+1) \left(-\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} \right) + c_2(x+1)$$

Summary

The solution(s) found are the following

$$y = (x + 1) \left(-\frac{\ln(x + 1)^2}{x + 1} + \frac{\ln(x + 1)}{2} - \frac{c_1}{2(x + 1)^2} - \frac{3}{4(x + 1)^2} \right) + c_2(x + 1) \quad (1)$$

Verification of solutions

$$y = (x + 1) \left(-\frac{\ln(x + 1)^2}{x + 1} + \frac{\ln(x + 1)}{2} - \frac{c_1}{2(x + 1)^2} - \frac{3}{4(x + 1)^2} \right) + c_2(x + 1)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((x+1)^2*diff(y(x),x$2)+(x+1)*diff(y(x),x)-y(x)=(ln(x+1))^2+x-1,y(x), singsol=all)
```

$$y(x) = \frac{(-4x - 4) \ln(x + 1)^2 + 2(x + 1)^2 \ln(x + 1) + 4c_2x^2 + 8c_2x + 4c_1 + 4c_2 - 3}{4x + 4}$$

✓ Solution by Mathematica

Time used: 0.262 (sec). Leaf size: 72

```
DSolve[(x+1)^2*y'[x]+(x+1)*y'[x]-y[x]==(Log[x+1])^2+x-1,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow \frac{(-1 + 2c_1 + 2ic_2)x^2 - 4(x + 1) \log^2(x + 1) + 2(x + 1)^2 \log(x + 1) + (-2 + 4c_1 + 4ic_2)x - 1 + 4c_1}{4(x + 1)}$$

12.6 problem 11

12.6.1 Solving as second order integrable as is ode	1894
12.6.2 Solving as type second_order_integrable_as_is (not using ABC version)	1896
12.6.3 Solving using Kovacic algorithm	1898
12.6.4 Solving as exact linear second order ode ode	1906

Internal problem ID [5411]

Internal file name [OUTPUT/4902_Sunday_February_04_2024_12_47_02_AM_76936695/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legendre). Supplementary problems. Page 110

Problem number: 11.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "exact linear second order ode", "second_order_integrable_as_is"

Maple gives the following as the ode type

```
[[_2nd_order , _exact , _linear , _nonhomogeneous]]
```

$$(1 + 2x)^2 y'' - 2(1 + 2x) y' - 12y = 6x$$

12.6.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(4 \left(\frac{1}{2} + x \right)^2 y'' + (-4x - 2) y' - 12y \right) dx = \int 6x dx$$
$$(-12x - 6)y + (4x^2 + 4x + 1) y' = 3x^2 + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{6}{1+2x}$$
$$q(x) = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

Hence the ode is

$$y' - \frac{6y}{1+2x} = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{1+2x} dx}$$
$$= \frac{1}{(1+2x)^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2} \right)$$
$$\frac{d}{dx} \left(\frac{y}{(1+2x)^3} \right) = \left(\frac{1}{(1+2x)^3} \right) \left(\frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2} \right)$$
$$d \left(\frac{y}{(1+2x)^3} \right) = \left(\frac{3x^2 + c_1}{(1+2x)^5} \right) dx$$

Integrating gives

$$\frac{y}{(1+2x)^3} = \int \frac{3x^2 + c_1}{(1+2x)^5} dx$$
$$\frac{y}{(1+2x)^3} = -\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$y = (1+2x)^3 \left(-\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} \right) + c_2(1+2x)^3$$

which simplifies to

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Summary

The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x} \quad (1)$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Verified OK.

12.6.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$4\left(\frac{1}{2} + x\right)^2 y'' + (-4x - 2)y' - 12y = 6x$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(4\left(\frac{1}{2} + x\right)^2 y'' + (-4x - 2)y' - 12y \right) dx = \int 6x dx$$
$$(-12x - 6)y + (4x^2 + 4x + 1)y' = 3x^2 + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{6}{1 + 2x}$$
$$q(x) = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

Hence the ode is

$$y' - \frac{6y}{1 + 2x} = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{6}{1+2x} dx} \\ &= \frac{1}{(1+2x)^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2} \right) \\ \frac{d}{dx} \left(\frac{y}{(1+2x)^3} \right) &= \left(\frac{1}{(1+2x)^3} \right) \left(\frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2} \right) \\ d \left(\frac{y}{(1+2x)^3} \right) &= \left(\frac{3x^2 + c_1}{(1+2x)^5} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{(1+2x)^3} &= \int \frac{3x^2 + c_1}{(1+2x)^5} dx \\ \frac{y}{(1+2x)^3} &= -\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$y = (1+2x)^3 \left(-\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} \right) + c_2(1+2x)^3$$

which simplifies to

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Summary

The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x} \quad (1)$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Verified OK.

12.6.3 Solving using Kovacic algorithm

Writing the ode as

$$4\left(\frac{1}{2} + x\right)^2 y'' + (-4x - 2)y' - 12y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4\left(\frac{1}{2} + x\right)^2 \\ B &= -4x - 2 \\ C &= -12 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{(1 + 2x)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= (1 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{(1 + 2x)^2}\right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1 + 2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(\frac{1}{2} + x\right)^2}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(\frac{1}{2}+x\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{(1 + 2x)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{5}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{3}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(1 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to

determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_{∞}^{\pm} . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{2\left(\frac{1}{2} + x\right)} + (-)(0) \\ &= -\frac{3}{2\left(\frac{1}{2} + x\right)} \\ &= -\frac{3}{1 + 2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\left(\frac{1}{2} + x\right)}\right)(0) + \left(\left(\frac{3}{2\left(\frac{1}{2} + x\right)}\right)^2 - \left(\frac{15}{(1 + 2x)^2}\right)\right) = 0$$

$0 = 0$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int -\frac{3}{2(\frac{1}{2}+x)} dx} \\ &= \frac{1}{(1+2x)^{\frac{3}{2}}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x-2}{4(\frac{1}{2}+x)^2} dx} \\ &= z_1 e^{\frac{\ln(1+2x)}{2}} \\ &= z_1 \left(\sqrt{1+2x} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{1+2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x-2}{4(\frac{1}{2}+x)^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} dx \\ &= y_1 \left(\frac{(1+2x)^4}{8} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{1+2x} \right) + c_2 \left(\frac{1}{1+2x} \left(\frac{(1+2x)^4}{8} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$4 \left(\frac{1}{2} + x \right)^2 y'' + (-4x - 2) y' - 12y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{1+2x} + \frac{c_2(1+2x)^3}{8}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{1}{1+2x} \\
y_2 &= \frac{(1+2x)^3}{8}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{1+2x} & \frac{(1+2x)^3}{8} \\ \frac{d}{dx} \left(\frac{1}{1+2x} \right) & \frac{d}{dx} \left(\frac{(1+2x)^3}{8} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{1+2x} & \frac{(1+2x)^3}{8} \\ -\frac{2}{(1+2x)^2} & \frac{3(1+2x)^2}{4} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{1+2x} \right) \left(\frac{3(1+2x)^2}{4} \right) - \left(\frac{(1+2x)^3}{8} \right) \left(-\frac{2}{(1+2x)^2} \right)$$

Which simplifies to

$$W = 1 + 2x$$

Which simplifies to

$$W = 1 + 2x$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{3x(1+2x)^3}{4}}{4 \left(\frac{1}{2} + x \right)^2 (1+2x)} dx$$

Which simplifies to

$$u_1 = - \int \frac{3x}{4} dx$$

Hence

$$u_1 = -\frac{3x^2}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{6x}{1+2x}}{4 \left(\frac{1}{2} + x \right)^2 (1+2x)} dx$$

Which simplifies to

$$u_2 = \int \frac{6x}{(1+2x)^4} dx$$

Hence

$$u_2 = -\frac{3}{4(1+2x)^2} + \frac{1}{2(1+2x)^3}$$

Which simplifies to

$$u_1 = -\frac{3x^2}{8}$$
$$u_2 = \frac{-6x-1}{4(1+2x)^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3x^2}{8(1+2x)} - \frac{3x}{16} - \frac{1}{32}$$

Which simplifies to

$$y_p(x) = \frac{-24x^2 - 8x - 1}{32 + 64x}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\frac{c_1}{1+2x} + \frac{c_2(1+2x)^3}{8} \right) + \left(\frac{-24x^2 - 8x - 1}{32 + 64x} \right)$$

Which simplifies to

$$y = \frac{2\left(\frac{1}{2} + x\right)^4 c_2 + c_1}{1+2x} + \frac{-24x^2 - 8x - 1}{32 + 64x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\left(\frac{1}{2} + x\right)^4 c_2 + c_1}{1+2x} + \frac{-24x^2 - 8x - 1}{32 + 64x} \quad (1)$$

Verification of solutions

$$y = \frac{2\left(\frac{1}{2} + x\right)^4 c_2 + c_1}{1 + 2x} + \frac{-24x^2 - 8x - 1}{32 + 64x}$$

Verified OK.

12.6.4 Solving as exact linear second order ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 4\left(\frac{1}{2} + x\right)^2 \\ q(x) &= -4x - 2 \\ r(x) &= -12 \\ s(x) &= 6x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 8 \\ q'(x) &= -4 \end{aligned}$$

Therefore (1) becomes

$$8 - (-4) + (-12) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x) y' + (q(x) - p'(x)) y)' = s(x)$$

Integrating gives

$$p(x) y' + (q(x) - p'(x)) y = \int s(x) dx$$

Substituting the above values for p, q, r, s gives

$$4\left(\frac{1}{2} + x\right)^2 y' + (-12x - 6) y = \int 6x dx$$

We now have a first order ode to solve which is

$$4\left(\frac{1}{2} + x\right)^2 y' + (-12x - 6)y = 3x^2 + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{6}{1+2x}$$
$$q(x) = \frac{3x^2 + c_1}{(1+2x)^2}$$

Hence the ode is

$$y' - \frac{6y}{1+2x} = \frac{3x^2 + c_1}{(1+2x)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{1+2x} dx}$$
$$= \frac{1}{(1+2x)^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{3x^2 + c_1}{(1+2x)^2} \right)$$
$$\frac{d}{dx} \left(\frac{y}{(1+2x)^3} \right) = \left(\frac{1}{(1+2x)^3} \right) \left(\frac{3x^2 + c_1}{(1+2x)^2} \right)$$
$$d \left(\frac{y}{(1+2x)^3} \right) = \left(\frac{3x^2 + c_1}{(1+2x)^5} \right) dx$$

Integrating gives

$$\frac{y}{(1+2x)^3} = \int \frac{3x^2 + c_1}{(1+2x)^5} dx$$
$$\frac{y}{(1+2x)^3} = -\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$y = (1 + 2x)^3 \left(-\frac{\frac{3}{8} + \frac{c_1}{2}}{4(1+2x)^4} - \frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} \right) + c_2(1+2x)^3$$

which simplifies to

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Summary

The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x} \quad (1)$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)x^2 + (256c_2 - 8)x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((2*x+1)^2*diff(y(x),x$2)-2*(2*x+1)*diff(y(x),x)-12*y(x)=6*x,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{2x + 1} + (2x + 1)^3 c_2 + \frac{-24x^2 - 8x - 1}{64x + 32}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 41

```
DSolve[(2*x+1)^2*y'[x]-2*(2*x+1)*y'[x]-12*y[x]==6*x,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \rightarrow \frac{-24x^2 - 8x + 32c_1(2x + 1)^4 - 1 + 32c_2}{32(2x + 1)}$$

13 Chapter 18. Linear equations with variable coefficients (Equations of second order).

Supplementary problems. Page 120

13.1 problem 21	1911
13.2 problem 22	1925
13.3 problem 23	1944
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13.5 problem 25	1973
13.6 problem 26	1980
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13.8 problem 28	1999
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13.1 problem 21

- 13.1.1 Solving using Kovacic algorithm 1911
- 13.1.2 Solving as second order ode lagrange adjoint equation method od4917
- 13.1.3 Maple step by step solution 1921

Internal problem ID [5412]

Internal file name [OUTPUT/4903_Sunday_February_04_2024_12_47_04_AM_2677231/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
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Problem number: 21.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**kovacic**"

Maple gives the following as the ode type

[_Laguerre]

$$xy'' - (x + 2)y' + 2y = 0$$

13.1.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-x - 2)y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x - 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for \sqrt{r} at ∞ . Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^v a_i x^i \\ &= \sum_{i=0}^0 a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of \sqrt{r} at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \quad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_{\infty} &= \sum_{i=0}^0 a_i x^i \\ &= \frac{1}{2} \end{aligned} \quad (10)$$

Now we need to find b , where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $([\sqrt{r}]_{\infty})^2$ where $[\sqrt{r}]_{\infty}$ was found above in Eq (10). Hence

$$([\sqrt{r}]_{\infty})^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r . How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t . Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4 . Dividing this by leading coefficient in t which is 4 gives -1 . Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_{\infty} &= \frac{1}{2} \\ \alpha_{\infty}^{+} &= \frac{1}{2} \left(\frac{b}{a} - v \right) = \frac{1}{2} \left(\frac{-1}{\frac{1}{2}} - 0 \right) = -1 \\ \alpha_{\infty}^{-} &= \frac{1}{2} \left(-\frac{b}{a} - v \right) = \frac{1}{2} \left(-\frac{-1}{\frac{1}{2}} - 0 \right) = 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^{\pm} have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated α_{∞}^{\pm} have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_{∞}^{\pm} . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_{\infty}^{+} = -1$ then

$$\begin{aligned} d &= \alpha_{\infty}^{+} - (\alpha_{c_1}^{-}) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{x} + \left(\frac{1}{2} \right) \\ &= \frac{1}{2} - \frac{1}{x} \\ &= \frac{-2 + x}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2 - 4x + 8}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\ &= \frac{e^{\frac{x}{2}}}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x-2}{x} dx} \\ &= z_1 e^{\frac{x}{2} + \ln(x)} \\ &= z_1 \left(x e^{\frac{x}{2}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{x+2\ln(x)}}{(y_1)^2} dx \\ &= y_1 (-(x^2 + 2x + 2) e^{-x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^x) + c_2 (e^x (-(x^2 + 2x + 2) e^{-x})) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 (-x^2 - 2x - 2) \quad (1)$$

Verification of solutions

$$y = c_1 e^x + c_2 (-x^2 - 2x - 2)$$

Verified OK.

13.1.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$xy'' + (-x - 2)y' + 2y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = r(x) \quad (2)$$

Where

$$\begin{aligned} p(x) &= \frac{-x-2}{x} \\ q(x) &= \frac{2}{x} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi p)' + \xi q &= 0 \\ \xi'' - \left(\frac{(-x-2)\xi(x)}{x} \right)' + \left(\frac{2\xi(x)}{x} \right) &= 0 \\ \xi''(x) - \frac{(-x-2)\xi'(x)}{x} + \left(\frac{3}{x} + \frac{-x-2}{x^2} \right) \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. In normal form the ode

$$\xi''(x) x^2 + (x^2 + 2x) \xi'(x) + (2x - 2) \xi(x) = 0 \quad (1)$$

Becomes

$$\xi''(x) + p(x) \xi'(x) + q(x) \xi(x) = 0 \quad (2)$$

Where

$$\begin{aligned} p(x) &= \frac{x+2}{x} \\ q(x) &= \frac{2x-2}{x^2} \end{aligned}$$

Applying change of variables on the dependent variable $\xi(x) = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$v''(x) + \left(\frac{2n}{x} + p \right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q \right) v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(x+2)}{x^2} + \frac{2x-2}{x^2} = 0 \quad (5)$$

Solving (5) for n gives

$$n = -2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{4}{x} + \frac{x+2}{x} \right) v'(x) &= 0 \\ v''(x) + \frac{(-2+x)v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2+x)u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{(-2+x)u}{x} \end{aligned}$$

Where $f(x) = -\frac{-2+x}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{-2+x}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{-2+x}{x} dx \\ \ln(u) &= -x + 2 \ln(x) + c_1 \\ u &= e^{-x+2 \ln(x)+c_1} \\ &= c_1 e^{-x+2 \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 x^2 e^{-x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\v(x) &= \int u(x) dx + c_2 \\&= -(x^2 + 2x + 2) c_1 e^{-x} + c_2\end{aligned}$$

Hence

$$\begin{aligned}\xi(x) &= v(x) x^n \\&= \frac{-(x^2 + 2x + 2) c_1 e^{-x} + c_2}{x^2} \\&= \frac{-(x^2 + 2x + 2) c_1 e^{-x} + c_2}{x^2}\end{aligned}$$

The original ode (2) now reduces to first order ode

$$\begin{aligned}\xi(x) y' - y \xi'(x) + \xi(x) p(x) y &= \int \xi(x) r(x) dx \\y' + y \left(p(x) - \frac{\xi'(x)}{\xi(x)} \right) &= \frac{\int \xi(x) r(x) dx}{\xi(x)} \\y' + y \left(\frac{-x - 2}{x} - \frac{\left(\frac{-(2+2x)c_3 e^{-x} + (x^2+2x+2)c_3 e^{-x}}{x^2} - \frac{2(-(x^2+2x+2)c_3 e^{-x} + c_2)}{x^3} \right) x^2}{-(x^2 + 2x + 2) c_3 e^{-x} + c_2} \right) &= 0\end{aligned}$$

Which is now a first order ode. This is now solved for y . In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\&= f(x)g(y) \\&= \frac{y(c_2 e^x - 2c_3 x - 2c_3)}{-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3}\end{aligned}$$

Where $f(x) = \frac{c_2 e^x - 2c_3 x - 2c_3}{-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y} dy &= \frac{c_2 e^x - 2c_3 x - 2c_3}{-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3} dx \\ \int \frac{1}{y} dy &= \int \frac{c_2 e^x - 2c_3 x - 2c_3}{-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3} dx \\ \ln(y) &= \ln(-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3) + c_3 \\ y &= e^{\ln(-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3) + c_3} \\ &= c_3(-c_3 x^2 + c_2 e^x - 2c_3 x - 2c_3)\end{aligned}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = e^x c_2 c_3 + c_3(-c_3 x^2 - 2c_3 x - 2c_3)$$

Summary

The solution(s) found are the following

$$y = e^x c_2 c_3 + c_3(-c_3 x^2 - 2c_3 x - 2c_3) \quad (1)$$

Verification of solutions

$$y = e^x c_2 c_3 + c_3(-c_3 x^2 - 2c_3 x - 2c_3)$$

Verified OK.

13.1.3 Maple step by step solution

Let's solve

$$y''x + (-x - 2)y' + 2y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{2y}{x} + \frac{(x+2)y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(x+2)y'}{x} + \frac{2y}{x} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{x+2}{x}, P_3(x) = \frac{2}{x}]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x + (-x - 2)y' + 2y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1} (k+1+r)(k+r-2) - a_k (k+r-2)) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)(a_{k+1}(k+1+r) - a_k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+1} \right]$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{a_k}{k+4}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = \frac{a_k}{k+4} \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3} \right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+4} \right]$$

Maple trace Kovacic algorithm successful

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Reducible group (found another exponential solution)
<- Kovacics algorithm successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*dif(y(x),x$2)-(x+2)*dif(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + c_2(x^2 + 2x + 2)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[x*y'[x]-(x+2)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 e^x - c_2 (x^2 + 2x + 2)$$

13.2 problem 22

- 13.2.1 Solving as second order change of variable on y method 2 ode . 1925
- 13.2.2 Solving as second order ode non constant coeff transformation
on B ode 1930
- 13.2.3 Solving using Kovacic algorithm 1935

Internal problem ID [5413]

Internal file name [OUTPUT/4904_Sunday_February_04_2024_12_47_05_AM_64049652/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 22.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$(x^2 + 1)y'' - 2xy' + 2y = 2$$

13.2.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 + 1$, $B = -2x$, $C = 2$, $f(x) = 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$(x^2 + 1)y'' - 2xy' + 2y = 0$$

In normal form the ode

$$(x^2 + 1) y'' - 2xy' + 2y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = -\frac{2x}{x^2 + 1}$$
$$q(x) = \frac{2}{x^2 + 1}$$

Applying change of variables on the dependent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2 + 1} + \frac{2}{x^2 + 1} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - \frac{2x}{x^2 + 1}\right) v'(x) = 0$$
$$v''(x) + \frac{2v'(x)}{x(x^2 + 1)} = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x(x^2 + 1)} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x(x^2 + 1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{2}{x(x^2 + 1)} dx \\ \int \frac{1}{u} du &= \int -\frac{2}{x(x^2 + 1)} dx \\ \ln(u) &= \ln(x^2 + 1) - 2 \ln(x) + c_1 \\ u &= e^{\ln(x^2+1) - 2 \ln(x) + c_1} \\ &= c_1 e^{\ln(x^2+1) - 2 \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{1}{x^2} \right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \left(x - \frac{1}{x} \right) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x) x^n \\ &= \left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x \\ &= c_1 x^2 + c_2 x - c_1 \end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2 + 1) y'' - 2xy' + 2y = 2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^2 - 1$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 - 1 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2 - 1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2 - 1)(1)$$

Which simplifies to

$$W = x^2 + 1$$

Which simplifies to

$$W = x^2 + 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{2x^2 - 2}{(x^2 + 1)^2} dx$$

Which simplifies to

$$u_1 = - \int \frac{2x^2 - 2}{(x^2 + 1)^2} dx$$

Hence

$$u_1 = \frac{2x}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x}{(x^2 + 1)^2} dx$$

Which simplifies to

$$u_2 = \int \frac{2x}{(x^2 + 1)^2} dx$$

Hence

$$u_2 = -\frac{1}{x^2 + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x^2}{x^2 + 1} - \frac{x^2 - 1}{x^2 + 1}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$\begin{aligned}
 y &= y_h + y_p \\
 &= \left(\left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x \right) + (1) \\
 &= 1 + \left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x
 \end{aligned}$$

Which simplifies to

$$y = c_1 x^2 + c_2 x - c_1 + 1$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x - c_1 + 1 \quad (1)$$

Verification of solutions

$$y = c_1 x^2 + c_2 x - c_1 + 1$$

Verified OK.

13.2.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
 y' &= B'v + v'B \\
 y'' &= B''v + B'v' + v''B + v'B' \\
 &= v''B + 2v' + B' + B''v
 \end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
 A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
 ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0
 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$A = x^2 + 1$$

$$B = -2x$$

$$C = 2$$

$$F = 2$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^2 + 1)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x(x^2 + 1)v'' + (-4)v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-2x^3 - 2x)u'(x) - 4u(x) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x(x^2 + 1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{2}{x(x^2+1)} dx \\ \int \frac{1}{u} du &= \int -\frac{2}{x(x^2+1)} dx \\ \ln(u) &= \ln(x^2+1) - 2\ln(x) + c_1 \\ u &= e^{\ln(x^2+1) - 2\ln(x) + c_1} \\ &= c_1 e^{\ln(x^2+1) - 2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{1}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 \left(1 + \frac{1}{x^2}\right)\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{(x^2+1)c_1}{x^2} dx \\ &= c_1 \left(x - \frac{1}{x}\right) + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-2x) \left(c_1 \left(x - \frac{1}{x}\right) + c_2\right) \\ &= -2c_1x^2 - 2c_2x + 2c_1\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = -2x^2 + 2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ 1 & -4x \end{vmatrix}$$

Therefore

$$W = (x)(-4x) - (-2x^2 + 2)(1) \quad (1)$$

Which simplifies to

$$W = -2x^2 - 2$$

Which simplifies to

$$W = -2x^2 - 2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{-4x^2 + 4}{(x^2 + 1)(-2x^2 - 2)} dx$$

Which simplifies to

$$u_1 = - \int \frac{2x^2 - 2}{(x^2 + 1)^2} dx$$

Hence

$$u_1 = \frac{2x}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x}{(x^2 + 1)(-2x^2 - 2)} dx$$

Which simplifies to

$$u_2 = \int -\frac{x}{(x^2 + 1)^2} dx$$

Hence

$$u_2 = \frac{1}{2x^2 + 2}$$

Which simplifies to

$$u_1 = \frac{2x}{x^2 + 1}$$
$$u_2 = \frac{1}{2x^2 + 2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x^2}{x^2 + 1} + \frac{-2x^2 + 2}{2x^2 + 2}$$

Which simplifies to

$$y_p(x) = 1$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2c_1x^2 - 2c_2x + 2c_1) + (1) \\ &= -2c_1x^2 - 2c_2x + 2c_1 + 1 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 - 2c_2x + 2c_1 + 1 \quad (1)$$

Verification of solutions

$$y = -2c_1x^2 - 2c_2x + 2c_1 + 1$$

Verified OK.

13.2.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 + 1)y'' - 2xy' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = x^2 + 1$$

$$B = -2x \quad (3)$$

$$C = 2$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2 + 1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$s = -3$$

$$t = (x^2 + 1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2 + 1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-i+x)^2} + \frac{3}{4(i+x)^2} + \frac{3i}{4(-i+x)} - \frac{3i}{4(i+x)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(-i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^+) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} + (-)(0) \\ &= -\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} \\ &= \frac{-2i+x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2 \left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} \right) (0) + \left(\left(\frac{1}{2(-i+x)^2} - \frac{3}{2(i+x)^2} \right) + \left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} \right)^2 \right) -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int \left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} \right) dx} \\ &= \frac{(x^2 + 1)^{\frac{3}{2}}}{(ix + 1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2+1} dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2 + 1} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 + 1)^2}{(ix + 1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2} dx \\ &= y_1 \left(-\frac{x}{(i+x)^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2 + 1)^2}{(ix + 1)^2} \right) + c_2 \left(\frac{(x^2 + 1)^2}{(ix + 1)^2} \left(-\frac{x}{(i + x)^2} \right) \right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 1)y'' - 2xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1(x^2 + 1)^2}{(ix + 1)^2} + \frac{c_2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{(x^2 + 1)^2}{(ix + 1)^2} \\
y_2 &= \frac{(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \\ \frac{d}{dx} \left(\frac{(x^2+1)^2}{(ix+1)^2} \right) & \frac{d}{dx} \left(\frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \\ \frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3} & \frac{4(x^2+1)x^2}{(-i+x)^2(i+x)^2} + \frac{(x^2+1)^2}{(-i+x)^2(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^3(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^2(i+x)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2+1)^2}{(ix+1)^2} \right) \left(\frac{4(x^2+1)x^2}{(-i+x)^2(i+x)^2} + \frac{(x^2+1)^2}{(-i+x)^2(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^3(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^2(i+x)^3} \right) - \left(\frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \right) \left(\frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3} \right)$$

Which simplifies to

$$W = \frac{(x^2+1)^3 (ix^5 + 3x^4 - 2ix^3 + 2x^2 - 3ix - 1)}{(ix+1)^3 (-x+i)^3 (i+x)^3}$$

Which simplifies to

$$W = \frac{(x^2+1)^4 (-x^3 + 3ix^2 + 3x - i)}{(-ix-1)^6 (i+x)^3}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{2(x^2+1)^2 x}{(-i+x)^2(i+x)^2}}{\frac{(x^2+1)^5 (-x^3 + 3ix^2 + 3x - i)}{(-ix-1)^6 (i+x)^3}} dx$$

Which simplifies to

$$u_1 = - \int \frac{2x(i+x)(-i+x)^4}{(x^3 - 3ix^2 - 3x + i)(x^2+1)^3} dx$$

Hence

$$u_1 = -\frac{x^2}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2(x^2+1)^2}{(ix+1)^2}}{\frac{(x^2+1)^5(-x^3+3ix^2+3x-i)}{(-ix-1)^6(i+x)^3}} dx$$

Which simplifies to

$$u_2 = \int \frac{2(i+x)^3(-i+x)^4}{(x^2+1)^3(-x^3+3ix^2+3x-i)} dx$$

Hence

$$u_2 = \frac{2ix}{-x+i}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(x^2+1)}{(ix+1)^2} + \frac{2ix^2(x^2+1)^2}{(-x+i)(-i+x)^2(i+x)^2}$$

Which simplifies to

$$y_p(x) = \frac{(x^2+1)^2 x^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(i+x)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \right) + \left(\frac{(x^2+1)^2 x^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(i+x)^2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(-i+x)^2(i+x)^2} + \frac{(x^2+1)^2 x^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(i+x)^2} \quad (1)$$

Verification of solutions

$$y = \frac{c_1(x^2 + 1)^2}{(ix + 1)^2} + \frac{c_2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2} + \frac{(x^2 + 1)^2 x^2 (-x^3 + 3ix^2 + 3x - i)}{(-x + i)^5 (i + x)^2}$$

Verified OK.

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=2,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + c_2 x - c_1 + 1$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 22

```
DSolve[(1+x^2)*y'[x]-2*x*y'[x]+2*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -c_1(x - i)^2 + c_2 x + 1$$

13.3 problem 23

- 13.3.1 Solving as second order change of variable on y method 2 ode . 1944
- 13.3.2 Solving as second order ode non constant coeff transformation on B ode 1949
- 13.3.3 Solving using Kovacic algorithm 1954

Internal problem ID [5414]

Internal file name [OUTPUT/4905_Sunday_February_04_2024_12_47_07_AM_82207300/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplementary problems. Page 120

Problem number: 23.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$(x^2 + 4) y'' - 2xy' + 2y = 8$$

13.3.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 + 4$, $B = -2x$, $C = 2$, $f(x) = 8$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$(x^2 + 4) y'' - 2xy' + 2y = 0$$

In normal form the ode

$$(x^2 + 4) y'' - 2xy' + 2y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = -\frac{2x}{x^2 + 4}$$
$$q(x) = \frac{2}{x^2 + 4}$$

Applying change of variables on the dependent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2 + 4} + \frac{2}{x^2 + 4} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - \frac{2x}{x^2 + 4}\right) v'(x) = 0$$
$$v''(x) + \frac{8v'(x)}{(x^2 + 4)x} = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{8u(x)}{(x^2 + 4)x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{8u}{(x^2 + 4)x} \end{aligned}$$

Where $f(x) = -\frac{8}{(x^2+4)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{8}{(x^2 + 4)x} dx \\ \int \frac{1}{u} du &= \int -\frac{8}{(x^2 + 4)x} dx \\ \ln(u) &= -2 \ln(x) + \ln(x^2 + 4) + c_1 \\ u &= e^{-2 \ln(x) + \ln(x^2 + 4) + c_1} \\ &= c_1 e^{-2 \ln(x) + \ln(x^2 + 4)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{4}{x^2} \right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \left(x - \frac{4}{x} \right) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x) x^n \\ &= \left(c_1 \left(x - \frac{4}{x} \right) + c_2 \right) x \\ &= c_1 x^2 + c_2 x - 4c_1 \end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2 + 4) y'' - 2xy' + 2y = 8$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^2 - 4$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 - 4 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2 - 4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 - 4 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2 - 4)(1)$$

Which simplifies to

$$W = x^2 + 4$$

Which simplifies to

$$W = x^2 + 4$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{8x^2 - 32}{(x^2 + 4)^2} dx$$

Which simplifies to

$$u_1 = - \int \frac{8x^2 - 32}{(x^2 + 4)^2} dx$$

Hence

$$u_1 = \frac{8x}{x^2 + 4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x}{(x^2 + 4)^2} dx$$

Which simplifies to

$$u_2 = \int \frac{8x}{(x^2 + 4)^2} dx$$

Hence

$$u_2 = -\frac{4}{x^2 + 4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8x^2}{x^2 + 4} - \frac{4(x^2 - 4)}{x^2 + 4}$$

Which simplifies to

$$y_p(x) = 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \left(x - \frac{4}{x} \right) + c_2 \right) x \right) + (4) \\ &= 4 + \left(c_1 \left(x - \frac{4}{x} \right) + c_2 \right) x \end{aligned}$$

Which simplifies to

$$y = c_1 x^2 + c_2 x - 4c_1 + 4$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x - 4c_1 + 4 \quad (1)$$

Verification of solutions

$$y = c_1 x^2 + c_2 x - 4c_1 + 4$$

Verified OK.

13.3.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$A = x^2 + 4$$

$$B = -2x$$

$$C = 2$$

$$F = 8$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^2 + 4)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2(x^2 + 4)xv'' + (-16)v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-2x^3 - 8x)u'(x) - 16u(x) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{8u}{(x^2 + 4)x} \end{aligned}$$

Where $f(x) = -\frac{8}{(x^2+4)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{8}{(x^2+4)x} dx \\ \int \frac{1}{u} du &= \int -\frac{8}{(x^2+4)x} dx \\ \ln(u) &= -2 \ln(x) + \ln(x^2+4) + c_1 \\ u &= e^{-2 \ln(x) + \ln(x^2+4) + c_1} \\ &= c_1 e^{-2 \ln(x) + \ln(x^2+4)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{4}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 \left(1 + \frac{4}{x^2}\right)\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{(x^2+4)c_1}{x^2} dx \\ &= c_1 \left(x - \frac{4}{x}\right) + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-2x) \left(c_1 \left(x - \frac{4}{x}\right) + c_2\right) \\ &= -2c_1x^2 - 2c_2x + 8c_1\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\y_2 &= -2x^2 + 8\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -2x^2 + 8 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 8) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -2x^2 + 8 \\ 1 & -4x \end{vmatrix}$$

Therefore

$$W = (x)(-4x) - (-2x^2 + 8) \quad (1)$$

Which simplifies to

$$W = -2x^2 - 8$$

Which simplifies to

$$W = -2x^2 - 8$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{-16x^2 + 64}{(x^2 + 4)(-2x^2 - 8)} dx$$

Which simplifies to

$$u_1 = - \int \frac{8x^2 - 32}{(x^2 + 4)^2} dx$$

Hence

$$u_1 = \frac{8x}{x^2 + 4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x}{(x^2 + 4)(-2x^2 - 8)} dx$$

Which simplifies to

$$u_2 = \int -\frac{4x}{(x^2 + 4)^2} dx$$

Hence

$$u_2 = \frac{2}{x^2 + 4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8x^2}{x^2 + 4} + \frac{-4x^2 + 16}{x^2 + 4}$$

Which simplifies to

$$y_p(x) = 4$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2c_1x^2 - 2c_2x + 8c_1) + (4) \\ &= -2c_1x^2 - 2c_2x + 8c_1 + 4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 - 2c_2x + 8c_1 + 4 \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 - 2c_2x + 8c_1 + 4$$

Verified OK.

13.3.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 + 4) y'' - 2xy' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 4 \\ B &= -2x \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-12}{(x^2 + 4)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -12 \\ t &= (x^2 + 4)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{12}{(x^2 + 4)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 4)^2$. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-2i+x)^2} + \frac{3}{4(x+2i)^2} + \frac{3i}{8(-2i+x)} - \frac{3i}{8(x+2i)}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(-2i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{12}{(x^2 + 4)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$2i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-2i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^+) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-) [\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+) [\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-) [\sqrt{r}]_\infty \\ &= -\frac{1}{2(-2i + x)} + \frac{3}{2(x + 2i)} + (-)(0) \\ &= -\frac{1}{2(-2i + x)} + \frac{3}{2(x + 2i)} \\ &= \frac{x - 4i}{x^2 + 4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2 \left(-\frac{1}{2(-2i + x)} + \frac{3}{2(x + 2i)} \right) (0) + \left(\left(\frac{1}{2(-2i + x)^2} - \frac{3}{2(x + 2i)^2} \right) + \left(-\frac{1}{2(-2i + x)} + \frac{3}{2(x + 2i)} \right)^2 \right) (0) = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(-\frac{1}{2(-2i + x)} + \frac{3}{2(x + 2i)} \right) dx} \\ &= \frac{(x^2 + 4)^{\frac{3}{2}}}{(ix + 2)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2+4} dx} \\ &= z_1 e^{\frac{\ln(x^2+4)}{2}} \\ &= z_1 \left(\sqrt{x^2+4} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+4)^2}{(ix+2)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+4} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{\ln(x^2+4)}}{(y_1)^2} dx \\ &= y_1 \left(-\frac{x}{(x+2i)^2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{(x^2+4)^2}{(ix+2)^2} \right) + c_2 \left(\frac{(x^2+4)^2}{(ix+2)^2} \left(-\frac{x}{(x+2i)^2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 4)y'' - 2xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1(x^2 + 4)^2}{(ix + 2)^2} + c_2x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x^2 + 4)^2}{(ix + 2)^2}$$

$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2+4)^2}{(ix+2)^2} & x \\ \frac{d}{dx} \left(\frac{(x^2+4)^2}{(ix+2)^2} \right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2+4)^2}{(ix+2)^2} & x \\ \frac{4(x^2+4)x}{(ix+2)^2} - \frac{2i(x^2+4)^2}{(ix+2)^3} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2+4)^2}{(ix+2)^2} \right) (1) - (x) \left(\frac{4(x^2+4)x}{(ix+2)^2} - \frac{2i(x^2+4)^2}{(ix+2)^3} \right)$$

Which simplifies to

$$W = -\frac{(x^2+4)(ix^3+6x^2-12ix-8)}{(ix+2)^3}$$

Which simplifies to

$$W = -\frac{(x^2+4)(ix^3+6x^2-12ix-8)}{(ix+2)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x}{-\frac{(x^2+4)^2(ix^3+6x^2-12ix-8)}{(ix+2)^3}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{8x(ix+2)^3}{(x^2+4)^2(ix^3+6x^2-12ix-8)} dx$$

Hence

$$u_1 = -\frac{x^2}{x^2+4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{8(x^2+4)^2}{(ix+2)^2}}{-\frac{(x^2+4)^2(ix^3+6x^2-12ix-8)}{(ix+2)^3}} dx$$

Which simplifies to

$$u_2 = \int \frac{-16i+8x}{-x^3+6ix^2+12x-8i} dx$$

Hence

$$u_2 = \frac{4ix}{2i - x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(x^2 + 4)x^2}{(ix + 2)^2} + \frac{4ix^2}{2i - x}$$

Which simplifies to

$$y_p(x) = \frac{(x^2 + 4)x^2}{(-2i + x)^2} + \frac{4ix^2}{2i - x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1(x^2 + 4)^2}{(ix + 2)^2} + c_2x \right) + \left(\frac{(x^2 + 4)x^2}{(-2i + x)^2} + \frac{4ix^2}{2i - x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2 + 4)^2}{(ix + 2)^2} + c_2x + \frac{(x^2 + 4)x^2}{(-2i + x)^2} + \frac{4ix^2}{2i - x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1(x^2 + 4)^2}{(ix + 2)^2} + c_2x + \frac{(x^2 + 4)x^2}{(-2i + x)^2} + \frac{4ix^2}{2i - x}$$

Verified OK.

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve((x^2+4)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=8,y(x), singsol=all)
```

$$y(x) = c_1x^2 + c_2x - 4c_1 + 4$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 22

```
DSolve[(x^2+4)*y'[x]-2*x*y'[x]+2*y[x]==8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1(x - 2i)^2 - c_2x + 4$$

13.4 problem 24

13.4.1 Solving using Kovacic algorithm 1963

Internal problem ID [5415]

Internal file name [OUTPUT/4906_Sunday_February_04_2024_12_47_08_AM_16356909/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplementary problems. Page 120

Problem number: 24.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$(x + 1)y'' - (2x + 3)y' + (x + 2)y = (x^2 + 2x + 1)e^{2x}$$

13.4.1 Solving using Kovacic algorithm

Writing the ode as

$$(x + 1)y'' + (-2x - 3)y' + (x + 2)y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = x + 1$$

$$B = -2x - 3 \tag{3}$$

$$C = x + 2$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4(x+1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$s = 3$$

$$t = 4(x+1)^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4(x+1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4(x+1)^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x+1)} + (-)(0) \\ &= -\frac{1}{2(x+1)} \\ &= -\frac{1}{2(x+1)} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x+1)^2}\right) + \left(-\frac{1}{2(x+1)}\right)^2 - \left(\frac{3}{4(x+1)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int -\frac{1}{2(x+1)} dx} \\ &= \frac{1}{\sqrt{x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-3}{x+1} dx} \\ &= z_1 e^{x + \frac{\ln(x+1)}{2}} \\ &= z_1 (\sqrt{x+1} e^x) \end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-3}{x+1} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{2x+\ln(x+1)}}{(y_1)^2} dx \\&= y_1 \left(\frac{x(x+2)}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 (e^x) + c_2 \left(e^x \left(\frac{x(x+2)}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x+1)y'' + (-2x-3)y' + (x+2)y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^x + \frac{c_2 x(x+2) e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$

$$y_2 = \frac{x(x+2)e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{x(x+2)e^x}{2} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{x(x+2)e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{x(x+2)e^x}{2} \\ e^x & \frac{(x+2)e^x}{2} + \frac{x e^x}{2} + \frac{x(x+2)e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^x) \left(\frac{(x+2)e^x}{2} + \frac{x e^x}{2} + \frac{x(x+2)e^x}{2} \right) - \left(\frac{x(x+2)e^x}{2} \right) (e^x)$$

Which simplifies to

$$W = e^{2x}x + e^{2x}$$

Which simplifies to

$$W = (x+1)e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x(x+2)e^x(x+1)^2 e^{2x}}{(x+1)^2 e^{2x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{x(x+2)e^x}{2} dx$$

Hence

$$u_1 = - \frac{x^2 e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x(x+1)^2 e^{2x}}{(x+1)^2 e^{2x}} dx$$

Which simplifies to

$$u_2 = \int e^x dx$$

Hence

$$u_2 = e^x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2 e^{2x}}{2} + \frac{x(x+2)e^{2x}}{2}$$

Which simplifies to

$$y_p(x) = e^{2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 x(x+2)e^x}{2} \right) + (e^{2x} x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 x(x+2) e^x}{2} + e^{2x} x \quad (1)$$

Verification of solutions

$$y = c_1 e^x + \frac{c_2 x(x+2) e^x}{2} + e^{2x} x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve((x+1)*diff(y(x),x$2)-(2*x+3)*diff(y(x),x)+(x+2)*y(x)=(x^2+2*x+1)*exp(2*x),y(x), sings
```

$$y(x) = e^{2x} x + e^x (c_1 x^2 + 2c_1 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 32

```
DSolve[(x+1)*y'[x]-(2*x+3)*y'[x]+(x+2)*y[x]==(x^2+2*x+1)*Exp[2*x],y[x],x,IncludeSingularSol
```

$$y(x) \rightarrow \frac{1}{2}e^x(2e^x x + ec_2(x+2)x + 2ec_1)$$

13.5 problem 25

13.5.1 Solving as second order change of variable on y method 1 ode . 1973

13.5.2 Solving using Kovacic algorithm 1976

Internal problem ID [5416]

Internal file name [OUTPUT/4907_Sunday_February_04_2024_12_47_09_AM_15884514/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 25.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_change_of_variable_on_y_method_1"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - 2 \tan(x) y' - 10y = 0$$

13.5.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = -2 \tan(x)$$

$$q(x) = -10$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
 Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
 &= -10 - \frac{(-2 \tan(x))'}{2} - \frac{(-2 \tan(x))^2}{4} \\
 &= -10 - \frac{(-2 - 2 \tan(x)^2)}{2} - \frac{(4 \tan(x)^2)}{4} \\
 &= -10 - (-1 - \tan(x)^2) - \tan(x)^2 \\
 &= -9
 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x) z(x) \quad (3)$$

is used to change the original ode to a constant coefficients ode in v . In (3) the term $z(x)$ is given by

$$\begin{aligned}
 z(x) &= e^{-\left(\int \frac{p(x)}{2} dx\right)} \\
 &= e^{-\int \frac{-2 \tan(x)}{2}} \\
 &= \sec(x)
 \end{aligned} \quad (5)$$

Hence (3) becomes

$$y = v(x) \sec(x) \quad (4)$$

Applying this change of variable to the original ode results in

$$\sec(x) (v''(x) - 9v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 9 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-9)} \\ &= \pm 3 \end{aligned}$$

Hence

$$\lambda_1 = +3$$

$$\lambda_2 = -3$$

Which simplifies to

$$\lambda_1 = 3$$

$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$v(x) = c_1 e^{(3)x} + c_2 e^{(-3)x}$$

Or

$$v(x) = e^{3x} c_1 + c_2 e^{-3x}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x) z(x) \\ &= (e^{3x} c_1 + c_2 e^{-3x}) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = (e^{3x} c_1 + c_2 e^{-3x}) \sec(x)$$

Summary

The solution(s) found are the following

$$y = (e^{3x}c_1 + c_2e^{-3x}) \sec(x) \quad (1)$$

Verification of solutions

$$y = (e^{3x}c_1 + c_2e^{-3x}) \sec(x)$$

Verified OK.

13.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2 \tan(x) y' - 10y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \tan(x) \\ C &= -10 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2 \tan(x)}{1} dx} \\ &= z_1 e^{-\ln(\cos(x))} \\ &= z_1 (\sec(x)) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-3x} \sec(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2 \tan(x)}{1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2 \ln(\cos(x))}}{(y_1)^2} dx \\ &= y_1 \left(\frac{e^{6x}}{6} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{-3x} \sec(x)) + c_2 \left(e^{-3x} \sec(x) \left(\frac{e^{6x}}{6} \right) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} \sec(x) + \frac{c_2 e^{3x} \sec(x)}{6} \quad (1)$$

Verification of solutions

$$y = c_1 e^{-3x} \sec(x) + \frac{c_2 e^{3x} \sec(x)}{6}$$

Verified OK.

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacics algorithm  
  A Liouvillian solution exists  
  Reducible group (found an exponential solution)  
  Reducible group (found another exponential solution)  
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*tan(x)*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sec(x) (c_1 \sinh(3x) + c_2 \cosh(3x))$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 29

```
DSolve[y''[x]-2*Tan[x]*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6} e^{-3x} (c_2 e^{6x} + 6c_1) \sec(x)$$

13.6 problem 26

13.6.1 Solving using Kovacic algorithm 1980

Internal problem ID [5417]

Internal file name [OUTPUT/4908_Sunday_February_04_2024_12_47_09_AM_12356748/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 26.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$x^2 y'' - x(2x + 3) y' + (x^2 + 3x + 3) y = (-x^2 + 6) e^x$$

13.6.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + (-2x^2 - 3x) y' + (x^2 + 3x + 3) y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2 - 3x \\ C &= x^2 + 3x + 3 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$

$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = 3$$

$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega dx}$$

$$= e^{\int -\frac{1}{2x} dx}$$

$$= \frac{1}{\sqrt{x}}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

$$= z_1 e^{-\int \frac{1}{2} \frac{-2x^2 - 3x}{x^2} dx}$$

$$= z_1 e^{x + \frac{3 \ln(x)}{2}}$$

$$= z_1 \left(x^{\frac{3}{2}} e^x\right)$$

Which simplifies to

$$y_1 = x e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
 y_2 &= y_1 \int \frac{e^{\int -\frac{2x^2-3x}{x^2} dx}}{(y_1)^2} dx \\
 &= y_1 \int \frac{e^{2x+3 \ln(x)}}{(y_1)^2} dx \\
 &= y_1 \left(\frac{x^2}{2} \right)
 \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 (x e^x) + c_2 \left(x e^x \left(\frac{x^2}{2} \right) \right)
 \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' + (-2x^2 - 3x) y' + (x^2 + 3x + 3) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x e^x + \frac{c_2 x^3 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x e^x$$

$$y_2 = \frac{x^3 e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x e^x & \frac{x^3 e^x}{2} \\ \frac{d}{dx}(x e^x) & \frac{d}{dx}\left(\frac{x^3 e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x e^x & \frac{x^3 e^x}{2} \\ x e^x + e^x & \frac{3x^2 e^x}{2} + \frac{x^3 e^x}{2} \end{vmatrix}$$

Therefore

$$W = (x e^x) \left(\frac{3x^2 e^x}{2} + \frac{x^3 e^x}{2} \right) - \left(\frac{x^3 e^x}{2} \right) (x e^x + e^x)$$

Which simplifies to

$$W = e^{2x} x^3$$

Which simplifies to

$$W = e^{2x} x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^3 e^{2x} (-x^2 + 6)}{x^5 e^{2x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{-x^2 + 6}{2x^2} dx$$

