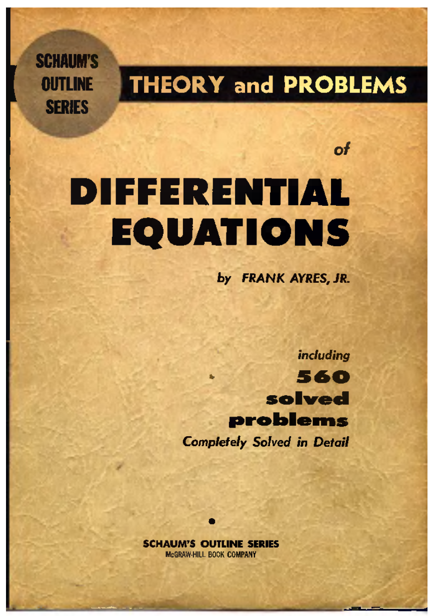
A Solution Manual For

Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952



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Contents

1 Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11 3

2 Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22 119

3 Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33 438

4 Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39 709

5 Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65 977

6 Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74 1100

7 Chapter 12. Linear equations of order n. Supplemetary problems. Page 81 1191

8 Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86 1283

9 Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92 1346

10 Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98 1480

11 Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107 1617

12 Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110 1781

13 Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120 1909

14 Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132 2177

15 Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163 2283

16 Chapter 25. Integration in series. Supplemetary problems. Page 205 2338

17 Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218 2445

18 Chapter 27. The Legendre, Bessel and Gauss Equations. Supplemetary problems. Page 230 2587

1 Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11

1.1 problem 13 4
1.2 problem 14 19
1.3 problem 15 34
1.4 problem 16 39
1.5 problem 17 55
1.6 problem 18 64
1.7 problem 19 79
1.8 problem 20 89
1.9 problem 21 100
1.10 problem 22 108

1.1 problem 13

1.1.1 Solving as separable ode 4
1.1.2 Solving as linear ode 6
1.1.3 Solving as homogeneousTypeD2 ode 7
1.1.4 Solving as first order ode lie symmetry lookup ode 9
1.1.5 Solving as exact ode 13
1.1.6 Maple step by step solution 17

Internal problem ID [5226]
Internal file name [`OUTPUT/4717_Friday_February_02_2024_05_10_30_AM_16279652/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - 2y = 0}$$

1.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{2y}{x} \end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln(y) &= 2\ln(x) + c_1 \\
y &= \mathrm{e}^{2\ln(x)+c_1} \\
&= c_1 x^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \tag{1}
$$

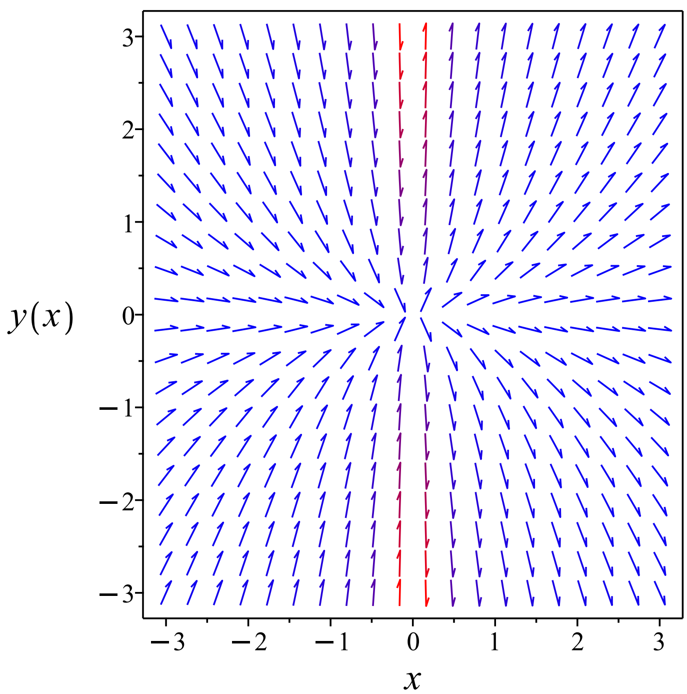


Figure 1: Slope field plot

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

1.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x^2} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = c_1 x^2$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

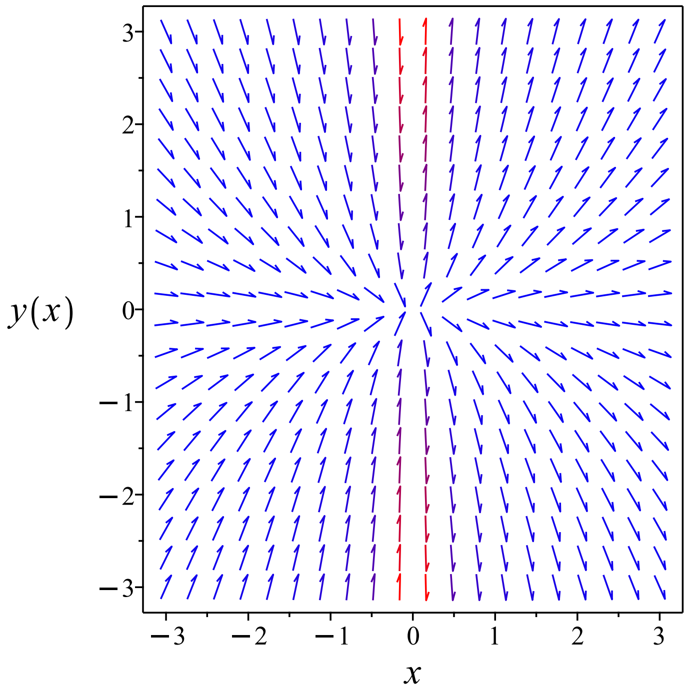


Figure 2: Slope field plot

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

1.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\, x + u(x)) - 2u(x)\, x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x) + c_2 \\
u &= e^{\ln(x)+c_2} \\
&= c_2 x
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= c_2 x^2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 \tag{1}$$

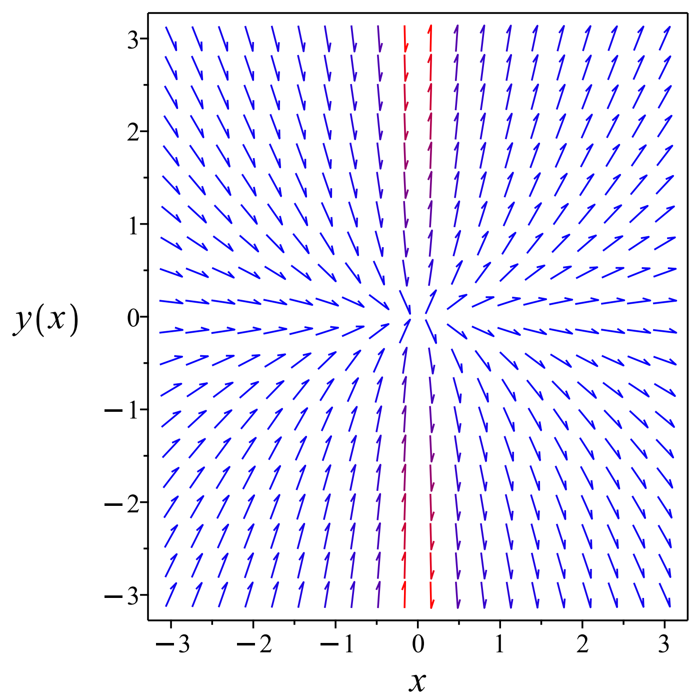


Figure 3: Slope field plot

Verification of solutions

$$y = c_2 x^2$$

Verified OK.

1.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 1: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = c_1$$

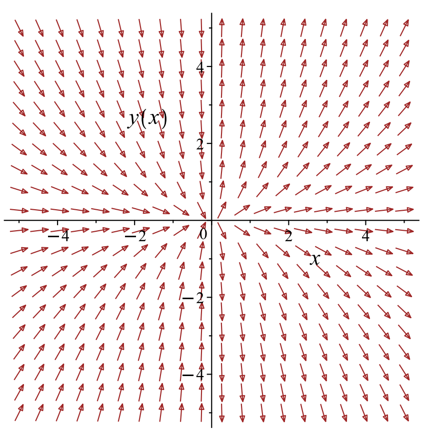
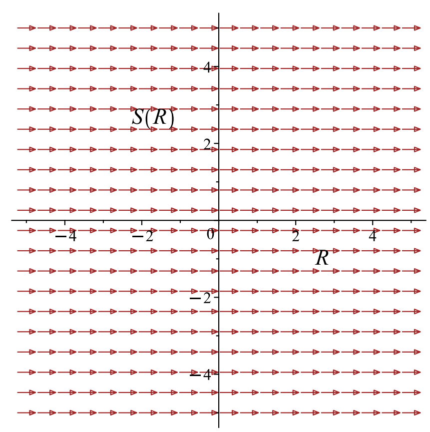
Which simplifies to

$$\frac{y}{x^2} = c_1$$

Which gives

$$y = c_1 x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{2y}{x}$

</td><td>$R = x$
$S = \frac{y}{x^2}$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = c_1 x^2 \tag{1}$$

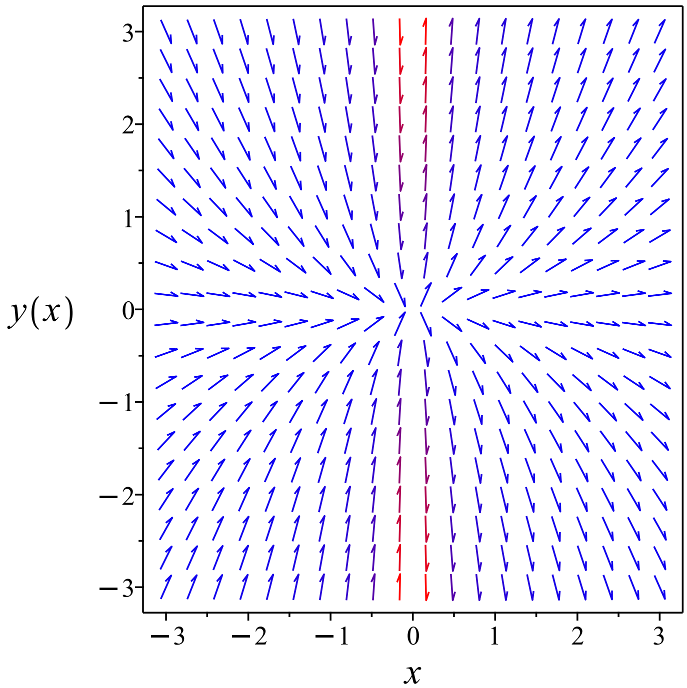


Figure 4: Slope field plot

Verification of solutions

$$y = c_1 x^2$$

Verified OK.

1.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{2y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y}$. Therefore equation (4) becomes

$$
\frac{1}{2y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{2y}\right) \mathrm{d}y \\
f(y) &= \frac{\ln(y)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{\ln(y)}{2}$$

The solution becomes

$$y = x^2 e^{2c_1}$$

Summary

The solution(s) found are the following

$$y = x^2 e^{2c_1} \tag{1}$$

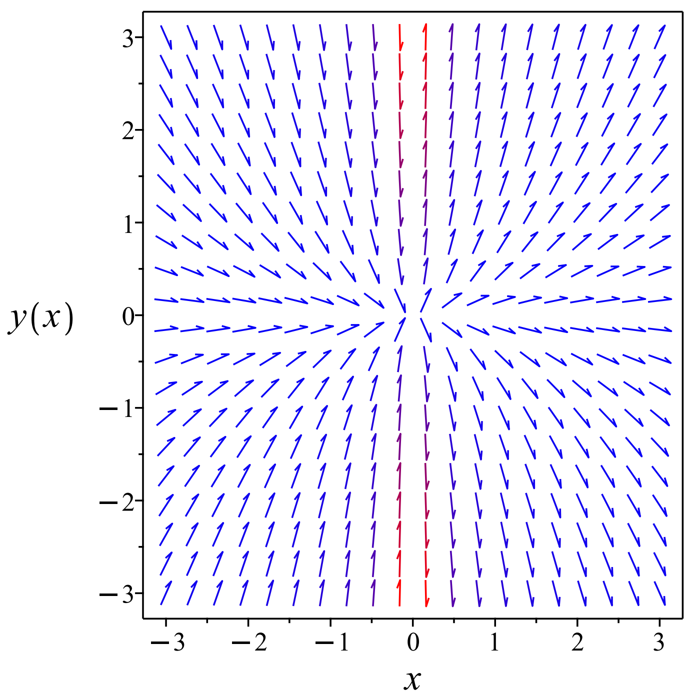


Figure 5: Slope field plot

Verification of solutions

$$y = x^2 e^{2c_1}$$

Verified OK.

1.1.6 Maple step by step solution

Let's solve

$xy' - 2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{2}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = 2\ln(x) + c_1$

- Solve for y

  $y = x^2 e^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)=2*y(x),y(x), singsol=all)
```

$$y(x) = c_1 x^2$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[x*y'[x]==2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2$$
$$y(x) \to 0$$

1.2 problem 14

1.2.1 Solving as separable ode 19
1.2.2 Solving as homogeneousTypeD2 ode 21
1.2.3 Solving as differentialType ode 23
1.2.4 Solving as first order ode lie symmetry lookup ode 24
1.2.5 Solving as exact ode 28
1.2.6 Maple step by step solution 32

Internal problem ID [5227]
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Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{yy' = -x}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{1}{y}}\, dy = \int -x\, dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c_1$$

Which results in

$$y = \sqrt{-x^2 + 2c_1}$$

$$y = -\sqrt{-x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-x^2 + 2c_1} \tag{1}$$

$$y = -\sqrt{-x^2 + 2c_1} \tag{2}$$

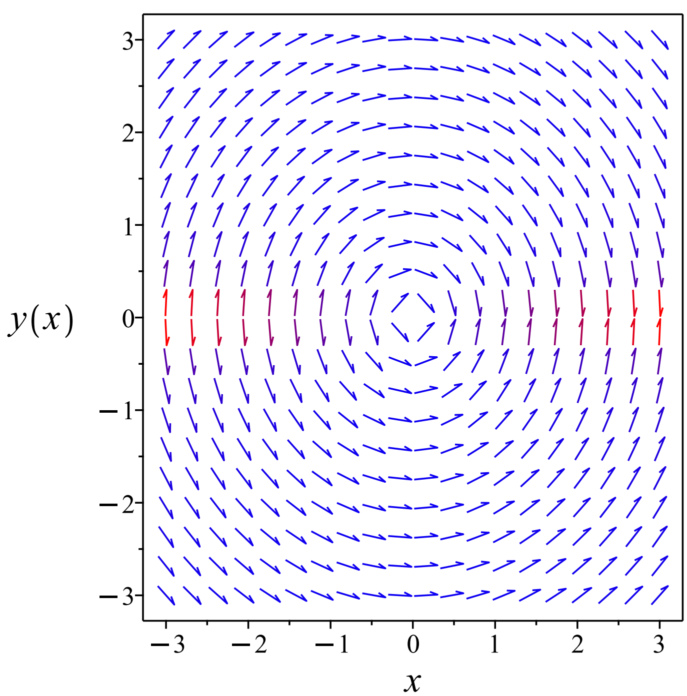


Figure 6: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1}$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1}$$

Verified OK.

1.2.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x(u'(x)\, x + u(x)) = -x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{u^2 + 1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{u^2+1}{u}}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{\frac{u^2+1}{u}}\, du &= \int -\frac{1}{x}\, dx \\ \frac{\ln (u^2 + 1)}{2} &= -\ln (x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 + 1} = \mathrm{e}^{-\ln(x) + c_2}$$

Which simplifies to

$$\sqrt{u^2 + 1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2 + 1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^{2}+1} = \frac{c_{3}\mathrm{e}^{c_{2}}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^{2}}{x^{2}}+1} = \frac{c_{3}\mathrm{e}^{c_{2}}}{x}$$
$$\sqrt{\frac{x^{2}+y^{2}}{x^{2}}} = \frac{c_{3}\mathrm{e}^{c_{2}}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{x^{2}+y^{2}}{x^{2}}} = \frac{c_{3}\mathrm{e}^{c_{2}}}{x} \tag{1}$$

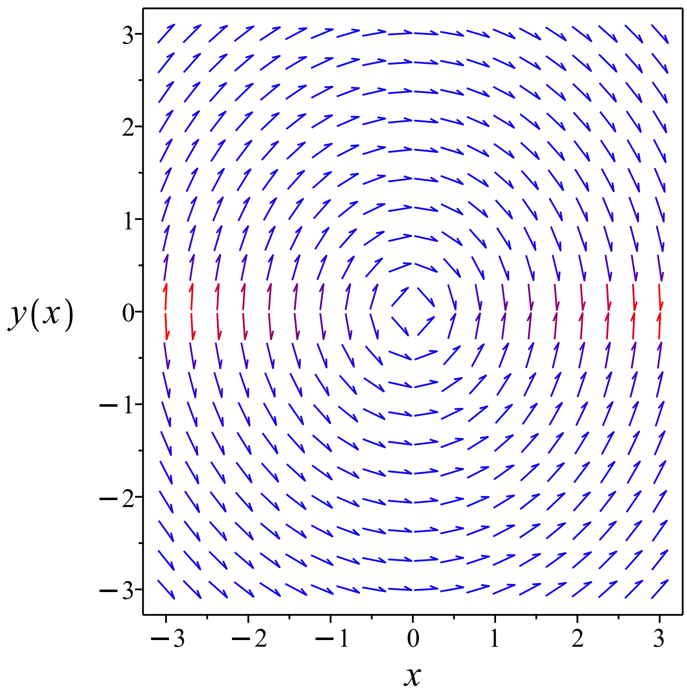


Figure 7: Slope field plot

Verification of solutions

$$\sqrt{\frac{x^{2}+y^{2}}{x^{2}}} = \frac{c_{3}\mathrm{e}^{c_{2}}}{x}$$

Verified OK.

1.2.3 Solving as differentialType ode

Writing the ode as

$$y' = -\frac{x}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (-x)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dx = d\left(-\frac{x^2}{2}\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(-\frac{x^2}{2}\right)$$

Integrating both sides gives gives these solutions

$$\begin{aligned} y &= \sqrt{-x^2+2c_1}+c_1 \\ y &= -\sqrt{-x^2+2c_1}+c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{-x^2+2c_1}+c_1 \tag{1}$$
$$y = -\sqrt{-x^2+2c_1}+c_1 \tag{2}$$

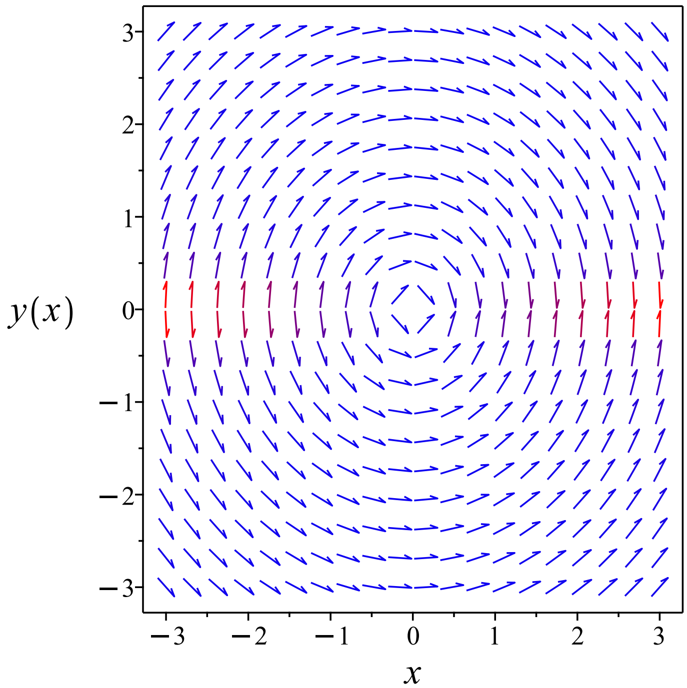


Figure 8: Slope field plot

Verification of solutions

$$y = \sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -\sqrt{-x^2 + 2c_1} + c_1$$

Verified OK.

1.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 4: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -\frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

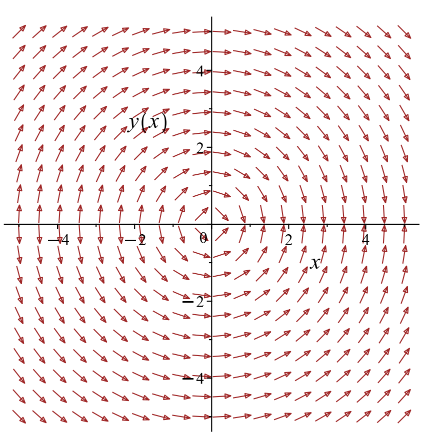
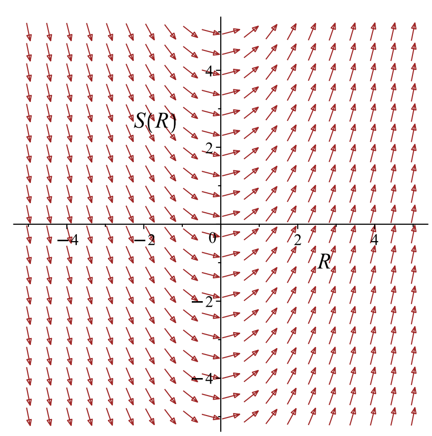
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = R$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1 \tag{1}$$

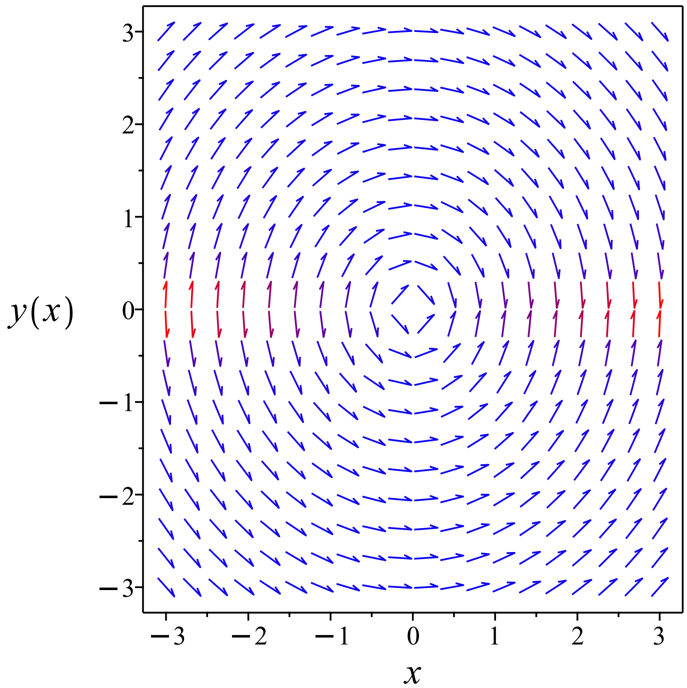


Figure 9: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{y^2}{2} + c_1$$

Verified OK.

1.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-y)\,\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + (-y)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= -y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y$. Therefore equation (4) becomes

$$-y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y) \, \mathrm{d}y$$
$$f(y) = -\frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{y^2}{2} - \frac{x^2}{2} = c_1 \tag{1}$$

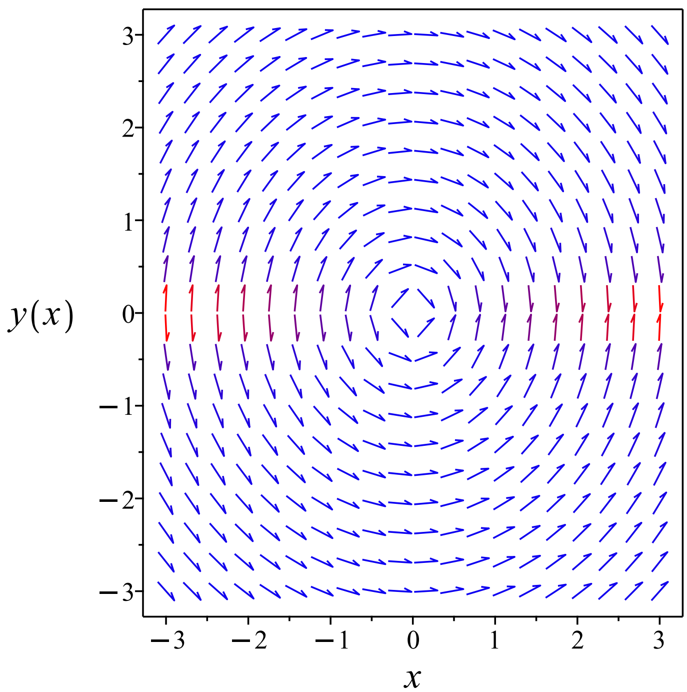


Figure 10: Slope field plot

Verification of solutions

$$-\frac{y^2}{2} - \frac{x^2}{2} = c_1$$

Verified OK.

1.2.6 Maple step by step solution

Let's solve

$yy' = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{-x^2 + 2c_1}, y = -\sqrt{-x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x^2 + c_1}$$
$$y(x) = -\sqrt{-x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 39

```
DSolve[y[x]*y'[x]+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x^2 + 2c_1}$$
$$y(x) \to \sqrt{-x^2 + 2c_1}$$

1.3 problem 15

1.3.1 Solving as clairaut ode 34

Internal problem ID [5228]
Internal file name [`OUTPUT/4719_Friday_February_02_2024_05_10_32_AM_47063498/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 15.
ODE order: 1.
ODE degree: 4.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' - y'^4 = 0}$$

1.3.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^4 - xp + y = 0$$

Solving for y from the above results in

$$y = p^4 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^4 + xp \\ &= p^4 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^4$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^4 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^4$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= 4p^3 + x \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= \frac{(-2x)^{\frac{1}{3}}}{2} \\
p_2 &= -\frac{(-2x)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(-2x)^{\frac{1}{3}}}{4} \\
p_3 &= -\frac{(-2x)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(-2x)^{\frac{1}{3}}}{4}
\end{aligned}
$$

Substituting the above back in (1) results in

$$
\begin{aligned}
y_1 &= -\frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}}{8} \\
y_2 &= \frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)}{16} \\
y_3 &= -\frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}\left(i\sqrt{3}-1\right)}{16}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1^4 + c_1 x \tag{1}$$

$$y = -\frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}}{8} \tag{2}$$

$$y = \frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)}{16} \tag{3}$$

$$y = -\frac{3\,2^{\frac{1}{3}}(-x)^{\frac{4}{3}}\left(i\sqrt{3}-1\right)}{16} \tag{4}$$

Verification of solutions

$$y = c_1^4 + c_1 x$$

Verified OK.

$$y = -\frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}}}{8}$$

Verified OK.

$$y = \frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}} \left(1 + i\sqrt{3}\right)}{16}$$

Verified OK.

$$y = -\frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}} \left(i\sqrt{3} - 1\right)}{16}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 66

```
dsolve(y(x)=x*diff(y(x),x)+diff(y(x),x)^4,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}}}{8} \\
y(x) &= \frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}} \left(1+i\sqrt{3}\right)}{16} \\
y(x) &= -\frac{3\,2^{\frac{1}{3}} (-x)^{\frac{4}{3}} \left(i\sqrt{3}-1\right)}{16} \\
y(x) &= c_1\left(c_1^3+x\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 75

```
DSolve[y[x]==x*y'[x]+(y'[x])^4,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to c_1\left(x+c_1{}^3\right) \\
y(x) &\to -\frac{3}{4}\left(-\frac{1}{2}\right)^{2/3} x^{4/3} \\
y(x) &\to -\frac{3x^{4/3}}{4\ 2^{2/3}} \\
y(x) &\to \frac{3\sqrt[3]{-1}x^{4/3}}{4\ 2^{2/3}}
\end{aligned}
$$

1.4 problem 16

1.4.1 Solving as homogeneousTypeD2 ode 39
1.4.2 Solving as first order ode lie symmetry lookup ode 41
1.4.3 Solving as bernoulli ode 45
1.4.4 Solving as exact ode 49

Internal problem ID [5229]
Internal file name [`OUTPUT/4720_Friday_February_02_2024_05_10_33_AM_57195971/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2y'x^3 - y(y^2 + 3x^2) = 0}$$

1.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2(u'(x)\,x + u(x))\,x^3 - u(x)\,x(u(x)^2\,x^2 + 3x^2) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(u^2+1)}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u(u^2+1)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u\left(u^2+1\right)}\, du &= \frac{1}{2x}\, dx \\
\int \frac{1}{u\left(u^2+1\right)}\, du &= \int \frac{1}{2x}\, dx \\
-\frac{\ln\left(u^2+1\right)}{2} + \ln\left(u\right) &= \frac{\ln\left(x\right)}{2} + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{-\frac{\ln\left(u^2+1\right)}{2}+\ln(u)} = \mathrm{e}^{\frac{\ln(x)}{2}+c_2}
$$

Which simplifies to

$$
\frac{u}{\sqrt{u^2+1}} = c_3\sqrt{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x - 1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x - 1}} \tag{1}
$$

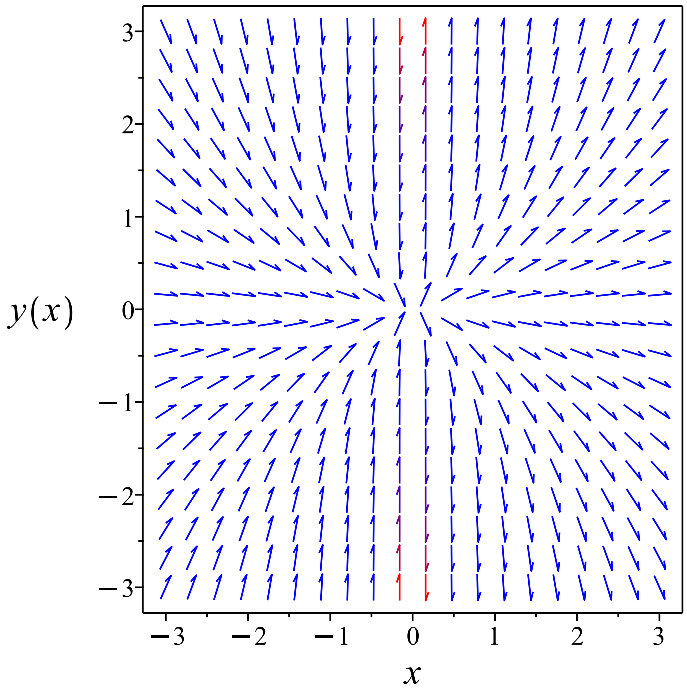


Figure 11: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_3 \sqrt{-\frac{1}{c_3^2 x - 1}}$$

Verified OK.

1.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(3x^2 + y^2)}{2x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 7: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^3}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^3}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^3}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(3x^2 + y^2)}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3x^2}{2y^2} \\ S_y &= \frac{x^3}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

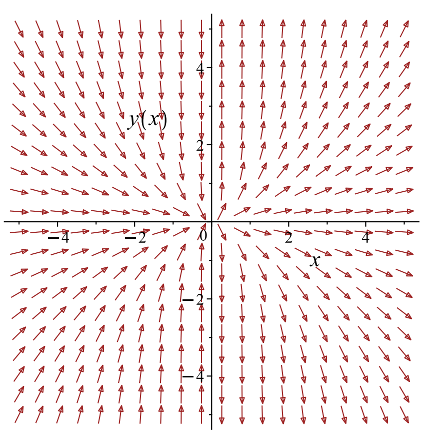
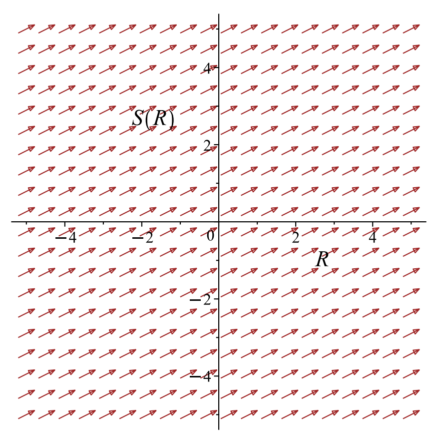
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

Which simplifies to

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(3x^2+y^2)}{2x^3}$ 	$R = x$ $S = -\frac{x^3}{2y^2}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary

The solution(s) found are the following

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1 \tag{1}$$

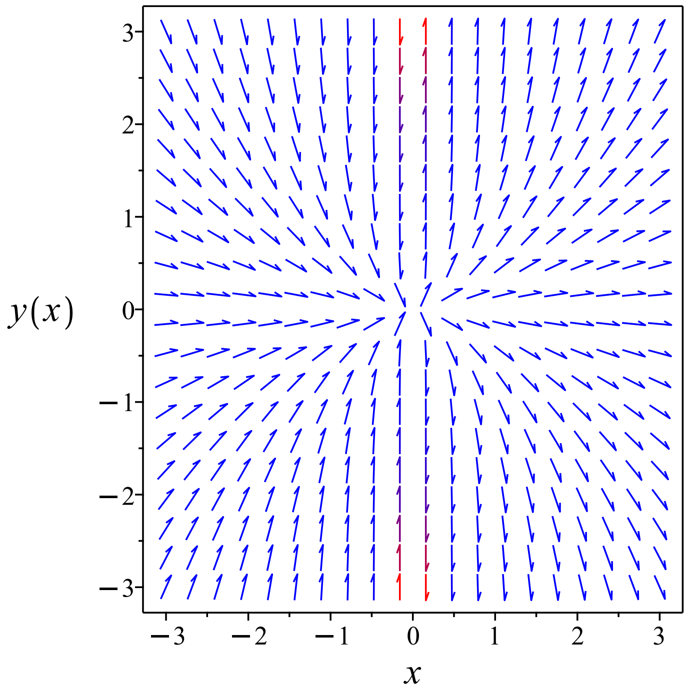


Figure 12: Slope field plot

Verification of solutions

$$-\frac{x^3}{2y^2} = \frac{x}{2} + c_1$$

Verified OK.

1.4.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(3x^2 + y^2)}{2x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{2x}y + \frac{1}{2x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= \frac{3}{2x} \\
f_1(x) &= \frac{1}{2x^3} \\
n &= 3
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{3}{2y^2 x} + \frac{1}{2x^3} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2} \tag{5}
\end{aligned}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-\frac{w'(x)}{2} &= \frac{3w(x)}{2x} + \frac{1}{2x^3} \\
w' &= -\frac{3w}{x} - \frac{1}{x^3} \tag{7}
\end{aligned}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= -\frac{1}{x^3}
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = -\frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{1}{x^3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 w\right) &= \left(x^3\right)\left(-\frac{1}{x^3}\right) \\ \mathrm{d}\left(x^3 w\right) &= -1\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int -1\,\mathrm{d}x \\ x^3 w &= -x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = -\frac{1}{x^2} + \frac{c_1}{x^3}$$

which simplifies to

$$w(x) = \frac{c_1 - x}{x^3}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{c_1 - x}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{x\,(c_1 - x)}\,x}{c_1 - x} \\ y(x) &= \frac{\sqrt{x\,(c_1 - x)}\,x}{-c_1 + x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x\left(c_1 - x\right)}\, x}{c_1 - x} \tag{1}$$
$$y = \frac{\sqrt{x\left(c_1 - x\right)}\, x}{-c_1 + x} \tag{2}$$

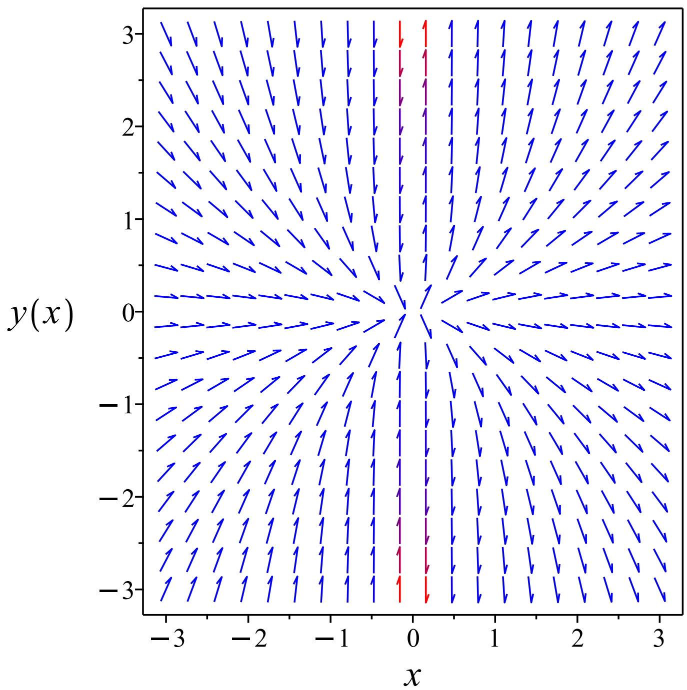


Figure 13: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x\left(c_1 - x\right)}\, x}{c_1 - x}$$

Verified OK.

$$y = \frac{\sqrt{x\left(c_1 - x\right)}\, x}{-c_1 + x}$$

Verified OK.

1.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2x^3\right)\mathrm{d}y&=\left(y\left(3x^2+y^2\right)\right)\mathrm{d}x\\ \left(-y\left(3x^2+y^2\right)\right)\mathrm{d}x+\left(2x^3\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\left(3x^2+y^2\right)\\ N(x,y)&=2x^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\left(3x^2+y^2\right)\right)\\ &= -3x^2-3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^3\right)\\ &= 6x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2x^3}\left(\left(-3x^2-3y^2\right)-\left(6x^2\right)\right)\\ &= \frac{-9x^2-3y^2}{2x^3}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{3x^2y+y^3}\left(\left(6x^2\right)-\left(-3x^2-3y^2\right)\right)\\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int -\frac{3}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(y)} \\
&= \frac{1}{y^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^3}\left(-y\left(3x^2+y^2\right)\right) \\
&= \frac{-3x^2-y^2}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}\left(2x^3\right) \\
&= \frac{2x^3}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-3x^2-y^2}{y^2}\right) + \left(\frac{2x^3}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-3x^2-y^2}{y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x(x^2+y^2)}{y^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{2x}{y} + \frac{2x(x^2+y^2)}{y^3} + f'(y) \qquad (4)\\ &= \frac{2x^3}{y^3} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2x^3}{y^3}$. Therefore equation (4) becomes

$$\frac{2x^3}{y^3} = \frac{2x^3}{y^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x^2+y^2)}{y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x^2+y^2)}{y^2}$$

<u>Summary</u>

The solution(s) found are the following

$$-\frac{x(x^2+y^2)}{y^2} = c_1 \tag{1}$$

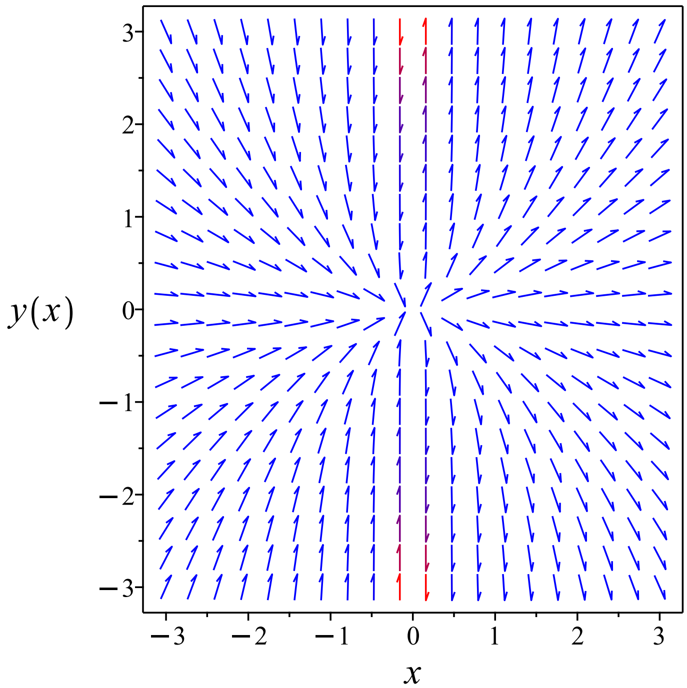


Figure 14: Slope field plot

Verification of solutions

$$
-\frac{x(x^2+y^2)}{y^2} = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(2*x^3*diff(y(x),x)=y(x)*(y(x)^2+3*x^2),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(-x+c_1)\,x}\,x}{-x+c_1}$$
$$y(x) = \frac{\sqrt{(-x+c_1)\,x}\,x}{x-c_1}$$

✓ Solution by Mathematica

Time used: 0.179 (sec). Leaf size: 47

```
DSolve[2*x^3*y'[x]==y[x]*(y[x]^2+3*x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^{3/2}}{\sqrt{-x+c_1}}$$
$$y(x) \to \frac{x^{3/2}}{\sqrt{-x+c_1}}$$
$$y(x) \to 0$$

1.5 problem 17

1.5.1 Solving as second order linear constant coeff ode 55
1.5.2 Solving as linear second order ode solved by an integrating factor ode 57
1.5.3 Solving using Kovacic algorithm 58
1.5.4 Maple step by step solution 62

Internal problem ID [5230]
Internal file name [`OUTPUT/4721_Friday_February_02_2024_05_10_37_AM_91798504/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

1.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

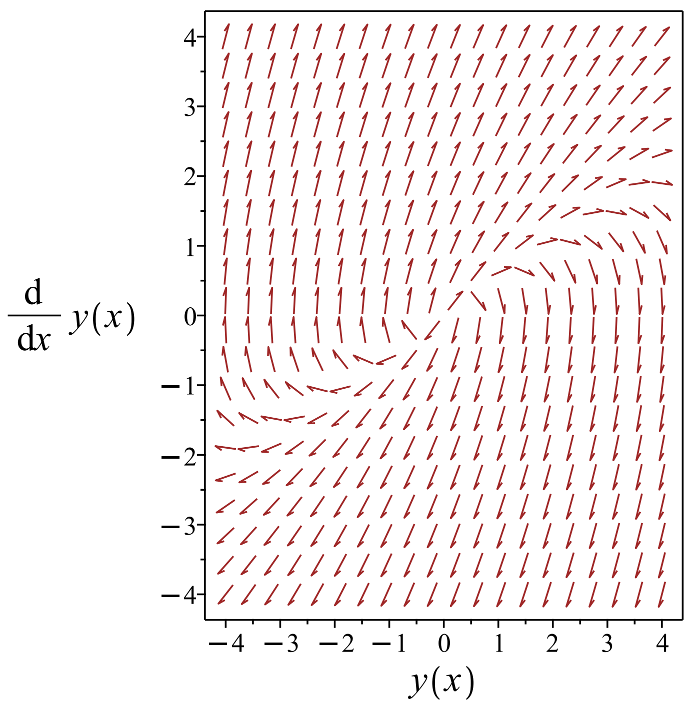


Figure 15: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 x\, e^x$$

Verified OK.

1.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int -2\, dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{-x} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-x} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-x} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{-x}}
$$

Or

$$
y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}
$$

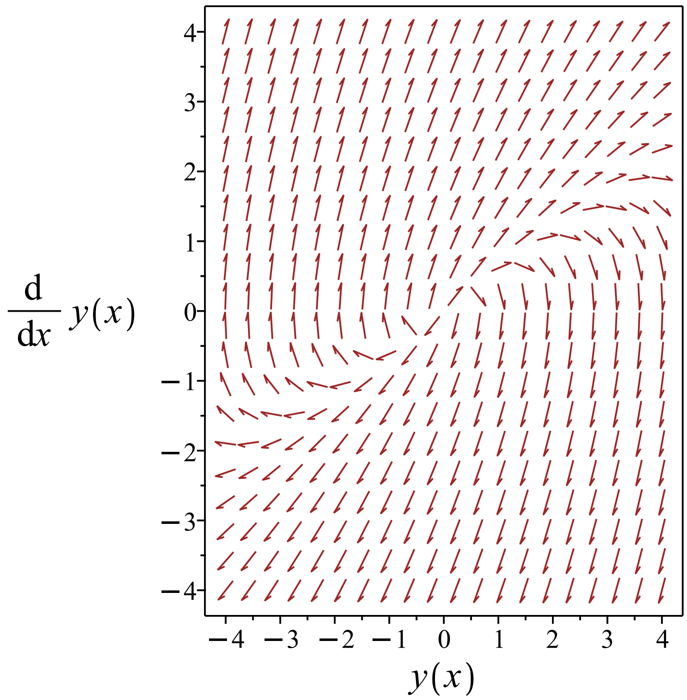


Figure 16: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

Verified OK.

1.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 9: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} \tag{1}
$$

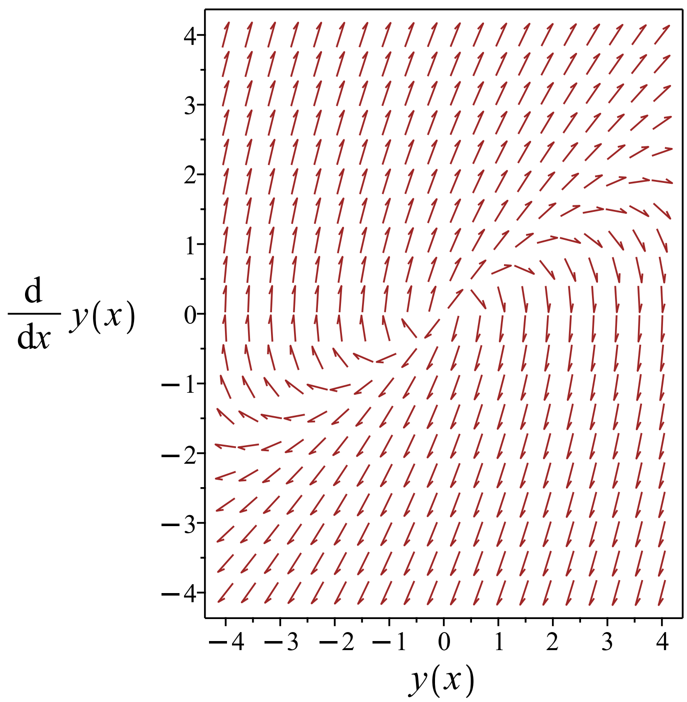


Figure 17: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x}
$$

Verified OK.

1.5.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 x\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 16

```
DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2x + c_1)$$

1.6 problem 18

1.6.1 Solving as second order change of variable on y method 2 ode . 64
1.6.2 Solving as second order ode non constant coeff transformation on B ode 67
1.6.3 Solving using Kovacic algorithm 69
1.6.4 Maple step by step solution 75

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Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x-1)\,y''-xy'+y=0}$$

1.6.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x-1)\,y''-xy'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{x}{x-1} \\ q(x) &= \frac{1}{x-1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x-1} + \frac{1}{x-1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \\ v''(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) v'(x) &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{x}{x-1}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2-2x+2)}{x\,(x-1)} \end{aligned}$$

Where $f(x) = \frac{x^2-2x+2}{x(x-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{x^2 - 2x + 2}{x\,(x-1)}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{x^2 - 2x + 2}{x\,(x-1)}\, dx \\
\ln(u) &= x + \ln(x-1) - 2\ln(x) + c_1 \\
u &= \mathrm{e}^{x+\ln(x-1)-2\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{x+\ln(x-1)-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1 \left(\frac{\mathrm{e}^x}{x} - \frac{\mathrm{e}^x}{x^2} \right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{\mathrm{e}^x c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{\mathrm{e}^x c_1}{x} + c_2 \right) x \\
&= c_1 \mathrm{e}^x + c_2 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{\mathrm{e}^x c_1}{x} + c_2 \right) x \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{\mathrm{e}^x c_1}{x} + c_2 \right) x
$$

Verified OK.

1.6.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x - 1)(0) + (-x)(-1) + (1)(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x(x-1)\,v''+\left(x^2-2x+2\right)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-x^2+x\right)u'(x)+\left(x^2-2x+2\right)u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{u(x^2-2x+2)}{x\,(x-1)}\end{aligned}$$

Where $f(x)=\frac{x^2-2x+2}{x(x-1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{x^2-2x+2}{x\,(x-1)}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{x^2-2x+2}{x\,(x-1)}\,dx\\ \ln(u) &= x+\ln(x-1)-2\ln(x)+c_1\\ u &= e^{x+\ln(x-1)-2\ln(x)+c_1}\\ &= c_1e^{x+\ln(x-1)-2\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{e^x}{x}-\frac{e^x}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u\\ &= c_1\left(\frac{e^x}{x}-\frac{e^x}{x^2}\right)\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}v(x) &= \int\frac{(x-1)\,c_1e^x}{x^2}\,\mathrm{d}x\\ &= \frac{e^xc_1}{x}+c_2\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-x)\left(\frac{e^x c_1}{x} + c_2\right) \\
&= -c_1 e^x - c_2 x
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = -c_1 e^x - c_2 x \tag{1}
$$

<u>Verification of solutions</u>

$$
y = -c_1 e^x - c_2 x
$$

Verified OK.

1.6.3 Solving using Kovacic algorithm

Writing the ode as

$$
(x-1)\, y'' - x y' + y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x - 1 \\
B &= -x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 6}{4(x-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 6 \\ t &= 4(x-1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 11: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x-1)^2$. There is a pole at $x = 1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} + \frac{3}{4(x-1)^2} - \frac{1}{2(x-1)}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{2x} + \frac{1}{x^3} + \frac{11}{4x^4} + \frac{21}{4x^5} + \frac{15}{2x^6} + \frac{6}{x^7} - \frac{117}{16x^8} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 6}{4x^2 - 8x + 4} \\ &= Q + \frac{R}{4x^2 - 8x + 4} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-2x + 5}{4x^2 - 8x + 4}\right) \\ &= \frac{1}{4} + \frac{-2x + 5}{4x^2 - 8x + 4} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -2. Dividing this by leading coefficient in t which is 4 gives $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (0) \\ &= -\frac{1}{2} \end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{1}{2} \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{1}{2}}{\frac{1}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{x^2 - 4x + 6}{4(x-1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x-1)} + \left(\frac{1}{2}\right) \\
&= -\frac{1}{2(x-1)} + \frac{1}{2} \\
&= \frac{-2+x}{2x-2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)(0) + \left(\left(\frac{1}{2(x-1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right)^2 - \left(\frac{x^2 - 4x + 6}{4(x-1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)} + \frac{1}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-x}{x-1} \, dx} \\
&= z_1 e^{\frac{x}{2} + \frac{\ln(x-1)}{2}} \\
&= z_1 \left(\sqrt{x-1}\, \mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x-1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x+\ln(x-1)}}{(y_1)^2}\, dx \\
&= y_1\left(-x\,e^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(-x\,e^{-x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x - c_2 x \tag{1}$$

Verification of solutions

$$y = c_1 e^x - c_2 x$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$(x-1)\,y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x-1} + \frac{xy'}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{xy'}{x-1} + \frac{y}{x-1} = 0$

- ☐ Check to see if $x_0 = 1$ is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{x}{x-1}, P_3(x) = \frac{1}{x-1}\right]$
  - $(x-1) \cdot P_2(x)$ is analytic at $x = 1$

    $\left((x-1) \cdot P_2(x)\right)\bigg|_{x=1} = -1$
  - $(x-1)^2 \cdot P_3(x)$ is analytic at $x = 1$

    $\left((x-1)^2 \cdot P_3(x)\right)\bigg|_{x=1} = 0$
  - $x = 1$is a regular singular point

    Check to see if $x_0 = 1$ is a regular singular point

    $x_0 = 1$

- Multiply by denominators

  $(x-1)\, y'' - xy' + y = 0$

- Change variables using $x = u + 1$ so that the regular singular point is at $u = 0$

  $u\left(\frac{d^2}{du^2} y(u)\right) + (-u-1)\left(\frac{d}{du} y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-1}$$

- Shift index using $k- >k+1$

$$u \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r-1) - a_k(k+r-1)\right) u^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r(-2+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 2\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-1)\left(a_{k+1}(k+1+r) - a_k\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_k}{k+1+r}$$

- Recursion relation for $r = 0$

$$a_{k+1} = \frac{a_k}{k+1}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^k, a_{k+1} = \frac{a_k}{k+1}\right]$$

- Recursion relation for $r = 2$

$$a_{k+1} = \frac{a_k}{k+3}$$

- Solution for $r = 2$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Revert the change of variables $u = x - 1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x-1)^{k+2}, a_{k+1} = \frac{a_k}{k+3}\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (x-1)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-1)^{k+2}\right), a_{k+1} = \frac{a_k}{k+1}, b_{k+1} = \frac{b_k}{k+3}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((x-1)*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + e^x c_2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 17

```
DSolve[(x-1)*y''[x]-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 x$$

1.7 problem 19

1.7.1 Solving as second order linear constant coeff ode 79
1.7.2 Solving as second order ode can be made integrable ode 81
1.7.3 Solving using Kovacic algorithm 84
1.7.4 Maple step by step solution 87

Internal problem ID [5232]
Internal file name [`OUTPUT/4723_Friday_February_02_2024_05_10_39_AM_67853078/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

1.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{0^2 - (4)(1)(-1)} \\
&= \pm 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-1)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \tag{1}
$$

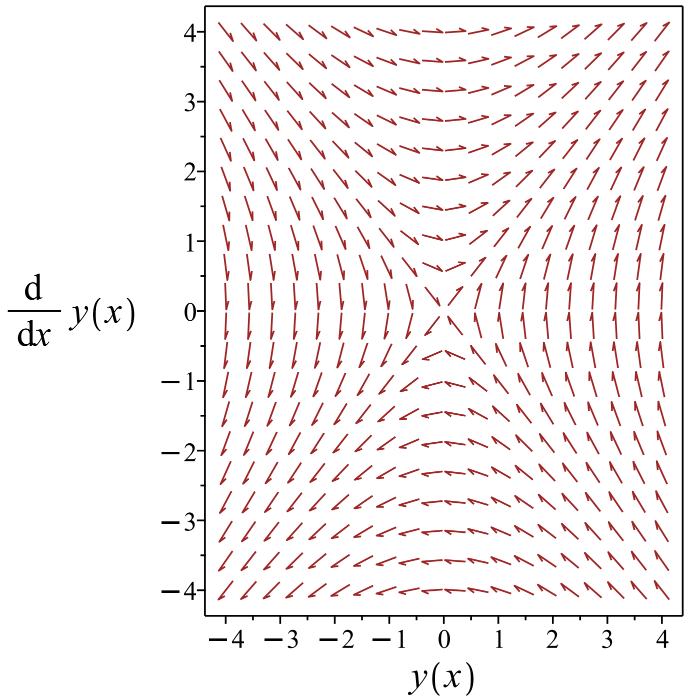


Figure 18: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2 e^{-x}$$

Verified OK.

1.7.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - yy')\, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\sqrt{y^2+2c_1}}dy &= \int dx \\
\ln\left(y+\sqrt{y^2+2c_1}\right) &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
y+\sqrt{y^2+2c_1} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$
y+\sqrt{y^2+2c_1} = c_3\mathrm{e}^{x}
$$

Solving for y gives

$$
y = \frac{\left(\mathrm{e}^{2x}c_3^2-2c_1\right)\mathrm{e}^{-x}}{2c_3}
$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$
\begin{aligned}
\int -\frac{1}{\sqrt{y^2+2c_1}}dy &= \int dx \\
-\ln\left(y+\sqrt{y^2+2c_1}\right) &= x + c_4
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{y+\sqrt{y^2+2c_1}} = \mathrm{e}^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{y+\sqrt{y^2+2c_1}} = c_5\mathrm{e}^{x}
$$

Solving for y gives

$$
y = -\frac{\left(2c_1c_5^2\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_5}
$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\, e^{-x}}{2c_3} \tag{1}$$

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\, e^{-x}}{2c_5} \tag{2}$$

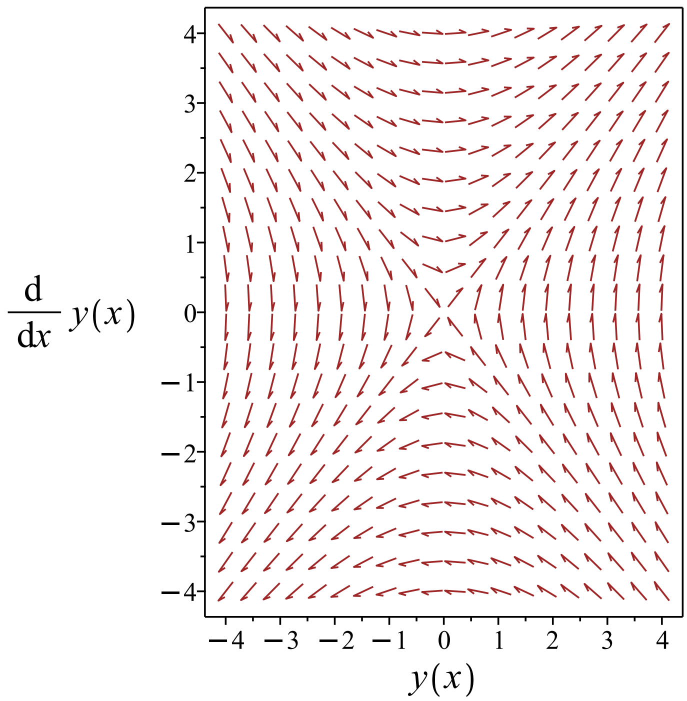


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\, e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\, e^{-x}}{2c_5}$$

Verified OK.

1.7.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1\\
B &= 0 \tag{3}\\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1\\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 13: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}
$$

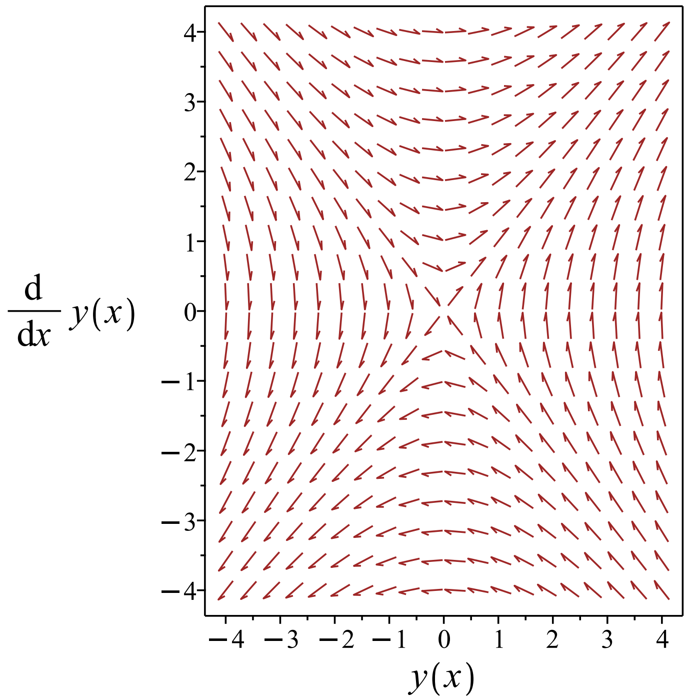


Figure 20: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

Verified OK.

1.7.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

  $$y_1(x) = \mathrm{e}^{-x}$$

- 2nd solution of the ODE

  $$y_2(x) = \mathrm{e}^{x}$$

- General solution of the ODE

  $$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

  $$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{x} c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

1.8 problem 20

1.8.1 Solving as second order linear constant coeff ode 89
1.8.2 Solving using Kovacic algorithm 92
1.8.3 Maple step by step solution 97

Internal problem ID [5233]
Internal file name [`OUTPUT/4724_Friday_February_02_2024_05_10_40_AM_27468358/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = -x + 4}$$

1.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = -x + 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x+1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 x - A_1 = -x + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + (x - 4) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + x - 4 \tag{1}$$

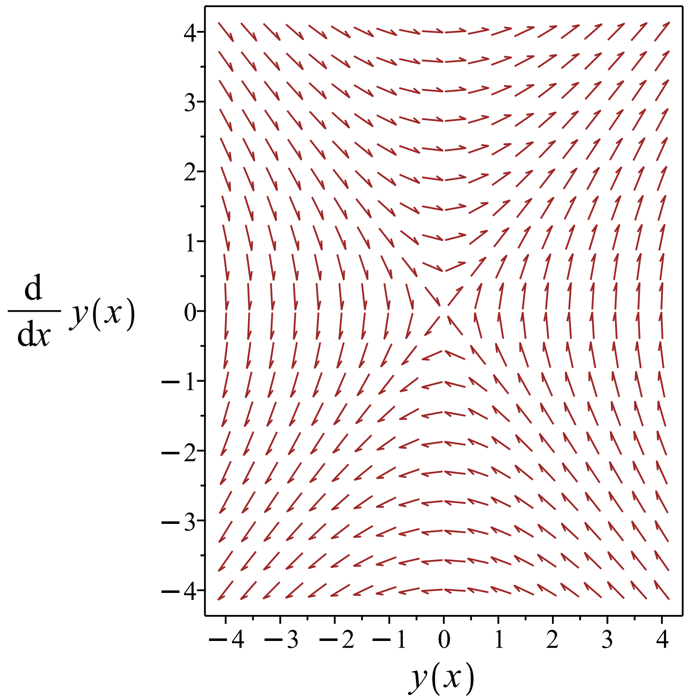


Figure 21: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + c_2 e^{-x} + x - 4$$

Verified OK.

1.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 15: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2x - A_1 = -x + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}\right) + (x - 4) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + x - 4 \tag{1}$$

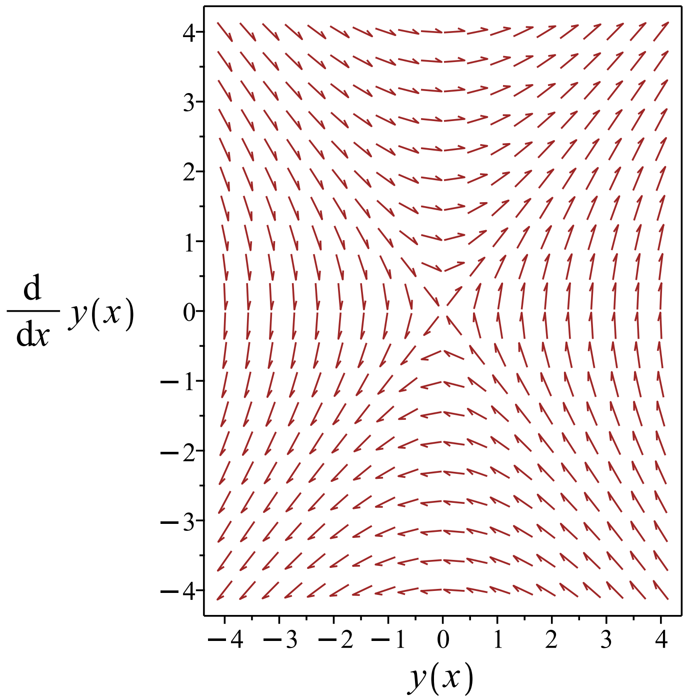


Figure 22: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + x - 4$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$y'' - y = -x + 4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -x + 4\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x}(x-4)dx\right)}{2} - \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x}(x-4)dx\right)}{2}$

  - Compute integrals

    $y_p(x) = x - 4$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + x - 4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-y(x)=4-x,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{x} c_1 + x - 4$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 22

```
DSolve[y''[x]-y[x]==4-x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^{x} + c_2 e^{-x} - 4$$

1.9 problem 21

1.9.1 Solving as second order linear constant coeff ode 100
1.9.2 Solving using Kovacic algorithm 102
1.9.3 Maple step by step solution 106

Internal problem ID [5234]
Internal file name [`OUTPUT/4725_Friday_February_02_2024_05_10_40_AM_9370276/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

1.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

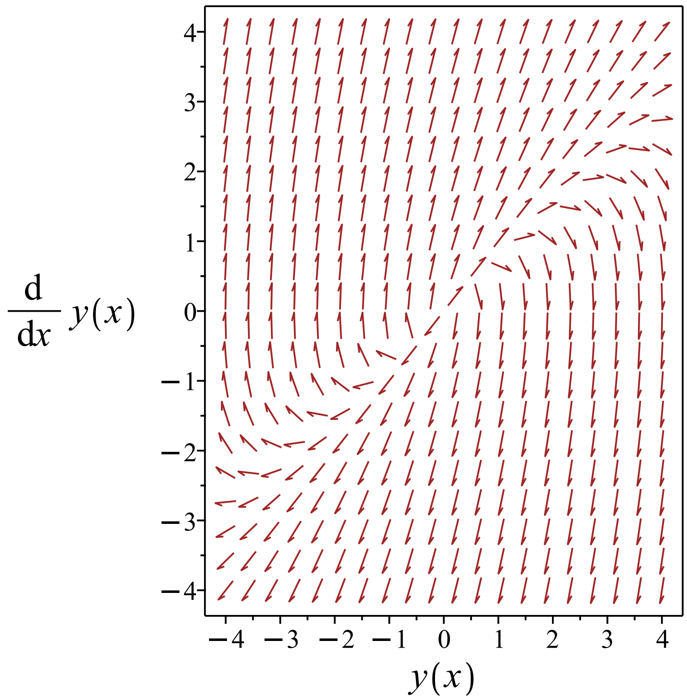


Figure 23: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

1.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 17: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(\mathrm{e}^{x}))
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} \tag{1}
$$

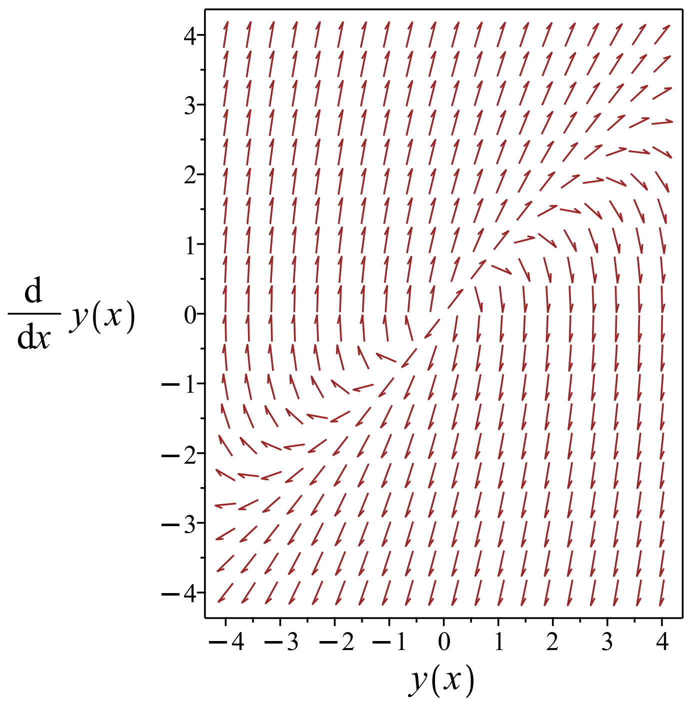


Figure 24: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}
$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 e^x + c_1)$$

1.10 problem 22

1.10.1 Solving as second order linear constant coeff ode 108
1.10.2 Solving using Kovacic algorithm 111
1.10.3 Maple step by step solution 116

Internal problem ID [5235]
Internal file name [`OUTPUT/4726_Friday_February_02_2024_05_10_40_AM_17761984/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 2. Solutions of differential equations. Supplemetary problems. Page 11
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = 2\,\mathrm{e}^{x}(1-x)}$$

1.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = -2(x-1)\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2(x-1)\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{2x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}, x^{2}\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2 x^{2}\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\mathrm{e}^{x} + 2A_2\mathrm{e}^{x} - 2A_2 x\,\mathrm{e}^{x} = -2(x-1)\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^{2}\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{2x} + c_2\mathrm{e}^{x}\right) + \left(x^{2}\mathrm{e}^{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{x} + x^2 e^{x} \tag{1}$$

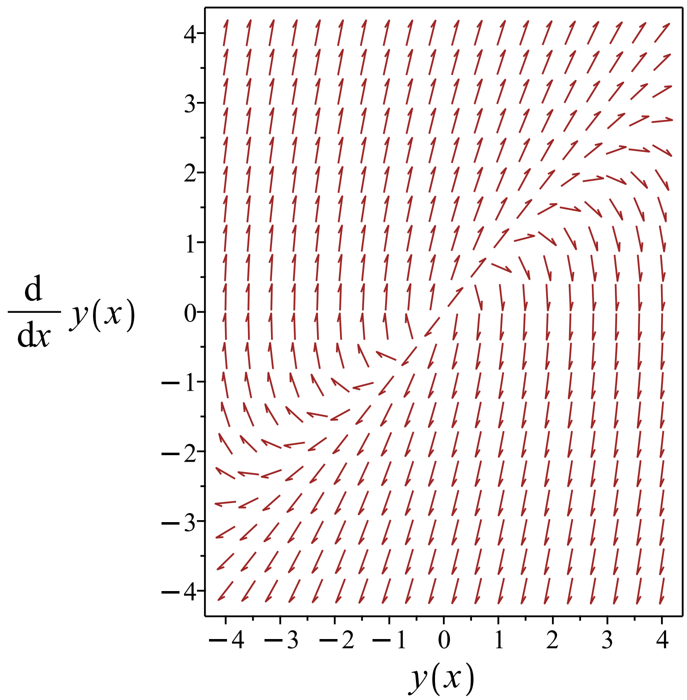


Figure 25: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x} + x^2 e^{x}$$

Verified OK.

1.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 19: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\
&= y_1(e^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^{x}) + c_2(e^{x}(e^{x}))
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{x} + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-2(x-1)\, e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{2x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{x}, x^2 e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{x} + A_2 x^2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{x} + 2A_2 e^{x} - 2A_2 x\,e^{x} = -2(x-1)\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{2x}\right) + \left(x^2 e^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{2x} + x^2 e^{x} \tag{1}$$

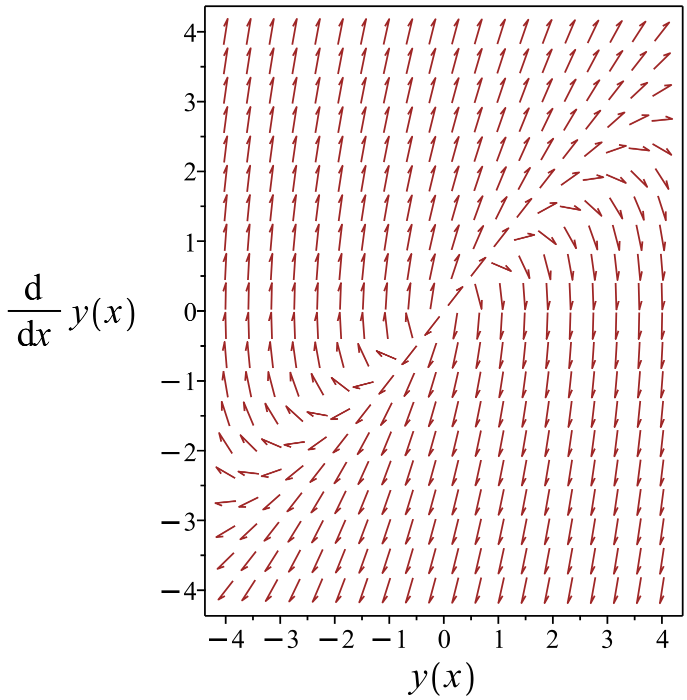


Figure 26: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{2x} + x^2 e^{x}$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = -2(x-1)\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)\,(r-2) = 0$

- Roots of the characteristic polynomial

$r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2(x-1)\,e^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 2\,e^x\left(\int (x-1)\, dx\right) - 2\,e^{2x}\left(\int (x-1)\,e^{-x} dx\right)$

  - Compute integrals

    $y_p(x) = x^2 e^x$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{2x} + x^2 e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=2*exp(x)*(1-x),y(x), singsol=all)
```

$$y(x) = \left(\mathrm{e}^{x} c_{1} + x^{2} + c_{2}\right) \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 21

```
DSolve[y''[x]-3*y'[x]+2*y[x]==2*Exp[x]*(1-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}\left(x^{2} + c_{2} e^{x} + c_{1}\right)$$

2 Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22

2.1 problem 24 120
2.2 problem 25 135
2.3 problem 26 149
2.4 problem 27 164
2.5 problem 28 182
2.6 problem 29 195
2.7 problem 30 210
2.8 problem 31 214
2.9 problem 32 227
2.10 problem 34 235
2.11 problem 35 251
2.12 problem 37 266
2.13 problem 38 269
2.14 problem 39 277
2.15 problem 40 288
2.16 problem 41 300
2.17 problem 42 315
2.18 problem 43 328
2.19 problem 44 340
2.20 problem 45 356
2.21 problem 46 364
2.22 problem 47 381
2.23 problem 48 398
2.24 problem 49 412
2.25 problem 51 425
2.26 problem 52 435

2.1 problem 24

2.1.1 Solving as separable ode 120
2.1.2 Solving as linear ode 122
2.1.3 Solving as homogeneousTypeD2 ode 123
2.1.4 Solving as first order ode lie symmetry lookup ode 125
2.1.5 Solving as exact ode 129
2.1.6 Maple step by step solution 133

Internal problem ID [5236]
Internal file name [`OUTPUT/4727_Friday_February_02_2024_05_10_41_AM_53066185/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{xy' + 4y = 0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{4y}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{4}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{4}{x}\,dx \\
\ln(y) &= -4\ln(x) + c_1 \\
y &= \mathrm{e}^{-4\ln(x)+c_1} \\
&= \frac{c_1}{x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^4} \tag{1}
$$

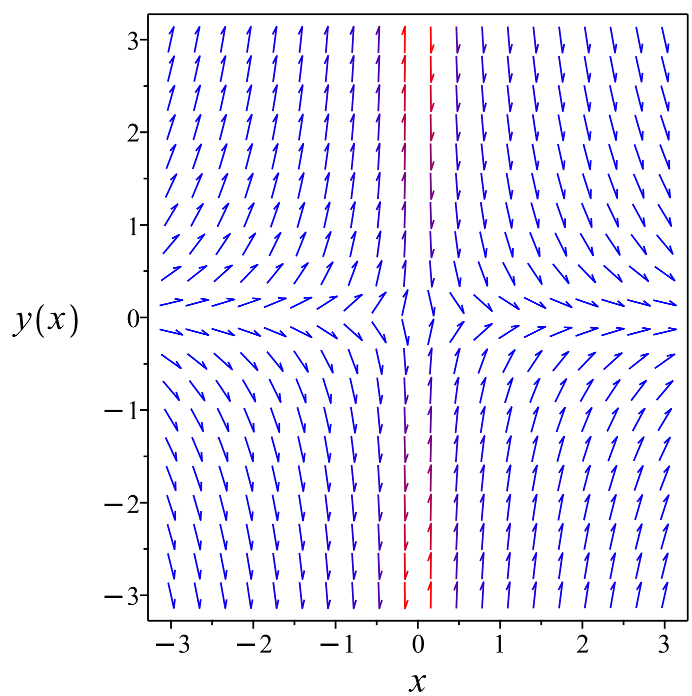


Figure 27: Slope field plot

Verification of solutions

$$
y = \frac{c_1}{x^4}
$$

Verified OK.

2.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{4y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{4}{x}dx} \\ &= x^4 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(x^4 y\right) &= 0 \end{aligned}$$

Integrating gives

$$x^4 y = c_1$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$y = \frac{c_1}{x^4}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^4} \tag{1}$$

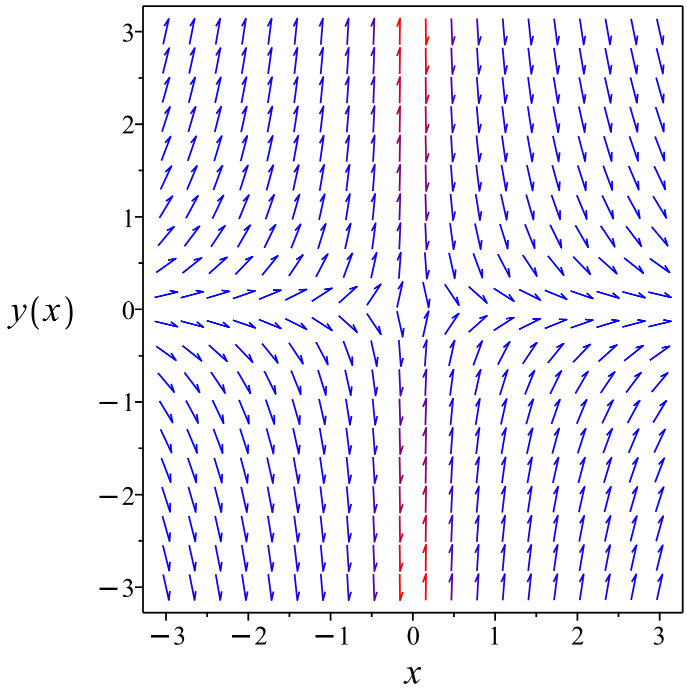


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^4}$$

Verified OK.

2.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) + 4u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{5u}{x}
\end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{5}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\
\ln(u) &= -5\ln(x) + c_2 \\
u &= e^{-5\ln(x)+c_2} \\
&= \frac{c_2}{x^5}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2}{x^4}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{x^4} \tag{1}
$$

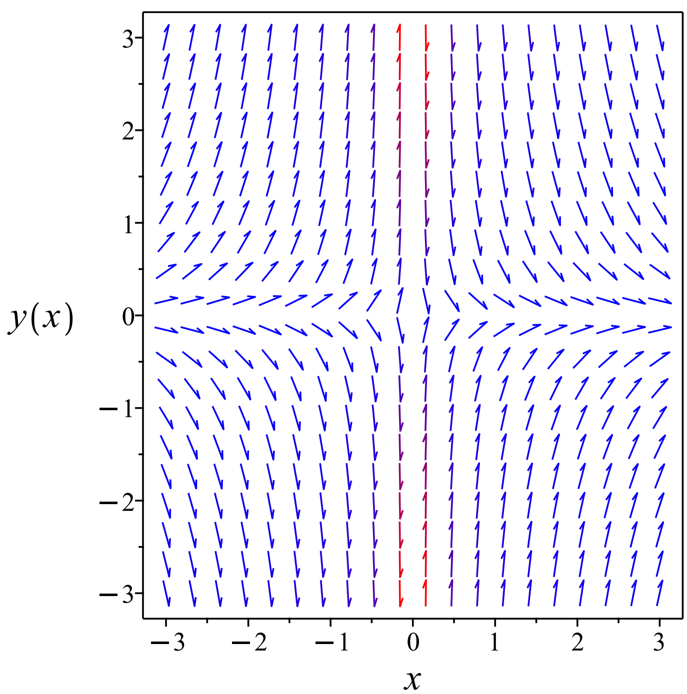


Figure 29: Slope field plot

Verification of solutions

$$
y = \frac{c_2}{x^4}
$$

Verified OK.

2.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{4y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^4}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^4}} dy \end{aligned}$$

Which results in

$$S = x^4 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{4y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 4y\,x^3 \\ S_y &= x^4 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^4 = c_1$$

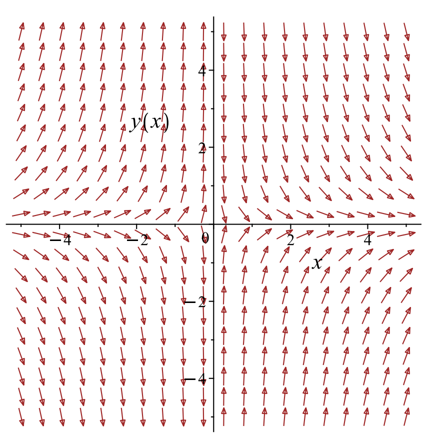
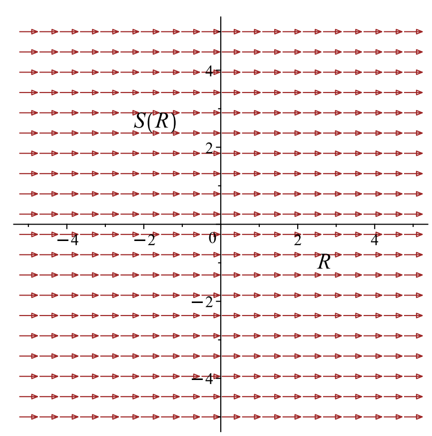
Which simplifies to

$$yx^4 = c_1$$

Which gives

$$y = \frac{c_1}{x^4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4y}{x}$ 	$R = x$ $S = x^4 y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} \tag{1}$$

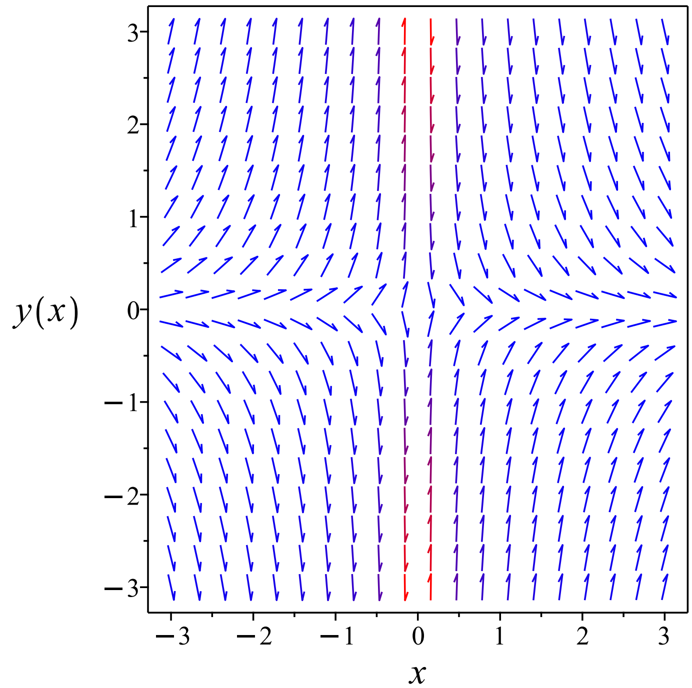


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{c_1}{x^4}$$

Verified OK.

2.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{4y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(-\frac{1}{4y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= -\frac{1}{4y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{4y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{4y}$. Therefore equation (4) becomes

$$
-\frac{1}{4y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{4y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{1}{4y}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y)}{4} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{4}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-4c_1}}{x^4}$$

Summary
The solution(s) found are the following

$$y = \frac{\mathrm{e}^{-4c_1}}{x^4} \tag{1}$$

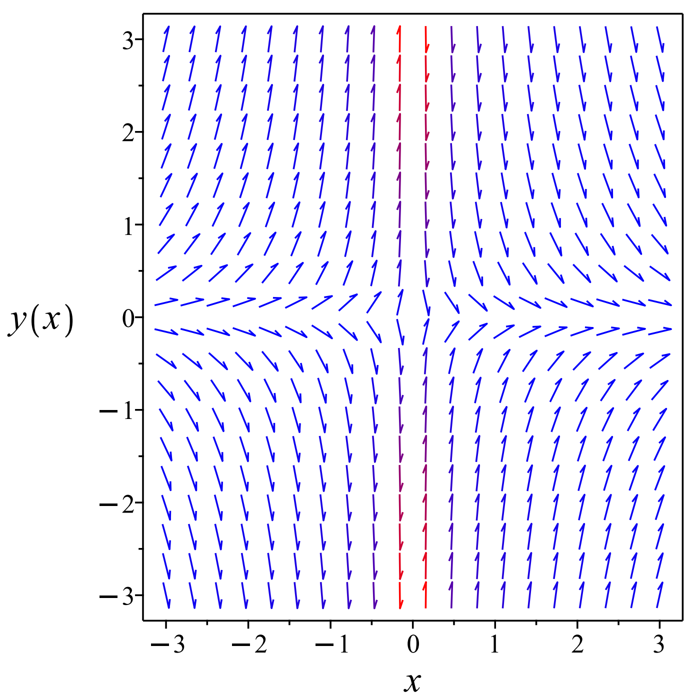


Figure 31: Slope field plot

Verification of solutions

$$
y = \frac{e^{-4c_1}}{x^4}
$$

Verified OK.

2.1.6 Maple step by step solution

Let's solve

$xy' + 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{4}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{4}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = -4\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x^4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(4*y(x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$
y(x) = \frac{c_1}{x^4}
$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[4*y[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^4}$$
$$y(x) \to 0$$

2.2 problem 25

2.2.1 Solving as separable ode 135
2.2.2 Solving as linear ode 137
2.2.3 Solving as first order ode lie symmetry lookup ode 139
2.2.4 Solving as exact ode 143
2.2.5 Maple step by step solution 147

Internal problem ID [5237]
Internal file name [`OUTPUT/4728_Friday_February_02_2024_05_10_42_AM_71893101/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y + \left(-x^2 + 4\right) y' = -1}$$

2.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1+2y}{x^2-4} \end{aligned}$$

Where $f(x) = \frac{1}{x^2-4}$ and $g(y) = 1 + 2y$. Integrating both sides gives

$$\frac{1}{1+2y}\,dy = \frac{1}{x^2-4}\,dx$$

$$\int \frac{1}{1+2y}\,dy = \int \frac{1}{x^2-4}\,dx$$

$$\frac{\ln(1+2y)}{2} = \frac{\ln(-2+x)}{4} - \frac{\ln(2+x)}{4} + c_1$$

Raising both side to exponential gives

$$\sqrt{1+2y} = \mathrm{e}^{\frac{\ln(-2+x)}{4} - \frac{\ln(2+x)}{4} + c_1}$$

Which simplifies to

$$\sqrt{1+2y} = c_2 \mathrm{e}^{\frac{\ln(-2+x)}{4} - \frac{\ln(2+x)}{4}}$$

Which simplifies to

$$y = \frac{c_2^2 \sqrt{-2+x}\,\mathrm{e}^{2c_1}}{2\sqrt{2+x}} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{c_2^2 \sqrt{-2+x}\,\mathrm{e}^{2c_1}}{2\sqrt{2+x}} - \frac{1}{2} \tag{1}$$

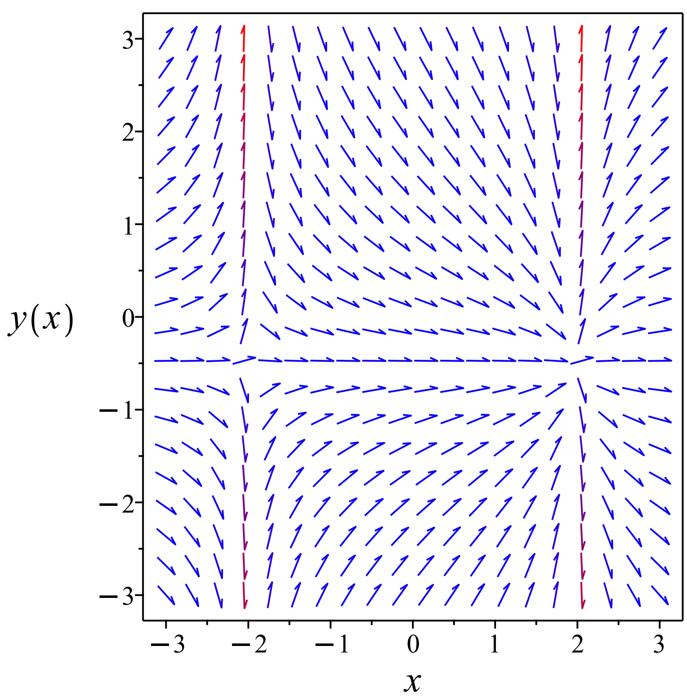


Figure 32: Slope field plot

Verification of solutions

$$
y=\frac{c_2^2\sqrt{-2+x}\,\mathrm{e}^{2c_1}}{2\sqrt{2+x}}-\frac{1}{2}
$$

Verified OK.

2.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{2}{x^2-4}\\
q(x) &= \frac{1}{x^2-4}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{2y}{x^2-4}=\frac{1}{x^2-4}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x^2-4}dx}\\
&= \mathrm{e}^{-\frac{\ln(-2+x)}{2}+\frac{\ln(2+x)}{2}}
\end{aligned}
$$

Which simplifies to

$$
\mu=\frac{\sqrt{2+x}}{\sqrt{-2+x}}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^2-4}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sqrt{2+x}\,y}{\sqrt{-2+x}}\right) &= \left(\frac{\sqrt{2+x}}{\sqrt{-2+x}}\right)\left(\frac{1}{x^2-4}\right)\\
\mathrm{d}\left(\frac{\sqrt{2+x}\,y}{\sqrt{-2+x}}\right) &= \left(\frac{\sqrt{2+x}}{\sqrt{-2+x}\,(x^2-4)}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{\sqrt{2+x}\, y}{\sqrt{-2+x}} &= \int \frac{\sqrt{2+x}}{\sqrt{-2+x}\,(x^2-4)}\, \mathrm{d}x \\
\frac{\sqrt{2+x}\, y}{\sqrt{-2+x}} &= -\frac{\sqrt{-2+x}\,(2+x)^{\frac{3}{2}}}{2\,(x^2-4)} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{\sqrt{2+x}}{\sqrt{-2+x}}$ results in

$$
y = -\frac{(-2+x)\,(2+x)}{2\,(x^2-4)} + \frac{c_1\sqrt{-2+x}}{\sqrt{2+x}}
$$

which simplifies to

$$
y = -\frac{1}{2} + \frac{c_1\sqrt{-2+x}}{\sqrt{2+x}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{1}{2} + \frac{c_1\sqrt{-2+x}}{\sqrt{2+x}} \tag{1}
$$

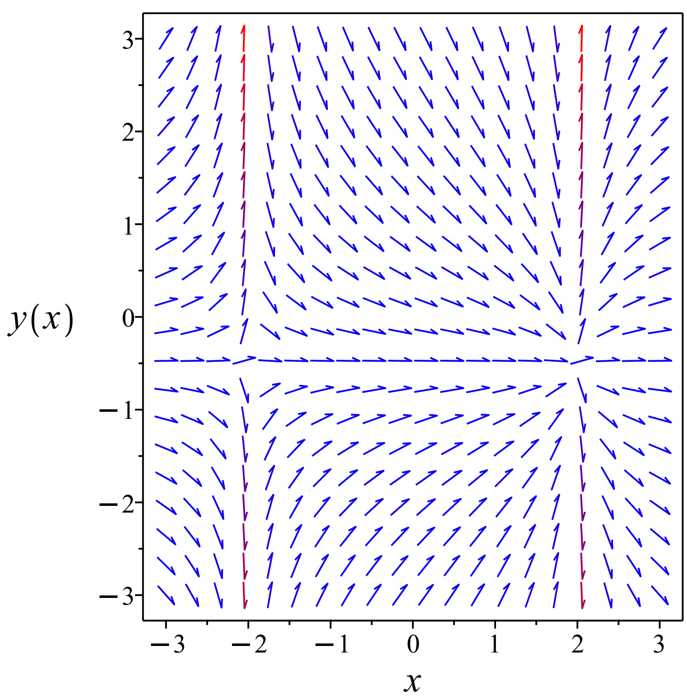


Figure 33: Slope field plot

Verification of solutions

$$
y = -\frac{1}{2} + \frac{c_1 \sqrt{-2+x}}{\sqrt{2+x}}
$$

Verified OK.

2.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{1+2y}{x^2-4} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\ln(-2+x)}{2} - \frac{\ln(2+x)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{\ln(-2+x)}{2} - \frac{\ln(2+x)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln\left(\frac{1}{\sqrt{-2+x}}\right) + \ln\left(\sqrt{2+x}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1+2y}{x^2-4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{(-2+x)^{\frac{3}{2}} \sqrt{2+x}} \\ S_y &= \frac{\sqrt{2+x}}{\sqrt{-2+x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(-2+x)^{\frac{3}{2}} \sqrt{2+x}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(-2+R)^{\frac{3}{2}} \sqrt{2+R}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\sqrt{2+R}}{2\sqrt{-2+R}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{2+x}\, y}{\sqrt{-2+x}} = -\frac{\sqrt{2+x}}{2\sqrt{-2+x}} + c_1$$

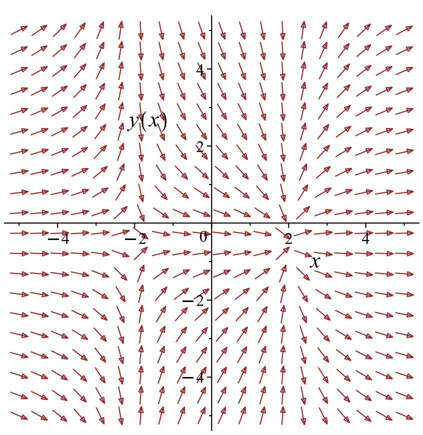
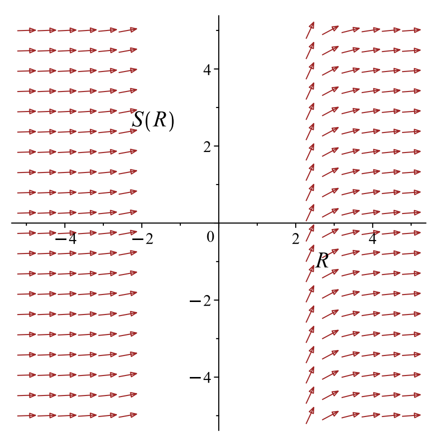
Which simplifies to

$$\frac{\sqrt{2+x}\, y}{\sqrt{-2+x}} = -\frac{\sqrt{2+x}}{2\sqrt{-2+x}} + c_1$$

Which gives

$$y = \frac{2c_1\sqrt{-2+x} - \sqrt{2+x}}{2\sqrt{2+x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1+2y}{x^2-4}$ 	$R = x$ $S = \frac{\sqrt{2+x}\, y}{\sqrt{-2+x}}$	$\frac{dS}{dR} = \frac{1}{(-2+R)^{\frac{3}{2}}\sqrt{2+R}}$ 

Summary
The solution(s) found are the following

$$y = \frac{2c_1\sqrt{-2+x} - \sqrt{2+x}}{2\sqrt{2+x}} \tag{1}$$

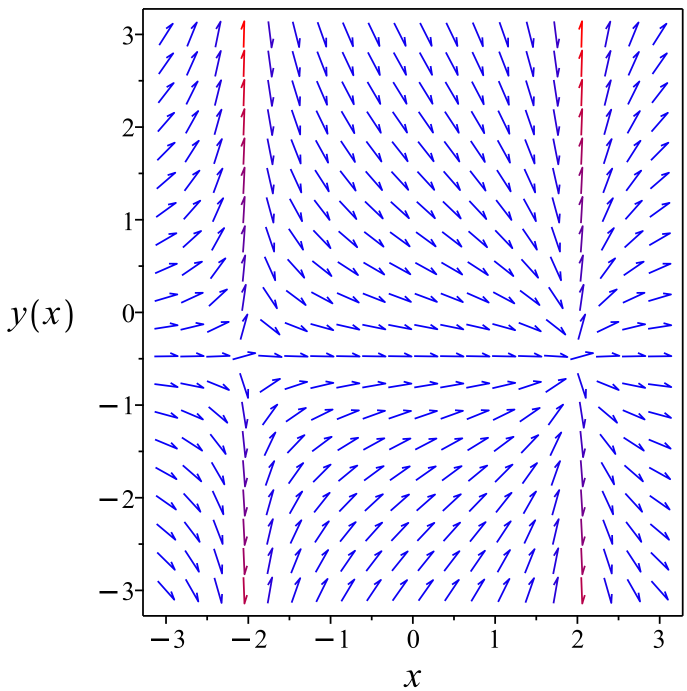


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{2c_1\sqrt{-2+x} - \sqrt{2+x}}{2\sqrt{2+x}}$$

Verified OK.

2.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1+2y}\right)\mathrm{d}y&=\left(\frac{1}{x^2-4}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2-4}\right)\mathrm{d}x+\left(\frac{1}{1+2y}\right)\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2-4}\\ N(x,y)&=\frac{1}{1+2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2-4}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1+2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{1}{x^2-4}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(-2+x)}{4} + \frac{\ln(2+x)}{4} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+2y}$. Therefore equation (4) becomes

$$
\frac{1}{1+2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{1+2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{1+2y}\right)\mathrm{d}y$$
$$f(y) = \frac{\ln(1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(-2+x)}{4} + \frac{\ln(2+x)}{4} + \frac{\ln(1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(-2+x)}{4} + \frac{\ln(2+x)}{4} + \frac{\ln(1+2y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{\frac{\ln(-2+x)}{2} - \frac{\ln(2+x)}{2} + 2c_1}}{2} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{e}^{\frac{\ln(-2+x)}{2} - \frac{\ln(2+x)}{2} + 2c_1}}{2} - \frac{1}{2} \tag{1}$$

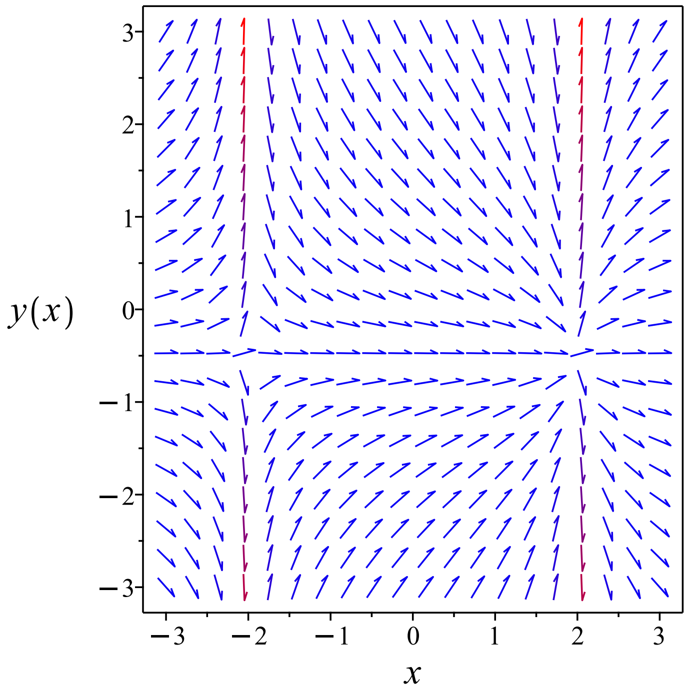


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{e^{\frac{\ln(-2+x)}{2} - \frac{\ln(2+x)}{2} + 2c_1}}{2} - \frac{1}{2}$$

Verified OK.