Hence

$$u_1 = \frac{x}{2} + \frac{3}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x e^{2x} (-x^2 + 6)}{x^5 e^{2x}} dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2 + 6}{x^4} dx$$

Hence

$$u_2 = -\frac{2}{x^3} + \frac{1}{x}$$

Which simplifies to

$$u_1 = \frac{x}{2} + \frac{3}{x}$$
$$u_2 = \frac{x^2 - 2}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{2} + \frac{3}{x} \right) x e^x + \frac{(x^2 - 2) e^x}{2}$$

Which simplifies to

$$y_p(x) = e^x (x^2 + 2)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 x e^x + \frac{c_2 x^3 e^x}{2} \right) + (e^x (x^2 + 2))\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x e^x + \frac{c_2 x^3 e^x}{2} + e^x (x^2 + 2) \quad (1)$$

Verification of solutions

$$y = c_1 x e^x + \frac{c_2 x^3 e^x}{2} + e^x (x^2 + 2)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x)-x*(2*x+3)*diff(y(x),x)+(x^2+3*x+3)*y(x)=(6-x^2)*exp(x),y(x),sings
```

$$y(x) = e^x(c_1x^3 + c_2x + x^2 + 2)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 30

```
DSolve[x^2*y'[x]-x*(2*x+3)*y'[x]+(x^2+3*x+3)*y[x]==(6-x^2)*Exp[x],y[x],x,IncludeSingularSol
```

$$y(x) \rightarrow \frac{1}{2}e^x(c_2x^3 + 2x^2 + 2c_1x + 4)$$

13.7 problem 27

13.7.1 Solving using Kovacic algorithm 1990

13.7.2 Maple step by step solution 1995

Internal problem ID [5418]

Internal file name [OUTPUT/4909_Sunday_February_04_2024_12_47_10_AM_63012554/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 27.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**kovacic**"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$4x^2y'' + 4y'x^3 + (x^2 + 1)^2 y = 0$$

13.7.1 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + 4y'x^3 + (x^2 + 1)^2 y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 4x^2$$

$$B = 4x^3 \tag{3}$$

$$C = (x^2 + 1)^2$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ , which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{4x^3}{4x^2} dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(e^{-\frac{x^2}{4}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x} e^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3}{4x^2} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2} dx \\&= y_1(\ln(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 \left(\sqrt{x} e^{-\frac{x^2}{4}} \right) + c_2 \left(\sqrt{x} e^{-\frac{x^2}{4}} (\ln(x)) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \sqrt{x} e^{-\frac{x^2}{4}} + c_2 \sqrt{x} e^{-\frac{x^2}{4}} \ln(x) \quad (1)$$

Verification of solutions

$$y = c_1 \sqrt{x} e^{-\frac{x^2}{4}} + c_2 \sqrt{x} e^{-\frac{x^2}{4}} \ln(x)$$

Verified OK.

13.7.2 Maple step by step solution

Let's solve

$$4x^2 y'' + 4y' x^3 + (x^2 + 1)^2 y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -xy' - \frac{(x^2+1)^2 y}{4x^2}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + xy' + \frac{(x^2+1)^2 y}{4x^2} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$\left[P_2(x) = x, P_3(x) = \frac{(x^2+1)^2}{4x^2} \right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = 0$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = \frac{1}{4}$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$4x^2 y'' + 4y' x^3 + (x^2 + 1)^2 y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..4$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k \rightarrow k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^3 \cdot y'$ to series expansion

$$x^3 \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r+2}$$

- Shift index using $k \rightarrow k - 2$

$$x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2} (k-2+r) x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1 + 2r)^2 x^r + a_1(1 + 2r)^2 x^{1+r} + (a_2(3 + 2r)^2 + 2a_0(1 + 2r)) x^{2+r} + (a_3(5 + 2r)^2 + 2a_1(3 + 2r)) x^{3+r} + \dots$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$(-1 + 2r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = \frac{1}{2}$$

- The coefficients of each power of x must be 0

$$[a_1(1 + 2r)^2 = 0, a_2(3 + 2r)^2 + 2a_0(1 + 2r) = 0, a_3(5 + 2r)^2 + 2a_1(3 + 2r) = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{ a_1 = 0, a_2 = -\frac{2a_0(1+2r)}{4r^2+12r+9}, a_3 = 0 \right\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(2k + 2r - 1)^2 + (4k + 4r - 6) a_{k-2} + a_{k-4} = 0$$

- Shift index using $k- > k + 4$

$$a_{k+4}(2k + 7 + 2r)^2 + (4k + 10 + 4r) a_{k+2} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4ka_{k+2} + 4ra_{k+2} + a_k + 10a_{k+2}}{(2k+7+2r)^2}$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+4} = -\frac{4ka_{k+2} + a_k + 12a_{k+2}}{(2k+8)^2}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{4ka_{k+2} + a_k + 12a_{k+2}}{(2k+8)^2}, a_1 = 0, a_2 = -\frac{a_0}{4}, a_3 = 0 \right]$$

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacics algorithm  
  A Liouvillian solution exists  
  Reducible group (found an exponential solution)  
  Group is reducible, not completely reducible  
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(4*x^2*diff(y(x),x$2)+4*x^3*diff(y(x),x)+(x^2+1)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x} e^{-\frac{x^2}{4}} (c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 28

```
DSolve[4*x^2*y'[x]+4*x^3*y'[x]+(x^2+1)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-\frac{x^2}{4}} \sqrt{x} (c_2 \log(x) + c_1)$$

13.8 problem 28

13.8.1 Solving using Kovacic algorithm 1999

Internal problem ID [5419]

Internal file name [OUTPUT/4910_Sunday_February_04_2024_12_47_11_AM_42613447/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 28.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$x^2 y'' + (-4x^2 + x) y' + (4x^2 - 2x + 1) y = (x^2 - x + 1) e^x$$

13.8.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + (-4x^2 + x) y' + (4x^2 - 2x + 1) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x^2 + x \\ C &= 4x^2 - 2x + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$

$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -5$$

$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ , which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i \right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-) [\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-) [\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - i}{x} + (-) (0) \\ &= \frac{\frac{1}{2} - i}{x} \\ &= \frac{\frac{1}{2} - i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int \frac{\frac{1}{2} - i}{x} dx} \\ &= x^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 + x}{x^2} dx} \\ &= z_1 e^{2x - \frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{e^{2x}}{\sqrt{x}} \right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{-i} e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
 y_2 &= y_1 \int \frac{e^{\int -\frac{4x^2+x}{x^2} dx}}{(y_1)^2} dx \\
 &= y_1 \int \frac{e^{4x-\ln(x)}}{(y_1)^2} dx \\
 &= y_1 \left(-\frac{ix^{2i}}{2} \right)
 \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 (x^{-i} e^{2x}) + c_2 \left(x^{-i} e^{2x} \left(-\frac{ix^{2i}}{2} \right) \right)
 \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' + (-4x^2 + x) y' + (4x^2 - 2x + 1) y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^{-i} e^{2x} - \frac{ic_2 x^i e^{2x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-i} e^{2x}$$

$$y_2 = -\frac{ix^i e^{2x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-i} e^{2x} & -\frac{ix^i e^{2x}}{2} \\ \frac{d}{dx}(x^{-i} e^{2x}) & \frac{d}{dx}\left(-\frac{ix^i e^{2x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-i} e^{2x} & -\frac{ix^i e^{2x}}{2} \\ -\frac{ix^{-i} e^{2x}}{x} + 2x^{-i} e^{2x} & \frac{x^i e^{2x}}{2x} - ix^i e^{2x} \end{vmatrix}$$

Therefore

$$W = (x^{-i} e^{2x}) \left(\frac{x^i e^{2x}}{2x} - ix^i e^{2x} \right) - \left(-\frac{ix^i e^{2x}}{2} \right) \left(-\frac{ix^{-i} e^{2x}}{x} + 2x^{-i} e^{2x} \right)$$

Which simplifies to

$$W = \frac{x^{-i} e^{4x} x^i}{x}$$

Which simplifies to

$$W = \frac{e^{4x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{-ix^i e^{2x} (x^2 - x + 1) e^x}{x e^{4x}} dx$$

Which simplifies to

$$u_1 = - \int - \frac{ix^{-1+i} e^{-x} (x^2 - x + 1)}{2} dx$$

Hence

$$u_1 = \text{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-i} e^{2x} (x^2 - x + 1) e^x}{x e^{4x}} dx$$

Which simplifies to

$$u_2 = \int x^{-1-i} e^{-x} (x^2 - x + 1) dx$$

Hence

$$u_2 = \text{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{-i} e^{2x} - i \text{undefined } x^i e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x} (ix^i + x^{-i}) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{-i} e^{2x} - \frac{ic_2 x^i e^{2x}}{2} \right) + (e^{2x} (ix^i + x^{-i}) \text{undefined}) \end{aligned}$$

Which simplifies to

$$y = e^{2x} \left(x^{-i} c_1 - \frac{i x^i c_2}{2} \right) + e^{2x} (i x^i + x^{-i})$$

Summary

The solution(s) found are the following

$$y = e^{2x} \left(x^{-i} c_1 - \frac{i x^i c_2}{2} \right) + e^{2x} (i x^i + x^{-i}) \quad (1)$$

Verification of solutions

$$y = e^{2x} \left(x^{-i} c_1 - \frac{i x^i c_2}{2} \right) + e^{2x} (i x^i + x^{-i})$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Group is reducible or imprimitive
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+(x-4*x^2)*diff(y(x),x)+(1-2*x+4*x^2)*y(x)=(x^2-x+1)*exp(x),y(x), s
```

$$y(x) = e^{2x} x^i c_2 + e^{2x} x^{-i} c_1 + e^x$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 104

```
DSolve[x^2*y'[x]+(x-4*x^2)*y'[x]+(1-2*x+4*x^2)*y[x]==(x^2-x+1)*Exp[x],y[x],x,IncludeSingular
```

$$y(x) \rightarrow \frac{1}{2}e^{2x}x^{-i}(ix^{2i}\Gamma(-i,x) - ix^{2i}\Gamma(1-i,x) + ix^{2i}\Gamma(2-i,x) - ic_2x^{2i} - i\Gamma(i,x) + i\Gamma(1+i,x) - i\Gamma(2+i,x) + 2c_1)$$

13.9 problem 29

- 13.9.1 Solving as second order change of variable on x method 2 ode . 2009
- 13.9.2 Solving as second order change of variable on x method 1 ode . 2012
- 13.9.3 Solving as second order besse ode 2014
- 13.9.4 Solving using Kovacic algorithm 2015
- 13.9.5 Maple step by step solution 2021

Internal problem ID [5420]

Internal file name [OUTPUT/4911_Sunday_February_04_2024_12_47_12_AM_81941235/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 29.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode",
"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, ` _with_symmetry_[0,F(x)]`]]
```

$$xy'' - y' + 4yx^3 = 0$$

13.9.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)} dx \\ &= \int e^{-(\int -\frac{1}{x}dx)} dx \\ &= \int e^{\ln(x)} dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{4x^2}{x^2} \\ &= 4 \end{aligned} \quad (7)$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4 e^{\lambda\tau} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i \end{aligned}$$

Hence

$$\lambda_1 = +2i$$

$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$

$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau} (c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0 (c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) \quad (1)$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

13.9.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{x^2}}\end{aligned}\tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0\end{aligned}\tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q} dx \\ &= \frac{\int 2\sqrt{x^2} dx}{c} \\ &= \frac{x\sqrt{x^2}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)\tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

13.9.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - xy' + 4yx^4 = 0 \quad (1)$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = 1$$

$$\beta = 1$$

$$n = \frac{1}{2}$$

$$\gamma = 2$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{2} \sin(x^2)}{\sqrt{\pi} \sqrt{x^2}} - \frac{c_2 x \sqrt{2} \cos(x^2)}{\sqrt{\pi} \sqrt{x^2}}$$

Verified OK.

13.9.4 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' + 4yx^3 = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 4x^3 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16x^4 + 3}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{-16x^4 + 3}{4x^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for \sqrt{r} at ∞ . Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^v a_i x^i \\ &= \sum_{i=0}^1 a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of \sqrt{r} at ∞ is

$$\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 2i$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^1 a_i x^i \\ &= 2ix \end{aligned} \tag{10}$$

Now we need to find b , where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $([\sqrt{r}]_\infty)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$([\sqrt{r}]_\infty)^2 = -4x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r . How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{-16x^4 + 3}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= (-4x^2) + \left(\frac{3}{4x^2}\right) \\ &= -4x^2 + \frac{3}{4x^2} \end{aligned}$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_{\infty} &= 2ix \\ \alpha_{\infty}^+ &= \frac{1}{2} \left(\frac{b}{a} - v \right) = \frac{1}{2} \left(\frac{0}{2i} - 1 \right) = -\frac{1}{2} \\ \alpha_{\infty}^- &= \frac{1}{2} \left(-\frac{b}{a} - v \right) = \frac{1}{2} \left(-\frac{0}{2i} - 1 \right) = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16x^4 + 3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^+	α_{∞}^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^{\pm} have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated α_{∞}^{\pm} have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(2ix) \\ &= -\frac{1}{2x} - 2ix \\ &= -\frac{1}{2x} - 2ix \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4 + 3}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\ &= \frac{e^{-ix^2}}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} dx} \\&= z_1 e^{\frac{\ln(x)}{2}} \\&= z_1 (\sqrt{x})\end{aligned}$$

Which simplifies to

$$y_1 = e^{-ix^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} dx \\&= y_1 \left(-\frac{ie^{2ix^2}}{4} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 (e^{-ix^2}) + c_2 \left(e^{-ix^2} \left(-\frac{ie^{2ix^2}}{4} \right) \right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-ix^2} - \frac{ic_2 e^{ix^2}}{4} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-ix^2} - \frac{ic_2 e^{ix^2}}{4}$$

Verified OK.

13.9.5 Maple step by step solution

Let's solve

$$y''x - y' + 4yx^3 = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{y'}{x} - 4yx^2$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{y'}{x} + 4yx^2 = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x - y' + 4yx^3 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

$$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- Shift index using $k \rightarrow k - 3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r) x^{-1+r} + a_1 (1+r)(-1+r) x^r + a_2 (2+r)r x^{1+r} + a_3 (3+r)(1+r) x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k \dots \right)$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$
- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$
- The coefficients of each power of x must be 0

$$[a_1(1+r)(-1+r) = 0, a_2(2+r)r = 0, a_3(3+r)(1+r) = 0]$$
- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0, a_3 = 0\}$$
- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r-1) + 4a_{k-3} = 0$$

- Shift index using $k \rightarrow k + 3$

$$a_{k+4}(k+4+r)(k+2+r) + 4a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0 \right]$$

- Recursion relation for $r = 2$

$$a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+2} \right), a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{k+4} = -\frac{4b_k}{(k+6)(k+4)} \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x^2) + c_2 \cos(x^2)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 20

```
DSolve[x*y'[x]-y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(x^2) + c_2 \sin(x^2)$$

13.10 problem 30

- 13.10.1 Solving as second order change of variable on x method 2 ode . 2025
- 13.10.2 Solving as second order change of variable on x method 1 ode . 2031
- 13.10.3 Solving as second order besel ode ode 2036
- 13.10.4 Solving using Kovacic algorithm 2039

Internal problem ID [5421]

Internal file name [OUTPUT/4912_Sunday_February_04_2024_12_47_13_AM_96616221/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 30.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode",
"second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$x^4 y'' + 2y'x^3 + y = \frac{x+1}{x}$$

13.10.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$x^4 y'' + 2y'x^3 + y = 0$$

In normal form the ode

$$x^4 y'' + 2y' x^3 + y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{1}{x^4}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x) \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x) dx)} dx \\ &= \int e^{-(\int \frac{2}{x} dx)} dx \\ &= \int e^{-2 \ln(x)} dx \\ &= \int \frac{1}{x^2} dx \\ &= -\frac{1}{x} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\ &= 1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$

$$\lambda_2 = -i$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau} (c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0 (c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos\left(\frac{1}{x}\right) \\ y_2 &= \sin\left(\frac{1}{x}\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{x}\right)\right) \left(-\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right) - \left(\sin\left(\frac{1}{x}\right)\right) \left(\frac{\sin\left(\frac{1}{x}\right)}{x^2}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{x}\right)^2 + \sin\left(\frac{1}{x}\right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin\left(\frac{1}{x}\right) \left(1 + \frac{1}{x}\right)}{-x^2} dx$$

Which simplifies to

$$u_1 = - \int -\frac{\sin\left(\frac{1}{x}\right) (x + 1)}{x^3} dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right)\left(1 + \frac{1}{x}\right)}{-x^2} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right)(x+1)}{x^3} dx$$

Hence

$$u_2 = \cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)\right) \cos\left(\frac{1}{x}\right) \\ &+ \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)\right) \sin\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{x+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)\right) + \left(\frac{x+1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x} \quad (1)$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x}$$

Verified OK.

13.10.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4$, $B = 2x^3$, $C = 1$, $f(x) = 1 + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^4 y'' + 2y'x^3 + y = 0$$

In normal form the ode

$$x^4 y'' + 2y'x^3 + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{1}{x^4}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^4}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^4}}x^5}\end{aligned}\tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^4}}x^5} + \frac{2}{x}\frac{\sqrt{\frac{1}{x^4}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^4}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0\end{aligned}\tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q} dx \\ &= \frac{\int \sqrt{\frac{1}{x^4}} dx}{c} \\ &= -\frac{x\sqrt{\frac{1}{x^4}}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

Now the particular solution to this ODE is found

$$x^4 y'' + 2y'x^3 + y = 1 + \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{x}\right)$$

$$y_2 = \sin\left(\frac{1}{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos \left(\frac{1}{x} \right) \right) \left(-\frac{\cos \left(\frac{1}{x} \right)}{x^2} \right) - \left(\sin \left(\frac{1}{x} \right) \right) \left(\frac{\sin \left(\frac{1}{x} \right)}{x^2} \right)$$

Which simplifies to

$$W = -\frac{\cos \left(\frac{1}{x} \right)^2 + \sin \left(\frac{1}{x} \right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin \left(\frac{1}{x} \right) \left(1 + \frac{1}{x} \right)}{-x^2} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin \left(\frac{1}{x} \right) (x + 1)}{x^3} dx$$

Hence

$$u_1 = -\sin \left(\frac{1}{x} \right) + \frac{\cos \left(\frac{1}{x} \right)}{x} + \cos \left(\frac{1}{x} \right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos \left(\frac{1}{x} \right) \left(1 + \frac{1}{x} \right)}{-x^2} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos \left(\frac{1}{x} \right) (x + 1)}{x^3} dx$$

Hence

$$u_2 = \cos \left(\frac{1}{x} \right) + \frac{\sin \left(\frac{1}{x} \right)}{x} + \sin \left(\frac{1}{x} \right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right) \right) \cos\left(\frac{1}{x}\right) \\ + \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right) \right) \sin\left(\frac{1}{x}\right)$$

Which simplifies to

$$y_p(x) = \frac{x+1}{x}$$

Therefore the general solution is

$$y = y_h + y_p \\ = \left(c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) \right) + \left(\frac{x+1}{x} \right) \\ = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x}$$

Which simplifies to

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x} \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x}$$

Verified OK.

13.10.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' + \frac{y}{x^2} = \frac{1 + \frac{1}{x}}{x^2} \quad (1)$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{1}{2} \\ \beta &= 1 \\ n &= \frac{1}{2} \\ \gamma &= -1 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2} \sin\left(\frac{1}{x}\right)}{\sqrt{x} \sqrt{\pi} \sqrt{\frac{1}{x}}} - \frac{c_2 \sqrt{2} \cos\left(\frac{1}{x}\right)}{\sqrt{x} \sqrt{\pi} \sqrt{\frac{1}{x}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sqrt{2} \sin\left(\frac{1}{x}\right)}{\sqrt{x} \sqrt{\pi} \sqrt{\frac{1}{x}}} - \frac{c_2 \sqrt{2} \cos\left(\frac{1}{x}\right)}{\sqrt{x} \sqrt{\pi} \sqrt{\frac{1}{x}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{x}\right)$$

$$y_2 = \sin\left(\frac{1}{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{x}\right)\right) \left(-\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right) - \left(\sin\left(\frac{1}{x}\right)\right) \left(\frac{\sin\left(\frac{1}{x}\right)}{x^2}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{x}\right)^2 + \sin\left(\frac{1}{x}\right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}{-1} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{x}\right)(x+1)}{x^3} dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}{-1} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right)(x+1)}{x^3} dx$$

Hence

$$u_2 = \cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)\right) \cos\left(\frac{1}{x}\right) \\ &+ \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)\right) \sin\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{x+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\sqrt{2} \sin\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2} \cos\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} \right) + \left(\frac{x+1}{x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2} \sin\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2} \cos\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} + \frac{x+1}{x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1\sqrt{2} \sin\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2} \cos\left(\frac{1}{x}\right)}{\sqrt{x}\sqrt{\pi}\sqrt{\frac{1}{x}}} + \frac{x+1}{x}$$

Verified OK.

13.10.4 Solving using Kovacic algorithm

Writing the ode as

$$x^4 y'' + 2y'x^3 + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= 2x^3 \\ C &= 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^4} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^4}\right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one). Then for each pole c , $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of \sqrt{r} expanded around each pole c . Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \quad (1B)$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2} \left(\frac{b}{a} + v \right) \\ \alpha_c^- &= \frac{1}{2} \left(-\frac{b}{a} + v \right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding \sqrt{r} as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i}{x^2} + \dots \quad (2B)$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i}{x^2} \quad (3B)$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i$$

Now we need to find b . let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i}{x^2} \\ \alpha_c^+ &= \frac{1}{2} \left(\frac{b}{a} + v \right) = \frac{1}{2} \left(\frac{0}{i} + 2 \right) = 1 \\ \alpha_c^- &= \frac{1}{2} \left(-\frac{b}{a} + v \right) = \frac{1}{2} \left(-\frac{0}{i} + 2 \right) = 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{i}{x^2} + \frac{1}{x} + (-)(0) \\ &= -\frac{i}{x^2} + \frac{1}{x} \\ &= \frac{-i + x}{x^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2 \left(-\frac{i}{x^2} + \frac{1}{x} \right) (0) + \left(\left(\frac{2i}{x^3} - \frac{1}{x^2} \right) + \left(-\frac{i}{x^2} + \frac{1}{x} \right)^2 - \left(-\frac{1}{x^4} \right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(-\frac{i}{x^2} + \frac{1}{x} \right) dx} \\ &= x e^{\frac{i}{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{2x^3}{x^4} dx} \\&= z_1 e^{-\ln(x)} \\&= z_1 \left(\frac{1}{x} \right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{i}{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{x^4} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-2 \ln(x)}}{(y_1)^2} dx \\&= y_1 \left(-\frac{ie^{-\frac{2i}{x}}}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 \left(e^{\frac{i}{x}} \right) + c_2 \left(e^{\frac{i}{x}} \left(-\frac{ie^{-\frac{2i}{x}}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^4 y'' + 2y' x^3 + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\frac{i}{x}} - \frac{ic_2 e^{-\frac{i}{x}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{i}{x}}$$

$$y_2 = -\frac{ie^{-\frac{i}{x}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{i}{x}} & -\frac{ie^{-\frac{i}{x}}}{2} \\ \frac{d}{dx} \left(e^{\frac{i}{x}} \right) & \frac{d}{dx} \left(-\frac{ie^{-\frac{i}{x}}}{2} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{i}{x}} & -\frac{ie^{-\frac{i}{x}}}{2} \\ -\frac{ie^{\frac{i}{x}}}{x^2} & \frac{e^{-\frac{i}{x}}}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{i}{x}}\right) \left(\frac{e^{-\frac{i}{x}}}{2x^2}\right) - \left(-\frac{ie^{-\frac{i}{x}}}{2}\right) \left(-\frac{ie^{\frac{i}{x}}}{x^2}\right)$$

Which simplifies to

$$W = \frac{e^{\frac{i}{x}} e^{-\frac{i}{x}}}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{-\frac{ie^{-\frac{i}{x}}(1+\frac{1}{x})}{2}}{x^2} dx$$

Which simplifies to

$$u_1 = - \int -\frac{ie^{-\frac{i}{x}}(x+1)}{2x^3} dx$$

Hence

$$u_1 = \text{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\frac{i}{x}}(1+\frac{1}{x})}{x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{\frac{i}{x}}(x+1)}{x^3} dx$$

Hence

$$u_2 = \text{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{i}{2} e^{\frac{i}{x}} - i \frac{i}{2} e^{-\frac{i}{x}}$$

Which simplifies to

$$y_p(x) = \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}} \right) \frac{i}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2} \right) + \left(\left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}} \right) \frac{i}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2} + \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}} \right) \frac{i}{2} \quad (1)$$

Verification of solutions

$$y = c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2} + \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}} \right) \frac{i}{2}$$

Verified OK.

Maple trace

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`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```


✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(x^4*diff(y(x),x$2)+2*x^3*diff(y(x),x)+y(x)=(1+x)/x,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{x}\right) c_2 + \cos\left(\frac{1}{x}\right) c_1 + \frac{x+1}{x}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 25

```
DSolve[x^4*y''[x]+2*x^3*y'[x]+y[x]==(1+x)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{x} + c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + 1$$

13.11 problem 31

- 13.11.1 Solving as second order change of variable on x method 2 ode . 2049
- 13.11.2 Solving as second order change of variable on x method 1 ode . 2055
- 13.11.3 Solving as second order besel ode ode 2059
- 13.11.4 Solving using Kovacic algorithm 2063

Internal problem ID [5422]

Internal file name [OUTPUT/4913_Sunday_February_04_2024_12_47_14_AM_75733031/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 31.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$x^8 y'' + 4y'x^7 + y = \frac{1}{x^3}$$

13.11.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$x^8 y'' + 4y'x^7 + y = 0$$

In normal form the ode

$$x^8 y'' + 4y'x^7 + y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{1}{x^8}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x) \tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x) \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x) dx)} dx \\ &= \int e^{-(\int \frac{4}{x} dx)} dx \\ &= \int e^{-4 \ln(x)} dx \\ &= \int \frac{1}{x^4} dx \\ &= -\frac{1}{3x^3} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{x^8}}{\frac{1}{x^8}} \\ &= 1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0 \end{aligned}$$

The above ode is now solved for $y(\tau)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$

$$\lambda_2 = -i$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau} (c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0 (c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos\left(\frac{1}{3x^3}\right) \\ y_2 &= \sin\left(\frac{1}{3x^3}\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right) \left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right) \left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin\left(\frac{1}{3x^3}\right)}{\frac{x^3}{-x^4}} dx$$

Which simplifies to

$$u_1 = - \int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3 \sin \left(\frac{1}{3x^3} \right) + \frac{\cos \left(\frac{1}{3x^3} \right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos \left(\frac{1}{3x^3} \right)}{-x^4} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos \left(\frac{1}{3x^3} \right)}{x^7} dx$$

Hence

$$u_2 = 3 \cos \left(\frac{1}{3x^3} \right) + \frac{\sin \left(\frac{1}{3x^3} \right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3 \sin \left(\frac{1}{3x^3} \right) + \frac{\cos \left(\frac{1}{3x^3} \right)}{x^3} \right) \cos \left(\frac{1}{3x^3} \right) + \left(3 \cos \left(\frac{1}{3x^3} \right) + \frac{\sin \left(\frac{1}{3x^3} \right)}{x^3} \right) \sin \left(\frac{1}{3x^3} \right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos \left(\frac{1}{3x^3} \right) - c_2 \sin \left(\frac{1}{3x^3} \right) \right) + \left(\frac{1}{x^3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos \left(\frac{1}{3x^3} \right) - c_2 \sin \left(\frac{1}{3x^3} \right) + \frac{1}{x^3} \quad (1)$$

Verification of solutions

$$y = c_1 \cos \left(\frac{1}{3x^3} \right) - c_2 \sin \left(\frac{1}{3x^3} \right) + \frac{1}{x^3}$$

Verified OK.

13.11.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^8$, $B = 4x^7$, $C = 1$, $f(x) = \frac{1}{x^3}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^8y'' + 4y'x^7 + y = 0$$

In normal form the ode

$$x^8y'' + 4y'x^7 + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{1}{x^8}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\tau' = \frac{1}{c}\sqrt{q}$$
$$= \frac{\sqrt{\frac{1}{x^8}}}{c}$$
$$\tau'' = -\frac{4}{c\sqrt{\frac{1}{x^8}}x^9} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
 p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\
 &= \frac{-\frac{4}{c\sqrt{\frac{1}{x^8}}x^9} + \frac{4}{x}\frac{\sqrt{\frac{1}{x^8}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^8}}}{c}\right)^2} \\
 &= 0
 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
 y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\
 \frac{d^2}{d\tau^2}y(\tau) + c^2y(\tau) &= 0 \tag{7}
 \end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
 \tau &= \int \frac{1}{c}\sqrt{q} dx \\
 &= \frac{\int \sqrt{\frac{1}{x^8}} dx}{c} \\
 &= -\frac{x\sqrt{\frac{1}{x^8}}}{3c}
 \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

Now the particular solution to this ODE is found

$$x^8y'' + 4y'x^7 + y = \frac{1}{x^3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{3x^3}\right)$$

$$y_2 = \sin\left(\frac{1}{3x^3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right) \left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right) \left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{1}{3x^3}\right)}{-x^4} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3 \sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{3x^3}\right)}{-x^4} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_2 = 3 \cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3 \sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}\right) \cos\left(\frac{1}{3x^3}\right) + \left(3 \cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}\right) \sin\left(\frac{1}{3x^3}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) \right) + \left(\frac{1}{x^3}\right) \\ &= c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3} \end{aligned}$$

Which simplifies to

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3} \quad (1)$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}$$

Verified OK.

13.11.3 Solving as second order Bessel ODE

Writing the ODE as

$$x^2 y'' + 4xy' + \frac{y}{x^6} = \frac{1}{x^9} \quad (1)$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ODE has the form

$$x^2 y'' + xy' + (-n^2 + x^2)y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= -\frac{3}{2} \\ \beta &= \frac{1}{3} \\ n &= \frac{1}{2} \\ \gamma &= -3 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sqrt{2} \sqrt{3} \sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} - \frac{c_2 \sqrt{2} \sqrt{3} \cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sqrt{2} \sqrt{3} \sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} - \frac{c_2 \sqrt{2} \sqrt{3} \cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{3x^3}\right)$$

$$y_2 = \sin\left(\frac{1}{3x^3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right) \left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right) \left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\sin\left(\frac{1}{3x^3}\right)}{\frac{x^9}{-\frac{1}{x^2}}} dx$$

Which simplifies to

$$u_1 = - \int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3 \sin \left(\frac{1}{3x^3} \right) + \frac{\cos \left(\frac{1}{3x^3} \right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos \left(\frac{1}{3x^3} \right)}{x^9}}{-\frac{1}{x^2}} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos \left(\frac{1}{3x^3} \right)}{x^7} dx$$

Hence

$$u_2 = 3 \cos \left(\frac{1}{3x^3} \right) + \frac{\sin \left(\frac{1}{3x^3} \right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3 \sin \left(\frac{1}{3x^3} \right) + \frac{\cos \left(\frac{1}{3x^3} \right)}{x^3} \right) \cos \left(\frac{1}{3x^3} \right) + \left(3 \cos \left(\frac{1}{3x^3} \right) + \frac{\sin \left(\frac{1}{3x^3} \right)}{x^3} \right) \sin \left(\frac{1}{3x^3} \right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \sqrt{2} \sqrt{3} \sin \left(\frac{1}{3x^3} \right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} - \frac{c_2 \sqrt{2} \sqrt{3} \cos \left(\frac{1}{3x^3} \right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} \right) + \left(\frac{1}{x^3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{2} \sqrt{3} \sin \left(\frac{1}{3x^3} \right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} - \frac{c_2 \sqrt{2} \sqrt{3} \cos \left(\frac{1}{3x^3} \right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} + \frac{1}{x^3} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \sqrt{2} \sqrt{3} \sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} - \frac{c_2 \sqrt{2} \sqrt{3} \cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}} \sqrt{\pi} \sqrt{\frac{1}{x^3}}} + \frac{1}{x^3}$$

Verified OK.

13.11.4 Solving using Kovacic algorithm

Writing the ode as

$$x^8 y'' + 4y' x^7 + y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^8 \\ B &= 4x^7 \\ C &= 1 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^6 - 1}{x^8} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^6 - 1 \\ t &= x^8 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^6 - 1}{x^8} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 8 - 6 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^8$. There is a pole at $x = 0$ of order 8. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one). Then for each pole c , $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of \sqrt{r} expanded around each pole c . Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \quad (1B)$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\alpha_c^+ = \frac{1}{2} \left(\frac{b}{a} + v \right)$$

$$\alpha_c^- = \frac{1}{2} \left(-\frac{b}{a} + v \right)$$

The partial fraction decomposition of r is

$$r = -\frac{1}{x^8} + \frac{2}{x^2}$$

There is pole in r at $x = 0$ of order 8, hence $v = 4$. Expanding \sqrt{r} as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i}{x^4} + \dots \quad (2B)$$

Using eq. (1B), taking the sum up to $v = 4$ the above becomes

$$[\sqrt{r}]_c = \frac{i}{x^4} \quad (3B)$$

The above shows that the coefficient of $\frac{1}{(x-0)^4}$ is

$$a = i$$

Now we need to find b . let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^5}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$b = (0) - (0)$$

$$= 0$$

Hence

$$\begin{aligned}
 [\sqrt{r}]_c &= \frac{i}{x^4} \\
 \alpha_c^+ &= \frac{1}{2} \left(\frac{b}{a} + v \right) = \frac{1}{2} \left(\frac{0}{i} + 4 \right) = 2 \\
 \alpha_c^- &= \frac{1}{2} \left(-\frac{b}{a} + v \right) = \frac{1}{2} \left(-\frac{0}{i} + 4 \right) = 2
 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2x^6 - 1}{x^8}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
 [\sqrt{r}]_\infty &= 0 \\
 \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
 \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2x^6 - 1}{x^8}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	8	$\frac{i}{x^4}$	2	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either + or - and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω .

Trying $\alpha_{\infty}^+ = 2$ then

$$\begin{aligned} d &= \alpha_{\infty}^+ - (\alpha_{c_1}^-) \\ &= 2 - (2) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\ &= -\frac{i}{x^4} + \frac{2}{x} + (0) \\ &= -\frac{i}{x^4} + \frac{2}{x} \\ &= \frac{2x^3 - i}{x^4} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2 \left(-\frac{i}{x^4} + \frac{2}{x} \right) (0) + \left(\left(\frac{4i}{x^5} - \frac{2}{x^2} \right) + \left(-\frac{i}{x^4} + \frac{2}{x} \right)^2 - \left(\frac{2x^6 - 1}{x^8} \right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(-\frac{i}{x^4} + \frac{2}{x} \right) dx} \\ &= x^2 e^{\frac{i}{3x^3}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{4x^7}{x^8} dx} \\&= z_1 e^{-2 \ln(x)} \\&= z_1 \left(\frac{1}{x^2} \right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{\frac{i}{3x^3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4x^7}{x^8} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-4 \ln(x)}}{(y_1)^2} dx \\&= y_1 \left(-\frac{ie^{-\frac{2i}{3x^3}}}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1 \left(e^{\frac{i}{3x^3}} \right) + c_2 \left(e^{\frac{i}{3x^3}} \left(-\frac{ie^{-\frac{2i}{3x^3}}}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^8 y'' + 4y'x^7 + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\frac{i}{3x^3}} - \frac{ic_2 e^{-\frac{i}{3x^3}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{i}{3x^3}}$$

$$y_2 = -\frac{ie^{-\frac{i}{3x^3}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{i}{3x^3}} & -\frac{ie^{-\frac{i}{3x^3}}}{2} \\ \frac{d}{dx} \left(e^{\frac{i}{3x^3}} \right) & \frac{d}{dx} \left(-\frac{ie^{-\frac{i}{3x^3}}}{2} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{i}{3x^3}} & -\frac{ie^{-\frac{i}{3x^3}}}{2} \\ -\frac{ie^{\frac{i}{3x^3}}}{x^4} & e^{-\frac{i}{3x^3}} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{i}{3x^3}} \right) \left(\frac{e^{-\frac{i}{3x^3}}}{2x^4} \right) - \left(-\frac{ie^{-\frac{i}{3x^3}}}{2} \right) \left(-\frac{ie^{\frac{i}{3x^3}}}{x^4} \right)$$

Which simplifies to

$$W = \frac{e^{\frac{i}{3x^3}} e^{-\frac{i}{3x^3}}}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{-\frac{ie^{-\frac{i}{3x^3}}}{2x^3}}{x^4} dx$$

Which simplifies to

$$u_1 = - \int -\frac{ie^{-\frac{i}{3x^3}}}{2x^7} dx$$

Hence

$$u_1 = \text{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{\frac{i}{3x^3}}}{x^3}}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{\frac{i}{3x^3}}}{x^7} dx$$

Hence

$$u_2 = \text{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{1}{2} e^{\frac{i}{3x^3}} - i \frac{1}{2} e^{-\frac{i}{3x^3}}$$

Which simplifies to

$$y_p(x) = \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}} \right) \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2} \right) + \left(\left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}} \right) \frac{1}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2} + \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}} \right) \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2} + \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}} \right) \frac{1}{2}$$

Verified OK.

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`

```


✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(x^8*diff(y(x),x$2)+4*x^7*diff(y(x),x)+y(x)=1/x^3,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{3x^3}\right) c_2 + \cos\left(\frac{1}{3x^3}\right) c_1 + \frac{1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 32

```
DSolve[x^8*y''[x]+4*x^7*y'[x]+y[x]==1/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{x^3} + c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

13.12 problem 32

13.12.1 Solving as second order change of variable on y method 2 ode . 2073

Internal problem ID [5423]

Internal file name [OUTPUT/4914_Sunday_February_04_2024_12_47_15_AM_24853985/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 32.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_change_of_variable_on_y_method_2"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$(\sin(x)x + \cos(x))y'' - \cos(x)y'x + y\cos(x) = x$$

13.12.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = \sin(x)x + \cos(x)$, $B = -\cos(x)x$, $C = \cos(x)$, $f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$(\sin(x)x + \cos(x))y'' - \cos(x)y'x + y\cos(x) = 0$$

In normal form the ode

$$(\sin(x)x + \cos(x))y'' - \cos(x)y'x + y\cos(x) = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{\cos(x)x}{\sin(x)x + \cos(x)}$$
$$q(x) = \frac{\cos(x)}{\sin(x)x + \cos(x)}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n\cos(x)}{\sin(x)x + \cos(x)} + \frac{\cos(x)}{\sin(x)x + \cos(x)} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - \frac{\cos(x)x}{\sin(x)x + \cos(x)}\right)v'(x) = 0$$
$$v''(x) + \left(\frac{2}{x} - \frac{\cos(x)x}{\sin(x)x + \cos(x)}\right)v'(x) = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{\cos(x)x}{\sin(x)x + \cos(x)} \right) u(x) = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u(2\sin(x)x + 2\cos(x) - \cos(x)x^2)}{x(\sin(x)x + \cos(x))} \end{aligned}$$

Where $f(x) = -\frac{2\sin(x)x + 2\cos(x) - \cos(x)x^2}{x(\sin(x)x + \cos(x))}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{2\sin(x)x + 2\cos(x) - \cos(x)x^2}{x(\sin(x)x + \cos(x))} dx \\ \int \frac{1}{u} du &= \int -\frac{2\sin(x)x + 2\cos(x) - \cos(x)x^2}{x(\sin(x)x + \cos(x))} dx \\ \ln(u) &= -2\ln(x) - \ln\left(1 + \tan\left(\frac{x}{2}\right)^2\right) + \ln\left(2\tan\left(\frac{x}{2}\right)x - \tan\left(\frac{x}{2}\right)^2 + 1\right) + c_1 \\ u &= e^{-2\ln(x) - \ln\left(1 + \tan\left(\frac{x}{2}\right)^2\right) + \ln\left(2\tan\left(\frac{x}{2}\right)x - \tan\left(\frac{x}{2}\right)^2 + 1\right) + c_1} \\ &= c_1 e^{-2\ln(x) - \ln\left(1 + \tan\left(\frac{x}{2}\right)^2\right) + \ln\left(2\tan\left(\frac{x}{2}\right)x - \tan\left(\frac{x}{2}\right)^2 + 1\right)} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= -\frac{c_1 \cos(x)}{x} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)x^n \\ &= \left(-\frac{c_1 \cos(x)}{x} + c_2\right)x \\ &= -\cos(x)c_1 + c_2x \end{aligned}$$

Now the particular solution to this ODE is found

$$(\sin(x)x + \cos(x))y'' - \cos(x)y'x + y\cos(x) = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \cos(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \cos(x) \\ \frac{d}{dx}(x) & \frac{d}{dx}(\cos(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \cos(x) \\ 1 & -\sin(x) \end{vmatrix}$$

Therefore

$$W = (x)(-\sin(x)) - (\cos(x))(1)$$

Which simplifies to

$$W = -\sin(x)x - \cos(x)$$

Which simplifies to

$$W = -\sin(x)x - \cos(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos(x)x}{(\sin(x)x + \cos(x))(-\sin(x)x - \cos(x))} dx$$

Which simplifies to

$$u_1 = -\int -\frac{x \cos(x)}{(\sin(x)x + \cos(x))^2} dx$$

Hence

$$u_1 = -\frac{1}{\sin(x)x + \cos(x)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{(\sin(x)x + \cos(x))(-\sin(x)x - \cos(x))} dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2}{(\sin(x)x + \cos(x))^2} dx$$

Hence

$$u_2 = \frac{x - x \tan\left(\frac{x}{2}\right)^2 - 2 \tan\left(\frac{x}{2}\right)}{2 \tan\left(\frac{x}{2}\right)x - \tan\left(\frac{x}{2}\right)^2 + 1}$$

Which simplifies to

$$u_1 = -\frac{1}{\sin(x)x + \cos(x)}$$
$$u_2 = \frac{\cos(x)x - \sin(x)}{\sin(x)x + \cos(x)}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{\sin(x)x + \cos(x)} + \frac{(\cos(x)x - \sin(x))\cos(x)}{\sin(x)x + \cos(x)}$$

Which simplifies to

$$y_p(x) = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1 \cos(x)}{x} + c_2 \right) x \right) + (-\sin(x)) \\ &= -\sin(x) + \left(-\frac{c_1 \cos(x)}{x} + c_2 \right) x \end{aligned}$$

Which simplifies to

$$y = -\cos(x)c_1 + c_2x - \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\cos(x)c_1 + c_2x - \sin(x) \tag{1}$$

Verification of solutions

$$y = -\cos(x)c_1 + c_2x - \sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
Try integration with the canonical coordinates of the symmetry [0, x]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (cos(_a)*_a^2*_b(_a)-2*sin(_a)*_a*_b(_a)
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying 1st order linear
  <- 1st order linear successful
<- differential order: 2; canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 17

```
dsolve((x*sin(x)+cos(x))*diff(y(x),x$2)-x*cos(x)*diff(y(x),x)+y(x)*cos(x)=x,y(x), singsol=all
```

$$y(x) = -\cos(x)c_1 + c_2x - \sin(x)$$

✓ Solution by Mathematica

Time used: 0.764 (sec). Leaf size: 20

```
DSolve[(x*SIN[x]+Cos[x])*y''[x]-x*COS[x]*y'[x]+y[x]*Cos[x]==x,y[x],x,IncludeSingularSolution
```

$$y(x) \rightarrow -\sin(x) + c_1x - c_2 \cos(x)$$

13.13 problem 33

13.13.1 Solving as second order euler ode	2081
13.13.2 Solving as second order change of variable on x method 2 ode .	2084
13.13.3 Solving as second order change of variable on x method 1 ode .	2090
13.13.4 Solving as second order change of variable on y method 2 ode .	2094
13.13.5 Solving as second order ode non constant coeff transformation on B ode	2099
13.13.6 Solving using Kovacic algorithm	2104

Internal problem ID [5424]

Internal file name [OUTPUT/4915_Sunday_February_04_2024_12_47_20_AM_85124973/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 33.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$xy'' - 3y' + \frac{3y}{x} = x + 2$$

The ode can be written as

$$x^2y'' - 3xy' + 3y = x(x + 2)$$

Which shows it is a Euler ODE.

13.13.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -3x$, $C = 3$, $f(x) = x(x + 2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3rx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)x^r - 3rx^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$

$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 3y = x(x + 2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)(3x^2) - (x^3)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^4(x+2)}{2x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{x+2}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(x+2)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x^2} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right)x + \left(-\frac{1}{2x^2} - \frac{1}{2x}\right)x^3$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2} \quad (1)$$

Verification of solutions

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2}$$

Verified OK.

13.13.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \quad (3)$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \quad (4)$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \quad (5)$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-(\int p(x)dx)} dx \\ &= \int e^{-(\int -\frac{3}{x} dx)} dx \\ &= \int e^{3\ln(x)} dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \quad (6)$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{x^2}}{x^6} \\ &= \frac{3}{x^8} \end{aligned} \quad (7)$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\tau^r + 0\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}(x^4)^{\frac{1}{4}}(c_2\sqrt{x^4} + 2c_1)}{4}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}(x^4)^{\frac{1}{4}}(c_2\sqrt{x^4} + 2c_1)}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^4)^{\frac{1}{4}}$$

$$y_2 = (x^4)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{d}{dx}((x^4)^{\frac{1}{4}}) & \frac{d}{dx}((x^4)^{\frac{3}{4}}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{x^3}{(x^4)^{\frac{3}{4}}} & \frac{3x^3}{(x^4)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((x^4)^{\frac{1}{4}} \right) \left(\frac{3x^3}{(x^4)^{\frac{1}{4}}} \right) - \left((x^4)^{\frac{3}{4}} \right) \left(\frac{x^3}{(x^4)^{\frac{3}{4}}} \right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^4)^{\frac{3}{4}} x(x+2)}{2x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^4)^{\frac{3}{4}} (x+2)}{2x^4} dx$$

Hence

$$u_1 = - \frac{(x^4)^{\frac{3}{4}}}{2x^2} - \frac{(x^4)^{\frac{3}{4}} \ln(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} x(x+2)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} (x+2)}{2x^4} dx$$

Hence

$$u_2 = - \frac{(x+1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Which simplifies to

$$u_1 = -\frac{(x^4)^{\frac{3}{4}}(x + 2 \ln(x))}{2x^3}$$
$$u_2 = -\frac{(x + 1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x(x + 2 \ln(x))}{2} - \frac{x(x + 1)}{2}$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(\frac{\sqrt{2}(x^4)^{\frac{1}{4}}(c_2\sqrt{x^4} + 2c_1)}{4} \right) + \left(-x^2 - \ln(x)x - \frac{x}{2} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}(x^4)^{\frac{1}{4}}(c_2\sqrt{x^4} + 2c_1)}{4} - x^2 - \ln(x)x - \frac{x}{2} \quad (1)$$

Verification of solutions

$$y = \frac{\sqrt{2}(x^4)^{\frac{1}{4}}(c_2\sqrt{x^4} + 2c_1)}{4} - x^2 - \ln(x)x - \frac{x}{2}$$

Verified OK.