2.2.5 Maple step by step solution

Let's solve

$2y + (-x^2 + 4)\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-2y-1} = \frac{1}{-x^2+4}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-2y-1} dx = \int \frac{1}{-x^2+4} dx + c_1$

- Evaluate integral

$$-\frac{\ln(-2y-1)}{2} = -\frac{\ln(-2+x)}{4} + \frac{\ln(2+x)}{4} + c_1$$

- Solve for y

$$\left\{y = \frac{-x-2+\sqrt{e^{-4c_1}x^2-4\,e^{-4c_1}}}{2(2+x)}, y = -\frac{x+2+\sqrt{e^{-4c_1}x^2-4\,e^{-4c_1}}}{2(2+x)}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve((1+2*y(x))+(4-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{2} + \frac{\sqrt{-2+x}\,c_1}{\sqrt{x+2}}$$

✓ Solution by Mathematica

Time used: 0.076 (sec). Leaf size: 35

```
DSolve[(1+2*y[x])+(4-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2} + \frac{c_1\sqrt{2-x}}{\sqrt{x+2}}$$
$$y(x) \to -\frac{1}{2}$$

2.3 problem 26

2.3.1 Solving as separable ode 149
2.3.2 Solving as homogeneousTypeD2 ode 151
2.3.3 Solving as first order ode lie symmetry lookup ode 152
2.3.4 Solving as exact ode 156
2.3.5 Solving as riccati ode 160
2.3.6 Maple step by step solution 162

Internal problem ID [5238]
Internal file name [`OUTPUT/4729_Friday_February_02_2024_05_10_42_AM_66874428/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 - x^2 y' = 0}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x) g(y) \\ &= \frac{y^2}{x^2} \end{aligned}$$

Where $f(x) = \frac{1}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{1}{x^2}\,dx$$
$$\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx$$
$$-\frac{1}{y} = c_1 - \frac{1}{x}$$

Which results in

$$y = -\frac{x}{c_1 x - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_1 x - 1} \tag{1}$$

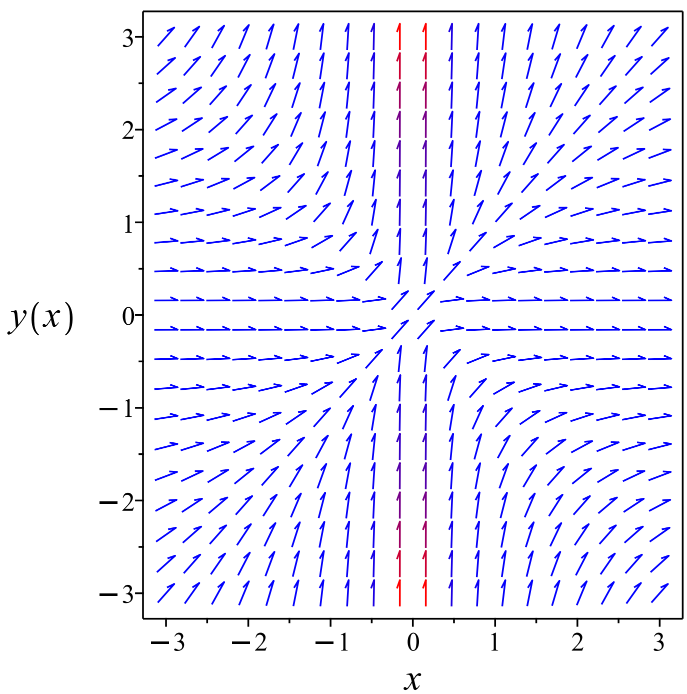


Figure 36: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1 x - 1}$$

Verified OK.

2.3.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 - x^2(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{u(u-1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u-1)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u\,(u-1)}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{u\,(u-1)}\,du &= \int \frac{1}{x}\,dx \\ \ln\,(u-1) - \ln\,(u) &= \ln\,(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)-\ln(u)} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{u-1}{u} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= -\frac{x}{c_3 x - 1} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{c_3 x - 1} \tag{1}$$

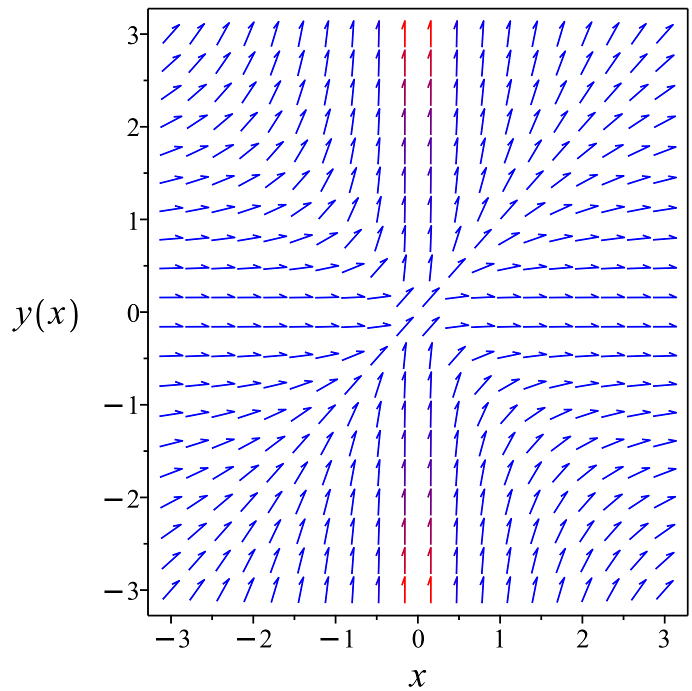


Figure 37: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_3 x - 1}$$

Verified OK.

2.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 27: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= x^2 \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2} dx \end{aligned}$$

Which results in

$$S = -\frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

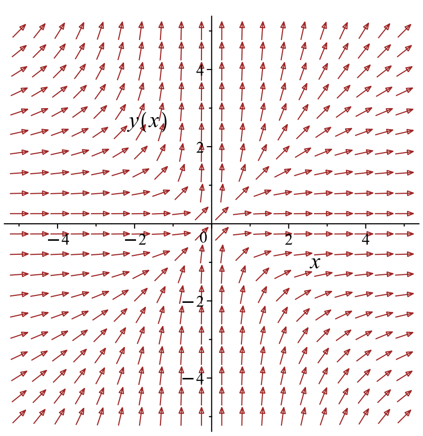
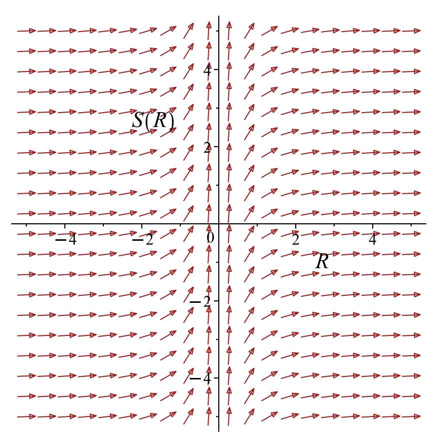
Which simplifies to

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{c_1 x + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x^2}$ 	$R = y$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary
The solution(s) found are the following

$$y = \frac{x}{c_1 x + 1} \tag{1}$$

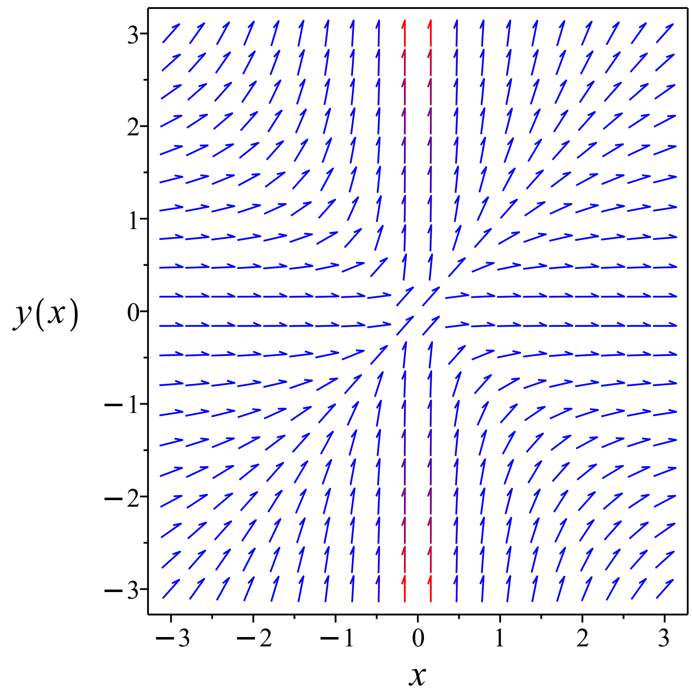


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{x}{c_1 x + 1}$$

Verified OK.

2.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y&=\left(\frac{1}{x^2}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2}\right)\mathrm{d}x+\left(\frac{1}{y^2}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2}\\ N(x,y)&=\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x^2}\,\mathrm{d}x\\ \phi &= \frac{1}{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^2}\right) \mathrm{d}y \\
f(y) &= -\frac{1}{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{1}{x} - \frac{1}{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{1}{x} - \frac{1}{y}
$$

The solution becomes

$$
y = -\frac{x}{c_1 x - 1}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{x}{c_1 x - 1} \tag{1}
$$

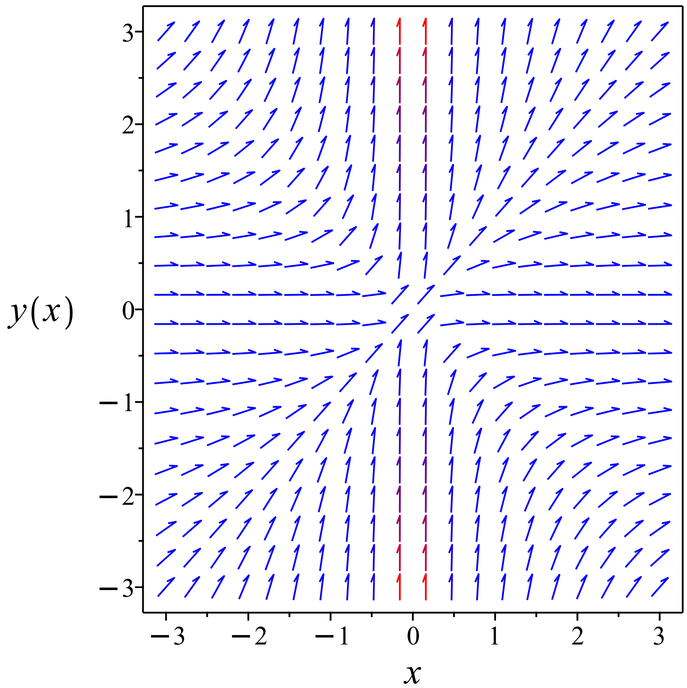


Figure 39: Slope field plot

Verification of solutions

$$y = -\frac{x}{c_1 x - 1}$$

Verified OK.

2.3.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^3} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{u''(x)}{x^2} + \frac{2u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \frac{c_2}{x}
$$

The above shows that

$$
u'(x) = -\frac{c_2}{x^2}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_2}{c_1 + \frac{c_2}{x}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{1}{c_3 + \frac{1}{x}}
$$

Summary

The solution(s) found are the following

$$y = \frac{1}{c_3 + \frac{1}{x}} \tag{1}$$

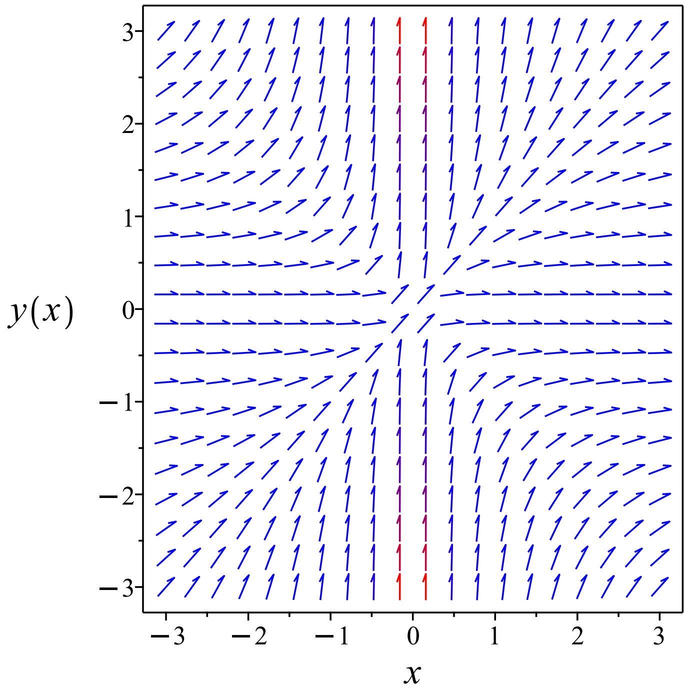


Figure 40: Slope field plot

Verification of solutions

$$y = \frac{1}{c_3 + \frac{1}{x}}$$

Verified OK.

2.3.6 Maple step by step solution

Let's solve

$y^2 - x^2 y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{y^2} = \frac{1}{x^2}$$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = c_1 - \frac{1}{x}$

- Solve for y

  $y = -\frac{x}{c_1 x - 1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(y(x)^2-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 x + 1}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 21

```
DSolve[y[x]^2-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{1 - c_1 x}$$
$$y(x) \to 0$$

2.4 problem 27

2.4.1 Solving as separable ode 164
2.4.2 Solving as linear ode 166
2.4.3 Solving as homogeneousTypeD2 ode 168
2.4.4 Solving as homogeneousTypeMapleC ode 169
2.4.5 Solving as first order ode lie symmetry lookup ode 172
2.4.6 Solving as exact ode 176
2.4.7 Maple step by step solution 180

Internal problem ID [5239]
Internal file name [`OUTPUT/4730_Friday_February_02_2024_05_10_43_AM_16493042/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-(x+1)\,y'+y=-1}$$

2.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{y+1}{x+1}\end{aligned}$$

Where $f(x) = \frac{1}{x+1}$ and $g(y) = y + 1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y+1}\, dy &= \frac{1}{x+1}\, dx \\
\int \frac{1}{y+1}\, dy &= \int \frac{1}{x+1}\, dx \\
\ln(y+1) &= \ln(x+1) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$y + 1 = \mathrm{e}^{\ln(x+1)+c_1}$$

Which simplifies to

$$y + 1 = c_2(x+1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2 \mathrm{e}^{\ln(x+1)+c_1} - 1 \tag{1}$$

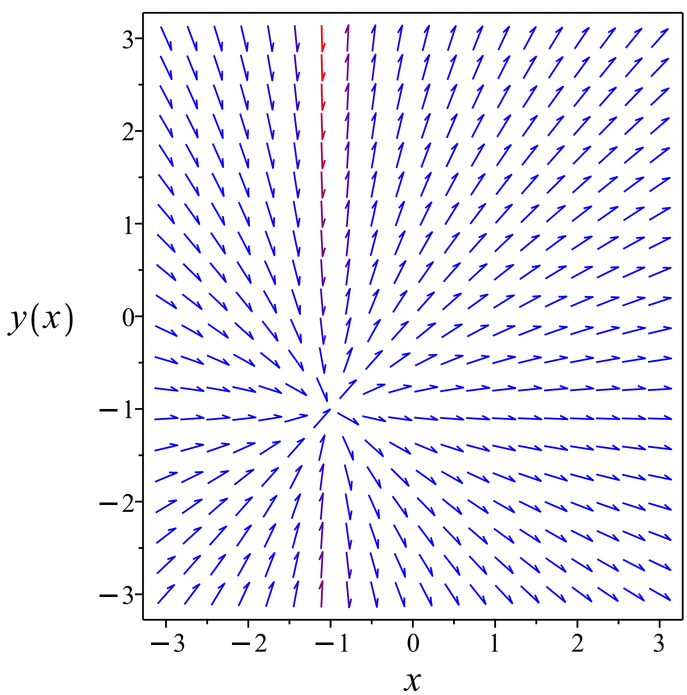


Figure 41: Slope field plot

Verification of solutions

$$y = c_2 e^{\ln(x+1)+c_1} - 1$$

Verified OK.

2.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x+1} \\ q(x) &= \frac{1}{x+1} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{1}{x+1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x+1}dx} \\ &= \frac{1}{x+1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x+1}\right) \\ \frac{d}{dx}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right)\left(\frac{1}{x+1}\right) \\ d\left(\frac{y}{x+1}\right) &= \frac{1}{(x+1)^2}\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x+1} &= \int \frac{1}{(x+1)^2}\, dx \\ \frac{y}{x+1} &= -\frac{1}{x+1} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = -1 + (x+1)\, c_1$$

which simplifies to

$$y = c_1 x + c_1 - 1$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 1 \tag{1}$$

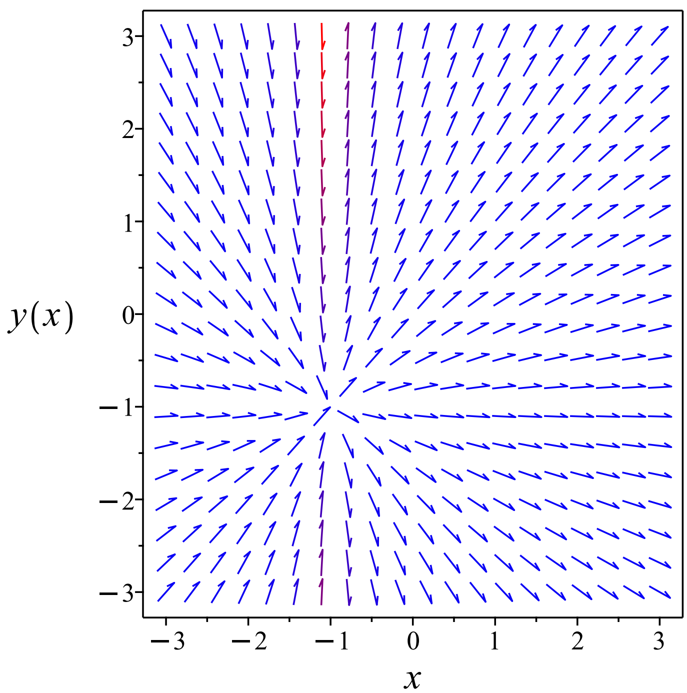


Figure 42: Slope field plot

Verification of solutions

$$y = c_1 x + c_1 - 1$$

Verified OK.

2.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-(x+1)\,(u'(x)\,x + u(x)) + u(x)\,x = -1$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{-u+1}{x\,(x+1)}
\end{aligned}$$

Where $f(x) = \frac{1}{x(x+1)}$ and $g(u) = -u+1$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{-u+1}\,du &= \frac{1}{x\,(x+1)}\,dx \\
\int \frac{1}{-u+1}\,du &= \int \frac{1}{x\,(x+1)}\,dx \\
-\ln(u-1) &= -\ln(x+1) + \ln(x) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{u-1} = \mathrm{e}^{-\ln(x+1)+\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{u-1} = c_3\mathrm{e}^{-\ln(x+1)+\ln(x)}$$

Which simplifies to

$$u(x) = \frac{\left(\frac{c_3\mathrm{e}^{c_2}x}{x+1}+1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3 x}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{\left(\frac{c_3\mathrm{e}^{c_2}x}{x+1}+1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(\frac{c_3 \mathrm{e}^{c_2 x}}{x+1}+1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3} \tag{1}$$

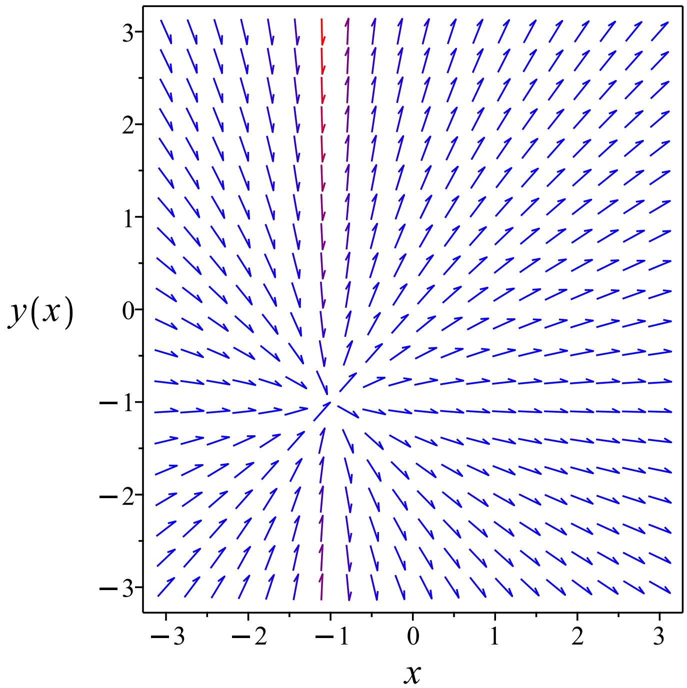


Figure 43: Slope field plot

Verification of solutions

$$y = \frac{\left(\frac{c_3 \mathrm{e}^{c_2 x}}{x+1}+1\right)(x+1)\,\mathrm{e}^{-c_2}}{c_3}$$

Verified OK.

2.4.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X)+y_0+1}{X+x_0+1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= -1 \\
y_0 &= -1
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = \frac{Y(X)}{X}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y}{X} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= 0
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. Integrating both sides gives

$$
\begin{aligned}
u(X) &= \int 0 \,\mathrm{d}X \\
&= c_2
\end{aligned}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = Xc_2$$

Using the solution for $Y(X)$

$$Y(X) = Xc_2$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x - 1$$

Then the solution in y becomes

$$y + 1 = c_2(x + 1)$$

Summary

The solution(s) found are the following

$$y + 1 = c_2(x + 1) \tag{1}$$

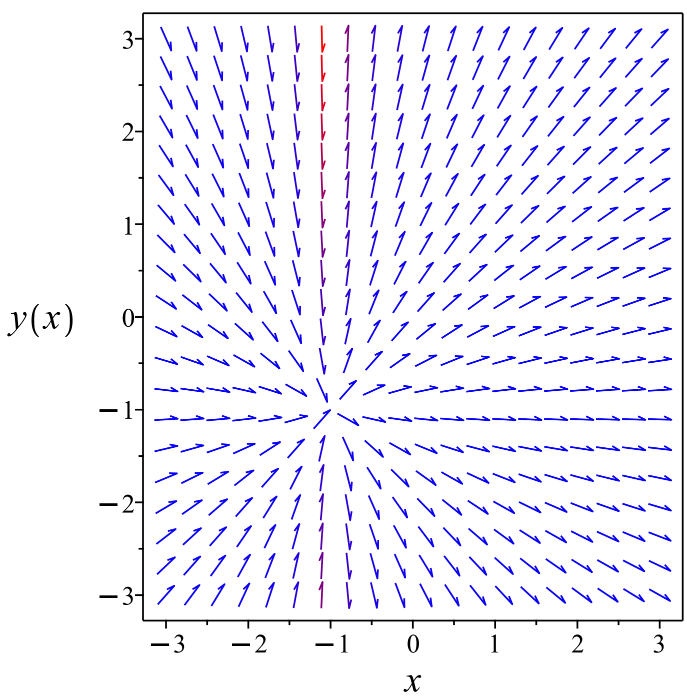


Figure 44: Slope field plot

Verification of solutions

$$y + 1 = c_2(x + 1)$$

Verified OK.

2.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y+1}{x+1} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x + 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+1}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(x+1)^2} \\ S_y &= \frac{1}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(x+1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x+1} = -\frac{1}{x+1} + c_1$$

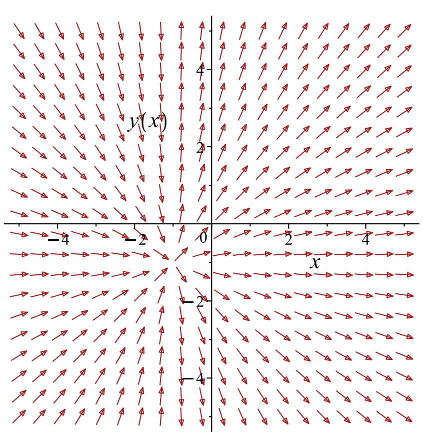
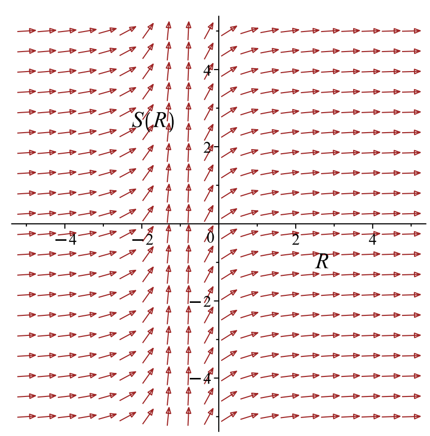
Which simplifies to

$$\frac{y}{x+1} = -\frac{1}{x+1} + c_1$$

Which gives

$$y = c_1 x + c_1 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+1}{x+1}$ 	$R = x$ $S = \frac{y}{x+1}$	$\frac{dS}{dR} = \frac{1}{(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x + c_1 - 1 \tag{1}$$

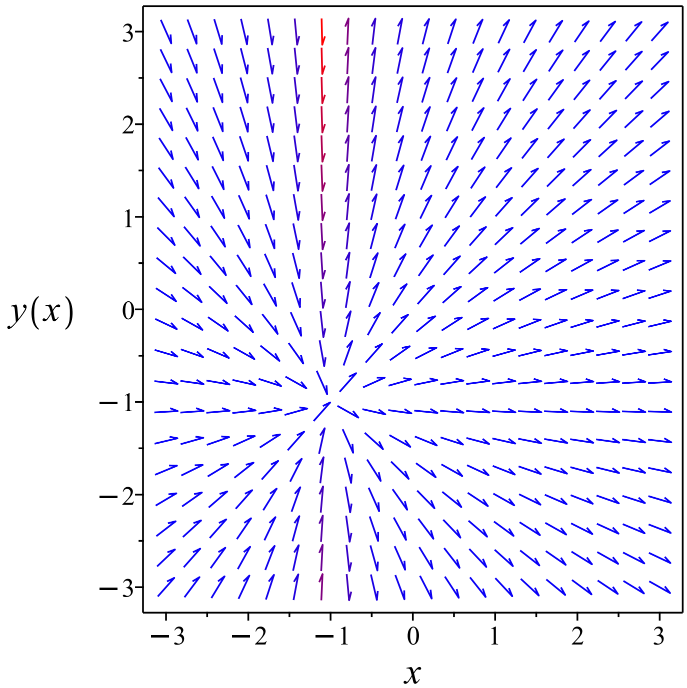


Figure 45: Slope field plot

Verification of solutions

$$y = c_1 x + c_1 - 1$$

Verified OK.

2.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y+1}\right) \mathrm{d}y &= \left(\frac{1}{x+1}\right) \mathrm{d}x \\
\left(-\frac{1}{x+1}\right) \mathrm{d}x + \left(\frac{1}{y+1}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x+1} \\
N(x, y) &= \frac{1}{y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x+1}\,\mathrm{d}x\\
\phi &= -\ln(x+1) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$
\frac{1}{y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y+1}\right)\mathrm{d}y\\
f(y) &= \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \ln(y+1)$$

The solution becomes

$$y = x\,e^{c_1} + e^{c_1} - 1$$

Summary

The solution(s) found are the following

$$y = x\,e^{c_1} + e^{c_1} - 1 \tag{1}$$

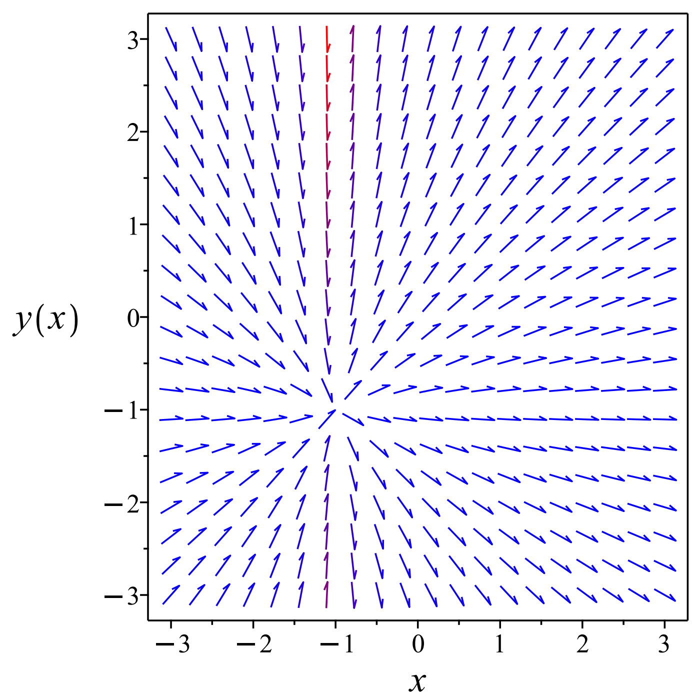


Figure 46: Slope field plot

Verification of solutions

$$y = x\,e^{c_1} + e^{c_1} - 1$$

Verified OK.

2.4.7 Maple step by step solution

Let's solve

$-(x+1)\, y' + y = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1-y} = -\frac{1}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1-y} dx = \int -\frac{1}{x+1} dx + c_1$

- Evaluate integral

  $-\ln(-1-y) = -\ln(x+1) + c_1$

- Solve for y

  $y = -\frac{e^{c_1} + x + 1}{e^{c_1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 10

```
dsolve((1+y(x))-(1+x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + c_1 - 1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 18

```
DSolve[(1+y[x])-(1+x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -1 + c_1(x+1)$$
$$y(x) \rightarrow -1$$

2.5 problem 28

2.5.1 Solving as first order ode lie symmetry calculated ode 182
2.5.2 Solving as exact ode 188

Internal problem ID [5240]
Internal file name [`OUTPUT/4731_Friday_February_02_2024_05_10_44_AM_120439/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{xy^2 + y + (x^2y - x)\, y' = 0}$$

2.5.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(xy+1)}{x\,(xy-1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{y(xy+1)(b_3-a_2)}{x(xy-1)} - \frac{y^2(xy+1)^2 a_3}{x^2(xy-1)^2} \\
&- \left(-\frac{y^2}{x(xy-1)} + \frac{y(xy+1)}{x^2(xy-1)} + \frac{y^2(xy+1)}{x(xy-1)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{xy+1}{x(xy-1)} - \frac{y}{xy-1} + \frac{y(xy+1)}{(xy-1)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^4y^2b_2 - 2x^2y^4a_3 + x^3y^2b_1 - x^2y^3a_1 - 4x^3yb_2 - 2x^2y^2a_2 - 2x^2y^2b_3 - 4x\,y^3a_3 - 2x^2yb_1 - 2x\,y^2a_1 - xb_1 +}{x^2(xy-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^4y^2b_2 - 2x^2y^4a_3 + x^3y^2b_1 - x^2y^3a_1 - 4x^3yb_2 - 2x^2y^2a_2 \\
&- 2x^2y^2b_3 - 4x\,y^3a_3 - 2x^2yb_1 - 2x\,y^2a_1 - xb_1 + ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2a_3v_1^2v_2^4 + 2b_2v_1^4v_2^2 - a_1v_1^2v_2^3 + b_1v_1^3v_2^2 - 2a_2v_1^2v_2^2 - 4a_3v_1v_2^3 \\
&- 4b_2v_1^3v_2 - 2b_3v_1^2v_2^2 - 2a_1v_1v_2^2 - 2b_1v_1^2v_2 + a_1v_2 - b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}2b_2v_1^4v_2^2 + b_1v_1^3v_2^2 - 4b_2v_1^3v_2 - 2a_3v_1^2v_2^4 - a_1v_1^2v_2^3 + (-2a_2 - 2b_3)\, v_1^2v_2^2 \\ - 2b_1v_1^2v_2 - 4a_3v_1v_2^3 - 2a_1v_1v_2^2 - b_1v_1 + a_1v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_1 &= 0\\ b_1 &= 0\\ -2a_1 &= 0\\ -a_1 &= 0\\ -4a_3 &= 0\\ -2a_3 &= 0\\ -2b_1 &= 0\\ -b_1 &= 0\\ -4b_2 &= 0\\ 2b_2 &= 0\\ -2a_2 - 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(xy+1)}{x\,(xy-1)}\right)(-x) \\ &= -\frac{2y}{xy-1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{2y}{xy-1}} dy\end{aligned}$$

Which results in

$$S = -\frac{xy}{2} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(xy+1)}{x\,(xy-1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y}{2} \\
S_y &= \frac{-xy + 1}{2y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln(R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{yx}{2} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1
$$

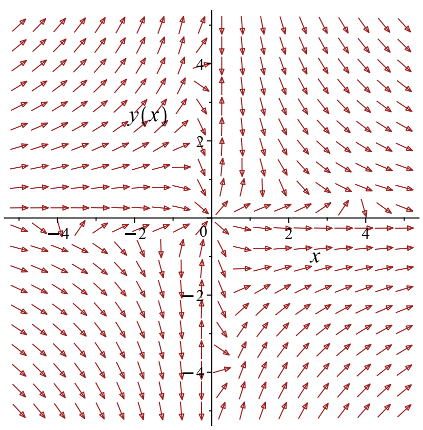
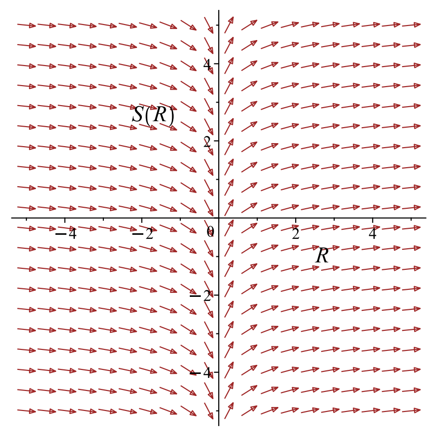
Which simplifies to

$$
-\frac{yx}{2} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_1
$$

Which gives

$$
y = -\frac{\operatorname{LambertW}\left(-x^2 e^{2c_1}\right)}{x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(xy+1)}{x(xy-1)}$ 	$R = x$ $S = -\frac{xy}{2} + \frac{\ln(y)}{2}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{2c_1}\right)}{x} \tag{1}$$

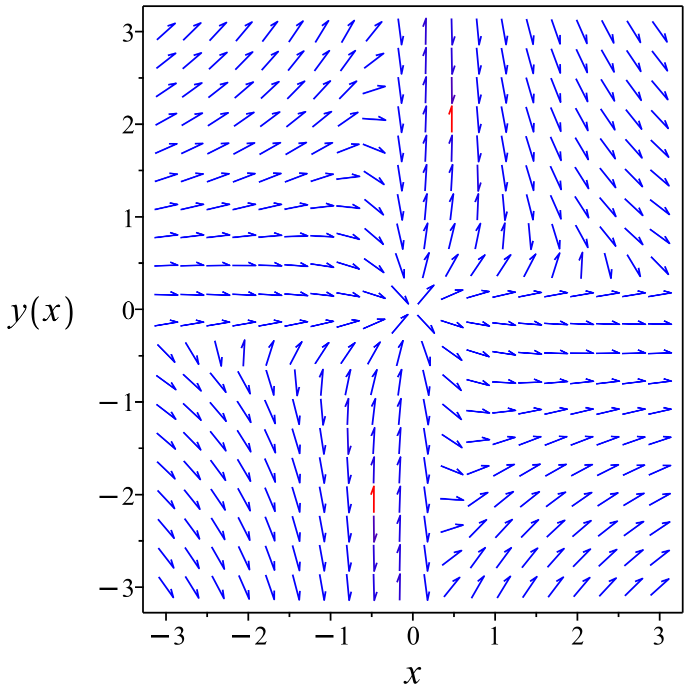


Figure 47: Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{LambertW}\left(-x^{2} e^{2c_{1}}\right)}{x}$$

Verified OK.

2.5.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2y - x)\,\mathrm{d}y &= (-y^2x - y)\,\mathrm{d}x\\ (y^2x + y)\,\mathrm{d}x + (x^2y - x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y^2x + y$$
$$N(x, y) = x^2y - x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2x + y)\\ &= 2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2y - x)\\ &= 2xy - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x\left(xy-1\right)}\left(\left(2xy+1\right) - \left(2xy-1\right)\right) \\ &= \frac{2}{x\left(xy-1\right)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y^2x+y}\left(\left(2xy-1\right) - \left(2xy+1\right)\right) \\ &= -\frac{2}{y\left(xy+1\right)} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{\left(2xy-1\right) - \left(2xy+1\right)}{x\left(y^2x+y\right) - y\left(x^2y-x\right)} \\ &= -\frac{1}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{1}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(t)}\\ &= \frac{1}{t}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{yx}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{yx}(y^2x + y)\\ &= \frac{xy+1}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{yx}(x^2y - x)\\ &= \frac{xy-1}{y}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{xy+1}{x}\right) + \left(\frac{xy-1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \frac{xy+1}{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = xy + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{xy-1}{y}$. Therefore equation (4) becomes

$$\frac{xy-1}{y} = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = xy + \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy + \ln(x) - \ln(y)$$

The solution becomes

$$y = -\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-c_1} x^2\right)}{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{LambertW}\left(-\mathrm{e}^{-c_1} x^2\right)}{x} \tag{1}$$

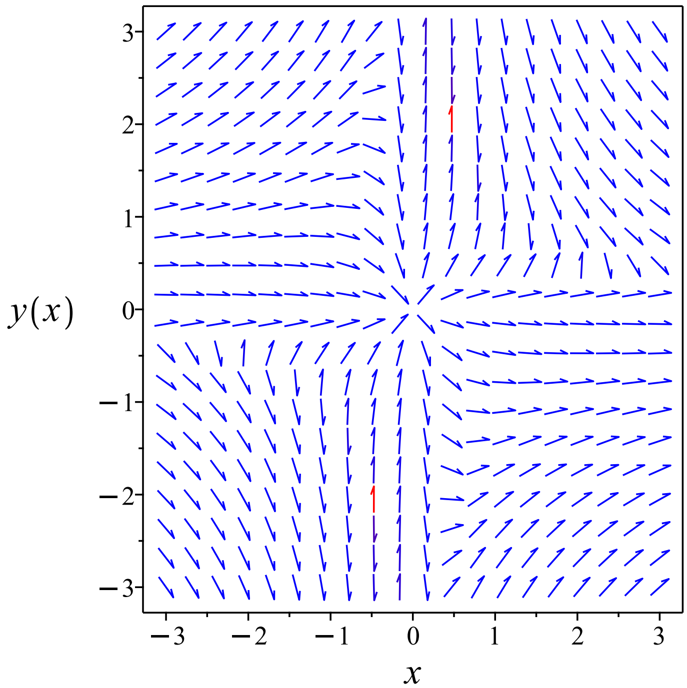


Figure 48: Slope field plot

Verification of solutions

$$y = -\frac{\text{LambertW}\left(-e^{-c_1} x^2\right)}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 19

```
dsolve((x*y(x)^2+y(x))+(x^2*y(x)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\operatorname{LambertW}\left(-x^2 \mathrm{e}^{-2c_1}\right)}{x}$$

✓ Solution by Mathematica

Time used: 13.386 (sec). Leaf size: 33

```
DSolve[(x*y[x]^2+y[x])+(x^2*y[x]-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{W\left(e^{-1+\frac{9c_1}{2^{2/3}}}x^2\right)}{x}$$
$$y(x) \to 0$$

2.6 problem 29

2.6.1 Solving as homogeneousTypeD2 ode 195
2.6.2 Solving as first order ode lie symmetry calculated ode 197
2.6.3 Solving as exact ode 204

Internal problem ID [5241]
Internal file name [`OUTPUT/4732_Friday_February_02_2024_05_10_45_AM_26690807/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{x \sin\left(\frac{y}{x}\right)-y \cos\left(\frac{y}{x}\right)+x \cos\left(\frac{y}{x}\right) y'=0}$$

2.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x \sin\left(u(x)\right)-u(x)\,x \cos\left(u(x)\right)+x \cos\left(u(x)\right)\left(u'(x)\,x+u(x)\right)=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{\tan\left(u\right)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\frac{1}{\tan(u)}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\tan(u)}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(\sin(u)) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x \arcsin\left(\frac{c_3 \mathrm{e}^{c_2}}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x \arcsin\left(\frac{c_3 \mathrm{e}^{c_2}}{x}\right) \tag{1}$$

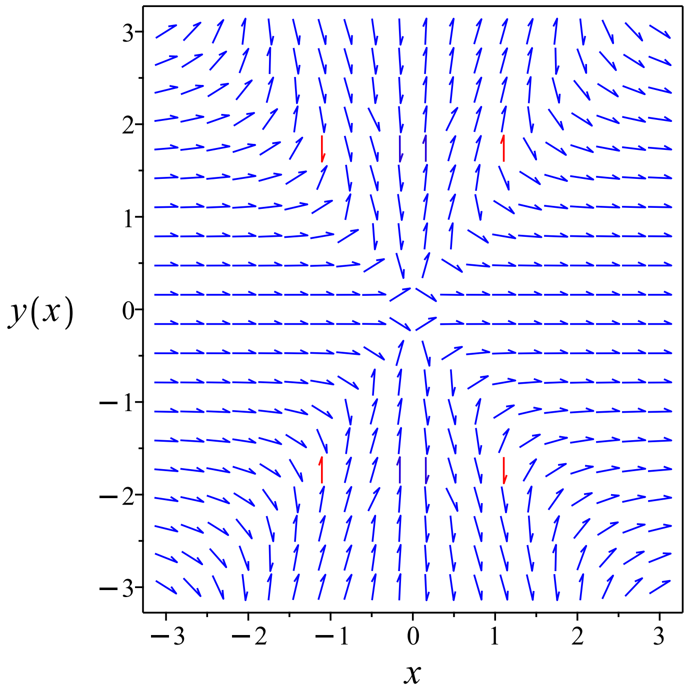


Figure 49: Slope field plot

Verification of solutions

$$y = x \arcsin\left(\frac{c_3 e^{c_2}}{x}\right)$$

Verified OK.

2.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right) x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right)(b_3 - a_2)}{\cos\left(\frac{y}{x}\right) x} - \frac{\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right)^2 a_3}{\cos\left(\frac{y}{x}\right)^2 x^2} \\
&- \left(-\frac{\sin\left(\frac{y}{x}\right) - \frac{y \cos\left(\frac{y}{x}\right)}{x} - \frac{y^2 \sin\left(\frac{y}{x}\right)}{x^2}}{\cos\left(\frac{y}{x}\right) x} + \frac{\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) y \sin\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)^2 x^3}\right. \\
&\left. + \frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right) x^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{y \sin\left(\frac{y}{x}\right)}{x^2 \cos\left(\frac{y}{x}\right)} - \frac{\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) \sin\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right)^2 x^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\sin\left(\frac{y}{x}\right)^2 x^2 a_3 - \sin\left(\frac{y}{x}\right)^2 x^2 b_2 + \sin\left(\frac{y}{x}\right)^2 x y a_2 - \sin\left(\frac{y}{x}\right)^2 x y b_3 + \sin\left(\frac{y}{x}\right)^2 y^2 a_3 - \sin\left(\frac{y}{x}\right) \cos\left(\frac{y}{x}\right) x^2 a_2 + \sin}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\sin\left(\frac{y}{x}\right)^2 x^2 a_3 + \sin\left(\frac{y}{x}\right)^2 x^2 b_2 - \sin\left(\frac{y}{x}\right)^2 x y a_2 + \sin\left(\frac{y}{x}\right)^2 x y b_3 \\
&- \sin\left(\frac{y}{x}\right)^2 y^2 a_3 + \sin\left(\frac{y}{x}\right) \cos\left(\frac{y}{x}\right) x^2 a_2 - \sin\left(\frac{y}{x}\right) \cos\left(\frac{y}{x}\right) x^2 b_3 \\
&+ 2 \sin\left(\frac{y}{x}\right) \cos\left(\frac{y}{x}\right) x y a_3 + b_2 \cos\left(\frac{y}{x}\right)^2 x^2 - \cos\left(\frac{y}{x}\right)^2 x y a_2 \\
&+ \cos\left(\frac{y}{x}\right)^2 x y b_3 - \cos\left(\frac{y}{x}\right)^2 y^2 a_3 + \sin\left(\frac{y}{x}\right)^2 x b_1 - \sin\left(\frac{y}{x}\right)^2 y a_1 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\frac{x\left(x^2 a_2 \sin\left(\frac{2y}{x}\right) - x^2 b_3 \sin\left(\frac{2y}{x}\right) + 2 x y a_3 \sin\left(\frac{2y}{x}\right) + x^2 a_3 \cos\left(\frac{2y}{x}\right) - x b_1 \cos\left(\frac{2y}{x}\right) + y a_1 \cos\left(\frac{2y}{x}\right) - x^2 a_3 + 2x\right.}{2} \tag{6E}$$

$$= 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \cos\left(\frac{2y}{x}\right), \sin\left(\frac{2y}{x}\right)\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \cos\left(\frac{2y}{x}\right) = v_3, \sin\left(\frac{2y}{x}\right) = v_4\right\}$$

The above PDE (6E) now becomes

$$\frac{v_1(v_1^2 a_2 v_4 + v_1^2 a_3 v_3 + 2v_1 v_2 a_3 v_4 - v_1^2 b_3 v_4 + v_2 a_1 v_3 - 2v_1 v_2 a_2 - v_1^2 a_3 - 2v_2^2 a_3 - v_1 b_1 v_3 + 2v_1^2 b_2 + 2v_1 v_2 b_3 -}{2} \tag{7E}$$
$$= 0$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&\left(-\frac{a_3}{2} + b_2\right) v_1^3 + \frac{a_3 v_3 v_1^3}{2} + \left(\frac{a_2}{2} - \frac{b_3}{2}\right) v_4 v_1^3 + \frac{b_1 v_1^2}{2} + (b_3 - a_2)\, v_2 v_1^2 \\
&- \frac{b_1 v_3 v_1^2}{2} + a_3 v_4 v_2 v_1^2 - \frac{a_1 v_2 v_1}{2} - a_3 v_2^2 v_1 + \frac{a_1 v_2 v_3 v_1}{2} = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0 \\
-\frac{a_1}{2} &= 0 \\
\frac{a_1}{2} &= 0 \\
-a_3 &= 0 \\
\frac{a_3}{2} &= 0 \\
-\frac{b_1}{2} &= 0 \\
\frac{b_1}{2} &= 0 \\
\frac{a_2}{2} - \frac{b_3}{2} &= 0 \\
-\frac{a_3}{2} + b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right) x}\right)(x) \\
&= \frac{\sin\left(\frac{y}{x}\right) x}{\cos\left(\frac{y}{x}\right)} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sin\left(\frac{y}{x}\right) x}{\cos\left(\frac{y}{x}\right)}} dy \end{aligned}$$

Which results in

$$S = \ln\left(\sin\left(\frac{y}{x}\right)\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right) x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y \cot\left(\frac{y}{x}\right)}{x^2} \\ S_y &= \frac{\cot\left(\frac{y}{x}\right)}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

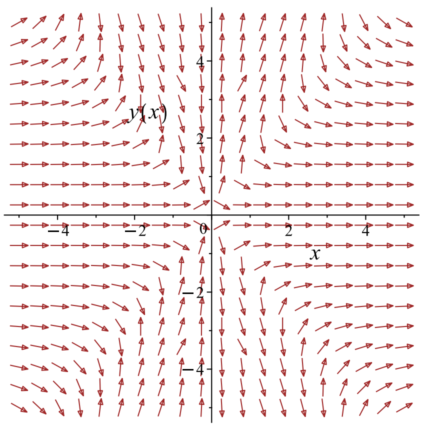
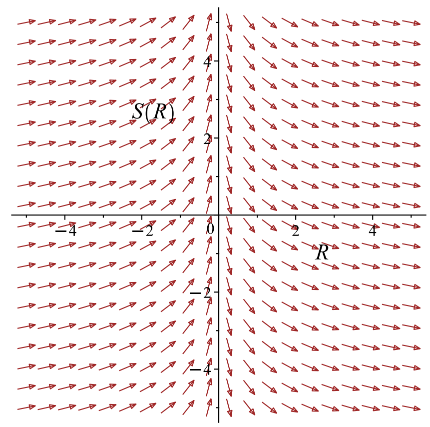
Which simplifies to

$$\ln\left(\sin\left(\frac{y}{x}\right)\right) = -\ln(x) + c_1$$

Which gives

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)}{\cos\left(\frac{y}{x}\right) x}$ 	$R = x$ $S = \ln\left(\sin\left(\frac{y}{x}\right)\right)$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right) \tag{1}$$

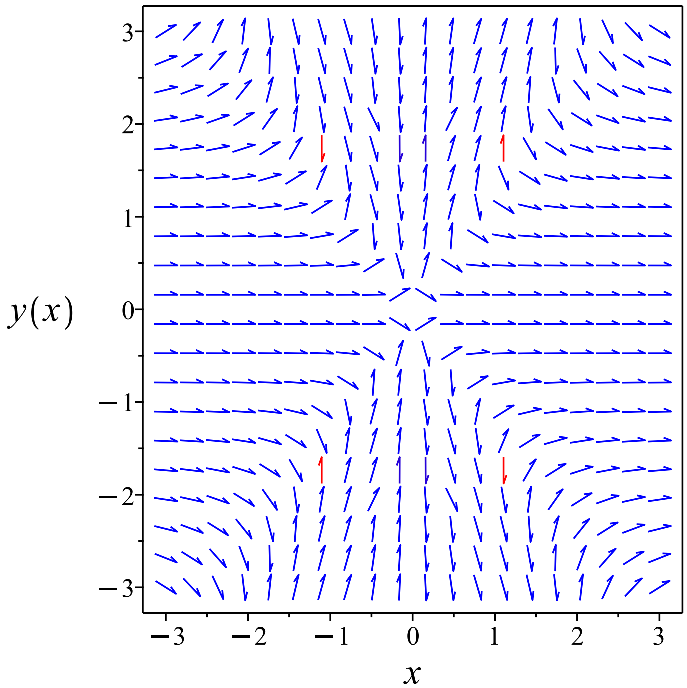


Figure 50: Slope field plot

Verification of solutions

$$y = x \arcsin\left(\frac{e^{c_1}}{x}\right)$$

Verified OK.

2.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\cos\left(\frac{y}{x}\right) x\right) \mathrm{d}y &= \left(-x \sin\left(\frac{y}{x}\right) + y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x \\
\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) \mathrm{d}x + \left(\cos\left(\frac{y}{x}\right) x\right) \mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right) \\
N(x,y) &= \cos\left(\frac{y}{x}\right) x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x \sin\left(\frac{y}{x}\right) - y \cos\left(\frac{y}{x}\right)\right) \\
&= \frac{y \sin\left(\frac{y}{x}\right)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\cos\left(\frac{y}{x}\right)x\right)\\
&= \frac{y\sin\left(\frac{y}{x}\right)}{x} + \cos\left(\frac{y}{x}\right)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= \frac{\sec\left(\frac{y}{x}\right)}{x}\left(\left(\frac{y\sin\left(\frac{y}{x}\right)}{x}\right) - \left(\frac{y\sin\left(\frac{y}{x}\right)}{x} + \cos\left(\frac{y}{x}\right)\right)\right)\\
&= -\frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x}\\
&= e^{\int -\frac{1}{x}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x)}\\
&= \frac{1}{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M\\
&= \frac{1}{x}\left(x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)\right)\\
&= \frac{x\sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N\\
&= \frac{1}{x}\left(\cos\left(\frac{y}{x}\right)x\right)\\
&= \cos\left(\frac{y}{x}\right)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)}{x}\right)+\left(\cos\left(\frac{y}{x}\right)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{x\sin\left(\frac{y}{x}\right)-y\cos\left(\frac{y}{x}\right)}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=x\sin\left(\frac{y}{x}\right)+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=\cos\left(\frac{y}{x}\right)+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\cos\left(\frac{y}{x}\right)$. Therefore equation (4) becomes

$$
\cos\left(\frac{y}{x}\right)=\cos\left(\frac{y}{x}\right)+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y)=0
$$

Therefore

$$
f(y)=c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x \sin\left(\frac{y}{x}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x \sin\left(\frac{y}{x}\right)$$

Summary

The solution(s) found are the following

$$x \sin\left(\frac{y}{x}\right) = c_1 \tag{1}$$

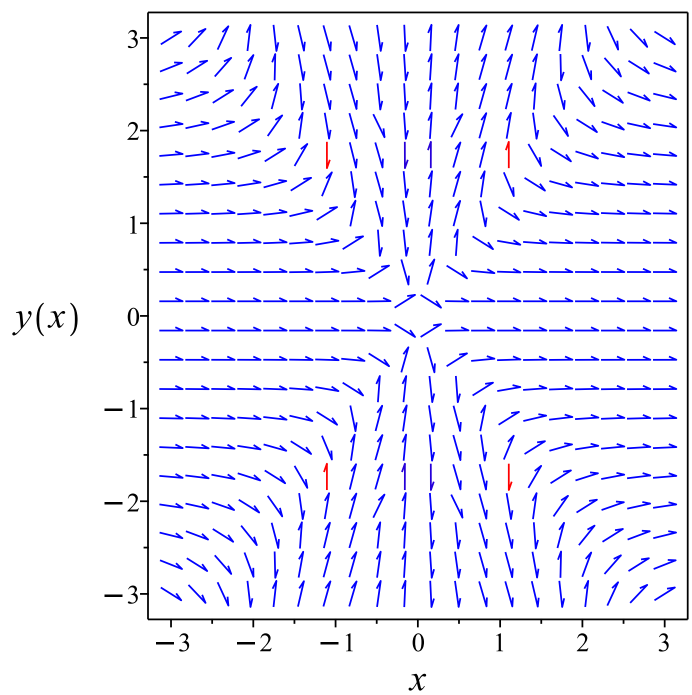


Figure 51: Slope field plot

Verification of solutions

$$x \sin\left(\frac{y}{x}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve((x*sin(y(x)/x)-y(x)*cos(y(x)/x))+(x*cos(y(x)/x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x \arcsin\left(\frac{1}{c_1 x}\right)$$

✓ Solution by Mathematica

Time used: 12.962 (sec). Leaf size: 21

```
DSolve[(x*Sin[y[x]/x]-y[x]*Cos[y[x]/x])+(x*Cos[y[x]/x])*y'[x]==0,y[x],x,IncludeSingularSolut
```

$$y(x) \to x \arcsin\left(\frac{e^{c_1}}{x}\right)$$
$$y(x) \to 0$$

2.7 problem 30

2.7.1 Solving as homogeneousTypeD2 ode 210

Internal problem ID [5242]
Internal file name [`OUTPUT/4733_Friday_February_02_2024_05_10_46_AM_84485799/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational]
```

$$\boxed{y^2(x^2+2)+(x^3+y^3)(-xy'+y)=0}$$

2.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2(x^2+2)+(x^3+u(x)^3\,x^3)(-x(u'(x)\,x+u(x))+u(x)\,x)=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2(x^2+2)}{(u^3+1)\,x^3} \end{aligned}$$

Where $f(x)=\frac{x^2+2}{x^3}$ and $g(u)=\frac{u^2}{u^3+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u^3+1}}\,du=\frac{x^2+2}{x^3}\,dx$$

$$\int \frac{1}{\frac{u^2}{u^3+1}}\, du = \int \frac{x^2+2}{x^3}\, dx$$
$$\frac{u^2}{2} - \frac{1}{u} = -\frac{1}{x^2} + \ln(x) + c_2$$

The solution is

$$\frac{u(x)^2}{2} - \frac{1}{u(x)} + \frac{1}{x^2} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{2x^2} - \frac{x}{y} + \frac{1}{x^2} - \ln(x) - c_2 = 0$$
$$\frac{y^2}{2x^2} - \frac{x}{y} + \frac{1}{x^2} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^2} - \frac{x}{y} + \frac{1}{x^2} - \ln(x) - c_2 = 0 \tag{1}$$

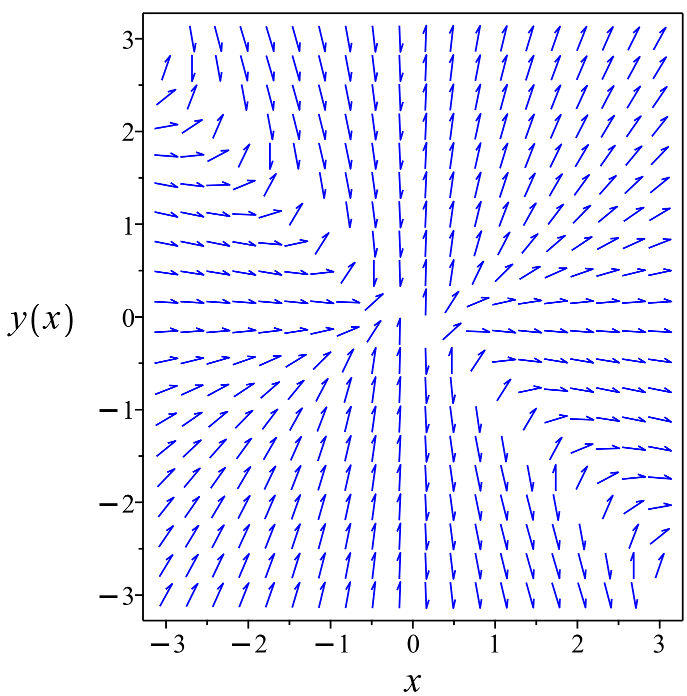


Figure 52: Slope field plot

Verification of solutions

$$\frac{y^2}{2x^2} - \frac{x}{y} + \frac{1}{x^2} - \ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 659

```
dsolve(y(x)^2*(x^2+2)+(x^3+y(x)^3)*(y(x)-x*diff(y(x),x))=0,y(x), singsol=all)
```

$$y(x) = \frac{6\ln(x)\,x^2 + 6c_1x^2 + \left(27x^3 + 3\sqrt{-24c_1^3x^6 - 72c_1^2x^6\ln(x) + 72c_1^2x^4 - 72c_1x^6\ln(x)^2 + 144c_1x^4\ln(x) - }\right.}{3\left(27x^3 + 3\sqrt{-24c_1^3x^6 - 72c_1^2x^6\ln(x) + 72c_1^2x^4 - 72c_1x^6\ln(x)^2 + 144c_1x^4\ln(x) - 72c_1x^2 }\right.}$$

$$y(x) = \frac{\frac{\left(-1-i\sqrt{3}\right)\left(27x^3+3\sqrt{-24c_1^3x^6-72c_1^2x^6\ln(x)+72c_1^2x^4-72c_1x^6\ln(x)^2+144c_1x^4\ln(x)-72c_1x^2-24\ln(x)^3x^6+72\ln(x)^2x^4-72\ln(x)x^2+24}\right.}{6}}{\left(27x^3 + 3\sqrt{-24c_1^3x^6 - 72c_1^2x^6\ln(x) + 72c_1^2x^4 - 72c_1x^6\ln(x)^2 + 144c_1x^4\ln(x) - 72c_1x^2 - 24\ln(x)^3x^6}\right.}$$

$$y(x) = \frac{\frac{\left(i\sqrt{3}-1\right)\left(27x^3+3\sqrt{-24c_1^3x^6-72c_1^2x^6\ln(x)+72c_1^2x^4-72c_1x^6\ln(x)^2+144c_1x^4\ln(x)-72c_1x^2-24\ln(x)^3x^6+72\ln(x)^2x^4-72\ln(x)x^2+24+8}\right.}{6}}{\left(27x^3 + 3\sqrt{-24c_1^3x^6 - 72c_1^2x^6\ln(x) + 72c_1^2x^4 - 72c_1x^6\ln(x)^2 + 144c_1x^4\ln(x) - 72c_1x^2 - 24\ln(x)^3x^6}\right.}$$

✓ Solution by Mathematica

Time used: 54.35 (sec). Leaf size: 396

```
DSolve[y[x]^2*(x^2+2)+(x^3+y[x]^3)*(y[x]-x*y'[x])==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{6x^2\log(x)+6c_1x^2+3^{2/3}\left(9x^3+\frac{1}{3}\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}\right)^{2/3}-6}{3\sqrt[3]{3}\sqrt[3]{9x^3+\frac{1}{3}\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}}}$$

$$y(x) \to \frac{i(\sqrt{3}+i)\sqrt[3]{9x^3+\frac{1}{3}\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}}}{2\ 3^{2/3}} - \frac{i\sqrt[3]{2}(\sqrt{3}-i)(x^2\log(x)+c_1x^2-1)}{\sqrt[3]{54x^3+2\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}}}$$

$$y(x) \to \frac{i\sqrt[3]{2}(\sqrt{3}+i)(x^2\log(x)+c_1x^2-1)}{\sqrt[3]{54x^3+2\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}}} - \frac{(1+i\sqrt{3})\sqrt[3]{54x^3+2\sqrt{729x^6+(-6x^2\log(x)-6c_1x^2+6)^3}}}{6\sqrt[3]{2}}$$

2.8 problem 31

2.8.1 Solving as first order ode lie symmetry calculated ode 214
2.8.2 Solving as exact ode 220

Internal problem ID [5243]
Internal file name [`OUTPUT/4734_Friday_February_02_2024_05_10_48_AM_20890870/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _dAlembert]
```

$$\boxed{\sqrt{x^2+y^2}\,y - x\left(x+\sqrt{x^2+y^2}\right)y' = 0}$$

2.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{\sqrt{x^2+y^2}\,y}{x\left(\sqrt{x^2+y^2}+x\right)} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\sqrt{x^2+y^2}\, y(b_3 - a_2)}{x\left(\sqrt{x^2+y^2}+x\right)} - \frac{(x^2+y^2)\, y^2 a_3}{x^2\left(\sqrt{x^2+y^2}+x\right)^2} \\
&- \left(\frac{y}{\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right)} - \frac{\sqrt{x^2+y^2}\, y}{x^2\left(\sqrt{x^2+y^2}+x\right)}\right. \\
&\left.- \frac{\sqrt{x^2+y^2}\, y\left(\frac{x}{\sqrt{x^2+y^2}}+1\right)}{x\left(\sqrt{x^2+y^2}+x\right)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(\frac{y^2}{\sqrt{x^2+y^2}\, x\left(\sqrt{x^2+y^2}+x\right)} + \frac{\sqrt{x^2+y^2}}{\left(\sqrt{x^2+y^2}+x\right) x}\right. \\
&\left.- \frac{y^2}{x\left(\sqrt{x^2+y^2}+x\right)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-(x^2+y^2)^{\frac{3}{2}}\, y^2 a_3 - \sqrt{x^2+y^2}\, x^4 b_2 + y^2 a_3 \sqrt{x^2+y^2}\, x^2 + y^4 a_3 \sqrt{x^2+y^2} - x^5 b_2 - x^3 y^2 a_3 - x^2 y^3 a_2 + x^2 y^3}{\sqrt{x^2+y^2}\,\left(\sqrt{x^2+y^2}+x\right.}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&\left(x^2+y^2\right)^{\frac{3}{2}} y^2 a_3 + \sqrt{x^2+y^2}\, x^4 b_2 - y^2 a_3 \sqrt{x^2+y^2}\, x^2 - y^4 a_3 \sqrt{x^2+y^2} \\
&+ x^5 b_2 + x^3 y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 + 2x\, y^4 a_3 - \left(x^2+y^2\right)^{\frac{3}{2}} x b_1 \\
&+ \left(x^2+y^2\right)^{\frac{3}{2}} y a_1 - x^4 b_1 + x^3 y a_1 - 2x^2 y^2 b_1 + 2x\, y^3 a_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&\left(x^2+y^2\right)^{\frac{3}{2}} y^2 a_3 + \left(x^2+y^2\right) x^3 b_2 + \left(x^2+y^2\right) x^2 y a_2 + 2\left(x^2+y^2\right) x\, y^2 a_3 \\
&+ \sqrt{x^2+y^2}\, x^4 b_2 - y^2 a_3 \sqrt{x^2+y^2}\, x^2 - y^4 a_3 \sqrt{x^2+y^2} - x^4 y a_2 \\
&- x^3 y^2 a_3 - x^3 y^2 b_2 - x^2 y^3 b_3 - \left(x^2+y^2\right)^{\frac{3}{2}} x b_1 + \left(x^2+y^2\right)^{\frac{3}{2}} y a_1 \\
&- \left(x^2+y^2\right) x^2 b_1 + 2\left(x^2+y^2\right) x y a_1 - x^3 y a_1 - x^2 y^2 b_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$x^5 b_2 + \sqrt{x^2+y^2}\, x^4 b_2 + x^3 y^2 a_3 + x^2 y^3 a_2 - x^2 y^3 b_3 + 2x\, y^4 a_3 - x^4 b_1 - \sqrt{x^2+y^2}\, x^3 b_1 + x^3 y a_1 + \sqrt{x^2+y^2}\, y a_1 x^2 - 2x^2 y^2 b_1 - \sqrt{x^2+y^2}\, x b_1 y^2 + 2x\, y^3 a_1 + \sqrt{x^2+y^2}\, y^3 a_1 = 0$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$v_1^2 v_2^3 a_2 + v_1^3 v_2^2 a_3 + 2v_1 v_2^4 a_3 + v_1^5 b_2 + v_3 v_1^4 b_2 - v_1^2 v_2^3 b_3 + v_1^3 v_2 a_1 + v_3 v_2 a_1 v_1^2 + 2v_1 v_2^3 a_1 + v_3 v_2^3 a_1 - v_1^4 b_1 - v_3 v_1^3 b_1 - 2v_1^2 v_2^2 b_1 - v_3 v_1 b_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$v_1^5 b_2 + v_3 v_1^4 b_2 - v_1^4 b_1 + v_1^3 v_2^2 a_3 + v_1^3 v_2 a_1 - v_3 v_1^3 b_1 + (-b_3 + a_2)\, v_1^2 v_2^3 - 2v_1^2 v_2^2 b_1 + v_3 v_2 a_1 v_1^2 + 2v_1 v_2^4 a_3 + 2v_1 v_2^3 a_1 - v_3 v_1 b_1 v_2^2 + v_3 v_2^3 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_2 &= 0 \\
2a_1 &= 0 \\
2a_3 &= 0 \\
-2b_1 &= 0 \\
-b_1 &= 0 \\
-b_3 + a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{\sqrt{x^2+y^2}\, y}{x\left(\sqrt{x^2+y^2}+x\right)}\right)(x) \\
&= \frac{xy}{\sqrt{x^2+y^2}+x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{xy}{\sqrt{x^2+y^2}+x}} dy
\end{aligned}
$$

Which results in

$$S=\frac{\sqrt{x^{2}+y^{2}}}{x}-\frac{x\ln\left(\frac{2x^{2}+2\sqrt{x^{2}}\,\sqrt{x^{2}+y^{2}}}{y}\right)}{\sqrt{x^{2}}}+\ln\left(y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{\sqrt{x^{2}+y^{2}}\,y}{x\left(\sqrt{x^{2}+y^{2}}+x\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{-\sqrt{x^{2}+y^{2}}-x}{x^{2}}\\
S_y &= \frac{2x\sqrt{x^{2}+y^{2}}+2x^{2}+y^{2}}{y\left(\sqrt{x^{2}+y^{2}}+x\right)x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{-2x\sqrt{x^{2}+y^{2}}-2x^{2}-y^{2}}{x\left(\sqrt{x^{2}+y^{2}}+x\right)^{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=-\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R)=-\ln\left(R\right)+c_1 \tag{4}$$

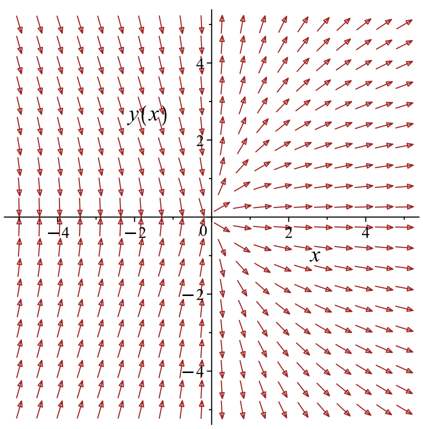
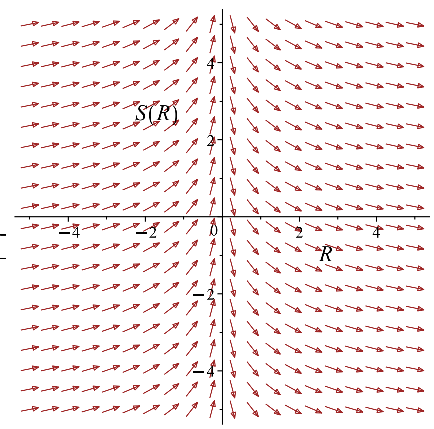
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-\ln(2)\,x - x\ln(x) - \ln\left(x + \sqrt{x^2+y^2}\right)x + 2x\ln(y) + \sqrt{x^2+y^2}}{x} = -\ln(x) + c_1$$

Which simplifies to

$$\frac{-\ln(2)\,x - \ln\left(x + \sqrt{x^2+y^2}\right)x + 2x\ln(y) - c_1 x + \sqrt{x^2+y^2}}{x} = 0$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}\,y}{x\left(\sqrt{x^2+y^2}+x\right)}$ 	$R = x$ $S = \frac{-\ln(2)\,x - x\ln(x) - \cdots x + \sqrt{x^{\cdots}}}{\cdots}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$\frac{-\ln(2)\,x - \ln\left(x + \sqrt{x^2+y^2}\right)x + 2x\ln(y) - c_1 x + \sqrt{x^2+y^2}}{x} = 0 \tag{1}$$

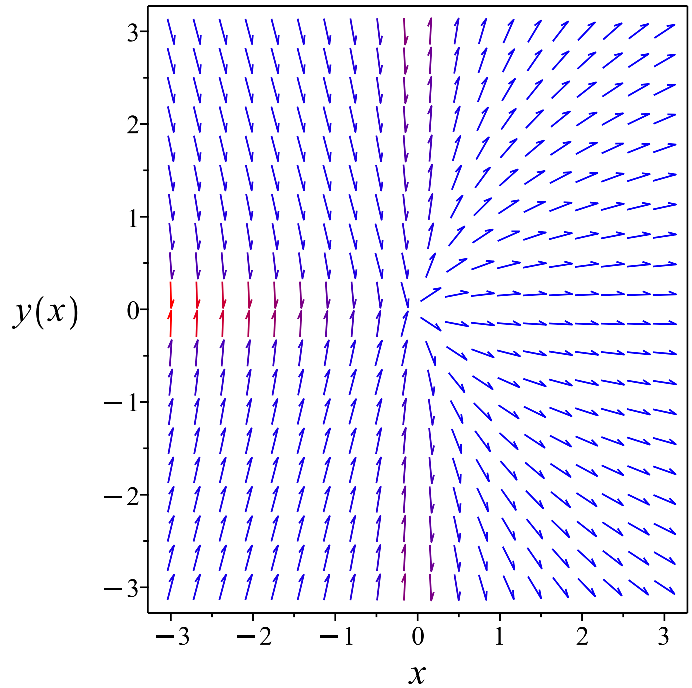


Figure 53: Slope field plot

Verification of solutions

$$\frac{-\ln(2)\, x-\ln\left(x+\sqrt{x^2+y^2}\right) x+2x\ln(y)-c_1x+\sqrt{x^2+y^2}}{x}=0$$

Verified OK.

2.8.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-x\left(\sqrt{x^2+y^2}+x\right)\right) \mathrm{d}y &= \left(-\sqrt{x^2+y^2}\, y\right) \mathrm{d}x \\
\left(\sqrt{x^2+y^2}\, y\right) \mathrm{d}x + \left(-x\left(\sqrt{x^2+y^2}+x\right)\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= \sqrt{x^2+y^2}\, y \\
N(x, y) &= -x\left(\sqrt{x^2+y^2}+x\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}\, y\right) \\
&= \frac{x^2+2y^2}{\sqrt{x^2+y^2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x\left(\sqrt{x^2+y^2}+x\right)\right) \\
&= -\frac{2\left(x^2+x\sqrt{x^2+y^2}+\frac{y^2}{2}\right)}{\sqrt{x^2+y^2}}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = \sqrt{x^2+y^2}\, y$ and $N = -x\left(x+\sqrt{x^2+y^2}\right)$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{\sqrt{x^2+y^2}}{x^2} \\
N &= -\frac{x+\sqrt{x^2+y^2}}{xy}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sqrt{x^2+y^2}+x}{xy}\right)\mathrm{d}y = \left(-\frac{\sqrt{x^2+y^2}}{x^2}\right)\mathrm{d}x$$

$$\left(\frac{\sqrt{x^2+y^2}}{x^2}\right)\mathrm{d}x + \left(-\frac{\sqrt{x^2+y^2}+x}{xy}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{\sqrt{x^2+y^2}}{x^2}$$
$$N(x,y) = -\frac{\sqrt{x^2+y^2}+x}{xy}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\sqrt{x^2+y^2}}{x^2}\right)\\ &= \frac{y}{\sqrt{x^2+y^2}\,x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sqrt{x^2+y^2}+x}{xy}\right)\\ &= \frac{y}{\sqrt{x^2+y^2}\,x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$
$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{\sqrt{x^2+y^2}}{x^2} \mathrm{d}x \\
\phi &= \frac{\ln\left(\sqrt{x^2+y^2}+x\right) x - \sqrt{x^2+y^2}}{x} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{\frac{yx}{\sqrt{x^2+y^2}\left(\sqrt{x^2+y^2}+x\right)} - \frac{y}{\sqrt{x^2+y^2}}}{x} + f'(y) \qquad (4) \\
&= -\frac{y}{x\left(\sqrt{x^2+y^2}+x\right)} + f'(y)
\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sqrt{x^2+y^2}+x}{xy}$. Therefore equation (4) becomes

$$-\frac{\sqrt{x^2+y^2}+x}{xy} = -\frac{y}{x\left(\sqrt{x^2+y^2}+x\right)} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(-\frac{2}{y}\right) \mathrm{d}y \\
f(y) &= -2\ln(y) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln\left(\sqrt{x^2+y^2}+x\right) x - \sqrt{x^2+y^2}}{x} - 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln\left(\sqrt{x^2+y^2}+x\right)x - \sqrt{x^2+y^2}}{x} - 2\ln\left(y\right)$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)x - \sqrt{x^2+y^2}}{x} - 2\ln\left(y\right) = c_1 \tag{1}$$

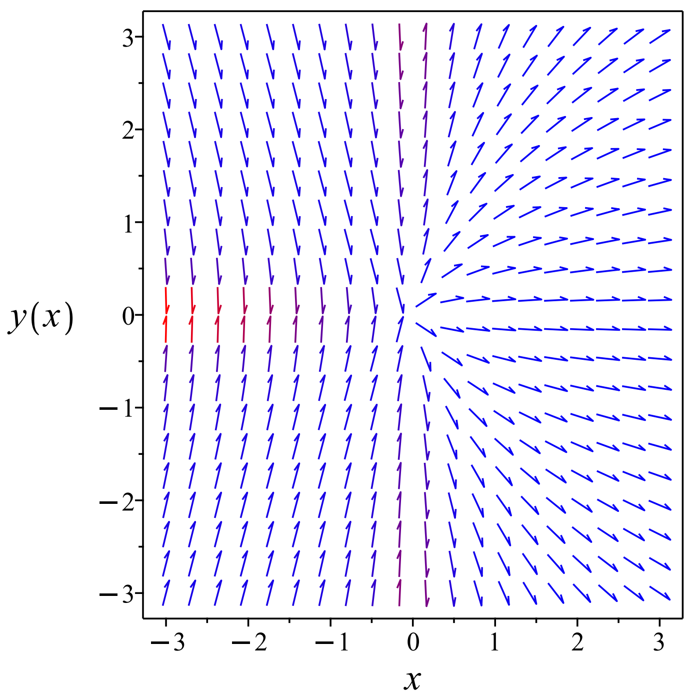


Figure 54: Slope field plot

Verification of solutions

$$\frac{\ln\left(x+\sqrt{x^2+y^2}\right)x - \sqrt{x^2+y^2}}{x} - 2\ln\left(y\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 60

```
dsolve(y(x)*sqrt(x^2+y(x)^2)-x*(x+sqrt(x^2+y(x)^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x\ln(2)+x\ln\left(\frac{x\left(x+\sqrt{x^2+y(x)^2}\right)}{y(x)}\right)-\ln(y(x))\,x-\ln(x)\,x-c_1x-\sqrt{x^2+y(x)^2}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.319 (sec). Leaf size: 43

```
DSolve[y[x]*Sqrt[x^2+y[x]^2]-x*(x+Sqrt[x^2+y[x]^2])*y'[x]==0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[\sqrt{\frac{y(x)^2}{x^2}+1}+\log\left(\sqrt{\frac{y(x)^2}{x^2}+1}-1\right)=-\log(x)+c_1,y(x)\right]$$

2.9 problem 32

2.9.1 Solving as first order ode lie symmetry calculated ode 227

Internal problem ID [5244]
Internal file name [`OUTPUT/4735_Friday_February_02_2024_05_10_50_AM_13746180/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (2x + 2y + 1)\, y' = -x - 1}$$

2.9.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x + y + 1}{2x + 2y + 1} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y+1)(b_3-a_2)}{2x+2y+1} - \frac{(x+y+1)^2 a_3}{(2x+2y+1)^2} \\ &- \left(-\frac{1}{2x+2y+1} + \frac{2x+2y+2}{(2x+2y+1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{1}{2x+2y+1} + \frac{2x+2y+2}{(2x+2y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 + 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 + 2xa_2}{(2x+2y+1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &2x^2a_2 - x^2a_3 + 4x^2b_2 - 2x^2b_3 + 4xya_2 - 2xya_3 + 8xyb_2 - 4xyb_3 \\ &+ 2y^2a_2 - y^2a_3 + 4y^2b_2 - 2y^2b_3 + 2xa_2 - 2xa_3 + 3xb_2 - 3xb_3 \\ &+ 3ya_2 - 3ya_3 + 4yb_2 - 4yb_3 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &2a_2v_1^2 + 4a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 4b_2v_1^2 + 8b_2v_1v_2 \\ &+ 4b_2v_2^2 - 2b_3v_1^2 - 4b_3v_1v_2 - 2b_3v_2^2 + 2a_2v_1 + 3a_2v_2 - 2a_3v_1 - 3a_3v_2 \\ &+ 3b_2v_1 + 4b_2v_2 - 3b_3v_1 - 4b_3v_2 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(2a_2 - a_3 + 4b_2 - 2b_3)\, v_1^2 + (4a_2 - 2a_3 + 8b_2 - 4b_3)\, v_1 v_2 \\ &+ (2a_2 - 2a_3 + 3b_2 - 3b_3)\, v_1 + (2a_2 - a_3 + 4b_2 - 2b_3)\, v_2^2 \\ &+ (3a_2 - 3a_3 + 4b_2 - 4b_3)\, v_2 - a_1 + a_2 - a_3 - b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}2a_2 - 2a_3 + 3b_2 - 3b_3 &= 0 \\ 2a_2 - a_3 + 4b_2 - 2b_3 &= 0 \\ 3a_2 - 3a_3 + 4b_2 - 4b_3 &= 0 \\ 4a_2 - 2a_3 + 8b_2 - 4b_3 &= 0 \\ -a_1 + a_2 - a_3 - b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_1 \\ a_2 &= -2b_3 \\ a_3 &= -2b_3 \\ b_1 &= b_1 \\ b_2 &= b_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -1 \\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= 1 - \left(-\frac{x + y + 1}{2x + 2y + 1}\right)(-1) \\ &= \frac{x + y}{2x + 2y + 1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x+y}{2x+2y+1}} dy \end{aligned}$$

Which results in

$$S = 2y + \ln(x + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y + 1}{2x + 2y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y} \\ S_y &= 2 + \frac{1}{x + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2y + \ln(x + y) = c_1 - x$$

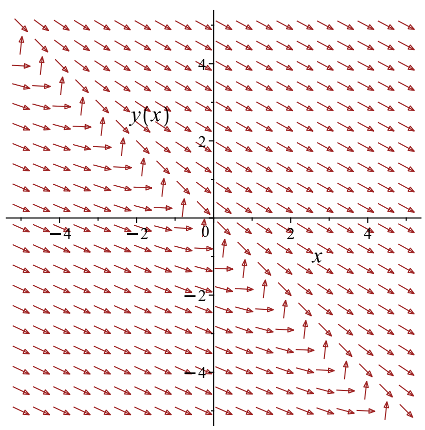
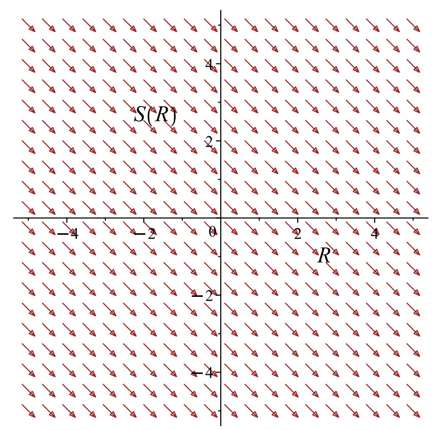
Which simplifies to

$$2y + \ln(x + y) = c_1 - x$$

Which gives

$$y = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x+c_1}\right)}{2} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+1}{2x+2y+1}$ 	$R = x$ $S = 2y + \ln(x + y)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{\mathrm{LambertW}\left(2\,\mathrm{e}^{x+c_1}\right)}{2} - x \tag{1}$$

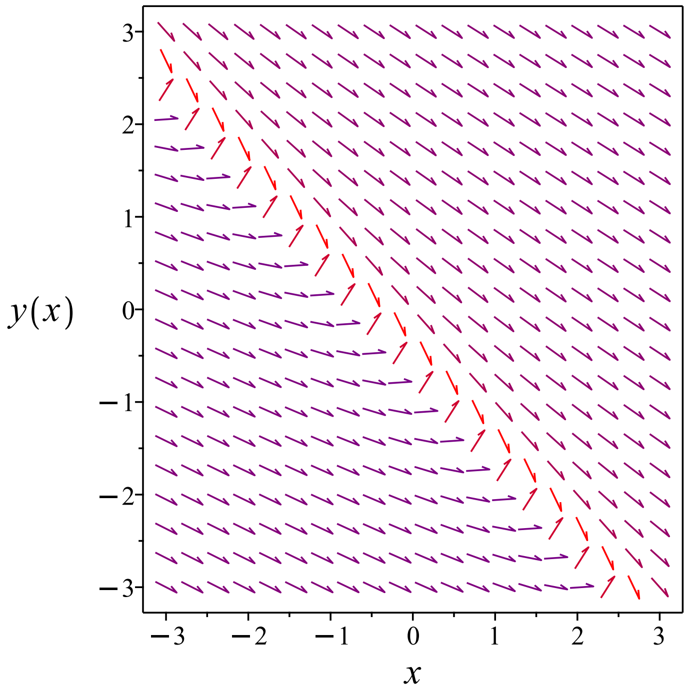


Figure 55: Slope field plot

Verification of solutions

$$y = \frac{\text{LambertW}\left(2\,\mathrm{e}^{x+c_1}\right)}{2} - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((x+y(x)+1)+(2*x+2*y(x)+1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\text{LambertW}\left(2\,\mathrm{e}^{x-c_1}\right)}{2} - x$$

✓ Solution by Mathematica

Time used: 4.251 (sec). Leaf size: 30

```
DSolve[(x+y[x]+1)+(2*x+2*y[x]+1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-2x + W\left(-e^{x-1+c_1}\right)\right)$$
$$y(x) \to -x$$

2.10 problem 34

2.10.1 Solving as separable ode 235
2.10.2 Solving as linear ode 237
2.10.3 Solving as homogeneousTypeMapleC ode 239
2.10.4 Solving as first order ode lie symmetry lookup ode 241
2.10.5 Solving as exact ode 245
2.10.6 Maple step by step solution 249

Internal problem ID [5245]
Internal file name [`OUTPUT/4736_Friday_February_02_2024_05_10_51_AM_38539046/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2y - y'(-x+4) = -1}$$

2.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-2y-1}{x-4} \end{aligned}$$

Where $f(x) = \frac{1}{x-4}$ and $g(y) = -2y - 1$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{-2y-1}\,dy &= \frac{1}{x-4}\,dx\\
\int \frac{1}{-2y-1}\,dy &= \int \frac{1}{x-4}\,dx\\
-\frac{\ln(-2y-1)}{2} &= \ln(x-4) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{\sqrt{-2y-1}} = \mathrm{e}^{\ln(x-4)+c_1}
$$

Which simplifies to

$$
\frac{1}{\sqrt{-2y-1}} = c_2(x-4)
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\left(\mathrm{e}^{2c_1} c_2^2 x^2 - 8\,\mathrm{e}^{2c_1} c_2^2 x + 16 c_2^2 \mathrm{e}^{2c_1} + 1\right)\mathrm{e}^{-2c_1}}{2c_2^2\,(x-4)^2} \tag{1}
$$

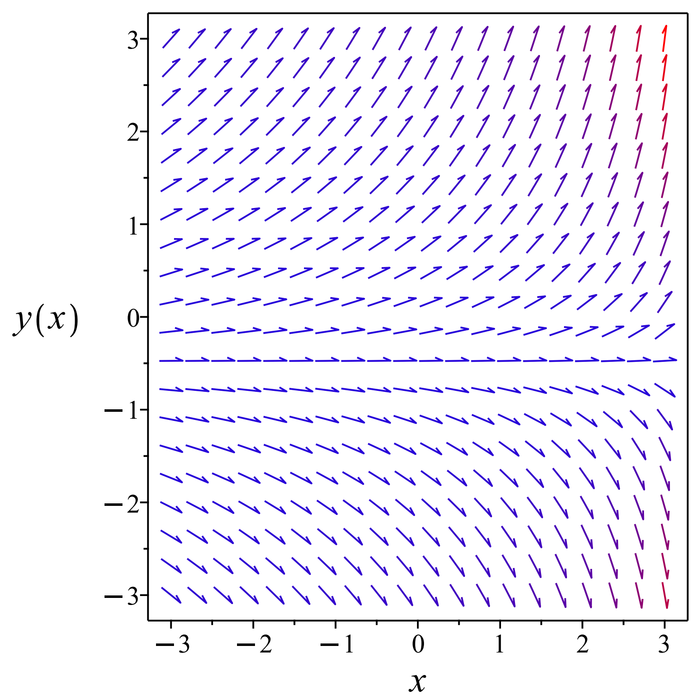


Figure 56: Slope field plot

Verification of solutions

$$y = -\frac{\left(\mathrm{e}^{2c_1} c_2^2 x^2 - 8\,\mathrm{e}^{2c_1} c_2^2 x + 16 c_2^2 \mathrm{e}^{2c_1} + 1\right) \mathrm{e}^{-2c_1}}{2c_2^2 \left(x-4\right)^2}$$

Verified OK.

2.10.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x-4} \\ q(x) &= -\frac{1}{x-4} \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x-4} = -\frac{1}{x-4}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{x-4}dx} \\ &= \left(x-4\right)^2 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{1}{x-4}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x-4\right)^2 y\right) &= \left(\left(x-4\right)^2\right)\left(-\frac{1}{x-4}\right) \\ \mathrm{d}\left(\left(x-4\right)^2 y\right) &= \left(-x+4\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \left(x-4\right)^2 y &= \int -x + 4\,\mathrm{d}x \\ \left(x-4\right)^2 y &= -\frac{1}{2}x^2 + 4x + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(x-4\right)^2$ results in

$$y = \frac{-\frac{1}{2}x^2 + 4x}{\left(x-4\right)^2} + \frac{c_1}{\left(x-4\right)^2}$$

which simplifies to

$$y = \frac{-x^2 + 2c_1 + 8x}{2(x-4)^2}$$

Summary
The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1 + 8x}{2(x-4)^2} \tag{1}$$

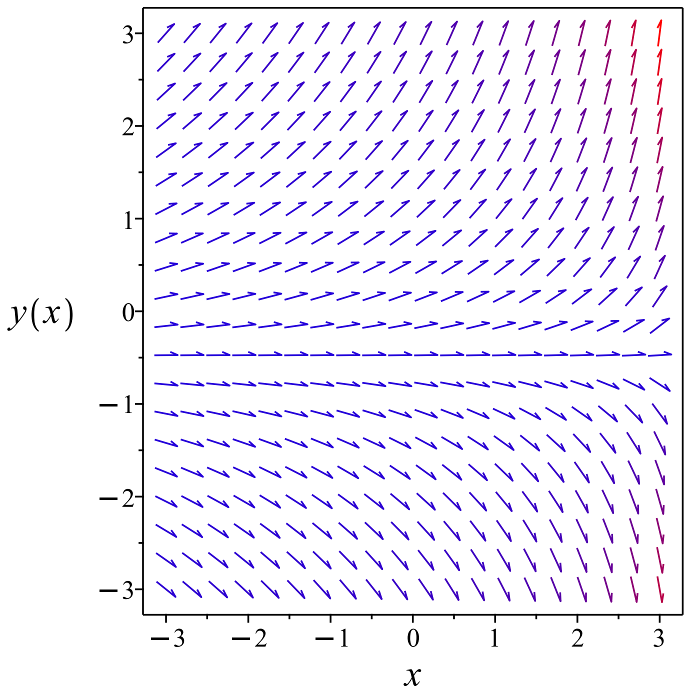


Figure 57: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1 + 8x}{2(x-4)^2}$$

Verified OK.

2.10.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{1 + 2Y(X) + 2y_0}{X + x_0 - 4}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 4 \\ y_0 &= -\frac{1}{2} \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= -\frac{3u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) + \frac{3u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X) = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= -\frac{3u}{X} \end{aligned}$$

Where $f(X) = -\frac{3}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{X}\,dX \\ \int \frac{1}{u}\,du &= \int -\frac{3}{X}\,dX \\ \ln(u) &= -3\ln(X) + c_2 \\ u &= \mathrm{e}^{-3\ln(X)+c_2} \\ &= \frac{c_2}{X^3} \end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{c_2}{X^2}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{c_2}{X^2}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y - \frac{1}{2} \\ X &= x + 4 \end{aligned}$$

Then the solution in y becomes

$$\frac{1}{2} + y = \frac{c_2}{(x-4)^2}$$

Summary

The solution(s) found are the following

$$\frac{1}{2}+y=\frac{c_2}{(x-4)^2} \tag{1}$$

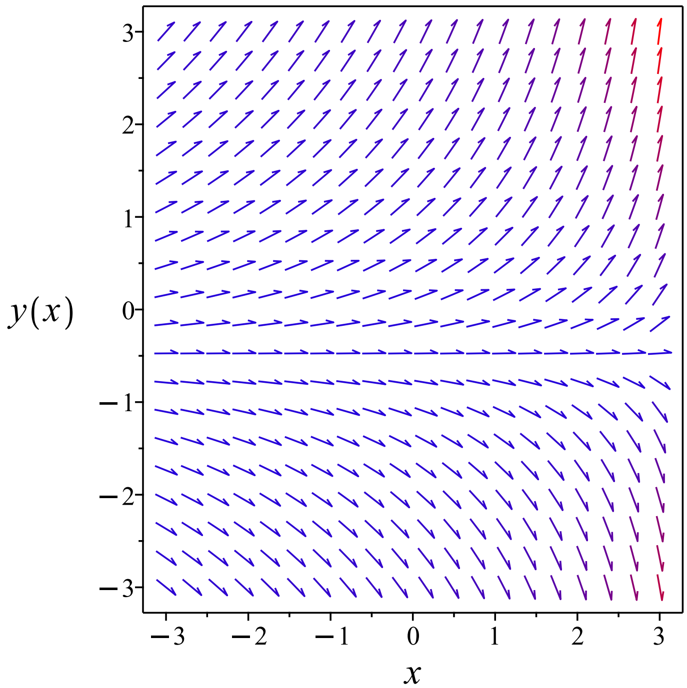


Figure 58: Slope field plot

Verification of solutions

$$\frac{1}{2}+y=\frac{c_2}{(x-4)^2}$$

Verified OK.

2.10.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{1+2y}{x-4}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 33: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{(x-4)^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x-4)^2}} dy \end{aligned}$$

Which results in

$$S = (x-4)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{1+2y}{x-4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2(x-4) y \\ S_y &= (x-4)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x + 4 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R + 4$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2}R^2 + 4R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-4)^2 y = -\frac{1}{2}x^2 + 4x + c_1$$

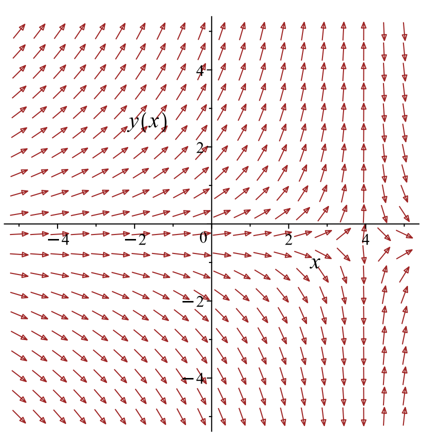
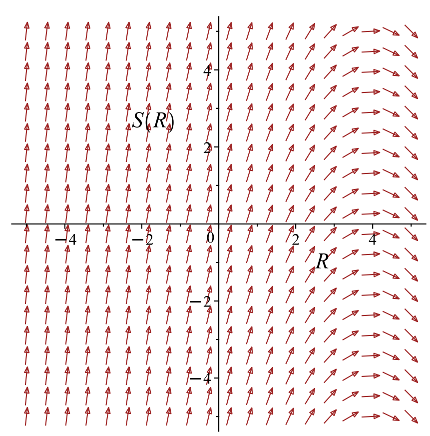
Which simplifies to

$$(x-4)^2 y = -\frac{1}{2}x^2 + 4x + c_1$$

Which gives

$$y = \frac{-x^2 + 2c_1 + 8x}{2(x-4)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{1+2y}{x-4}$ 	$R = x$ $S = (x-4)^2 y$	$\frac{dS}{dR} = -R + 4$ 

Summary
The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1 + 8x}{2 (x-4)^2} \tag{1}$$

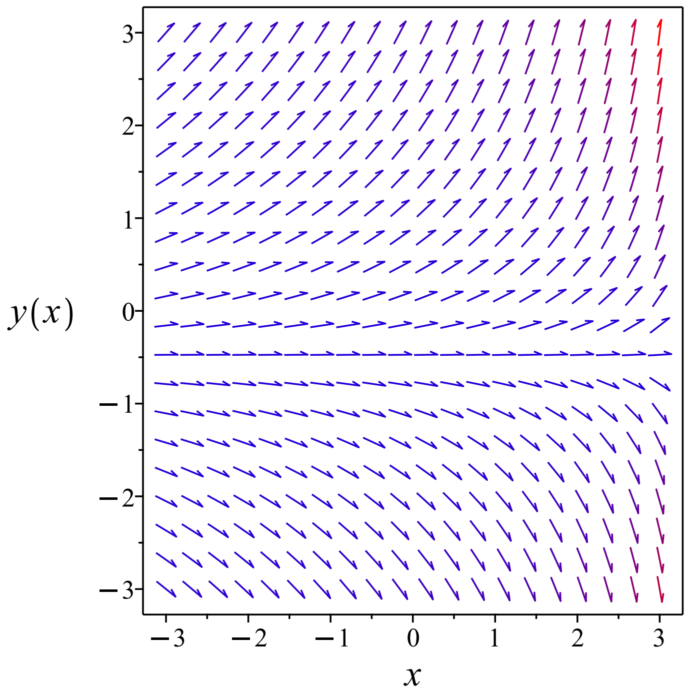


Figure 59: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1 + 8x}{2 (x-4)^2}$$

Verified OK.

2.10.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-2y-1}\right)\mathrm{d}y = \left(\frac{1}{x-4}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x-4}\right)\mathrm{d}x + \left(\frac{1}{-2y-1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x-4}$$
$$N(x,y) = \frac{1}{-2y-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x-4}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-2y-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{1}{x-4} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x-4) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-2y-1}$. Therefore equation (4) becomes

$$
\frac{1}{-2y-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{1+2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{1+2y}\right)\mathrm{d}y$$
$$f(y) = -\frac{\ln(1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x-4) - \frac{\ln(1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x-4) - \frac{\ln(1+2y)}{2}$$

The solution becomes

$$y = \frac{-x^2 + \mathrm{e}^{-2c_1} + 8x - 16}{2x^2 - 16x + 32}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + \mathrm{e}^{-2c_1} + 8x - 16}{2x^2 - 16x + 32} \tag{1}$$

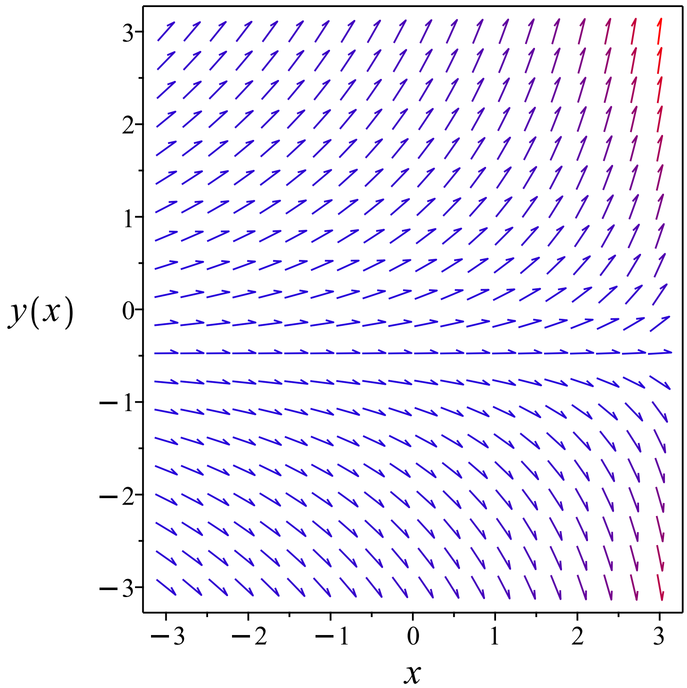


Figure 60: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x^2 + e^{-2c_1} + 8x - 16}{2x^2 - 16x + 32}$$

Verified OK.

2.10.6 Maple step by step solution

Let's solve

$2y - y'(-x+4) = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-2y-1} = -\frac{1}{-x+4}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-2y-1} dx = \int -\frac{1}{-x+4} dx + c_1$

- Evaluate integral

$$-\frac{\ln(-2y-1)}{2} = \ln(-x+4) + c_1$$

- Solve for y

$$y = -\frac{x^2 + e^{-2c_1} - 8x + 16}{2(x^2 - 8x + 16)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((1+2*y(x))-(4-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{4x - \frac{1}{2}x^2 + c_1}{(x-4)^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 34

```
DSolve[(1+2*y[x])-(4-x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-x^2 + 8x + 2c_1}{2(x-4)^2}$$
$$y(x) \to -\frac{1}{2}$$

2.11 problem 35

2.11.1 Solving as separable ode 251
2.11.2 Solving as linear ode 253
2.11.3 Solving as homogeneousTypeD2 ode 254
2.11.4 Solving as first order ode lie symmetry lookup ode 256
2.11.5 Solving as exact ode 260
2.11.6 Maple step by step solution 264

Internal problem ID [5246]
Internal file name [`OUTPUT/4737_Friday_February_02_2024_05_10_52_AM_63507069/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yx + \left(x^2+1\right) y' = 0}$$

2.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{xy}{x^2+1} \end{aligned}$$

Where $f(x) = -\frac{x}{x^2+1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -\frac{x}{x^2+1}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{x}{x^2+1}\,dx \\
\ln(y) &= -\frac{\ln(x^2+1)}{2} + c_1 \\
y &= \mathrm{e}^{-\frac{\ln(x^2+1)}{2}+c_1} \\
&= \frac{c_1}{\sqrt{x^2+1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

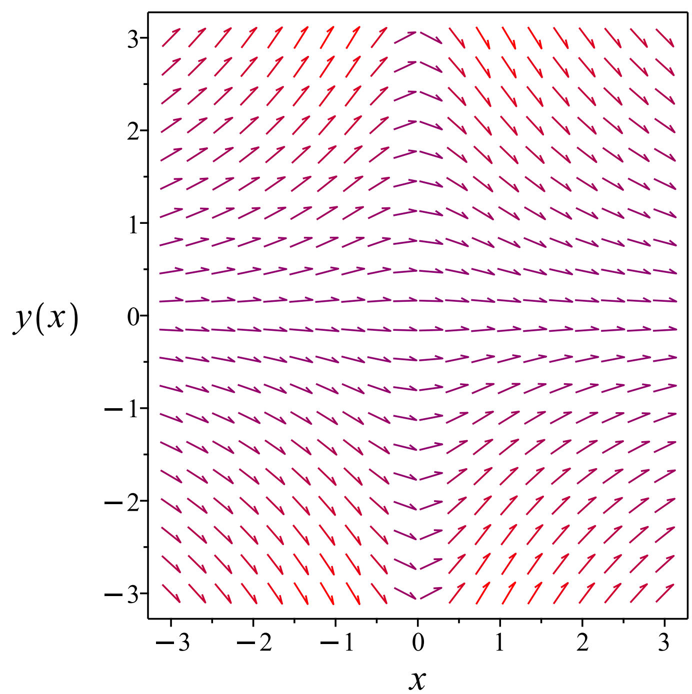


Figure 61: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

2.11.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{x}{x^2+1} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{xy}{x^2+1} = 0
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x}{x^2+1}dx} \\
&= \sqrt{x^2+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^2+1}\, y\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\sqrt{x^2+1}\, y = c_1
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$
y = \frac{c_1}{\sqrt{x^2+1}}
$$

Summary
The solution(s) found are the following

$$
y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}
$$

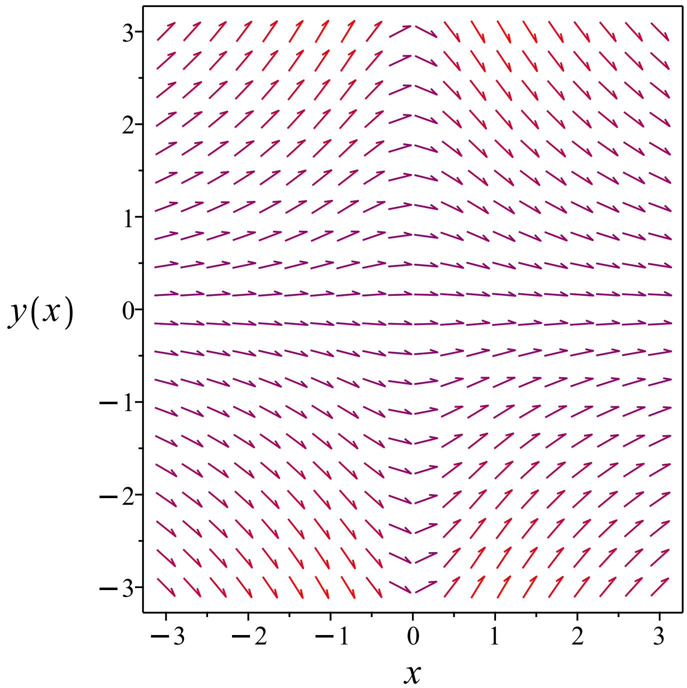


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

Verified OK.

2.11.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 + \left(x^2+1\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(2x^2+1)}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2x^2+1}{x\,(x^2+1)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2x^2+1}{x\,(x^2+1)}\,dx \\
\ln(u) &= -\frac{\ln(x^2+1)}{2} - \ln(x) + c_2 \\
u &= \mathrm{e}^{-\frac{\ln(x^2+1)}{2} - \ln(x) + c_2} \\
&= c_2 \mathrm{e}^{-\frac{\ln(x^2+1)}{2} - \ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2}{\sqrt{x^2+1}\,x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2}{\sqrt{x^2+1}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_2}{\sqrt{x^2+1}} \tag{1}
$$

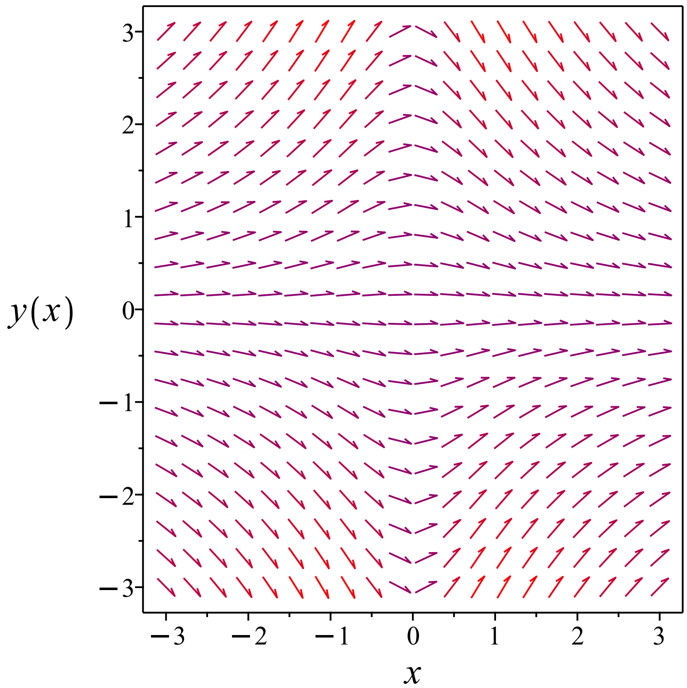


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{c_2}{\sqrt{x^2 + 1}}$$

Verified OK.

2.11.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{xy}{x^2 + 1} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 36: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x^2 + 1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2 + 1}} \\ S_y &= \sqrt{x^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1}\, y = c_1$$

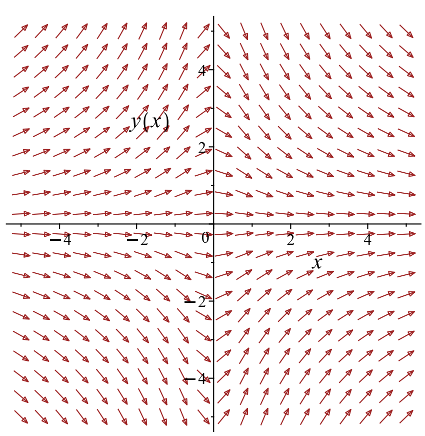
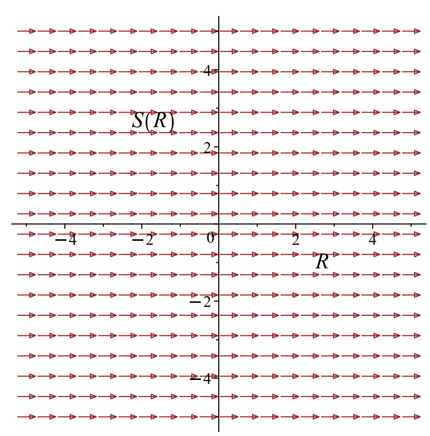
Which simplifies to

$$\sqrt{x^2+1}\, y = c_1$$

Which gives

$$y = \frac{c_1}{\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy}{x^2+1}$ 	$R = x$ $S = \sqrt{x^2+1}\, y$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1}{\sqrt{x^2+1}} \tag{1}$$

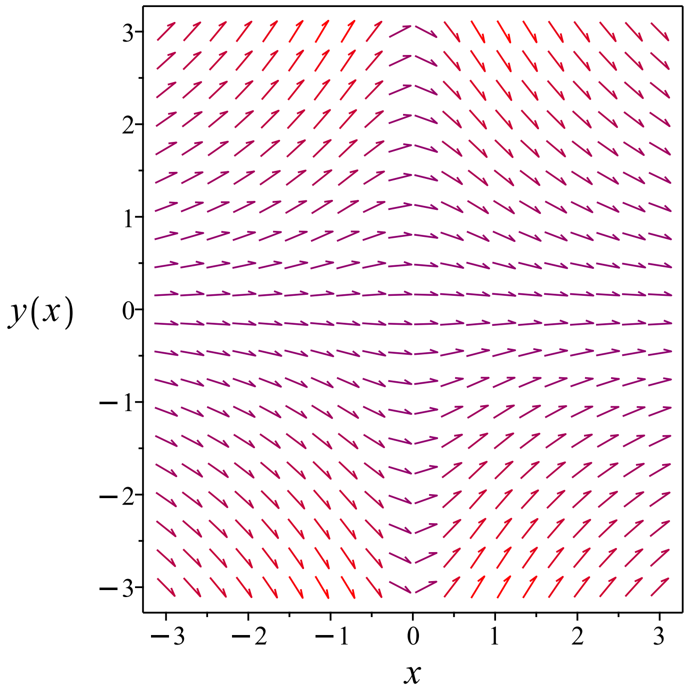


Figure 64: Slope field plot

Verification of solutions

$$y = \frac{c_1}{\sqrt{x^2 + 1}}$$

Verified OK.

2.11.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{x}{x^2+1}\right)\mathrm{d}x \\
\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(-\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2+1} \\
N(x,y) &= -\frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\frac{x}{x^2+1}\, \mathrm{d}x\\ \phi &= -\frac{\ln(x^2+1)}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y\\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}-c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\ln\left(x^2+1\right)}{2}-c_1} \tag{1}$$

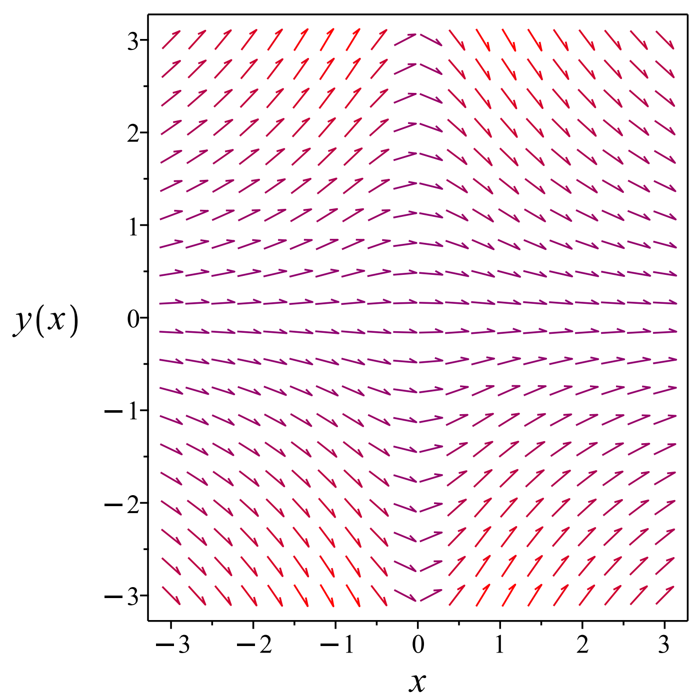


Figure 65: Slope field plot

Verification of solutions

$$y = e^{-\frac{\ln\left(x^2+1\right)}{2}-c_1}$$

Verified OK.

2.11.6 Maple step by step solution

Let's solve

$yx + (x^2+1)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{x}{x^2+1} dx + c_1$

- Evaluate integral

  $\ln(y) = -\frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = e^{-\frac{\ln\left(x^2+1\right)}{2}+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve((x*y(x))+(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{\sqrt{x^2+1}}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 22