13.13.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -3x$, $C = 3$, $f(x) = x(x + 2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$

$$q(x) = \frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
 p_1(\tau) &= \frac{\tau''(x) + p(x)\tau'(x)}{\tau'(x)^2} \\
 &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}x^3} - \frac{3}{x}\frac{\sqrt{3}}{c}\sqrt{\frac{1}{x^2}}}{\left(\frac{\sqrt{3}}{c}\sqrt{\frac{1}{x^2}}\right)^2} \\
 &= -\frac{4c\sqrt{3}}{3}
 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
 y(\tau)'' + p_1y(\tau)' + q_1y(\tau) &= 0 \\
 \frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}}{3}\left(\frac{d}{d\tau}y(\tau)\right) + c^2y(\tau) &= 0
 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{\frac{2\sqrt{3}c\tau}{3}} \left(c_1 \cosh\left(\frac{\sqrt{3}c\tau}{3}\right) + ic_2 \sinh\left(\frac{\sqrt{3}c\tau}{3}\right) \right)$$

Now from (6)

$$\begin{aligned}
 \tau &= \int \frac{1}{c}\sqrt{q} dx \\
 &= \frac{\int \sqrt{3}\sqrt{\frac{1}{x^2}} dx}{c} \\
 &= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}} x \ln(x)}{c}
 \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{x((ic_2 + c_1)x^2 - ic_2 + c_1)}{2}$$

Now the particular solution to this ODE is found

$$x^2y'' - 3xy' + 3y = x(x + 2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = (x^4)^{\frac{1}{4}}$$

$$y_2 = (x^4)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{d}{dx} \left((x^4)^{\frac{1}{4}} \right) & \frac{d}{dx} \left((x^4)^{\frac{3}{4}} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^4)^{\frac{1}{4}} & (x^4)^{\frac{3}{4}} \\ \frac{x^3}{(x^4)^{\frac{3}{4}}} & \frac{3x^3}{(x^4)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((x^4)^{\frac{1}{4}} \right) \left(\frac{3x^3}{(x^4)^{\frac{1}{4}}} \right) - \left((x^4)^{\frac{3}{4}} \right) \left(\frac{x^3}{(x^4)^{\frac{3}{4}}} \right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^4)^{\frac{3}{4}} x(x+2)}{2x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^4)^{\frac{3}{4}} (x+2)}{2x^4} dx$$

Hence

$$u_1 = - \frac{(x^4)^{\frac{3}{4}}}{2x^2} - \frac{(x^4)^{\frac{3}{4}} \ln(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} x(x+2)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} (x+2)}{2x^4} dx$$

Hence

$$u_2 = - \frac{(x+1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Which simplifies to

$$u_1 = - \frac{(x^4)^{\frac{3}{4}} (x+2 \ln(x))}{2x^3}$$

$$u_2 = - \frac{(x+1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = - \frac{x(x+2 \ln(x))}{2} - \frac{x(x+1)}{2}$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\&= \left(\frac{x((ic_2 + c_1)x^2 - ic_2 + c_1)}{2} \right) + \left(-x^2 - \ln(x)x - \frac{x}{2} \right) \\&= -x^2 - \ln(x)x - \frac{x}{2} + \frac{x((ic_2 + c_1)x^2 - ic_2 + c_1)}{2}\end{aligned}$$

Which simplifies to

$$y = \frac{(-2 \ln(x) + (ic_2 + c_1)x^2 - 2x - ic_2 + c_1 - 1)x}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2 \ln(x) + (ic_2 + c_1)x^2 - 2x - ic_2 + c_1 - 1)x}{2} \quad (1)$$

Verification of solutions

$$y = \frac{(-2 \ln(x) + (ic_2 + c_1)x^2 - 2x - ic_2 + c_1 - 1)x}{2}$$

Verified OK.

13.13.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2$, $B = -3x$, $C = 3$, $f(x) = x(x + 2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2 y'' - 3xy' + 3y = 0 \quad (1)$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \quad (2)$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables on the dependent variable $y = v(x)x^n$ to (2) gives the following ode where the dependent variable is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 3 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \frac{3v'(x)}{x} = 0$$
$$v''(x) + \frac{3v'(x)}{x} = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{3}{x} dx \\ \int \frac{1}{u} du &= \int -\frac{3}{x} dx \\ \ln(u) &= -3 \ln(x) + c_1 \\ u &= e^{-3 \ln(x) + c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x) x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2 \right) x^3 \\ &= c_2 x^3 - \frac{1}{2} c_1 x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 3y = x(x + 2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)(3x^2) - (x^3)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^4(x+2)}{2x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{x+2}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(x+2)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x^2} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right)x + \left(-\frac{1}{2x^2} - \frac{1}{2x}\right)x^3$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right)x^3\right) + \left(-x^2 - \ln(x)x - \frac{x}{2}\right) \\ &= -x^2 - \ln(x)x - \frac{x}{2} + \left(-\frac{c_1}{2x^2} + c_2\right)x^3 \end{aligned}$$

Which simplifies to

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2} \quad (1)$$

Verification of solutions

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2}$$

Verified OK.

13.13.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= x^2 \\B &= -3x \\C &= 3 \\F &= x^2 + 2x\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (x^2)(0) + (-3x)(-3) + (3)(-3x) \\&= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-3x^3v'' + (3x^2)v' = 0$$

Now by applying $v' = u$ the above becomes

$$-3x^2(u'(x)x - u(x)) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\&= f(x)g(u) \\&= \frac{u}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{1}{x} dx \\ \int \frac{1}{u} du &= \int \frac{1}{x} dx \\ \ln(u) &= \ln(x) + c_1 \\ u &= e^{\ln(x)+c_1} \\ &= c_1x\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1x\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int c_1x \, dx \\ &= \frac{c_1x^2}{2} + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-3x) \left(\frac{c_1x^2}{2} + c_2 \right) \\ &= -\frac{3x(c_1x^2 + 2c_2)}{2}\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= x^3\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)(3x^2) - (x^3)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^3(x^2 + 2x)}{2x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{x + 2}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2 + 2x)}{2x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x^2} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right)x + \left(-\frac{1}{2x^2} - \frac{1}{2x}\right)x^3$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{3x(c_1x^2 + 2c_2)}{2}\right) + \left(-x^2 - \ln(x)x - \frac{x}{2}\right) \\ &= -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2} \quad (1)$$

Verification of solutions

$$y = -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2}$$

Verified OK.

13.13.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 3y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ , which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω .

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} dx} \\&= z_1 e^{\frac{3 \ln(x)}{2}} \\&= z_1 \left(x^{\frac{3}{2}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{3 \ln(x)}}{(y_1)^2} dx \\&= y_1 \left(\frac{x^2}{2} \right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1(x) + c_2 \left(x \left(\frac{x^2}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$x^2 y'' - 3xy' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1x + \frac{1}{2}c_2x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{x^3}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ 1 & \frac{3x^2}{2} \end{vmatrix}$$

Therefore

$$W = (x) \left(\frac{3x^2}{2} \right) - \left(\frac{x^3}{2} \right) \quad (1)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^3(x^2+2x)}{x^5} dx$$

Which simplifies to

$$u_1 = - \int \frac{x+2}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2+2x)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{x+2}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{x^2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x) \right) x + \frac{\left(-\frac{1}{x^2} - \frac{1}{x} \right) x^3}{2}$$

Which simplifies to

$$y_p(x) = -x^2 - \ln(x)x - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1x + \frac{1}{2}c_2x^3 \right) + \left(-x^2 - \ln(x)x - \frac{x}{2} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + \frac{c_2x^3}{2} - x^2 - \ln(x)x - \frac{x}{2} \quad (1)$$

Verification of solutions

$$y = c_1x + \frac{c_2x^3}{2} - x^2 - \ln(x)x - \frac{x}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]  
trying a double symmetry of the form [xi=0, eta=F(x)]  
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)-3*diff(y(x),x)+3*y(x)/x=x+2,y(x), singsol=all)
```

$$y(x) = \frac{(-2x - 2 \ln(x) + c_1x^2 + 2c_2)x}{2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 30

```
DSolve[x*y'[x]-3*y'[x]+3*y[x]/x==x+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}x(2c_2x^2 - 2x - 2\log(x) - 1 + 2c_1)$$

13.14 problem 35

13.14.1 Solving as second order ode non constant coeff transformation on B ode	2113
13.14.2 Solving using Kovacic algorithm	2118

Internal problem ID [5425]

Internal file name [OUTPUT/4916_Sunday_February_04_2024_12_47_22_AM_74213593/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 35.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$(x + 1)y'' - (3x + 4)y' + 3y = (2 + 3x)e^{3x}$$

13.14.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}y' &= B'v + v'B \\y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v\end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x + 1 \\ B &= -3x - 4 \\ C &= 3 \\ F &= (2 + 3x)e^{3x} \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x + 1)(0) + (-3x - 4)(-3) + (3)(-3x - 4) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-3x^2 - 7x - 4v'' + (9x^2 + 18x + 10)v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-3x^2 - 7x - 4)u'(x) + 9u(x)\left(x^2 + 2x + \frac{10}{9}\right) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(9x^2 + 18x + 10)}{3x^2 + 7x + 4} \end{aligned}$$

Where $f(x) = \frac{9x^2+18x+10}{3x^2+7x+4}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{9x^2 + 18x + 10}{3x^2 + 7x + 4} dx \\ \int \frac{1}{u} du &= \int \frac{9x^2 + 18x + 10}{3x^2 + 7x + 4} dx \\ \ln(u) &= 3x + \ln(x+1) - 2\ln(3x+4) + c_1 \\ u &= e^{3x+\ln(x+1)-2\ln(3x+4)+c_1} \\ &= c_1 e^{3x+\ln(x+1)-2\ln(3x+4)}\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 e^{3x+\ln(x+1)-2\ln(3x+4)}\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int c_1 e^{3x+\ln(x+1)-2\ln(3x+4)} dx \\ &= \frac{(3x+4) c_1 e^{3x+\ln(x+1)-2\ln(3x+4)}}{9+9x} + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-3x-4) \left(\frac{(3x+4) c_1 e^{3x+\ln(x+1)-2\ln(3x+4)}}{9+9x} + c_2 \right) \\ &= -\frac{e^{3x} c_1}{9} + (-3x-4) c_2\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= e^{3x} \\ y_2 &= -3x - 4\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & -3x - 4 \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}(-3x - 4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & -3x - 4 \\ 3e^{3x} & -3 \end{vmatrix}$$

Therefore

$$W = (e^{3x})(-3) - (-3x - 4)(3e^{3x})$$

Which simplifies to

$$W = 9e^{3x}x + 9e^{3x}$$

Which simplifies to

$$W = 9(x + 1)e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(-3x - 4)(2 + 3x)e^{3x}}{9(x + 1)^2 e^{3x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{-9x^2 - 18x - 8}{9(x + 1)^2} dx$$

Hence

$$u_1 = x + \frac{1}{9 + 9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(e^{3x})^2 (2 + 3x)}{9(x + 1)^2 e^{3x}} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{3x}(2 + 3x)}{9(x + 1)^2} dx$$

Hence

$$u_2 = \frac{e^{3x}}{9 + 9x}$$

Which simplifies to

$$u_1 = x + \frac{1}{9 + 9x}$$
$$u_2 = \frac{e^{3x}}{9 + 9x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x + \frac{1}{9 + 9x}\right) e^{3x} + \frac{e^{3x}(-3x - 4)}{9 + 9x}$$

Which simplifies to

$$y_p(x) = \frac{(3x - 1)e^{3x}}{3}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{e^{3x}c_1}{9} + (-3x - 4)c_2\right) + \left(\frac{(3x - 1)e^{3x}}{3}\right) \\ &= \frac{(9x - c_1 - 3)e^{3x}}{9} + (-3x - 4)c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(9x - c_1 - 3) e^{3x}}{9} + (-3x - 4) c_2 \quad (1)$$

Verification of solutions

$$y = \frac{(9x - c_1 - 3) e^{3x}}{9} + (-3x - 4) c_2$$

Verified OK.

13.14.2 Solving using Kovacic algorithm

Writing the ode as

$$(x + 1) y'' + (-3x - 4) y' + 3y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$A = x + 1$$

$$B = -3x - 4 \quad (3)$$

$$C = 3$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 6}{4(x + 1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9x^2 + 12x + 6 \\ t &= 4(x + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{9x^2 + 12x + 6}{4(x + 1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x + 1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order

pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} - \frac{3}{2(x+1)} + \frac{3}{4(x+1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for \sqrt{r} at ∞ . Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^v a_i x^i \\ &= \sum_{i=0}^0 a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of \sqrt{r} at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{2}{3x^2} - \frac{7}{9x^3} + \frac{91}{108x^4} - \frac{283}{324x^5} + \frac{143}{162x^6} - \frac{214}{243x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_{\infty} &= \sum_{i=0}^0 a_i x^i \\ &= \frac{3}{2} \end{aligned} \tag{10}$$

Now we need to find b , where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $([\sqrt{r}]_{\infty})^2$ where $[\sqrt{r}]_{\infty}$ was found above in Eq (10). Hence

$$([\sqrt{r}]_{\infty})^2 = \frac{9}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r . How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t . Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{9x^2 + 12x + 6}{4x^2 + 8x + 4} \\ &= Q + \frac{R}{4x^2 + 8x + 4} \\ &= \left(\frac{9}{4}\right) + \left(\frac{-3 - 6x}{4x^2 + 8x + 4}\right) \\ &= \frac{9}{4} + \frac{-3 - 6x}{4x^2 + 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6 . Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{3}{2}\right) - (0) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned}
 [\sqrt{r}]_{\infty} &= \frac{3}{2} \\
 \alpha_{\infty}^{+} &= \frac{1}{2} \left(\frac{b}{a} - v \right) = \frac{1}{2} \left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0 \right) = -\frac{1}{2} \\
 \alpha_{\infty}^{-} &= \frac{1}{2} \left(-\frac{b}{a} - v \right) = \frac{1}{2} \left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0 \right) = \frac{1}{2}
 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9x^2 + 12x + 6}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_{\infty}$	α_{∞}^{+}	α_{∞}^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^{\pm} have been determined for all the poles in the set Γ and $[\sqrt{r}]_{\infty}$ and its associated α_{∞}^{\pm} have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_{∞}^{\pm} . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_{\infty}^{+} = -\frac{1}{2}$ then

$$\begin{aligned}
 d &= \alpha_{\infty}^{+} - (\alpha_{c_1}^{-}) \\
 &= -\frac{1}{2} - \left(-\frac{1}{2} \right) \\
 &= 0
 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

Substituting the above values in the above results in

$$\begin{aligned}
 \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
 &= -\frac{1}{2(x+1)} + \left(\frac{3}{2} \right) \\
 &= -\frac{1}{2(x+1)} + \frac{3}{2} \\
 &= \frac{2+3x}{2+2x}
 \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
 (0) + 2 \left(-\frac{1}{2(x+1)} + \frac{3}{2} \right) (0) + \left(\left(\frac{1}{2(x+1)^2} \right) + \left(-\frac{1}{2(x+1)} + \frac{3}{2} \right)^2 - \left(\frac{9x^2 + 12x + 6}{4(x+1)^2} \right) \right) &= 0 \\
 0 &= 0
 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
 z_1(x) &= p e^{\int \omega dx} \\
 &= e^{\int \left(-\frac{1}{2(x+1)} + \frac{3}{2} \right) dx} \\
 &= \frac{e^{\frac{3x}{2}}}{\sqrt{x+1}}
 \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
 y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\
 &= z_1 e^{-\int \frac{1}{2} \frac{-3x-4}{x+1} dx} \\
 &= z_1 e^{\frac{3x}{2} + \frac{\ln(x+1)}{2}} \\
 &= z_1 \left(\sqrt{x+1} e^{\frac{3x}{2}} \right)
 \end{aligned}$$

Which simplifies to

$$y_1 = e^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-4}{x+1} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{3x+\ln(x+1)}}{(y_1)^2} dx \\ &= y_1 \left(-\frac{(3x+4)e^{-3x}}{9} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 (e^{3x}) + c_2 \left(e^{3x} \left(-\frac{(3x+4)e^{-3x}}{9} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x+1)y'' + (-3x-4)y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{3x} c_1 + c_2 \left(-\frac{x}{3} - \frac{4}{9} \right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{3x}$$

$$y_2 = -\frac{x}{3} - \frac{4}{9}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & -\frac{x}{3} - \frac{4}{9} \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}\left(-\frac{x}{3} - \frac{4}{9}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & -\frac{x}{3} - \frac{4}{9} \\ 3e^{3x} & -\frac{1}{3} \end{vmatrix}$$

Therefore

$$W = (e^{3x}) \left(-\frac{1}{3}\right) - \left(-\frac{x}{3} - \frac{4}{9}\right) (3e^{3x})$$

Which simplifies to

$$W = e^{3x} x + e^{3x}$$

Which simplifies to

$$W = (x + 1) e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\left(-\frac{x}{3} - \frac{4}{9}\right) (2 + 3x) e^{3x}}{(x + 1)^2 e^{3x}} dx$$

Which simplifies to

$$u_1 = - \int \frac{-9x^2 - 18x - 8}{9(x + 1)^2} dx$$

Hence

$$u_1 = x + \frac{1}{9 + 9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{6x}(2 + 3x)}{(x + 1)^2 e^{3x}} dx$$

Which simplifies to

$$u_2 = \int \frac{e^{3x}(2 + 3x)}{(x + 1)^2} dx$$

Hence

$$u_2 = \frac{e^{3x}}{x + 1}$$

Which simplifies to

$$u_1 = x + \frac{1}{9 + 9x}$$

$$u_2 = \frac{e^{3x}}{x + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x + \frac{1}{9 + 9x}\right) e^{3x} + \frac{e^{3x}\left(-\frac{x}{3} - \frac{4}{9}\right)}{x + 1}$$

Which simplifies to

$$y_p(x) = \frac{(3x - 1)e^{3x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x}c_1 + c_2 \left(-\frac{x}{3} - \frac{4}{9} \right) \right) + \left(\frac{(3x - 1)e^{3x}}{3} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2 \left(-\frac{x}{3} - \frac{4}{9} \right) + \frac{(3x - 1)e^{3x}}{3} \quad (1)$$

Verification of solutions

$$y = e^{3x}c_1 + c_2 \left(-\frac{x}{3} - \frac{4}{9} \right) + \frac{(3x - 1)e^{3x}}{3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Reducible group (found another exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```


✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x+1)*diff(y(x),x$2)-(3*x+4)*diff(y(x),x)+3*y(x)=(3*x+2)*exp(3*x),y(x), singsol=all)
```

$$y(x) = (x + c_1) e^{3x} + \frac{(3x + 4) c_2}{3}$$

✓ Solution by Mathematica

Time used: 0.364 (sec). Leaf size: 48

```
DSolve[(x+1)*y''[x]-(3*x+4)*y'[x]+3*y[x]==(3*x+2)*Exp[3*x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \rightarrow e^{3x} \left(x + \frac{2}{3} \right) + \frac{c_1 e^{3x+3}}{\sqrt{2}} - \frac{1}{9} \sqrt{2} c_2 (3x + 4)$$

13.15 problem 36

13.15.1 Solving as second order change of variable on y method 1 ode .	2129
13.15.2 Solving as second order bessel ode ode	2132
13.15.3 Solving using Kovacic algorithm	2133
13.15.4 Maple step by step solution	2136

Internal problem ID [5426]

Internal file name [OUTPUT/4917_Sunday_February_04_2024_12_47_23_AM_18303722/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 36.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode",
"second_order_change_of_variable_on_y_method_1"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$x^2 y'' - 4xy' + (9x^2 + 6)y = 0$$

13.15.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{9x^2 + 6}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
 Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
 &= \frac{9x^2 + 6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
 &= \frac{9x^2 + 6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
 &= \frac{9x^2 + 6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
 &= 9
 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x) z(x) \quad (3)$$

is used to change the original ode to a constant coefficients ode in v . In (3) the term $z(x)$ is given by

$$\begin{aligned}
 z(x) &= e^{-\left(\int \frac{p(x)}{2} dx\right)} \\
 &= e^{-\int \frac{-4}{2} dx} \\
 &= x^2
 \end{aligned} \quad (5)$$

Hence (3) becomes

$$y = v(x) x^2 \quad (4)$$

Applying this change of variable to the original ode results in

$$x^4(v''(x) + 9v(x)) = 0$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= +3i \\ \lambda_2 &= -3i \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= 3i \\ \lambda_2 &= -3i \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x} (c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0 (c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$v(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x) z(x) \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 \cos(3x) + c_2 \sin(3x)) x^2$$

Summary

The solution(s) found are the following

$$y = (c_1 \cos(3x) + c_2 \sin(3x)) x^2 \quad (1)$$

Verification of solutions

$$y = (c_1 \cos(3x) + c_2 \sin(3x)) x^2$$

Verified OK.

13.15.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 4xy' + (9x^2 + 6)y = 0 \quad (1)$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2)y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2)y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{5}{2}$$

$$\beta = 3$$

$$n = -\frac{1}{2}$$

$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^2 \sqrt{2} \sqrt{3} \cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2} \sqrt{3} \sin(3x)}{3\sqrt{\pi}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 \sqrt{2} \sqrt{3} \cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2} \sqrt{3} \sin(3x)}{3\sqrt{\pi}} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 x^2 \sqrt{2} \sqrt{3} \cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2} \sqrt{3} \sin(3x)}{3\sqrt{\pi}}$$

Verified OK.

13.15.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4xy' + (9x^2 + 6)y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 9x^2 + 6 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-4x}{x^2} dx} \\ &= z_1 e^{2 \ln(x)} \\ &= z_1 (x^2) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{4 \ln(x)}}{(y_1)^2} dx \\ &= y_1 \left(\frac{\tan(3x)}{3} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 (x^2 \cos(3x)) + c_2 \left(x^2 \cos(3x) \left(\frac{\tan(3x)}{3} \right) \right)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 \cos(3x) + \frac{c_2 x^2 \sin(3x)}{3} \quad (1)$$

Verification of solutions

$$y = c_1 x^2 \cos(3x) + \frac{c_2 x^2 \sin(3x)}{3}$$

Verified OK.

13.15.4 Maple step by step solution

Let's solve

$$x^2 y'' - 4xy' + (9x^2 + 6)y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{3(3x^2+2)y}{x^2} + \frac{4y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{4y'}{x} + \frac{3(3x^2+2)y}{x^2} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{3(3x^2+2)}{x^2} \right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -4$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 6$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$x^2 y'' - 4xy' + (9x^2 + 6)y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k \rightarrow k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)(-3+r)x^r + a_1(-1+r)(-2+r)x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r-2)(k+r-3) + 9a_{k-2}) \right) x^{k+r}$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$(-2+r)(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{2, 3\}$$

- Each term must be 0

$$a_1(-1+r)(-2+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k+r-2)(k+r-3) + 9a_{k-2} = 0$$

- Shift index using $k \rightarrow k+2$

$$a_{k+2}(k+r)(k+r-1) + 9a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{9a_k}{(k+r)(k+r-1)}$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{9a_k}{(k+2)(k+1)}$$

- Solution for $r = 2$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = -\frac{9a_k}{(k+2)(k+1)}, a_1 = 0 \right]$$

- Recursion relation for $r = 3$

$$a_{k+2} = -\frac{9a_k}{(k+3)(k+2)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2} = -\frac{9a_k}{(k+3)(k+2)}, a_1 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+2} \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+3} \right), a_{k+2} = -\frac{9a_k}{(k+2)(k+1)}, a_1 = 0, b_{k+2} = -\frac{9b_k}{(k+3)(k+2)}, b_1 = 0 \right]$$

Maple trace Kovacic algorithm successful

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Group is reducible or imprimitive
<- Kovacics algorithm successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(6+9*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-4*x*y'[x]+(6+9*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{6}e^{-3ix}x^2(6c_1 - ic_2e^{6ix})$$

13.16 problem 37

- 13.16.1 Solving as second order change of variable on y method 1 ode . 2140
- 13.16.2 Solving as second order bessel ode ode 2147
- 13.16.3 Solving using Kovacic algorithm 2150

Internal problem ID [5427]

Internal file name [OUTPUT/4918_Sunday_February_04_2024_12_47_23_AM_15867267/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 37.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_bessel_ode",
"second_order_change_of_variable_on_y_method_1"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$xy'' + 2y' + 4xy = 4$$

13.16.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.
 y_h is the solution to

$$xy'' + 2y' + 4xy = 0$$

In normal form the given ode is written as

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 4$$

Calculating the Liouville ode invariant Q given by

$$Q = q - \frac{p'}{2} - \frac{p^2}{4}$$
$$= 4 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4}$$
$$= 4 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4}$$
$$= 4 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2}$$
$$= 4$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x) z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v . In (3) the term $z(x)$ is given by

$$z(x) = e^{-\left(\int \frac{p(x)}{2} dx\right)}$$
$$= e^{-\int \frac{2}{x}}$$
$$= \frac{1}{x} \tag{5}$$

Hence (3) becomes

$$y = \frac{v(x)}{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) + 4v(x) = 4$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 4$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x) + Bv'(x) + Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x) + Bv'(x) + Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + 4v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \quad (1)$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \quad (2)$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form. Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A} \sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i \end{aligned}$$

Hence

$$\lambda_1 = +2i$$

$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$

$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$v(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution v_p , gives the particular solution

$$v_p = 1$$

Therefore the general solution is

$$\begin{aligned}v &= v_h + v_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + 1\end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)z(x) \\ &= (c_1 \cos(2x) + c_2 \sin(2x) + 1)(z(x))\end{aligned}\tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2\tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(2x)}{x}$$

$$y_2 = \frac{\sin(2x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ \frac{d}{dx} \left(\frac{\cos(2x)}{x} \right) & \frac{d}{dx} \left(\frac{\sin(2x)}{x} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ -\frac{2 \sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{2 \cos(2x)}{x} - \frac{\sin(2x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x} \right) \left(\frac{2 \cos(2x)}{x} - \frac{\sin(2x)}{x^2} \right) - \left(\frac{\sin(2x)}{x} \right) \left(-\frac{2 \sin(2x)}{x} - \frac{\cos(2x)}{x^2} \right)$$

Which simplifies to

$$W = \frac{2 \cos(2x)^2 + 2 \sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{4 \sin(2x)}{\frac{x}{\frac{2}{x}}} dx$$

Which simplifies to

$$u_1 = - \int 2 \sin(2x) dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4 \cos(2x)}{x}}{\frac{2}{x}} dx$$

Which simplifies to

$$u_2 = \int 2 \cos(2x) dx$$

Hence

$$u_2 = \sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x} \right) + \left(\frac{1}{x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x} + \frac{1}{x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x} + \frac{1}{x}$$

Warning, solution could not be verified

13.16.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 2xy' + 4yx^2 = 4x \quad (1)$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2) y = 0 \quad (2)$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2) y = 0 \quad (3)$$

With the standard solution

$$y = x^\alpha (c_1 \text{BesselJ}(n, \beta x^\gamma) + c_2 \text{BesselY}(n, \beta x^\gamma)) \quad (4)$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = -\frac{1}{2}$$

$$\beta = 2$$

$$n = \frac{1}{2}$$

$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(2x)}{x}$$

$$y_2 = \frac{\sin(2x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ \frac{d}{dx} \left(\frac{\cos(2x)}{x} \right) & \frac{d}{dx} \left(\frac{\sin(2x)}{x} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ -\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x} \right) \left(\frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2} \right) - \left(\frac{\sin(2x)}{x} \right) \left(-\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} \right)$$

Which simplifies to

$$W = \frac{2\cos(2x)^2 + 2\sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{4 \sin(2x)}{2} dx$$

Which simplifies to

$$u_1 = - \int 2 \sin(2x) dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{4 \cos(2x)}{2} dx$$

Which simplifies to

$$u_2 = \int 2 \cos(2x) dx$$

Hence

$$u_2 = \sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}} \right) + \left(\frac{1}{x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}} + \frac{1}{x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}} + \frac{1}{x}$$

Verified OK.

13.16.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' + 4xy = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r . Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x , then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2}{x} dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x} \right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{x} dx}}{(y_1)^2} dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx \\ &= y_1 \left(\frac{\tan(2x)}{2} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{\cos(2x)}{x} \right) + c_2 \left(\frac{\cos(2x)}{x} \left(\frac{\tan(2x)}{2} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' + 4xy = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(2x)}{x}$$

$$y_2 = \frac{\sin(2x)}{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{2x} \\ \frac{d}{dx} \left(\frac{\cos(2x)}{x} \right) & \frac{d}{dx} \left(\frac{\sin(2x)}{2x} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{2x} \\ -\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{\cos(2x)}{x} - \frac{\sin(2x)}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x} \right) \left(\frac{\cos(2x)}{x} - \frac{\sin(2x)}{2x^2} \right) - \left(\frac{\sin(2x)}{2x} \right) \left(-\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} \right)$$

Which simplifies to

$$W = \frac{\cos(2x)^2 + \sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{2\sin(2x)}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_1 = - \int 2\sin(2x) dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\cos(2x)}{x}}{\frac{1}{x}} dx$$

Which simplifies to

$$u_2 = \int 4 \cos(2x) dx$$

Hence

$$u_2 = 2 \sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x} \right) + \left(\frac{1}{x} \right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}}{x} + \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}}{x} + \frac{1}{x} \quad (1)$$

Verification of solutions

$$y = \frac{c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}}{x} + \frac{1}{x}$$

Verified OK.

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Group is reducible or imprimitive
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=4,y(x), singsol=all)
```

$$y(x) = \frac{1 + \sin(2x) c_2 + \cos(2x) c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 38

```
DSolve[x*y'[x]+2*y'[x]+4*x*y[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{4c_1 e^{-2ix} - ic_2 e^{2ix} + 4}{4x}$$

13.17 problem 38

- 13.17.1 Solving as second order change of variable on y method 2 ode . 2157
- 13.17.2 Solving as second order ode non constant coeff transformation
on B ode 2162
- 13.17.3 Solving using Kovacic algorithm 2167

Internal problem ID [5428]

Internal file name [OUTPUT/4919_Sunday_February_04_2024_12_47_25_AM_26764961/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 18. Linear equations with variable coefficients (Equations of second order).
Supplementary problems. Page 120

Problem number: 38.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$(x^2 + 1)y'' - 2xy' + 2y = \frac{-x^2 + 1}{x}$$

13.17.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2 + 1$, $B = -2x$, $C = 2$, $f(x) = -x + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

Solving for y_h from

$$(x^2 + 1) y'' - 2xy' + 2y = 0$$

In normal form the ode

$$(x^2 + 1) y'' - 2xy' + 2y = 0 \quad (1)$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \quad (2)$$

Where

$$p(x) = -\frac{2x}{x^2 + 1}$$
$$q(x) = \frac{2}{x^2 + 1}$$

Applying change of variables on the dependent variable $y = v(x) x^n$ to (2) gives the following ode where the dependent variable is $v(x)$ and not y .

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2 + 1} + \frac{2}{x^2 + 1} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - \frac{2x}{x^2 + 1}\right) v'(x) = 0$$
$$v''(x) + \frac{2v'(x)}{x(x^2 + 1)} = 0 \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x(x^2 + 1)} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x(x^2 + 1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= -\frac{2}{x(x^2 + 1)} dx \\ \int \frac{1}{u} du &= \int -\frac{2}{x(x^2 + 1)} dx \\ \ln(u) &= \ln(x^2 + 1) - 2 \ln(x) + c_1 \\ u &= e^{\ln(x^2+1) - 2 \ln(x) + c_1} \\ &= c_1 e^{\ln(x^2+1) - 2 \ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{1}{x^2} \right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \left(x - \frac{1}{x} \right) + c_2 \end{aligned}$$

Hence

$$\begin{aligned}y &= v(x) x^n \\ &= \left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x \\ &= c_1 x^2 + c_2 x - c_1\end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2 + 1) y'' - 2xy' + 2y = -x + \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= x \\ y_2 &= x^2 - 1\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 - 1 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2 - 1) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2 - 1)(1)$$

Which simplifies to

$$W = x^2 + 1$$

Which simplifies to

$$W = x^2 + 1$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(x^2 - 1)(-x + \frac{1}{x})}{(x^2 + 1)^2} dx$$

Which simplifies to

$$u_1 = - \int - \frac{(x^2 - 1)^2}{x(x^2 + 1)^2} dx$$

Hence

$$u_1 = \frac{2}{x^2 + 1} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(-x + \frac{1}{x})}{(x^2 + 1)^2} dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2 + 1}{(x^2 + 1)^2} dx$$

Hence

$$u_2 = \frac{x}{x^2 + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{2}{x^2 + 1} + \ln(x) \right) x + \frac{x(x^2 - 1)}{x^2 + 1}$$

Which simplifies to

$$y_p(x) = (1 + \ln(x)) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x \right) + ((1 + \ln(x)) x) \\ &= (1 + \ln(x)) x + \left(c_1 \left(x - \frac{1}{x} \right) + c_2 \right) x \end{aligned}$$

Which simplifies to

$$y = c_1 x^2 + \ln(x) x + c_2 x - c_1 + x$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + \ln(x) x + c_2 x - c_1 + x \quad (1)$$

Verification of solutions

$$y = c_1 x^2 + \ln(x) x + c_2 x - c_1 + x$$

Verified OK.

13.17.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}y' &= B'v + v'B \\y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0\end{aligned}\tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= x^2 + 1 \\B &= -2x \\C &= 2 \\F &= -x + \frac{1}{x}\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (x^2 + 1)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x(x^2 + 1)v'' + (-4)v' = 0$$

Now by applying $v' = u$ the above becomes

$$(-2x^3 - 2x)u'(x) - 4u(x) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x(x^2 + 1)}\end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= -\frac{2}{x(x^2 + 1)} dx \\ \int \frac{1}{u} du &= \int -\frac{2}{x(x^2 + 1)} dx \\ \ln(u) &= \ln(x^2 + 1) - 2 \ln(x) + c_1 \\ u &= e^{\ln(x^2+1)-2\ln(x)+c_1} \\ &= c_1 e^{\ln(x^2+1)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(1 + \frac{1}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 \left(1 + \frac{1}{x^2}\right)\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int \frac{(x^2 + 1)c_1}{x^2} dx \\ &= c_1 \left(x - \frac{1}{x}\right) + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-2x) \left(c_1 \left(x - \frac{1}{x}\right) + c_2\right) \\ &= -2c_1x^2 - 2c_2x + 2c_1\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= -2x^2 + 2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ 1 & -4x \end{vmatrix}$$

Therefore

$$W = (x)(-4x) - (-2x^2 + 2) \quad (1)$$

Which simplifies to

$$W = -2x^2 - 2$$

Which simplifies to

$$W = -2x^2 - 2$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{(-2x^2 + 2) \left(-x + \frac{1}{x}\right)}{(x^2 + 1)(-2x^2 - 2)} dx$$

Which simplifies to

$$u_1 = - \int - \frac{(x^2 - 1)^2}{x(x^2 + 1)^2} dx$$

Hence

$$u_1 = \frac{2}{x^2 + 1} + \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x \left(-x + \frac{1}{x}\right)}{(x^2 + 1)(-2x^2 - 2)} dx$$

Which simplifies to

$$u_2 = \int \frac{x^2 - 1}{2(x^2 + 1)^2} dx$$

Hence

$$u_2 = - \frac{x}{2(x^2 + 1)}$$

Which simplifies to

$$u_1 = \frac{2}{x^2 + 1} + \ln(x)$$

$$u_2 = - \frac{x}{2x^2 + 2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{2}{x^2 + 1} + \ln(x) \right) x - \frac{x(-2x^2 + 2)}{2x^2 + 2}$$

Which simplifies to

$$y_p(x) = (1 + \ln(x)) x$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2c_1x^2 - 2c_2x + 2c_1) + ((1 + \ln(x)) x) \\ &= -2c_1x^2 + \ln(x) x - 2c_2x + 2c_1 + x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 + \ln(x) x - 2c_2x + 2c_1 + x \quad (1)$$

Verification of solutions

$$y = -2c_1x^2 + \ln(x) x - 2c_2x + 2c_1 + x$$

Verified OK.

13.17.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 + 1) y'' - 2xy' + 2y = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= -2x \\ C &= 2 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2 + 1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2 + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2 + 1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} \mathcal{O}(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-i+x)^2} + \frac{3}{4(i+x)^2} + \frac{3i}{4(-i+x)} - \frac{3i}{4(i+x)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(-i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2 + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^- + \alpha_{c_2}^+) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-) [\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + \left((+) [\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-) [\sqrt{r}]_\infty \\ &= -\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} + (-)(0) \\ &= -\frac{1}{2(-i+x)} + \frac{3}{2(i+x)} \\ &= \frac{-2i+x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)}\right)(0) + \left(\left(\frac{1}{2(-i+x)^2} - \frac{3}{2(i+x)^2}\right) + \left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)}\right)^2\right) -$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega dx} \\ &= e^{\int \left(-\frac{1}{2(-i+x)} + \frac{3}{2(i+x)}\right) dx} \\ &= \frac{(x^2 + 1)^{\frac{3}{2}}}{(ix + 1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2+1} dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2 + 1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2 + 1)^2}{(ix + 1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}
 y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1} dx}}{(y_1)^2} dx \\
 &= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2} dx \\
 &= y_1 \left(-\frac{x}{(i+x)^2} \right)
 \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
 y &= c_1 y_1 + c_2 y_2 \\
 &= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2} \right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2} \left(-\frac{x}{(i+x)^2} \right) \right)
 \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 1) y'' - 2xy' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(-i+x)^2 (i+x)^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x^2 + 1)^2}{(ix + 1)^2}$$

$$y_2 = \frac{(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \\ \frac{d}{dx} \left(\frac{(x^2+1)^2}{(ix+1)^2} \right) & \frac{d}{dx} \left(\frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(-i+x)^2(i+x)^2} \\ \frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3} & \frac{4(x^2+1)x^2}{(-i+x)^2(i+x)^2} + \frac{(x^2+1)^2}{(-i+x)^2(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^3(i+x)^2} - \frac{2(x^2+1)^2 x}{(-i+x)^2(i+x)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2 + 1)^2}{(ix + 1)^2} \right) \left(\frac{4(x^2 + 1)x^2}{(-i + x)^2 (i + x)^2} + \frac{(x^2 + 1)^2}{(-i + x)^2 (i + x)^2} - \frac{2(x^2 + 1)^2 x}{(-i + x)^3 (i + x)^2} - \frac{2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^3} \right) - \left(\frac{(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2} \right) \left(\frac{4(x^2 + 1)x}{(ix + 1)^2} - \frac{2i(x^2 + 1)^2}{(ix + 1)^3} \right)$$

Which simplifies to

$$W = \frac{(x^2 + 1)^3 (ix^5 + 3x^4 - 2ix^3 + 2x^2 - 3ix - 1)}{(ix + 1)^3 (-x + i)^3 (i + x)^3}$$

Which simplifies to

$$W = \frac{(x^2 + 1)^4 (-x^3 + 3ix^2 + 3x - i)}{(-ix - 1)^6 (i + x)^3}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{(x^2+1)^2 x (-x + \frac{1}{x})}{(-i+x)^2 (i+x)^2}}{\frac{(x^2+1)^5 (-x^3+3ix^2+3x-i)}{(-ix-1)^6 (i+x)^3}} dx$$

Which simplifies to

$$u_1 = - \int \frac{(x^2 - 1) (i + x) (-i + x)^4}{(x^2 + 1)^3 (-x^3 + 3ix^2 + 3x - i)} dx$$

Hence

$$u_1 = - \frac{x}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{(x^2+1)^2 (-x + \frac{1}{x})}{(ix+1)^2}}{\frac{(x^2+1)^5 (-x^3+3ix^2+3x-i)}{(-ix-1)^6 (i+x)^3}} dx$$

Which simplifies to

$$u_2 = \int - \frac{(i + x)^3 (ix + 1)^4 (x^2 - 1)}{(-x^3 + 3ix^2 + 3x - i) x (x^2 + 1)^3} dx$$

Hence

$$u_2 = \int_0^x - \frac{(i + \alpha)^3 (i\alpha + 1)^4 (\alpha^2 - 1)}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i) \alpha (\alpha^2 + 1)^3} d\alpha$$

Which simplifies to

$$u_1 = -\frac{x}{x^2 + 1}$$

$$u_2 = -\left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3(i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(x^2 + 1)x}{(ix + 1)^2} - \frac{\left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3(i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha\right) (x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2}$$

Which simplifies to

$$y_p(x) = \frac{2(x^2 + 1)^2 x \left(\left(-\frac{1}{2}x^2 + \frac{1}{2} + ix\right) \left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3(i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha\right) + \frac{x^2}{2} + \frac{1}{2} \right)}{(-x + i)^4 (i + x)^2}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(\frac{c_1(x^2 + 1)^2}{(ix + 1)^2} + \frac{c_2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2} \right)$$

$$+ \left(\frac{2(x^2 + 1)^2 x \left(\left(-\frac{1}{2}x^2 + \frac{1}{2} + ix\right) \left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3(i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha\right) + \frac{x^2}{2} + \frac{1}{2} \right)}{(-x + i)^4 (i + x)^2} \right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2 + 1)^2}{(ix + 1)^2} + \frac{c_2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2}$$

$$+ \frac{2(x^2 + 1)^2 x \left(\left(-\frac{1}{2}x^2 + \frac{1}{2} + ix\right) \left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3(i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha\right) + \frac{x^2}{2} + \frac{1}{2} \right)}{(-x + i)^4 (i + x)^2} \quad (1)$$

Verification of solutions

$$y = \frac{c_1(x^2 + 1)^2}{(ix + 1)^2} + \frac{c_2(x^2 + 1)^2 x}{(-i + x)^2 (i + x)^2} + \frac{2(x^2 + 1)^2 x \left(-\frac{1}{2}x^2 + \frac{1}{2} + ix \right) \left(\int_0^x \frac{(\alpha^2 - 1)(i + \alpha)^3 (i - \alpha)^4}{(-\alpha^3 + 3i\alpha^2 + 3\alpha - i)\alpha(\alpha^2 + 1)^3} d\alpha \right) + \frac{x^2}{2} + \frac{1}{2}}{(-x + i)^4 (i + x)^2}$$

Verified OK.

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
<- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=(1-x^2)/x,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + \ln(x) x + c_2 x - c_1 + x$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 27

```
DSolve[(1+x^2)*y'[x]-2*x*y'[x]+2*y[x]==(1-x^2)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x(\log(x) + 1) - c_1(x - i)^2 + c_2x$$

14 Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

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14.1 problem 22

- 14.1.1 Solving as second order ode missing y ode 2179
- 14.1.2 Solving as second order ode missing x ode 2180
- 14.1.3 Maple step by step solution 2182

Internal problem ID [5429]

Internal file name [OUTPUT/4920_Tuesday_February_06_2024_10_14_16_PM_16279652/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 22.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x",
"second_order_ode_missing_y"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$y'' + y'^2 = -1$$

14.1.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y . Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2 - 1} dp = x + c_1$$
$$- \arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(x + c_1) \, dx \\ &= -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2 \quad (1)$$

Verification of solutions

$$y = -\frac{\ln(1 + \tan(x + c_1)^2)}{2} + c_2$$

Verified OK.

14.1.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y) \left(\frac{d}{dy} p(y) \right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2 + 1} dp = \int dy$$

$$-\frac{\ln(p^2 + 1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p^2 + 1}} = e^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p^2 + 1}} = c_2 e^y$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \text{RootOf}(-Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1)$$

Integrating both sides gives

$$\int \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1)} dy = \int dx$$

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3 \quad (1)$$

Verification of solutions

$$\int^y \frac{1}{\text{RootOf}(-Z^2 c_2^2 e^{2-a} + c_2^2 e^{2-a} - 1)} d_a = x + c_3$$

Verified OK.

14.1.3 Maple step by step solution

Let's solve

$$y'' + y'^2 = -1$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Make substitution $u = y'$ to reduce order of ODE

$$u'(x) + u(x)^2 = -1$$

- Separate variables

$$\frac{u'(x)}{-u(x)^2-1} = 1$$

- Integrate both sides with respect to x

$$\int \frac{u'(x)}{-u(x)^2-1} dx = \int 1 dx + c_1$$

- Evaluate integral

$$-\arctan(u(x)) = x + c_1$$

- Solve for $u(x)$

$$u(x) = -\tan(x + c_1)$$

- Solve 1st ODE for $u(x)$

$$u(x) = -\tan(x + c_1)$$

- Make substitution $u = y'$

$$y' = -\tan(x + c_1)$$

- Integrate both sides to solve for y

$$\int y' dx = \int -\tan(x + c_1) dx + c_2$$

- Compute integrals

$$y = -\frac{\ln(1+\tan(x+c_1)^2)}{2} + c_2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying 2nd order Liouville  
trying 2nd order WeierstrassP  
trying 2nd order JacobiSN  
differential order: 2; trying a linearization to 3rd order  
trying 2nd order ODE linearizable_by_differentiation  
trying 2nd order, 2 integrating factors of the form mu(x,y)  
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \ln(\cos(x) c_2 - c_1 \sin(x))$$

✓ Solution by Mathematica

Time used: 1.79 (sec). Leaf size: 16

```
DSolve[y''[x]+y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \log(\cos(x - c_1)) + c_2$$

14.2 problem 23

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14.2.2 Solving as second order ode missing y ode	2185
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14.2.5 Solving as exact linear second order ode ode	2195
14.2.6 Maple step by step solution	2196

Internal problem ID [5430]

Internal file name [OUTPUT/4921_Tuesday_February_06_2024_10_14_17_PM_52449057/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 23.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**kovacic**", "**exact linear second order ode**", "**second_order_integrable_as_is**", "**second_order_ode_missing_y**"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$(x^2 + 1)y'' + 2xy' = \frac{2}{x^3}$$

14.2.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((x^2 + 1)y'' + 2xy') dx = \int \frac{2}{x^3} dx$$
$$(x^2 + 1)y' = -\frac{1}{x^2} + c_1$$

Which is now solved for y . Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_1 x^2 - 1}{(x^2 + 1)x^2} dx \\ &= (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2 \quad (1)$$

Verification of solutions

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y . Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$(x^2 + 1)p'(x) + 2xp(x) - \frac{2}{x^3} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{2x}{x^2 + 1} \\ q(x) &= \frac{2}{(x^2 + 1)x^3}\end{aligned}$$

Hence the ode is

$$p'(x) + \frac{2xp(x)}{x^2 + 1} = \frac{2}{(x^2 + 1)x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2x}{x^2+1} dx} \\ &= x^2 + 1\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu) \left(\frac{2}{(x^2 + 1)x^3} \right) \\ \frac{d}{dx}((x^2 + 1)p) &= (x^2 + 1) \left(\frac{2}{(x^2 + 1)x^3} \right) \\ d((x^2 + 1)p) &= \left(\frac{2}{x^3} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}(x^2 + 1)p &= \int \frac{2}{x^3} dx \\ (x^2 + 1)p &= -\frac{1}{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 + 1$ results in

$$p(x) = -\frac{1}{(x^2 + 1)x^2} + \frac{c_1}{x^2 + 1}$$

which simplifies to

$$p(x) = \frac{c_1 x^2 - 1}{(x^2 + 1)x^2}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1 x^2 - 1}{(x^2 + 1)x^2}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_1 x^2 - 1}{(x^2 + 1)x^2} dx \\ &= (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2 \quad (1)$$

Verification of solutions

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2 + 1) y'' + 2xy' = \frac{2}{x^3}$$

Integrating both sides of the ODE w.r.t x gives

$$\int ((x^2 + 1) y'' + 2xy') dx = \int \frac{2}{x^3} dx$$
$$(x^2 + 1) y' = -\frac{1}{x^2} + c_1$$

Which is now solved for y . Integrating both sides gives

$$y = \int \frac{c_1 x^2 - 1}{(x^2 + 1) x^2} dx$$
$$= (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Summary

The solution(s) found are the following

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2 \quad (1)$$

Verification of solutions

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.4 Solving using Kovacic algorithm

Writing the ode as

$$(x^2 + 1) y'' + 2xy' = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= 2x \\ C &= 0 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{(x^2 + 1)^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= (x^2 + 1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{(x^2 + 1)^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(-i+x)^2} - \frac{1}{4(i+x)^2} - \frac{i}{4(-i+x)} + \frac{i}{4i+4x}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(-i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{(x^2 + 1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^+ + \alpha_{c_2}^+) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x - c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x - c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2i + 2x} + \frac{1}{-2i + 2x} + (-)(0) \\ &= \frac{1}{2i + 2x} + \frac{1}{-2i + 2x} \\ &= \frac{x}{x^2 + 1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2 \left(\frac{1}{2i + 2x} + \frac{1}{-2i + 2x} \right) (0) + \left(\left(-\frac{1}{2(i+x)^2} - \frac{1}{2(-i+x)^2} \right) + \left(\frac{1}{2i + 2x} + \frac{1}{-2i + 2x} \right)^2 - \left(\frac{1}{x^2} \right) \right) (1) = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega dx} \\ &= e^{\int \left(\frac{1}{2i+2x} + \frac{1}{-2i+2x} \right) dx} \\ &= \sqrt{x^2 + 1} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx} \\&= z_1 e^{-\int \frac{1}{2} \frac{2x}{x^2+1} dx} \\&= z_1 e^{-\frac{\ln(x^2+1)}{2}} \\&= z_1 \left(\frac{1}{\sqrt{x^2+1}} \right)\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1} dx}}{(y_1)^2} dx \\&= y_1 \int \frac{e^{-\ln(x^2+1)}}{(y_1)^2} dx \\&= y_1 (\arctan(x))\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= c_1(1) + c_2(1(\arctan(x)))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x^2 + 1) y'' + 2xy' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2 \arctan(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \arctan(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \arctan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\arctan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \arctan(x) \\ 0 & \frac{1}{x^2+1} \end{vmatrix}$$

Therefore

$$W = (1) \left(\frac{1}{x^2+1} \right) - (\arctan(x))(0)$$

Which simplifies to

$$W = \frac{1}{x^2 + 1}$$

Which simplifies to

$$W = \frac{1}{x^2 + 1}$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{\frac{2 \arctan(x)}{x^3}}{1} dx$$

Which simplifies to

$$u_1 = - \int \frac{2 \arctan(x)}{x^3} dx$$

Hence

$$u_1 = \frac{\arctan(x)}{x^2} + \arctan(x) + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2}{x^3}}{1} dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^3} dx$$

Hence

$$u_2 = -\frac{1}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \arctan(x) + \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \arctan(x)) + \left(\arctan(x) + \frac{1}{x} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \arctan(x) + \arctan(x) + \frac{1}{x} \quad (1)$$

Verification of solutions

$$y = c_1 + c_2 \arctan(x) + \arctan(x) + \frac{1}{x}$$

Verified OK.