```
DSolve[(x*y[x])+(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{\sqrt{x^2+1}}$$
$$y(x) \to 0$$

2.12 problem 37

Internal problem ID [5247]
Internal file name [`OUTPUT/4738_Friday_February_02_2024_05_10_53_AM_14454408/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[_rational, [_Abel, `2nd type`, `class A`]]
```

Unable to solve or complete the solution.

$$\boxed{2yx + (3y + 2x)\,y' = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve((x*2*y(x))+(2*x+3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[(x*2*y[x])+(2*x+3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

2.13 problem 38

2.13.1 Solving as first order ode lie symmetry calculated ode 269

Internal problem ID [5248]
Internal file name [`OUTPUT/4739_Friday_February_02_2024_05_10_53_AM_118797/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{2xy' - 2y - \sqrt{x^2 + 4y^2} = 0}$$

2.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{2y + \sqrt{x^2 + 4y^2}}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(2y+\sqrt{x^2+4y^2}\right)(b_3-a_2)}{2x} - \frac{\left(2y+\sqrt{x^2+4y^2}\right)^2 a_3}{4x^2} \\ & - \left(\frac{1}{2\sqrt{x^2+4y^2}} - \frac{2y+\sqrt{x^2+4y^2}}{2x^2}\right)(xa_2+ya_3+a_1) \\ & - \frac{\left(2+\frac{4y}{\sqrt{x^2+4y^2}}\right)(xb_2+yb_3+b_1)}{2x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(x^2+4y^2\right)^{\frac{3}{2}} a_3 + 2x^3a_2 - 2x^3b_3 + 4x^2ya_3 + 8x^2yb_2 + 8y^3a_3 + 4\sqrt{x^2+4y^2}\,xb_1 - 4\sqrt{x^2+4y^2}\,ya_1 + 8xyb}{4\sqrt{x^2+4y^2}\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -\left(x^2+4y^2\right)^{\frac{3}{2}} a_3 - 2x^3a_2 + 2x^3b_3 - 4x^2ya_3 - 8x^2yb_2 - 8y^3a_3 \\ & - 4\sqrt{x^2+4y^2}\,xb_1 + 4\sqrt{x^2+4y^2}\,ya_1 - 8xyb_1 + 8y^2a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -\left(x^2+4y^2\right)^{\frac{3}{2}} a_3 + 2\left(x^2+4y^2\right)xb_3 - 2\left(x^2+4y^2\right)ya_3 \\ & - 2x^3a_2 - 2x^2ya_3 - 8x^2yb_2 - 8x\,y^2b_3 + 2\left(x^2+4y^2\right)a_1 \\ & - 4\sqrt{x^2+4y^2}\,xb_1 + 4\sqrt{x^2+4y^2}\,ya_1 - 2x^2a_1 - 8xyb_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} & -2x^3a_2 + 2x^3b_3 - x^2\sqrt{x^2+4y^2}\,a_3 - 4x^2ya_3 - 8x^2yb_2 - 4\sqrt{x^2+4y^2}\,y^2a_3 \\ & - 8y^3a_3 - 4\sqrt{x^2+4y^2}\,xb_1 - 8xyb_1 + 4\sqrt{x^2+4y^2}\,ya_1 + 8y^2a_1 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+4y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+4y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2v_1^3a_2 - 4v_1^2v_2a_3 - v_1^2v_3a_3 - 8v_2^3a_3 - 4v_3v_2^2a_3 - 8v_1^2v_2b_2 \\ &+ 2v_1^3b_3 + 8v_2^2a_1 + 4v_3v_2a_1 - 8v_1v_2b_1 - 4v_3v_1b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 + 2b_3)\, v_1^3 + (-4a_3 - 8b_2)\, v_1^2v_2 - v_1^2v_3a_3 - 8v_1v_2b_1 \\ &- 4v_3v_1b_1 - 8v_2^3a_3 - 4v_3v_2^2a_3 + 8v_2^2a_1 + 4v_3v_2a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 4a_1 &= 0 \\ 8a_1 &= 0 \\ -8a_3 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ -8b_1 &= 0 \\ -4b_1 &= 0 \\ -2a_2 + 2b_3 &= 0 \\ -4a_3 - 8b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{2y + \sqrt{x^2 + 4y^2}}{2x}\right)(x) \\
&= -\frac{\sqrt{x^2 + 4y^2}}{2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{\sqrt{x^2+4y^2}}{2}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\ln\left(\sqrt{4}\, y + \sqrt{x^2+4y^2}\right)\sqrt{4}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y + \sqrt{x^2+4y^2}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2+4y^2}\,\left(2y+\sqrt{x^2+4y^2}\right)} \\
S_y &= -\frac{2}{\sqrt{x^2+4y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(2\sqrt{x^2+4y^2}\, y + x^2 + 4y^2\right)}{x\sqrt{x^2+4y^2}\,\left(2y+\sqrt{x^2+4y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(2y + \sqrt{x^2+4y^2}\right) = -2\ln(x) + c_1$$

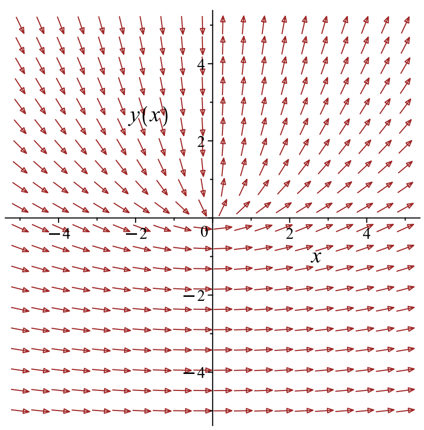
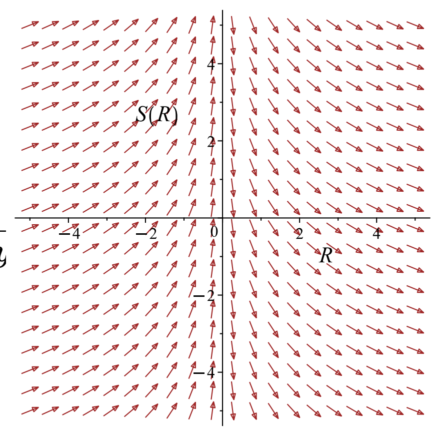
Which simplifies to

$$-\ln\left(2y+\sqrt{x^2+4y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{4}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+\sqrt{x^2+4y^2}}{2x}$ 	$R = x$ $S = -\ln\left(2y + \sqrt{x^2 + 4y}\right.$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{4} \tag{1}$$

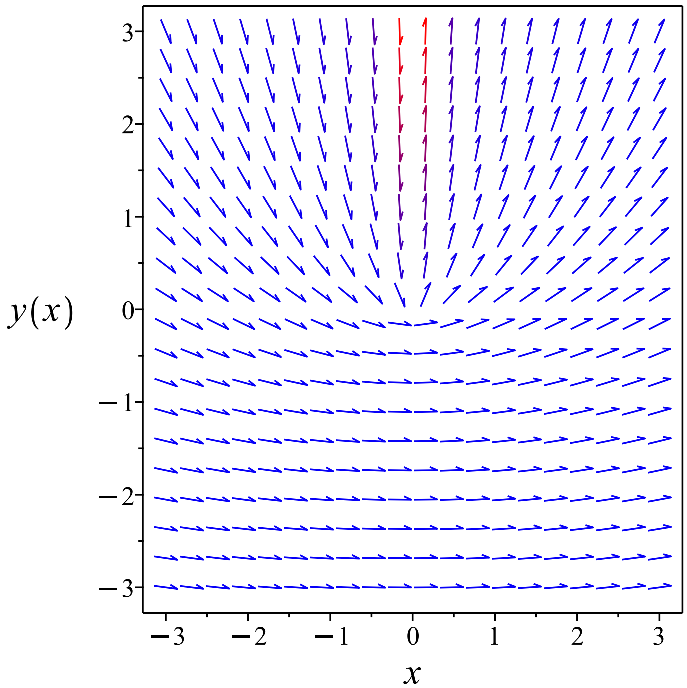


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve(2*x*diff(y(x),x)-2*y(x)= sqrt(x^2+4*y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1x^2 + 2y(x) + \sqrt{x^2 + 4y\left(x\right)^2}}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.394 (sec). Leaf size: 27

```
DSolve[2*x*y'[x]-2*y[x]== Sqrt[x^2+4*y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-2c_1}\left(-1 + e^{4c_1}x^2\right)$$

2.14 problem 39

2.14.1 Solving as homogeneousTypeMapleC ode 277
2.14.2 Solving as first order ode lie symmetry calculated ode 281

Internal problem ID [5249]
Internal file name [`OUTPUT/4740_Friday_February_02_2024_05_10_56_AM_45477591/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (7y - 3x + 3)\, y' = 7x - 7}$$

2.14.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 - 7X - 7x_0 + 7}{7Y(X) + 7y_0 - 3X - 3x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) - 7X}{7Y(X) - 3X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{3Y - 7X}{7Y - 3X} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = -7Y + 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{-3u + 7}{7u - 3} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX} u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0
$$

Or

$$
7\left(\frac{d}{dX} u(X)\right) X u(X) - 3\left(\frac{d}{dX} u(X)\right) X + 7u(X)^2 - 7 = 0
$$

Or

$$
-7 + X(7u(X) - 3)\left(\frac{d}{dX} u(X)\right) + 7u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X) g(u) \\
&= -\frac{7(u^2 - 1)}{X\,(7u - 3)}
\end{aligned}
$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}}\,du = -\frac{7}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{7u-3}}\,du = \int -\frac{7}{X}\,dX$$

$$2\ln(u-1) + 5\ln(u+1) = -7\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+5\ln(u+1)} = e^{-7\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \mathrm{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$y = \mathrm{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + \left(x^2 - 2x + 1\right)\_Z^5 + \left(-5x^3 + 15x^2 - 15x + 5\right)\_Z^4 + \left(-5x^4 + 20x\right.\right.$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = \operatorname{RootOf}\big(&\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7 - 7x^6 + 21x^5 - 35x^4 \\
&+ 35x^3 - 21x^2 - c_3 + 7x - 1\big)
\end{aligned} \tag{1}
$$

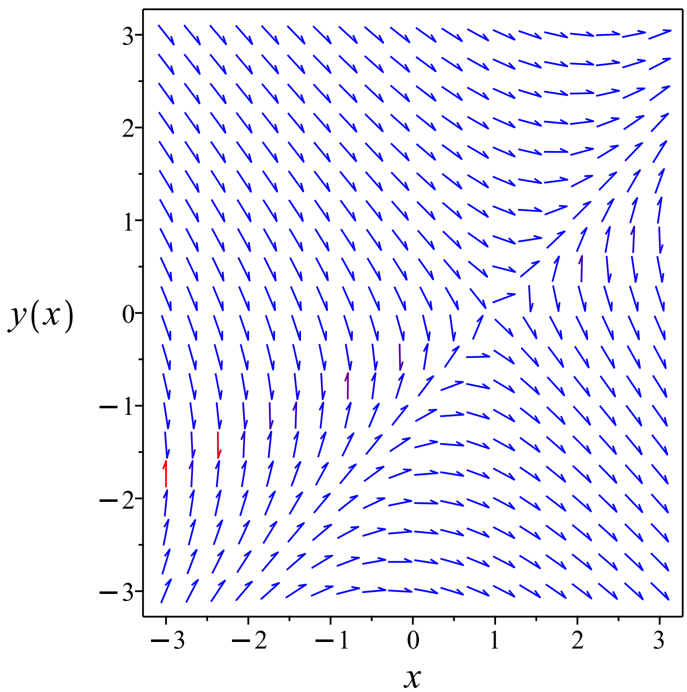


Figure 67: Slope field plot

Verification of solutions

$$
\begin{aligned}
y = \operatorname{RootOf}\big(&\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7 - 7x^6 + 21x^5 - 35x^4 \\
&+ 35x^3 - 21x^2 - c_3 + 7x - 1\big)
\end{aligned}
$$

Verified OK.

2.14.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{3y-7x+7}{7y-3x+3}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(3y-7x+7)(b_3-a_2)}{7y-3x+3} - \frac{(3y-7x+7)^2 a_3}{(7y-3x+3)^2} \\ &- \left(\frac{7}{7y-3x+3} - \frac{3(3y-7x+7)}{(7y-3x+3)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(-\frac{3}{7y-3x+3} + \frac{21y-49x+49}{(7y-3x+3)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 -}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 \\ &+21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3 \\ &-40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\ - 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\ - 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\ + 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_1^2 + (-98a_2 + 42a_3 - 42b_2 + 98b_3)\, v_1v_2 \\ + (-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3)\, v_1 + (21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_2^2 \\ + (-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3)\, v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-98a_2 + 42a_3 - 42b_2 + 98b_3 &= 0 \\ 21a_2 - 49a_3 + 49b_2 - 21b_3 &= 0 \\ -40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3 &= 0 \\ -42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3 &= 0 \\ 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_3 \\ a_2 &= b_3 \\ a_3 &= b_2 \\ b_1 &= -b_2 \\ b_2 &= b_2 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = y$$
$$\eta = x - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 - \left(-\frac{3y-7x+7}{7y-3x+3}\right)(y) \\
&= \frac{3x^2-3y^2-6x+3}{-7y+3x-3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2-3y^2-6x+3}{-7y+3x-3}} dy
\end{aligned}$$

Which results in

$$S = \frac{2\ln(y-x+1)}{3} + \frac{5\ln(x+y-1)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y-7x+7}{7y-3x+3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{3x-3-3y} + \frac{5}{3x-3+3y} \\
S_y &= -\frac{2}{3x-3-3y} + \frac{5}{3x-3+3y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

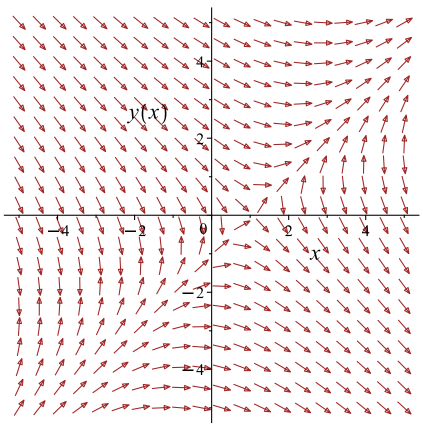
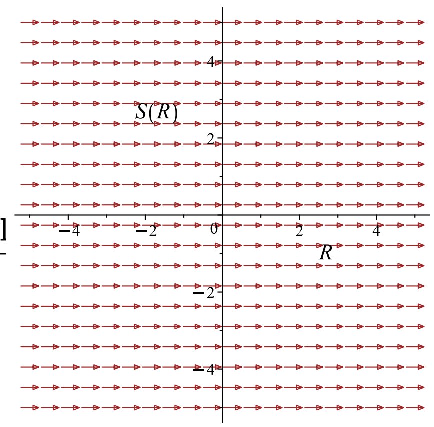
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y-x+1)}{3} + \frac{5\ln(x+y-1)}{3} = c_1$$

Which simplifies to

$$\frac{2\ln(y-x+1)}{3} + \frac{5\ln(x+y-1)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$ 	$R = x$ $S = \frac{2\ln(y-x+1)}{3} + \frac{5\ln(x+y-1)}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y-x+1)}{3} + \frac{5\ln(x+y-1)}{3} = c_1 \tag{1}$$

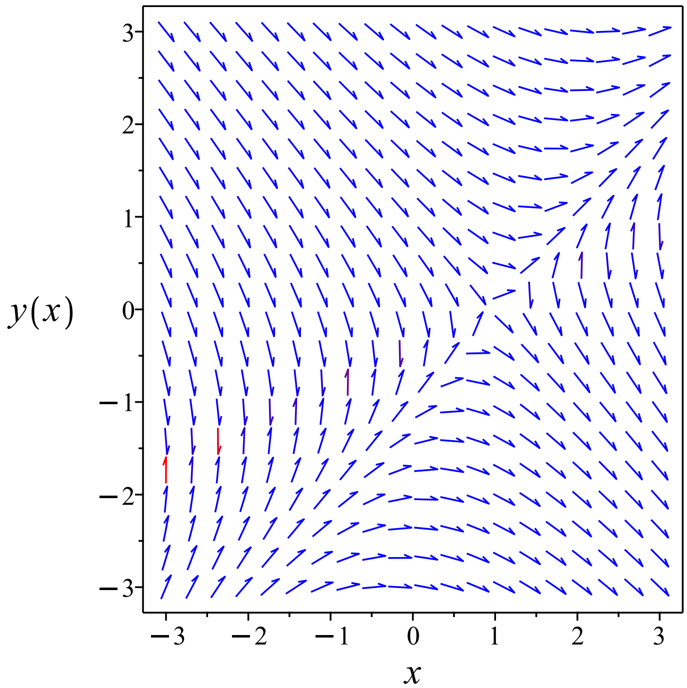


Figure 68: Slope field plot

Verification of solutions

$$\frac{2\ln(y - x + 1)}{3} + \frac{5\ln(x + y - 1)}{3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 1814

```
dsolve((3*y(x)-7*x+7)+(7*y(x)-3*x+3)*diff(y(x),x)= 0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.746 (sec). Leaf size: 7785

```
DSolve[(3*y[x]-7*x+7)+(7*y[x]-3*x+3)*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.15 problem 40

2.15.1 Solving as separable ode 288
2.15.2 Solving as first order ode lie symmetry lookup ode 290
2.15.3 Solving as exact ode 294
2.15.4 Maple step by step solution 298

Internal problem ID [5250]
Internal file name [`OUTPUT/4741_Friday_February_02_2024_05_10_58_AM_59247198/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yy'x-(y+1)(1-x)=0}$$

2.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{(x-1)(-y-1)}{xy}\end{aligned}$$

Where $f(x)=\frac{x-1}{x}$ and $g(y)=\frac{-y-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{-y-1}{y}}\,dy=\frac{x-1}{x}\,dx$$

$$\int \frac{1}{\frac{-y-1}{y}} dy = \int \frac{x-1}{x} dx$$

$$-y + \ln(y+1) = x - \ln(x) + c_1$$

Which results in

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. Which simplifies to

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

gives

$$y = -\operatorname{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1$$

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1 \tag{1}$$

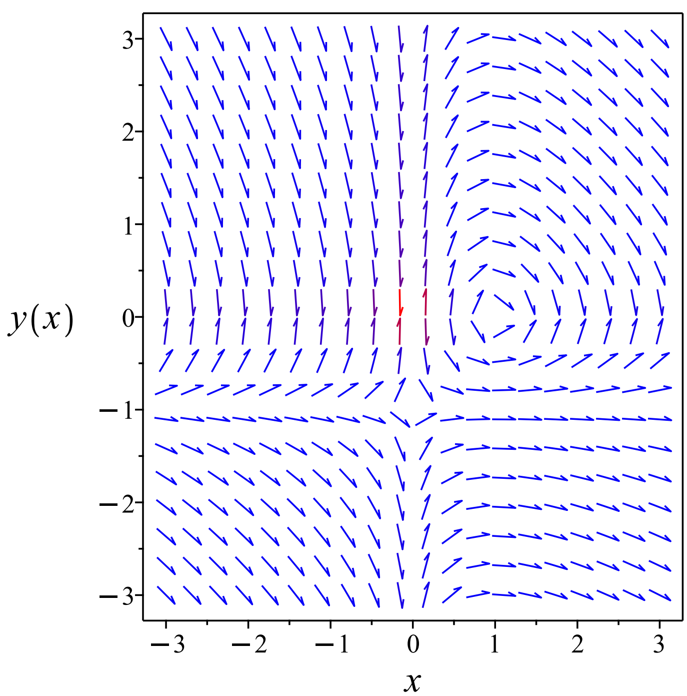


Figure 69: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{c_1 \mathrm{e}^{x-1}}{x}\right) - 1$$

Verified OK.

2.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{xy + x - y - 1}{xy}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x}{x-1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{x-1}} dx \end{aligned}$$

Which results in

$$S = x - \ln(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{xy + x - y - 1}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x-1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y}{y+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{R+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -R + \ln(R+1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x - \ln(x) = -y + \ln(y+1) + c_1$$

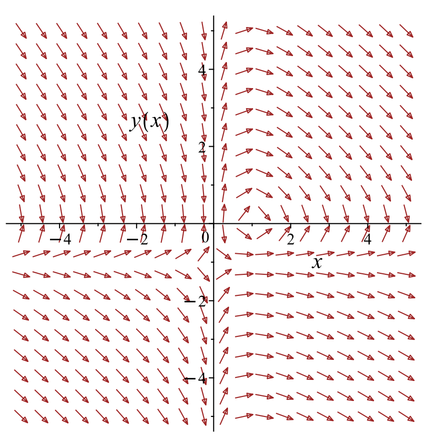
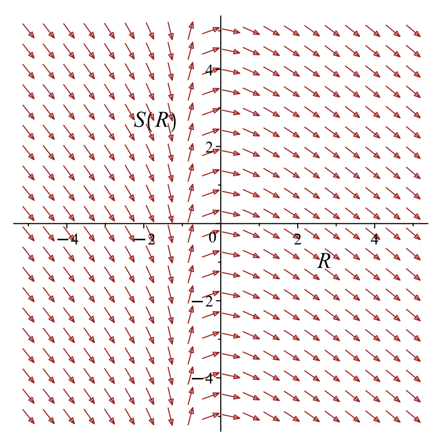
Which simplifies to

$$x - \ln(x) = -y + \ln(y+1) + c_1$$

Which gives

$$y = -\operatorname{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+x-y-1}{xy}$ 	$R = y$ $S = x - \ln(x)$	$\frac{dS}{dR} = -\frac{R}{R+1}$ 

Summary

The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1 \tag{1}$$

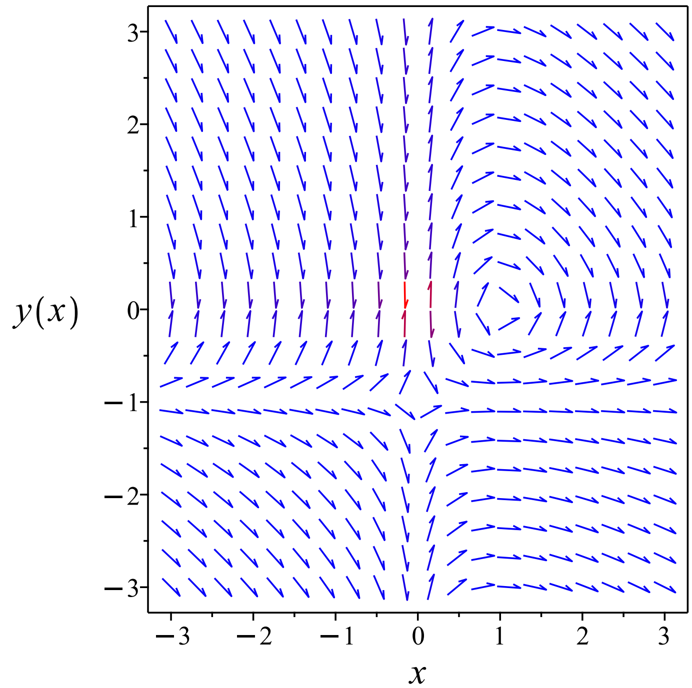


Figure 70: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$$

Verified OK.

2.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{-y-1}\right)\mathrm{d}y &= \left(\frac{x-1}{x}\right)\mathrm{d}x \\ \left(-\frac{x-1}{x}\right)\mathrm{d}x + \left(\frac{y}{-y-1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{x-1}{x} \\ N(x,y) &= \frac{y}{-y-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x-1}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{-y-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x-1}{x} \, \mathrm{d}x \\
\phi &= -x + \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{-y-1}$. Therefore equation (4) becomes

$$
\frac{y}{-y-1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{y+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{y}{y+1}\right) \mathrm{d}y \\
f(y) &= -y + \ln(y+1) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \ln(x) - y + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \ln(x) - y + \ln(y+1)$$

The solution becomes

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1 \tag{1}$$

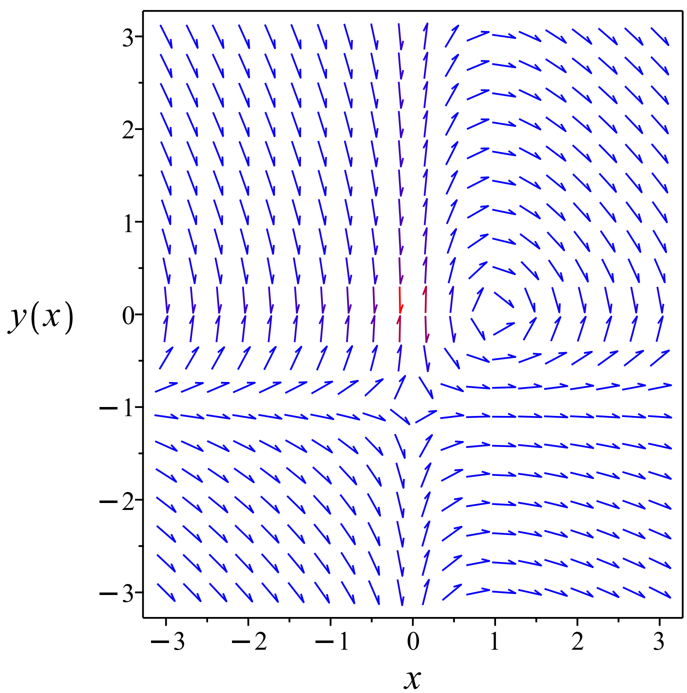


Figure 71: Slope field plot

Verification of solutions

$$y = -\operatorname{LambertW}\left(-\frac{e^{-1+x+c_1}}{x}\right) - 1$$

Verified OK.

2.15.4 Maple step by step solution

Let's solve

$yy'x - (y+1)(1-x) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{y+1} = \frac{1-x}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{y+1}dx = \int \frac{1-x}{x}dx + c_1$

- Evaluate integral

  $y - \ln(y+1) = -x + \ln(x) + c_1$

- Solve for y

  $y = -\mathit{LambertW}\left(-\frac{e^{x-1-c_1}}{x}\right) - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*y(x)*diff(y(x),x)= (y(x)+1)*(1-x),y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{c_1 e^{x-1}}{x}\right) - 1$$

✓ Solution by Mathematica

Time used: 6.202 (sec). Leaf size: 29

```
DSolve[x*y[x]*y'[x]== (y[x]+1)*(1-x),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 - W\left(-\frac{e^{x-1-c_1}}{x}\right)$$
$$y(x) \to -1$$

2.16 problem 41

2.16.1 Solving as homogeneousTypeD2 ode 300
2.16.2 Solving as first order ode lie symmetry lookup ode 302
2.16.3 Solving as bernoulli ode 306
2.16.4 Solving as exact ode 309

Internal problem ID [5251]
Internal file name [`OUTPUT/4742_Friday_February_02_2024_05_10_59_AM_6697373/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 + yy'x = x^2}$$

2.16.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + u(x)\,x^2(u'(x)\,x + u(x)) = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2-1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2-1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^2-1}{u}}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{2u^2-1}{u}}\,du &= \int -\frac{1}{x}\,dx \\
\frac{\ln\left(2u^2-1\right)}{4} &= -\ln\left(x\right) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(2u^2-1\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\left(2u^2-1\right)^{\frac{1}{4}} = \frac{c_3}{x}
$$

Which simplifies to

$$
\left(2u(x)^2-1\right)^{\frac{1}{4}} = \frac{c_3\,\mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\left(2u(x)^2-1\right)^{\frac{1}{4}} = \frac{c_3\,\mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{2y^2}{x^2}-1\right)^{\frac{1}{4}} &= \frac{c_3\,\mathrm{e}^{c_2}}{x} \\
\left(\frac{2y^2-x^2}{x^2}\right)^{\frac{1}{4}} &= \frac{c_3\,\mathrm{e}^{c_2}}{x}
\end{aligned}
$$

Which simplifies to

$$
\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\,\mathrm{e}^{c_2}}{x}
$$

Summary

The solution(s) found are the following

$$
\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\,\mathrm{e}^{c_2}}{x} \tag{1}
$$

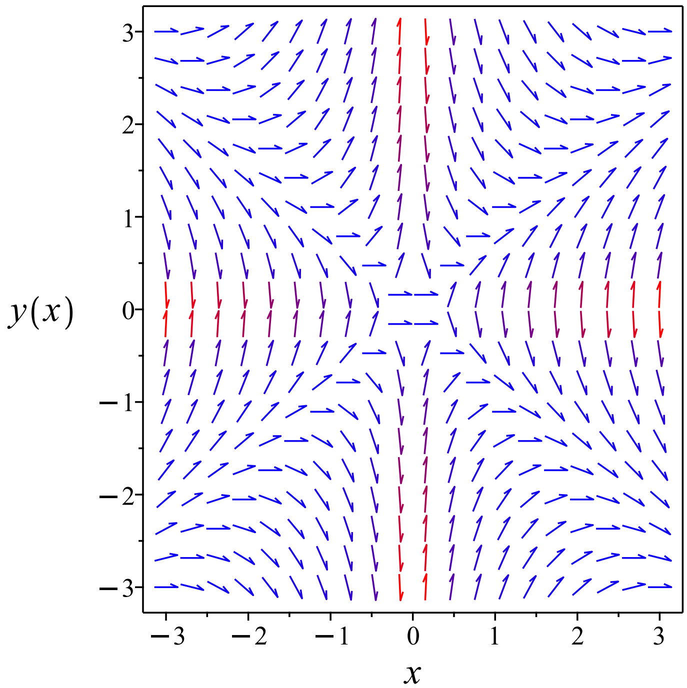


Figure 72: Slope field plot

Verification of solutions

$$\left(-\frac{-2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Verified OK.

2.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-x^2+y^2}{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y\, x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y\,x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^2 + y^2}{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^2 x \\ S_y &= x^2 y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^4}{4} + c_1 \tag{4}$$

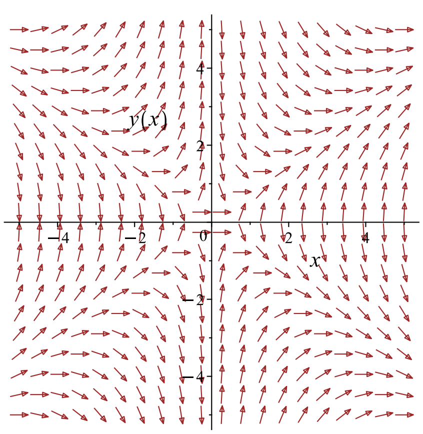
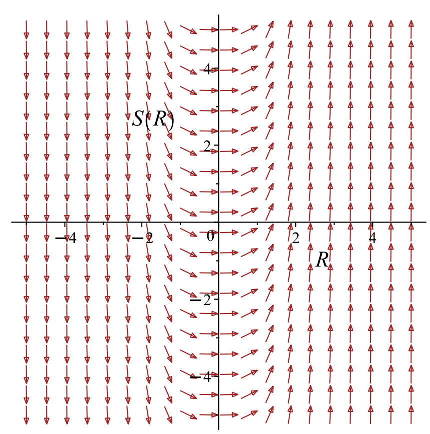
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y^2}{yx}$ 	$R = x$ $S = \frac{x^2y^2}{2}$	$\frac{dS}{dR} = R^3$ 

Summary
The solution(s) found are the following

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1 \tag{1}$$

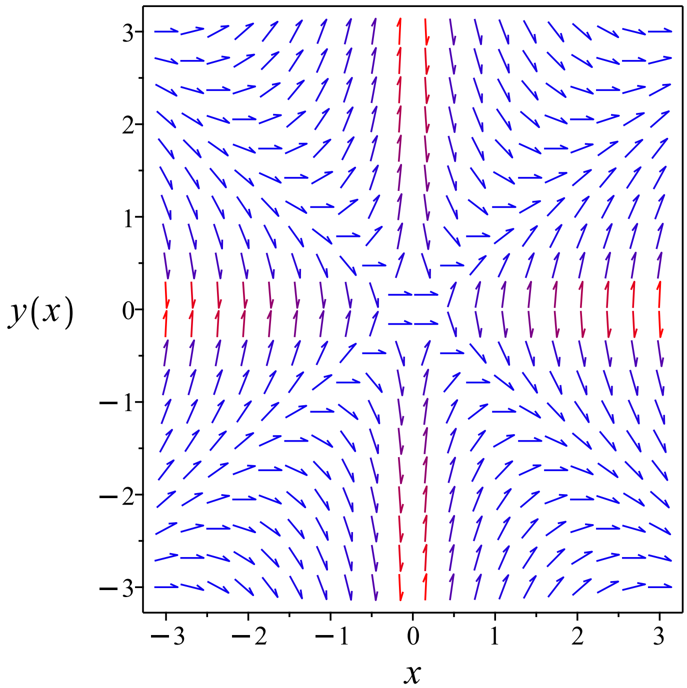


Figure 73: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2} = \frac{x^4}{4} + c_1$$

Verified OK.

2.16.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x^2+y^2}{yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} + x \\
w' &= -\frac{2w}{x} + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(2x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)(2x) \\ \mathrm{d}(x^2 w) &= (2x^3)\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int 2x^3 \,\mathrm{d}x \\ x^2 w &= \frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x} \tag{2}$$

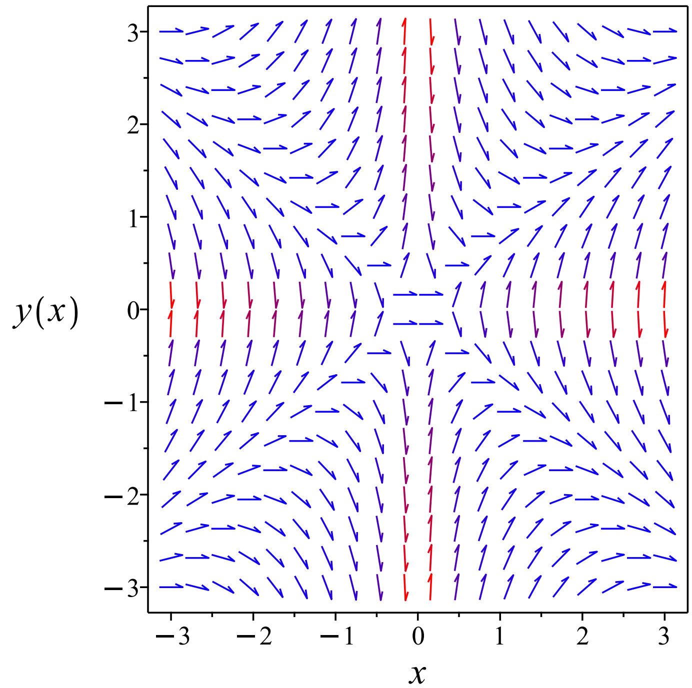


Figure 74: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{2x^4 + 4c_1}}{2x}$$

Verified OK.

2.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y&=\left(x^2-y^2\right)\mathrm{d}x\\ \left(-x^2+y^2\right)\mathrm{d}x+(xy)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2+y^2\\ N(x,y)&=xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^2+y^2\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{yx}((2y) - (y))\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x\left(-x^2 + y^2\right)\\ &= -x^3 + y^2 x\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= x(xy)\\ &= x^2 y\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-x^3 + y^2 x\right) + \left(x^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -x^3 + y^2 x \, \mathrm{d}x$$
$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 y$. Therefore equation (4) becomes

$$x^2 y = x^2 y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^4 + \frac{1}{2}x^2 y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^4 + \frac{1}{2}x^2 y^2$$

Summary

The solution(s) found are the following

$$\frac{x^2 y^2}{2} - \frac{x^4}{4} = c_1 \tag{1}$$

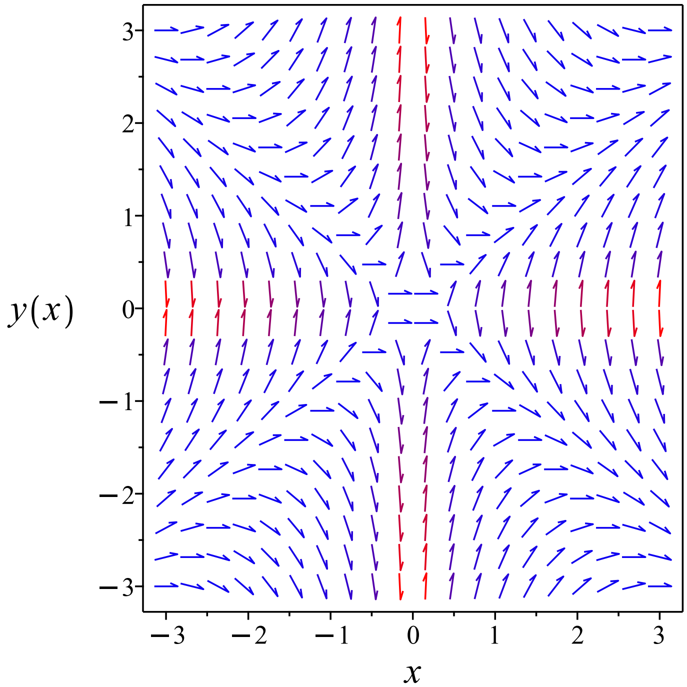


Figure 75: Slope field plot

Verification of solutions

$$\frac{x^2 y^2}{2} - \frac{x^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((y(x)^2-x^2)+x*y(x)*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2x^4+4c_1}}{2x}$$
$$y(x) = \frac{\sqrt{2x^4+4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.199 (sec). Leaf size: 46

```
DSolve[(y[x]^2-x^2)+x*y[x]*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{\frac{x^4}{2}+c_1}}{x}$$
$$y(x) \to \frac{\sqrt{\frac{x^4}{2}+c_1}}{x}$$

2.17 problem 42

2.17.1 Solving as first order ode lie symmetry calculated ode 315
2.17.2 Solving as exact ode 321

Internal problem ID [5252]
Internal file name [`OUTPUT/4743_Friday_February_02_2024_05_11_00_AM_59621178/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y(1+2yx)+x(1-yx)\,y'=0}$$

2.17.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(2xy+1)}{x\,(xy-1)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{y(2xy+1)(b_3-a_2)}{x(xy-1)} - \frac{y^2(2xy+1)^2 a_3}{x^2(xy-1)^2} \\
&- \left(\frac{2y^2}{x(xy-1)} - \frac{y(2xy+1)}{x^2(xy-1)} - \frac{y^2(2xy+1)}{x(xy-1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2xy+1}{x(xy-1)} + \frac{2y}{xy-1} - \frac{y(2xy+1)}{(xy-1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4y^2b_2 + 2x^2y^4a_3 + 2x^3y^2b_1 - 2x^2y^3a_1 - 2x^3yb_2 - 3x^2y^2a_2 - 3x^2y^2b_3 + 2x\,y^3a_3 - 4x^2yb_1 - 2x\,y^2a_1 - 2}{x^2(xy-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-x^4y^2b_2 - 2x^2y^4a_3 - 2x^3y^2b_1 + 2x^2y^3a_1 + 2x^3yb_2 + 3x^2y^2a_2 + 3x^2y^2b_3 \\
&- 2x\,y^3a_3 + 4x^2yb_1 + 2x\,y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2a_3v_1^2v_2^4 - b_2v_1^4v_2^2 + 2a_1v_1^2v_2^3 - 2b_1v_1^3v_2^2 + 3a_2v_1^2v_2^2 - 2a_3v_1v_2^3 + 2b_2v_1^3v_2 \\
&+ 3b_3v_1^2v_2^2 + 2a_1v_1v_2^2 + 4b_1v_1^2v_2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} -b_2v_1^4v_2^2 - 2b_1v_1^3v_2^2 + 2b_2v_1^3v_2 - 2a_3v_1^2v_2^4 + 2a_1v_1^2v_2^3 + (3a_2 + 3b_3)\,v_1^2v_2^2 \\ + 4b_1v_1^2v_2 + 2b_2v_1^2 - 2a_3v_1v_2^3 + 2a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ 2a_1 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ 4b_1 &= 0 \\ -b_2 &= 0 \\ 2b_2 &= 0 \\ 3a_2 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(\frac{y(2xy+1)}{x\,(xy-1)}\right)(-x) \\ &= \frac{3y^2x}{xy-1} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy \\ &= \int \frac{1}{\frac{3y^2x}{xy-1}}dy\end{aligned}$$

Which results in

$$S = \frac{1}{3yx} + \frac{\ln(y)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2xy+1)}{x\,(xy-1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{3y\,x^2} \\
S_y &= \frac{xy-1}{3y^2 x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{3x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{3R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2\ln(R)}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{xy\ln(y)+1}{3xy} = \frac{2\ln(x)}{3} + c_1
$$

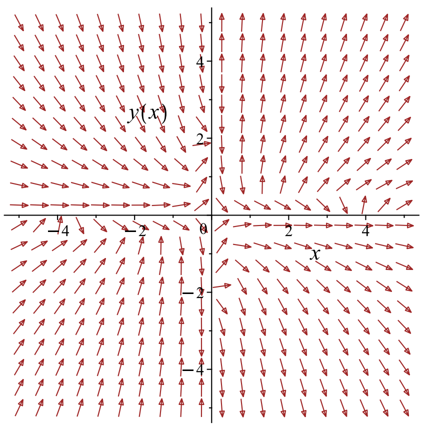
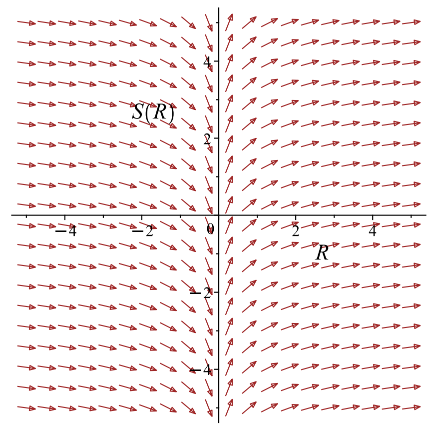
Which simplifies to

$$
\frac{xy\ln(y)+1}{3xy} = \frac{2\ln(x)}{3} + c_1
$$

Which gives

$$
y = -\frac{1}{x\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-3c_1}}{x^3}\right)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(2xy+1)}{x(xy-1)}$ 	$R = x$ $S = \frac{\ln(y)\, xy + 1}{3xy}$	$\frac{dS}{dR} = \frac{2}{3R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-3c_1}}{x^3}\right)} \tag{1}$$

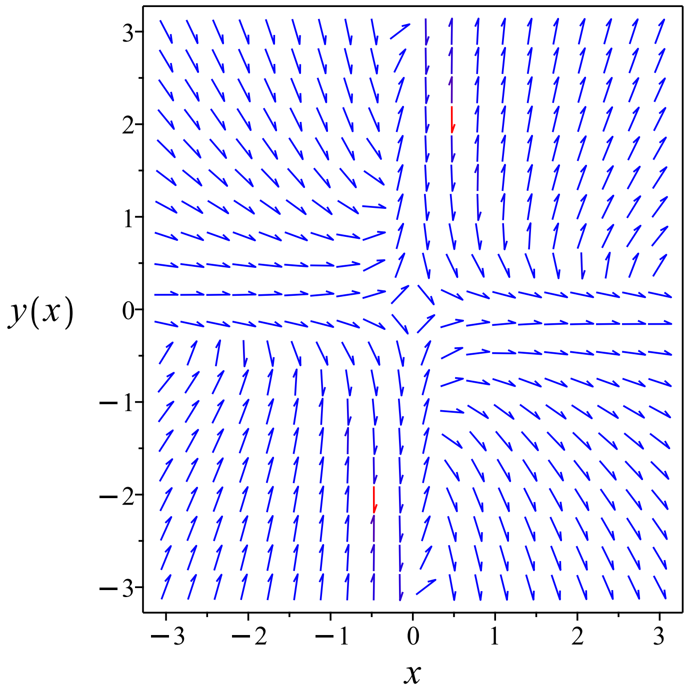


Figure 76: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \operatorname{LambertW}\left(-\frac{e^{-3c_1}}{x^3}\right)}$$

Verified OK.

2.17.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x(-xy + 1)) \, \mathrm{d}y &= (-y(2xy + 1)) \, \mathrm{d}x \\
(y(2xy + 1)) \, \mathrm{d}x + (x(-xy + 1)) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y(2xy + 1) \\
N(x, y) &= x(-xy + 1)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(2xy + 1)) \\
&= 4xy + 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(-xy + 1)) \\
&= -2xy + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x\left(xy-1\right)}\left(\left(4xy+1\right) - \left(-2xy+1\right)\right) \\
&= -\frac{6y}{xy-1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{2y^2x+y}\left(\left(-2xy+1\right) - \left(4xy+1\right)\right) \\
&= -\frac{6x}{2xy+1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(-2xy+1\right) - \left(4xy+1\right)}{x\left(y\left(2xy+1\right)\right) - y\left(x\left(-xy+1\right)\right)} \\
&= -\frac{2}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{x^2y^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2y^2}(y(2xy+1)) \\
&= \frac{2xy+1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2y^2}(x(-xy+1)) \\
&= \frac{-xy+1}{x\,y^2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2xy+1}{y\,x^2}\right) + \left(\frac{-xy+1}{x\,y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{2xy+1}{y\,x^2} \mathrm{d}x \\
\phi &= -\frac{1}{yx} + 2\ln(x) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x} + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-xy+1}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{-xy+1}{x\,y^2} = \frac{1}{y^2 x} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{yx} + 2\ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{yx} + 2\ln(x) - \ln(y)$$

The solution becomes

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x^3}\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x^3}\right)} \tag{1}$$

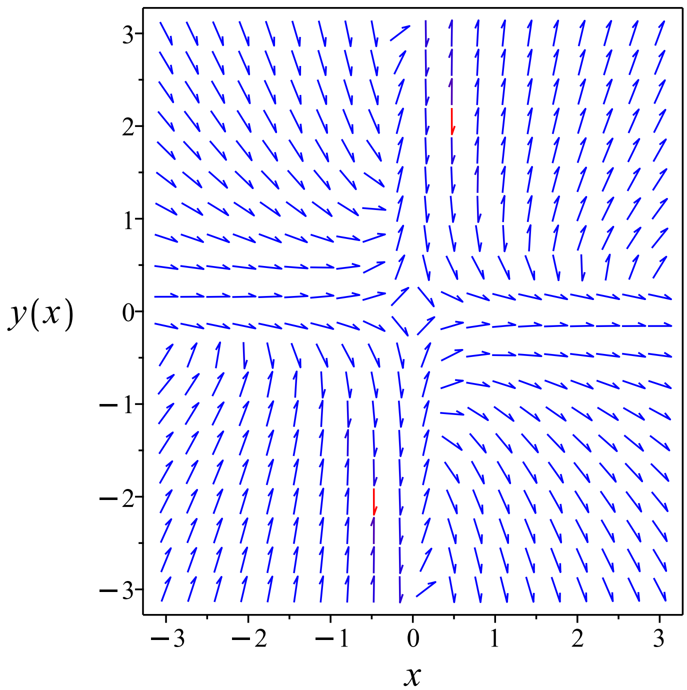


Figure 77: Slope field plot

Verification of solutions

$$y = -\frac{1}{x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x^3}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 18

```
dsolve(y(x)*(1+2*x*y(x))+x*(1-x*y(x))*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{\operatorname{LambertW}\left(-\frac{c_1}{x^3}\right) x}$$

✓ Solution by Mathematica

Time used: 6.645 (sec). Leaf size: 35

```
DSolve[y[x]*(1+2*x*y[x])+x*(1-x*y[x])*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{xW\left(\frac{e^{-1+\frac{9c_1}{2^{2/3}}}}{x^3}\right)}$$

$$y(x) \to 0$$

2.18 problem 43

2.18.1 Solving as separable ode 328
2.18.2 Solving as first order ode lie symmetry lookup ode 330
2.18.3 Solving as exact ode 334
2.18.4 Maple step by step solution 338

Internal problem ID [5253]
Internal file name [`OUTPUT/4744_Friday_February_02_2024_05_11_02_AM_57669137/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\left(-x^{2}+1\right)\cot\left(y\right)y' = -1}$$

2.18.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1}{\cot\left(y\right)\left(x^{2}-1\right)} \end{aligned}$$

Where $f(x)=\frac{1}{x^{2}-1}$ and $g(y)=\frac{1}{\cot(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\cot(y)}}\,dy = \frac{1}{x^{2}-1}\,dx$$

$$\int \frac{1}{\frac{1}{\cot(y)}} dy = \int \frac{1}{x^2 - 1} dx$$

$$\ln(\sin(y)) = -\operatorname{arctanh}(x) + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{-\operatorname{arctanh}(x)+c_1}$$

Which simplifies to

$$\sin(y) = \frac{c_2\sqrt{-x^2+1}}{x+1}$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(c_2 e^{-\operatorname{arctanh}(x)+c_1}\right) \tag{1}$$

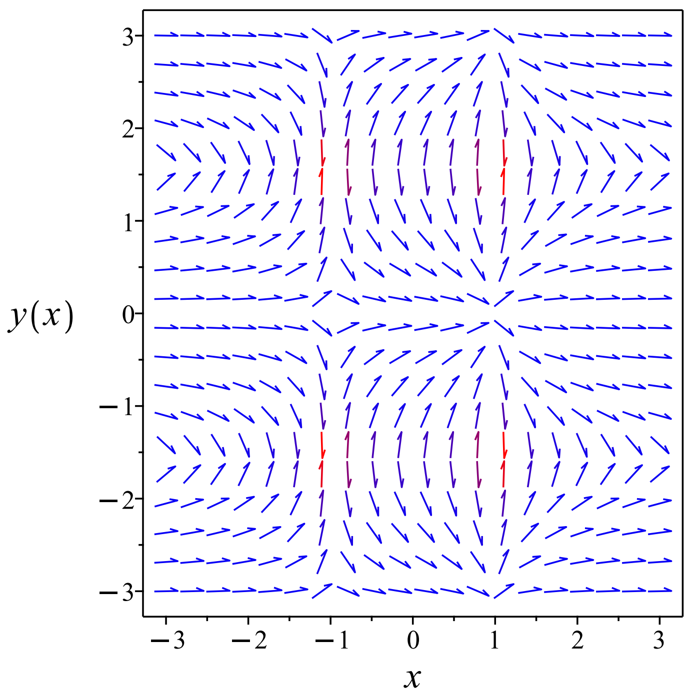


Figure 78: Slope field plot

Verification of solutions

$$y = \arcsin\left(c_2 e^{-\operatorname{arctanh}(x)+c_1}\right)$$

Verified OK.

2.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{1}{\cot(y)(x^2-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 44: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2 - 1\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{x^2 - 1} dx\end{aligned}$$

Which results in

$$S = -\operatorname{arctanh}(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1}{\cot(y)\,(x^2 - 1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2 - 1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cot(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cot(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\sin(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\operatorname{arctanh}(x) = \ln(\sin(y)) + c_1
$$

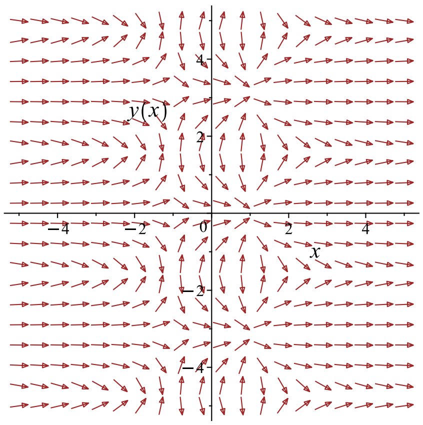
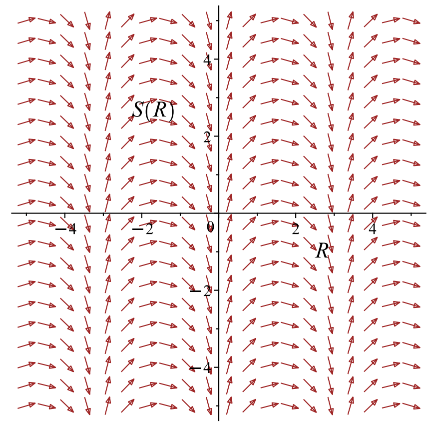
Which simplifies to

$$
-\operatorname{arctanh}(x) = \ln(\sin(y)) + c_1
$$

Which gives

$$
y = \arcsin\left(\mathrm{e}^{-\operatorname{arctanh}(x) - c_1}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1}{\cot(y)(x^2-1)}$ 	$R = y$ $S = -\operatorname{arctanh}(x)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(\mathrm{e}^{-\operatorname{arctanh}(x)-c_1}\right) \tag{1}$$

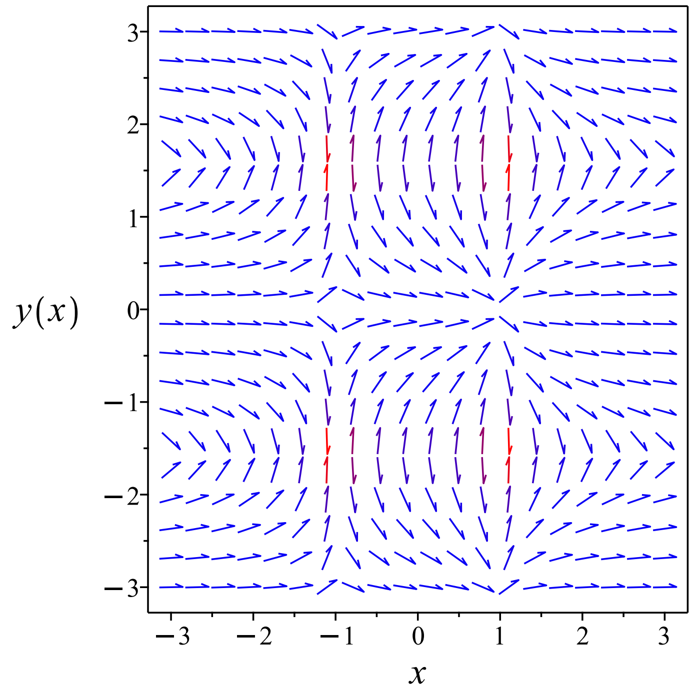


Figure 79: Slope field plot

Verification of solutions

$$y = \arcsin\left(e^{-\operatorname{arctanh}(x)-c_1}\right)$$

Verified OK.

2.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\cot(y))\,\mathrm{d}y = \left(\frac{1}{x^2-1}\right)\mathrm{d}x$$
$$\left(-\frac{1}{x^2-1}\right)\mathrm{d}x + (\cot(y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{x^2-1}$$
$$N(x,y) = \cot(y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x^2-1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cot(y)) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x^2-1} \,\mathrm{d}x \\ \phi &= \operatorname{arctanh}(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cot(y)$. Therefore equation (4) becomes

$$\cot(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \cot(y)$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y) \,\mathrm{d}y &= \int (\cot(y)) \,\mathrm{d}y \\ f(y) &= \ln(\sin(y)) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \operatorname{arctanh}(x) + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \operatorname{arctanh}(x) + \ln(\sin(y))$$

Summary

The solution(s) found are the following

$$\operatorname{arctanh}(x) + \ln(\sin(y)) = c_1 \tag{1}$$

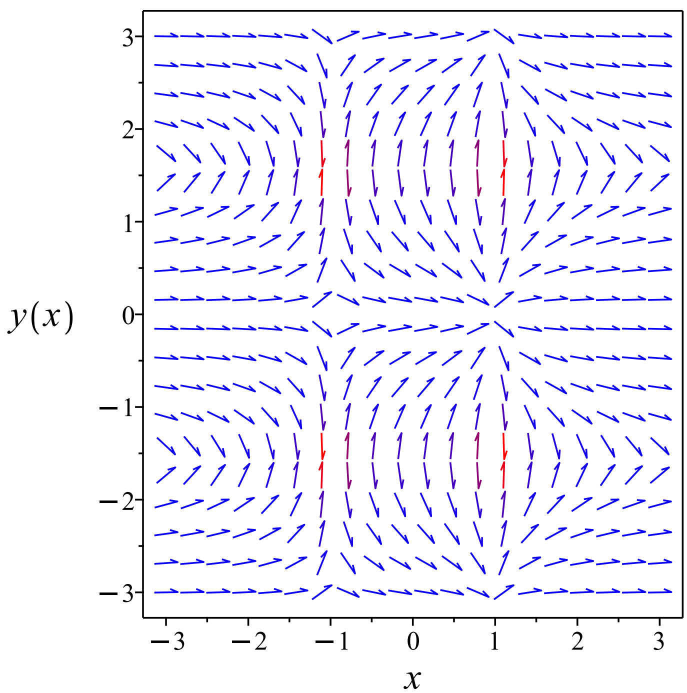


Figure 80: Slope field plot

Verification of solutions

$$\operatorname{arctanh}(x) + \ln(\sin(y)) = c_1$$

Verified OK.

2.18.4 Maple step by step solution

Let's solve

$(-x^2+1)\cot(y)\,y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\cot(y)\,y' = -\frac{1}{-x^2+1}$

- Integrate both sides with respect to x

  $\int \cot(y)\,y' dx = \int -\frac{1}{-x^2+1} dx + c_1$

- Evaluate integral

  $\ln(\sin(y)) = -\mathrm{arctanh}(x) + c_1$

- Solve for y

  $y = \arcsin\left(\mathrm{e}^{-\mathrm{arctanh}(x)+c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(1+(1-x^2)*cot(y(x))*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{\sqrt{-x^2+1}\,c_1}{x+1}\right)$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 27

```
DSolve[1+(1-x^2)*Cot[y[x]]*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{e^{c_1}\sqrt{1-x}}{\sqrt{x+1}}\right)$$

2.19 problem 44

2.19.1 Solving as homogeneousTypeD2 ode 340
2.19.2 Solving as first order ode lie symmetry lookup ode 342
2.19.3 Solving as bernoulli ode 346
2.19.4 Solving as exact ode 350
2.19.5 Maple step by step solution 353

Internal problem ID [5254]
Internal file name [`OUTPUT/4745_Friday_February_02_2024_05_11_02_AM_13641118/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{y^3 + 3y^2 y' x = -x^3}$$

2.19.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)^3 x^3 + 3u(x)^2 x^3 (u'(x)\, x + u(x)) = -x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{4u^3+1}{3u^2 x} \end{aligned}$$

Where $f(x) = -\frac{1}{3x}$ and $g(u) = \frac{4u^3+1}{u^2}$. Integrating both sides gives

$$\frac{1}{\frac{4u^3+1}{u^2}}\, du = -\frac{1}{3x}\, dx$$
$$\int \frac{1}{\frac{4u^3+1}{u^2}}\, du = \int -\frac{1}{3x}\, dx$$
$$\frac{\ln\left(4u^3+1\right)}{12} = -\frac{\ln\left(x\right)}{3} + c_2$$

Raising both side to exponential gives

$$\left(4u^3+1\right)^{\frac{1}{12}} = \mathrm{e}^{-\frac{\ln(x)}{3}+c_2}$$

Which simplifies to

$$\left(4u^3+1\right)^{\frac{1}{12}} = \frac{c_3}{x^{\frac{1}{3}}}$$

Which simplifies to

$$\left(4u(x)^3+1\right)^{\frac{1}{12}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

The solution is

$$\left(4u(x)^3+1\right)^{\frac{1}{12}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{4y^3}{x^3}+1\right)^{\frac{1}{12}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$
$$\left(\frac{4y^3+x^3}{x^3}\right)^{\frac{1}{12}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\left(\frac{4y^3+x^3}{x^3}\right)^{\frac{1}{12}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}} \tag{1}$$

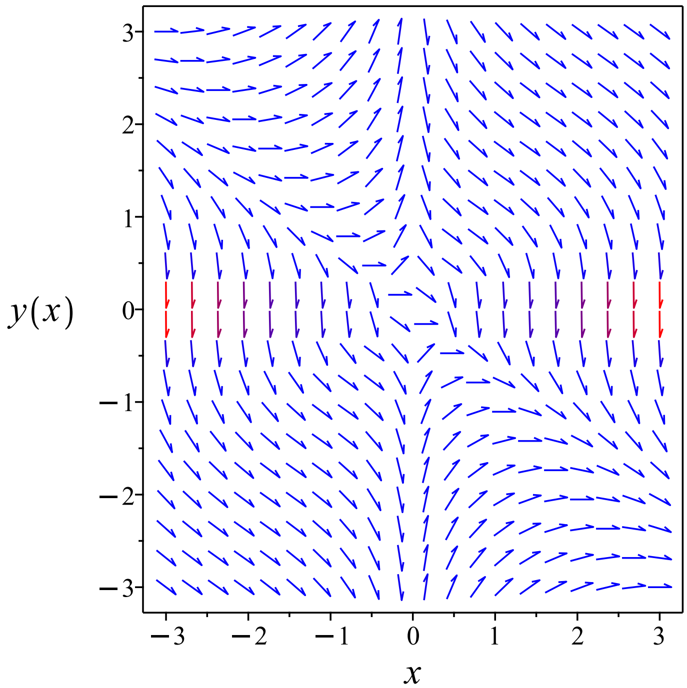


Figure 81: Slope field plot

Verification of solutions

$$\left(\frac{4y^3+x^3}{x^3}\right)^{\frac{1}{12}} = \frac{c_3 e^{c_2}}{x^{\frac{1}{3}}}$$

Verified OK.

2.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x^3+y^3}{3y^2x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{y^2 x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x}} dy \end{aligned}$$

Which results in

$$S = \frac{x\, y^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^3 + y^3}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3}{3} \\ S_y &= y^2 x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^3}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^3}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{12} + c_1 \tag{4}$$

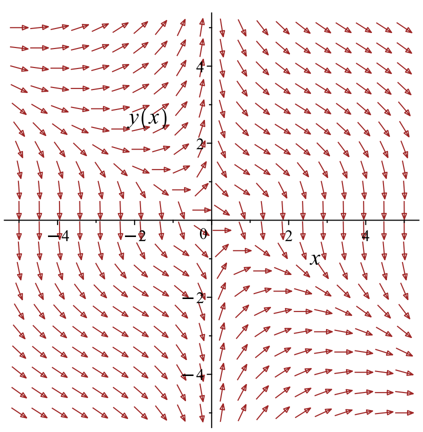
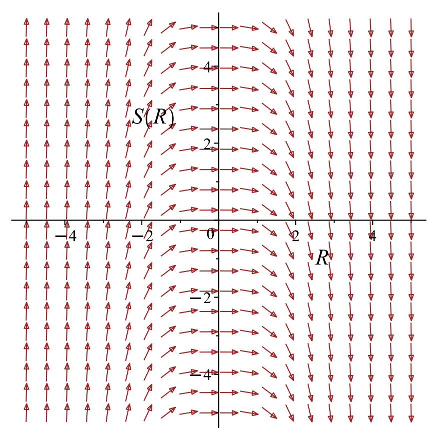
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy^3}{3} = -\frac{x^4}{12} + c_1$$

Which simplifies to

$$\frac{xy^3}{3} = -\frac{x^4}{12} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^3+y^3}{3y^2x}$ 	$R = x$ $S = \frac{x\,y^3}{3}$	$\frac{dS}{dR} = -\frac{R^3}{3}$ 

Summary

The solution(s) found are the following

$$\frac{xy^3}{3} = -\frac{x^4}{12} + c_1 \tag{1}$$

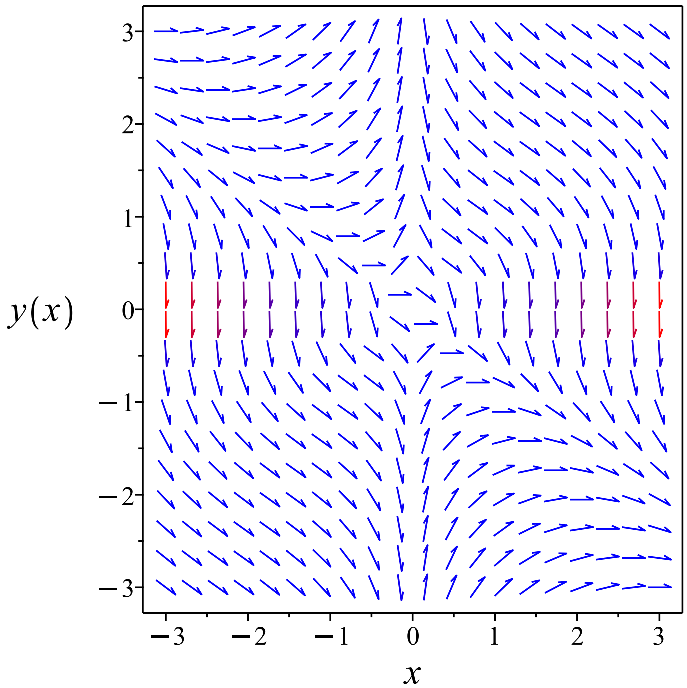


Figure 82: Slope field plot

Verification of solutions

$$\frac{xy^3}{3} = -\frac{x^4}{12} + c_1$$

Verified OK.

2.19.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^3 + y^3}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{3x}y - \frac{x^2}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{3x} \\
f_1(x) &= -\frac{x^2}{3} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = -\frac{y^3}{3x} - \frac{x^2}{3} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2 y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= -\frac{w(x)}{3x} - \frac{x^2}{3} \\
w' &= -\frac{w}{x} - x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(xw) &= (x)\left(-x^2\right) \\ \mathrm{d}(xw) &= \left(-x^3\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^3\,\mathrm{d}x \\ xw &= -\frac{x^4}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^3}{4} + \frac{c_1}{x}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -\frac{x^3}{4} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{2^{\frac{1}{3}}\left(-\left(x^4 - 4c_1\right)x^2\right)^{\frac{1}{3}}}{2x} \\ y(x) &= \frac{2^{\frac{1}{3}}\left(-\left(x^4 - 4c_1\right)x^2\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{4x} \\ y(x) &= -\frac{2^{\frac{1}{3}}\left(-\left(x^4 - 4c_1\right)x^2\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{4x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}}}{2x} \tag{1}$$

$$y = \frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4x} \tag{2}$$

$$y = -\frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x} \tag{3}$$

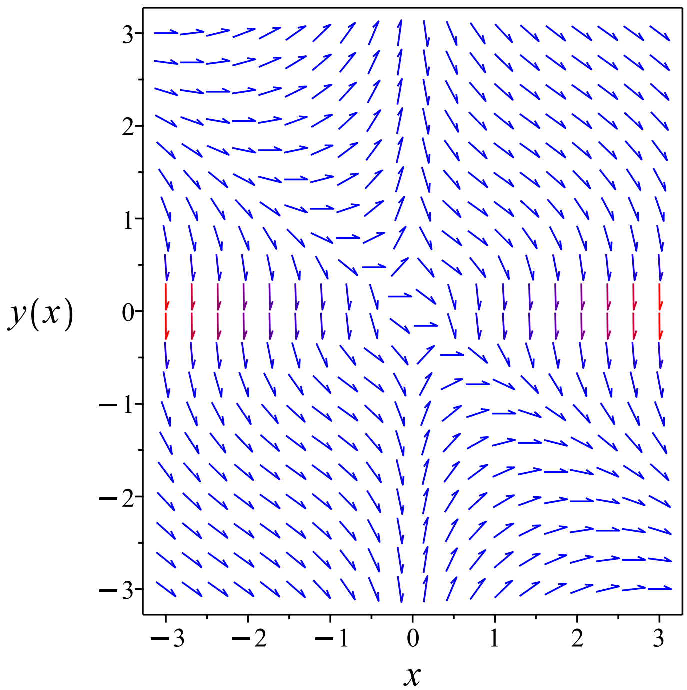


Figure 83: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}}}{2x}$$

Verified OK.

$$y = \frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{4x}$$

Verified OK.

$$y = -\frac{2^{\frac{1}{3}} \left(-\left(x^4 - 4c_1\right) x^2\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{4x}$$

Verified OK.

2.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(3y^2 x\right) \mathrm{d}y &= \left(-x^3 - y^3\right) \mathrm{d}x \\
\left(x^3 + y^3\right) \mathrm{d}x + \left(3y^2 x\right) \mathrm{d}y &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= x^3 + y^3 \\
N(x, y) &= 3y^2 x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3 + y^3\right) \\
&= 3y^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^2 x\right) \\
&= 3y^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int x^3 + y^3 \mathrm{d}x \\
\phi &= \frac{1}{4}x^4 + x\,y^3 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2x + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2x$. Therefore equation (4) becomes

$$3y^2x = 3y^2x + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{4}x^4 + x\,y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{4}x^4 + x\,y^3$$

Summary
The solution(s) found are the following

$$\frac{x^4}{4} + xy^3 = c_1 \qquad (1)$$

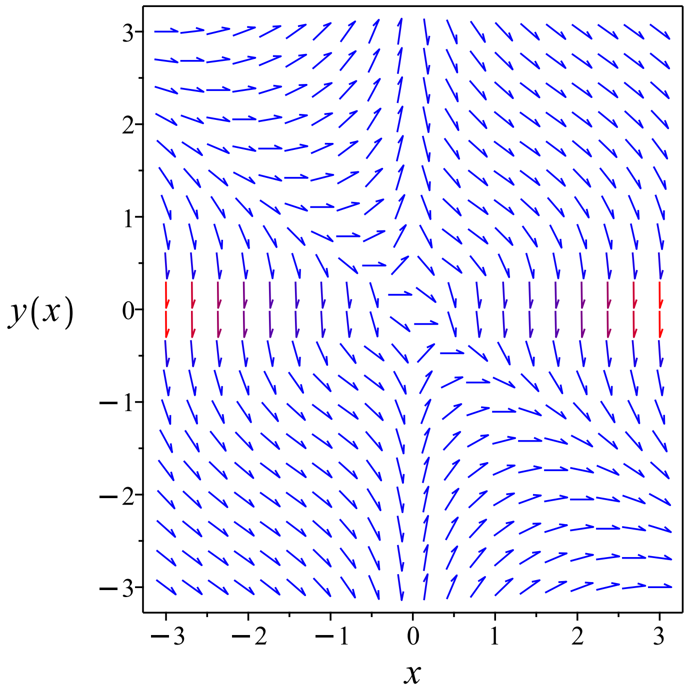


Figure 84: Slope field plot

Verification of solutions

$$\frac{x^4}{4} + xy^3 = c_1$$

Verified OK.

2.19.5 Maple step by step solution

Let's solve

$y^3 + 3y^2y'x = -x^3$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ◦ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ◦ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $3y^2 = 3y^2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x^3+y^3)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^4}{4}+x\,y^3+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3y^2x=3y^2x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{1}{4}x^4+x\,y^3$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{4}x^4+x\,y^3=c_1$

- Solve for y

  $\left\{y=\frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{2x}, y=-\frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x}-\frac{\mathrm{I}\sqrt{3}\,((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x}, y=-\frac{((-2x^4+8c_1)x^2)^{\frac{1}{3}}}{4x}+\frac{\mathrm{I}\sqrt{3}\,((-2x^4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 92

```
dsolve((x^3+y(x)^3)+3*x*y(x)^2*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{1}{3}} \left(-\left(x^{4}-4 c_{1}\right) x^{2}\right)^{\frac{1}{3}}}{2 x}$$
$$y(x) = -\frac{2^{\frac{1}{3}} \left(-\left(x^{4}-4 c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(1+i \sqrt{3}\right)}{4 x}$$
$$y(x) = \frac{2^{\frac{1}{3}} \left(-\left(x^{4}-4 c_{1}\right) x^{2}\right)^{\frac{1}{3}} \left(i \sqrt{3}-1\right)}{4 x}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 99

```
DSolve[(x^3+y[x]^3)+3*x*y[x]^2*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt[3]{-x^{4}+4 c_{1}}}{2^{2/3} \sqrt[3]{x}}$$
$$y(x) \to -\frac{\sqrt[3]{-1} \sqrt[3]{-x^{4}+4 c_{1}}}{2^{2/3} \sqrt[3]{x}}$$
$$y(x) \to \frac{(-1)^{2/3} \sqrt[3]{-x^{4}+4 c_{1}}}{2^{2/3} \sqrt[3]{x}}$$

2.20 problem 45

2.20.1 Solving as first order ode lie symmetry calculated ode 356

Internal problem ID [5255]
Internal file name [`OUTPUT/4746_Friday_February_02_2024_05_11_10_AM_65773297/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{2y-(3x+2y-1)\,y'=-3x-1}$$

2.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{3x+2y+1}{3x+2y-1}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(3x+2y+1)(b_3-a_2)}{3x+2y-1} - \frac{(3x+2y+1)^2 a_3}{(3x+2y-1)^2} \\
& - \left(\frac{3}{3x+2y-1} - \frac{3(3x+2y+1)}{(3x+2y-1)^2}\right)(xa_2+ya_3+a_1) \\
& - \left(\frac{2}{3x+2y-1} - \frac{2(3x+2y+1)}{(3x+2y-1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{9x^2a_2+9x^2a_3-9x^2b_2-9x^2b_3+12xya_2+12xya_3-12xyb_2-12xyb_3+4y^2a_2+4y^2a_3-4y^2b_2-4y^2b_3}{(3x+2y-1)^2}
$$
$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
& -9x^2a_2-9x^2a_3+9x^2b_2+9x^2b_3-12xya_2-12xya_3+12xyb_2 \\
& +12xyb_3-4y^2a_2-4y^2a_3+4y^2b_2+4y^2b_3+6xa_2-6xa_3-2xb_2 \\
& +2ya_3-4yb_2+4yb_3+6a_1+a_2-a_3+4b_1+b_2-b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -9a_2v_1^2-12a_2v_1v_2-4a_2v_2^2-9a_3v_1^2-12a_3v_1v_2-4a_3v_2^2+9b_2v_1^2 \\
& +12b_2v_1v_2+4b_2v_2^2+9b_3v_1^2+12b_3v_1v_2+4b_3v_2^2+6a_2v_1-6a_3v_1 \\
& +2a_3v_2-2b_2v_1-4b_2v_2+4b_3v_2+6a_1+a_2-a_3+4b_1+b_2-b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-9a_2 - 9a_3 + 9b_2 + 9b_3)\, v_1^2 + (-12a_2 - 12a_3 + 12b_2 + 12b_3)\, v_1 v_2 \\
&+ (6a_2 - 6a_3 - 2b_2)\, v_1 + (-4a_2 - 4a_3 + 4b_2 + 4b_3)\, v_2^2 \\
&+ (2a_3 - 4b_2 + 4b_3)\, v_2 + 6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_2 - 6a_3 - 2b_2 &= 0 \\
2a_3 - 4b_2 + 4b_3 &= 0 \\
-12a_2 - 12a_3 + 12b_2 + 12b_3 &= 0 \\
-9a_2 - 9a_3 + 9b_2 + 9b_3 &= 0 \\
-4a_2 - 4a_3 + 4b_2 + 4b_3 &= 0 \\
6a_1 + a_2 - a_3 + 4b_1 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= -9a_1 - 6b_1 \\
a_3 &= -6a_1 - 4b_1 \\
b_1 &= b_1 \\
b_2 &= -9a_1 - 6b_1 \\
b_3 &= -6a_1 - 4b_1
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -6x - 4y \\
\eta &= -6x - 4y + 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -6x - 4y + 1 - \left(\frac{3x + 2y + 1}{3x + 2y - 1}\right)(-6x - 4y) \\
&= \frac{15x + 10y - 1}{3x + 2y - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{15x+10y-1}{3x+2y-1}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y}{5} - \frac{2 \ln (15x + 10y - 1)}{25}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{3x + 2y + 1}{3x + 2y - 1}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{6}{75x + 50y - 5} \\
S_y &= \frac{3x + 2y - 1}{15x + 10y - 1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{5} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{5}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R}{5} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{5} - \frac{2\ln\left(15x + 10y - 1\right)}{25} = \frac{x}{5} + c_1
$$

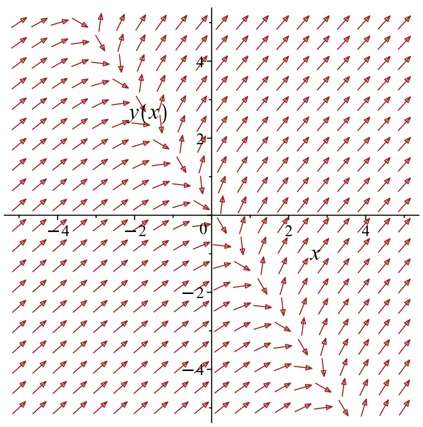
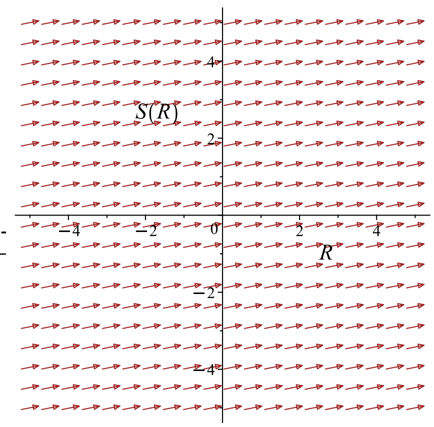
Which simplifies to

$$
\frac{y}{5} - \frac{2\ln\left(15x + 10y - 1\right)}{25} = \frac{x}{5} + c_1
$$

Which gives

$$
y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{25x}{4} + \frac{1}{4} - \frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3x+2y+1}{3x+2y-1}$ 	$R = x$ $S = \frac{y}{5} - \frac{2\ln\left(15x + 10y -\right.}{25}$	$\frac{dS}{dR} = \frac{1}{5}$ 

Summary

The solution(s) found are the following

$$y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{e^{-\frac{25x}{4}+\frac{1}{4}-\frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10} \tag{1}$$

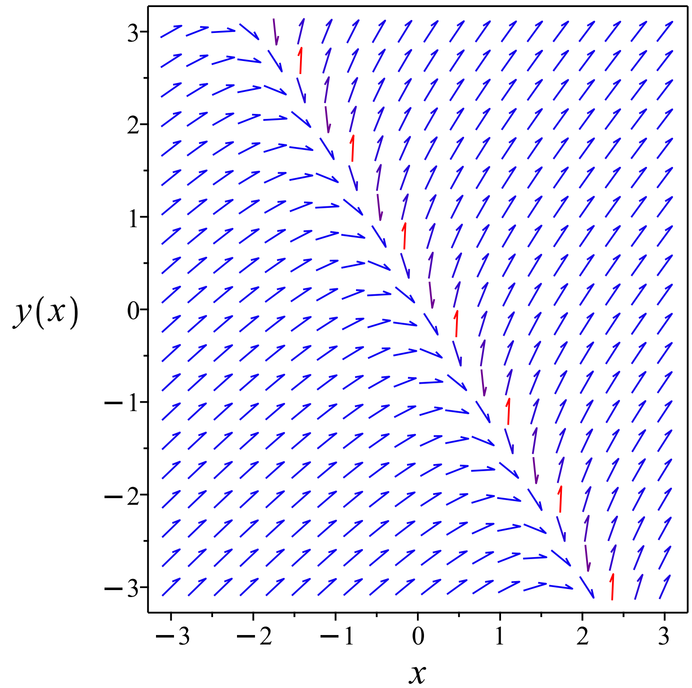


Figure 85: Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{25x}{4}+\frac{1}{4}-\frac{25c_1}{2}}}{4}\right)}{5} + \frac{1}{10}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -3/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve((3*x+2*y(x)+1)-(3*x+2*y(x)-1)*diff(y(x),x)= 0,y(x), singsol=all)
```

$$y(x) = -\frac{3x}{2} - \frac{2\,\mathrm{LambertW}\left(-\frac{c_1 e^{\frac{1}{4}-\frac{25x}{4}}}{4}\right)}{5} + \frac{1}{10}$$

✓ Solution by Mathematica

Time used: 4.841 (sec). Leaf size: 43

```
DSolve[(3*x+2*y[x]+1)-(3*x+2*y[x]-1)*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}\left(-4W\left(-e^{-\frac{25x}{4}-1+c_1}\right) - 15x + 1\right)$$
$$y(x) \to \frac{1}{10} - \frac{3x}{2}$$

2.21 problem 46

2.21.1 Existence and uniqueness analysis 365
2.21.2 Solving as separable ode 365
2.21.3 Solving as linear ode 367
2.21.4 Solving as homogeneousTypeD2 ode 368
2.21.5 Solving as first order ode lie symmetry lookup ode 370
2.21.6 Solving as exact ode 375
2.21.7 Maple step by step solution 379

Internal problem ID [5256]
Internal file name [`OUTPUT/4747_Friday_February_02_2024_05_11_12_AM_99165738/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' + 2y = 0}$$

With initial conditions

$$[y(2) = 1]$$

2.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 0
\end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x} = 0$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. Hence solution exists and is unique.

2.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{2y}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= -\frac{2}{x}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{2}{x}\, dx \\
\ln(y) &= -2\ln(x) + c_1 \\
y &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

The solutions are

$$c_1 = 4$$

Trying the constant

$$c_1 = 4$$

Substituting this in the general solution gives

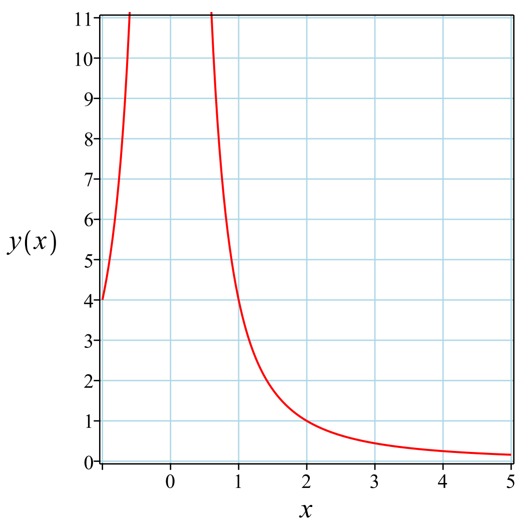
$$y = \frac{4}{x^2}$$

The constant $c_1 = 4$ gives valid solution.

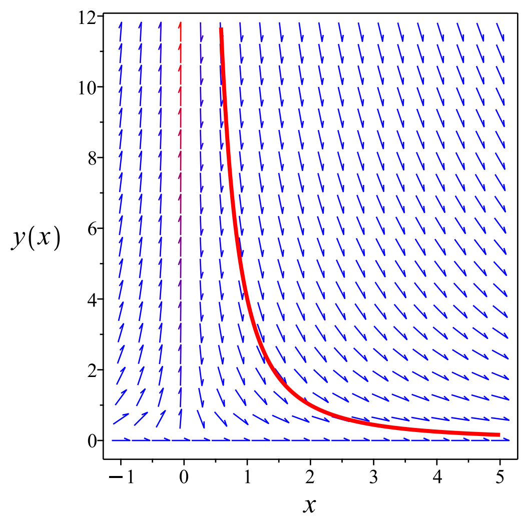
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(x^2 y\right) &= 0\end{aligned}$$

Integrating gives

$$x^2 y = c_1$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

The solutions are

$$c_1 = 4$$

Trying the constant

$$c_1 = 4$$

Substituting this in the general solution gives

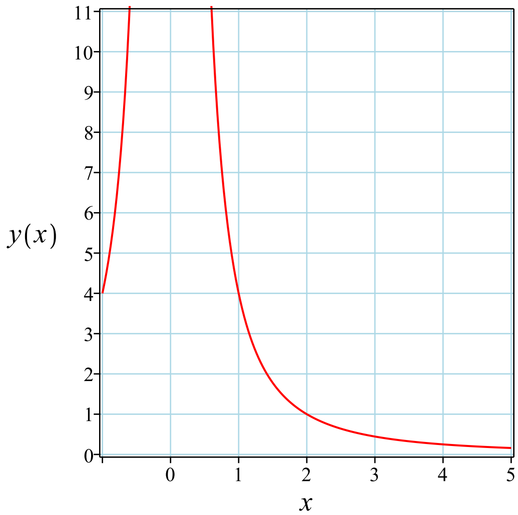
$$y = \frac{4}{x^2}$$

The constant $c_1 = 4$ gives valid solution.

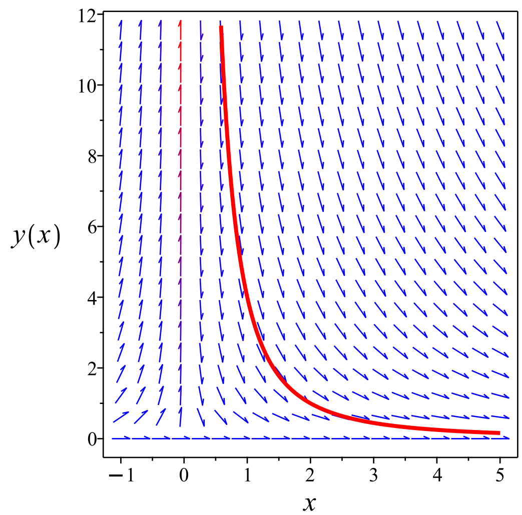
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) + 2u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_2 \\
u &= \mathrm{e}^{-3\ln(x)+c_2} \\
&= \frac{c_2}{x^3}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{c_2}{x^2}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2}{4}$$

The solutions are

$$c_2 = 4$$

Trying the constant

$$c_2 = 4$$

Substituting this in the general solution gives

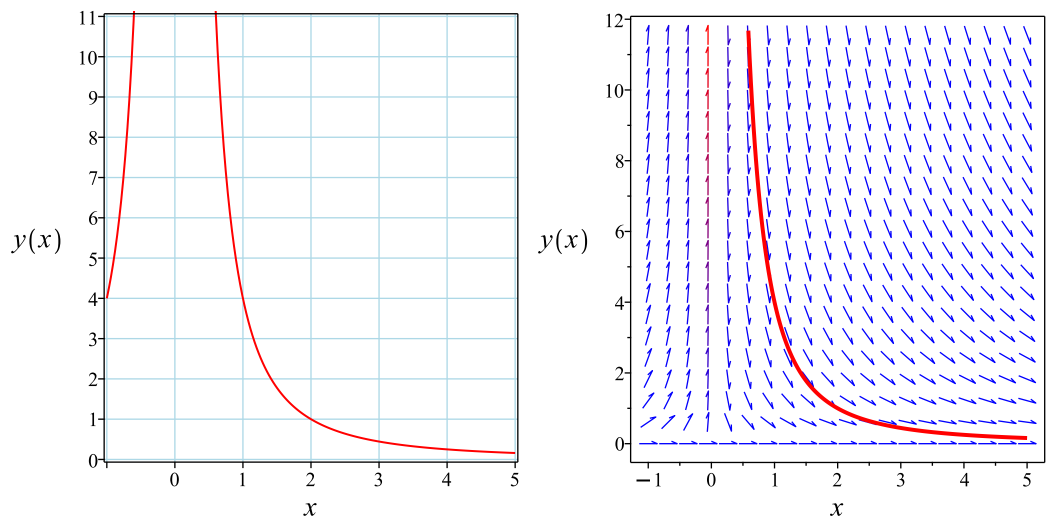
$$y = \frac{4}{x^2}$$

The constant $c_2 = 4$ gives valid solution.

Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = x^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = c_1$$

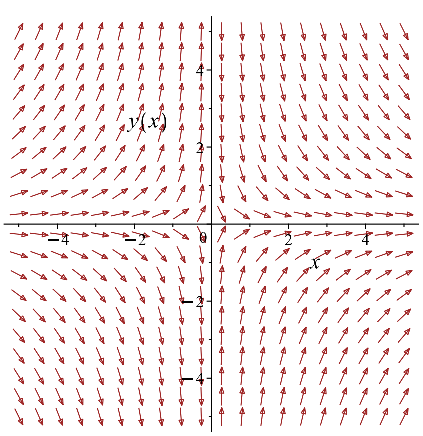
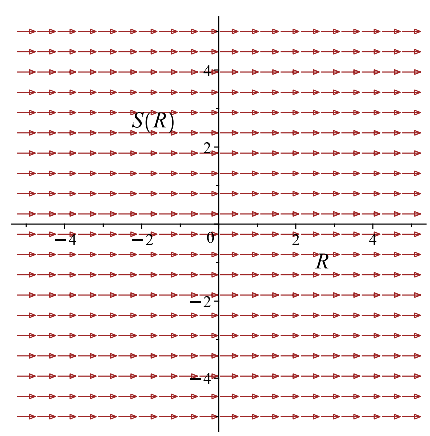
Which simplifies to

$$x^2 y = c_1$$

Which gives

$$y = \frac{c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{x}$ 	$R = x$ $S = x^2 y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4}$$

The solutions are

$$c_1 = 4$$

Trying the constant

$$c_1 = 4$$

Substituting this in the general solution gives

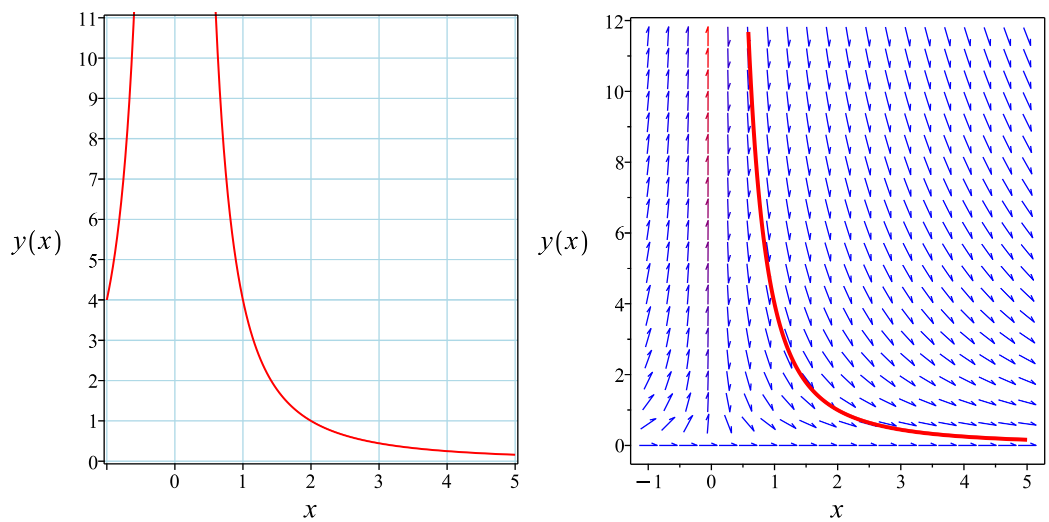
$$y = \frac{4}{x^2}$$

The constant $c_1 = 4$ gives valid solution.

Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{2y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{1}{2y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= -\frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x \\
\phi &= -\ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$
-\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \,\mathrm{d}y = \int \left(-\frac{1}{2y}\right) \mathrm{d}y
$$

$$
f(y) = -\frac{\ln(y)}{2} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\ln(x) - \frac{\ln(y)}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\ln(x) - \frac{\ln(y)}{2}
$$

The solution becomes

$$
y = \frac{\mathrm{e}^{-2c_1}}{x^2}
$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{\mathrm{e}^{-2c_1}}{4}
$$

The solutions are

$$
c_1 = -\ln(2)
$$

Trying the constant

$$c_1 = -\ln(2)$$

Substituting this in the general solution gives

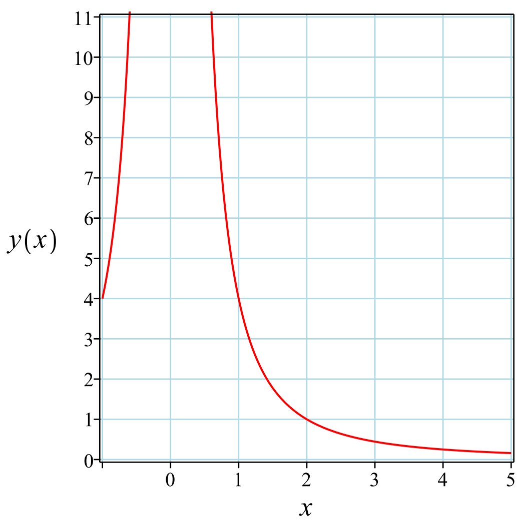
$$y = \frac{4}{x^2}$$

The constant $c_1 = -\ln(2)$ gives valid solution.

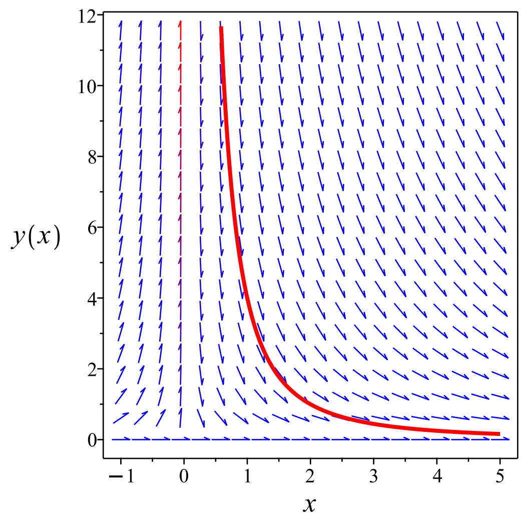
Summary

The solution(s) found are the following

$$y = \frac{4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{4}{x^2}$$

Verified OK.

2.21.7 Maple step by step solution

Let's solve

$[xy' + 2y = 0, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int -\frac{2}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = -2\ln(x) + c_1$

- Solve for y

  $y = \frac{e^{c_1}}{x^2}$

- Use initial condition $y(2) = 1$

  $1 = \frac{e^{c_1}}{4}$

- Solve for c_1

  $c_1 = 2\ln(2)$

- Substitute $c_1 = 2\ln(2)$ into general solution and simplify

  $y = \frac{4}{x^2}$

- Solution to the IVP

  $y = \frac{4}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)+2*y(x)= 0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{4}{x^2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 10

```
DSolve[{x*y'[x]+2*y[x]==0,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{x^2}$$

2.22 problem 47

2.22.1 Existence and uniqueness analysis 381
2.22.2 Solving as homogeneousTypeD2 ode 382
2.22.3 Solving as first order ode lie symmetry lookup ode 384
2.22.4 Solving as bernoulli ode 389
2.22.5 Solving as exact ode 392

Internal problem ID [5257]
Internal file name [`OUTPUT/4748_Friday_February_02_2024_05_11_13_AM_48587445/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 + yy'x = -x^2}$$

With initial conditions

$$[y(1) = -1]$$

2.22.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2 + y^2}{xy}\right) \\ &= -\frac{2}{x} + \frac{x^2 + y^2}{x\,y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

2.22.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + u(x)\,x^2(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u^2 + 1}{ux}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{2u^2+1}{u}}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\frac{2u^2+1}{u}}\,du &= \int -\frac{1}{x}\,dx \\ \frac{\ln(2u^2 + 1)}{4} &= -\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\left(2u^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(2u^2+1\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$
$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Substituting initial conditions and solving for c_2 gives $c_2 = \frac{\ln\left(\frac{3}{c_3^4}\right)}{4}$. Hence the solution becomes Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$3^{\frac{1}{4}} = c_3 \sqrt{\sqrt{3}\,\sqrt{\frac{1}{c_3^4}}}$$

The solutions are

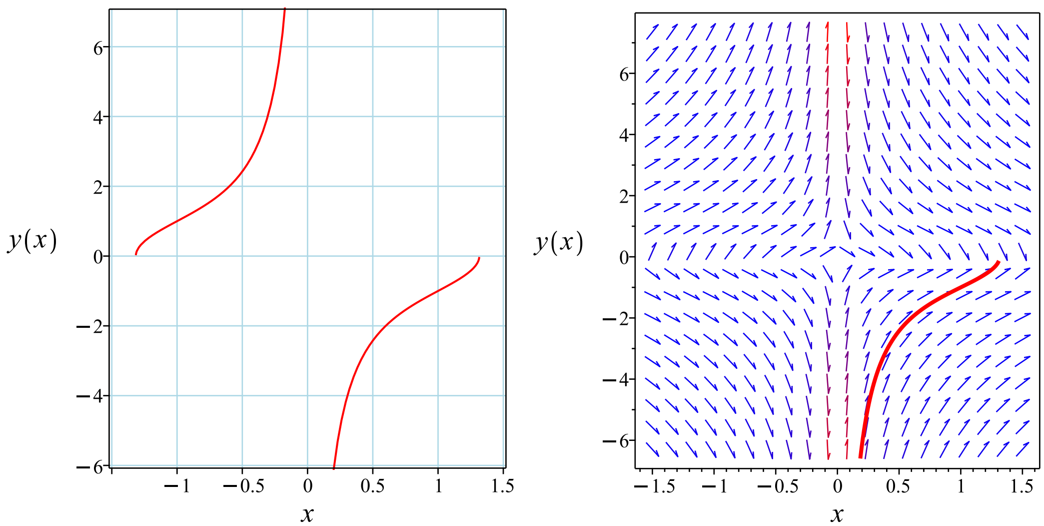
$$c_3 = c_3$$

Since $\lim_{c_1 \to \infty}$ gives $\left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \sqrt{\sqrt{3}\,\sqrt{\frac{1}{c_3^4}}}}{x} = \left(\frac{x^2+2y^2}{x^2}\right)^{\frac{1}{4}} = \frac{3^{\frac{1}{4}}}{x}$ and this result satisfies the given initial condition. Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4+6}}{2x}$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4+6}}{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4+6}}{2x}$$

Verified OK.