14.2.5 Solving as exact linear second order ode

An ode of the form

$$p(x)y'' + q(x)y' + r(x)y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \quad (1)$$

For the given ode we have

$$p(x) = x^2 + 1$$

$$q(x) = 2x$$

$$r(x) = 0$$

$$s(x) = \frac{2}{x^3}$$

Hence

$$p''(x) = 2$$

$$q'(x) = 2$$

Therefore (1) becomes

$$2 - (2) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)y' + (q(x) - p'(x))y)' = s(x)$$

Integrating gives

$$p(x)y' + (q(x) - p'(x))y = \int s(x) dx$$

Substituting the above values for p, q, r, s gives

$$(x^2 + 1) y' = \int \frac{2}{x^3} dx$$

We now have a first order ode to solve which is

$$(x^2 + 1) y' = -\frac{1}{x^2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1 x^2 - 1}{(x^2 + 1) x^2} dx \\ &= (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2 \quad (1)$$

Verification of solutions

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.6 Maple step by step solution

Let's solve

$$(x^2 + 1) y'' + 2xy' = \frac{2}{x^3}$$

- Highest derivative means the order of the ODE is 2

y''

- Make substitution $u = y'$ to reduce order of ODE

$$(x^2 + 1) u'(x) + 2xu(x) = \frac{2}{x^3}$$

- Isolate the derivative

$$u'(x) = -\frac{2xu(x)}{x^2+1} + \frac{2}{(x^2+1)x^3}$$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$$u'(x) + \frac{2xu(x)}{x^2+1} = \frac{2}{(x^2+1)x^3}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(u'(x) + \frac{2xu(x)}{x^2+1} \right) = \frac{2\mu(x)}{(x^2+1)x^3}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)u(x))$

$$\mu(x) \left(u'(x) + \frac{2xu(x)}{x^2+1} \right) = \mu'(x)u(x) + \mu(x)u'(x)$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{2\mu(x)x}{x^2+1}$$

- Solve to find the integrating factor

$$\mu(x) = x^2 + 1$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)u(x)) \right) dx = \int \frac{2\mu(x)}{(x^2+1)x^3} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)u(x) = \int \frac{2\mu(x)}{(x^2+1)x^3} dx + c_1$$

- Solve for $u(x)$

$$u(x) = \frac{\int \frac{2\mu(x)}{(x^2+1)x^3} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x^2 + 1$

$$u(x) = \frac{\int \frac{2}{x^3} dx + c_1}{x^2+1}$$

- Evaluate the integrals on the rhs

$$u(x) = \frac{-\frac{1}{x^2} + c_1}{x^2+1}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{-\frac{1}{x^2} + c_1}{x^2+1}$$

- Make substitution $u = y'$

$$y' = \frac{-\frac{1}{x^2} + c_1}{x^2+1}$$

- Integrate both sides to solve for y

$$\int y' dx = \int \frac{-\frac{1}{x^2} + c_1}{x^2+1} dx + c_2$$

- Compute integrals

$$y = (c_1 + 1) \arctan(x) + \frac{1}{x} + c_2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*(_b(_a)*_a^4-1)/(_a^3*(  
Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful  
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)=2*x^(-3),y(x), singsol=all)
```

$$y(x) = \frac{1}{x} + (c_1 + 1) \arctan(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y'[x]+2*x*y'[x]==2*x^(-3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow (1 + c_1) \arctan(x) + \frac{1}{x} + c_2$$

14.3 problem 24

14.3.1 Solving as second order integrable as is ode	2200
14.3.2 Solving as second order ode missing y ode	2201
14.3.3 Solving as second order ode non constant coeff transformation on B ode	2203
14.3.4 Solving as type second_order_integrable_as_is (not using ABC version)	2207
14.3.5 Solving using Kovacic algorithm	2209
14.3.6 Solving as exact linear second order ode ode	2217
14.3.7 Maple step by step solution	2219

Internal problem ID [5431]

Internal file name [OUTPUT/4922_Tuesday_February_06_2024_10_14_18_PM_47983974/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 24.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$xy'' - y' = -\frac{2}{x} - \ln(x)$$

14.3.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y') dx = \int \left(-\frac{2}{x} - \ln(x) \right) dx$$
$$xy' - 2y = -2 \ln(x) - \ln(x)x + x + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{(-x - 2) \ln(x) + x + c_1}{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x - 2) \ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{(-x - 2) \ln(x) + x + c_1}{x} \right)$$
$$\frac{d}{dx} \left(\frac{y}{x^2} \right) = \left(\frac{1}{x^2} \right) \left(\frac{(-x - 2) \ln(x) + x + c_1}{x} \right)$$
$$d \left(\frac{y}{x^2} \right) = \left(\frac{(-x - 2) \ln(x) + x + c_1}{x^3} \right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{(-x - 2) \ln(x) + x + c_1}{x^3} dx$$
$$\frac{y}{x^2} = \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} \right) + c_2 x^2$$

which simplifies to

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y . Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) x - p(x) + \frac{2}{x} + \ln(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{-\ln(x)x - 2}{x^2}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = \frac{-\ln(x)x - 2}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu) \left(\frac{-\ln(x)x - 2}{x^2} \right) \\ \frac{d}{dx} \left(\frac{p}{x} \right) &= \left(\frac{1}{x} \right) \left(\frac{-\ln(x)x - 2}{x^2} \right) \\ d \left(\frac{p}{x} \right) &= \left(\frac{-\ln(x)x - 2}{x^3} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int \frac{-\ln(x)x - 2}{x^3} dx \\ \frac{p}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = x \left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2} \right) + c_1 x$$

which simplifies to

$$p(x) = \frac{c_1 x^2 + \ln(x)x + x + 1}{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1 x^2 + \ln(x)x + x + 1}{x}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_1 x^2 + \ln(x)x + x + 1}{x} dx \\ &= \frac{c_1 x^2}{2} + \ln(x)x + \ln(x) + c_2\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} + \ln(x) x + \ln(x) + c_2 \quad (1)$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + \ln(x) x + \ln(x) + c_2$$

Verified OK.

14.3.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)v' + (AB'' + BB' + CB)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)u = 0$$

The above ode is first order ode which is solved for u . Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}A &= x \\B &= -1 \\C &= 0 \\F &= -\frac{2}{x} - \ln(x)\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (x)(0) + (-1)(0) + (0)(-1) \\&= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-xv'' + (1)v' = 0$$

Now by applying $v' = u$ the above becomes

$$-xu'(x) + u(x) = 0$$

Which is now solved for u . In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\&= f(x)g(u) \\&= \frac{u}{x}\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{1}{x} dx \\ \int \frac{1}{u} du &= \int \frac{1}{x} dx \\ \ln(u) &= \ln(x) + c_1 \\ u &= e^{\ln(x)+c_1} \\ &= c_1 x\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}v' &= u \\ &= c_1 x\end{aligned}$$

Which is now solved for v . Integrating both sides gives

$$\begin{aligned}v(x) &= \int c_1 x \, dx \\ &= \frac{c_1 x^2}{2} + c_2\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= Bv \\ &= (-1) \left(\frac{c_1 x^2}{2} + c_2 \right) \\ &= -\frac{c_1 x^2}{2} - c_2\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \quad (1)$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = -1$$

$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \quad (2)$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \quad (3)$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2 \left(-\frac{2}{x} - \ln(x) \right)}{-2x^2} dx$$

Which simplifies to

$$u_1 = - \int \left(\frac{1}{x} + \frac{\ln(x)}{2} \right) dx$$

Hence

$$u_1 = -\ln(x) - \frac{\ln(x)x}{2} + \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2}{x} + \ln(x)}{-2x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{-\ln(x)x - 2}{2x^3} dx$$

Hence

$$u_2 = \frac{\ln(x)}{2x} + \frac{1}{2x} + \frac{1}{2x^2}$$

Which simplifies to

$$u_1 = -\ln(x) - \frac{\ln(x)x}{2} + \frac{x}{2}$$
$$u_2 = \frac{\ln(x)x + x + 1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x) + \ln(x)x + \frac{1}{2}$$

Hence the complete solution is

$$y(x) = y_h + y_p$$
$$= \left(-\frac{c_1x^2}{2} - c_2\right) + \left(\ln(x) + \ln(x)x + \frac{1}{2}\right)$$
$$= -\frac{c_1x^2}{2} - c_2 + \ln(x) + \ln(x)x + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1x^2}{2} - c_2 + \ln(x) + \ln(x)x + \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = -\frac{c_1x^2}{2} - c_2 + \ln(x) + \ln(x)x + \frac{1}{2}$$

Verified OK.

14.3.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xy'' - y' = -\frac{2}{x} - \ln(x)$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y') dx = \int \left(-\frac{2}{x} - \ln(x)\right) dx$$
$$xy' - 2y = -2\ln(x) - \ln(x)x + x + c_1$$

Which is now solved for y .

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{(-x-2)\ln(x) + x + c_1}{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x-2)\ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{(-x-2)\ln(x) + x + c_1}{x} \right)$$
$$\frac{d}{dx} \left(\frac{y}{x^2} \right) = \left(\frac{1}{x^2} \right) \left(\frac{(-x-2)\ln(x) + x + c_1}{x} \right)$$
$$d \left(\frac{y}{x^2} \right) = \left(\frac{(-x-2)\ln(x) + x + c_1}{x^3} \right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{(-x-2)\ln(x) + x + c_1}{x^3} dx$$
$$\frac{y}{x^2} = \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} \right) + c_2 x^2$$

which simplifies to

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \quad (1)$$

$$Ay'' + By' + Cy = 0 \quad (2)$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \quad (3)$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \quad (4)$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \quad (5)$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \quad (6)$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2} \right) z(x) \quad (7)$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞ . The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \dots\}$	$\{\dots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \dots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}, \{1, 3\}, \{2\}, \{3\}, \{3, 4\}, \{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \dots\}$

Table 250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s . Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole

larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞ . which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated α_c^\pm have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated α_∞^\pm have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of α_∞^\pm . This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω . Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c) [\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty) [\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-) [\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-) [\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r) p = 0 \quad (1A)$$

Let

$$p(x) = 1 \quad (2A)$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$z_1(x) = pe^{\int \omega dx}$$

$$= e^{\int -\frac{1}{2x} dx}$$

$$= \frac{1}{\sqrt{x}}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} dx}$$

$$= z_1 e^{-\int \frac{1}{2} \frac{-1}{x} dx}$$

$$= z_1 e^{\frac{\ln(x)}{2}}$$

$$= z_1 (\sqrt{x})$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{-1}{x} dx}}{(y_1)^2} dx$$

$$= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2} dx$$

$$= y_1 \left(\frac{x^2}{2}\right)$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2 \left(1 \left(\frac{x^2}{2} \right) \right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$xy'' - y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + \frac{c_2 x^2}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= 1 \\ y_2 &= \frac{x^2}{2}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = - \int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = - \int \frac{x^2 \left(-\frac{2}{x} - \ln(x)\right)}{x^2} dx$$

Which simplifies to

$$u_1 = - \int \left(-\frac{1}{x} - \frac{\ln(x)}{2}\right) dx$$

Hence

$$u_1 = \ln(x) + \frac{\ln(x)x}{2} - \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{2}{x} - \ln(x)}{x^2} dx$$

Which simplifies to

$$u_2 = \int \frac{-\ln(x)x - 2}{x^3} dx$$

Hence

$$u_2 = \frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x) + \frac{\ln(x)x}{2} - \frac{x}{2} + \frac{\left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2}\right)x^2}{2}$$

Which simplifies to

$$y_p(x) = \ln(x) + \ln(x)x + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\ln(x) + \ln(x)x + \frac{1}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 x^2}{2} + \ln(x) + \ln(x)x + \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = c_1 + \frac{c_2 x^2}{2} + \ln(x) + \ln(x)x + \frac{1}{2}$$

Verified OK.

14.3.6 Solving as exact linear second order ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= -1 \\ r(x) &= 0 \\ s(x) &= -\frac{2}{x} - \ln(x) \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x) y' + (q(x) - p'(x)) y)' = s(x)$$

Integrating gives

$$p(x) y' + (q(x) - p'(x)) y = \int s(x) dx$$

Substituting the above values for p, q, r, s gives

$$xy' - 2y = \int -\frac{2}{x} - \ln(x) dx$$

We now have a first order ode to solve which is

$$xy' - 2y = -2 \ln(x) - \ln(x) x + x + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{(-x - 2) \ln(x) + x + c_1}{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x - 2) \ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x} dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu) \left(\frac{(-x - 2) \ln(x) + x + c_1}{x} \right)$$
$$\frac{d}{dx} \left(\frac{y}{x^2} \right) = \left(\frac{1}{x^2} \right) \left(\frac{(-x - 2) \ln(x) + x + c_1}{x} \right)$$
$$d \left(\frac{y}{x^2} \right) = \left(\frac{(-x - 2) \ln(x) + x + c_1}{x^3} \right) dx$$

Integrating gives

$$\frac{y}{x^2} = \int \frac{(-x - 2) \ln(x) + x + c_1}{x^3} dx$$
$$\frac{y}{x^2} = \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} \right) + c_2 x^2$$

which simplifies to

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \quad (1)$$

Verification of solutions

$$y = c_2 x^2 + \ln(x) x + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.7 Maple step by step solution

Let's solve

$$y''x - y' = -\frac{2}{x} - \ln(x)$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Make substitution $u = y'$ to reduce order of ODE

$$u'(x)x - u(x) = -\frac{2}{x} - \ln(x)$$

- Isolate the derivative

$$u'(x) = \frac{u(x)}{x} - \frac{\ln(x)x+2}{x^2}$$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$$u'(x) - \frac{u(x)}{x} = -\frac{\ln(x)x+2}{x^2}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(u'(x) - \frac{u(x)}{x} \right) = -\frac{\mu(x)(\ln(x)x+2)}{x^2}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)u(x))$

$$\mu(x) \left(u'(x) - \frac{u(x)}{x} \right) = \mu'(x)u(x) + \mu(x)u'(x)$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)u(x)) \right) dx = \int -\frac{\mu(x)(\ln(x)x+2)}{x^2} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) u(x) = \int -\frac{\mu(x)(\ln(x)x+2)}{x^2} dx + c_1$$

- Solve for $u(x)$

$$u(x) = \frac{\int -\frac{\mu(x)(\ln(x)x+2)}{x^2} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x}$

$$u(x) = x \left(\int -\frac{\ln(x)x+2}{x^3} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$u(x) = x \left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2} + c_1 \right)$$

- Simplify

$$u(x) = \frac{c_1 x^2 + \ln(x)x + x + 1}{x}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{c_1 x^2 + \ln(x)x + x + 1}{x}$$

- Make substitution $u = y'$

$$y' = \frac{c_1 x^2 + \ln(x)x + x + 1}{x}$$

- Integrate both sides to solve for y

$$\int y' dx = \int \frac{c_1 x^2 + \ln(x)x + x + 1}{x} dx + c_2$$

- Compute integrals

$$y = \frac{c_1 x^2}{2} + \ln(x)x + \ln(x) + c_2$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying high order exact linear fully integrable  
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)*_a-ln(_a)*_a-2)/_a^2, _b(_a)`  
  Methods for first order ODEs:  
    --- Trying classification methods ---  
    trying a quadrature  
    trying 1st order linear  
    <- 1st order linear successful  
  <- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)=-2/x-ln(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{2} + \ln(x) x + \ln(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 23

```
DSolve[x*y'[x]-y'[x]==-2/x-Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{c_1 x^2}{2} + (x + 1) \log(x) + c_2$$

14.4 problem 25

Internal problem ID [5432]

Internal file name [OUTPUT/4923_Tuesday_February_06_2024_10_14_19_PM_17103159/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 25.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**higher_order_linear_constant_coefficients_ODE**"

Maple gives the following as the ode type

```
[[_3rd_order , _missing_y]]
```

$$y''' + y'' = x^2$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$

$$\lambda_2 = 0$$

$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$

$$y_2 = 1$$

$$y_3 = x$$

Now the particular solution to the given ODE is found

$$y''' + y'' = x^2$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x . The UC_set becomes

$$[\{x^2, x^3, x^4\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3x^4 + A_2x^3 + A_1x^2$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12x^2A_3 + 6xA_2 + 24xA_3 + 2A_1 + 6A_2 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{1}{3}, A_3 = \frac{1}{12} \right]$$

Substituting the above back in the above trial solution y_p , gives the particular solution

$$y_p = \frac{1}{12}x^4 - \frac{1}{3}x^3 + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1e^{-x} + c_2 + c_3x) + \left(\frac{1}{12}x^4 - \frac{1}{3}x^3 + x^2 \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^{-x} + c_2 + c_3x + \frac{x^4}{12} - \frac{x^3}{3} + x^2 \quad (1)$$

Verification of solutions

$$y = c_1e^{-x} + c_2 + c_3x + \frac{x^4}{12} - \frac{x^3}{3} + x^2$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a), _b(_a)` *** Sublevel 2 *
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying 1st order linear
  <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^4}{12} + x^2 - \frac{x^3}{3} + c_1 e^{-x} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 37

```
DSolve[y''''[x]+y'''[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^4}{12} - \frac{x^3}{3} + x^2 + c_3 x + c_1 e^{-x} + c_2$$

14.5 problem 26

14.5.1 Solving as second order ode missing x ode 2226

14.5.2 Maple step by step solution 2228

Internal problem ID [5433]

Internal file name [OUTPUT/4924_Tuesday_February_06_2024_10_14_19_PM_55034848/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 26.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**second_order_ode_missing_x**"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],  
 [_2nd_order, _reducible, _mu_y_y1]]
```

$$yy'' + y'^3 = 0$$

14.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable.

Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y)g(p) \\ &= -\frac{p^2}{y} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p^2} dp &= -\frac{1}{y} dy \\ \int \frac{1}{p^2} dp &= \int -\frac{1}{y} dy \\ -\frac{1}{p} &= -\ln(y) + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{p(y)} + \ln(y) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} + \ln(y) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} \int (\ln(y) - c_1) dy &= c_2 + x \\ -c_1 y + \ln(y) y - y &= c_2 + x \end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = e^{\text{LambertW}((c_2+x)e^{-1-c_1})+1+c_1} \quad (1)$$

Verification of solutions

$$y = e^{\text{LambertW}((c_2+x)e^{-1-c_1})+1+c_1}$$

Verified OK.

14.5.2 Maple step by step solution

Let's solve

$$yy'' + y'^3 = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Define new dependent variable u

$$u(x) = y'$$

- Compute y''

$$u'(x) = y''$$

- Use chain rule on the lhs

$$y' \left(\frac{d}{dy} u(y) \right) = y''$$

- Substitute in the definition of u

$$u(y) \left(\frac{d}{dy} u(y) \right) = y''$$

- Make substitutions $y' = u(y)$, $y'' = u(y) \left(\frac{d}{dy} u(y) \right)$ to reduce order of ODE

$$yu(y) \left(\frac{d}{dy} u(y) \right) + u(y)^3 = 0$$

- Separate variables

$$\frac{\frac{d}{dy} u(y)}{u(y)^2} = -\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy} u(y)}{u(y)^2} dy = \int -\frac{1}{y} dy + c_1$$

- Evaluate integral

$$-\frac{1}{u(y)} = -\ln(y) + c_1$$

- Solve for $u(y)$

$$u(y) = \frac{1}{\ln(y) - c_1}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{1}{\ln(y) - c_1}$$
- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{1}{\ln(y) - c_1}$$
- Separate variables

$$y'(\ln(y) - c_1) = 1$$
- Integrate both sides with respect to x

$$\int y'(\ln(y) - c_1) dx = \int 1 dx + c_2$$
- Evaluate integral

$$-yc_1 + y \ln(y) - y = c_2 + x$$
- Solve for y

$$y = e^{\text{LambertW}((c_2+x)e^{-1-c_1})+1+c_1}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`

```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = c_1$$

$$y(x) = \frac{x + c_2}{\text{LambertW}((x + c_2) e^{c_1 - 1})}$$

✓ Solution by Mathematica

Time used: 60.092 (sec). Leaf size: 26

```
DSolve[y[x]*y'[x]+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x + c_2}{W(e^{-1-c_1}(x + c_2))}$$

14.6 problem 27

14.6.1 Solving as second order integrable as is ode	2231
14.6.2 Solving as second order ode missing x ode	2232
14.6.3 Solving as type second_order_integrable_as_is (not using ABC version)	2234
14.6.4 Solving as exact nonlinear second order ode ode	2235
14.6.5 Maple step by step solution	2236

Internal problem ID [5434]

Internal file name [OUTPUT/4925_Tuesday_February_06_2024_10_14_20_PM_13102016/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 27.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],  
_Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,  
_reducible, _mu_xy]]
```

$$yy'' + y'^2 = 0$$

14.6.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + y'^2) dx = 0$$
$$y'y = c_1$$

Which is now solved for y . Integrating both sides gives

$$\int \frac{y}{c_1} dy = c_2 + x$$
$$\frac{y^2}{2c_1} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \quad (1)$$

$$y = -\sqrt{2c_1c_2 + 2c_1x} \quad (2)$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$y'' = \frac{dp}{dx}$$
$$= \frac{dy}{dx} \frac{dp}{dy}$$
$$= p \frac{dp}{dy}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y)g(p) \\ &= -\frac{p}{y} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p} dp &= -\frac{1}{y} dy \\ \int \frac{1}{p} dp &= \int -\frac{1}{y} dy \\ \ln(p) &= -\ln(y) + c_1 \\ p &= e^{-\ln(y)+c_1} \\ &= \frac{c_1}{y} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{c_1}{y}$$

Integrating both sides gives

$$\begin{aligned} \int \frac{y}{c_1} dy &= c_2 + x \\ \frac{y^2}{2c_1} &= c_2 + x \end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned} y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\ y_2 &= -\sqrt{2c_1c_2 + 2c_1x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$

$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (yy'' + y'^2) dx = 0$$

$$y'y = c_1$$

Which is now solved for y . Integrating both sides gives

$$\int \frac{y}{c_1} dy = c_2 + x$$

$$\frac{y^2}{2c_1} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$

$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$

$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.4 Solving as exact nonlinear second order ode

An exact non-linear second order ode has the form

$$a_2(x, y, y') y'' + a_1(x, y, y') y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= y \\ a_1 &= y' \\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2 dy' + \int a_1 dy + \int a_0 dx &= c_1 \\ \int y dy' + \int y' dy + \int 0 dx &= c_1\end{aligned}$$

Which results in

$$2y'y = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{2y}{c_1} dy &= c_2 + x \\ \frac{y^2}{c_1} &= c_2 + x\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= \sqrt{c_1 c_2 + c_1 x} \\ y_2 &= -\sqrt{c_1 c_2 + c_1 x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 c_2 + c_1 x} \quad (1)$$

$$y = -\sqrt{c_1 c_2 + c_1 x} \quad (2)$$

Verification of solutions

$$y = \sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

$$y = -\sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

14.6.5 Maple step by step solution

Let's solve

$$yy'' + y'^2 = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Define new dependent variable u

$$u(x) = y'$$

- Compute y''

$$u'(x) = y''$$

- Use chain rule on the lhs

$$y' \left(\frac{d}{dy} u(y) \right) = y''$$

- Substitute in the definition of u

$$u(y) \left(\frac{d}{dy} u(y) \right) = y''$$

- Make substitutions $y' = u(y)$, $y'' = u(y) \left(\frac{d}{dy} u(y) \right)$ to reduce order of ODE

$$yu(y) \left(\frac{d}{dy} u(y) \right) + u(y)^2 = 0$$

- Separate variables

$$\frac{\frac{d}{dy} u(y)}{u(y)} = -\frac{1}{y}$$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy}u(y)}{u(y)} dy = \int -\frac{1}{y} dy + c_1$$

- Evaluate integral

$$\ln(u(y)) = -\ln(y) + c_1$$

- Solve for $u(y)$

$$u(y) = \frac{e^{c_1}}{y}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \frac{e^{c_1}}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{e^{c_1}}{y}$$

- Separate variables

$$y'y = e^{c_1}$$

- Integrate both sides with respect to x

$$\int y'y dx = \int e^{c_1} dx + c_2$$

- Evaluate integral

$$\frac{y^2}{2} = x e^{c_1} + c_2$$

- Solve for y

$$\left\{ y = \sqrt{2x e^{c_1} + 2c_2}, y = -\sqrt{2x e^{c_1} + 2c_2} \right\}$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \sqrt{2c_1x + 2c_2}$$

$$y(x) = -\sqrt{2c_1x + 2c_2}$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 20

```
DSolve[y[x]*y'[x]+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_2\sqrt{2x - c_1}$$

14.7 problem 28

14.7.1 Solving as second order ode missing x ode 2239

Internal problem ID [5435]

Internal file name [OUTPUT/4926_Tuesday_February_06_2024_10_14_20_PM_11197644/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 28.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x], [_2nd_order , _reducible , _mu_y_y1]]
```

$$yy'' - y'^2(1 - y' \cos(y) + yy' \sin(y)) = 0$$

14.7.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) + (-y \sin(y) p(y)^2 + \cos(y) p(y)^2 - p(y)) p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)y$ on the above ode results in new ode in $u(y)$

$$y^2 u(y) \left(\left(\frac{d}{dy} u(y) \right) y + u(y) \right) + (-y^3 \sin(y) u(y)^2 + \cos(y) u(y)^2 y^2 - u(y)y) u(y)y = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(y, u) \\ &= f(y)g(u) \\ &= u^2(y \sin(y) - \cos(y)) \end{aligned}$$

Where $f(y) = y \sin(y) - \cos(y)$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u^2} du &= y \sin(y) - \cos(y) dy \\ \int \frac{1}{u^2} du &= \int y \sin(y) - \cos(y) dy \\ -\frac{1}{u} &= -y \cos(y) + c_2 \end{aligned}$$

The solution is

$$-\frac{1}{u(y)} + y \cos(y) - c_2 = 0$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\begin{aligned} -\frac{y}{p(y)} + y \cos(y) - c_2 &= 0 \\ -\frac{y}{p(y)} + y \cos(y) - c_2 &= 0 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{y}{y'} + y \cos(y) - c_2 = 0$$

Integrating both sides gives

$$\begin{aligned} \int \frac{y \cos(y) - c_2}{y} dy &= \int dx \\ \int \frac{-a \cos(-a) - c_2}{-a} d_{-a} &= x + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$\int^y \frac{-a \cos(-a) - c_2}{-a} d_a = x + c_3 \quad (1)$$

Verification of solutions

$$\int^y \frac{-a \cos(-a) - c_2}{-a} d_a = x + c_3$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2*(-sin(_a)*_a*_b(_a)+cos
  Methods for first order ODEs:
  --- Trying classification methods ---
  trying a quadrature
  trying 1st order linear
  trying Bernoulli
  <- Bernoulli successful
  <- differential order: 2; canonical coordinates successful
  <- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 22

```
dsolve(y(x)*diff(y(x),x$2)=diff(y(x),x)^2*(1-diff(y(x),x)*cos(y(x))+y(x)*diff(y(x),x)*sin(y(x)
```

$$\begin{aligned} y(x) &= c_1 \\ \sin(y(x)) + c_1 \ln(y(x)) - x - c_2 &= 0 \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.425 (sec). Leaf size: 63

```
DSolve[y[x]*y'[x]==y'[x]^2*(1-y'[x]*Cos[y[x]]+y[x]*y'[x]*Sin[y[x]] ),y[x],x,IncludeSingular
```

$$y(x) \rightarrow \text{InverseFunction}[\sin(\#1) + c_1 \log(\#1)\&][x + c_2]$$

$$y(x) \rightarrow \text{InverseFunction}[\sin(\#1) - c_1 \log(\#1)\&][x + c_2]$$

$$y(x) \rightarrow \text{InverseFunction}[\sin(\#1) + c_1 \log(\#1)\&][x + c_2]$$

14.8 problem 29

14.8.1 Maple step by step solution 2243

Internal problem ID [5436]

Internal file name [OUTPUT/4927_Tuesday_February_06_2024_10_14_22_PM_63877284/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 29.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[[_3rd_order , _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$(2x - 3)y''' - (6x - 7)y'' + 4xy' - 4y = 8$$

Unable to solve this ODE.

14.8.1 Maple step by step solution

Let's solve

$$(2x - 3)y''' - (6x - 7)y'' + 4xy' - 4y = 8$$

- Highest derivative means the order of the ODE is 3
 y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
Equation is the LCLM of -1/x*y(x)+diff(y(x),x), -y(x)+diff(y(x),x), -2*y(x)+diff(y(x),x)
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((2*x-3)*diff(y(x),x$3)-(6*x-7)*diff(y(x),x$2)+4*x*diff(y(x),x)-4*y(x)=8,y(x), singsol
```

$$y(x) = -2 + c_1x + e^x c_2 + c_3e^{2x}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 26

```
DSolve[(2*x-3)*y'''[x]-(6*x-7)*y''[x]+4*x*y'[x]-4*y[x]==8,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \rightarrow c_1x + c_2e^x - c_3e^{2x} - 2$$

14.9 problem 30

14.9.1 Maple step by step solution 2248

Internal problem ID [5437]

Internal file name [OUTPUT/4928_Tuesday_February_06_2024_10_14_22_PM_25156565/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 30.

ODE order: 3.

ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

`[[_3rd_order , _missing_y]]`

$$(2x^3 - 1) y''' - 6x^2 y'' + 6xy' = 0$$

Since y is missing from the ode then we can use the substitution $y' = v(x)$ to reduce the order by one. The ODE becomes

$$(2x^3 - 1) v''(x) - 6x^2 v'(x) + 6xv(x) = 0$$

In normal form the ode

$$(2x^3 - 1) v''(x) - 6x^2 v'(x) + 6xv(x) = 0 \tag{1}$$

Becomes

$$v''(x) + p(x) v'(x) + q(x) v(x) = 0 \tag{2}$$

Where

$$p(x) = -\frac{6x^2}{2x^3 - 1}$$
$$q(x) = \frac{6x}{2x^3 - 1}$$

Applying change of variables on the dependent variable $v(x) = v(x) x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $v(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \quad (3)$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \quad (4)$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{6nx}{2x^3-1} + \frac{6x}{2x^3-1} = 0 \quad (5)$$

Solving (5) for n gives

$$n = 1 \quad (6)$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{6x^2}{2x^3-1}\right) v'(x) &= 0 \\ v''(x) + \frac{(-2x^3-2)v'(x)}{2x^4-x} &= 0 \end{aligned} \quad (7)$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2x^3-2)u(x)}{2x^4-x} = 0 \quad (8)$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{2u(x^3+1)}{x(2x^3-1)} \end{aligned}$$

Where $f(x) = \frac{2x^3+2}{x(2x^3-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u} du &= \frac{2x^3 + 2}{x(2x^3 - 1)} dx \\ \int \frac{1}{u} du &= \int \frac{2x^3 + 2}{x(2x^3 - 1)} dx \\ \ln(u) &= \ln(2x^3 - 1) - 2\ln(x) + c_1 \\ u &= e^{\ln(2x^3-1)-2\ln(x)+c_1} \\ &= c_1 e^{\ln(2x^3-1)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x) = c_1 \left(2x - \frac{1}{x^2} \right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\ v(x) &= \int u(x) dx + c_2 \\ &= c_1 \left(x^2 + \frac{1}{x} \right) + c_2\end{aligned}$$

Hence

$$\begin{aligned}v(x) &= v(x) x^n \\ &= \left(c_1 \left(x^2 + \frac{1}{x} \right) + c_2 \right) x \\ &= c_1 x^3 + c_2 x + c_1\end{aligned}$$

But since $y' = v(x)$ then we now need to solve the ode $y' = (c_1(x^2 + \frac{1}{x}) + c_2) x$. Integrating both sides gives

$$\begin{aligned}y &= \int c_1 x^3 + c_2 x + c_1 dx \\ &= \frac{1}{4} c_1 x^4 + \frac{1}{2} c_2 x^2 + c_1 x + c_3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{4} c_1 x^4 + \frac{1}{2} c_2 x^2 + c_1 x + c_3 \quad (1)$$

Verification of solutions

$$y = \frac{1}{4}c_1x^4 + \frac{1}{2}c_2x^2 + c_1x + c_3$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$$(2x^3 - 1)y''' - 6x^2y'' + 6xy' = 0$$

- Highest derivative means the order of the ODE is 3
 y'''

□ Check to see if x_0 is a regular singular point

- Define functions

$$\left[P_2(x) = -\frac{6x^2}{2x^3-1}, P_3(x) = \frac{6x}{2x^3-1}, P_4(x) = 0 \right]$$

- $\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right) \cdot P_2(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$

$$\left(\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right) \cdot P_2(x) \right) \Big|_{x=-\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}} = 0$$

- $\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right)^2 \cdot P_3(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$

$$\left(\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right)^2 \cdot P_3(x) \right) \Big|_{x=-\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}} = 0$$

- $\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right)^3 \cdot P_4(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$

$$\left(\left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\sqrt{3}2^{\frac{2}{3}}}{4}\right)^3 \cdot P_4(x) \right) \Big|_{x=-\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}} = 0$$

- $x = -\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$ is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = -\frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$$

- Change variables using $x = u - \frac{2^{\frac{2}{3}}}{4} - \frac{\sqrt{3}2^{\frac{2}{3}}}{4}$ so that the regular singular point is at $u = 0$

$$\left(2u^3 - \frac{3u^2 2^{\frac{2}{3}}}{2} - \frac{3\sqrt{3}u 2^{\frac{2}{3}}}{2} - \frac{3u 2^{\frac{1}{3}}}{2} + \frac{3\sqrt{3} 2^{\frac{1}{3}}}{2}\right) \left(\frac{d^3}{du^3}y(u)\right) + \left(-6u^2 + 3u 2^{\frac{2}{3}} + 3\sqrt{3} 2^{\frac{2}{3}} + \frac{3 2^{\frac{1}{3}}}{2} - 3\right) y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k \rightarrow k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)(k+r-2) u^{k+r-3+m}$$

- Shift index using $k \rightarrow k+3-m$

$$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=-3+m}^{\infty} a_{k+3-m} (k+3-m+r)(k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$\frac{3 \cdot 2^{\frac{1}{3}} (1 + \sqrt{3} - 1) (-3+r)r(r-1) a_0 u^{-2+r}}{2} + \left(\frac{3 \cdot 2^{\frac{1}{3}} (1 + \sqrt{3} - 1) (-2+r)(1+r) r a_1}{2} - \frac{3 \cdot 2^{\frac{2}{3}} (1 + \sqrt{3}) r (r^2 - 5r + 5) a_0}{2} \right) u^{r-1} + \left(\sum_{k=0}^{\infty} \right)$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$\frac{3 \cdot 2^{\frac{1}{3}} (1 + \sqrt{3} - 1) (-3+r)r(r-1)}{2} = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$\frac{3 \cdot 2^{\frac{1}{3}} (1 + \sqrt{3} - 1) (k+r-1)(k+2+r)(k+1+r) a_{k+2}}{2} - \frac{3(k+1+r)(k^2 + (2r-3)k + r^2 - 3r + 1) (1 + \sqrt{3}) a_{k+1} 2^{\frac{2}{3}}}{2} + 2a_k (k+r)(k+r)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{(-4k^3 a_k - 4r^3 a_k + 24k^2 a_k + 24r^2 a_k - 32ka_k + 3I2^{\frac{2}{3}} \sqrt{3} a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} r^2 a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 3I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(2 + I\sqrt{3} k^3 + 2I\sqrt{3} k^2 - 2I\sqrt{3} - k^3 - I\sqrt{3} k - 2k^2 + k)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{(-4k^3 a_k + 24k^2 a_k - 32ka_k + 3I2^{\frac{2}{3}} \sqrt{3} a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 3I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(2 + I\sqrt{3} k^3 + 2I\sqrt{3} k^2 - 2I\sqrt{3} - k^3 - I\sqrt{3} k - 2k^2 + k)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = \frac{(-4k^3 a_k + 24k^2 a_k - 32ka_k + 3I2^{\frac{2}{3}} \sqrt{3} a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 6I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 3I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(2 + I\sqrt{3} k^3 + 2I\sqrt{3} k^2 - 2I\sqrt{3} - k^3 - I\sqrt{3} k - 2k^2 + k)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{(-4k^3 a_k + 12k^2 a_k + 4ka_k - 6I2^{\frac{2}{3}} \sqrt{3} a_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} - 12a_k + 3I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 9I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} - 6I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(I\sqrt{3} k^3 + 5I\sqrt{3} k^2 + 6I\sqrt{3} k - k^3 - 5k^2 - 6k)}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{(-4k^3 a_k + 12k^2 a_k + 4ka_k - 6I2^{\frac{2}{3}} \sqrt{3} a_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} - 12a_k + 3I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} - 9I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} - 6I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(I\sqrt{3} k^3 + 5I\sqrt{3} k^2 + 6I\sqrt{3} k - k^3 - 5k^2 - 6k)}$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{(-4k^3 a_k - 12k^2 a_k + 4ka_k + 12I2^{\frac{2}{3}} \sqrt{3} a_{k+1} + 21I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} + 39I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 12a_k + 12I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(I\sqrt{3} k^3 + 11I\sqrt{3} k^2 + 40I\sqrt{3} + 38I\sqrt{3} k - 40 - k^3 - 11k^2 - 38k)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+2} = \frac{(-4k^3 a_k - 12k^2 a_k + 4ka_k + 12I2^{\frac{2}{3}} \sqrt{3} a_{k+1} + 21I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} + 39I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 12a_k + 12I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(I\sqrt{3} k^3 + 11I\sqrt{3} k^2 + 40I\sqrt{3} + 38I\sqrt{3} k - 40 - k^3 - 11k^2 - 38k)} \right]$$

- Revert the change of variables $u = x + \frac{2^{\frac{2}{3}}}{4} + \frac{I\sqrt{3} 2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{I\sqrt{3} 2^{\frac{2}{3}}}{4} \right)^{k+3}, a_{k+2} = \frac{(-4k^3 a_k - 12k^2 a_k + 4ka_k + 12I2^{\frac{2}{3}} \sqrt{3} a_{k+1} + 21I2^{\frac{2}{3}} \sqrt{3} k^2 a_{k+1} + 39I2^{\frac{2}{3}} \sqrt{3} ka_{k+1} + 3I2^{\frac{2}{3}} \sqrt{3} k^3 a_{k+1} + 12a_k + 12I2^{\frac{2}{3}} a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1} + 3I2^{\frac{2}{3}} k^3 a_{k+1})}{6(I\sqrt{3} k^3 + 11I\sqrt{3} k^2 + 40I\sqrt{3} + 38I\sqrt{3} k - 40 - k^3 - 11k^2 - 38k)} \right]$$

Maple trace Kovacic algorithm successful

```
`Methods for third order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying high order exact linear fully integrable  
trying to convert to a linear ODE with constant coefficients  
trying differential order: 3; missing the dependent variable  
  trying a quadrature  
  checking if the LODE has constant coefficients  
  checking if the LODE is of Euler type  
  trying a symmetry of the form [xi=0, eta=F(x)]  
  checking if the LODE is missing y  
  -> Trying a Liouvillian solution using Kovacics algorithm  
    A Liouvillian solution exists  
    Reducible group (found an exponential solution)  
  <- Kovacics algorithm successful  
<- missing the dependent variable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve((2*x^3-1)*diff(y(x),x$3)-6*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_3x^4 + c_2x^2 + 4c_3x + c_1$$

✓ Solution by Mathematica

Time used: 1.357 (sec). Leaf size: 31

```
DSolve[(2*x^3-1)*y'''[x]-6*x^2*y''[x]+6*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{c_2x^4}{4} + \frac{c_1x^2}{2} - c_2x + c_3$$

14.10 problem 31

14.10.1 Solving as second order ode missing x ode 2252

Internal problem ID [5438]

Internal file name [OUTPUT/4929_Tuesday_February_06_2024_10_14_22_PM_49348237/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 31.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x], [_2nd_order , _reducible , _mu_xy]]
```

$$yy'' - y'^2 - y^2 \ln(y) = 0$$

14.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y) \left(\frac{d}{dy} p(y) \right) - p(y)^2 - \ln(y) y^2 = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)y$ on the above ode results in new ode in $u(y)$

$$y^2 u(y) \left(\left(\frac{d}{dy} u(y) \right) y + u(y) \right) - u(y)^2 y^2 = \ln(y) y^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(y, u) \\ &= f(y)g(u) \\ &= \frac{\ln(y)}{uy} \end{aligned}$$

Where $f(y) = \frac{\ln(y)}{y}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} du &= \frac{\ln(y)}{y} dy \\ \int \frac{1}{u} du &= \int \frac{\ln(y)}{y} dy \\ \frac{u^2}{2} &= \frac{\ln(y)^2}{2} + c_2 \end{aligned}$$

The solution is

$$\frac{u(y)^2}{2} - \frac{\ln(y)^2}{2} - c_2 = 0$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\begin{aligned} \frac{p(y)^2}{2y^2} - \frac{\ln(y)^2}{2} - c_2 &= 0 \\ \frac{p(y)^2}{2y^2} - \frac{\ln(y)^2}{2} - c_2 &= 0 \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2y^2} - \frac{\ln(y)^2}{2} - c_2 = 0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{\ln(y)^2 + 2c_2 y} \quad (1)$$

$$y' = -\sqrt{\ln(y)^2 + 2c_2 y} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{\ln(y)^2 + 2c_2 y}} dy = \int dx$$

$$\ln \left(\ln(y) + \sqrt{\ln(y)^2 + 2c_2} \right) = x + c_3$$

Raising both side to exponential gives

$$\ln(y) + \sqrt{\ln(y)^2 + 2c_2} = e^{x+c_3}$$

Which simplifies to

$$\ln(y) + \sqrt{\ln(y)^2 + 2c_2} = c_4 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{\ln(y)^2 + 2c_2 y}} dy = \int dx$$

$$-\ln \left(\ln(y) + \sqrt{\ln(y)^2 + 2c_2} \right) = x + c_5$$

Raising both side to exponential gives

$$\frac{1}{\ln(y) + \sqrt{\ln(y)^2 + 2c_2}} = e^{x+c_5}$$

Which simplifies to

$$\frac{1}{\ln(y) + \sqrt{\ln(y)^2 + 2c_2}} = c_6 e^x$$

Summary

The solution(s) found are the following

$$y = e^{\frac{(e^{2x}c_4^2 - 2c_2)e^{-x}}{2c_4}} \quad (1)$$

$$y = e^{-\frac{(2c_2c_6^2e^{2x} - 1)e^{-x}}{2c_6}} \quad (2)$$

Verification of solutions

$$y = e^{\frac{(e^{2x}c_4^2 - 2c_2)e^{-x}}{2c_4}}$$

Verified OK.

$$y = e^{-\frac{(2c_2c_6^2e^{2x} - 1)e^{-x}}{2c_6}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2=y(x)^2*ln(y(x)),y(x), singsol=all)
```

$$y(x) = e^{\frac{c_1e^{-x}}{2} - \frac{e^xc_2}{2}}$$

✓ Solution by Mathematica

Time used: 2.66 (sec). Leaf size: 73

```
DSolve[y[x]*y'[x]-y'[x]^2==y[x]^2*Log[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \exp\left(-\frac{1}{2}\sqrt{c_1}e^{-x-c_2}(-1 + e^{2(x+c_2)})\right)$$

$$y(x) \rightarrow \exp\left(\frac{1}{2}\sqrt{c_1}e^{-x-c_2}(-1 + e^{2(x+c_2)})\right)$$

14.11 problem 32

- 14.11.1 Solving as second order integrable as is ode 2257
14.11.2 Solving as type second_order_integrable_as_is (not using ABC
version) 2262

Internal problem ID [5439]

Internal file name [OUTPUT/4930_Tuesday_February_06_2024_10_14_24_PM_44523640/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 32.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_integrable_as_is"