2.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y\, x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y\,x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^2 x \\ S_y &= x^2 y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \tag{4}$$

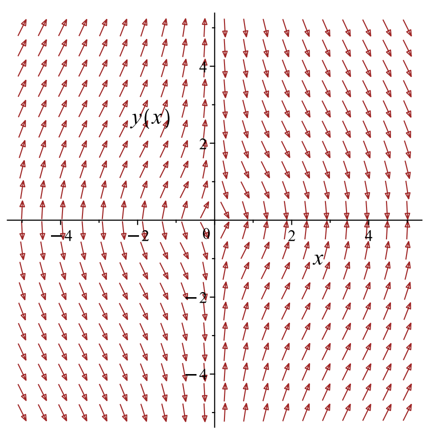
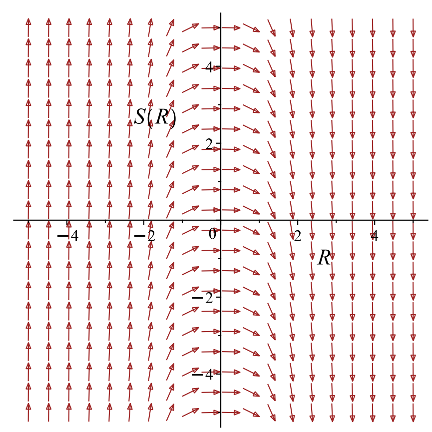
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2 y^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{x^2 y^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{x^2 y^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = -\frac{1}{4} + c_1$$

The solutions are

$$c_1 = \frac{3}{4}$$

Trying the constant

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + \frac{3}{4}$$

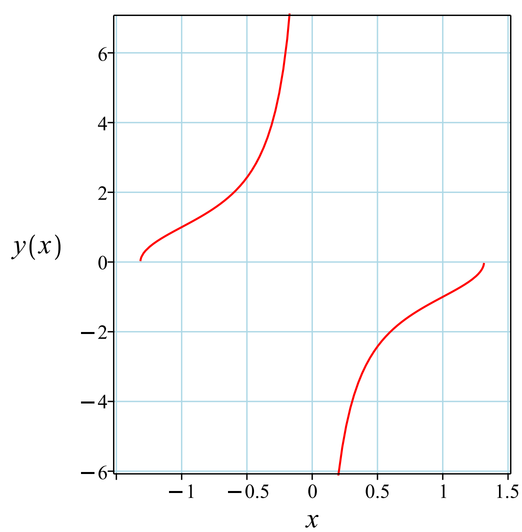
The constant $c_1 = \frac{3}{4}$ gives valid solution.

Solving for y from the above gives

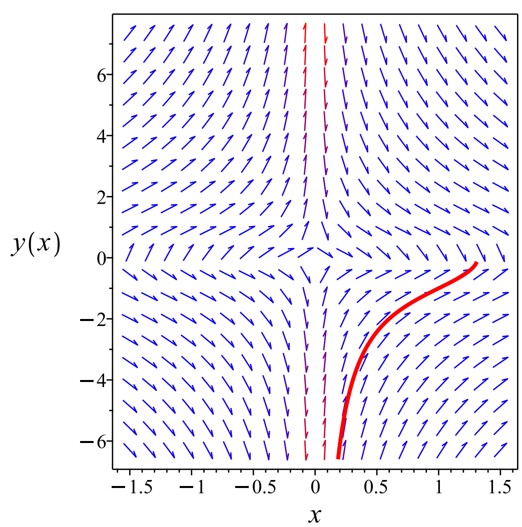
$$y = -\frac{\sqrt{-2x^4+6}}{2x}$$

Summary
The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4+6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4+6}}{2x}$$

Verified OK.

2.22.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{x} - x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\
w' &= -\frac{2w}{x} - 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{2w(x)}{x} = -2x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{2}{x} dx} \\
&= x^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)(-2x) \\
\mathrm{d}(x^2 w) &= (-2x^3)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^2 w &= \int -2x^3 \,\mathrm{d}x \\
x^2 w &= -\frac{x^4}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$
w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{2} + c_1$$

The solutions are

$$c_1 = \frac{3}{2}$$

Trying the constant

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$y^2 = -\frac{x^4 - 3}{2x^2}$$

The above simplifies to

$$x^4 + 2x^2y^2 - 3 = 0$$

The constant $c_1 = \frac{3}{2}$ gives valid solution.

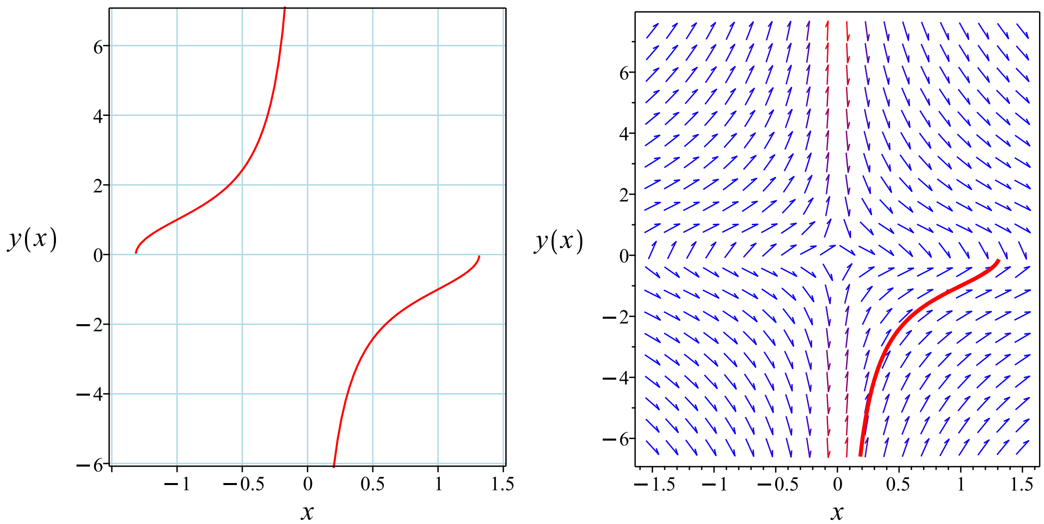
Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y=-\frac{\sqrt{-2x^4+6}}{2x}$$

Verified OK.

2.22.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x \\ \left(x^2 + y^2\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 + y^2 \\ N(x, y) &= xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) \\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{yx}((2y) - (y)) \\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x(x^2+y^2) \\ &= x(x^2+y^2)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(xy) \\ &= x^2 y\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(x(x^2+y^2)\right) + \left(x^2 y\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int x(x^2+y^2) \, \mathrm{d}x\end{aligned}$$

$$\phi = \frac{(x^2+y^2)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \left(x^2 + y^2\right) y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 y$. Therefore equation (4) becomes

$$x^2 y = \left(x^2 + y^2\right) y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(-y^3\right) \mathrm{d}y$$

$$f(y) = -\frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(x^2 + y^2\right)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\left(x^2 + y^2\right)^2}{4} - \frac{y^4}{4}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{4} = c_1$$

The solutions are

$$c_1 = \frac{3}{4}$$

Trying the constant

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = \frac{3}{4}$$

The constant $c_1 = \frac{3}{4}$ gives valid solution.

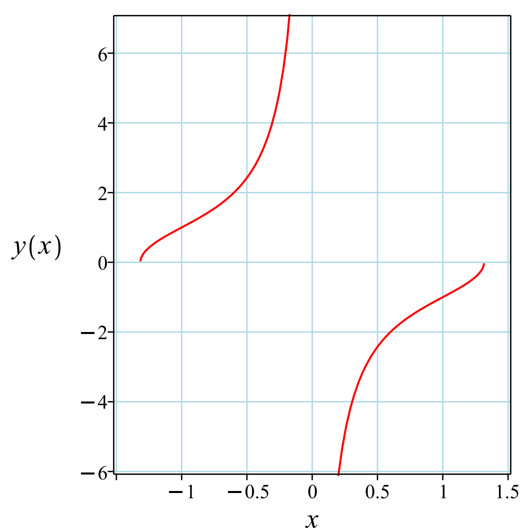
Solving for y from the above gives

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

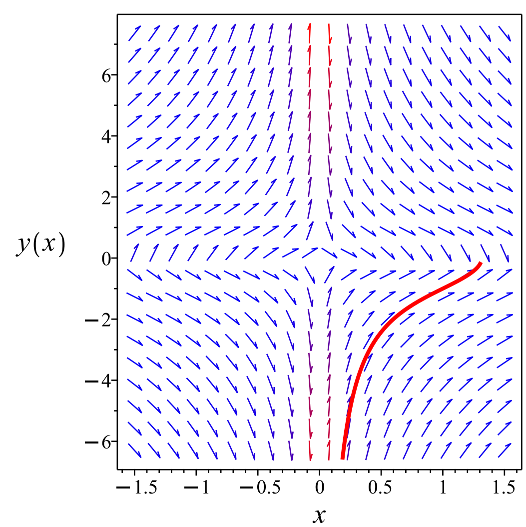
Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{-2x^4 + 6}}{2x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 18

```
dsolve([(x^2+y(x)^2)+x*y(x)*diff(y(x),x)= 0,y(1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^4+6}}{2x}$$

✓ Solution by Mathematica

Time used: 0.202 (sec). Leaf size: 26

```
DSolve[{(x^2+y[x]^2)+x*y[x]*y'[x]==0,{y[1]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{3-x^4}}{\sqrt{2}x}$$

2.23 problem 48

2.23.1 Existence and uniqueness analysis 398
2.23.2 Solving as separable ode 399
2.23.3 Solving as first order ode lie symmetry lookup ode 401
2.23.4 Solving as exact ode 405
2.23.5 Maple step by step solution 409

Internal problem ID [5258]
Internal file name [`OUTPUT/4749_Friday_February_02_2024_05_11_15_AM_62043747/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos(y) + (1 + e^{-x}) \sin(y)\, y' = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4}\right]$$

2.23.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{\cos(y)}{(1 + e^{-x}) \sin(y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{\pi}{4}$ is

$$\{-2i\pi\_Z173 - i\pi < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.23.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{\cot(y)}{1+\mathrm{e}^{-x}} \end{aligned}$$

Where $f(x) = -\frac{1}{1+\mathrm{e}^{-x}}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\cot(y)}\, dy &= -\frac{1}{1+\mathrm{e}^{-x}}\, dx \\ \int \frac{1}{\cot(y)}\, dy &= \int -\frac{1}{1+\mathrm{e}^{-x}}\, dx \\ -\ln(\cos(y)) &= -\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = \mathrm{e}^{-\ln(1+\mathrm{e}^{-x})+\ln(\mathrm{e}^{-x})+c_1}$$

Which simplifies to

$$\sec(y) = c_2\mathrm{e}^{-\ln(1+\mathrm{e}^{-x})+\ln(\mathrm{e}^{-x})}$$

Initial conditions are used to solve for c_1. Substituting $x=0$ and $y=\frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\frac{2\,\mathrm{e}^{-c_1}}{c_2}\right)$$

The solutions are

$$c_1 = -\frac{\ln\left(\frac{c_2^2}{8}\right)}{2}$$

Trying the constant

$$c_1 = -\frac{\ln\left(\frac{c_2^2}{8}\right)}{2}$$

Substituting this in the general solution gives

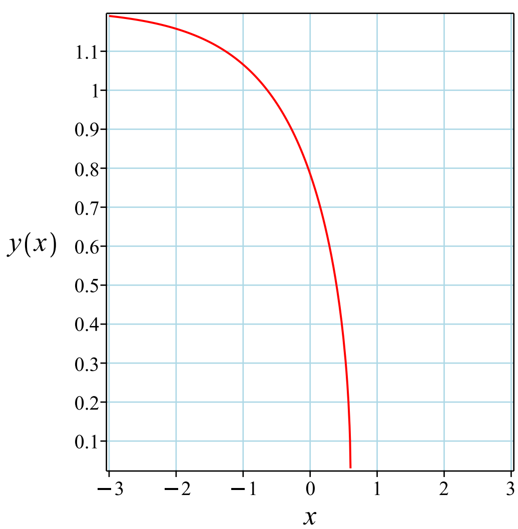
$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$$

The constant $c_1 = -\frac{\ln\left(\frac{c_2^2}{8}\right)}{2}$ gives valid solution.

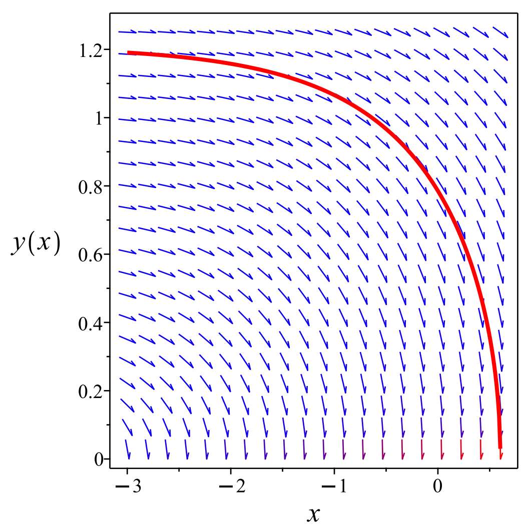
Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^x)\sqrt{2}}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$$

Verified OK. `{positive}`

2.23.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\cos(y)}{(1+\mathrm{e}^{-x})\sin(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= -1-\mathrm{e}^{-x}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R=y$$

S is found from

$$\begin{aligned}S &= \int\frac{1}{\xi}dx\\ &= \int\frac{1}{-1-\mathrm{e}^{-x}}dx\end{aligned}$$

Which results in

$$S=-\ln\left(1+\mathrm{e}^{-x}\right)+\ln\left(\mathrm{e}^{-x}\right)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=-\frac{\cos(y)}{(1+\mathrm{e}^{-x})\sin(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{1+\mathrm{e}^{-x}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \tan(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \tan(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(\cos(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln\left(1+\mathrm{e}^{-x}\right) - x = -\ln(\cos(y)) + c_1
$$

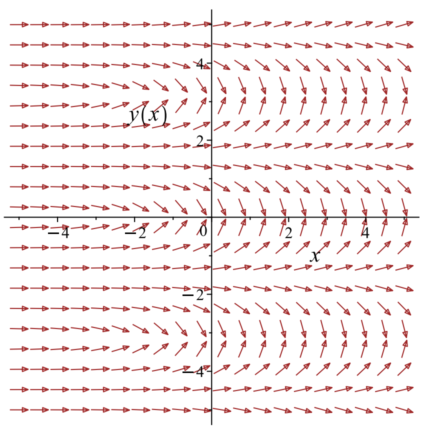
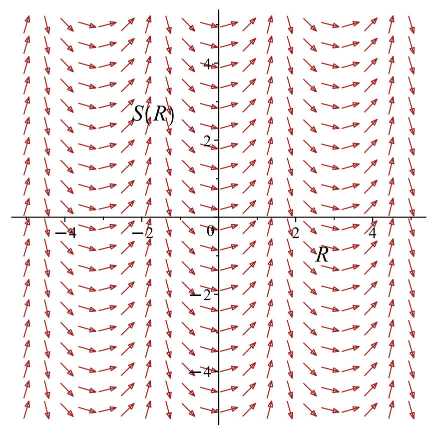
Which simplifies to

$$
-\ln\left(1+\mathrm{e}^{-x}\right) - x = -\ln(\cos(y)) + c_1
$$

Which gives

$$
y = \arccos\left(\mathrm{e}^{x+c_1}\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{-x}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\cos(y)}{(1+e^{-x})\sin(y)}$ 	$R = y$ $S = -\ln(1 + e^{-x}) - x$	$\frac{dS}{dR} = \tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(2\,e^{c_1}\right)$$

The solutions are

$$c_1 = -\frac{3\ln(2)}{2}$$

Trying the constant

$$c_1 = -\frac{3\ln(2)}{2}$$

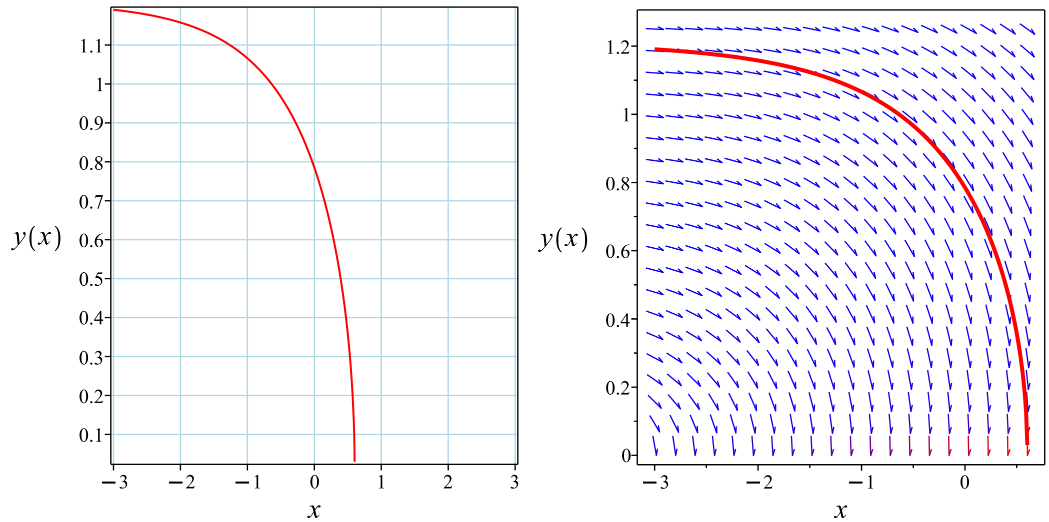
Substituting this in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right)$$

The constant $c_1 = -\frac{3\ln(2)}{2}$ gives valid solution.

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + e^x)\sqrt{2}}{4}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^{x})\sqrt{2}}{4}\right)$$

Verified OK. `{positive}`

2.23.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{\sin(y)}{\cos(y)}\right) \mathrm{d}y = \left(\frac{1}{1+\mathrm{e}^{-x}}\right) \mathrm{d}x$$
$$\left(-\frac{1}{1+\mathrm{e}^{-x}}\right) \mathrm{d}x + \left(-\frac{\sin(y)}{\cos(y)}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{1}{1+\mathrm{e}^{-x}}$$
$$N(x,y) = -\frac{\sin(y)}{\cos(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{1+\mathrm{e}^{-x}}\right) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{\sin(y)}{\cos(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{1+\mathrm{e}^{-x}}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{\sin(y)}{\cos(y)}$. Therefore equation (4) becomes

$$-\frac{\sin(y)}{\cos(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= -\frac{\sin(y)}{\cos(y)}\\ &= -\tan(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (-\tan(y))\,\mathrm{d}y$$
$$f(y) = \ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + \ln(\cos(y))$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{3\ln(2)}{2} = c_1$$

The solutions are

$$c_1 = -\frac{3\ln(2)}{2}$$

Trying the constant

$$c_1 = -\frac{3\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\ln\left(1+\mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + \ln(\cos(y)) = -\frac{3\ln(2)}{2}$$

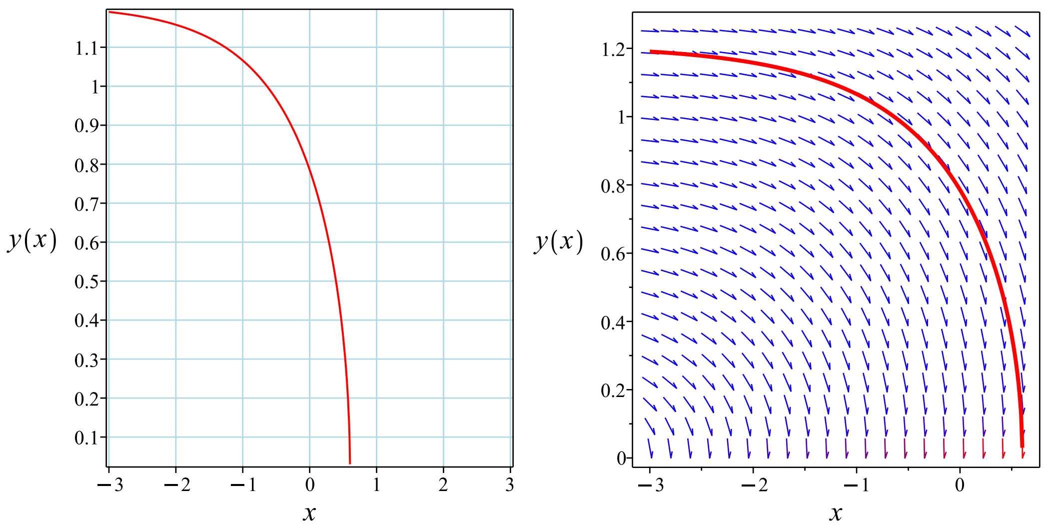
The constant $c_1 = -\frac{3\ln(2)}{2}$ gives valid solution.

Solving for y from the above gives

$$y = \arccos\left(\frac{(1+\mathrm{e}^{x})\sqrt{2}}{4}\right)$$

Summary
The solution(s) found are the following

$$y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$$

Verified OK. `{positive}`

2.23.5 Maple step by step solution

Let's solve

$\left[\cos(y) + (1+e^{-x})\sin(y)\, y' = 0, y(0) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{\cos(y)} = -\frac{1}{1+e^{-x}}$

- Integrate both sides with respect to x

$\int \frac{y' \sin(y)}{\cos(y)} dx = \int -\frac{1}{1+e^{-x}} dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = -\ln(1+e^{-x}) + \ln(e^{-x}) + c_1$

- Solve for y

  $y = \arccos\left(\frac{e^{-c_1+x}(1+e^x)}{e^x}\right)$

- Use initial condition $y(0) = \frac{\pi}{4}$

  $\frac{\pi}{4} = \arccos(2\,e^{-c_1})$

- Solve for c_1

  $c_1 = \frac{3\ln(2)}{2}$

- Substitute $c_1 = \frac{3\ln(2)}{2}$ into general solution and simplify

  $y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$

- Solution to the IVP

  $y = \arccos\left(\frac{(1+e^x)\sqrt{2}}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 14

```
dsolve([cos(y(x))+(1+exp(-x))*sin(y(x))*diff(y(x),x)= 0,y(0) = 1/4*Pi],y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\sqrt{2}\,(e^x + 1)}{4}\right)$$

✓ Solution by Mathematica

Time used: 46.229 (sec). Leaf size: 20

```
DSolve[{Cos[y[x]]+(1+Exp[-x])*Sin[y[x]]*y'[x]== 0,{y[0]==Pi/4}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to \arccos\left(\frac{e^x+1}{2\sqrt{2}}\right)$$

2.24 problem 49

2.24.1 Existence and uniqueness analysis 412
2.24.2 Solving as first order ode lie symmetry lookup ode 413
2.24.3 Solving as bernoulli ode 418
2.24.4 Solving as riccati ode 421

Internal problem ID [5259]
Internal file name [`OUTPUT/4750_Friday_February_02_2024_05_11_16_AM_8396201/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_rational, _Bernoulli]
```

$$\boxed{y^2 + yx - xy' = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.24.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y(x+y)}{x} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y(x+y)}{x}\right)\\ &= \frac{x+y}{x} + \frac{y}{x}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y(x+y)}{x}\\ y' &= \omega(x, y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x} y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x} y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x+y)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\mathrm{e}^x}{y} \\ S_y &= \frac{\mathrm{e}^x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^x}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\mathrm{e}^R}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\operatorname{expIntegral}_1(-R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^x}{y} = -\operatorname{expIntegral}_1(-x) + c_1$$

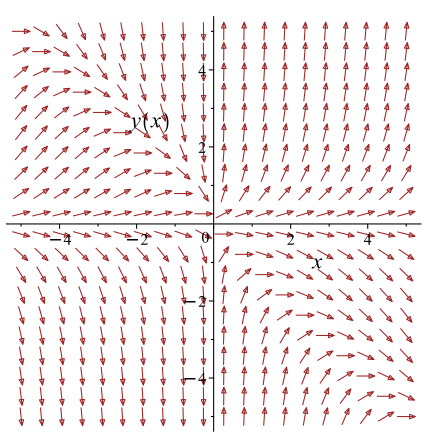
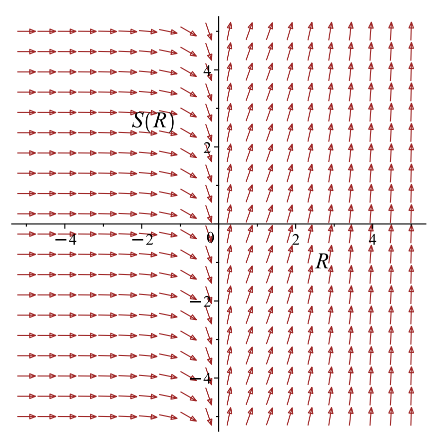
Which simplifies to

$$-\frac{e^x}{y} = -\operatorname{expIntegral}_1(-x) + c_1$$

Which gives

$$y = \frac{e^x}{\operatorname{expIntegral}_1(-x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x+y)}{x}$ 	$R = x$ $S = -\frac{e^x}{y}$	$\frac{dS}{dR} = \frac{e^R}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\mathrm{e}}{\mathrm{expIntegral}_1(-1) - c_1}$$

The solutions are

$$c_1 = \mathrm{expIntegral}_1(-1) - \mathrm{e}$$

Trying the constant

$$c_1 = \mathrm{expIntegral}_1(-1) - \mathrm{e}$$

Substituting this in the general solution gives

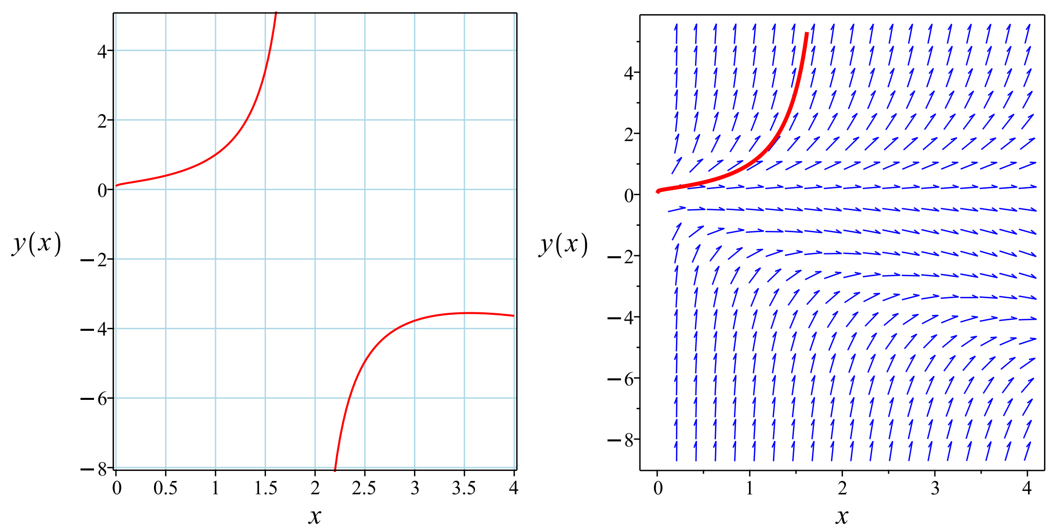
$$y = -\frac{\mathrm{e}^x}{-\mathrm{expIntegral}_1(-x) + \mathrm{expIntegral}_1(-1) - \mathrm{e}}$$

The constant $c_1 = \mathrm{expIntegral}_1(-1) - \mathrm{e}$ gives valid solution.

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^x}{-\mathrm{expIntegral}_1(-x) + \mathrm{expIntegral}_1(-1) - \mathrm{e}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^x}{-\mathrm{expIntegral}_1(-x) + \mathrm{expIntegral}_1(-1) - \mathrm{e}}$$

Verified OK.

2.24.3 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(x+y)}{x}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = y + \frac{1}{x}y^2 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= \frac{1}{x} \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{1}{y} + \frac{1}{x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= w(x) + \frac{1}{x} \\
w' &= -w - \frac{1}{x}
\end{aligned}
\tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -\frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + w(x) = -\frac{1}{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1dx} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{1}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x} w) &= (\mathrm{e}^{x})\left(-\frac{1}{x}\right) \\
\mathrm{d}(\mathrm{e}^{x} w) &= \left(-\frac{\mathrm{e}^{x}}{x}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x} w &= \int -\frac{\mathrm{e}^{x}}{x} \, \mathrm{d}x \\
\mathrm{e}^{x} w &= \mathrm{expIntegral}_1(-x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$
w(x) = \mathrm{e}^{-x} \,\mathrm{expIntegral}_1(-x) + c_1 \mathrm{e}^{-x}
$$

which simplifies to

$$
w(x) = \mathrm{e}^{-x}(\operatorname{expIntegral}_1(-x) + c_1)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \mathrm{e}^{-x}(\operatorname{expIntegral}_1(-x) + c_1)
$$

Or

$$
y = \frac{\mathrm{e}^{x}}{\operatorname{expIntegral}_1(-x) + c_1}
$$

Which is simplified to

$$
y = \frac{\mathrm{e}^{x}}{\operatorname{expIntegral}_1(-x) + c_1}
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = \frac{\mathrm{e}}{\operatorname{expIntegral}_1(-1) + c_1}
$$

The solutions are

$$
c_1 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}
$$

Trying the constant

$$
c_1 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}
$$

Substituting this in the general solution gives

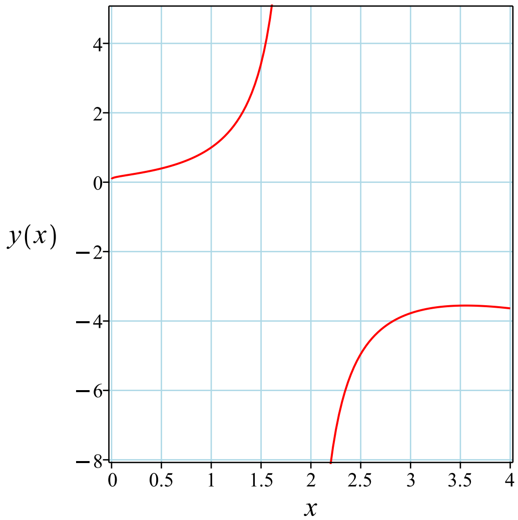
$$
y = -\frac{\mathrm{e}^{x}}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - \mathrm{e}}
$$

The constant $c_1 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}$ gives valid solution.

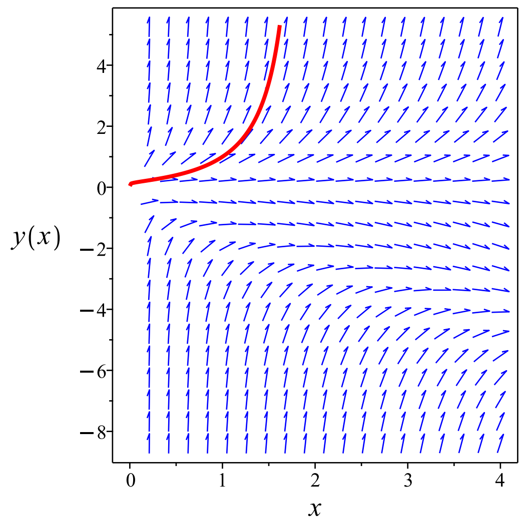
Summary

The solution(s) found are the following

$$
y = -\frac{\mathrm{e}^{x}}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - \mathrm{e}} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{e^x}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - e}$$

Verified OK.

2.24.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x+y)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y + \frac{y^2}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 1$ and $f_2(x) = \frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{1}{x^2} \\ f_1 f_2 &= \frac{1}{x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x} - \left(-\frac{1}{x^2} + \frac{1}{x}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \text{expIntegral}_1\,(-x)\, c_2$$

The above shows that

$$u'(x) = -\frac{e^x c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{e^x c_2}{c_1 + \text{expIntegral}_1\,(-x)\, c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{e^x}{c_3 + \text{expIntegral}_1\,(-x)}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{e}{c_3 + \text{expIntegral}_1\,(-1)}$$

The solutions are

$$c_3 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}$$

Trying the constant

$$c_3 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}$$

Substituting this in the general solution gives

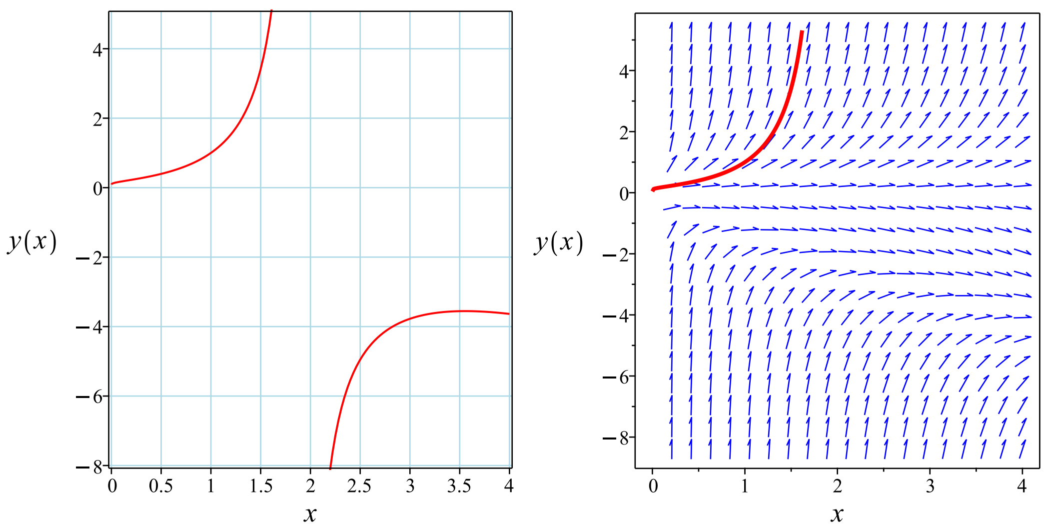
$$y = -\frac{\mathrm{e}^x}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - \mathrm{e}}$$

The constant $c_3 = -\operatorname{expIntegral}_1(-1) + \mathrm{e}$ gives valid solution.

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^x}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - \mathrm{e}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^x}{-\operatorname{expIntegral}_1(-x) + \operatorname{expIntegral}_1(-1) - \mathrm{e}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 22

```
dsolve([(y(x)^2+x*y(x))-x*diff(y(x),x)= 0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{x}}{\mathrm{expIntegral}_1\left(-x\right) + \mathrm{e} - \mathrm{expIntegral}_1\left(-1\right)}$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 19

```
DSolve[{(y[x]^2+x*y[x])-x*y'[x]== 0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{x}}{-\operatorname{ExpIntegralEi}(x) + \operatorname{ExpIntegralEi}(1) + e}$$

2.25 problem 51

2.25.1 Solving as first order ode lie symmetry calculated ode 425
2.25.2 Solving as riccati ode 431

Internal problem ID [5260]
Internal file name [`OUTPUT/4751_Friday_February_02_2024_05_11_32_AM_46710413/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Supplemetary problems. Page 22
Problem number: 51.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' + 2(3y + 2x)^2 = 0}$$

2.25.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -2(3y + 2x)^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - 2(3y+2x)^2 (b_3 - a_2) - 4(3y+2x)^4 a_3 \\
&\quad - (-16x - 24y)(xa_2 + ya_3 + a_1) - (-36y - 24x)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\begin{aligned}
&-64x^4 a_3 - 384x^3 y a_3 - 864x^2 y^2 a_3 - 864x\, y^3 a_3 - 324y^4 a_3 + 24x^2 a_2 \\
&+ 24x^2 b_2 - 8x^2 b_3 + 48xya_2 + 16xya_3 + 36xyb_2 + 18y^2 a_2 \\
&+ 24y^2 a_3 + 18y^2 b_3 + 16xa_1 + 24xb_1 + 24ya_1 + 36yb_1 + b_2 = 0
\end{aligned}
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-64x^4 a_3 - 384x^3 y a_3 - 864x^2 y^2 a_3 - 864x\, y^3 a_3 - 324y^4 a_3 + 24x^2 a_2 \\
&+ 24x^2 b_2 - 8x^2 b_3 + 48xya_2 + 16xya_3 + 36xyb_2 + 18y^2 a_2 \\
&+ 24y^2 a_3 + 18y^2 b_3 + 16xa_1 + 24xb_1 + 24ya_1 + 36yb_1 + b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-64a_3 v_1^4 - 384a_3 v_1^3 v_2 - 864a_3 v_1^2 v_2^2 - 864a_3 v_1 v_2^3 - 324a_3 v_2^4 + 24a_2 v_1^2 \\
&+ 48a_2 v_1 v_2 + 18a_2 v_2^2 + 16a_3 v_1 v_2 + 24a_3 v_2^2 + 24b_2 v_1^2 + 36b_2 v_1 v_2 \\
&- 8b_3 v_1^2 + 18b_3 v_2^2 + 16a_1 v_1 + 24a_1 v_2 + 24b_1 v_1 + 36b_1 v_2 + b_2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-64a_3v_1^4 - 384a_3v_1^3v_2 - 864a_3v_1^2v_2^2 + (24a_2 + 24b_2 - 8b_3)\,v_1^2 \\
&- 864a_3v_1v_2^3 + (48a_2 + 16a_3 + 36b_2)\,v_1v_2 + (16a_1 + 24b_1)\,v_1 \\
&- 324a_3v_2^4 + (18a_2 + 24a_3 + 18b_3)\,v_2^2 + (24a_1 + 36b_1)\,v_2 + b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-864a_3 &= 0 \\
-384a_3 &= 0 \\
-324a_3 &= 0 \\
-64a_3 &= 0 \\
16a_1 + 24b_1 &= 0 \\
24a_1 + 36b_1 &= 0 \\
18a_2 + 24a_3 + 18b_3 &= 0 \\
24a_2 + 24b_2 - 8b_3 &= 0 \\
48a_2 + 16a_3 + 36b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{3b_1}{2} \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{3}{2} \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(-2(3y+2x)^2\right)\left(-\frac{3}{2}\right) \\
&= -12x^2 - 36xy - 27y^2 + 1 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-12x^2 - 36xy - 27y^2 + 1} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\sqrt{3}\,\operatorname{arctanh}\left(\frac{(54y+36x)\sqrt{3}}{18}\right)}{9}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -2(3y+2x)^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{2}{3\left(1-3\left(3y+2x\right)^2\right)} \\
S_y &= \frac{1}{1-3\left(3y+2x\right)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{2}{3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{2}{3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2R}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\sqrt{3}\,\operatorname{arctanh}\left(\left(3y+2x\right)\sqrt{3}\right)}{9} = \frac{2x}{3} + c_1
$$

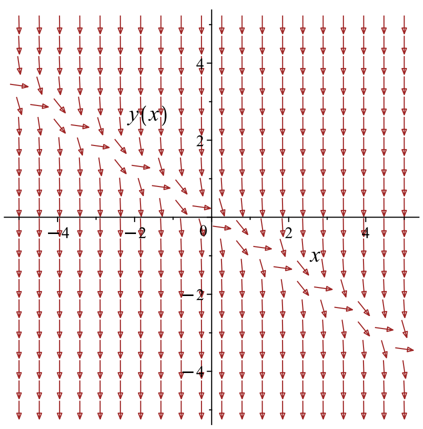
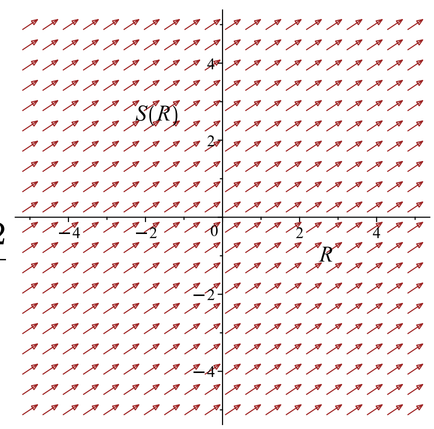
Which simplifies to

$$
\frac{\sqrt{3}\,\operatorname{arctanh}\left(\left(3y+2x\right)\sqrt{3}\right)}{9} = \frac{2x}{3} + c_1
$$

Which gives

$$
y = -\frac{\left(2\sqrt{3}\,x - \tanh\left(\left(3c_1+2x\right)\sqrt{3}\right)\right)\sqrt{3}}{9}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2(3y+2x)^2$ 	$R = x$ $S = \frac{\sqrt{3}\, \operatorname{arctanh}\left((3y+2\right.}{9}$	$\frac{dS}{dR} = \frac{2}{3}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\left(2\sqrt{3}\, x - \tanh\left(\left(3c_1 + 2x\right)\sqrt{3}\right)\right)\sqrt{3}}{9} \tag{1}$$

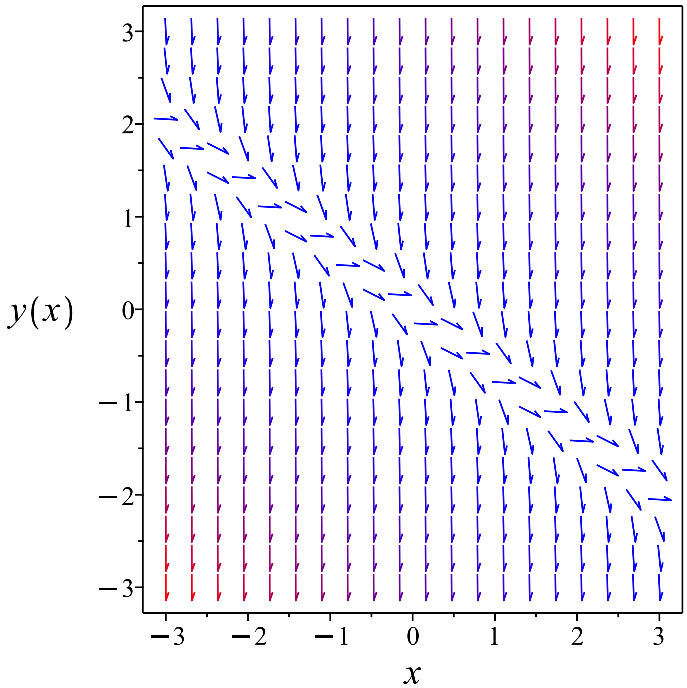


Figure 101: Slope field plot

Verification of solutions

$$y = -\frac{\left(2\sqrt{3}\, x - \tanh\left(\left(3c_1 + 2x\right)\sqrt{3}\right)\right)\sqrt{3}}{9}$$

Verified OK.

2.25.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= -2(3y + 2x)^2
\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -8x^2 - 24xy - 18y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -8x^2$, $f_1(x) = -24x$ and $f_2(x) = -18$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-18u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 0 \\
f_1 f_2 &= 432x \\
f_2^2 f_0 &= -2592x^2
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-18u''(x) - 432xu'(x) - 2592x^2 u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x} + c_2 \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x}
$$

The above shows that

$$
u'(x) = -12\left(x - \frac{\sqrt{3}}{6}\right) c_1 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x} - 12\left(x + \frac{\sqrt{3}}{6}\right) c_2 \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x}
$$

Using the above in (1) gives the solution

$$
y = \frac{-12\left(x - \frac{\sqrt{3}}{6}\right) c_1 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x} - 12\left(x + \frac{\sqrt{3}}{6}\right) c_2 \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x}}{18 c_1 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x} + 18 c_2 \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{\left(-6x - \sqrt{3}\right) \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x} - 6\left(x - \frac{\sqrt{3}}{6}\right) c_3 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x}}{9 c_3 \mathrm{e}^{-6x^2 + 2\sqrt{3}\, x} + 9\, \mathrm{e}^{-6x^2 - 2\sqrt{3}\, x}}
$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-6x-\sqrt{3}\right) e^{-6x^2-2\sqrt{3}\,x} - 6\left(x-\frac{\sqrt{3}}{6}\right) c_3 e^{-6x^2+2\sqrt{3}\,x}}{9c_3 e^{-6x^2+2\sqrt{3}\,x} + 9\, e^{-6x^2-2\sqrt{3}\,x}} \tag{1}$$

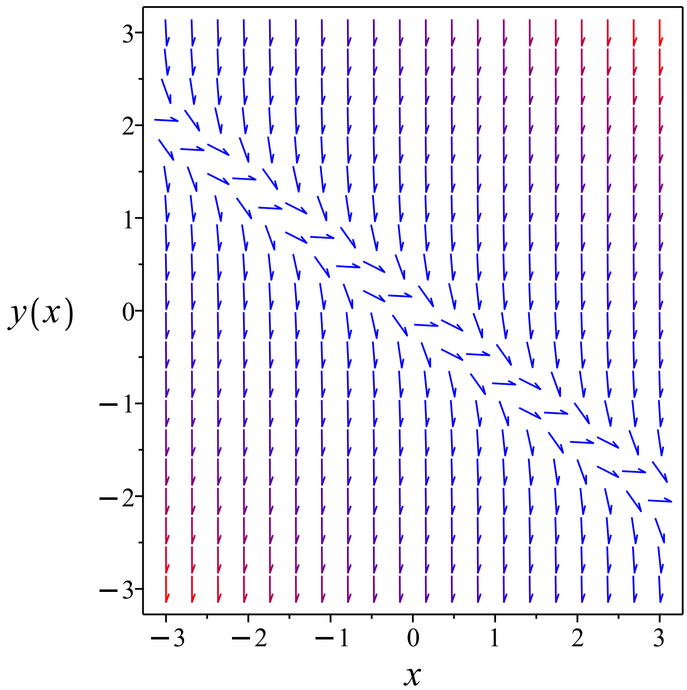


Figure 102: Slope field plot

Verification of solutions

$$y = \frac{\left(-6x-\sqrt{3}\right) e^{-6x^2-2\sqrt{3}\,x} - 6\left(x-\frac{\sqrt{3}}{6}\right) c_3 e^{-6x^2+2\sqrt{3}\,x}}{9c_3 e^{-6x^2+2\sqrt{3}\,x} + 9\, e^{-6x^2-2\sqrt{3}\,x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x)= -2*(2*x+3*y(x))^2,y(x), singsol=all)
```

$$y(x) = -\frac{2x}{3} - \frac{\sqrt{3}\tanh\left(2(-x+c_1)\sqrt{3}\right)}{9}$$

✓ Solution by Mathematica

Time used: 0.142 (sec). Leaf size: 59

```
DSolve[y'[x]==-2*(2*x+3*y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}\left(-6x - \frac{6}{\sqrt{3}+12c_1e^{4\sqrt{3}x}} + \sqrt{3}\right)$$
$$y(x) \to \frac{1}{9}\left(\sqrt{3}-6x\right)$$

2.26 problem 52

Internal problem ID [5261]
Internal file name [`OUTPUT/4752_Friday_February_02_2024_05_11_34_AM_63088358/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 4. Equations of first order and first degree (Variable separable). Suplemetary problems. Page 22
Problem number: 52.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

Unable to solve or complete the solution.

$$\boxed{-2\sin(y) + (2x - 4\sin(y) - 3)\cos(y)\, y' = -x - 3}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5`[0, (-3+8*sin(y)-4*x)/(-2*x+4*sin(y)+3)/cos(y)]
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 22

```
dsolve((x-2*sin(y(x))+3)+(2*x-4*sin(y(x))-3)*cos(y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\frac{9\,\mathrm{LambertW}\left(\frac{c_1 \mathrm{e}^{-\frac{1}{3}-\frac{8x}{9}}}{9}\right)}{8}+\frac{3}{8}+\frac{x}{2}\right)$$