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order,
  _with_linear_symmetries], [_2nd_order, _reducible, _mu_x_y1],
  [_2nd_order, _reducible, _mu_y_y1], [_2nd_order, _reducible,
  _mu_xy]]
```

$$(2y + x)y'' + 2y'^2 + 2y' = 2$$

14.11.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((2y + x)y'' + (2y' + 2)y') dx = \int 2dx$$
$$y + (2y + x)y' = 2x + c_1$$

Which is now solved for y . Writing the ode as

$$y' = -\frac{y - c_1 - 2x}{2y + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (\text{1E})$$

$$\eta = xb_2 + yb_3 + b_1 \quad (\text{2E})$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(y - c_1 - 2x)(b_3 - a_2)}{2y + x} - \frac{(y - c_1 - 2x)^2 a_3}{(2y + x)^2} \\ - \left(\frac{2}{2y + x} + \frac{y - c_1 - 2x}{(2y + x)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{1}{2y + x} + \frac{2y - 2c_1 - 4x}{(2y + x)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (\text{5E})$$

Putting the above in normal form gives

$$\frac{-c_1^2 a_3 + 4c_1 x a_3 - 2c_1 x b_2 - c_1 x b_3 + 2c_1 y a_2 - 3c_1 y a_3 - 4c_1 y b_3 + 2x^2 a_2 + 4x^2 a_3 - 6x^2 b_2 - 2x^2 b_3 + 8xy a_2 - 8xy a_3 - 8xy b_2 - 8xy b_3 + 2y^2 a_2 + 4y^2 a_3 - 2y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 + 5xb_1 - 5ya_1}{(2y + x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -c_1^2 a_3 - 4c_1 x a_3 + 2c_1 x b_2 + c_1 x b_3 - 2c_1 y a_2 + 3c_1 y a_3 + 4c_1 y b_3 \\ - 2x^2 a_2 - 4x^2 a_3 + 6x^2 b_2 + 2x^2 b_3 - 8xy a_2 + 4xy a_3 + 4xy b_2 + 8xy b_3 \\ + 2y^2 a_2 - 6y^2 a_3 + 4y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 + 5xb_1 - 5ya_1 = 0 \end{aligned} \quad (\text{6E})$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -c_1^2 a_3 - 2c_1 a_2 v_2 - 4c_1 a_3 v_1 + 3c_1 a_3 v_2 + 2c_1 b_2 v_1 + c_1 b_3 v_1 + 4c_1 b_3 v_2 \\ & - 2a_2 v_1^2 - 8a_2 v_1 v_2 + 2a_2 v_2^2 - 4a_3 v_1^2 + 4a_3 v_1 v_2 - 6a_3 v_2^2 + 6b_2 v_1^2 + 4b_2 v_1 v_2 \\ & + 4b_2 v_2^2 + 2b_3 v_1^2 + 8b_3 v_1 v_2 - 2b_3 v_2^2 + c_1 a_1 + 2c_1 b_1 - 5a_1 v_2 + 5b_1 v_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & (-2a_2 - 4a_3 + 6b_2 + 2b_3) v_1^2 + (-8a_2 + 4a_3 + 4b_2 + 8b_3) v_1 v_2 \\ & + (-4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1) v_1 + (2a_2 - 6a_3 + 4b_2 - 2b_3) v_2^2 \\ & + (-2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1) v_2 - c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_2 + 4a_3 + 4b_2 + 8b_3 &= 0 \\ -2a_2 - 4a_3 + 6b_2 + 2b_3 &= 0 \\ 2a_2 - 6a_3 + 4b_2 - 2b_3 &= 0 \\ -c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 &= 0 \\ -2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1 &= 0 \\ -4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= \frac{1}{5} c_1 b_2 + \frac{2}{5} c_1 b_3 \\ a_2 &= b_2 + b_3 \\ a_3 &= b_2 \\ b_1 &= \frac{2}{5} c_1 b_2 - \frac{1}{5} c_1 b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x + \frac{2c_1}{5}$$

$$\eta = y - \frac{c_1}{5}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \frac{c_1}{5} - \left(-\frac{y - c_1 - 2x}{2y + x} \right) \left(x + \frac{2c_1}{5} \right) \\ &= \frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{10y + 5x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{10y + 5x}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(-c_1^2 - 5c_1x - 5x^2 + 5xy + 5y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - c_1 - 2x}{2y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-5y + 5c_1 + 10x}{10x^2 + (-10y + 10c_1)x - 10y^2 + 2c_1^2} \\ S_y &= \frac{-10y - 5x}{10x^2 + (-10y + 10c_1)x - 10y^2 + 2c_1^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1 \quad (1)$$

Verification of solutions

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Verified OK.

14.11.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(2y + x)y'' + (2y' + 2)y' = 2$$

Integrating both sides of the ODE w.r.t x gives

$$\int ((2y + x)y'' + (2y' + 2)y') dx = \int 2dx$$
$$y + (2y + x)y' = 2x + c_1$$

Which is now solved for y . Writing the ode as

$$y' = -\frac{y - c_1 - 2x}{2y + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (A)$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \quad (1E)$$

$$\eta = xb_2 + yb_3 + b_1 \quad (2E)$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(y - c_1 - 2x)(b_3 - a_2)}{2y + x} - \frac{(y - c_1 - 2x)^2 a_3}{(2y + x)^2} \\ - \left(\frac{2}{2y + x} + \frac{y - c_1 - 2x}{(2y + x)^2} \right) (xa_2 + ya_3 + a_1) \\ - \left(-\frac{1}{2y + x} + \frac{2y - 2c_1 - 4x}{(2y + x)^2} \right) (xb_2 + yb_3 + b_1) = 0 \end{aligned} \quad (5E)$$

Putting the above in normal form gives

$$\frac{-c_1^2 a_3 + 4c_1 x a_3 - 2c_1 x b_2 - c_1 x b_3 + 2c_1 y a_2 - 3c_1 y a_3 - 4c_1 y b_3 + 2x^2 a_2 + 4x^2 a_3 - 6x^2 b_2 - 2x^2 b_3 + 8xy a_2 - 8xy a_3 - 4xy b_2 - 4xy b_3 + 2y^2 a_2 - 2y^2 a_3 + 4y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 - 5x b_1 - 5y a_1}{(2y + x)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} -c_1^2 a_3 - 4c_1 x a_3 + 2c_1 x b_2 + c_1 x b_3 - 2c_1 y a_2 + 3c_1 y a_3 + 4c_1 y b_3 \\ - 2x^2 a_2 - 4x^2 a_3 + 6x^2 b_2 + 2x^2 b_3 - 8xy a_2 + 4xy a_3 + 4xy b_2 + 8xy b_3 \\ + 2y^2 a_2 - 6y^2 a_3 + 4y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 + 5x b_1 - 5y a_1 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -c_1^2 a_3 - 2c_1 a_2 v_2 - 4c_1 a_3 v_1 + 3c_1 a_3 v_2 + 2c_1 b_2 v_1 + c_1 b_3 v_1 + 4c_1 b_3 v_2 \\ - 2a_2 v_1^2 - 8a_2 v_1 v_2 + 2a_2 v_2^2 - 4a_3 v_1^2 + 4a_3 v_1 v_2 - 6a_3 v_2^2 + 6b_2 v_1^2 + 4b_2 v_1 v_2 \\ + 4b_2 v_2^2 + 2b_3 v_1^2 + 8b_3 v_1 v_2 - 2b_3 v_2^2 + c_1 a_1 + 2c_1 b_1 - 5a_1 v_2 + 5b_1 v_1 = 0 \end{aligned} \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 - 4a_3 + 6b_2 + 2b_3)v_1^2 + (-8a_2 + 4a_3 + 4b_2 + 8b_3)v_1v_2 \\ &+ (-4c_1a_3 + 2c_1b_2 + c_1b_3 + 5b_1)v_1 + (2a_2 - 6a_3 + 4b_2 - 2b_3)v_2^2 \\ &+ (-2c_1a_2 + 3c_1a_3 + 4c_1b_3 - 5a_1)v_2 - c_1^2a_3 + c_1a_1 + 2c_1b_1 = 0 \end{aligned} \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_2 + 4a_3 + 4b_2 + 8b_3 &= 0 \\ -2a_2 - 4a_3 + 6b_2 + 2b_3 &= 0 \\ 2a_2 - 6a_3 + 4b_2 - 2b_3 &= 0 \\ -c_1^2a_3 + c_1a_1 + 2c_1b_1 &= 0 \\ -2c_1a_2 + 3c_1a_3 + 4c_1b_3 - 5a_1 &= 0 \\ -4c_1a_3 + 2c_1b_2 + c_1b_3 + 5b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= \frac{1}{5}c_1b_2 + \frac{2}{5}c_1b_3 \\ a_2 &= b_2 + b_3 \\ a_3 &= b_2 \\ b_1 &= \frac{2}{5}c_1b_2 - \frac{1}{5}c_1b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x + \frac{2c_1}{5} \\ \eta &= y - \frac{c_1}{5} \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y) \xi \\ &= y - \frac{c_1}{5} - \left(-\frac{y - c_1 - 2x}{2y + x} \right) \left(x + \frac{2c_1}{5} \right) \\ &= \frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{10y + 5x} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \quad (1)$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{10y + 5x}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln(-c_1^2 - 5c_1x - 5x^2 + 5xy + 5y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - c_1 - 2x}{2y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-5y + 5c_1 + 10x}{10x^2 + (-10y + 10c_1)x - 10y^2 + 2c_1^2} \\ S_y &= \frac{-10y - 5x}{10x^2 + (-10y + 10c_1)x - 10y^2 + 2c_1^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Summary

The solution(s) found are the following

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1 \quad (1)$$

Verification of solutions

$$\frac{\ln(-5x^2 + (5y - 5c_1)x + 5y^2 - c_1^2)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying 2nd order Liouville  
trying 2nd order WeierstrassP  
trying 2nd order JacobiSN  
differential order: 2; trying a linearization to 3rd order  
trying 2nd order ODE linearizable_by_differentiation  
trying 2nd order, 2 integrating factors of the form mu(x,y)  
trying a quadrature  
<- quadrature successful  
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve((x+2*y(x))*diff(y(x),x$2)+2*diff(y(x),x)^2+2*diff(y(x),x)=2,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} - \frac{\sqrt{-4c_1x + 5x^2 + 4c_2}}{2}$$
$$y(x) = -\frac{x}{2} + \frac{\sqrt{-4c_1x + 5x^2 + 4c_2}}{2}$$

✓ Solution by Mathematica

Time used: 0.645 (sec). Leaf size: 77

```
DSolve[(x+2*y[x])*y'[x]+2*y'[x]^2+2*y'[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2}x \left(1 + \sqrt{\frac{1}{x^2} \sqrt{5x^2 + 4c_2x + 4c_1}} \right)$$
$$y(x) \rightarrow \frac{1}{2}x \left(-1 + \sqrt{\frac{1}{x^2} \sqrt{5x^2 + 4c_2x + 4c_1}} \right)$$

14.12 problem 33

Internal problem ID [5440]

Internal file name [OUTPUT/4931_Tuesday_February_06_2024_10_14_25_PM_21281142/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 33.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[[_3rd_order , _exact , _nonlinear]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*(1+2*_b(_a)+3*_b(_a)^2)+6*_b(
  Methods for second order ODEs:
  --- Trying classification methods ---
  trying 2nd order Liouville
  trying 2nd order WeierstrassP
  trying 2nd order JacobiSN
  differential order: 2; trying a linearization to 3rd order
  trying 2nd order ODE linearizable_by_differentiation
  trying 2nd order, 2 integrating factors of the form mu(x,y)
  trying a quadrature
  <- quadrature successful
  <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 1110

`dsolve((1+2*y(x)+3*y(x)^2)*diff(y(x),x$3)+6*diff(y(x),x)*(diff(y(x),x$2)+diff(y(x),x)^2+3*y(x)^2),x$3))`

$y(x)$

$$= \frac{\left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2}\right)}{3 \left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2}\right) - \frac{1}{3}}$$

$y(x) =$

$$\frac{(1 + i\sqrt{3}) \left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 - 432c_2x^5 + (1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2)}\right)}{3 \left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2}\right) - \frac{1}{3}}$$

$y(x)$

$$= \frac{(i\sqrt{3} - 1) \left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 - 432c_2x^5 + (1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2)}\right)}{3 \left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3x^2}\right) - \frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 523

`DSolve[(1+2*y[x]+3*y[x]^2)*y''[x]+6*y'[x]*(y''[x]+y'[x]^2+3*y[x]*y''[x])]==x,y[x],x,Include`

$$y(x) \rightarrow \frac{2^{2/3} \left(9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2 \right)^{2/3} - 4}{12 \sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2}}$$

$$y(x) \rightarrow \frac{1}{24} \left(i 2^{2/3} (\sqrt{3} + i) \sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2} + \frac{16 \sqrt[3]{2} (1 + i\sqrt{3})}{\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2}} - 8 \right)$$

$$y(x) \rightarrow \frac{1}{24} \left(-2^{2/3} (1 + i\sqrt{3}) \sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2} + \frac{16 \sqrt[3]{2} (1 - i\sqrt{3})}{\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2}} - 8 \right)$$

14.13 problem 34

Internal problem ID [5441]

Internal file name [OUTPUT/4932_Tuesday_February_06_2024_10_14_25_PM_15270978/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 34.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[[_3rd_order , _exact , _nonlinear] , [_3rd_order ,  
_with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*_b(_a)^2*_a+2*_a*_b(_a)*(diff
  Methods for second order ODEs:
  --- Trying classification methods ---
  trying 2nd order Liouville
  trying 2nd order WeierstrassP
  trying 2nd order JacobiSN
  differential order: 2; trying a linearization to 3rd order
  trying 2nd order ODE linearizable_by_differentiation
  trying 2nd order, 2 integrating factors of the form mu(x,y)
  trying a quadrature
  checking if the LODE has constant coefficients
  checking if the LODE is of Euler type
  <- LODE of Euler type successful
  <- 2nd order, 2 integrating factors of the form mu(x,y) successful
  <- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 104

```
dsolve(3*x*( y(x)^2* diff(y(x),x$3)+6*y(x)*diff(y(x),x)*diff(y(x),x$2)+2*diff(y(x),x)^3 )-
```

$$y(x) = \frac{(8 \ln(x) x - 8c_3 x^3 + (12c_1 - 4) x + 8c_2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{(8 \ln(x) x - 8c_3 x^3 + (12c_1 - 4) x + 8c_2)^{\frac{1}{3}} (1 + i\sqrt{3})}{4}$$
$$y(x) = \frac{(8 \ln(x) x - 8c_3 x^3 + (12c_1 - 4) x + 8c_2)^{\frac{1}{3}} (i\sqrt{3} - 1)}{4}$$

✓ Solution by Mathematica

Time used: 0.297 (sec). Leaf size: 121

```
DSolve[3*x*( y[x]^2* y'''[x]+6*y[x]*y'[x]*y''[x]+2*y'[x]^3 )-3*y[x]*(y[x]*y''[x]+2*y'[x]
```

$$y(x) \rightarrow -\sqrt[3]{-\frac{1}{6}\sqrt[3]{6c_3x^3 + 6x \log(x) + (3 + 9c_2)x + 2c_1}}$$

$$y(x) \rightarrow \sqrt[3]{c_3x^3 + x \log(x) + \frac{1}{2}(1 + 3c_2)x + \frac{c_1}{3}}$$

$$y(x) \rightarrow (-1)^{2/3}\sqrt[3]{c_3x^3 + x \log(x) + \frac{1}{2}(1 + 3c_2)x + \frac{c_1}{3}}$$

14.14 problem 35

Internal problem ID [5442]

Internal file name [OUTPUT/4933_Tuesday_February_06_2024_10_14_25_PM_14447332/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 35.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "**unknown**"

Maple gives the following as the ode type

```
[[_3rd_order , _exact , _nonlinear] , [_3rd_order ,  
_with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (1/2)*_b(_a)^2+(diff(_b(_a), _a))^2-2*(diff(_b(_a), _a))*
  Methods for second order ODEs:
  --- Trying classification methods ---
  trying 2nd order Liouville
  trying 2nd order WeierstrassP
  trying 2nd order JacobiSN
  differential order: 2; trying a linearization to 3rd order
  trying 2nd order ODE linearizable_by_differentiation
  trying 2nd order, 2 integrating factors of the form mu(x,y)
  trying a quadrature
  checking if the LODE has constant coefficients
  <- constant coefficients successful
  <- 2nd order, 2 integrating factors of the form mu(x,y) successful
  <- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
dsolve(y(x)*diff(y(x),x$3)+3*diff(y(x),x)*diff(y(x),x$2)-2*y(x)*diff(y(x),x$2)-2*diff(y(x),x
```

$$y(x) = \sqrt{-2c_3x e^x + e^{2x} + 2e^x c_2 - 2c_1}$$
$$y(x) = -\sqrt{e^{2x} + (-2c_3x + 2c_2)e^x - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.387 (sec). Leaf size: 65

```
DSolve[y[x]*y''[x]+3*y'[x]*y''[x]-2*y[x]*y''[x]-2*y'[x]^2+y[x]*y'[x]==Exp[2*x],y[x],x,Inclu
```

$$y(x) \rightarrow -\sqrt{e^{2x} + e^x(c_3x + 2c_2) + 2c_1}$$
$$y(x) \rightarrow \sqrt{e^{2x} + e^x(c_3x + 2c_2) + 2c_1}$$

14.15 problem 36

14.15.1 Solving as second order ode missing x ode 2277

Internal problem ID [5443]

Internal file name [OUTPUT/4934_Tuesday_February_06_2024_10_14_26_PM_76002521/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplementary problems. Page 132

Problem number: 36.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_ode_missing_x"

Maple gives the following as the ode type

```
[[_2nd_order , _missing_x], [_2nd_order , _reducible , _mu_xy]]
```

$$2(1 + y)y'' + 2y'^2 + y^2 + 2y = 0$$

14.15.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned}y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx} \frac{dp}{dy} \\ &= p \frac{dp}{dy}\end{aligned}$$

Hence the ode becomes

$$(2y + 2)p(y) \left(\frac{d}{dy} p(y) \right) + 2p(y)^2 + y(y + 2) = 0$$

Which is now solved as first order ode for $p(y)$. Writing the ode as

$$\frac{d}{dy}p(y) = -\frac{2p^2 + y^2 + 2y}{2(1+y)p}$$

$$\frac{d}{dy}p(y) = \omega(y, p)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_p - \xi_y) - \omega^2 \xi_p - \omega_y \xi - \omega_p \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **Bernoulli**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 256: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(y, p) &= 0 \\ \eta(y, p) &= \frac{1}{p(1+y)^2}\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(y, p) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dy}{\xi} = \frac{dp}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial p}\right) S(y, p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{p(1+y)^2}} dy\end{aligned}$$

Which results in

$$S = \frac{(1+y)^2 p^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, p)S_p}{R_y + \omega(y, p)R_p}\tag{2}$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y, p)$ is the right hand side of the original ode given by

$$\omega(y, p) = -\frac{2p^2 + y^2 + 2y}{2(1+y)p}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_y &= 1 \\R_p &= 0 \\S_y &= (1 + y) p^2 \\S_p &= (1 + y)^2 p\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2}y^3 - \frac{3}{2}y^2 - y \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}R^3 - \frac{3}{2}R^2 - R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{8}R^4 - \frac{1}{2}R^3 - \frac{1}{2}R^2 + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$\frac{p(y)^2 (1 + y)^2}{2} = -\frac{1}{8}y^4 - \frac{1}{2}y^3 - \frac{1}{2}y^2 + c_1$$

Which simplifies to

$$\frac{p(y)^2 (1 + y)^2}{2} = -\frac{1}{8}y^4 - \frac{1}{2}y^3 - \frac{1}{2}y^2 + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2(1 + y)^2}{2} = -\frac{y^4}{8} - \frac{y^3}{2} - \frac{y^2}{2} + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}}{2y + 2} \quad (1)$$

$$y' = -\frac{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}}{2(1 + y)} \quad (2)$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2y + 2}{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}} dy = \int dx$$

$$-i \ln \left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right) = c_2 + x$$

Raising both side to exponential gives

$$e^{-i \ln \left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right)} = e^{c_2 + x}$$

Which simplifies to

$$\left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right)^{-i} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2(1 + y)}{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}} dy = \int dx$$

$$i \ln \left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right) = x + c_4$$

Raising both side to exponential gives

$$e^{i \ln \left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right)} = e^{x + c_4}$$

Which simplifies to

$$\left(iy^2 + 2iy + \sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1} \right)^i = c_5 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{i \left(2i(c_3 e^x)^i - \sqrt{2i(c_3 e^x)^i (c_3 e^x)^{2i} - 16i(c_3 e^x)^i c_1 - 4(c_3 e^x)^{2i}} \right) (c_3 e^x)^{-i}}{2} \quad (1)$$

$$y = \frac{i \left(2i(c_5 e^x)^{-i} - \sqrt{2i(c_5 e^x)^{-i} (c_5 e^x)^{-2i} - 16i(c_5 e^x)^{-i} c_1 - 4(c_5 e^x)^{-2i}} \right) (c_5 e^x)^i}{2} \quad (2)$$

Verification of solutions

$$y = \frac{i \left(2i(c_3 e^x)^i - \sqrt{2i(c_3 e^x)^i (c_3 e^x)^{2i} - 16i(c_3 e^x)^i c_1 - 4(c_3 e^x)^{2i}} \right) (c_3 e^x)^{-i}}{2}$$

Verified OK.

$$y = \frac{i \left(2i(c_5 e^x)^{-i} - \sqrt{2i(c_5 e^x)^{-i} (c_5 e^x)^{-2i} - 16i(c_5 e^x)^{-i} c_1 - 4(c_5 e^x)^{-2i}} \right) (c_5 e^x)^i}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying 2nd order Liouville  
trying 2nd order WeierstrassP  
trying 2nd order JacobiSN  
differential order: 2; trying a linearization to 3rd order  
trying 2nd order ODE linearizable_by_differentiation  
trying 2nd order, 2 integrating factors of the form mu(x,y)  
trying a quadrature  
checking if the LODE has constant coefficients  
<- constant coefficients successful  
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
dsolve(2*(y(x)+1)*diff(y(x),x$2)+2*diff(y(x),x)^2+y(x)^2+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -1 - \sqrt{1 + 2 \cos(x) c_2 - 2c_1 \sin(x)}$$

$$y(x) = -1 + \sqrt{1 + 2 \cos(x) c_2 - 2c_1 \sin(x)}$$

✓ Solution by Mathematica

Time used: 24.469 (sec). Leaf size: 5629

```
DSolve[2*(y[x]+1)*y'[x]+2*y'[x]^2+y[x]^2+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

15 Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

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15.2 problem 11	2296
15.3 problem 12	2307
15.4 problem 13	2319
15.5 problem 17	2322

15.1 problem 10

- 15.1.1 Solution using Matrix exponential method 2284
- 15.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2286
- 15.1.3 Maple step by step solution 2291

Internal problem ID [5444]

Internal file name [OUTPUT/4935_Tuesday_February_06_2024_10_14_27_PM_74109112/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

Problem number: 10.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**system of linear ODEs**"

Solve

$$\begin{aligned}x'(t) &= -x(t) + e^{2t} - e^t \\y'(t) &= -x(t) + y(t) + e^{2t}\end{aligned}$$

15.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} already. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A , the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}c_1 \\ \left(-\frac{e^t}{2} + \frac{e^{-t}}{2}\right)c_1 + e^tc_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}c_1 \\ \frac{e^{-t}c_1}{2} - \frac{e^t(c_1-2c_2)}{2} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^t & 0 \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & e^{-t} \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \int \begin{bmatrix} e^t & 0 \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix} dt \\ &= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \begin{bmatrix} \frac{e^{3t}}{3} - \frac{e^{2t}}{2} \\ \frac{t}{2} + \frac{e^{3t}}{6} - \frac{e^{2t}}{4} + \frac{e^t}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} \\ \frac{2e^{2t}}{3} + \frac{(2t-1)e^t}{4} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} e^{-t}c_1 + \frac{e^{2t}}{3} - \frac{e^t}{2} \\ \frac{e^{-t}c_1}{2} + \frac{2e^{2t}}{3} + \frac{(-1+2t-2c_1+4c_2)e^t}{4} \end{bmatrix}\end{aligned}$$

15.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A . This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1 - \lambda & 0 \\ -1 & 1 - \lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1 - \lambda)(1 - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$

$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 2 & 0 \end{array} \right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m , and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m , and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^t \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where \vec{x}_i are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{e^t}{2} & 0 \\ -\frac{e^{-t}}{2} & e^{-t} \end{bmatrix}$$

Hence

$$\begin{aligned}
 \vec{x}_p(t) &= \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix} \int \begin{bmatrix} \frac{e^t}{2} & 0 \\ -\frac{e^{-t}}{2} & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix} dt \\
 &= \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix} \int \begin{bmatrix} \frac{e^{2t}(e^t-1)}{2} \\ \frac{1}{2} + \frac{e^t}{2} \end{bmatrix} dt \\
 &= \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix} \begin{bmatrix} \frac{e^{3t}}{6} - \frac{e^{2t}}{4} \\ \frac{t}{2} + \frac{e^t}{2} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} \\ \frac{2e^{2t}}{3} + \frac{(2t-1)e^t}{4} \end{bmatrix}
 \end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
 \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 2c_1e^{-t} \\ c_1e^{-t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2e^t \end{bmatrix} + \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} \\ \frac{2e^{2t}}{3} + \frac{(2t-1)e^t}{4} \end{bmatrix}
 \end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1e^{-t} + \frac{e^{2t}}{3} - \frac{e^t}{2} \\ c_1e^{-t} + \frac{2e^{2t}}{3} + \frac{(-1+2t+4c_2)e^t}{4} \end{bmatrix}$$

15.1.3 Maple step by step solution

Let's solve

$$\left[x'(t) = -x(t) + (e^t)^2 - e^t, y'(t) = -x(t) + y(t) + (e^t)^2 \right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right], \left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p(t)$

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \vec{x}_p(t)$$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(t) = \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 2e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(t)$

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} + \frac{e^{-t}}{6} \\ \frac{e^{-t}}{12} + \frac{2e^{2t}}{3} + \frac{(2t-3)e^t}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} + \frac{e^{-t}}{6} \\ \frac{e^{-t}}{12} + \frac{2e^{2t}}{3} + \frac{(2t-3)e^t}{4} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t}c_1 + \frac{e^{2t}}{3} - \frac{e^t}{2} + \frac{e^{-t}}{6} \\ \frac{(12c_1+1)e^{-t}}{12} + \frac{2e^{2t}}{3} + \frac{(-3+2t+4c_2)e^t}{4} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ x(t) = 2e^{-t}c_1 + \frac{e^{2t}}{3} - \frac{e^t}{2} + \frac{e^{-t}}{6}, y(t) = \frac{(12c_1+1)e^{-t}}{12} + \frac{2e^{2t}}{3} + \frac{(-3+2t+4c_2)e^t}{4} \right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)-diff(y(t),t)+y(t)=-exp(t),x(t)+diff(y(t),t)-y(t)=exp(2*t)],singsol=all)
```

$$x(t) = -\frac{e^t}{2} + \frac{e^{2t}}{3} + c_2 e^{-t}$$

$$y(t) = \frac{c_2 e^{-t}}{2} + \frac{2e^{2t}}{3} + c_1 e^t + \frac{e^t t}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 72

```
DSolve[{x'[t]-y'[t]+y[t]==-Exp[t],x[t]+y'[t]-y[t]==Exp[2*t]},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \rightarrow \frac{1}{6}e^t(2e^t - 3) + c_1e^{-t}$$
$$y(t) \rightarrow \frac{2e^{2t}}{3} + \frac{c_1e^{-t}}{2} + \frac{1}{4}e^t(2t - 1 - 2c_1 + 4c_2)$$

15.2 problem 11

15.2.1 Solution using Matrix exponential method	2296
15.2.2 Solution using explicit Eigenvalue and Eigenvector method . . .	2298
15.2.3 Maple step by step solution	2303

Internal problem ID [5445]

Internal file name [OUTPUT/4936_Tuesday_February_06_2024_10_14_28_PM_79830252/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

Problem number: 11.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**system of linear ODEs**"

Solve

$$\begin{aligned}x'(t) &= 3x(t) - t^2 + 2y(t) + t \\y'(t) &= t^2 - 3y(t) - 5x(t)\end{aligned}$$

15.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} already. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A , the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 3 \sin(t) & 2 \sin(t) \\ -5 \sin(t) & \cos(t) - 3 \sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At} \vec{c} \\ &= \begin{bmatrix} \cos(t) + 3 \sin(t) & 2 \sin(t) \\ -5 \sin(t) & \cos(t) - 3 \sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t) + 3 \sin(t)) c_1 + 2 \sin(t) c_2 \\ -5 \sin(t) c_1 + (\cos(t) - 3 \sin(t)) c_2 \end{bmatrix} \\ &= \begin{bmatrix} (3c_1 + 2c_2) \sin(t) + c_1 \cos(t) \\ (-5c_1 - 3c_2) \sin(t) + c_2 \cos(t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) - 3 \sin(t) & -2 \sin(t) \\ 5 \sin(t) & \cos(t) + 3 \sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) + 3 \sin(t) & 2 \sin(t) \\ -5 \sin(t) & \cos(t) - 3 \sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - 3 \sin(t) & -2 \sin(t) \\ 5 \sin(t) & \cos(t) + 3 \sin(t) \end{bmatrix} \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix} dt \\ &= \begin{bmatrix} \cos(t) + 3 \sin(t) & 2 \sin(t) \\ -5 \sin(t) & \cos(t) - 3 \sin(t) \end{bmatrix} \begin{bmatrix} (-t^2 + t + 3) \cos(t) - \sin(t) (t^2 - 3t + 1) \\ (2t^2 - 3t - 4) \cos(t) + \sin(t) (t - 1) (t - 3) \end{bmatrix} \\ &= \begin{bmatrix} -t^2 + t + 3 \\ 2t^2 - 3t - 4 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} (3c_1 + 2c_2) \sin(t) + c_1 \cos(t) - t^2 + t + 3 \\ (-5c_1 - 3c_2) \sin(t) + c_2 \cos(t) + 2t^2 - 3t - 4 \end{bmatrix}\end{aligned}$$

15.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A . This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 3 - \lambda & 2 \\ -5 & -3 - \lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$

$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (-i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3+i & 2 \\ -5 & -3+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} 3+i & 2 & 0 \\ -5 & -3+i & 0 \end{array} \right]$$

$$R_2 = R_2 + \left(\frac{3}{2} - \frac{i}{2} \right) R_1 \implies \left[\begin{array}{cc|c} 3+i & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-\frac{3}{5} + \frac{i}{5})t\}$

Hence the solution is

$$\begin{bmatrix} (-\frac{3}{5} + \frac{i}{5})t \\ t \end{bmatrix} = \begin{bmatrix} (-\frac{3}{5} + \frac{i}{5})t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-\frac{3}{5} + \frac{i}{5})t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-\frac{3}{5} + \frac{i}{5}) \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} (-\frac{3}{5} + \frac{i}{5}) \\ 1 \end{bmatrix} = \begin{bmatrix} -3 + i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-i & 2 \\ -5 & -3-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} 3-i & 2 & 0 \\ -5 & -3-i & 0 \end{array} \right]$$

$$R_2 = R_2 + \left(\frac{3}{2} + \frac{i}{2}\right) R_1 \implies \left[\begin{array}{cc|c} 3-i & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-i & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-\frac{3}{5} - \frac{i}{5}) t\}$

Hence the solution is

$$\begin{bmatrix} (-\frac{3}{5} - \frac{i}{5}) t \\ t \end{bmatrix} = \begin{bmatrix} (-\frac{3}{5} - \frac{i}{5}) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-\frac{3}{5} - \frac{i}{5}) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-\frac{3}{5} - \frac{i}{5}) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} (-\frac{3}{5} - \frac{i}{5}) t \\ t \end{bmatrix} = \begin{bmatrix} -3 - i \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m , and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m , and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
i	1	1	No	$\begin{bmatrix} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{bmatrix}$
$-i$	1	1	No	$\begin{bmatrix} -\frac{3}{5} + \frac{i}{5} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \dots \end{bmatrix}$$

Where \vec{x}_i are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{5ie^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5ie^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{5ie^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5ie^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix} dt \\ &= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -e^{-it} \left(-\frac{5i}{2} + \left(-\frac{1}{2} + i\right) t\right) t \\ \frac{e^{it} t(2it+t-5i)}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{(-t^2+(2+i)t+1-2i)e^{-it}(-5+(2+i)t)}{2t-4+2i} \\ \frac{e^{it}(t^2+(-2+i)t-1-2i)(5+(-2+i)t)}{-2t+4+2i} \end{bmatrix} \\ &= \begin{bmatrix} \frac{t^4-5t^3+6t^2+7t-15}{(-t+2+i)(t-2+i)} \\ \frac{-2t^4+11t^3-18t^2-t+20}{(-t+2+i)(t-2+i)} \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) c_1 e^{it} \\ c_1 e^{it} \end{bmatrix} + \begin{bmatrix} \left(-\frac{3}{5} + \frac{i}{5}\right) c_2 e^{-it} \\ c_2 e^{-it} \end{bmatrix} + \begin{bmatrix} \frac{t^4 - 5t^3 + 6t^2 + 7t - 15}{(-t+2+i)(t-2+i)} \\ \frac{-2t^4 + 11t^3 - 18t^2 - t + 20}{(-t+2+i)(t-2+i)} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(t^2 - 4t + 5)((3-i)c_2 e^{-it} + (3+i)c_1 e^{it} + 5t^2 - 5t - 15)}{5(-t+2+i)(t-2+i)} \\ -\frac{(t^2 - 4t + 5)(c_2 e^{-it} + c_1 e^{it} + 2t^2 - 3t - 4)}{(-t+2+i)(t-2+i)} \end{bmatrix}$$

15.2.3 Maple step by step solution

Let's solve

$$[x'(t) = 3x(t) - t^2 + 2y(t) + t, y'(t) = t^2 - 3y(t) - 5x(t)]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-I, \begin{bmatrix} -\frac{3}{5} + \frac{I}{5} \\ 1 \end{bmatrix} \right], \left[I, \begin{bmatrix} -\frac{3}{5} - \frac{I}{5} \\ 1 \end{bmatrix} \right] \right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-I, \begin{bmatrix} -\frac{3}{5} + \frac{I}{5} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{-It} \cdot \begin{bmatrix} -\frac{3}{5} + \frac{I}{5} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(t) - I \sin(t)) \cdot \begin{bmatrix} -\frac{3}{5} + \frac{I}{5} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} \left(-\frac{3}{5} + \frac{I}{5}\right) (\cos(t) - I \sin(t)) \\ \cos(t) - I \sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -\frac{3 \cos(t)}{5} + \frac{\sin(t)}{5} \\ \cos(t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \frac{3 \sin(t)}{5} + \frac{\cos(t)}{5} \\ -\sin(t) \end{bmatrix} \right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p(t)$

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \vec{x}_p(t)$$

- Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(t) = \begin{bmatrix} -\frac{3 \cos(t)}{5} + \frac{\sin(t)}{5} & \frac{3 \sin(t)}{5} + \frac{\cos(t)}{5} \\ \cos(t) & -\sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} -\frac{3\cos(t)}{5} + \frac{\sin(t)}{5} & \frac{3\sin(t)}{5} + \frac{\cos(t)}{5} \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix}$$

- Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(t)$

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -t^2 - 3 \cos(t) - \sin(t) + t + 3 \\ 2t^2 + 4 \cos(t) + 3 \sin(t) - 3t - 4 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \begin{bmatrix} -t^2 - 3 \cos(t) - \sin(t) + t + 3 \\ 2t^2 + 4 \cos(t) + 3 \sin(t) - 3t - 4 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(-3c_1 + c_2 - 15) \cos(t)}{5} + \frac{(c_1 + 3c_2 - 5) \sin(t)}{5} - t^2 + t + 3 \\ (4 + c_1) \cos(t) + (-c_2 + 3) \sin(t) + 2t^2 - 3t - 4 \end{bmatrix}$$

- Solution to the system of ODEs

$$\begin{cases} x(t) = \frac{(-3c_1 + c_2 - 15) \cos(t)}{5} + \frac{(c_1 + 3c_2 - 5) \sin(t)}{5} - t^2 + t + 3, \\ y(t) = (4 + c_1) \cos(t) + (-c_2 + 3) \sin(t) - 4 \end{cases}$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
dsolve([diff(x(t),t)+2*x(t)+diff(y(t),t)+y(t)=t,5*x(t)+diff(y(t),t)+3*y(t)=t^2],singsol=all)
```

$$\begin{aligned} x(t) &= c_2 \sin(t) + c_1 \cos(t) - t^2 + t + 3 \\ y(t) &= 2t^2 - \frac{3c_2 \sin(t)}{2} - \frac{3c_1 \cos(t)}{2} - 3t - 4 + \frac{c_2 \cos(t)}{2} - \frac{c_1 \sin(t)}{2} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 61

```
DSolve[{x'[t]+2*x[t]+y'[t]+y[t]==t,5*x[t]+y'[t]+3*y[t]==t^2},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$\begin{aligned} x(t) &\rightarrow -t^2 + t + c_1 \cos(t) + (3c_1 + 2c_2) \sin(t) + 3 \\ y(t) &\rightarrow 2t^2 - 3t + c_2 \cos(t) - (5c_1 + 3c_2) \sin(t) - 4 \end{aligned}$$

15.3 problem 12

15.3.1 Solution using Matrix exponential method	2307
15.3.2 Solution using explicit Eigenvalue and Eigenvector method . . .	2309
15.3.3 Maple step by step solution	2314

Internal problem ID [5446]

Internal file name [OUTPUT/4937_Tuesday_February_06_2024_10_14_29_PM_15147259/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

Problem number: 12.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**system of linear ODEs**"

Solve

$$\begin{aligned}x'(t) &= 4 - 5x(t) - y(t) - e^t \\y'(t) &= 2x(t) - 3y(t) + e^t - 1\end{aligned}$$

15.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} already. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A , the matrix exponential can be found to be

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{-4t} \cos(t) - e^{-4t} \sin(t) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t} \cos(t) + e^{-4t} \sin(t) \end{bmatrix} \\ &= \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At} \vec{c} \\ &= \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t))c_1 - e^{-4t} \sin(t)c_2 \\ 2e^{-4t} \sin(t)c_1 + e^{-4t}(\sin(t) + \cos(t))c_2 \end{bmatrix} \\ &= \begin{bmatrix} ((-c_2 - c_1) \sin(t) + c_1 \cos(t)) e^{-4t} \\ ((2c_1 + c_2) \sin(t) + c_2 \cos(t)) e^{-4t} \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{4t}(\sin(t) + \cos(t)) & e^{4t} \sin(t) \\ -2e^{4t} \sin(t) & e^{4t}(\cos(t) - \sin(t)) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \int \begin{bmatrix} e^{4t}(\sin(t) + \cos(t)) & e^{4t} \sin(t) \\ -2e^{4t} \sin(t) & e^{4t}(\cos(t) - \sin(t)) \end{bmatrix} \\ &= \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} \frac{(16 \sin(t) + 13 \cos(t))e^{4t}}{17} - \frac{5(\cos(t) + \frac{\sin(t)}{5})e^{5t}}{26} \\ \frac{(-29 \sin(t) + 3 \cos(t))e^{4t}}{17} + \frac{2(\cos(t) + \frac{3 \sin(t)}{2})e^{5t}}{13} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5e^t}{26} + \frac{13}{17} \\ \frac{2e^t}{13} + \frac{3}{17} \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$\begin{aligned}\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \left[\begin{array}{c} \left(\frac{13e^{4t}}{17} - \frac{5e^{5t}}{26} + (-c_2 - c_1) \sin(t) + c_1 \cos(t) \right) e^{-4t} \\ \frac{(39e^{4t} + 34e^{5t} + 221(2c_1 + c_2) \sin(t) + 221c_2 \cos(t)) e^{-4t}}{221} \end{array} \right]\end{aligned}$$

15.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A . This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det \left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det \left(\begin{bmatrix} -5 - \lambda & -1 \\ 2 & -3 - \lambda \end{bmatrix} \right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 17 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4 + i$$

$$\lambda_2 = -4 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-4 - i$	1	complex eigenvalue
$-4 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - (-4 - i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 + i & -1 \\ 2 & 1 + i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} -1 + i & -1 & 0 \\ 2 & 1 + i & 0 \end{array} \right]$$

$$R_2 = R_2 + (1 + i)R_1 \implies \left[\begin{array}{cc|c} -1 + i & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 + i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-\frac{1}{2} - \frac{i}{2})t\}$

Hence the solution is

$$\begin{bmatrix} (-\frac{1}{2} - \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2} - \frac{i}{2})t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-\frac{1}{2} - \frac{i}{2})t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-\frac{1}{2} - \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} (-\frac{1}{2} - \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -4 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - (-4 + i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i & -1 \\ 2 & 1 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - i & -1 & 0 \\ 2 & 1 - i & 0 \end{array} \right]$$

$$R_2 = R_2 + (1 - i)R_1 \implies \left[\begin{array}{cc|c} -1 - i & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 - i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = (-\frac{1}{2} + \frac{i}{2})t\}$

Hence the solution is

$$\begin{bmatrix} (-\frac{1}{2} + \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} (-\frac{1}{2} + \frac{i}{2})t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} (-\frac{1}{2} + \frac{i}{2})t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} (-\frac{1}{2} + \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} (-\frac{1}{2} + \frac{i}{2})t \\ t \end{bmatrix} = \begin{bmatrix} -1 + i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m , and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m , and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-4 + i$	1	1	No	$\begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$
$-4 - i$	1	1	No	$\begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-4+i)t} \\ e^{(-4+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-4-i)t} \\ e^{(-4-i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \dots \end{bmatrix}$$

Where \vec{x}_i are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-4+i)t} & \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-4-i)t} \\ e^{(-4+i)t} & e^{(-4-i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -ie^{(4-i)t} & \left(\frac{1}{2} - \frac{i}{2}\right) e^{(4-i)t} \\ ie^{(4+i)t} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{(4+i)t} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-4+i)t} & \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-4-i)t} \\ e^{(-4+i)t} & e^{(-4-i)t} \end{bmatrix} \int \begin{bmatrix} -ie^{(4-i)t} & \left(\frac{1}{2} - \frac{i}{2}\right) e^{(4-i)t} \\ ie^{(4+i)t} & \left(\frac{1}{2} + \frac{i}{2}\right) e^{(4+i)t} \end{bmatrix} \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-4+i)t} & \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-4-i)t} \\ e^{(-4+i)t} & e^{(-4-i)t} \end{bmatrix} \int \begin{bmatrix} \left(-\frac{1}{2} - \frac{7i}{2}\right) e^{(4-i)t} + \left(\frac{1}{2} + \frac{i}{2}\right) e^{(5-i)t} \\ \left(-\frac{1}{2} + \frac{7i}{2}\right) e^{(4+i)t} + \left(\frac{1}{2} - \frac{i}{2}\right) e^{(5+i)t} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) e^{(-4+i)t} & \left(-\frac{1}{2} - \frac{i}{2}\right) e^{(-4-i)t} \\ e^{(-4+i)t} & e^{(-4-i)t} \end{bmatrix} \begin{bmatrix} \left(\frac{3}{34} - \frac{29i}{34}\right) e^{(4-i)t} + \left(\frac{1}{13} + \frac{3i}{26}\right) e^{(5-i)t} \\ \left(\frac{3}{34} + \frac{29i}{34}\right) e^{(4+i)t} + \left(\frac{1}{13} - \frac{3i}{26}\right) e^{(5+i)t} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5e^t}{26} + \frac{13}{17} \\ \frac{2e^t}{13} + \frac{3}{17} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) c_1 e^{(-4+i)t} \\ c_1 e^{(-4+i)t} \end{bmatrix} + \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right) c_2 e^{(-4-i)t} \\ c_2 e^{(-4-i)t} \end{bmatrix} + \begin{bmatrix} -\frac{5e^t}{26} + \frac{13}{17} \\ \frac{2e^t}{13} + \frac{3}{17} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) c_1 e^{(-4+i)t} + \left(-\frac{1}{2} - \frac{i}{2}\right) c_2 e^{(-4-i)t} - \frac{5e^t}{26} + \frac{13}{17} \\ c_1 e^{(-4+i)t} + c_2 e^{(-4-i)t} + \frac{2e^t}{13} + \frac{3}{17} \end{bmatrix}$$

15.3.3 Maple step by step solution

Let's solve

$$[x'(t) = 4 - 5x(t) - y(t) - e^t, y'(t) = 2x(t) - 3y(t) + e^t - 1]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-4 - I, \begin{bmatrix} -\frac{1}{2} - \frac{I}{2} \\ 1 \end{bmatrix} \right], \left[-4 + I, \begin{bmatrix} -\frac{1}{2} + \frac{I}{2} \\ 1 \end{bmatrix} \right] \right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-4 - I, \begin{bmatrix} -\frac{1}{2} - \frac{I}{2} \\ 1 \end{bmatrix} \right]$$

- Solution from eigenpair

$$e^{(-4-I)t} \cdot \begin{bmatrix} -\frac{1}{2} - \frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-4t} \cdot (\cos(t) - I \sin(t)) \cdot \begin{bmatrix} -\frac{1}{2} - \frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-4t} \cdot \begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right) (\cos(t) - I \sin(t)) \\ \cos(t) - I \sin(t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} = e^{-4t} \cdot \begin{bmatrix} -\frac{\cos(t)}{2} - \frac{\sin(t)}{2} \\ \cos(t) \end{bmatrix}, \begin{bmatrix} \vec{x}_2(t) \\ \vec{x}_1(t) \end{bmatrix} = e^{-4t} \cdot \begin{bmatrix} \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ -\sin(t) \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p(t)$
 $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \vec{x}_p(t)$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(t) = \begin{bmatrix} e^{-4t} \left(-\frac{\cos(t)}{2} - \frac{\sin(t)}{2} \right) & e^{-4t} \left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2} \right) \\ e^{-4t} \cos(t) & -e^{-4t} \sin(t) \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} e^{-4t} \left(-\frac{\cos(t)}{2} - \frac{\sin(t)}{2} \right) & e^{-4t} \left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2} \right) \\ e^{-4t} \cos(t) & -e^{-4t} \sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t} \sin(t) \\ 2e^{-4t} \sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(t)$

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{e^{-4t}(338e^{4t} - 85e^{5t} + 399\sin(t) - 253\cos(t))}{442} \\ \frac{e^{-4t}(39e^{4t} + 34e^{5t} - 326\sin(t) - 73\cos(t))}{221} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \begin{bmatrix} \frac{e^{-4t}(338e^{4t} - 85e^{5t} + 399\sin(t) - 253\cos(t))}{442} \\ \frac{e^{-4t}(39e^{4t} + 34e^{5t} - 326\sin(t) - 73\cos(t))}{221} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(-\frac{26e^{4t}}{17} + \frac{5e^{5t}}{13} + (c_1 + c_2 + \frac{253}{221})\cos(t) + (c_1 - c_2 - \frac{399}{221})\sin(t)\right)e^{-4t}}{2} \\ \frac{(39e^{4t} + 34e^{5t} + (221c_1 - 73)\cos(t) + (-221c_2 - 326)\sin(t))e^{-4t}}{221} \end{bmatrix}$$

- Solution to the system of ODEs

$$\begin{cases} x(t) = -\frac{\left(-\frac{26e^{4t}}{17} + \frac{5e^{5t}}{13} + (c_1 + c_2 + \frac{253}{221})\cos(t) + (c_1 - c_2 - \frac{399}{221})\sin(t)\right)e^{-4t}}{2}, & y(t) = \frac{(39e^{4t} + 34e^{5t} + (221c_1 - 73)\cos(t) + (-221c_2 - 326)\sin(t))e^{-4t}}{221} \end{cases}$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 71

```
dsolve([diff(x(t),t)+x(t)+2*diff(y(t),t)+7*y(t)=exp(t)+2,-2*x(t)+diff(y(t),t)+3*y(t)=exp(t)-
```

$$x(t) = e^{-4t} \sin(t) c_2 + e^{-4t} \cos(t) c_1 + \frac{13}{17} - \frac{5e^t}{26}$$

$$y(t) = -e^{-4t} \sin(t) c_2 - e^{-4t} \cos(t) c_1 + e^{-4t} \sin(t) c_1 + \frac{2e^t}{13} + \frac{3}{17}$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 79

```
DSolve[{x'[t]+x[t]+2*y'[t]+7*y[t]==Exp[t]+2,-2*x[t]+y'[t]+3*y[t]==Exp[t]-1},{x[t],y[t]},t,In
```

$$x(t) \rightarrow -\frac{5e^t}{26} + c_1 e^{-4t} \cos(t) - (c_1 + c_2) e^{-4t} \sin(t) + \frac{13}{17}$$
$$y(t) \rightarrow \frac{2e^t}{13} + c_2 e^{-4t} \cos(t) + (2c_1 + c_2) e^{-4t} \sin(t) + \frac{3}{17}$$

15.4 problem 13

Internal problem ID [5447]

Internal file name [OUTPUT/4938_Tuesday_February_06_2024_10_14_29_PM_76667389/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

Problem number: 13.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned}x'(t) + y'(t) &= x(t) - 3y(t) - 1 + e^{-t} \\x'(t) + y'(t) &= -2x(t) - 3y(t) + e^{2t} + 1\end{aligned}$$

The system is

$$x'(t) + y'(t) = x(t) - 3y(t) - 1 + e^{-t} \quad (1)$$

$$x'(t) + y'(t) = -2x(t) - 3y(t) + e^{2t} + 1 \quad (2)$$

Since the left side is the same, this implies

$$\begin{aligned}x(t) - 3y(t) - 1 + e^{-t} &= -2x(t) - 3y(t) + e^{2t} + 1 \\x(t) &= \frac{(e^{3t} + 2e^t - 1)e^{-t}}{3}\end{aligned} \quad (3)$$

Taking derivative of the above w.r.t. t gives

$$x'(t) = \frac{(3e^{3t} + 2e^t)e^{-t}}{3} - \frac{(e^{3t} + 2e^t - 1)e^{-t}}{3} \quad (4)$$

Substituting (3,4) in (1) to eliminate $x(t)$, $x'(t)$ gives

$$\begin{aligned}\frac{(3e^{3t} + 2e^t)e^{-t}}{3} - \frac{(e^{3t} + 2e^t - 1)e^{-t}}{3} + y'(t) &= \frac{(e^{3t} + 2e^t - 1)e^{-t}}{3} - 3y(t) - 1 + e^{-t} \\x'(t) &= -\frac{(e^{3t} + 9e^ty(t) + e^t - 1)e^{-t}}{3}\end{aligned} \quad (5)$$

Which is now solved for $y(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'(t) + p(t)y(t) = q(t)$$

Where here

$$p(t) = 3$$
$$q(t) = -\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3}$$

Hence the ode is

$$y'(t) + 3y(t) = -\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3}$$

The integrating factor μ is

$$\mu = e^{\int 3dt}$$
$$= e^{3t}$$

The ode becomes

$$\frac{d}{dt}(\mu y) = (\mu) \left(-\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3} \right)$$
$$\frac{d}{dt}(e^{3t}y) = (e^{3t}) \left(-\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3} \right)$$
$$d(e^{3t}y) = \left(-\frac{e^{5t}}{3} - \frac{e^{3t}}{3} + \frac{e^{2t}}{3} \right) dt$$

Integrating gives

$$e^{3t}y = \int -\frac{e^{5t}}{3} - \frac{e^{3t}}{3} + \frac{e^{2t}}{3} dt$$
$$e^{3t}y = -\frac{e^{3t}}{9} - \frac{e^{5t}}{15} + \frac{e^{2t}}{6} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$y(t) = e^{-3t} \left(-\frac{e^{3t}}{9} - \frac{e^{5t}}{15} + \frac{e^{2t}}{6} \right) + c_1 e^{-3t}$$

which simplifies to

$$y(t) = \frac{(-6e^{5t} - 10e^{3t} + 15e^{2t} + 90c_1)e^{-3t}}{90}$$

Given now that we have the solution

$$y(t) = \frac{(-6 e^{5t} - 10 e^{3t} + 15 e^{2t} + 90c_1) e^{-3t}}{90} \quad (6)$$

Then substituting (6) into (3) gives

$$x(t) = \frac{(e^{3t} + 2 e^t - 1) e^{-t}}{3} \quad (7)$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
dsolve([diff(x(t),t)-x(t)+diff(y(t),t)+3*y(t)=exp(-t)-1,diff(x(t),t)+2*x(t)+diff(y(t),t)+3*y
```

$$x(t) = \frac{e^{2t}}{3} + \frac{2}{3} - \frac{e^{-t}}{3}$$

$$y(t) = -\frac{1}{9} - \frac{e^{2t}}{15} + \frac{e^{-t}}{6} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 62

```
DSolve[{x'[t]-x[t]+y'[t]+3*y[t]==Exp[-t]-1,x'[t]+2*x[t]+y'[t]+3*y[t]==Exp[2*t]+1},{x[t],y[t]
```

$$x(t) \rightarrow \frac{1}{3} e^{-t} (2e^t + e^{3t} - 1)$$

$$y(t) \rightarrow \frac{e^{-t}}{6} - \frac{e^{2t}}{15} + \frac{1}{16} c_1 e^{-3t} - \frac{1}{9}$$

15.5 problem 17

- 15.5.1 Solution using Matrix exponential method 2322
- 15.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2324
- 15.5.3 Maple step by step solution 2333

Internal problem ID [5448]

Internal file name [OUTPUT/4939_Wednesday_February_14_2024_02_05_46_AM_5854370/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 21. System of simultaneous linear equations. Supplementary problems. Page 163

Problem number: 17.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**system of linear ODEs**"

Solve

$$\begin{aligned}x'(t) &= 1 + x(t) + \frac{e^t}{2} \\y'(t) &= -2y(t) + \frac{e^t}{2} \\z'(t) &= 2 - z(t) + \frac{e^t}{2}\end{aligned}$$

15.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} already. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A , the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^t c_1 \\ e^{-2t} c_2 \\ e^{-t} c_3 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t) dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned}
 \vec{x}_p(t) &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix} dt \\
 &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{t}{2} - e^{-t} \\ \frac{e^{3t}}{6} \\ 2e^t + \frac{e^{2t}}{4} \end{bmatrix} \\
 &= \begin{bmatrix} \frac{te^t}{2} - 1 \\ \frac{e^t}{6} \\ 2 + \frac{e^t}{4} \end{bmatrix}
 \end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
 \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
 &= \begin{bmatrix} -1 + \frac{(t+2c_1)e^t}{2} \\ \frac{(e^{3t}+6c_2)e^{-2t}}{6} \\ e^{-t}c_3 + 2 + \frac{e^t}{4} \end{bmatrix}
 \end{aligned}$$

15.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A . This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

Therefore

$$\det \left(\begin{bmatrix} 1 - \lambda & 0 & 0 \\ 0 & -2 - \lambda & 0 \\ 0 & 0 & -1 - \lambda \end{bmatrix} \right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1 - \lambda)(-2 - \lambda)(-1 - \lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

$$\lambda_3 = -2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector \vec{v} . The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right]$$

Since the current pivot $A(1,2)$ is zero, then the current pivot row is replaced with a

row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array} \right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m , and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m , and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^t \\ &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{-2t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t}\end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ e^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{-2t} \\ 0 \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \dots \end{bmatrix}$$

Where \vec{x}_i are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & e^{-2t} \\ 0 & e^{-t} & 0 \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) dt$$

But

$$\Phi^{-1} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^{2t} & 0 \end{bmatrix}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & e^{-2t} \\ 0 & e^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 0 & e^t \\ 0 & e^{2t} & 0 \end{bmatrix} \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & e^{-2t} \\ 0 & e^{-t} & 0 \end{bmatrix} \int \begin{bmatrix} e^{-t} + \frac{1}{2} \\ \frac{e^t(4+e^t)}{2} \\ \frac{e^{3t}}{2} \end{bmatrix} dt \\ &= \begin{bmatrix} e^t & 0 & 0 \\ 0 & 0 & e^{-2t} \\ 0 & e^{-t} & 0 \end{bmatrix} \begin{bmatrix} \frac{t}{2} - e^{-t} \\ 2e^t + \frac{e^{2t}}{4} \\ \frac{e^{3t}}{6} \end{bmatrix} \\ &= \begin{bmatrix} \frac{te^t}{2} - 1 \\ \frac{e^t}{6} \\ 2 + \frac{e^t}{4} \end{bmatrix} \end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} &= \begin{bmatrix} c_1 e^t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_2 e^{-t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 e^{-2t} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{te^t}{2} - 1 \\ \frac{e^t}{6} \\ 2 + \frac{e^t}{4} \end{bmatrix} \end{aligned}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -1 + \frac{(t+2c_1)e^t}{2} \\ \frac{(e^{3t}+6c_3)e^{-2t}}{6} \\ c_2 e^{-t} + 2 + \frac{e^t}{4} \end{bmatrix}$$

15.5.3 Maple step by step solution

Let's solve

$$\left[x'(t) = 1 + x(t) + \frac{e^t}{2}, y'(t) = -2y(t) + \frac{e^t}{2}, z'(t) = 2 - z(t) + \frac{e^t}{2} \right]$$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right], \left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^{-t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p(t)$

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 + \vec{x}_p(t)$$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous system

$$\phi(t) = \begin{bmatrix} 0 & 0 & e^t \\ e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is the identity matrix

$$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 0 & 0 & e^t \\ e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

- Take the derivative of the particular solution

$$\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$$

- Substitute particular solution and its derivative into the system of ODEs

$$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- The fundamental matrix has columns that are solutions to the homogeneous system so its derivative is $A \cdot \Phi(t)$

$$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$$

- Cancel like terms

$$\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$$

- Multiply by the inverse of the fundamental matrix

$$\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$$

- Integrate to solve for $\vec{v}(t)$

$$\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds$$

- Plug $\vec{v}(t)$ into the equation for the particular solution

$$\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s) ds \right)$$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} \frac{\ln(e^t)e^t}{2} + e^t - 1 \\ \frac{(e^{3t}-1)e^{-2t}}{6} \\ \frac{e^t}{4} + 2 - \frac{9e^{-t}}{4} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3 + \begin{bmatrix} \frac{\ln(e^t)e^t}{2} + e^t - 1 \\ \frac{(e^{3t}-1)e^{-2t}}{6} \\ \frac{e^t}{4} + 2 - \frac{9e^{-t}}{4} \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} c_3 e^t + \frac{\ln(e^t)e^t}{2} + e^t - 1 \\ \frac{(e^{3t}+6c_1-1)e^{-2t}}{6} \\ c_2 e^{-t} + \frac{e^t}{4} + 2 - \frac{9e^{-t}}{4} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ x(t) = c_3 e^t + \frac{\ln(e^t)e^t}{2} + e^t - 1, y(t) = \frac{(e^{3t}+6c_1-1)e^{-2t}}{6}, z(t) = c_2 e^{-t} + \frac{e^t}{4} + 2 - \frac{9e^{-t}}{4} \right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)-x(t)+diff(y(t),t)+2*y(t)=1+exp(t),diff(y(t),t)+2*y(t)+diff(z(t),t)+z(t)
```

$$x(t) = -1 + \frac{e^t(2c_3 + t)}{2}$$

$$y(t) = \frac{e^t}{6} + c_2 e^{-2t}$$

$$z(t) = 2 + \frac{e^t}{4} + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 60

```
DSolve[{x'[t]-x[t]+y'[t]+2*y[t]==1+Exp[t],y'[t]+2*y[t]+z'[t]+z[t]==2+Exp[t],x'[t]-x[t]+z'[t]
```

$$x(t) \rightarrow -1 + e^t \left(\frac{t}{2} + c_1 \right)$$

$$y(t) \rightarrow \frac{e^t}{6} + c_2 e^{-2t}$$

$$z(t) \rightarrow \frac{e^t}{4} + (4 + c_3) e^{-t} + 2$$

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16.1 problem 9

- 16.1.1 Solving as series ode 2339
- 16.1.2 Maple step by step solution 2346

Internal problem ID [5449]

Internal file name [OUTPUT/4940_Wednesday_February_14_2024_02_05_47_AM_53084472/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 9.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$(1 - x)y' + y = x^2$$

With the expansion point for the power series method at $x = 0$.

16.1.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0 . We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots \\ &= y_0 + xf + \frac{x^2}{2} \left. \frac{df}{dx} \right|_{x_0, y_0} + \frac{x^3}{3!} \left. \frac{d^2f}{dx^2} \right|_{x_0, y_0} + \dots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \quad (1)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \quad (4)$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) F_0 \end{aligned} \quad (5)$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0, y_0} \quad (6)$$

Hence

$$\begin{aligned} F_0 &= \frac{-x^2 + y}{x - 1} \\ F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\ &= -\frac{2x}{x - 1} \\ F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\ &= \frac{2}{(x - 1)^2} \\ F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\ &= -\frac{4}{(x - 1)^3} \\ F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\ &= \frac{12}{(x - 1)^4} \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$F_0 = -y(0)$$

$$F_1 = 0$$

$$F_2 = 2$$

$$F_3 = 4$$

$$F_4 = 12$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (1 - x)y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$y' + q(x)y = p(x)$$

$$y' - \frac{y}{x-1} = -\frac{x^2}{x-1}$$

Where

$$q(x) = -\frac{1}{x-1}$$

$$p(x) = -\frac{x^2}{x-1}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x - 1)y' - y = -x^2$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x - 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^n \right) = -x^2 \quad (1)$$

Expanding $-x^2$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -x^2 &= -x^2 + \dots \\ &= -x^2 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(x - 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^n \right) = -x^2 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n \right) + \sum_{n=1}^{\infty} (-n a_n x^{n-1}) + \sum_{n=0}^{\infty} (-a_n x^n) = -x^2 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} (-n a_n x^{n-1}) = \sum_{n=0}^{\infty} (-(n+1) a_{n+1} x^n)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=1}^{\infty} n a_n x^n \right) + \sum_{n=0}^{\infty} (-(n+1) a_{n+1} x^n) + \sum_{n=0}^{\infty} (-a_n x^n) = -x^2 \quad (3)$$

$n = 0$ gives

$$\begin{aligned} -a_1 - a_0 &= 0 \\ a_1 &= -a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$(n a_n - (n+1) a_{n+1} - a_n) x^n = -x^2 \quad (4)$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(-2a_2)x &= 0 \\ -2a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = 0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(a_2 - 3a_3)x^2 &= -x^2 \\ a_2 - 3a_3 &= -1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{1}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(2a_3 - 4a_4)x^3 &= 0 \\ 2a_3 - 4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{6}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(3a_4 - 5a_5)x^4 &= 0 \\ 3a_4 - 5a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{10}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(4a_5 - 6a_6)x^5 &= 0 \\ 4a_5 - 6a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{15}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 - a_0 x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (1 - x)a_0 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6) \quad (3)$$

Summary

The solution(s) found are the following

$$y = (1 - x)y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6) \quad (1)$$

$$y = (1 - x)c_1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6) \quad (2)$$

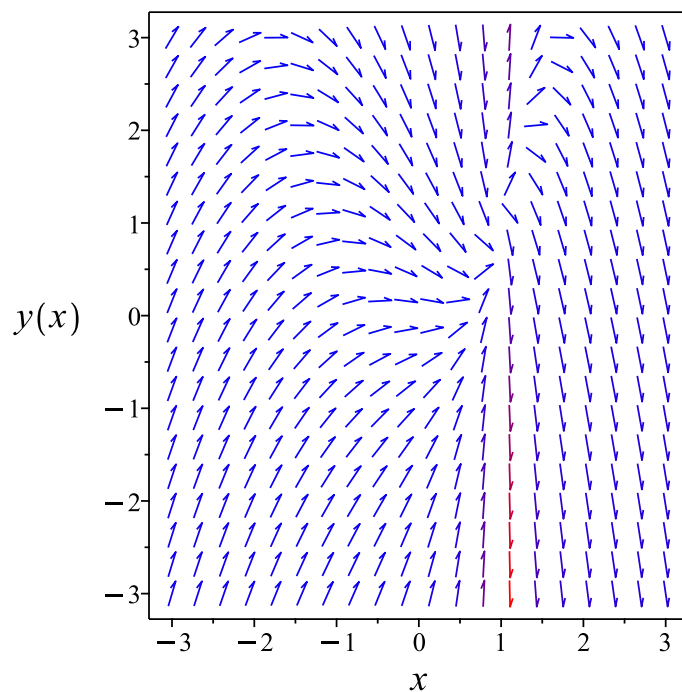


Figure 283: Slope field plot

Verification of solutions

$$y = (1 - x)y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Verified OK.

$$y = (1 - x)c_1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Verified OK.

16.1.2 Maple step by step solution

Let's solve

$$(x - 1)y' - y = -x^2$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{y}{x-1} - \frac{x^2}{x-1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{y}{x-1} = -\frac{x^2}{x-1}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{y}{x-1} \right) = -\frac{\mu(x)x^2}{x-1}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{y}{x-1} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{x-1}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x-1}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int -\frac{\mu(x)x^2}{x-1} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int -\frac{\mu(x)x^2}{x-1} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)x^2}{x-1} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x-1}$

$$y = (x-1) \left(\int -\frac{x^2}{(x-1)^2} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = (x-1) \left(-x - 2 \ln(x-1) + \frac{1}{x-1} + c_1 \right)$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=6;  
dsolve((1-x)*diff(y(x),x)=x^2-y(x),y(x),type='series',x=0);
```

$$y(x) = (1 - x)y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 33

```
AsymptoticDSolveValue[(1-x)*y'[x]==x^2-y[x],y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{x^5}{10} + \frac{x^4}{6} + \frac{x^3}{3} + c_1(1 - x)$$

16.2 problem 10A

16.2.1 Solving as series ode	2349
16.2.2 Maple step by step solution	2357

Internal problem ID [5450]

Internal file name [OUTPUT/4941_Wednesday_February_14_2024_02_05_48_AM_84468907/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 10A.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$xy' - 2y = 1 - x$$

With the expansion point for the power series method at $x = 1$.

16.2.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expansion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t . This results in

$$(t + 1) \left(\frac{d}{dt} y(t) \right) - 2y(t) = -t$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0 . We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots \\ &= y_0 + xf + \frac{x^2}{2} \left. \frac{df}{dx} \right|_{x_0, y_0} + \frac{x^3}{3!} \left. \frac{d^2f}{dx^2} \right|_{x_0, y_0} + \dots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned} \frac{d^2f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{d^3f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2f}{dx^2} \right) f \end{aligned} \tag{3}$$

⋮

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) F_0 \end{aligned} \tag{5}$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) f \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0, y_0} \quad (6)$$

Hence

$$\begin{aligned}F_0 &= \frac{2y(t) - t}{t + 1} \\F_1 &= \frac{dF_0}{dt} \\&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\&= \frac{-1 + 2y(t) - 2t}{(t + 1)^2} \\F_2 &= \frac{dF_1}{dt} \\&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\&= 0 \\F_3 &= \frac{dF_2}{dt} \\&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\&= 0 \\F_4 &= \frac{dF_3}{dt} \\&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\&= 0\end{aligned}$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}F_0 &= 2y(0) \\F_1 &= -1 + 2y(0) \\F_2 &= 0 \\F_3 &= 0 \\F_4 &= 0\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y(t) = (t^2 + 2t + 1) y(0) - \frac{t^2}{2} + O(t^6)$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used.

Writing the ODE as

$$\begin{aligned}\frac{d}{dt}y(t) + q(t)y(t) &= p(t) \\ \frac{d}{dt}y(t) - \frac{2y(t)}{t+1} &= -\frac{t}{t+1}\end{aligned}$$

Where

$$\begin{aligned}q(t) &= -\frac{2}{t+1} \\ p(t) &= -\frac{t}{t+1}\end{aligned}$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(t+1) \left(\frac{d}{dt}y(t) \right) - 2y(t) = -t$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$(t+1) \left(\sum_{n=1}^{\infty} n a_n t^{n-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n t^n \right) = -t \quad (1)$$

Expanding $-t$ as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -t &= -t + \dots \\ &= -t \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(t + 1) \left(\sum_{n=1}^{\infty} n a_n t^{n-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n t^n \right) = -t \quad (1)$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^n \right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1} \right) + \sum_{n=0}^{\infty} (-2 a_n t^n) = -t \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n .

$$\left(\sum_{n=1}^{\infty} n a_n t^n \right) + \left(\sum_{n=0}^{\infty} (n + 1) a_{n+1} t^n \right) + \sum_{n=0}^{\infty} (-2 a_n t^n) = -t \quad (3)$$

$n = 0$ gives

$$\begin{aligned} a_1 - 2a_0 &= 0 \\ a_1 &= 2a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$(n a_n + (n + 1) a_{n+1} - 2 a_n) t^n = -t \quad (4)$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned} (-a_1 + 2a_2) t &= -t \\ -a_1 + 2a_2 &= -1 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{1}{2} + a_0$$

For $n = 2$ the recurrence equation gives

$$(3a_3)t^2 = 0$$
$$3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$(a_3 + 4a_4)t^3 = 0$$
$$a_3 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$(2a_4 + 5a_5)t^4 = 0$$
$$2a_4 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$(3a_5 + 6a_6)t^5 = 0$$
$$3a_5 + 6a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

And so on. Therefore the solution is

$$\begin{aligned}y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + 2a_0 t + \left(-\frac{1}{2} + a_0\right) t^2 + \dots$$

Collecting terms, the solution becomes

$$y(t) = (t^2 + 2t + 1) a_0 - \frac{t^2}{2} + O(t^6) \quad (3)$$

Replacing t in the above with the original independent variable x s using $t = x - 1$ results in

$$y = ((x - 1)^2 + 2x - 1) y(1) - \frac{(x - 1)^2}{2} + O((x - 1)^6)$$

Summary

The solution(s) found are the following

$$y = ((x - 1)^2 + 2x - 1) y(1) - \frac{(x - 1)^2}{2} + O((x - 1)^6) \quad (1)$$

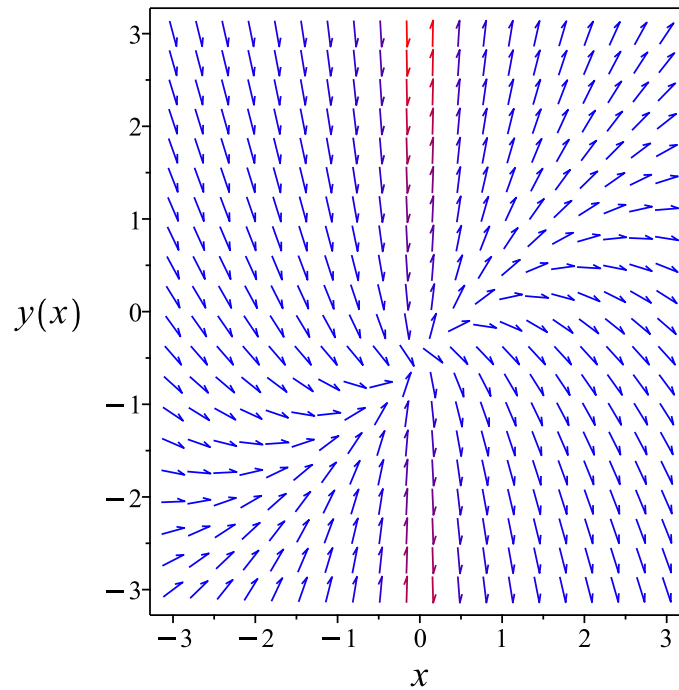


Figure 284: Slope field plot

Verification of solutions

$$y = ((x - 1)^2 + 2x - 1) y(1) - \frac{(x - 1)^2}{2} + O((x - 1)^6)$$

Verified OK.

16.2.2 Maple step by step solution

Let's solve

$$xy' - 2y = 1 - x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{2y}{x} - \frac{x-1}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{2y}{x} = -\frac{x-1}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = -\frac{\mu(x)(x-1)}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x^2}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)(x-1)}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^2}$

$$y = x^2 \left(\int -\frac{x-1}{x^3} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x^2 \left(-\frac{1}{2x^2} + \frac{1}{x} + c_1 \right)$$

- Simplify

$$y = -\frac{1}{2} + x + c_1 x^2$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;  
dsolve(x*diff(y(x),x)=1-x+2*y(x),y(x),type='series',x=1);
```

$$y(x) = y(1)x^2 - \frac{(x-1)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 28

```
AsymptoticDSolveValue[x*y'[x]==1-x+2*y[x],y[x],{x,1,5}]
```

$$y(x) \rightarrow -\frac{1}{2}(x-1)^2 + c_1((x-1)^2 + 2(x-1) + 1)$$

16.3 problem 10B

16.3.1 Solving as linear ode	2360
16.3.2 Solving as homogeneousTypeMapleC ode	2362
16.3.3 Solving as first order ode lie symmetry lookup ode	2365
16.3.4 Solving as exact ode	2369
16.3.5 Maple step by step solution	2374

Internal problem ID [5451]

Internal file name [OUTPUT/4942_Wednesday_February_14_2024_02_05_48_AM_69199443/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 10B.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : "**linear**", "**homogeneousTypeMapleC**", "**exactWithIntegrationFactor**", "**first_order_ode_lie_symmetry_lookup**"

Maple gives the following as the ode type

[_linear]

$$xy' - 2y = 1 - x$$

16.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{1-x}{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{1-x}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{1-x}{x} \right) \\ \frac{d}{dx} \left(\frac{y}{x^2} \right) &= \left(\frac{1}{x^2} \right) \left(\frac{1-x}{x} \right) \\ d \left(\frac{y}{x^2} \right) &= \left(\frac{1-x}{x^3} \right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{1-x}{x^3} dx \\ \frac{y}{x^2} &= -\frac{1}{2x^2} + \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(-\frac{1}{2x^2} + \frac{1}{x} \right) + c_1 x^2$$

which simplifies to

$$y = -\frac{1}{2} + x + c_1 x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + x + c_1 x^2 \tag{1}$$

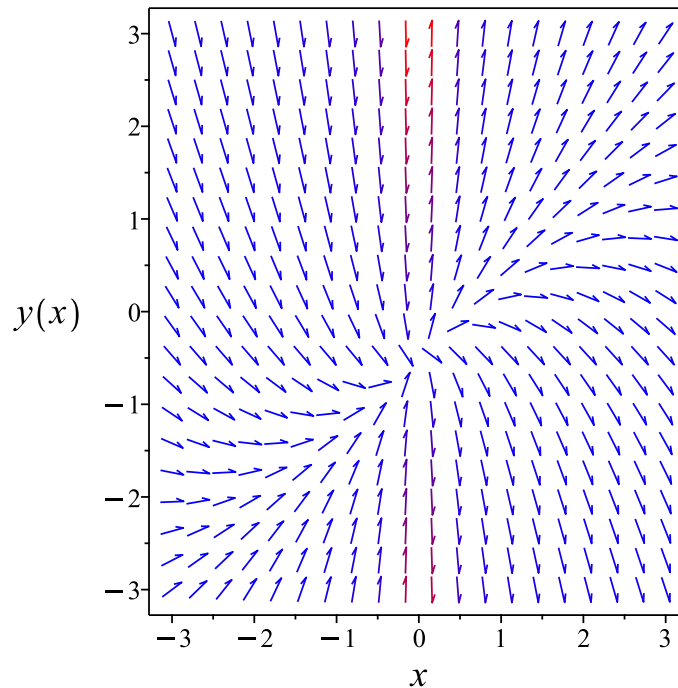


Figure 285: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + x + c_1x^2$$

Verified OK.

16.3.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{1 - X - x_0 + 2Y(X) + 2y_0}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 0 \\ y_0 &= -\frac{1}{2} \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + 2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{-X + 2Y}{X} \end{aligned} \quad (1)$$

An ode of the form $Y' = \frac{M(X, Y)}{N(X, Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -X + 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= -1 + 2u \\ \frac{du}{dX} &= \frac{-1 + u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-1 + u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X) \right) X - u(X) + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X)g(u) \\ &= \frac{-1 + u}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = -1 + u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-1 + u} du &= \frac{1}{X} dX \\ \int \frac{1}{-1 + u} du &= \int \frac{1}{X} dX \\ \ln(-1 + u) &= \ln(X) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$-1 + u = e^{\ln(X)+c_2}$$

Which simplifies to

$$-1 + u = c_3 X$$

Which simplifies to

$$u(X) = c_3 X e^{c_2} + 1$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X(c_3 X e^{c_2} + 1)$$

Using the solution for $Y(X)$

$$Y(X) = X(c_3 X e^{c_2} + 1)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$

$$X = x + x_0$$

Or

$$Y = y - \frac{1}{2}$$

$$X = x$$

Then the solution in y becomes

$$y + \frac{1}{2} = x(c_3 x e^{c_2} + 1)$$

Summary

The solution(s) found are the following

$$y + \frac{1}{2} = x(c_3 x e^{c_2} + 1) \quad (1)$$

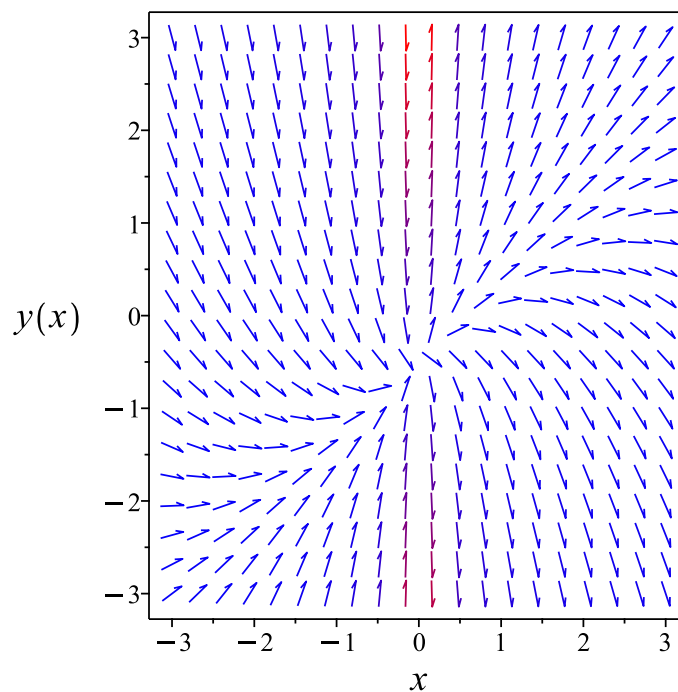


Figure 286: Slope field plot

Verification of solutions

$$y + \frac{1}{2} = x(c_3 x e^{c_2} + 1)$$

Verified OK.

16.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1 - x + 2y}{x}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \quad (\text{A})$$

The type of this ode is known. It is of type **linear**. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 264: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)y + g(x)y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)y + f_2(x)y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x^2\end{aligned}\tag{A1}$$

The next step is to determine the canonical coordinates R, S . The canonical coordinates map $(x, y) \rightarrow (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS\tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \quad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 - x + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1 - x}{x^3} \quad (2A)$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1 - R}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S . Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + \frac{1}{R} + c_1 \quad (4)$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = -\frac{1}{2x^2} + \frac{1}{x} + c_1$$

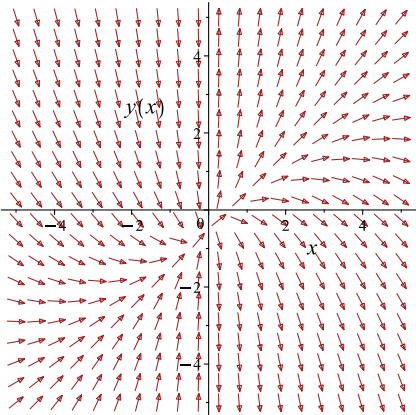
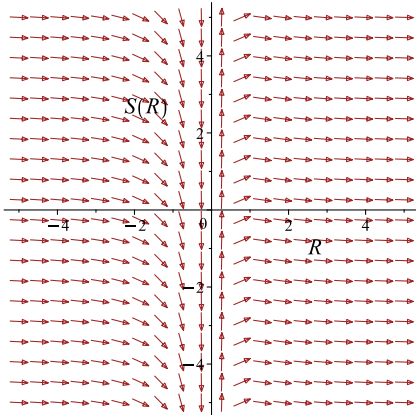
Which simplifies to

$$\frac{y}{x^2} = -\frac{1}{2x^2} + \frac{1}{x} + c_1$$

Which gives

$$y = -\frac{1}{2} + x + c_1 x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1-x+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = \frac{1-R}{R^3}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + x + c_1x^2 \quad (1)$$

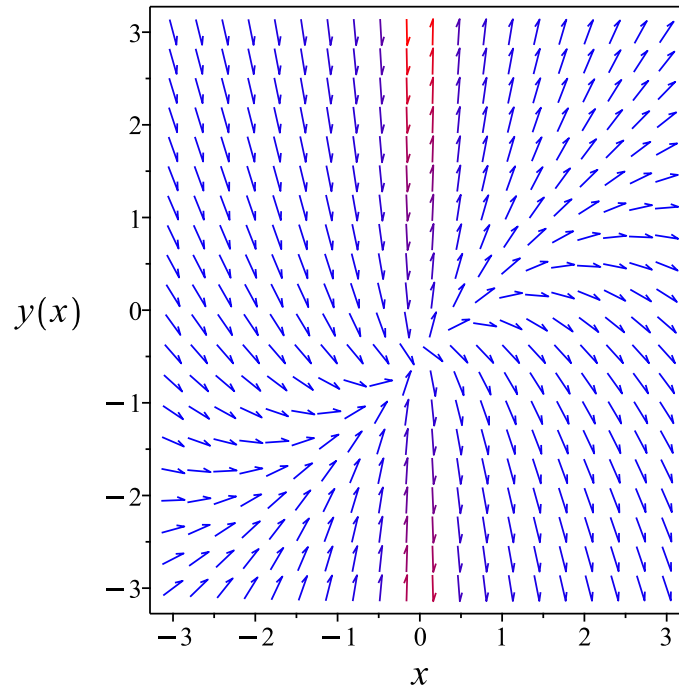


Figure 287: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + x + c_1x^2$$

Verified OK.

16.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \quad (\text{A})$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \quad (\text{B})$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) dx + N(x, y) dy = 0 \quad (\text{1A})$$

Therefore

$$\begin{aligned} (x) dy &= (1 - x + 2y) dx \\ (-1 + x - 2y) dx + (x) dy &= 0 \end{aligned} \quad (\text{2A})$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -1 + x - 2y \\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 + x - 2y) \\ &= -2 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) \\ &= \frac{1}{x} ((-2) - (1)) \\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y , then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{3}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3 \ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called \bar{M} and \bar{N} for now so not to confuse them with the original M and N .

$$\begin{aligned}\bar{M} &= \mu M \\ &= \frac{1}{x^3}(-1 + x - 2y) \\ &= \frac{-1 + x - 2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\bar{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is obtained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\ \left(\frac{-1 + x - 2y}{x^3} \right) + \left(\frac{1}{x^2} \right) \frac{dy}{dx} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\ \int \frac{\partial \phi}{\partial x} dx &= \int \frac{-1 + x - 2y}{x^3} dx \\ \phi &= \frac{1 + 2y - 2x}{2x^2} + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y . Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1 + 2y - 2x}{2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1 + 2y - 2x}{2x^2}$$

The solution becomes

$$y = -\frac{1}{2} + x + c_1x^2$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{2} + x + c_1x^2 \tag{1}$$

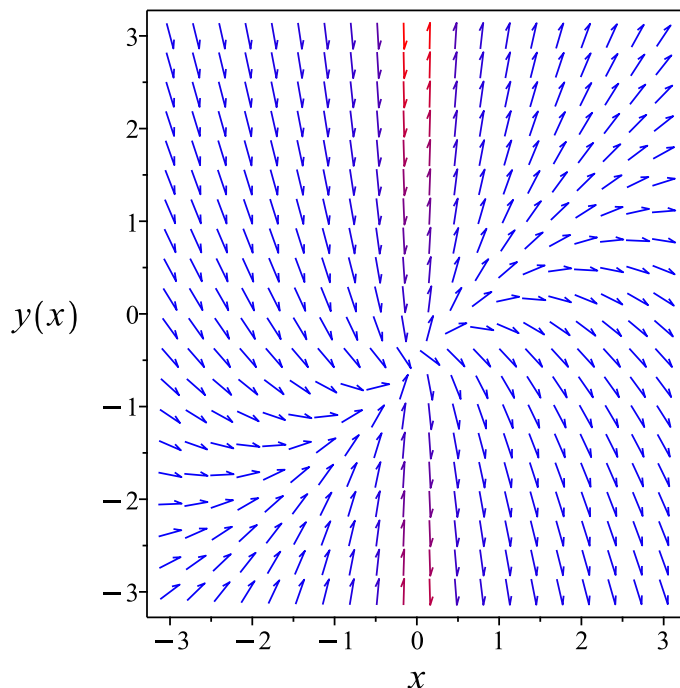


Figure 288: Slope field plot

Verification of solutions

$$y = -\frac{1}{2} + x + c_1x^2$$

Verified OK.

16.3.5 Maple step by step solution

Let's solve

$$xy' - 2y = 1 - x$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{2y}{x} - \frac{x-1}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{2y}{x} = -\frac{x-1}{x}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = -\frac{\mu(x)(x-1)}{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{2y}{x} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{2\mu(x)}{x}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x^2}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$$

- Solve for y

$$y = \frac{\int -\frac{\mu(x)(x-1)}{x} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x^2}$

$$y = x^2 \left(\int -\frac{x-1}{x^3} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = x^2 \left(-\frac{1}{2x^2} + \frac{1}{x} + c_1 \right)$$

- Simplify

$$y = -\frac{1}{2} + x + c_1 x^2$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)=1-x+2*y(x),y(x), singsol=all)
```

$$y(x) = x - \frac{1}{2} + c_1 x^2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 16

```
DSolve[x*y'[x]==1-x+2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 x^2 + x - \frac{1}{2}$$

16.4 problem 11

16.4.1 Solving as series ode	2376
16.4.2 Maple step by step solution	2383

Internal problem ID [5452]

Internal file name [OUTPUT/4943_Wednesday_February_14_2024_02_05_49_AM_48988675/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 11.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$y' - 3y = 2x^2$$

With the expansion point for the power series method at $x = 0$.

16.4.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0 . We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx} \Big|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2f}{dx^2} \Big|_{x_0, y_0} + \dots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \quad (1)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \quad (4)$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) F_0 \end{aligned} \quad (5)$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0, y_0} \quad (6)$$

Hence

$$F_0 = 2x^2 + 3y$$

$$\begin{aligned} F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\ &= 4x + 6x^2 + 9y \end{aligned}$$

$$\begin{aligned} F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\ &= 4 + 12x + 18x^2 + 27y \end{aligned}$$

$$\begin{aligned} F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\ &= 12 + 36x + 54x^2 + 81y \end{aligned}$$

$$\begin{aligned} F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\ &= 36 + 108x + 162x^2 + 243y \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$F_0 = 3y(0)$$

$$F_1 = 9y(0)$$

$$F_2 = 27y(0) + 4$$

$$F_3 = 81y(0) + 12$$

$$F_4 = 243y(0) + 36$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned}y' + q(x)y &= p(x) \\y' - 3y &= 2x^2\end{aligned}$$

Where

$$\begin{aligned}q(x) &= -3 \\p(x) &= 2x^2\end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - 3 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 2x^2 \quad (1)$$

Expanding $2x^2$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}2x^2 &= 2x^2 + \dots \\ &= 2x^2\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - 3 \left(\sum_{n=0}^{\infty} a_n x^n \right) = 2x^2 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + \sum_{n=0}^{\infty} (-3a_n x^n) = 2x^2 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \right) + \sum_{n=0}^{\infty} (-3a_n x^n) = 2x^2 \quad (3)$$

For $0 \leq n$, the recurrence equation is

$$((n+1) a_{n+1} - 3a_n) x^n = 2x^2 \quad (4)$$

For $n = 0$ the recurrence equation gives

$$\begin{aligned} (a_1 - 3a_0) 1 &= 0 \\ a_1 - 3a_0 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_1 = 3a_0$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned} (2a_2 - 3a_1) x &= 0 \\ 2a_2 - 3a_1 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{9a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3 - 3a_2)x^2 &= 2x^2 \\ 3a_3 - 3a_2 &= 2\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2}{3} + \frac{9a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(4a_4 - 3a_3)x^3 &= 0 \\ 4a_4 - 3a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{2} + \frac{27a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(5a_5 - 3a_4)x^4 &= 0 \\ 5a_5 - 3a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3}{10} + \frac{81a_0}{40}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(6a_6 - 3a_5)x^5 &= 0 \\ 6a_6 - 3a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{3}{20} + \frac{81a_0}{80}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + 3a_0 x + \frac{9a_0 x^2}{2} + \left(\frac{2}{3} + \frac{9a_0}{2}\right) x^3 + \left(\frac{1}{2} + \frac{27a_0}{8}\right) x^4 + \left(\frac{3}{10} + \frac{81a_0}{40}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) a_0 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \quad (3)$$

Summary

The solution(s) found are the following

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \quad (1)$$

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) c_1 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \quad (2)$$

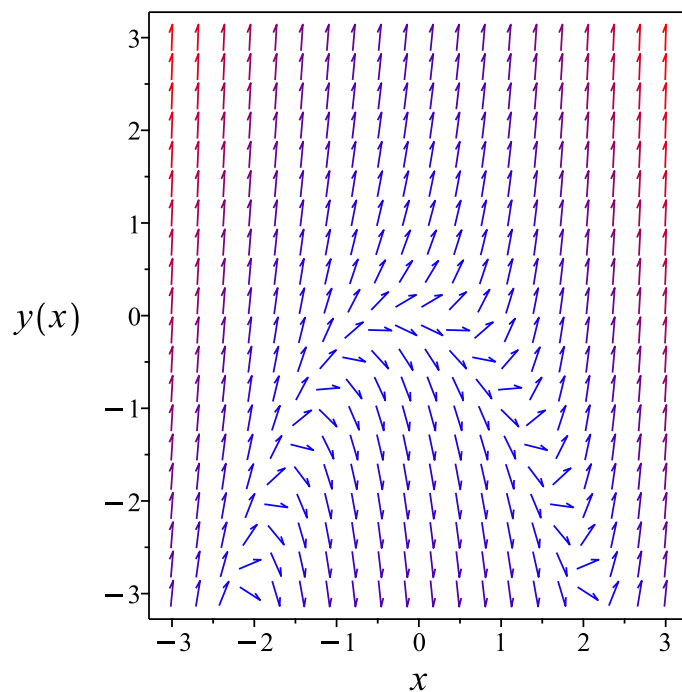


Figure 289: Slope field plot

Verification of solutions

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Verified OK.

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) c_1 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Verified OK.

16.4.2 Maple step by step solution

Let's solve

$$y' - 3y = 2x^2$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = 2x^2 + 3y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - 3y = 2x^2$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) (y' - 3y) = 2\mu(x) x^2$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

$$\mu(x) (y' - 3y) = \mu'(x) y + \mu(x) y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -3\mu(x)$$

- Solve to find the integrating factor

$$\mu(x) = e^{-3x}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x) y) \right) dx = \int 2\mu(x) x^2 dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x) y = \int 2\mu(x) x^2 dx + c_1$$

- Solve for y

$$y = \frac{\int 2\mu(x)x^2 dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = e^{-3x}$

$$y = \frac{\int 2e^{-3x}x^2 dx + c_1}{e^{-3x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{2(9x^2+6x+2)e^{-3x}}{27} + c_1}{e^{-3x}}$$

- Simplify

$$y = e^{3x} c_1 - \frac{2x^2}{3} - \frac{4x}{9} - \frac{4}{27}$$

Maple trace

```

`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
Order:=6;  
dsolve(diff(y(x),x)=2*x^2+3*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right)y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 61

```
AsymptoticDSolveValue[y'[x]==2*x^2+3*y[x],y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{3x^5}{10} + \frac{x^4}{2} + \frac{2x^3}{3} + c_1 \left(\frac{81x^5}{40} + \frac{27x^4}{8} + \frac{9x^3}{2} + \frac{9x^2}{2} + 3x + 1 \right)$$

16.5 problem 12

16.5.1 Solving as series ode	2386
16.5.2 Maple step by step solution	2393

Internal problem ID [5453]

Internal file name [OUTPUT/4944_Wednesday_February_14_2024_02_05_50_AM_32247685/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 12.

ODE order: 1.

ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$(x + 1)y' - y = x^2 - 2x$$

With the expansion point for the power series method at $x = 0$.

16.5.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0 . We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \dots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx} \Big|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2f}{dx^2} \Big|_{x_0, y_0} + \dots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \quad (1)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \quad (4)$$

$$\begin{aligned} F_n &= \frac{d}{dx}(F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) F_0 \end{aligned} \quad (5)$$

For example, for $n = 1$ we see that

$$\begin{aligned} F_1 &= \frac{d}{dx}(F_0) \\ &= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) F_0 \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned} F_2 &= \frac{d}{dx}(F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f \end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0, y_0} \quad (6)$$

Hence

$$\begin{aligned} F_0 &= \frac{x^2 - 2x + y}{x + 1} \\ F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\ &= \frac{2x - 2}{x + 1} \\ F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\ &= \frac{4}{(x + 1)^2} \\ F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\ &= -\frac{8}{(x + 1)^3} \\ F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\ &= \frac{24}{(x + 1)^4} \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$F_0 = y(0)$$

$$F_1 = -2$$

$$F_2 = 4$$

$$F_3 = -8$$

$$F_4 = 24$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (x + 1)y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{x+1} &= \frac{x(-2+x)}{x+1} \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{x+1} \\ p(x) &= \frac{x(-2+x)}{x+1} \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x + 1)y' - y = x(-2 + x)$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^n \right) = x(-2 + x) \quad (1)$$

Expanding $x(-2 + x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x(-2 + x) &= x^2 - 2x + \dots \\ &= x^2 - 2x \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(x + 1) \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^n \right) = x^2 - 2x \quad (1)$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n \right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + \sum_{n=0}^{\infty} (-a_n x^n) = x^2 - 2x \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=1}^{\infty} n a_n x^n \right) + \left(\sum_{n=0}^{\infty} (n + 1) a_{n+1} x^n \right) + \sum_{n=0}^{\infty} (-a_n x^n) = x^2 - 2x \quad (3)$$

$n = 0$ gives

$$\begin{aligned} a_1 - a_0 &= 0 \\ a_1 &= a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$(na_n + (n + 1)a_{n+1} - a_n)x^n = x^2 - 2x \quad (4)$$

For $n = 1$ the recurrence equation gives

$$(2a_2)x = -2x$$

$$2a_2 = -2$$

Which after substituting the earlier terms found becomes

$$a_2 = -1$$

For $n = 2$ the recurrence equation gives

$$(a_2 + 3a_3)x^2 = x^2$$

$$a_2 + 3a_3 = 1$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2}{3}$$

For $n = 3$ the recurrence equation gives

$$(2a_3 + 4a_4)x^3 = 0$$

$$2a_3 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{3}$$

For $n = 4$ the recurrence equation gives

$$(3a_4 + 5a_5)x^4 = 0$$

$$3a_4 + 5a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{5}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(4a_5 + 6a_6)x^5 &= 0 \\ 4a_5 + 6a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{2}{15}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (x + 1) a_0 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \quad (3)$$

Summary

The solution(s) found are the following

$$y = (x + 1) y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \quad (1)$$

$$y = (x + 1) c_1 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \quad (2)$$

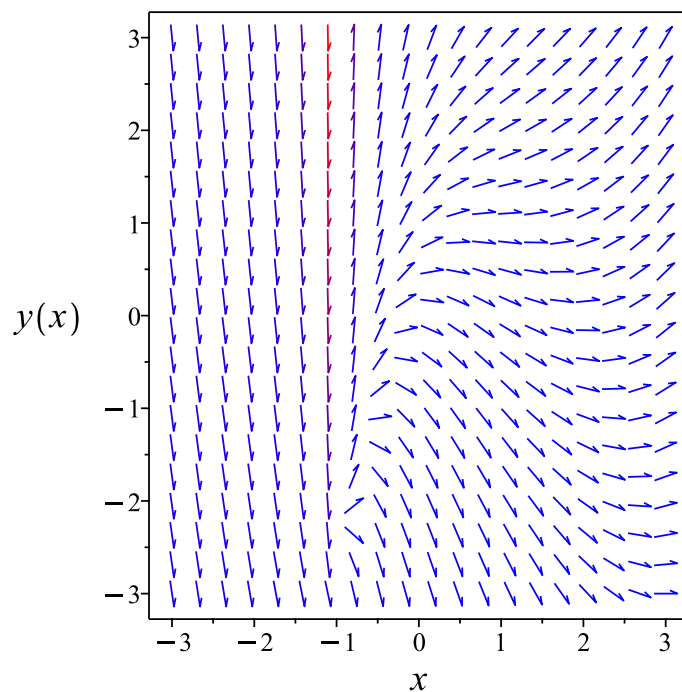


Figure 290: Slope field plot

Verification of solutions

$$y = (x + 1)y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

$$y = (x + 1)c_1 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

16.5.2 Maple step by step solution

Let's solve

$$(x + 1)y' - y = x(-2 + x)$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y' = \frac{y}{x+1} + \frac{x(-2+x)}{x+1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y' - \frac{y}{x+1} = \frac{x(-2+x)}{x+1}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y' - \frac{y}{x+1} \right) = \frac{\mu(x)x(-2+x)}{x+1}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)y)$

$$\mu(x) \left(y' - \frac{y}{x+1} \right) = \mu'(x)y + \mu(x)y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = -\frac{\mu(x)}{x+1}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{1}{x+1}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)y) \right) dx = \int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)y = \int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{1}{x+1}$

$$y = (x+1) \left(\int \frac{x(-2+x)}{(x+1)^2} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

$$y = (x+1) \left(x - 4 \ln(x+1) - \frac{3}{x+1} + c_1 \right)$$

- Simplify

$$y = (-4x - 4) \ln(x+1) + x^2 + (c_1 + 1)x + c_1 - 3$$

Maple trace

```
`Methods for first order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
trying 1st order linear  
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=6;  
dsolve((x+1)*diff(y(x),x)=x^2-2*x+y(x),y(x),type='series',x=0);
```

$$y(x) = (x + 1)y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(x+1)*y'[x]==x^2-2*x+y[x],y[x],{x,0,5}]
```

$$y(x) \rightarrow \frac{x^5}{5} - \frac{x^4}{3} + \frac{2x^3}{3} - x^2 + c_1(x + 1)$$

16.6 problem 13

16.6.1 Maple step by step solution 2403

Internal problem ID [5454]

Internal file name [OUTPUT/4945_Wednesday_February_14_2024_02_05_50_AM_84918379/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 13.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$y'' + xy = 0$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0, y_0, y'_0} + \frac{x^3}{3!}f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \quad (1)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad (463)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (464)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \tag{4}$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \end{aligned} \tag{5}$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \end{aligned}$$

⋮

$$\begin{aligned} F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \tag{6}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
 F_0 &= -xy \\
 F_1 &= \frac{dF_0}{dx} \\
 &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
 &= -y - xy' \\
 F_2 &= \frac{dF_1}{dx} \\
 &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
 &= -2y' + yx^2 \\
 F_3 &= \frac{dF_2}{dx} \\
 &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
 &= x(xy' + 4y) \\
 F_4 &= \frac{dF_3}{dx} \\
 &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
 &= -yx^3 + 6xy' + 4y
 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
 F_0 &= 0 \\
 F_1 &= -y(0) \\
 F_2 &= -2y'(0) \\
 F_3 &= 0 \\
 F_4 &= 4y(0)
 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right)y(0) + \left(x - \frac{1}{12}x^4\right)y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -x \left(\sum_{n=0}^{\infty} a_n x^n \right) \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n \right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} a_n = \sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (1+n) x^n \right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n \right) = 0 \quad (3)$$

For $1 \leq n$, the recurrence equation is

$$(n+2) a_{n+2} (1+n) + a_{n-1} = 0 \quad (4)$$

Solving for a_{n+2} , gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \quad (5)$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \quad (1)$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

16.6.1 Maple step by step solution

Let's solve

$$y'' = -xy$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + xy = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

- Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$$

- Shift index using $k \rightarrow k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k \rightarrow k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} (a_{k+2}(k+2)(k+1) + a_{k-1}) x^k \right) = 0$$

- Each term must be 0

$$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} + a_{k-1} = 0$$

- Shift index using $k \rightarrow k + 1$

$$((k+1)^2 + 3k + 5) a_{k+3} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k}{k^2+5k+6}, 2a_2 = 0 \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exist
-> Trying a solution in terms of special functions:
    -> Bessel
    <- Bessel successful
<- special function solution successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```

Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);

```

$$y(x) = \left(1 - \frac{x^3}{6}\right) y(0) + \left(x - \frac{1}{12}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```

AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]

```

$$y(x) \rightarrow c_2 \left(x - \frac{x^4}{12}\right) + c_1 \left(1 - \frac{x^3}{6}\right)$$

16.7 problem 14

16.7.1 Maple step by step solution 2411

Internal problem ID [5455]

Internal file name [OUTPUT/4946_Wednesday_February_14_2024_02_05_51_AM_18391819/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 14.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$y'' + 2yx^2 = 0$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0, y_0, y'_0} + \frac{x^3}{3!}f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \quad (1)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad (466)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (467)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \quad (4)$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \end{aligned} \quad (5)$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \quad (6)$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \quad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
 F_0 &= -2yx^2 \\
 F_1 &= \frac{dF_0}{dx} \\
 &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \\
 &= -2x(xy' + 2y) \\
 F_2 &= \frac{dF_1}{dx} \\
 &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}y' + \frac{\partial F_1}{\partial y'}F_1 \\
 &= 4yx^4 - 8xy' - 4y \\
 F_3 &= \frac{dF_2}{dx} \\
 &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}y' + \frac{\partial F_2}{\partial y'}F_2 \\
 &= 4y'x^4 + 32yx^3 - 12y' \\
 F_4 &= \frac{dF_3}{dx} \\
 &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y}y' + \frac{\partial F_3}{\partial y'}F_3 \\
 &= 48y'x^3 - 8x^2y(x^4 - 15)
 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
 F_0 &= 0 \\
 F_1 &= 0 \\
 F_2 &= -4y(0) \\
 F_3 &= -12y'(0) \\
 F_4 &= 0
 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{x^4}{6}\right)y(0) + \left(x - \frac{1}{10}x^5\right)y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -2 \left(\sum_{n=0}^{\infty} a_n x^n \right) x^2 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n \right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n \right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n \right) = 0 \quad (3)$$

For $2 \leq n$, the recurrence equation is

$$(n+2) a_{n+2} (n+1) + 2a_{n-2} = 0 \quad (4)$$

Solving for a_{n+2} , gives

$$a_{n+2} = -\frac{2a_{n-2}}{(n+2)(n+1)} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x - \frac{1}{6}a_0x^4 - \frac{1}{10}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{6}\right) a_0 + \left(x - \frac{1}{10}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10}x^5\right) y'(0) + O(x^6) \quad (1)$$

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10}x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10}x^5\right) c_2 + O(x^6)$$

Verified OK.

16.7.1 Maple step by step solution

Let's solve

$$y'' = -2yx^2$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + 2yx^2 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

- Rewrite ODE with series expansions

- Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- Shift index using $k \rightarrow k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k \rightarrow k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$6a_3x + 2a_2 + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) + 2a_{k-2}) x^k \right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} + 2a_{k-2} = 0$$

- Shift index using $k \rightarrow k + 2$

$$((k+2)^2 + 3k + 8) a_{k+4} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2a_k}{k^2 + 7k + 12}, a_2 = 0, a_3 = 0 \right]$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacic's algorithm  
<- No Liouvillian solutions exist  
-> Trying a solution in terms of special functions:  
    -> Bessel  
    <- Bessel successful  
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;  
dsolve(diff(y(x),x$2)+2*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+2*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2 \left(x - \frac{x^5}{10}\right) + c_1 \left(1 - \frac{x^4}{6}\right)$$

16.8 problem 15

16.8.1 Maple step by step solution 2421

Internal problem ID [5456]

Internal file name [OUTPUT/4947_Wednesday_February_14_2024_02_05_51_AM_85677221/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 15.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$y'' - xy' + yx^2 = 0$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \quad (1)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad (469)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (470)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \tag{4}$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \end{aligned} \tag{5}$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \\ &\vdots \\ F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \tag{6}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
 F_0 &= xy' - yx^2 \\
 F_1 &= \frac{dF_0}{dx} \\
 &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
 &= y' + (-x^3 - 2x)y \\
 F_2 &= \frac{dF_1}{dx} \\
 &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
 &= -y'x^3 - 4yx^2 - xy' - 2y \\
 F_3 &= \frac{dF_2}{dx} \\
 &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
 &= (-x^4 - 8x^2 - 3)y' + yx(x^4 + x^2 - 8) \\
 F_4 &= \frac{dF_3}{dx} \\
 &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
 &= (-11x^3 - 27x)y' + y(x^6 + 13x^4 + 6x^2 - 8)
 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
 F_0 &= 0 \\
 F_1 &= y'(0) \\
 F_2 &= -2y(0) \\
 F_3 &= -3y'(0) \\
 F_4 &= -8y(0)
 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{12}x^4 - \frac{1}{90}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^n \right) x^2 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \sum_{n=1}^{\infty} (-n x^n a_n) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n \right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n \right) + \sum_{n=1}^{\infty} (-n x^n a_n) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n \right) = 0 \quad (3)$$

$n = 1$ gives

$$6a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)a_{n+2}(n+1) - na_n + a_{n-2} = 0 \quad (4)$$

Solving for a_{n+2} , gives

$$\begin{aligned} a_{n+2} &= \frac{na_n - a_{n-2}}{(n+2)(n+1)} \\ (5) \quad &= \frac{na_n}{(n+2)(n+1)} - \frac{a_{n-2}}{(n+2)(n+1)} \end{aligned}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 3a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 4a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{90}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 5a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{144}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_1 x^3 - \frac{1}{12} a_0 x^4 - \frac{1}{40} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{12} x^4 - \frac{1}{90} x^6\right) y(0) + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) y'(0) + O(x^6) \quad (1)$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{1}{12} x^4 - \frac{1}{90} x^6\right) y(0) + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6)$$

Verified OK.

16.8.1 Maple step by step solution

Let's solve

$$y'' = xy' - yx^2$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - xy' + yx^2 = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

- Rewrite ODE with series expansions

- Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- Shift index using $k \rightarrow k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2}$$

- Shift index using $k \rightarrow k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^k$$

Rewrite ODE with series expansions

$$2a_2 + (6a_3 - a_1)x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - a_k k + a_{k-2}) x^k \right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 - a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = \frac{a_1}{6}\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2) a_{k+2} - a_k k + a_{k-2} = 0$$

- Shift index using $k \rightarrow k + 2$

$$((k + 2)^2 + 3k + 8) a_{k+4} - a_{k+2}(k + 2) + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{k a_{k+2} - a_k + 2 a_{k-2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_1}{6} \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exist
-> Trying a solution in terms of special functions:
  -> Bessel
  -> elliptic
  -> Legendre
  -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
  <- Kummer successful
<- special function solution successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
Order:=6;  
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \left(1 - \frac{x^4}{12}\right) + c_2 \left(-\frac{x^5}{40} + \frac{x^3}{6} + x\right)$$

16.9 problem 16

16.9.1 Maple step by step solution 2432

Internal problem ID [5457]

Internal file name [OUTPUT/4948_Wednesday_February_14_2024_02_05_52_AM_58150107/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

[_Gegenbauer]

$$(-x^2 + 1) y'' - 2xy' + p(p + 1) y = 0$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \quad (1)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad (472)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (473)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \tag{4}$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \end{aligned} \tag{5}$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \end{aligned}$$

⋮

$$\begin{aligned} F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \tag{6}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$F_0 = \frac{p^2 y + py - 2xy'}{x^2 - 1}$$

$$\begin{aligned} F_1 &= \frac{dF_0}{dx} \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\ &= \frac{(x^2 p^2 + x^2 p - p^2 + 6x^2 - p + 2) y' - 4xpy(p+1)}{(x^2 - 1)^2} \end{aligned}$$

$$\begin{aligned} F_2 &= \frac{dF_1}{dx} \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\ &= \frac{(x+1)(-8((p^2+p+3)x^2 - p^2 - p + 3)xy' + ((p^2+p+18)x^2 - p^2 - p + 6)y(p+1)p)(x-1)}{(x^2 - 1)^4} \end{aligned}$$

$$\begin{aligned} F_3 &= \frac{dF_2}{dx} \\ &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\ &= \frac{(((p^4 + 2p^3 + 59p^2 + 58p + 120)x^4 + (-2p^4 - 4p^3 - 46p^2 - 44p + 240)x^2 + p^4 + 2p^3 - 13p^2 - 14p)(x-1)^2)}{(x^2 - 1)^5} \end{aligned}$$

$$\begin{aligned} F_4 &= \frac{dF_3}{dx} \\ &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\ &= \frac{(-18x((p^4 + 2p^3 + \frac{77}{3}p^2 + \frac{74}{3}p + 40)x^4 - 2(p^2 + p - \frac{20}{3})(p^2 + p + 10)x^2 + p^4 + 2p^3 - 17p^2 - 18p)(x-1)^3)}{(x^2 - 1)^6} \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$F_0 = -y(0)(p+1)p$$

$$F_1 = -y'(0)p^2 - y'(0)p + 2y'(0)$$

$$F_2 = y(0)p^4 + 2y(0)p^3 - 5y(0)p^2 - 6y(0)p$$

$$F_3 = y'(0)p^4 + 2y'(0)p^3 - 13y'(0)p^2 - 14y'(0)p + 24y'(0)$$

$$F_4 = -y(0)p^6 - 3y(0)p^5 + 23y(0)p^4 + 51y(0)p^3 - 94y(0)p^2 - 120y(0)p$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2p^2 - \frac{1}{2}x^2p + \frac{1}{24}p^4x^4 + \frac{1}{12}p^3x^4 - \frac{5}{24}p^2x^4 - \frac{1}{4}px^4 - \frac{1}{720}x^6p^6 - \frac{1}{240}x^6p^5 \right. \\ \left. + \frac{23}{720}x^6p^4 + \frac{17}{240}x^6p^3 - \frac{47}{360}x^6p^2 - \frac{1}{6}x^6p \right) y(0) \\ + \left(x - \frac{1}{6}p^2x^3 - \frac{1}{6}px^3 + \frac{1}{3}x^3 + \frac{1}{120}x^5p^4 + \frac{1}{60}x^5p^3 - \frac{13}{120}x^5p^2 - \frac{7}{60}x^5p + \frac{1}{5}x^5 \right) y'(0) \\ + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(-x^2 + 1)y'' - 2xy' + (p^2 + p)y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(-x^2 + 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) - 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1} \right) + (p^2 + p) \left(\sum_{n=0}^{\infty} a_n x^n \right) = 0 \quad (1)$$

Which simplifies to

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) \\ + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} (p^2 + p) a_n x^n \right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\begin{aligned} \sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n \right) \\ + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} (p^2 + p) a_n x^n \right) = 0 \end{aligned} \quad (3)$$

$n = 0$ gives

$$2a_2 + a_0 p(p+1) = 0$$

$$a_2 = -\frac{1}{2} a_0 p^2 - \frac{1}{2} a_0 p$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1 p(p+1) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 p^2 - \frac{1}{6} a_1 p + \frac{1}{3} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n (n-1) + (n+2) a_{n+2} (n+1) - 2n a_n + a_n p(p+1) = 0 \quad (4)$$

Solving for a_{n+2} , gives

$$a_{n+2} = \frac{a_n (n^2 - p^2 + n - p)}{(n+2)(n+1)} \quad (5)$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 12a_4 + a_2p(p + 1) = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5}{24}a_0p^2 - \frac{1}{4}a_0p + \frac{1}{24}a_0p^4 + \frac{1}{12}a_0p^3$$

For $n = 3$ the recurrence equation gives

$$-12a_3 + 20a_5 + a_3p(p + 1) = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13}{120}a_1p^2 - \frac{7}{60}a_1p + \frac{1}{5}a_1 + \frac{1}{120}a_1p^4 + \frac{1}{60}a_1p^3$$

For $n = 4$ the recurrence equation gives

$$-20a_4 + 30a_6 + a_4p(p + 1) = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{47}{360}a_0p^2 - \frac{1}{6}a_0p + \frac{23}{720}a_0p^4 + \frac{17}{240}a_0p^3 - \frac{1}{720}a_0p^6 - \frac{1}{240}a_0p^5$$

For $n = 5$ the recurrence equation gives

$$-30a_5 + 42a_7 + a_5p(p + 1) = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5}{63}a_1p^2 - \frac{37}{420}a_1p + \frac{1}{7}a_1 + \frac{41}{5040}a_1p^4 + \frac{29}{1680}a_1p^3 - \frac{1}{5040}a_1p^6 - \frac{1}{1680}a_1p^5$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}
 y &= a_0 + a_1x + \left(-\frac{1}{2}a_0p^2 - \frac{1}{2}a_0p\right)x^2 + \left(-\frac{1}{6}a_1p^2 - \frac{1}{6}a_1p + \frac{1}{3}a_1\right)x^3 \\
 &+ \left(-\frac{5}{24}a_0p^2 - \frac{1}{4}a_0p + \frac{1}{24}a_0p^4 + \frac{1}{12}a_0p^3\right)x^4 \\
 &+ \left(-\frac{13}{120}a_1p^2 - \frac{7}{60}a_1p + \frac{1}{5}a_1 + \frac{1}{120}a_1p^4 + \frac{1}{60}a_1p^3\right)x^5 + \dots
 \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}
 y &= \left(1 + \left(-\frac{1}{2}p^2 - \frac{1}{2}p\right)x^2 + \left(-\frac{5}{24}p^2 - \frac{1}{4}p + \frac{1}{24}p^4 + \frac{1}{12}p^3\right)x^4\right)a_0 \\
 &+ \left(x + \left(-\frac{1}{6}p^2 - \frac{1}{6}p + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}p^2 - \frac{7}{60}p + \frac{1}{5} + \frac{1}{120}p^4 + \frac{1}{60}p^3\right)x^5\right)a_1 + O(x^6)
 \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned}
 y &= \left(1 + \left(-\frac{1}{2}p^2 - \frac{1}{2}p\right)x^2 + \left(-\frac{5}{24}p^2 - \frac{1}{4}p + \frac{1}{24}p^4 + \frac{1}{12}p^3\right)x^4\right)c_1 \\
 &+ \left(x + \left(-\frac{1}{6}p^2 - \frac{1}{6}p + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}p^2 - \frac{7}{60}p + \frac{1}{5} + \frac{1}{120}p^4 + \frac{1}{60}p^3\right)x^5\right)c_2 + O(x^6)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
 y &= \left(1 - \frac{1}{2}x^2p^2 - \frac{1}{2}x^2p + \frac{1}{24}p^4x^4 + \frac{1}{12}p^3x^4 - \frac{5}{24}p^2x^4 - \frac{1}{4}px^4 - \frac{1}{720}x^6p^6 - \frac{1}{240}x^6p^5\right. \\
 &+ \left.\frac{23}{720}x^6p^4 + \frac{17}{240}x^6p^3 - \frac{47}{360}x^6p^2 - \frac{1}{6}x^6p\right)y(0) + \left(x - \frac{1}{6}p^2x^3 - \frac{1}{6}px^3 + \frac{1}{3}\right. \\
 &+ \left.\frac{1}{120}x^5p^4 + \frac{1}{60}x^5p^3 - \frac{13}{120}x^5p^2 - \frac{7}{60}x^5p + \frac{1}{5}x^5\right)y'(0) + O(x^6)
 \end{aligned}$$

$$\begin{aligned}
 y &= \left(1 + \left(-\frac{1}{2}p^2 - \frac{1}{2}p\right)x^2 + \left(-\frac{5}{24}p^2 - \frac{1}{4}p + \frac{1}{24}p^4 + \frac{1}{12}p^3\right)x^4\right)c_1 \\
 &+ \left(x + \left(-\frac{1}{6}p^2 - \frac{1}{6}p + \frac{1}{3}\right)x^3 + \left(-\frac{13}{120}p^2 - \frac{7}{60}p + \frac{1}{5} + \frac{1}{120}p^4 + \frac{1}{60}p^3\right)x^5\right)c_2 \\
 &+ O(x^6)
 \end{aligned} \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2p^2 - \frac{1}{2}x^2p + \frac{1}{24}p^4x^4 + \frac{1}{12}p^3x^4 - \frac{5}{24}p^2x^4 - \frac{1}{4}px^4 - \frac{1}{720}x^6p^6 - \frac{1}{240}x^6p^5 \right. \\ \left. + \frac{23}{720}x^6p^4 + \frac{17}{240}x^6p^3 - \frac{47}{360}x^6p^2 - \frac{1}{6}x^6p \right) y(0) \\ + \left(x - \frac{1}{6}p^2x^3 - \frac{1}{6}px^3 + \frac{1}{3}x^3 + \frac{1}{120}x^5p^4 + \frac{1}{60}x^5p^3 - \frac{13}{120}x^5p^2 - \frac{7}{60}x^5p + \frac{1}{5}x^5 \right) y'(0) \\ + O(x^6)$$

Verified OK.

$$y = \left(1 + \left(-\frac{1}{2}p^2 - \frac{1}{2}p \right) x^2 + \left(-\frac{5}{24}p^2 - \frac{1}{4}p + \frac{1}{24}p^4 + \frac{1}{12}p^3 \right) x^4 \right) c_1 \\ + \left(x + \left(-\frac{1}{6}p^2 - \frac{1}{6}p + \frac{1}{3} \right) x^3 + \left(-\frac{13}{120}p^2 - \frac{7}{60}p + \frac{1}{5} + \frac{1}{120}p^4 + \frac{1}{60}p^3 \right) x^5 \right) c_2 + O(x^6)$$

Verified OK.

16.9.1 Maple step by step solution

Let's solve

$$(-x^2 + 1)y'' - 2xy' + (p^2 + p)y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{p(p+1)y}{x^2-1} - \frac{2xy'}{x^2-1}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{2xy'}{x^2-1} - \frac{p(p+1)y}{x^2-1} = 0$$

- Check to see if x_0 is a regular singular point

- Define functions

$$\left[P_2(x) = \frac{2x}{x^2-1}, P_3(x) = -\frac{p(p+1)}{x^2-1} \right]$$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

$$\left. ((x+1) \cdot P_2(x)) \right|_{x=-1} = 1$$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left. ((x+1)^2 \cdot P_3(x)) \right|_{x=-1} = 0$$

- $x = -1$ is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = -1$$

- Multiply by denominators

$$(x^2 - 1)y'' + 2xy' - p(p+1)y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(u^2 - 2u) \left(\frac{d^2}{du^2} y(u) \right) + (2u - 2) \left(\frac{d}{du} y(u) \right) + (-p^2 - p)y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u) \right)$ to series expansion for $m = 0..1$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u) \right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k \rightarrow k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r^2 u^{-1+r} + \left(\sum_{k=0}^{\infty} (-2a_{k+1} (k+1+r)^2 + a_k (r+1+p+k)(r-p+k)) u^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$-2r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- Each term in the series must be 0, giving the recursion relation

$$-2a_{k+1}(k+1)^2 + a_k(1+p+k)(-p+k) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2} \right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^k, a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2} \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exist
-> Trying a solution in terms of special functions:
    -> Bessel
    -> elliptic
    -> Legendre
    <- Legendre successful
<- special function solution successful`

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 101

Order:=6;

```
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+p*(p+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{p(p+1)x^2}{2} + \frac{p(p^3+2p^2-5p-6)x^4}{24}\right) y(0) + \left(x - \frac{(p^2+p-2)x^3}{6} + \frac{(p^4+2p^3-13p^2-14p+24)x^5}{120}\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 120

```
AsymptoticDSolveValue[(1-x^2)*y'[x]-2*x*y'[x]+p*(p+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2 \left(\frac{1}{120} (p^2+p)^2 x^5 + \frac{7}{60} (-p^2-p) x^5 + \frac{1}{6} (-p^2-p) x^3 + \frac{x^5}{5} + \frac{x^3}{3} + x \right) + c_1 \left(\frac{1}{24} (p^2+p)^2 x^4 + \frac{1}{4} (-p^2-p) x^4 + \frac{1}{2} (-p^2-p) x^2 + 1 \right)$$

16.10 problem 17

Internal problem ID [5458]

Internal file name [OUTPUT/4949_Wednesday_February_14_2024_02_05_52_AM_71386219/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 25. Integration in series. Supplementary problems. Page 205

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"

Maple gives the following as the ode type

```
[[_2nd_order , _linear , _nonhomogeneous]]
```

$$y'' + yx^2 = x^2 + x + 1$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \quad (1)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \quad (475)$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \quad (476)$$

$$\begin{aligned} \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \end{aligned} \quad (3)$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \tag{4}$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \end{aligned} \tag{5}$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \end{aligned}$$

⋮

$$\begin{aligned} F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \tag{6}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
 F_0 &= -yx^2 + x^2 + x + 1 \\
 F_1 &= \frac{dF_0}{dx} \\
 &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
 &= -x^2 y' - 2xy + 2x + 1 \\
 F_2 &= \frac{dF_1}{dx} \\
 &= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
 &= yx^4 - x^4 - x^3 - 4xy' - x^2 - 2y + 2 \\
 F_3 &= \frac{dF_2}{dx} \\
 &= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
 &= (x^4 - 6) y' + 8yx^3 - 8x^3 - 7x^2 - 6x \\
 F_4 &= \frac{dF_3}{dx} \\
 &= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
 &= -yx^6 + x^6 + x^5 + 12y'x^3 + x^4 + 30yx^2 - 30x^2 - 20x - 12
 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
 F_0 &= 1 \\
 F_1 &= 1 \\
 F_2 &= -2y(0) + 2 \\
 F_3 &= -6y'(0) \\
 F_4 &= -12
 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = - \left(\sum_{n=0}^{\infty} a_n x^n \right) x^2 + x^2 + x + 1 \quad (1)$$

Expanding $x^2 + x + 1$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x^2 + x + 1 &= x^2 + x + 1 + \dots \\ &= x^2 + x + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} a_n x^n \right) x^2 = x^2 + x + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n \right) = x^2 + x + 1 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n$$

$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) x^n \right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n \right) = x^2 + x + 1 \quad (3)$$

$n = 0$ gives

$$(2a_2) 1 = 1$$

$$2a_2 = 1$$

Or

$$a_2 = \frac{1}{2}$$

$n = 1$ gives

$$(6a_3) x = x$$

$$6a_3 = 1$$

Which after substituting earlier equations, simplifies to

$$6a_3 = 1$$

Or

$$a_3 = \frac{1}{6}$$

For $2 \leq n$, the recurrence equation is

$$((n+2) a_{n+2} (n+1) + a_{n-2}) x^n = x^2 + x + 1 \quad (4)$$

For $n = 2$ the recurrence equation gives

$$(12a_4 + a_0) x^2 = x^2$$

$$12a_4 + a_0 = 1$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{12} - \frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 + a_1)x^3 &= 0 \\ 20a_5 + a_1 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 + a_2)x^4 &= 0 \\ 30a_6 + a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{60}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 + a_3)x^5 &= 0 \\ 42a_7 + a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{252}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{x^2}{2} + \frac{x^3}{6} + \left(\frac{1}{12} - \frac{a_0}{12}\right)x^4 - \frac{a_1 x^5}{20} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20}x^5\right) a_1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O(x^6) \quad (1)$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6) \quad (2)$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
    -> Trying a Liouvillian solution using Kovacic's algorithm
    <- No Liouvillian solutions exists
    -> Trying a solution in terms of special functions:
        -> Bessel
        <- Bessel successful
    <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=1+x+x^2,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) D(y)(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 49

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==1+x+x^2,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2 \left(x - \frac{x^5}{20}\right) + \frac{x^4}{12} + c_1 \left(1 - \frac{x^4}{12}\right) + \frac{x^3}{6} + \frac{x^2}{2}$$

17 Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

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17.1 problem 11

17.1.1 Maple step by step solution 2455

Internal problem ID [5459]

Internal file name [OUTPUT/4950_Wednesday_February_14_2024_02_05_53_AM_19838076/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 11.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$2(x^3 + x^2) y'' - (-3x^2 + x) y' + y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(2x^3 + 2x^2) y'' + (3x^2 - x) y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x - 1}{2x(x + 1)}$$
$$q(x) = \frac{1}{2x^2(x + 1)}$$

Table 273: Table $p(x), q(x)$ singularities.

$p(x) = \frac{3x-1}{2x(x+1)}$		$q(x) = \frac{1}{2x^2(x+1)}$	
singularity	type	singularity	type
$x = -1$	“regular”	$x = -1$	“regular”
$x = 0$	“regular”	$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[\]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2y''x^2(x+1) + (3x^2 - x)y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x^2(x+1) + (3x^2 - x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1) \right) \\ & + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r) (n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r} \\ \sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r} \end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} 2a_{n-1} (n+r-1) (n+r-2) x^{n+r} \right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1) \right) \\ & + \left(\sum_{n=1}^{\infty} 3a_{n-1} (n+r-1) x^{n+r} \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r) (n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r (-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(2x^r r (-1+r) - x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 - 3r + 1) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ + 3a_{n-1}(n+r-1) - a_n(n+r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -1$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = 1$$

Which for the root $r = 1$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -1$$

Which for the root $r = 1$ becomes

$$a_3 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$a_4 = 1$$

Which for the root $r = 1$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1

For $n = 5$, using the above recursive equation gives

$$a_5 = -1$$

Which for the root $r = 1$ becomes

$$a_5 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1
a_5	-1	-1

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} 2b_{n-1}(n+r-1)(n+r-2) + 2b_n(n+r)(n+r-1) \\ + 3b_{n-1}(n+r-1) - b_n(n+r) + b_n = 0 \end{aligned} \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -b_{n-1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -b_{n-1} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1

For $n = 3$, using the above recursive equation gives

$$b_3 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$b_4 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1

For $n = 5$, using the above recursive equation gives

$$b_5 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1
b_5	-1	-1

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= \sqrt{x}(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) + c_2\sqrt{x}(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) + c_2\sqrt{x}(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) + c_2\sqrt{x}(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6))$$

Verification of solutions

$$y = c_1x(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)) + c_2\sqrt{x}(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6))$$

Verified OK.

17.1.1 Maple step by step solution

Let's solve

$$2y''x^2(x+1) + (3x^2 - x)y' + y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{y}{2x^2(x+1)} - \frac{(3x-1)y'}{2x(x+1)}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{(3x-1)y'}{2x(x+1)} + \frac{y}{2x^2(x+1)} = 0$$

- Check to see if x_0 is a regular singular point

- Define functions

$$\left[P_2(x) = \frac{3x-1}{2x(x+1)}, P_3(x) = \frac{1}{2x^2(x+1)} \right]$$

- $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

$$\left. ((x+1) \cdot P_2(x)) \right|_{x=-1} = 2$$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left. ((x+1)^2 \cdot P_3(x)) \right|_{x=-1} = 0$$

- $x = -1$ is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = -1$$

- Multiply by denominators

$$2y''x^2(x+1) + x(3x-1)y' + y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(2u^3 - 4u^2 + 2u) \left(\frac{d^2}{du^2} y(u) \right) + (3u^2 - 7u + 4) \left(\frac{d}{du} y(u) \right) + y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r) u^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k \rightarrow k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(1+r) u^{-1+r} + (2a_1(1+r)(2+r) - a_0(1+r)(-1+4r)) u^r + \left(\sum_{k=1}^{\infty} (2a_{k+1}(k+r+1)(k+r) - a_k(k+r)(k+r-1)) u^{k+r}\right)$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$2r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$2a_1(1+r)(2+r) - a_0(1+r)(-1+4r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k + 2a_{k-1} + 2a_{k+1})k^2 + ((-8a_k + 4a_{k-1} + 4a_{k+1})r - 3a_k - 3a_{k-1} + 6a_{k+1})k + (-4a_k + 2a_{k-1} + 2a_{k+1})k = 0$$

- Shift index using $k \rightarrow k+1$

$$(-4a_{k+1} + 2a_k + 2a_{k+2})(k+1)^2 + ((-8a_{k+1} + 4a_k + 4a_{k+2})r - 3a_{k+1} - 3a_k + 6a_{k+2})(k+1) + (-4a_{k+1} + 2a_k + 2a_{k+2})k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} + 4kra_k - 8kra_{k+1} + 2r^2a_k - 4r^2a_{k+1} + ka_k - 11ka_{k+1} + ra_k - 11ra_{k+1} - 6a_{k+1}}{2(k^2 + 2kr + r^2 + 5k + 5r + 6)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{2k^2a_k - 4k^2a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}$$

- Solution for $r = -1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-1}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0 \right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x + 1)^{k-1}, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0 \right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0 \right]$$

- Revert the change of variables $u = x + 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x + 1)^k, a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} + ka_k - 11ka_{k+1} - 6a_{k+1}}{2(k^2 + 5k + 6)}, 4a_1 + a_0 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x + 1)^{k-1} \right) + \left(\sum_{k=0}^{\infty} b_k (x + 1)^k \right), a_{k+2} = -\frac{2k^2 a_k - 4k^2 a_{k+1} - 3ka_k - 3ka_{k+1} + a_k + a_{k+1}}{2(k^2 + 3k + 2)}, 0 = 0, \right]$$

Maple trace Kovacic algorithm successful

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Reducible group (found another exponential solution)
<- Kovacics algorithm successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

Order:=6;

```
dsolve(2*(x^2+x^3)*diff(y(x),x$2)-(x-3*x^2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-x^5 + x^4 - x^3 + x^2 - x + 1) (c_1\sqrt{x} + c_2x) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 58

```
AsymptoticDSolveValue[2*(x^2+x^3)*y'[x]-(x-3*x^2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1x(-x^5 + x^4 - x^3 + x^2 - x + 1) + c_2\sqrt{x}(-x^5 + x^4 - x^3 + x^2 - x + 1)$$

17.2 problem 12

17.2.1 Maple step by step solution 2469

Internal problem ID [5460]

Internal file name [OUTPUT/4951_Wednesday_February_14_2024_02_05_54_AM_44143317/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 12.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$4xy'' + 2(1 - x)y' - y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + (-2x + 2)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{2x}$$
$$q(x) = -\frac{1}{4x}$$

Table 275: Table $p(x), q(x)$ singularities.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	“regular”

$q(x) = -\frac{1}{4x}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + (-2x + 2)y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x + (-2x + 2) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-2x^{n+r} a_n (n+r)) \\ & + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} \right) + \sum_{n=0}^{\infty} (-a_n x^{n+r}) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=0}^{\infty} (-2x^{n+r} a_n (n+r)) &= \sum_{n=1}^{\infty} (-2a_{n-1} (n+r-1) x^{n+r-1}) \\ \sum_{n=0}^{\infty} (-a_n x^{n+r}) &= \sum_{n=1}^{\infty} (-a_{n-1} x^{n+r-1}) \end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r) (n+r-1) \right) + \sum_{n=1}^{\infty} (-2a_{n-1} (n+r-1) x^{n+r-1}) \\ & + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} \right) + \sum_{n=1}^{\infty} (-a_{n-1} x^{n+r-1}) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1} a_n (n+r) (n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$4x^{-1+r} a_0 r (-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$(4x^{-1+r} r (-1+r) + 2r x^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(4r^2 - 2r) x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 2r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$

$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(4r^2 - 2r) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n+2r} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n+1} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(1+r)(2+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(1+r)(2+r)(3+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16(1+r)(2+r)(3+r)(4+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$
a_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{10395}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
 y_1(x) &= \sqrt{x}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\
 &= \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6) \right)
 \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) + 2(n+r)b_n - b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n+2r} \quad (4)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{2n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2 + 2r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8(1+r)(2+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16(1+r)(2+r)(3+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$
b_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6) \right) \\ &\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6) \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6) \right) \\ &\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6) \right) \\ &\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6) \right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6) \right) \\ + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6) \right)$$

Verified OK.

17.2.1 Maple step by step solution

Let's solve

$$4y''x + (-2x + 2)y' - y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{y}{4x} + \frac{(x-1)y'}{2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{(x-1)y'}{2x} - \frac{y}{4x} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{x-1}{2x}, P_3(x) = -\frac{1}{4x}]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = \frac{1}{2}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$4y''x + (-2x + 2)y' - y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

$$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$2a_0 r(-1+2r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)(2k+2r+1) - a_k(2k+2r+1)) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$2r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{ 0, \frac{1}{2} \right\}$$

- Each term in the series must be 0, giving the recursion relation

$$4(a_{k+1}(k+1+r) - \frac{a_k}{2})(k+r+\frac{1}{2}) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{2(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{2(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{2(k+1)} \right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+1} = \frac{a_k}{2(k+\frac{3}{2})}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2(k+\frac{3}{2})} \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}} \right), a_{k+1} = \frac{a_k}{2(k+1)}, b_{k+1} = \frac{b_k}{2(k+\frac{3}{2})} \right]$$

Maple trace Kovacic algorithm successful

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
  A Liouvillian solution exists
  Reducible group (found an exponential solution)
  Group is reducible, not completely reducible
  Solution has integrals. Trying a special function solution free of integrals...
-> Trying a solution in terms of special functions:
  -> Bessel
  -> elliptic
  -> Legendre
  <- Kummer successful
<- special function solution successful
  -> Trying to convert hypergeometric functions to elementary form...
  <- elementary form is not straightforward to achieve - returning special function solu
<- Kovacics algorithm successful`

```


✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;  
dsolve(4*x*diff(y(x),x$2)+2*(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x} \left(1 + \frac{1}{3}x + \frac{1}{15}x^2 + \frac{1}{105}x^3 + \frac{1}{945}x^4 + \frac{1}{10395}x^5 + O(x^6) \right) \\ + c_2 \left(1 + \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4 + \frac{1}{3840}x^5 + O(x^6) \right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 85

```
AsymptoticDSolveValue[4*x*y'[x]+2*(1-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1\sqrt{x} \left(\frac{x^5}{10395} + \frac{x^4}{945} + \frac{x^3}{105} + \frac{x^2}{15} + \frac{x}{3} + 1 \right) + c_2 \left(\frac{x^5}{3840} + \frac{x^4}{384} + \frac{x^3}{48} + \frac{x^2}{8} + \frac{x}{2} + 1 \right)$$

17.3 problem 13

17.3.1 Maple step by step solution 2481

Internal problem ID [5461]

Internal file name [OUTPUT/4952_Wednesday_February_14_2024_02_05_55_AM_88251085/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 13.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$2x^2y'' - xy' + (-x^2 + 1)y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + (-x^2 + 1)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2 - 1}{2x^2}$$

Table 277: Table $p(x), q(x)$ singularities.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	“regular”

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) - x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (-x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) \\ & + \sum_{n=0}^{\infty} (-x^{n+r+2} a_n) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} (-x^{n+r+2} a_n) = \sum_{n=2}^{\infty} (-a_{n-2} x^{n+r})$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) \\ & + \sum_{n=2}^{\infty} (-a_{n-2} x^{n+r}) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r) (n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r (-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(2x^r r (-1+r) - x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 - 3r + 1) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6) \right) + c_2\sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6) \right) + c_2\sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6) \right) + c_2\sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right) \quad (1)$$

Verification of solutions

$$y = c_1x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6) \right) + c_2\sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right)$$

Verified OK.

17.3.1 Maple step by step solution

Let's solve

$$2x^2y'' - xy' + (-x^2 + 1)y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{(x^2-1)y}{2x^2} + \frac{y'}{2x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{y'}{2x} - \frac{(x^2-1)y}{2x^2} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{x^2-1}{2x^2} \right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -\frac{1}{2}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = \frac{1}{2}$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$2x^2 y'' - xy' + (-x^2 + 1)y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k \rightarrow k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-1+r)x^r + a_1(1+2r)rx^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r-1)(k+r-1) - a_{k-2})x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$(-1+2r)(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{ 1, \frac{1}{2} \right\}$$

- Each term must be 0

$$a_1(1+2r)r = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2\left(k+r-\frac{1}{2}\right)(k+r-1)a_k - a_{k-2} = 0$$

- Shift index using $k \rightarrow k+2$

$$2\left(k+\frac{3}{2}+r\right)(k+1+r)a_{k+2} - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{a_k}{(2k+3+2r)(k+1+r)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{a_k}{(2k+5)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = \frac{a_k}{(2k+5)(k+2)}, a_1 = 0 \right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = \frac{a_k}{(2k+4)\left(k+\frac{3}{2}\right)}$$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{a_k}{(2k+4)\left(k+\frac{3}{2}\right)}, a_1 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k+1} \right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}} \right), a_{k+2} = \frac{a_k}{(2k+5)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k}{(2k+4)\left(k+\frac{3}{2}\right)}, b_1 = 0 \right]$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacic's algorithm  
<- No Liouvillian solutions exist  
-> Trying a solution in terms of special functions:  
  -> Bessel  
  <- Bessel successful  
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
Order:=6;  
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x} \left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + O(x^6) \right) + c_2x \left(1 + \frac{1}{10}x^2 + \frac{1}{360}x^4 + O(x^6) \right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*y'[x]-x*y'[x]+(1-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1x \left(\frac{x^4}{360} + \frac{x^2}{10} + 1 \right) + c_2\sqrt{x} \left(\frac{x^4}{168} + \frac{x^2}{6} + 1 \right)$$

17.4 problem 14

17.4.1 Maple step by step solution 2492

Internal problem ID [5462]

Internal file name [OUTPUT/4953_Wednesday_February_14_2024_02_05_55_AM_39957459/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 14.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

[_Lienard]

$$xy'' + y' + xy = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Table 279: Table $p(x), q(x)$ singularities.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	“regular”

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + x \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n \right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \right) = 0 \quad (2B)$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r (-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$(x^{-1+r} r (-1+r) + r x^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$

$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (1A)$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \quad (1B)$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitrary constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2} \quad (4)$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2(r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(r+2)^3(r+4)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\ &\quad + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$y = c_1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) + c_2 \left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right)$$

Verified OK.

17.4.1 Maple step by step solution

Let's solve

$$y''x + y' + xy = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{y'}{x} - y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{y'}{x} + y = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = \frac{1}{x}, P_3(x) = 1]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = 1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x + y' + xy = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k \rightarrow k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + \left(\sum_{k=1}^{\infty} (a_{k+1} (k+r+1)^2 + a_{k-1}) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$
- Values of r that satisfy the indicial equation

$$r = 0$$
- Each term must be 0

$$a_1 (1+r)^2 = 0$$
- Each term in the series must be 0, giving the recursion relation

$$a_{k+1} (k+1)^2 + a_{k-1} = 0$$
- Shift index using $k \rightarrow k + 1$

$$a_{k+2} (k+2)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2)^2}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0 \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exist
-> Trying a solution in terms of special functions:
    -> Bessel
    <- Bessel successful
<- special function solution successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```

Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);

```

$$y(x) = (c_2 \ln(x) + c_1) \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + O(x^6) \right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + O(x^6) \right) c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) + c_2 \left(-\frac{3x^4}{128} + \frac{x^2}{4} + \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) \log(x) \right)$$

17.5 problem 15

17.5.1 Maple step by step solution 2503

Internal problem ID [5463]

Internal file name [OUTPUT/4954_Wednesday_February_14_2024_02_05_56_AM_55353319/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 15.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$x^2y'' - xy' + (x^2 + 1)y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy' + (x^2 + 1)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^2 + 1}{x^2}$$

Table 281: Table $p(x), q(x)$ singularities.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	“regular”

$q(x) = \frac{x^2+1}{x^2}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - x y' + (x^2 + 1) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) \\ & - x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + (x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) \\ & + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-x^{n+r} a_n (n+r)) \\ & + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r (-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$(x^r r (-1+r) - x^r r + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$

$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1 + r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (1A)$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \quad (1B)$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1} \right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} + a_n = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2(r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$	$\frac{2}{(1+r)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$	$\frac{-4r-8}{(1+r)^3(r+3)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \\ &\quad + c_2 \left(x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right) \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned}
 y &= y_h \\
 &= c_1 x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \\
 &\quad + c_2 \left(x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right) \right)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
 y &= c_1 x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \\
 &\quad + c_2 \left(x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right) \right)
 \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned}
 y &= c_1 x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \\
 &\quad + c_2 \left(x \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \right) \ln(x) + x \left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6) \right) \right)
 \end{aligned}$$

Verified OK.

17.5.1 Maple step by step solution

Let's solve

$$x^2 y'' - x y' + (x^2 + 1) y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{(x^2+1)y}{x^2} + \frac{y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{y'}{x} + \frac{(x^2+1)y}{x^2} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{x^2+1}{x^2} \right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -1$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 1$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$x^2 y'' - x y' + (x^2 + 1) y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..2$

$$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

$$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r}$$

- Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+r)^2 x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k (k+r-1)^2 + a_{k-2}) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$(-1 + r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 1$$

- Each term must be 0

$$a_1 r^2 = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_k(k + r - 1)^2 + a_{k-2} = 0$$

- Shift index using $k \rightarrow k + 2$

$$a_{k+2}(k + 1 + r)^2 + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+1+r)^2}$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{a_k}{(k+2)^2}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0 \right]$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacic's algorithm  
<- No Liouvillian solutions exist  
-> Trying a solution in terms of special functions:  
  -> Bessel  
  <- Bessel successful  
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;  
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x \left((c_2 \ln(x) + c_1) \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + O(x^6) \right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + O(x^6) \right) c_2 \right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*y'[x]-x*y'[x]+(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 x \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) + c_2 \left(x \left(\frac{x^2}{4} - \frac{3x^4}{128} \right) + x \left(\frac{x^4}{64} - \frac{x^2}{4} + 1 \right) \log(x) \right)$$

17.6 problem 16

17.6.1 Maple step by step solution 2519

Internal problem ID [5464]

Internal file name [OUTPUT/4955_Wednesday_February_14_2024_02_05_56_AM_11051350/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 16.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

[[_Emden , _Fowler]]

$$xy'' - 2y' + y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{1}{x}$$

Table 283: Table $p(x), q(x)$ singularities.

$p(x) = -\frac{2}{x}$	
singularity	type
$x = 0$	“regular”

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 2y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x - 2 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1) \right) + \sum_{n=0}^{\infty} (-2(n+r) a_n x^{n+r-1}) + \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1) \right) + \sum_{n=0}^{\infty} (-2(n+r) a_n x^{n+r-1}) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} \right) = 0 \quad (2B)$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r (-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$(x^{-1+r} r (-1+r) - 2r x^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r x^{-1+r} (-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$

$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r x^{-1+r} (-3+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x^3 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+3}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1 . Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 2a_n(n+r) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 - 3n - 3r} \quad (4)$$

Which for the root $r = 3$ becomes

$$a_n = -\frac{a_{n-1}}{n(n+3)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 - r - 2}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^4 - 5r^2 + 4}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r^4 - 5r^2 + 4) r (r + 3)}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^4 - 5r^2 + 4) r (r + 3) (r^2 + 5r + 4)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r+1)^2 (r-2) (r+2)^2 (-1+r) r (r+3) (r+4) (r+5)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{1}{806400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$
a_5	$-\frac{1}{(r+1)^2(r-2)(r+2)^2(-1+r)r(r+3)(r+4)(r+5)}$	$-\frac{1}{806400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\
&= x^3\left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_3 \\
&= -\frac{1}{(r^4 - 5r^2 + 4)r(r+3)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \rightarrow r_2} -\frac{1}{(r^4 - 5r^2 + 4)r(r+3)} &= \lim_{r \rightarrow 0} -\frac{1}{(r^4 - 5r^2 + 4)r(r+3)} \\
&= \text{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Therefore

$$\begin{aligned} \frac{d}{dx}y_2(x) &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x} \right) \\ &= Cy_1'(x) \ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2) \right) \\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2} \right) \\ &= Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2) \right) \end{aligned}$$

Substituting these back into the given ode $xy'' - 2y' + y = 0$ gives

$$\begin{aligned} &\left(Cy_1''(x) \ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2} \right) \right) x - 2Cy_1'(x) \ln(x) \\ &\quad - \frac{2Cy_1(x)}{x} - 2 \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x} \right) + Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \end{aligned}$$

Which can be written as

$$\begin{aligned} &\left((y_1''(x)x + y_1(x) - 2y_1'(x)) \ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) x - \frac{2y_1(x)}{x} \right) C \\ &\quad + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2} \right) \right) x \\ &\quad - 2 \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x} \right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \end{aligned} \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)x + y_1(x) - 2y_1'(x) = 0$$

Eq (7) simplifies to

$$\begin{aligned} & \left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) x - \frac{2y_1(x)}{x} \right) C \\ & + \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) x \\ & - 2 \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2} (n+r_2)}{x} \right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \end{aligned} \quad (8)$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned} & \frac{\left(2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1) \right) x - 3 \left(\sum_{n=0}^{\infty} a_n x^{n+r_1} \right) \right) C}{x} \\ & + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2) (-1+n+r_2) \right) x^2 + \left(-2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2) \right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) \right) x}{x} \\ & = 0 \end{aligned} \quad (9)$$

Since $r_1 = 3$ and $r_2 = 0$ then the above becomes

$$\begin{aligned} & \frac{\left(2 \left(\sum_{n=0}^{\infty} x^{2+n} a_n (n+3) \right) x - 3 \left(\sum_{n=0}^{\infty} a_n x^{n+3} \right) \right) C}{x} \\ & + \frac{\left(\sum_{n=0}^{\infty} x^{-2+n} b_n n (n-1) \right) x^2 + \left(-2 \left(\sum_{n=0}^{\infty} x^{n-1} b_n n \right) + \left(\sum_{n=0}^{\infty} b_n x^n \right) \right) x}{x} = 0 \end{aligned} \quad (10)$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} 2C x^{2+n} a_n (n+3) \right) + \sum_{n=0}^{\infty} (-3C x^{2+n} a_n) \\ & + \left(\sum_{n=0}^{\infty} n x^{n-1} b_n (n-1) \right) + \sum_{n=0}^{\infty} (-2x^{n-1} b_n n) + \left(\sum_{n=0}^{\infty} b_n x^n \right) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=0}^{\infty} 2C x^{2+n} a_n (n+3) &= \sum_{n=3}^{\infty} 2C a_{n-3} n x^{n-1} \\ \sum_{n=0}^{\infty} (-3C x^{2+n} a_n) &= \sum_{n=3}^{\infty} (-3C a_{n-3} x^{n-1}) \\ \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n - 1$.

$$\begin{aligned}\left(\sum_{n=3}^{\infty} 2C a_{n-3} n x^{n-1} \right) + \sum_{n=3}^{\infty} (-3C a_{n-3} x^{n-1}) + \left(\sum_{n=0}^{\infty} n x^{n-1} b_n (n-1) \right) \\ + \sum_{n=0}^{\infty} (-2x^{n-1} b_n n) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1} \right) = 0\end{aligned} \quad (2B)$$

For $n = 0$ in Eq. (2B), we choose arbitrary value for b_0 as $b_0 = 1$. For $n = 1$, Eq (2B) gives

$$-2b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C , gives

$$-2b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n = 2$, Eq (2B) gives

$$-2b_2 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C , gives

$$-2b_2 + \frac{1}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{4}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C + \frac{1}{4} = 0$$

Which is solved for C . Solving for C gives

$$C = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$5Ca_1 + b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C , gives

$$4b_4 + \frac{5}{48} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{192}$$

For $n = 5$, Eq (2B) gives

$$7Ca_2 + b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C , gives

$$10b_5 - \frac{13}{320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{13}{3200}$$

Now that we found all b_n and C , we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Using the above value found for $C = -\frac{1}{12}$ and all b_n , then the second solution becomes

$$y_2(x) = -\frac{1}{12} \left(x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \right) \ln(x) \\ + 1 + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned}
 y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
 &= c_1 x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \\
 &\quad + c_2 \left(-\frac{1}{12} \left(x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \right) \ln(x) + 1 \right. \\
 &\quad \left. + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6) \right)
 \end{aligned}$$

Hence the final solution is

$$\begin{aligned}
 y &= y_h \\
 &= c_1 x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \\
 &\quad + c_2 \left(-\frac{x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \ln(x)}{12} + 1 + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} \right. \\
 &\quad \left. + \frac{13x^5}{3200} + O(x^6) \right)
 \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
 y &= c_1 x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \\
 &\quad + c_2 \left(-\frac{x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \ln(x)}{12} + 1 + \frac{x}{2} + \frac{x^2}{4} \right. \\
 &\quad \left. - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6) \right) \quad (1)
 \end{aligned}$$

Verification of solutions

$$y = c_1 x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \\ + c_2 \left(-\frac{x^3 \left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6) \right) \ln(x)}{12} + 1 + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} \right. \\ \left. + \frac{13x^5}{3200} + O(x^6) \right)$$

Verified OK.

17.6.1 Maple step by step solution

Let's solve

$$y''x - 2y' + y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{y}{x} + \frac{2y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' - \frac{2y'}{x} + \frac{y}{x} = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{1}{x}]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = -2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x - 2y' + y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k- > k+1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k- > k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-3+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} (a_{k+1} (k+1+r)(k-2+r) + a_k) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1} (k+1+r)(k-2+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = -\frac{a_k}{(k+1+r)(k-2+r)}$$

- Recursion relation for $r = 0$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-2)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k}{(k+4)(k+1)} \right]$$

Maple trace

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacic's algorithm
<- No Liouvillian solutions exist
-> Trying a solution in terms of special functions:
    -> Bessel
    <- Bessel successful
<- special function solution successful`

```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 56

```

Order:=6;
dsolve(x*difff(y(x),x$2)-2*difff(y(x),x)+y(x)=0,y(x),type='series',x=0);

```

$$\begin{aligned}
 y(x) = & c_1 x^3 \left(1 - \frac{1}{4}x + \frac{1}{40}x^2 - \frac{1}{720}x^3 + \frac{1}{20160}x^4 - \frac{1}{806400}x^5 + O(x^6) \right) \\
 & + c_2 \left(\ln(x) \left(-x^3 + \frac{1}{4}x^4 - \frac{1}{40}x^5 + O(x^6) \right) \right. \\
 & \quad \left. + \left(12 + 6x + 3x^2 - \frac{5}{16}x^4 + \frac{39}{800}x^5 + O(x^6) \right) \right)
 \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 79

```
AsymptoticDSolveValue[x*y''[x]-2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \left(\frac{1}{48} (x-4)x^3 \log(x) + \frac{1}{576} (-19x^4 + 16x^3 + 144x^2 + 288x + 576) \right) \\ + c_2 \left(\frac{x^7}{20160} - \frac{x^6}{720} + \frac{x^5}{40} - \frac{x^4}{4} + x^3 \right)$$

17.7 problem 17

17.7.1 Maple step by step solution 2532

Internal problem ID [5465]

Internal file name [OUTPUT/4956_Wednesday_February_14_2024_02_05_58_AM_87988338/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 17.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

[_Lienard]

$$xy'' + 2y' + xy = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 285: Table $p(x), q(x)$ singularities.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	“regular”

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x + 2 \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + x \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1) \right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n \right) = 0 \quad (2A)$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n + r - 1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} \right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} \right) = 0 \quad (2B)$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r) (n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$x^{-1+r} a_0 r (-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$(x^{-1+r} r (-1+r) + 2r x^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r x^{-1+r} (1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$

$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r x^{-1+r} (1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right)$$

Where C above can be zero. We start by finding y_1 . Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-2} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \quad (4)$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(1+n)} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow r_2} 0 &= \lim_{r \rightarrow -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)b_n + b_{n-2} = 0 \quad (4)$$

Which for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)b_n + b_{n-2} = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \quad (5)$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - n} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$y = y_h$$

$$= c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \right)}{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \right)}{x} \quad (1)$$

Verification of solutions

$$y = c_1 \left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \right) + \frac{c_2 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6) \right)}{x}$$

Verified OK.

17.7.1 Maple step by step solution

Let's solve

$$y''x + 2y' + xy = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = -\frac{2y'}{x} - y$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{2y'}{x} + y = 0$$

- Check to see if $x_0 = 0$ is a regular singular point

- Define functions

$$[P_2(x) = \frac{2}{x}, P_3(x) = 1]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x)) \Big|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x)) \Big|_{x=0} = 0$$

- $x = 0$ is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$y''x + 2y' + xy = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

$$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k \rightarrow k - 1$

$$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

$$y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1) x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-1}$$

- Shift index using $k \rightarrow k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+r+1)(k+r) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(1+r) x^{-1+r} + a_1 (1+r)(2+r) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1} (k+r+1)(k+2+r) + a_{k-1}) x^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{-1, 0\}$$

- Each term must be 0

$$a_1(1+r)(2+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$$

- Shift index using $k \rightarrow k+1$

$$a_{k+2}(k+2+r)(k+3+r) + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$$

- Recursion relation for $r = -1$

$$a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$$

- Solution for $r = -1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0 \right]$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1} \right) + \left(\sum_{k=0}^{\infty} b_k x^k \right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0 \right]$$

Maple trace Kovacic algorithm successful

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacics algorithm  
  A Liouvillian solution exists  
  Group is reducible or imprimitive  
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;  
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 \left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O(x^6) \right) + \frac{c_2 \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6) \right)}{x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \rightarrow c_1 \left(\frac{x^3}{24} - \frac{x}{2} + \frac{1}{x} \right) + c_2 \left(\frac{x^4}{120} - \frac{x^2}{6} + 1 \right)$$

17.8 problem 18

17.8.1 Maple step by step solution 2546

Internal problem ID [5466]

Internal file name [OUTPUT/4957_Wednesday_February_14_2024_02_05_58_AM_3640080/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 18.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$x^2(x+1)y'' + x(x+1)y' - y = 0$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^3 + x^2)y'' + (x^2 + x)y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2(x+1)}$$

Table 287: Table $p(x), q(x)$ singularities.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	“regular”

$q(x) = -\frac{1}{x^2(x+1)}$	
singularity	type
$x = -1$	“regular”
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -1, \infty]$

Irregular singular points : $[\]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)y'' + (x^2+x)y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x^2(x+1) \\ & + (x^2+x) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1) \right) \\ & + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} (-a_n x^{n+r}) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) (n+r-1) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r} \\ \sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} \end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} & \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) (n+r-2) x^{n+r} \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1) \right) \\ & + \left(\sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r} \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} (-a_n x^{n+r}) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r (-1+r) + x^r a_0 r - a_0 x^r = 0$$

Or

$$(x^r r (-1+r) + x^r r - x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 - 1) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= x \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1} \right) \end{aligned}$$

Where C above can be zero. We start by finding y_1 . Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + a_n(n+r)(n+r-1) \\ + a_{n-1}(n+r-1) + a_n(n+r) - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-1)}{1+n+r} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}n}{2+n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{2+r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+1)r}{(2+r)(3+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(r+1)}{(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+1)}{(4+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$
a_4	$\frac{r(r+1)}{(4+r)(5+r)}$	$\frac{1}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(r+1)}{(5+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$
a_4	$\frac{r(r+1)}{(4+r)(5+r)}$	$\frac{1}{15}$
a_5	$-\frac{r(r+1)}{(5+r)(6+r)}$	$-\frac{1}{21}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \rightarrow r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{(r+1)r}{(2+r)(3+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \rightarrow r_2} \frac{(r+1)r}{(2+r)(3+r)} &= \lim_{r \rightarrow -1} \frac{(r+1)r}{(2+r)(3+r)} \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) + b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + b_n(n+r) - b_n = 0 \quad (4)$$

Which for the root $r = -1$ becomes

$$b_{n-1}(n-2)(n-3) + b_n(n-1)(n-2) + b_{n-1}(n-2) + b_n(n-1) - b_n = 0 \quad (4A)$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}(n+r-1)}{1+n+r} \quad (5)$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-1}(n-2)}{n} \quad (6)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{r}{2+r}$$

Which for the root $r = -1$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r+1)r}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r(r+1)}{(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r+1)}{(4+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0
b_4	$\frac{r(r+1)}{(4+r)(5+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r(r+1)}{(5+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0
b_4	$\frac{r(r+1)}{(4+r)(5+r)}$	0
b_5	$-\frac{r(r+1)}{(5+r)(6+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots) \\ &= \frac{1 + x + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x \left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6) \right) + \frac{c_2(1 + x + O(x^6))}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1x \left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6) \right) + \frac{c_2(1 + x + O(x^6))}{x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x \left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6) \right) + \frac{c_2(1 + x + O(x^6))}{x} \quad (1)$$

Verification of solutions

$$y = c_1x \left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6) \right) + \frac{c_2(1 + x + O(x^6))}{x}$$

Verified OK.

17.8.1 Maple step by step solution

Let's solve

$$y''x^2(1+x) + (x^2+x)y' - y = 0$$

- Highest derivative means the order of the ODE is 2

$$y''$$

- Isolate 2nd derivative

$$y'' = \frac{y}{x^2(1+x)} - \frac{y'}{x}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + \frac{y'}{x} - \frac{y}{x^2(1+x)} = 0$$

- Check to see if x_0 is a regular singular point

- Define functions

$$\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{1}{x^2(1+x)} \right]$$

- $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

$$\left. ((1+x) \cdot P_2(x)) \right|_{x=-1} = 0$$

- $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

$$\left. ((1+x)^2 \cdot P_3(x)) \right|_{x=-1} = 0$$

- $x = -1$ is a regular singular point

Check to see if x_0 is a regular singular point

$$x_0 = -1$$

- Multiply by denominators

$$y''x^2(1+x) + x(1+x)y' - y = 0$$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

$$(u^3 - 2u^2 + u) \left(\frac{d^2}{du^2} y(u) \right) + (u^2 - u) \left(\frac{d}{du} y(u) \right) - y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

- Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du} y(u) \right)$ to series expansion for $m = 1..2$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r) u^{k+r-1+m}$$

- Shift index using $k \rightarrow k+1-m$

$$u^m \cdot \left(\frac{d}{du} y(u) \right) = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r) u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u) \right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) u^{k+r-2+m}$$

- Shift index using $k \rightarrow k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u) \right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r) (k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+r) u^{-1+r} + (a_1(1+r)r - a_0(2r^2 - r + 1)) u^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r) - a_k(2k^2 + 4kr + 2r^2 - k - r + 1)) u^{k+r} \right) = 0$$

- a_0 cannot be 0 by assumption, giving the indicial equation

$$r(-1+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 1\}$$

- Each term must be 0

$$a_1(1+r)r - a_0(2r^2 - r + 1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1+r)(k+r) - a_k(2k^2 + 4kr + 2r^2 - k - r + 1) + a_{k-1}(k+r-1)^2 = 0$$

- Shift index using $k \rightarrow k + 1$

$$a_{k+2}(k+2+r)(k+1+r) - a_{k+1}(2(k+1)^2 + 4(k+1)r + 2r^2 - k - r) + a_k(k+r)^2 = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} + 2k r a_k - 4k r a_{k+1} + r^2 a_k - 2r^2 a_{k+1} - 3k a_{k+1} - 3r a_{k+1} - 2a_{k+1}}{(k+2+r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_{k+1} - 2a_{k+1}}{(k+2)(k+1)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_{k+1} - 2a_{k+1}}{(k+2)(k+1)}, -a_0 = 0 \right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^k, a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_{k+1} - 2a_{k+1}}{(k+2)(k+1)}, -a_0 = 0 \right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 7k a_{k+1} + a_k - 7a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 1$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+1}, a_{k+2} = - \frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 7k a_{k+1} + a_k - 7a_{k+1}}{(k+3)(k+2)}, 2a_1 - 2a_0 = 0 \right]$$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2ka_k - 7ka_{k+1} + a_k - 7a_{k+1}}{(k+3)(k+2)}, 2a_1 - 2a_0 = 0 \right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k \right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+1} \right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3ka_{k+1} - 2a_{k+1}}{(k+2)(k+1)}, -a_0 = 0, b_{k+2} \right]$$

Maple trace Kovacic algorithm successful

```

Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Group is reducible, not completely reducible
<- Kovacics algorithm successful

```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```

Order:=6;
dsolve(x^2*(x+1)*diff(y(x),x$2)+x*(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);

```

$$y(x) = c_1 x \left(1 - \frac{1}{3}x + \frac{1}{6}x^2 - \frac{1}{10}x^3 + \frac{1}{15}x^4 - \frac{1}{21}x^5 + O(x^6) \right) + \frac{c_2(-2 - 2x + O(x^6))}{x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 45

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AsymptoticDSolveValue[x^2*(x+1)*y'[x]+x*(x+1)*y'[x]-y[x]==0,y[x],{x,0,5}]

```

$$y(x) \rightarrow c_2 \left(\frac{x^5}{15} - \frac{x^4}{10} + \frac{x^3}{6} - \frac{x^2}{3} + x \right) + c_1 \left(\frac{1}{x} + 1 \right)$$

17.9 problem 19

Internal problem ID [5467]

Internal file name [OUTPUT/4958_Wednesday_February_14_2024_02_05_59_AM_96706282/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 19.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$2xy'' + y' - y = 1 + x$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x}$$

Table 289: Table $p(x), q(x)$ singularities.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	“regular”

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' - y = 1 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2xy'' + y' - y = 0$, and y_p is a particular solution to the inhomogeneous ode. which is found using the balance equation generated from indicial equation

First, we solve for y_h . Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \sum_{n=0}^{\infty} (-a_n x^{n+r}) = 0 \quad (2A)$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} (-a_n x^{n+r}) = \sum_{n=1}^{\infty} (-a_{n-1} x^{n+r-1})$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \right) + \sum_{n=1}^{\infty} (-a_{n-1} x^{n+r-1}) = 0 \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r) (n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r (-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$(2x^{-1+r} r (-1+r) + r x^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r x^{-1+r} (-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= 0 \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^{-1+m}m(-1+m) + mx^{-1+m})c_0 = 1 + x$$

This equation will be used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$rx^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as. Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \quad (4)$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n^2 + n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 + 72600r + 10560}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots) \\ &= \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6) \right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + (n+r)b_n - b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \quad (4)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{n(2n-1)} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 - \dots}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6) \right) \\ &\quad + c_2 \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6) \right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^{-1+m}m(-1+m) + mx^{-1+m})c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. In order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m .

The following are the values of a_n found in terms of the indicial root r .

$$\begin{aligned}
 a_1 &= \frac{a_0}{2r^2+3r+1} \\
 a_2 &= \frac{a_0}{4r^4+20r^3+35r^2+25r+6} \\
 a_3 &= \frac{a_0}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90} \\
 a_4 &= \frac{a_0}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520} \\
 a_5 &= \frac{a_0}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}
 \end{aligned}$$

Since the $F = 1 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$(2x^{-1+m}m(-1+m) + mx^{-1+m})c_0 = 1$$

For c_0 and x . This results in

$$c_0 = 1$$

$$m = 1$$

The particular solution is therefore

$$\begin{aligned}
 y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
 &= \sum_{n=0}^{\infty} c_n x^{n+1}
 \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$c_0 = 1$$

$$c_1 = \frac{1}{6}$$

$$c_2 = \frac{1}{90}$$

$$c_3 = \frac{1}{2520}$$

$$c_4 = \frac{1}{113400}$$

$$c_5 = \frac{1}{7484400}$$

The particular solution is now found using

$$\begin{aligned}
 y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
 &= x \sum_{n=0}^{\infty} c_n x^n
 \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned}
 y_p &= x \left(1 + \frac{1}{6}x + \frac{1}{90}x^2 + \frac{1}{2520}x^3 + \frac{1}{113400}x^4 + \frac{1}{7484400}x^5 \right) \\
 &= x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2520}x^4 + \frac{1}{113400}x^5 + \frac{1}{7484400}x^6
 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(2x^{-1+m}m(-1+m) + mx^{-1+m})c_0 = x$$

For c_0 and x . This results in

$$\begin{aligned}
 c_0 &= \frac{1}{6} \\
 m &= 2
 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned}
 y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
 &= \sum_{n=0}^{\infty} c_n x^{n+2}
 \end{aligned}$$

Where in the above $c_0 = \frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{6}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{6}$ $c_1 = \frac{1}{90}$ $c_2 = \frac{1}{2520}$ $c_3 = \frac{1}{113400}$ $c_4 = \frac{1}{7484400}$ $c_5 = \frac{1}{681080400}$
--

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n$$

$$= x^2 \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$y_p = x^2 \left(\frac{1}{6} + \frac{1}{90}x + \frac{1}{2520}x^2 + \frac{1}{113400}x^3 + \frac{1}{7484400}x^4 + \frac{1}{681080400}x^5 \right)$$

$$= \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2520}x^4 + \frac{1}{113400}x^5 + \frac{1}{7484400}x^6 + \frac{1}{681080400}x^7$$

Adding all the above particular solution(s) gives

$$y_p = x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + \frac{x^6}{3742200} + \frac{x^7}{681080400} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6)$$

Hence the final solution is

$$y = y_h + y_p$$

$$= x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6)$$

$$+ c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6) \right)$$

$$+ c_2 \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6) \right)$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = & x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6) \\ & + c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6) \right) \\ & + c_2 \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6) \right) \end{aligned} \quad (1)$$

Verification of solutions

$$\begin{aligned} y = & x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6) \\ & + c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6) \right) \\ & + c_2 \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6) \right) \end{aligned}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
    checking if the LODE has constant coefficients
    checking if the LODE is of Euler type
    trying a symmetry of the form [xi=0, eta=F(x)]
    <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

Order:=6;

dsolve(2*x*diff(y(x),x\$2)+diff(y(x),x)-y(x)=x+1,y(x),type='series',x=0);

$$\begin{aligned}
 y(x) = & c_1 \sqrt{x} \left(1 + \frac{1}{3}x + \frac{1}{30}x^2 + \frac{1}{630}x^3 + \frac{1}{22680}x^4 + \frac{1}{1247400}x^5 + O(x^6) \right) \\
 & + c_2 \left(1 + x + \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2520}x^4 + \frac{1}{113400}x^5 + O(x^6) \right) \\
 & + x \left(1 + \frac{1}{3}x + \frac{1}{45}x^2 + \frac{1}{1260}x^3 + \frac{1}{56700}x^4 + O(x^5) \right)
 \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 246

AsymptoticDSolveValue[2*x*y''[x]+y'[x]-y[x]==x+1,y[x],{x,0,5}]

$$\begin{aligned}
 y(x) \rightarrow & c_1 \left(\frac{x^5}{113400} + \frac{x^4}{2520} + \frac{x^3}{90} + \frac{x^2}{6} + x + 1 \right) \\
 & + c_2 \sqrt{x} \left(\frac{x^5}{1247400} + \frac{x^4}{22680} + \frac{x^3}{630} + \frac{x^2}{30} + \frac{x}{3} + 1 \right) + \sqrt{x} \left(\frac{x^5}{1247400} + \frac{x^4}{22680} + \frac{x^3}{630} \right. \\
 & \quad \left. + \frac{x^2}{30} + \frac{x}{3} + 1 \right) \left(\frac{23x^{11/2}}{311850} + \frac{29x^{9/2}}{11340} + \frac{16x^{7/2}}{315} + \frac{7x^{5/2}}{15} + \frac{4x^{3/2}}{3} \right. \\
 & \left. + 2\sqrt{x} \right) + \left(\frac{x^5}{113400} + \frac{x^4}{2520} + \frac{x^3}{90} + \frac{x^2}{6} + x + 1 \right) \left(-\frac{x^6}{133650} - \frac{37x^5}{113400} - \frac{11x^4}{1260} - \frac{11x^3}{90} - \frac{2x^2}{3} - x \right)
 \end{aligned}$$

17.10 problem 20

Internal problem ID [5468]

Internal file name [OUTPUT/4959_Wednesday_February_14_2024_02_06_00_AM_35920087/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

[[_Emden , _Fowler]]

$$2x^3y'' + x^2y' + y = 0$$

With the expansion point for the power series method at $x = \infty$.

Since expansion is around ∞ , then the independent variable x is replaced by $\frac{1}{t}$ and the expansion is made around $t = 0$ and after solving, the solution is changed back to x using $x = \frac{1}{t}$. Changing variables results in the new ode

$$-\frac{2\left(-\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 2t\left(\frac{d}{dt}y(t)\right)\right)}{t} - \frac{d}{dt}y(t) + y(t) = 0$$

The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2\left(\frac{d^2}{dt^2}y(t)\right)t + 3\frac{d}{dt}y(t) + y(t) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$p(t) = \frac{3}{2t}$$

$$q(t) = \frac{1}{2t}$$

Table 290: Table $p(t), q(t)$ singularities.

$p(t) = \frac{3}{2t}$	
singularity	type
$t = 0$	“regular”

$q(t) = \frac{1}{2t}$	
singularity	type
$t = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2 \left(\frac{d^2}{dt^2} y(t) \right) t + 3 \frac{d}{dt} y(t) + y(t) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$\frac{d}{dt} y(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}$$

$$\frac{d^2}{dt^2} y(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} \right) t + 3 \left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n t^{n+r} \right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n t^{n+r-1} \right) + \left(\sum_{n=0}^{\infty} a_n t^{n+r} \right) = 0 \quad (2A)$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n t^{n+r} = \sum_{n=1}^{\infty} a_{n-1} t^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r) (n+r-1) \right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n t^{n+r-1} \right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^{n+r-1} \right) = 0 \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2t^{n+r-1} a_n (n+r) (n+r-1) + 3(n+r) a_n t^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2t^{-1+r} a_0 r (-1+r) + 3r a_0 t^{-1+r} = 0$$

Or

$$(2t^{-1+r} r (-1+r) + 3r t^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r^2 + r) t^{-1+r} = 0$$

Since the above is true for all t then the indicial equation becomes

$$2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -\frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 + r)t^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(t) &= t^{r_1} \left(\sum_{n=0}^{\infty} a_n t^n \right) \\ y_2(t) &= t^{r_2} \left(\sum_{n=0}^{\infty} b_n t^n \right) \end{aligned}$$

Or

$$\begin{aligned} y_1(t) &= \sum_{n=0}^{\infty} a_n t^n \\ y_2(t) &= \sum_{n=0}^{\infty} b_n t^{n-\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 3a_n(n+r) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \quad (4)$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 878630r^2 + 32000r + 1280}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(t)$ is

$$\begin{aligned}
y_1(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 \dots \\
&= 1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6)
\end{aligned}$$

Now the second solution $y_2(t)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 3(n+r)b_n + b_{n-1} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \quad (4)$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{b_{n-1}}{n(2n-1)} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 8786367r^2 + 5039190r + 1247400}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(t)$ is

$$\begin{aligned} y_2(t) &= 1(b_0 + b_1t + b_2t^2 + b_3t^3 + b_4t^4 + b_5t^5 + b_6t^6 \dots) \\ &= \frac{1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6)}{\sqrt{t}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(t) &= c_1y_1(t) + c_2y_2(t) \\ &= c_1 \left(1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6) \right) \\ &\quad + \frac{c_2 \left(1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6) \right)}{\sqrt{t}} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y(t) &= y_h \\ &= c_1 \left(1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6) \right) \\ &\quad + \frac{c_2 \left(1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6) \right)}{\sqrt{t}} \end{aligned}$$

Replacing t by $\frac{1}{x}$ gives

$$y = c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right) \right) + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \frac{1}{113400x^5} + O\left(\frac{1}{x^6}\right) \right)}{\sqrt{\frac{1}{x}}}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right) \right) \\ &\quad + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \frac{1}{113400x^5} + O\left(\frac{1}{x^6}\right) \right)}{\sqrt{\frac{1}{x}}} \end{aligned} \quad (1)$$

Verification of solutions

$$y = c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right) \right) \\ + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \frac{1}{113400x^5} + O\left(\frac{1}{x^6}\right) \right)}{\sqrt{\frac{1}{x}}}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacics algorithm  
    A Liouvillian solution exists  
    Group is reducible or imprimitive  
<- Kovacics algorithm successful`
```


✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 117

Order:=6;

dsolve(2*x^3*diff(y(x),x\$2)+x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=infinity);

$$\begin{aligned}
 y(x) = & \left(1 - \frac{(x - \text{Infinity})^2}{4 \text{Infinity}^3} + \frac{7(x - \text{Infinity})^3}{24 \text{Infinity}^4} + \frac{(-59 \text{Infinity} + 2)(x - \text{Infinity})^4}{192 \text{Infinity}^6} \right. \\
 & \left. + \frac{(605 \text{Infinity} - 52)(x - \text{Infinity})^5}{1920 \text{Infinity}^7} \right) y(\text{Infinity}) \\
 & + \left(x - \text{Infinity} - \frac{(x - \text{Infinity})^2}{4 \text{Infinity}} + \frac{(3 \text{Infinity}^2 - 2 \text{Infinity})(x - \text{Infinity})^3}{24 \text{Infinity}^4} \right. \\
 & \left. - \frac{5(\text{Infinity} - \frac{28}{15})(x - \text{Infinity})^4}{64 \text{Infinity}^4} \right. \\
 & \left. + \frac{(105 \text{Infinity}^3 - 370 \text{Infinity}^2 + 4 \text{Infinity})(x - \text{Infinity})^5}{1920 \text{Infinity}^7} \right) D(y)(\text{Infinity}) \\
 & + O(x^6)
 \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 96

AsymptoticDSolveValue[2*x^3*y''[x]+x^2*y'[x]+y[x]==0,y[x],{x,infinity,5}]

$$\begin{aligned}
 y(x) \rightarrow & c_2 \left(\frac{1}{6x^{3/2}} - \frac{1}{90x^{5/2}} + \frac{1}{2520x^{7/2}} - \frac{1}{113400x^{9/2}} + \sqrt{x} \right. \\
 & \left. - \frac{1}{\sqrt{x}} \right) + c_1 \left(-\frac{1}{1247400x^5} + \frac{1}{22680x^4} - \frac{1}{630x^3} + \frac{1}{30x^2} - \frac{1}{3x} + 1 \right)
 \end{aligned}$$

17.11 problem 21

Internal problem ID [5469]

Internal file name [OUTPUT/4960_Wednesday_February_14_2024_02_06_01_AM_85208353/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 26. Integration in series (singular points). Supplementary problems. Page 218

Problem number: 21.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$x^3 y'' + (x^2 + x) y' - y = 0$$

With the expansion point for the power series method at $x = \infty$.

Since expansion is around ∞ , then the independent variable x is replaced by $\frac{1}{t}$ and the expansion is made around $t = 0$ and after solving, the solution is changed back to x using $x = \frac{1}{t}$. Changing variables results in the new ode

$$-\frac{\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 2t\left(\frac{d}{dt}y(t)\right)}{t} - \left(\frac{1}{t^2} + \frac{1}{t}\right)\left(\frac{d}{dt}y(t)\right)t^2 - y(t) = 0$$

The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(\frac{d^2}{dt^2}y(t)\right)t + (-t + 1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$p(t) = -\frac{t-1}{t}$$

$$q(t) = -\frac{1}{t}$$

Table 291: Table $p(t), q(t)$ singularities.

$p(t) = -\frac{t-1}{t}$	
singularity	type
$t = 0$	“regular”

$q(t) = -\frac{1}{t}$	
singularity	type
$t = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)t + (-t + 1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}$$

$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} \right) t \\ & + (-t+1) \left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \right) - \left(\sum_{n=0}^{\infty} a_n t^{n+r} \right) = 0 \end{aligned} \quad (1)$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} t^{n+r-1} a_n (n+r)(n+r-1) \right) + \sum_{n=0}^{\infty} (-t^{n+r} a_n (n+r)) \\ & + \left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \right) + \sum_{n=0}^{\infty} (-a_n t^{n+r}) = 0 \end{aligned} \quad (2A)$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=0}^{\infty} (-t^{n+r} a_n (n+r)) &= \sum_{n=1}^{\infty} (-a_{n-1} (n+r-1) t^{n+r-1}) \\ \sum_{n=0}^{\infty} (-a_n t^{n+r}) &= \sum_{n=1}^{\infty} (-a_{n-1} t^{n+r-1}) \end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} t^{n+r-1} a_n (n+r)(n+r-1) \right) + \sum_{n=1}^{\infty} (-a_{n-1} (n+r-1) t^{n+r-1}) \\ & + \left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \right) + \sum_{n=1}^{\infty} (-a_{n-1} t^{n+r-1}) = 0 \end{aligned} \quad (2B)$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$t^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n t^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$t^{-1+r} a_0 r(-1+r) + r a_0 t^{-1+r} = 0$$

Or

$$(t^{-1+r} r(-1+r) + r t^{-1+r}) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$t^{-1+r} r^2 = 0$$

Since the above is true for all t then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$

$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$t^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as. Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \quad (1A)$$

Now the second solution y_2 is found using

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r} \right) \quad (1B)$$

Then the general solution will be

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitrary constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case

$n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \quad (4)$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n} \quad (5)$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(3+r)(2+r)(1+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(2+r)(1+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(2+r)(1+r)(4+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(3+r)(2+r)(1+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(2+r)(1+r)(4+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(3+r)(2+r)(1+r)(4+r)(5+r)}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(t)$ becomes

$$\begin{aligned} y_1(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 \dots \\ &= t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{1+r}$	1	$-\frac{1}{(1+r)^2}$	-1
b_2	$\frac{1}{(2+r)(1+r)}$	$\frac{1}{2}$	$\frac{-3-2r}{(2+r)^2(1+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(3+r)(2+r)(1+r)}$	$\frac{1}{6}$	$\frac{-3r^2-12r-11}{(3+r)^2(2+r)^2(1+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(3+r)(2+r)(1+r)(4+r)}$	$\frac{1}{24}$	$\frac{-4r^3-30r^2-70r-50}{(3+r)^2(2+r)^2(1+r)^2(4+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(3+r)(2+r)(1+r)(4+r)(5+r)}$	$\frac{1}{120}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{(3+r)^2(2+r)^2(1+r)^2(4+r)^2(5+r)^2}$	$-\frac{137}{7200}$

The above table gives all values of b_n needed. Hence the second solution is

$$y_2(t) = y_1(t) \ln(t) + b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 \dots$$

$$= \left(t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \right) \ln(t) - t - \frac{3t^2}{4} - \frac{11t^3}{36} - \frac{25t^4}{288} - \frac{137t^5}{7200} + O(t^6)$$

Therefore the homogeneous solution is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$= c_1 \left(t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \right)$$

$$+ c_2 \left(\left(t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \right) \ln(t) - t - \frac{3t^2}{4} - \frac{11t^3}{36} - \frac{25t^4}{288} - \frac{137t^5}{7200} + O(t^6) \right)$$

Hence the final solution is

$$y(t) = y_h$$

$$= c_1 \left(t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \right)$$

$$+ c_2 \left(\left(t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \right) \ln(t) - t - \frac{3t^2}{4} - \frac{11t^3}{36} - \frac{25t^4}{288} - \frac{137t^5}{7200} + O(t^6) \right)$$

Replacing t by $\frac{1}{x}$ gives

$$y = c_1 \left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right) \right) + c_2 \left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + \frac{1}{720x^6} + O\left(\frac{1}{x^7}\right) \right) \ln\left(\frac{1}{x}\right) - \frac{1}{x} - \frac{3}{4x^2} - \frac{11}{36x^3} - \frac{25}{288x^4} - \frac{137}{7200x^5} + O\left(\frac{1}{x^6}\right) \right)$$

Summary

The solution(s) found are the following

$$y = c_1 \left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right) \right) + c_2 \left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right) \right) \ln\left(\frac{1}{x}\right) - \frac{1}{x} - \frac{3}{4x^2} - \frac{11}{36x^3} - \frac{25}{288x^4} - \frac{137}{7200x^5} + O\left(\frac{1}{x^6}\right) \right) \quad (1)$$

Verification of solutions

$$y = c_1 \left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right) \right) + c_2 \left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right) \right) \ln\left(\frac{1}{x}\right) - \frac{1}{x} - \frac{3}{4x^2} - \frac{11}{36x^3} - \frac{25}{288x^4} - \frac{137}{7200x^5} + O\left(\frac{1}{x^6}\right) \right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```

`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
    A Liouvillian solution exists
    Reducible group (found an exponential solution)
    Group is reducible, not completely reducible
<- Kovacics algorithm successful`

```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 179

Order:=6;

dsolve(x^3*diff(y(x),x\$2)+(x^2+x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=infinity);

$$\begin{aligned}
 y(x) = & \left(1 + \frac{(x - \text{Infinity})^2}{2 \text{Infinity}^3} + \frac{(-4 \text{Infinity} - 1)(x - \text{Infinity})^3}{6 \text{Infinity}^5} \right. \\
 & + \frac{(18 \text{Infinity}^2 + 10 \text{Infinity} + 1)(x - \text{Infinity})^4}{24 \text{Infinity}^7} \\
 & \left. + \frac{(-96 \text{Infinity}^3 - 86 \text{Infinity}^2 - 18 \text{Infinity} - 1)(x - \text{Infinity})^5}{120 \text{Infinity}^9} \right) y(\text{Infinity}) \\
 & + \left(x - \text{Infinity} + \frac{(- \text{Infinity}^2 - \text{Infinity})(x - \text{Infinity})^2}{2 \text{Infinity}^3} \right. \\
 & + \frac{(2 \text{Infinity}^3 + 5 \text{Infinity}^2 + \text{Infinity})(x - \text{Infinity})^3}{6 \text{Infinity}^5} \\
 & + \frac{(-6 \text{Infinity}^4 - 26 \text{Infinity}^3 - 11 \text{Infinity}^2 - \text{Infinity})(x - \text{Infinity})^4}{24 \text{Infinity}^7} \\
 & \left. + \frac{(24 \text{Infinity}^5 + 154 \text{Infinity}^4 + 102 \text{Infinity}^3 + 19 \text{Infinity}^2 + \text{Infinity})(x - \text{Infinity})^5}{120 \text{Infinity}^9} \right) D(y)(\text{Infinity}) \\
 & + O(x^6)
 \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 124

AsymptoticDSolveValue[x^3*y''[x]+(x^2+x)*y'[x]-y[x]==0,y[x],{x,infinity,5}]

$$\begin{aligned}
 y(x) \rightarrow & c_1 \left(\frac{1}{120x^5} + \frac{1}{24x^4} + \frac{1}{6x^3} + \frac{1}{2x^2} + \frac{1}{x} + 1 \right) + c_2 \left(-\frac{137}{7200x^5} - \frac{\log(x)}{120x^5} - \frac{25}{288x^4} \right. \\
 & \left. - \frac{\log(x)}{24x^4} - \frac{11}{36x^3} - \frac{\log(x)}{6x^3} - \frac{3}{4x^2} - \frac{\log(x)}{2x^2} - \frac{1}{x} - \frac{\log(x)}{x} - \log(x) \right)
 \end{aligned}$$

**18 Chapter 27. The Legendre, Bessel and Gauss
Equations. Supplementary problems. Page 230**

18.1 problem 20 2588

18.1 problem 20

18.1.1 Maple step by step solution 2595

Internal problem ID [5470]

Internal file name [OUTPUT/4961_Wednesday_February_14_2024_02_06_02_AM_23127676/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952

Section: Chapter 27. The Legendre, Bessel and Gauss Equations. Supplementary problems. Page 230

Problem number: 20.

ODE order: 2.

ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order , _with_linear_symmetries]]
```

$$z'' + tz' + \left(t^2 - \frac{1}{9}\right)z = 0$$

With the expansion point for the power series method at $t = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
 y(x) &= y(x_0) + (x - x_0)y'(x_0) + \frac{(x - x_0)^2}{2}y''(x_0) + \frac{(x - x_0)^3}{3!}y'''(x_0) + \cdots \\
 &= y_0 + xy'_0 + \frac{x^2}{2}f|_{x_0, y_0, y'_0} + \frac{x^3}{3!}f'|_{x_0, y_0, y'_0} + \cdots \\
 &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n} \Big|_{x_0, y_0, y'_0}
 \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} + \frac{\partial f}{\partial y'} \frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \tag{489}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{490}$$

$$\begin{aligned}
 \frac{d^2 f}{dx^2} &= \frac{d}{dx} \left(\frac{df}{dx} \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) y' + \frac{\partial}{\partial y'} \left(\frac{df}{dx} \right) f \tag{2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{d^3 f}{dx^3} &= \frac{d}{dx} \left(\frac{d^2 f}{dx^2} \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{d^2 f}{dx^2} \right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2} \right) y' + \frac{\partial}{\partial y'} \left(\frac{d^2 f}{dx^2} \right) f \tag{3}
 \end{aligned}$$

⋮

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$F_0 = f(x, y, y') \tag{4}$$

$$\begin{aligned} F_1 &= \frac{df}{dx} \\ &= \frac{dF_0}{dx} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \\ &= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0 \end{aligned} \tag{5}$$

$$\begin{aligned} F_2 &= \frac{d}{dx} \left(\frac{d}{dx} f \right) \\ &= \frac{d}{dx} (F_1) \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) y' + \left(\frac{\partial F_1}{\partial y'} \right) F_0 \end{aligned}$$

⋮

$$\begin{aligned} F_n &= \frac{d}{dx} (F_{n-1}) \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) y'' \\ &= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y} \right) y' + \left(\frac{\partial F_{n-1}}{\partial y'} \right) F_0 \end{aligned} \tag{6}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
 F_0 &= -zt^2 - tz' + \frac{z}{9} \\
 F_1 &= \frac{dF_0}{dt} \\
 &= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial z} z' + \frac{\partial F_0}{\partial z'} F_0 \\
 &= -\frac{8z'}{9} + \frac{(9t^3 - 19t)z}{9} \\
 F_2 &= \frac{dF_1}{dt} \\
 &= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial z} z' + \frac{\partial F_1}{\partial z'} F_1 \\
 &= z't^3 + \frac{35zt^2}{9} - \frac{11tz'}{9} - \frac{179z}{81} \\
 F_3 &= \frac{dF_2}{dt} \\
 &= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial z} z' + \frac{\partial F_2}{\partial z'} F_2 \\
 &= \frac{(-81t^4 + 657t^2 - 278)z'}{81} - zt \left(t^4 - \frac{4}{3}t^2 - \frac{619}{81} \right) \\
 F_4 &= \frac{dF_3}{dt} \\
 &= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial z} z' + \frac{\partial F_3}{\partial z'} F_3 \\
 &= \frac{(-291t^3 + 737t)z'}{27} + \left(t^6 - \frac{119}{9}t^4 + \frac{25}{3}t^2 + \frac{5293}{729} \right) z
 \end{aligned}$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $z(0) = z(0)$ and $z'(0) = z'(0)$ gives

$$\begin{aligned}
 F_0 &= \frac{z(0)}{9} \\
 F_1 &= -\frac{8z'(0)}{9} \\
 F_2 &= -\frac{179z(0)}{81} \\
 F_3 &= -\frac{278z'(0)}{81} \\
 F_4 &= \frac{5293z(0)}{729}
 \end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4 + \frac{5293}{524880}t^6\right) z(0) + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) z'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$z = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$z' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

$$z'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = - \left(\sum_{n=0}^{\infty} a_n t^n \right) t^2 - t \left(\sum_{n=1}^{\infty} n a_n t^{n-1} \right) + \frac{\left(\sum_{n=0}^{\infty} a_n t^n \right)}{9} \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \right) + \left(\sum_{n=1}^{\infty} n t^n a_n \right) + \left(\sum_{n=0}^{\infty} t^{n+2} a_n \right) + \sum_{n=0}^{\infty} \left(-\frac{a_n t^n}{9} \right) = 0 \quad (2)$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) t^n$$

$$\sum_{n=0}^{\infty} t^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n .

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2} (n+1) t^n \right) + \left(\sum_{n=1}^{\infty} n t^n a_n \right) + \left(\sum_{n=2}^{\infty} a_{n-2} t^n \right) + \sum_{n=0}^{\infty} \left(-\frac{a_n t^n}{9} \right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 - \frac{a_0}{9} = 0$$

$$a_2 = \frac{a_0}{18}$$

$n = 1$ gives

$$6a_3 + \frac{8a_1}{9} = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_1}{27}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)a_{n+2}(n+1) + na_n + a_{n-2} - \frac{a_n}{9} = 0 \quad (4)$$

Solving for a_{n+2} , gives

$$\begin{aligned} a_{n+2} &= -\frac{9na_n - a_n + 9a_{n-2}}{9(n+2)(n+1)} \\ (5) \quad &= -\frac{(9n-1)a_n}{9(n+2)(n+1)} - \frac{a_{n-2}}{(n+2)(n+1)} \end{aligned}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + \frac{17a_2}{9} + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{179a_0}{1944}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + \frac{26a_3}{9} + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{139a_1}{4860}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + \frac{35a_4}{9} + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{5293a_0}{524880}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + \frac{44a_5}{9} + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3149a_1}{459270}$$

And so on. Therefore the solution is

$$\begin{aligned} z &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$z = a_0 + a_1 t + \frac{1}{18} a_0 t^2 - \frac{4}{27} a_1 t^3 - \frac{179}{1944} a_0 t^4 - \frac{139}{4860} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$z = \left(1 + \frac{1}{18} t^2 - \frac{179}{1944} t^4\right) a_0 + \left(t - \frac{4}{27} t^3 - \frac{139}{4860} t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$z = \left(1 + \frac{1}{18} t^2 - \frac{179}{1944} t^4\right) c_1 + \left(t - \frac{4}{27} t^3 - \frac{139}{4860} t^5\right) c_2 + O(t^6)$$

Summary

The solution(s) found are the following

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4 + \frac{5293}{524880}t^6\right) z(0) + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) z'(0) + O(t^6) \quad (1)$$

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4\right) c_1 + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) c_2 + O(t^6) \quad (2)$$

Verification of solutions

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4 + \frac{5293}{524880}t^6\right) z(0) + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) z'(0) + O(t^6)$$

Verified OK.

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4\right) c_1 + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) c_2 + O(t^6)$$

Verified OK.

18.1.1 Maple step by step solution

Let's solve

$$z'' = -zt^2 - tz' + \frac{z}{9}$$

- Highest derivative means the order of the ODE is 2

$$z''$$

- Isolate 2nd derivative

$$z'' = \left(-t^2 + \frac{1}{9}\right) z - tz'$$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$z'' + tz' + \left(t^2 - \frac{1}{9}\right) z = 0$$

- Multiply by denominators

$$9z'' + 9tz' + (9t^2 - 1)z = 0$$

- Assume series solution for z

$$z = \sum_{k=0}^{\infty} a_k t^k$$

- Rewrite ODE with series expansions

- Convert $t^m \cdot z$ to series expansion for $m = 0..2$

$$t^m \cdot z = \sum_{k=\max(0,-m)}^{\infty} a_k t^{k+m}$$

- Shift index using $k \rightarrow k - m$

$$t^m \cdot z = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^k$$

- Convert $t \cdot z'$ to series expansion

$$t \cdot z' = \sum_{k=0}^{\infty} a_k k t^k$$

- Convert z'' to series expansion

$$z'' = \sum_{k=2}^{\infty} a_k k(k-1) t^{k-2}$$

- Shift index using $k \rightarrow k + 2$

$$z'' = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) t^k$$

Rewrite ODE with series expansions

$$18a_2 - a_0 + (54a_3 + 8a_1)t + \left(\sum_{k=2}^{\infty} (9a_{k+2}(k+2)(k+1) + a_k(9k-1) + 9a_{k-2}) t^k \right) = 0$$

- The coefficients of each power of t must be 0
- Solve for the dependent coefficient(s)
- Each term in the series must be 0, giving the recursion relation
- Shift index using $k \rightarrow k + 2$
- Recursion relation that defines the series solution to the ODE

$$[18a_2 - a_0 = 0, 54a_3 + 8a_1 = 0]$$

$$\left\{ a_2 = \frac{a_0}{18}, a_3 = -\frac{4a_1}{27} \right\}$$

$$9(k^2 + 3k + 2) a_{k+2} + 9a_k k - a_k + 9a_{k-2} = 0$$

$$9((k+2)^2 + 3k + 8) a_{k+4} + 9a_{k+2}(k+2) - a_{k+2} + 9a_k = 0$$

$$\left[z = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = -\frac{9ka_{k+2} + 9a_k + 17a_{k+2}}{9(k^2 + 7k + 12)}, a_2 = \frac{a_0}{18}, a_3 = -\frac{4a_1}{27} \right]$$

Maple trace

```
`Methods for second order ODEs:  
--- Trying classification methods ---  
trying a quadrature  
checking if the LODE has constant coefficients  
checking if the LODE is of Euler type  
trying a symmetry of the form [xi=0, eta=F(x)]  
checking if the LODE is missing y  
-> Trying a Liouvillian solution using Kovacic's algorithm  
<- No Liouvillian solutions exist  
-> Trying a solution in terms of special functions:  
  -> Bessel  
  -> elliptic  
  -> Legendre  
  -> Kummer  
    -> hyper3: Equivalence to 1F1 under a power @ Moebius  
    <- hyper3 successful: received ODE is equivalent to the 1F1 ODE  
  <- Kummer successful  
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;  
dsolve(diff(z(t),t$2)+t*diff(z(t),t)+(t^2-1/9)*z(t)=0,z(t),type='series',t=0);
```

$$z(t) = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4\right) z(0) + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) D(z)(0) + O(t^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[z'[t]+t*z'[t]+(t^2-1/9)*z[t]==0,z[t],{t,0,5}]
```

$$z(t) \rightarrow c_2 \left(-\frac{139t^5}{4860} - \frac{4t^3}{27} + t \right) + c_1 \left(-\frac{179t^4}{1944} + \frac{t^2}{18} + 1 \right)$$