✓ Solution by Mathematica

Time used: 60.95 (sec). Leaf size: 73

```
DSolve[(x-2*Sin[y[x]]+3)+(2*x-4*Sin[y[x]]-3)*Cos[y[x]]*y'[x]==0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to \arcsin\left(\frac{1}{8}\left(9W\left(-\frac{1}{9}e^{-\frac{2}{9}(4x+3-8c_1)}\right)+4x+3\right)\right)$$
$$y(x) \to \arcsin\left(\frac{1}{8}\left(9W\left(-\frac{1}{9}e^{-\frac{2}{9}(4x+3-8c_1)}\right)+4x+3\right)\right)$$

3 Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33

3.1 problem 23 (a) 440
3.2 problem 23 (d) 446
3.3 problem 23 (e) 452
3.4 problem 23 (h) 458
3.5 problem 23 (i) 465
3.6 problem 23 (j) 472
3.7 problem 23 (k) 480
3.8 problem 23 (m) 486
3.9 problem 23 (o) 493
3.10 problem 23 (p) 500
3.11 problem 24 (p) 506
3.12 problem 24 (c) 524
3.13 problem 24 (d) 539
3.14 problem 24 (g) 554
3.15 problem 24 (L) 557
3.16 problem 25 (a) 568
3.17 problem 25 (b) 574
3.18 problem 25 (c) 581
3.19 problem 25 (d) 587
3.20 problem 25 (e) 594
3.21 problem 25 (f) 601
3.22 problem 25 (g) 607
3.23 problem 25 (h) 614
3.24 problem 25 (L) 621
3.25 problem 25 (j) 628
3.26 problem 25 (k) 636
3.27 problem 26 (a) 643
3.28 problem 26 (b) 650
3.29 problem 26 (c) 656
3.30 problem 26 (d) 663
3.31 problem 26 (e) 669
3.32 problem 26 (f) 676
3.33 problem 26 (g) 683
3.34 problem 26 (h) 690

3.35 problem 26 (i) 696
3.36 problem 27 703

3.1 problem 23 (a)

3.1.1 Solving as exact ode 440
3.1.2 Maple step by step solution 444

Internal problem ID [5262]
Internal file name [`OUTPUT/4753_Friday_February_02_2024_05_11_37_AM_91689604/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-xy' - y = -x^2}$$

3.1.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= \left(-x^2 + y\right)\mathrm{d}x\\ \left(x^2 - y\right)\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^2 - y\\ N(x, y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 - y\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x^2 - y\,\mathrm{d}x\\
\phi &= \frac{1}{3}x^3 - xy + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-x$. Therefore equation (4) becomes

$$
-x = -x + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{1}{3}x^3 - xy + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{1}{3}x^3 - xy
$$

The solution becomes

$$y = -\frac{-x^3 + 3c_1}{3x}$$

Summary
The solution(s) found are the following

$$y = -\frac{-x^3 + 3c_1}{3x} \tag{1}$$

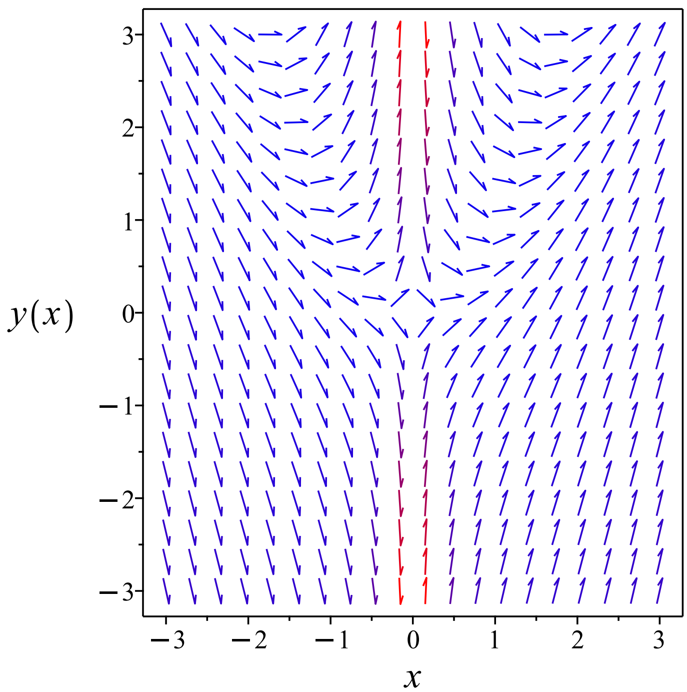


Figure 103: Slope field plot

Verification of solutions

$$y = -\frac{-x^3 + 3c_1}{3x}$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$-xy' - y = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\,x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int x^2 dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^3}{3} + c_1}{x}$

- Simplify

$y = \frac{x^3+3c_1}{3x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve((x^2-y(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1}{3x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[(x^2-y[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{3} + \frac{c_1}{x}$$

3.2 problem 23 (d)

3.2.1 Solving as exact ode 446
3.2.2 Maple step by step solution 449

Internal problem ID [5263]
Internal file name [`OUTPUT/4754_Friday_February_02_2024_05_11_38_AM_91193230/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{y^2 + 2yy'x = -x^2}$$

3.2.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2xy)\, \mathrm{d}y &= \left(-x^2 - y^2\right) \mathrm{d}x \\
\left(x^2 + y^2\right) \mathrm{d}x + (2xy)\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x^2 + y^2 \\
N(x, y) &= 2xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\
&= 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x^2 + y^2 \, \mathrm{d}x$$

$$\phi = \frac{1}{3}x^3 + y^2 x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y^2 x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y^2 x$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + x y^2 = c_1 \tag{1}$$

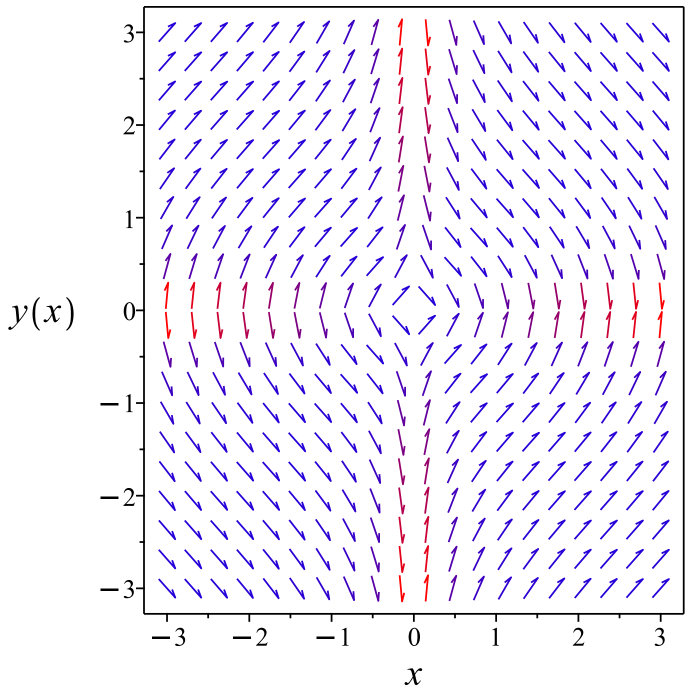


Figure 104: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + x y^2 = c_1$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$y^2 + 2yy'x = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

- Evaluate derivatives

  $2y = 2y$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2+y^2)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + y^2x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2xy = 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = \frac{1}{3}x^3 + y^2x$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + y^2x = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x}$$
$$y(x) = \frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.233 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$

3.3 problem 23 (e)

3.3.1 Solving as exact ode 452
3.3.2 Maple step by step solution 456

Internal problem ID [5264]
Internal file name [`OUTPUT/4755_Friday_February_02_2024_05_11_38_AM_77911395/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y\cos(x) + y'\sin(x) = -x}$$

3.3.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(\sin(x)) \, \mathrm{d}y &= (-y \cos(x) - x) \, \mathrm{d}x \\
(x + y \cos(x)) \, \mathrm{d}x + (\sin(x)) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x + y \cos(x) \\
N(x, y) &= \sin(x)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y \cos(x)) \\
&= \cos(x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\sin(x)) \\
&= \cos(x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int x + y\cos(x)\,\mathrm{d}x$$

$$\phi = \frac{x^2}{2} + y\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2}{2} + y\sin(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2}{2} + y\sin(x)$$

The solution becomes

$$y = \frac{-x^2 + 2c_1}{2\sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2\sin(x)} \tag{1}$$

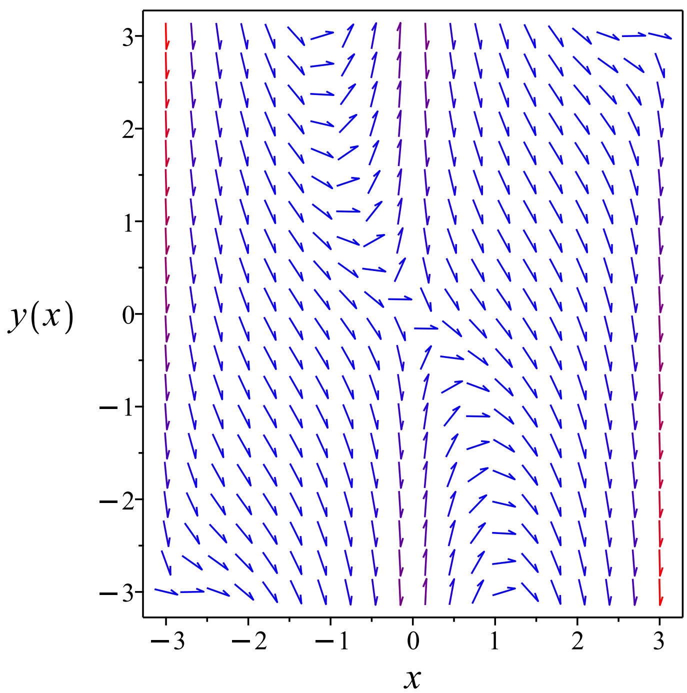


Figure 105: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2\sin(x)}$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$y\cos(x) + y'\sin(x) = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\cos(x)y}{\sin(x)} - \frac{x}{\sin(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\cos(x)y}{\sin(x)} = -\frac{x}{\sin(x)}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{\cos(x)y}{\sin(x)}\right) = -\frac{\mu(x)x}{\sin(x)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{\cos(x)y}{\sin(x)}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)\cos(x)}{\sin(x)}$

- Solve to find the integrating factor

  $\mu(x) = \sin(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int -\frac{\mu(x)x}{\sin(x)}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)x}{\sin(x)}dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)x}{\sin(x)}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int -x\,dx + c_1}{\sin(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{-\frac{x^2}{2} + c_1}{\sin(x)}$

- Simplify

  $y = -\frac{(x^2-2c_1)\csc(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x+y(x)*cos(x))+sin(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{(x^2-2c_1)\csc(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 19

```
DSolve[(x+y[x]*Cos[x])+Sin[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{1}{2}(x^2-2c_1)\csc(x)$$

3.4 problem 23 (h)

3.4.1 Solving as exact ode 458
3.4.2 Maple step by step solution 462

Internal problem ID [5265]
Internal file name [`OUTPUT/4756_Friday_February_02_2024_05_11_38_AM_9973994/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{3y + (3x + 4y + 5)\, y' = -2x - 4}$$

3.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x + 4y + 5)\, \mathrm{d}y &= (-2x - 3y - 4)\, \mathrm{d}x\\ (2x + 3y + 4)\, \mathrm{d}x + (3x + 4y + 5)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2x + 3y + 4\\ N(x, y) &= 3x + 4y + 5\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + 3y + 4)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x + 4y + 5)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x + 3y + 4 \, \mathrm{d}x$$

$$\phi = x(x + 3y + 4) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x + 4y + 5$. Therefore equation (4) becomes

$$3x + 4y + 5 = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y + 5$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (4y + 5) \, \mathrm{d}y$$

$$f(y) = 2y^2 + 5y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x + 3y + 4) + 2y^2 + 5y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x + 3y + 4) + 2y^2 + 5y$$

Summary

The solution(s) found are the following

$$x(x + 3y + 4) + 2y^2 + 5y = c_1 \tag{1}$$

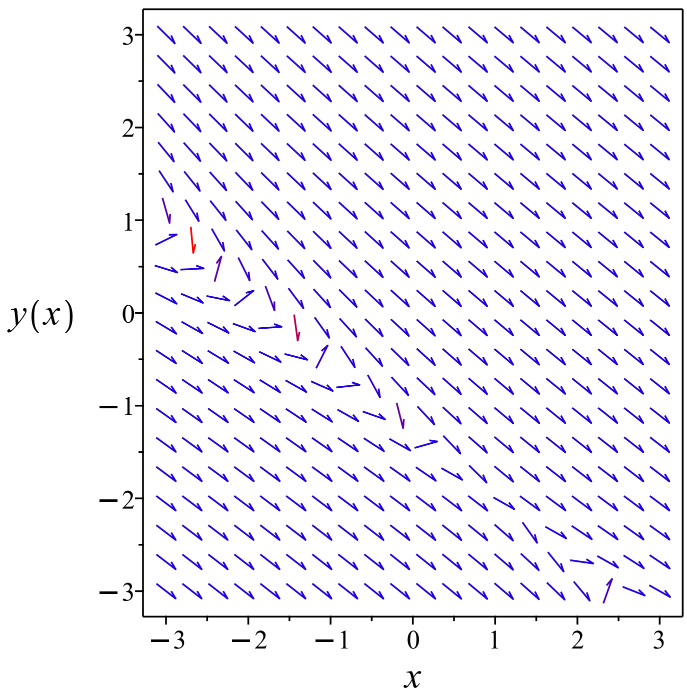


Figure 106: Slope field plot

Verification of solutions

$$x(x + 3y + 4) + 2y^2 + 5y = c_1$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$3y + (3x + 4y + 5)\, y' = -2x - 4$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ○ ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x, y) = 0$

- ○ Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- ○ Evaluate derivatives

  $3 = 3$

- ○ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x + 3y + 4)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^2 + 3xy + 4x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $3x + 4y + 5 = 3x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 4y + 5$

- Solve for $f_1(y)$

  $f_1(y) = 2y^2 + 5y$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = x^2 + 3xy + 2y^2 + 4x + 5y$$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^2 + 3xy + 2y^2 + 4x + 5y = c_1$
- Solve for y

  $\left\{y = -\frac{3x}{4} - \frac{5}{4} - \frac{\sqrt{x^2+8c_1-2x+25}}{4}, y = -\frac{3x}{4} - \frac{5}{4} + \frac{\sqrt{x^2+8c_1-2x+25}}{4}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.218 (sec). Leaf size: 32

```
dsolve((2*x+3*y(x)+4)+(3*x+4*y(x)+5)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{(x-1)^2 c_1^2 + 8} + (-3x-5)\, c_1}{4c_1}$$

✓ Solution by Mathematica

Time used: 0.134 (sec). Leaf size: 61

```
DSolve[(2*x+3*y[x]+4)+(3*x+4*y[x]+5)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{x^2 - 2x + 25 + 16c_1} - 3x - 5\right)$$
$$y(x) \to \frac{1}{4}\left(\sqrt{x^2 - 2x + 25 + 16c_1} - 3x - 5\right)$$

3.5 problem 23 (i)

3.5.1 Solving as exact ode 465
3.5.2 Maple step by step solution 469

Internal problem ID [5266]
Internal file name [`OUTPUT/4757_Friday_February_02_2024_05_11_39_AM_99484607/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _exact, _rational]
```

$$\boxed{4x^3y^3 + \left(3x^4y^2 - \frac{1}{y}\right) y' = -\frac{1}{x}}$$

3.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(3x^4y^2 - \frac{1}{y}\right)\mathrm{d}y = \left(-4x^3y^3 - \frac{1}{x}\right)\mathrm{d}x$$
$$\left(4x^3y^3 + \frac{1}{x}\right)\mathrm{d}x + \left(3x^4y^2 - \frac{1}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = 4x^3y^3 + \frac{1}{x}$$
$$N(x,y) = 3x^4y^2 - \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4x^3y^3 + \frac{1}{x}\right)\\ &= 12x^3y^2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x^4y^2 - \frac{1}{y}\right) \\
&= 12x^3y^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 4x^3y^3 + \frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(x) + x^4y^3 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3x^4y^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^4y^2 - \frac{1}{y}$. Therefore equation (4) becomes

$$
3x^4y^2 - \frac{1}{y} = 3x^4y^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= -\ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + x^4 y^3 - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + x^4 y^3 - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-3x^7 \mathrm{e}^{-3c_1}\right)}{3} - c_1} x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-3x^7 \mathrm{e}^{-3c_1}\right)}{3} - c_1} x \tag{1}$$

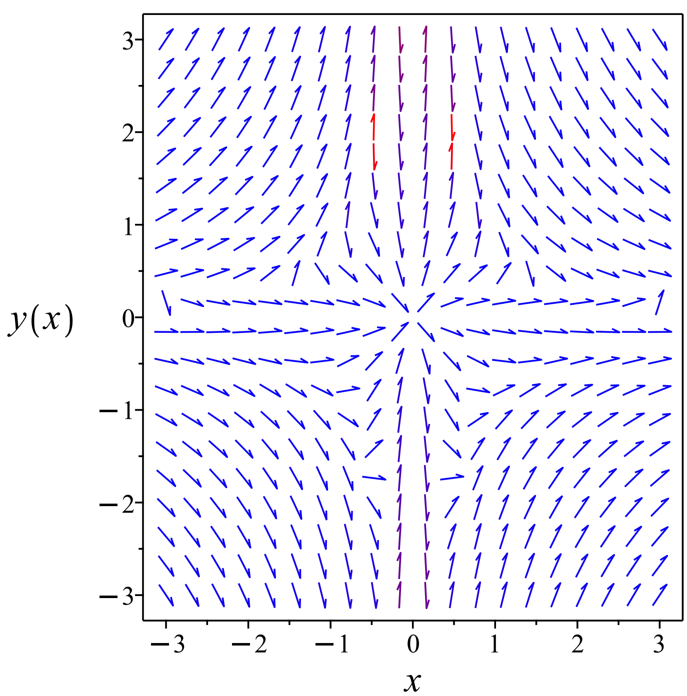


Figure 107: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{\mathrm{LambertW}\left(-3x^7 \mathrm{e}^{-3c_1}\right)}{3} - c_1} x$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$4x^3y^3 + \left(3x^4y^2 - \frac{1}{y}\right) y' = -\frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $12x^3y^2 = 12x^3y^2$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \left(4x^3y^3 + \frac{1}{x}\right) dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \ln(x) + x^4y^3 + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3x^4y^2 - \frac{1}{y} = 3x^4y^2 + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = -\frac{1}{y}$$

- Solve for $f_1(y)$

$$f_1(y) = -\ln(y)$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = \ln(x) + x^4y^3 - \ln(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\ln(x) + x^4y^3 - \ln(y) = c_1$$

- Solve for y

$$y = \frac{x}{e^{\frac{\mathit{LambertW}\left(-3x^7 e^{-3c_1}\right)}{3} + c_1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 25

```
dsolve((4*x^3*y(x)^3+1/x)+(3*x^4*y(x)^2-1/y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{2}{3}}}{3\left(-\frac{x^4}{\mathrm{LambertW}(-3c_1x^7)}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 4.154 (sec). Leaf size: 108

```
DSolve[(4*x^3*y[x]^3+1/x)+(3*x^4*y[x]^2-1/y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{W\left(-3e^{-3c_1}x^7\right)}}{x^{4/3}}$$
$$y(x) \to -\frac{\sqrt[3]{W\left(-3e^{-3c_1}x^7\right)}}{\sqrt[3]{3}x^{4/3}}$$
$$y(x) \to -\frac{(-1)^{2/3}\sqrt[3]{W\left(-3e^{-3c_1}x^7\right)}}{\sqrt[3]{3}x^{4/3}}$$
$$y(x) \to 0$$

3.6 problem 23 (j)

3.6.1 Solving as exact ode 472
3.6.2 Maple step by step solution 476

Internal problem ID [5267]
Internal file name [`OUTPUT/4758_Friday_February_02_2024_05_11_39_AM_54772160/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2uv + (u^2 + v^2)\, v' = -2u^2}$$

3.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(u, v)\,\mathrm{d}u + N(u, v)\,\mathrm{d}v = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(u^2 + v^2)\,\mathrm{d}v &= (-2u^2 - 2uv)\,\mathrm{d}u \\
(2u^2 + 2uv)\,\mathrm{d}u + (u^2 + v^2)\,\mathrm{d}v &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(u, v) &= 2u^2 + 2uv \\
N(u, v) &= u^2 + v^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}(2u^2 + 2uv) \\
&= 2u
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial u} &= \frac{\partial}{\partial u}(u^2 + v^2) \\
&= 2u
\end{aligned}
$$

Since $\frac{\partial M}{\partial v} = \frac{\partial N}{\partial u}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(u, v)$

$$\frac{\partial \phi}{\partial u} = M \tag{1}$$
$$\frac{\partial \phi}{\partial v} = N \tag{2}$$

Integrating (1) w.r.t. u gives

$$\int \frac{\partial \phi}{\partial u} \, \mathrm{d}u = \int M \, \mathrm{d}u$$
$$\int \frac{\partial \phi}{\partial u} \, \mathrm{d}u = \int 2u^2 + 2uv \, \mathrm{d}u$$
$$\phi = \frac{u^2(2u + 3v)}{3} + f(v) \tag{3}$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both u and v. Taking derivative of equation (3) w.r.t v gives

$$\frac{\partial \phi}{\partial v} = u^2 + f'(v) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = u^2 + v^2$. Therefore equation (4) becomes

$$u^2 + v^2 = u^2 + f'(v) \tag{5}$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = v^2$$

Integrating the above w.r.t v gives

$$\int f'(v) \, \mathrm{d}v = \int (v^2) \, \mathrm{d}v$$
$$f(v) = \frac{v^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(v)$ into equation (3) gives ϕ

$$\phi = \frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{u^2(2u + 3v)}{3} + \frac{v^3}{3}$$

Summary
The solution(s) found are the following

$$\frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} = c_1 \tag{1}$$

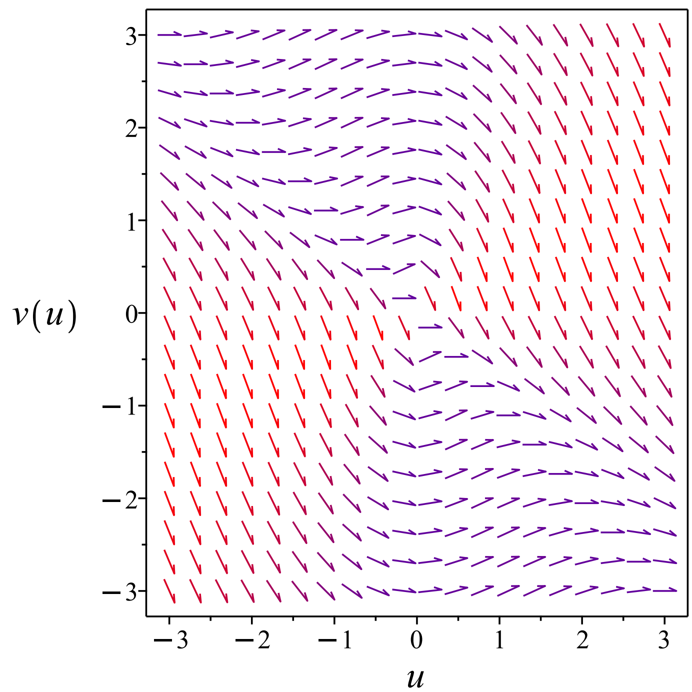


Figure 108: Slope field plot

Verification of solutions

$$\frac{u^2(2u + 3v)}{3} + \frac{v^3}{3} = c_1$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$2uv + (u^2 + v^2)\, v' = -2u^2$

- Highest derivative means the order of the ODE is 1

  v'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(u, v) = 0$

  - Compute derivative of lhs

    $F'(u, v) + \left(\frac{\partial}{\partial v} F(u, v)\right) v' = 0$

  - Evaluate derivatives

    $2u = 2u$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(u, v) = c_1, M(u, v) = F'(u, v)\,, N(u, v) = \frac{\partial}{\partial v} F(u, v)\right]$

- Solve for $F(u, v)$ by integrating $M(u, v)$ with respect to u

  $F(u, v) = \int (2u^2 + 2uv)\, du + f_1(v)$

- Evaluate integral

  $F(u, v) = \frac{2u^3}{3} + u^2 v + f_1(v)$

- Take derivative of $F(u, v)$ with respect to v

  $N(u, v) = \frac{\partial}{\partial v} F(u, v)$

- Compute derivative

  $u^2 + v^2 = u^2 + \frac{d}{dv} f_1(v)$

- Isolate for $\frac{d}{dv} f_1(v)$

  $\frac{d}{dv} f_1(v) = v^2$

- Solve for $f_1(v)$

  $f_1(v) = \frac{v^3}{3}$

- Substitute $f_1(v)$ into equation for $F(u, v)$

  $F(u, v) = \frac{2}{3}u^3 + u^2 v + \frac{1}{3}v^3$

- Substitute $F(u,v)$ into the solution of the ODE

  $\frac{2}{3}u^3 + u^2 v + \frac{1}{3}v^3 = c_1$

- Solve for v

$$\left\{v = \frac{\left(-8u^3+12c_1+4\sqrt{8u^6-12u^3c_1+9c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2u^2}{\left(-8u^3+12c_1+4\sqrt{8u^6-12u^3c_1+9c_1^2}\right)^{\frac{1}{3}}}, v = -\frac{\left(-8u^3+12c_1+4\sqrt{8u^6-12u\right.}}{4}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 321

```
dsolve(2*(u^2+u*v(u))+(u^2+v(u)^2)*diff(v(u),u)=0,v(u), singsol=all)
```

$$v(u) = -\frac{2\left(u^2c_1 - \frac{\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}}{4}\right)}{\sqrt{c_1}\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$v(u) = -\frac{\left(1+i\sqrt{3}\right)\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}{4\sqrt{c_1}} - \frac{u^2\sqrt{c_1}\left(i\sqrt{3}-1\right)}{\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}}$$

$$v(u) = \frac{4i\sqrt{3}\,c_1u^2 + i\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{2}{3}}\sqrt{3} + 4u^2c_1 - \left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+\right.}{4\left(4-8u^3c_1^{\frac{3}{2}}+4\sqrt{8u^6c_1^3-4u^3c_1^{\frac{3}{2}}+1}\right)^{\frac{1}{3}}\sqrt{c_1}}$$

✓ Solution by Mathematica

Time used: 15.565 (sec). Leaf size: 593

```
DSolve[2*(u^2+u*v[u])+(u^2+v[u]^2)*v'[u]==0,v[u],u,IncludeSingularSolutions -> True]
```

$$v(u) \to \frac{\sqrt[3]{-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}}}{\sqrt[3]{2}} - \frac{\sqrt[3]{2}u^2}{\sqrt[3]{-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}}}$$

$$v(u) \to \frac{\sqrt[3]{2}\left(2+2i\sqrt{3}\right)u^2+i2^{2/3}\left(\sqrt{3}+i\right)\left(-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}\right)^{2/3}}{4\sqrt[3]{-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}}}$$

$$v(u) \to \frac{\left(1-i\sqrt{3}\right)u^2}{2^{2/3}\sqrt[3]{-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{-2u^3+\sqrt{8u^6-4e^{3c_1}u^3+e^{6c_1}}+e^{3c_1}}}{2\sqrt[3]{2}}$$

$$v(u) \to \sqrt[3]{\sqrt{2}\sqrt{u^6}-u^3} - \frac{u^2}{\sqrt[3]{\sqrt{2}\sqrt{u^6}-u^3}}$$

$$v(u) \to \frac{\left(1-i\sqrt{3}\right)u^2+\left(-1-i\sqrt{3}\right)\left(\sqrt{2}\sqrt{u^6}-u^3\right)^{2/3}}{2\sqrt[3]{\sqrt{2}\sqrt{u^6}-u^3}}$$

$$v(u) \to \frac{\left(1+i\sqrt{3}\right)u^2+i\left(\sqrt{3}+i\right)\left(\sqrt{2}\sqrt{u^6}-u^3\right)^{2/3}}{2\sqrt[3]{\sqrt{2}\sqrt{u^6}-u^3}}$$

3.7 problem 23 (k)

3.7.1 Solving as exact ode 480
3.7.2 Maple step by step solution 484

Internal problem ID [5268]
Internal file name [`OUTPUT/4759_Friday_February_02_2024_05_11_39_AM_80796220/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (k).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{x\sqrt{x^2+y^2}-y+\left(\sqrt{x^2+y^2}\,y-x\right)y'=0}$$

3.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\sqrt{x^2 + y^2}\, y - x\right) \mathrm{d}y &= \left(-x\sqrt{x^2 + y^2} + y\right) \mathrm{d}x \\
\left(x\sqrt{x^2 + y^2} - y\right) \mathrm{d}x + \left(\sqrt{x^2 + y^2}\, y - x\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x\sqrt{x^2 + y^2} - y \\
N(x, y) &= \sqrt{x^2 + y^2}\, y - x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x\sqrt{x^2 + y^2} - y\right) \\
&= \frac{yx}{\sqrt{x^2 + y^2}} - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^2+y^2}\, y - x\right) \\
&= \frac{yx}{\sqrt{x^2+y^2}} - 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int x\sqrt{x^2+y^2} - y \,\mathrm{d}x \\
\phi &= \frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sqrt{x^2+y^2}\, y - x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{x^2+y^2}\, y - x$. Therefore equation (4) becomes

$$
\sqrt{x^2+y^2}\, y - x = \sqrt{x^2+y^2}\, y - x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - xy$$

<u>Summary</u>

The solution(s) found are the following

$$\frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - yx = c_1 \tag{1}$$

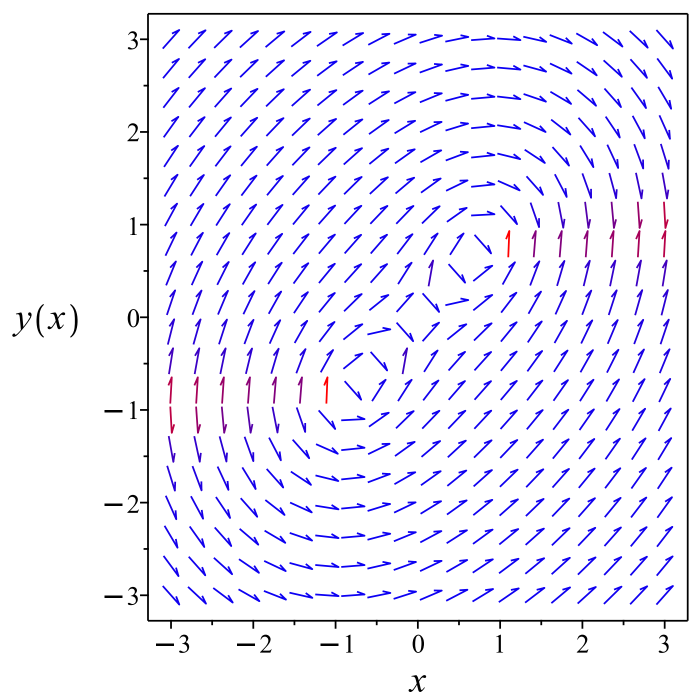


Figure 109: Slope field plot

<u>Verification of solutions</u>

$$\frac{(x^2 + y^2)^{\frac{3}{2}}}{3} - yx = c_1$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$x\sqrt{x^2+y^2}-y+\left(\sqrt{x^2+y^2}\, y-x\right) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $\frac{yx}{\sqrt{x^2+y^2}}-1 = \frac{yx}{\sqrt{x^2+y^2}}-1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\,, N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int \left(x\sqrt{x^2+y^2}-y\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{(x^2+y^2)^{\frac{3}{2}}}{3}-xy+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $\sqrt{x^2+y^2}\, y-x=\sqrt{x^2+y^2}\, y-x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=0$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = \frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{(x^2+y^2)^{\frac{3}{2}}}{3} - xy = c_1$$

- Solve for y

$$y = \frac{\mathit{RootOf}\left(\_Z^6 - 6c_1\_Z^3 - 9\_Z^2x^2 + 9x^4 + 9c_1^2\right)^3 - 3c_1}{3x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve((x*sqrt(x^2+y(x)^2)-y(x))+(y(x)*sqrt(x^2+y(x)^2)-x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{(x^2+y(x)^2)^{\frac{3}{2}}}{3} - xy(x) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 30.753 (sec). Leaf size: 319

```
DSolve[(x*Sqrt[x^2+y[x]^2]-y[x])+(y[x]*Sqrt[x^2+y[x]^2]-x)*y'[x]==0,y[x],x,IncludeSingularSo
```

$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 1\right]$$
$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 2\right]$$
$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 3\right]$$
$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 4\right]$$
$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 5\right]$$
$$y(x) \to \text{Root}\left[\#1^6 + 3\#1^4x^2 + \#1^2\left(3x^4 - 9x^2\right) - 18\#1c_1x + x^6 - 9c_1{}^2\&, 6\right]$$

3.8 problem 23 (m)

3.8.1 Solving as exact ode 486
3.8.2 Maple step by step solution 490

Internal problem ID [5269]
Internal file name [`OUTPUT/4760_Friday_February_02_2024_05_11_40_AM_79254509/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (m).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{y-(y-x+3)\,y'=-x-1}$$

3.8.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-y + x - 3)\,\mathrm{d}y &= (-x - 1 - y)\,\mathrm{d}x \\
(x + y + 1)\,\mathrm{d}x + (-y + x - 3)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x + y + 1 \\
N(x,y) &= -y + x - 3
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + y + 1) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-y + x - 3) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int x+y+1\,\mathrm{d}x$$
$$\phi=\frac{x(x+2y+2)}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-y+x-3$. Therefore equation (4) becomes

$$-y+x-3=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-3-y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (-3-y)\,\mathrm{d}y$$
$$f(y)=-3y-\frac{1}{2}y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=\frac{x(x+2y+2)}{2}-3y-\frac{y^2}{2}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y+2)}{2} - 3y - \frac{y^2}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{x(x+2y+2)}{2} - 3y - \frac{y^2}{2} = c_1 \tag{1}$$

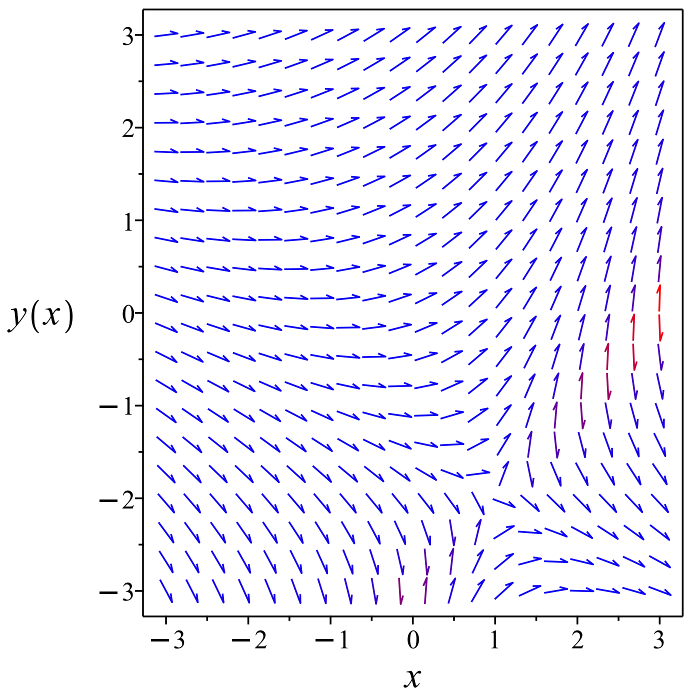


Figure 110: Slope field plot

<u>Verification of solutions</u>

$$\frac{x(x+2y+2)}{2} - 3y - \frac{y^2}{2} = c_1$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$y - (y - x + 3)\, y' = -x - 1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  ◦ Evaluate derivatives

    $1 = 1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (x + y + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \frac{x^2}{2} + xy + x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-y + x - 3 = x + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -3 - y$

- Solve for $f_1(y)$

  $f_1(y) = -3y - \frac{1}{2}y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$F(x,y) = \frac{1}{2}x^2 + xy + x - 3y - \frac{1}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + xy + x - 3y - \frac{1}{2}y^2 = c_1$

- Solve for y

  $\left\{y = x - 3 - \sqrt{2x^2 - 2c_1 - 4x + 9}, y = x - 3 + \sqrt{2x^2 - 2c_1 - 4x + 9}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 30

```
dsolve((x+y(x)+1)-(y(x)-x+3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-\sqrt{2\left(x-1\right)^2 c_1^2 + 1} + \left(x-3\right)c_1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.125 (sec). Leaf size: 59

```
DSolve[(x+y[x]+1)-(y[x]-x+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -i\sqrt{-2x^2+4x-9-c_1}+x-3 \\
y(x) &\to i\sqrt{-2x^2+4x-9-c_1}+x-3
\end{aligned}
$$

3.9 problem 23 (o)

3.9.1 Solving as exact ode 493
3.9.2 Maple step by step solution 497

Internal problem ID [5270]
Internal file name [`OUTPUT/4761_Friday_February_02_2024_05_11_40_AM_31601424/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (o).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{y^2 - \frac{y}{x(x+y)} + \left(\frac{1}{x+y} + 2y(x+1)\right) y' = -2}$$

3.9.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{x+y} + 2y(x+1)\right)\mathrm{d}y &= \left(-y^2 + \frac{y}{x\left(x+y\right)} - 2\right)\mathrm{d}x \\
\left(y^2 - \frac{y}{x\left(x+y\right)} + 2\right)\mathrm{d}x + \left(\frac{1}{x+y} + 2y(x+1)\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y^2 - \frac{y}{x\left(x+y\right)} + 2 \\
N(x, y) &= \frac{1}{x+y} + 2y(x+1)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 - \frac{y}{x\left(x+y\right)} + 2\right) \\
&= 2y - \frac{1}{x\left(x+y\right)} + \frac{y}{x\left(x+y\right)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{x+y} + 2y(x+1)\right) \\
&= -\frac{1}{(x+y)^2} + 2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y^2 - \frac{y}{x(x+y)} + 2\,\mathrm{d}x \\
\phi &= y^2x + 2x + \ln(x+y) - \ln(x) + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2xy + \frac{1}{x+y} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x+y} + 2y(x+1)$. Therefore equation (4) becomes

$$
\frac{1}{x+y} + 2y(x+1) = 2xy + \frac{1}{x+y} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 2y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (2y)\,\mathrm{d}y \\
f(y) &= y^2 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2x + 2x + \ln(x+y) - \ln(x) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2x + 2x + \ln(x+y) - \ln(x) + y^2$$

Summary

The solution(s) found are the following

$$xy^2 + 2x + \ln(x+y) - \ln(x) + y^2 = c_1 \tag{1}$$

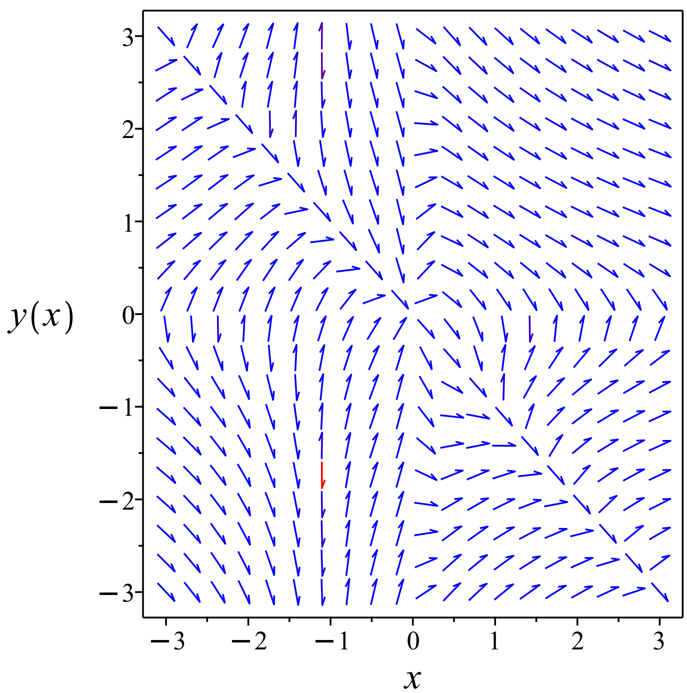


Figure 111: Slope field plot

Verification of solutions

$$xy^2 + 2x + \ln(x+y) - \ln(x) + y^2 = c_1$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$y^2 - \frac{y}{x(x+y)} + \left(\frac{1}{x+y} + 2y(x+1)\right) y' = -2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y} F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $2y - \frac{1}{x(x+y)} + \frac{y}{x(x+y)^2} = -\frac{1}{(x+y)^2} + 2y$

  - Simplify

    $2y - \frac{1}{x(x+y)} + \frac{y}{x(x+y)^2} = -\frac{1}{(x+y)^2} + 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y} F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int \left(y^2 - \frac{y}{x(x+y)} + 2\right) dx + f_1(y)$

- Evaluate integral

  $F(x,y) = y^2 x + 2x + \ln(x+y) - \ln(x) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y} F(x,y)$

- Compute derivative

  $\frac{1}{x+y} + 2y(x+1) = 2xy + \frac{1}{x+y} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -2xy + 2y(x+1)$

- Solve for $f_1(y)$

  $$f_1(y) = y^2$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $$F(x, y) = y^2x + 2x + \ln(x + y) - \ln(x) + y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

  $$y^2x + 2x + \ln(x + y) - \ln(x) + y^2 = c_1$$

- Solve for y

  $$y = \frac{-x\,\mathrm{e}^{\mathit{RootOf}\left(-x^3\left(\mathrm{e}^{\_Z}\right)^2 - x^2\left(\mathrm{e}^{\_Z}\right)^2 + 2x^3\mathrm{e}^{\_Z} + \left(\mathrm{e}^{\_Z}\right)^2 c_1 + \_Z\left(\mathrm{e}^{\_Z}\right)^2 - 2x\left(\mathrm{e}^{\_Z}\right)^2 + 2x^2\mathrm{e}^{\_Z} - x^3 - x^2\right)} + x}{\mathrm{e}^{\mathit{RootOf}\left(-x^3\left(\mathrm{e}^{\_Z}\right)^2 - x^2\left(\mathrm{e}^{\_Z}\right)^2 + 2x^3\mathrm{e}^{\_Z} + \left(\mathrm{e}^{\_Z}\right)^2 c_1 + \_Z\left(\mathrm{e}^{\_Z}\right)^2 - 2x\left(\mathrm{e}^{\_Z}\right)^2 + 2x^2\mathrm{e}^{\_Z} - x^3 - x^2\right)}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 129

```
dsolve((y(x)^2- y(x)/(x*(x+y(x)))+2)+( 1/(x+y(x)) + 2*y(x)*(1+x))*diff(y(x),x)=0,y(x), sings
```

$$y(x) = -x\left(\mathrm{e}^{\mathrm{RootOf}\left(x^3\mathrm{e}^{2\_Z} + x^2\mathrm{e}^{2\_Z} - 2x^3\mathrm{e}^{\_Z} + c_1\mathrm{e}^{2\_Z} - \_Z\mathrm{e}^{2\_Z} + 2x\,\mathrm{e}^{2\_Z} - 2x^2\mathrm{e}^{\_Z} + x^3 + x^2\right)} - 1\right)\mathrm{e}^{-\mathrm{RootOf}\left(x^3\mathrm{e}^{2\_Z} + x^2\mathrm{e}^{2\_Z} - 2x^3\mathrm{e}^{\_Z} + c_1\mathrm{e}^{2\_Z} - \_Z\mathrm{e}^{2\_Z} + 2x\,\mathrm{e}^{2\_Z} - 2x^2\mathrm{e}^{\_Z} + x^3 + x^2\right)}$$

✓ Solution by Mathematica

Time used: 0.43 (sec). Leaf size: 29

```
DSolve[(y[x]^2- y[x]/(x*(x+y[x]))+2)+( 1/(x+y[x]) + 2*y[x]*(1+x))*y'[x]==0,y[x],x,IncludeSin
```

$$\text{Solve}\left[xy(x)^2 + y(x)^2 + \log(y(x) + x) + 2x - \log(x) = c_1, y(x)\right]$$

3.10 problem 23 (p)

3.10.1 Solving as exact ode 500
3.10.2 Maple step by step solution 504

Internal problem ID [5271]
Internal file name [`OUTPUT/4762_Friday_February_02_2024_05_11_40_AM_81253565/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 23 (p).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2xy\,\mathrm{e}^{x^2y}+y^2\mathrm{e}^{x y^2}+\left(x^2\mathrm{e}^{x^2y}+2xy\,\mathrm{e}^{x y^2}-2y\right)y'=-1}$$

3.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(x^2 \mathrm{e}^{x^2 y} + 2xy\, \mathrm{e}^{y^2 x} - 2y\right) \mathrm{d}y = \left(-2xy\, \mathrm{e}^{x^2 y} - y^2 \mathrm{e}^{y^2 x} - 1\right) \mathrm{d}x$$
$$\left(2xy\, \mathrm{e}^{x^2 y} + y^2 \mathrm{e}^{y^2 x} + 1\right) \mathrm{d}x + \left(x^2 \mathrm{e}^{x^2 y} + 2xy\, \mathrm{e}^{y^2 x} - 2y\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2xy\, \mathrm{e}^{x^2 y} + y^2 \mathrm{e}^{y^2 x} + 1$$
$$N(x, y) = x^2 \mathrm{e}^{x^2 y} + 2xy\, \mathrm{e}^{y^2 x} - 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2xy\, \mathrm{e}^{x^2 y} + y^2 \mathrm{e}^{y^2 x} + 1\right) \\ &= \left(2x\, y^3 + 2y\right) \mathrm{e}^{y^2 x} + 2x\, \mathrm{e}^{x^2 y}\left(x^2 y + 1\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 \mathrm{e}^{x^2 y} + 2xy\, \mathrm{e}^{y^2 x} - 2y\right) \\ &= \left(2x\, y^3 + 2y\right) \mathrm{e}^{y^2 x} + 2x\, \mathrm{e}^{x^2 y}\left(x^2 y + 1\right)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2xy \, \mathrm{e}^{x^2 y} + y^2 \mathrm{e}^{y^2 x} + 1 \, \mathrm{d}x$$
$$\phi = x + \mathrm{e}^{y^2 x} + \mathrm{e}^{x^2 y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy \, \mathrm{e}^{y^2 x} + x^2 \mathrm{e}^{x^2 y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 \mathrm{e}^{x^2 y} + 2xy \, \mathrm{e}^{y^2 x} - 2y$. Therefore equation (4) becomes

$$x^2 \mathrm{e}^{x^2 y} + 2xy \, \mathrm{e}^{y^2 x} - 2y = 2xy \, \mathrm{e}^{y^2 x} + x^2 \mathrm{e}^{x^2 y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-2y) \, \mathrm{d}y$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \mathrm{e}^{y^2 x} + \mathrm{e}^{x^2 y} - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + e^{y^2 x} + e^{x^2 y} - y^2$$

Summary

The solution(s) found are the following

$$x + e^{x y^2} + e^{x^2 y} - y^2 = c_1 \tag{1}$$

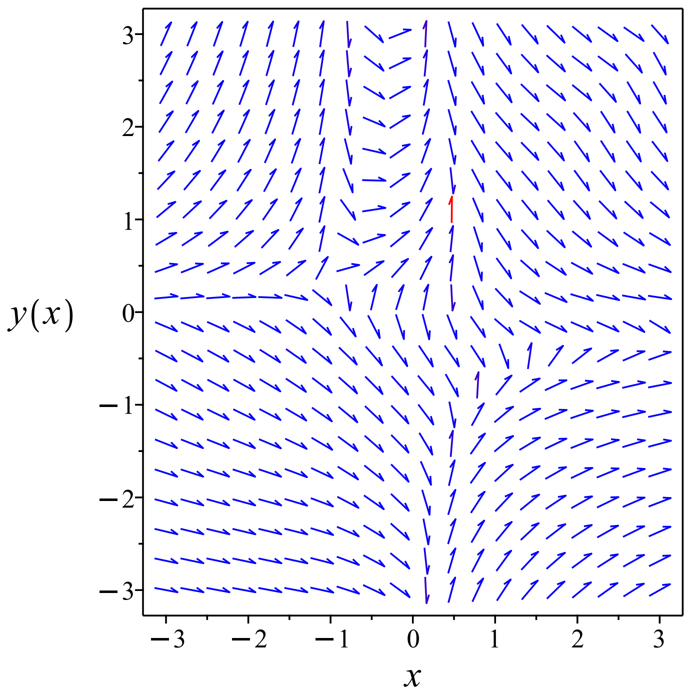


Figure 112: Slope field plot

Verification of solutions

$$x + e^{x y^2} + e^{x^2 y} - y^2 = c_1$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$2xy\,e^{x^2y}+y^2e^{xy^2}+\left(x^2e^{x^2y}+2xy\,e^{xy^2}-2y\right)y'=-1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ∘ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ∘ Evaluate derivatives

    $2x\,e^{x^2y}+2x^3y\,e^{x^2y}+2y\,e^{y^2x}+2y^3x\,e^{y^2x}=2x\,e^{x^2y}+2x^3y\,e^{x^2y}+2y\,e^{y^2x}+2y^3x\,e^{y^2x}$

  ∘ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(2xy\,e^{x^2y}+y^2e^{y^2x}+1\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=x+e^{y^2x}+e^{x^2y}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x^2e^{x^2y}+2xy\,e^{y^2x}-2y=2xy\,e^{y^2x}+x^2e^{x^2y}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-2y$

- Solve for $f_1(y)$

  $f_1(y)=-y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y) = x + e^{y^2 x} + e^{x^2 y} - y^2$$

- Substitute $F(x,y)$ into the solution of the ODE

$$x + e^{y^2 x} + e^{x^2 y} - y^2 = c_1$$

- Solve for y

$$y = \frac{\mathit{RootOf}\left(-e^{\frac{\_Z^2}{x^3}} x^4 - e^{\_Z} x^4 + c_1 x^4 - x^5 + \_Z^2\right)}{x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 41

```
dsolve((2*x*y(x)*exp(x^2*y(x))+ y(x)^2*exp(x*y(x)^2)+1)+(x^2*exp(x^2*y(x))+ 2*x*y(x)*exp(x*y
```

$$y(x) = \frac{\mathrm{RootOf}\left(e^{\_Z} x^4 + e^{\frac{\_Z^2}{x^3}} x^4 + c_1 x^4 + x^5 - \_Z^2\right)}{x^2}$$

✓ Solution by Mathematica

Time used: 0.385 (sec). Leaf size: 30

```
DSolve[(2*x*y[x]*Exp[x^2*y[x]]+ y[x]^2*Exp[x*y[x]^2]+1)+(x^2*Exp[x^2*y[x]]+ 2*x*y[x]*Exp[x*y
```

$$\mathrm{Solve}\left[e^{x^2 y(x)} - y(x)^2 + e^{x y(x)^2} + x = c_1, y(x)\right]$$

3.11 problem 24 (p)

3.11.1 Solving as homogeneousTypeD2 ode 506
3.11.2 Solving as first order ode lie symmetry lookup ode 508
3.11.3 Solving as bernoulli ode 512
3.11.4 Solving as exact ode 516
3.11.5 Solving as riccati ode 521

Internal problem ID [5272]
Internal file name [`OUTPUT/4763_Friday_February_02_2024_05_11_41_AM_4963611/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 24 (p).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y(x-2y)-x^2y'=0}$$

3.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(x-2u(x)\,x)-x^2(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u^2}{x}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u^2}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u^2}\, du &= \int -\frac{2}{x}\, dx \\
-\frac{1}{u} &= -2\ln(x) + c_2
\end{aligned}$$

The solution is

$$-\frac{1}{u(x)} + 2\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\begin{aligned}
-\frac{x}{y} + 2\ln(x) - c_2 &= 0 \\
-\frac{x}{y} + 2\ln(x) - c_2 &= 0
\end{aligned}$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} + 2\ln(x) - c_2 = 0 \tag{1}$$

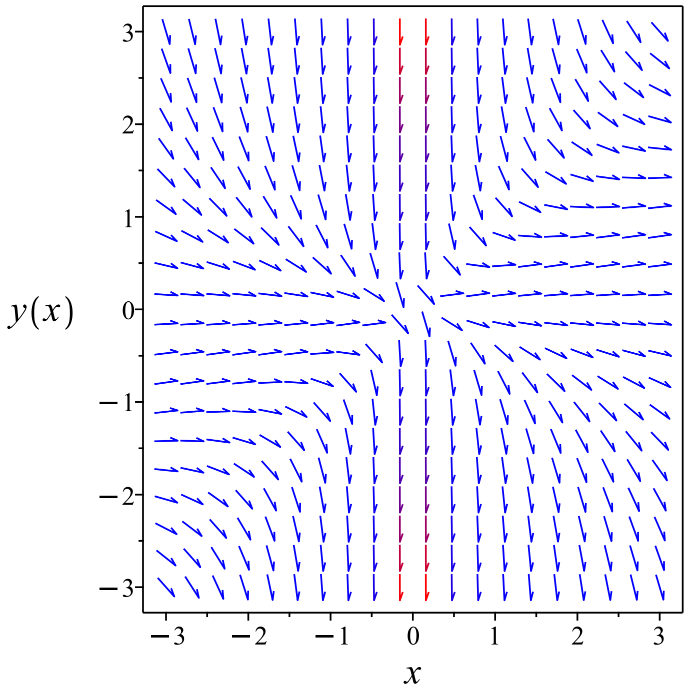


Figure 113: Slope field plot

Verification of solutions

$$-\frac{x}{y}+2\ln(x)-c_2=0$$

Verified OK.

3.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y(-x+2y)}{x^2}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(-x+2y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -2\ln(x) + c_1$$

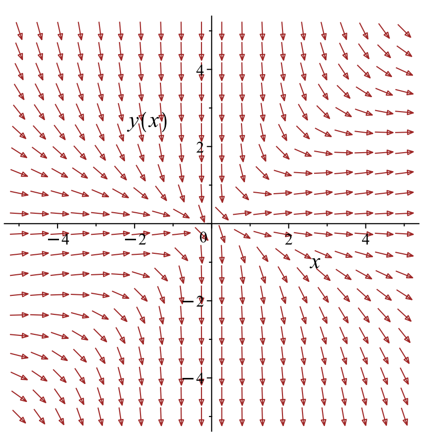
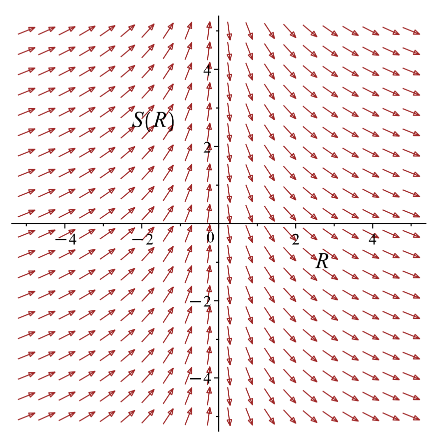
Which simplifies to

$$-\frac{x}{y} = -2\ln(x) + c_1$$

Which gives

$$y = \frac{x}{2\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(-x+2y)}{x^2}$
</td><td>$R = x$
$S = -\frac{x}{y}$</td><td>$\frac{dS}{dR} = -\frac{2}{R}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{x}{2 \ln (x) - c_1} \tag{1}$$

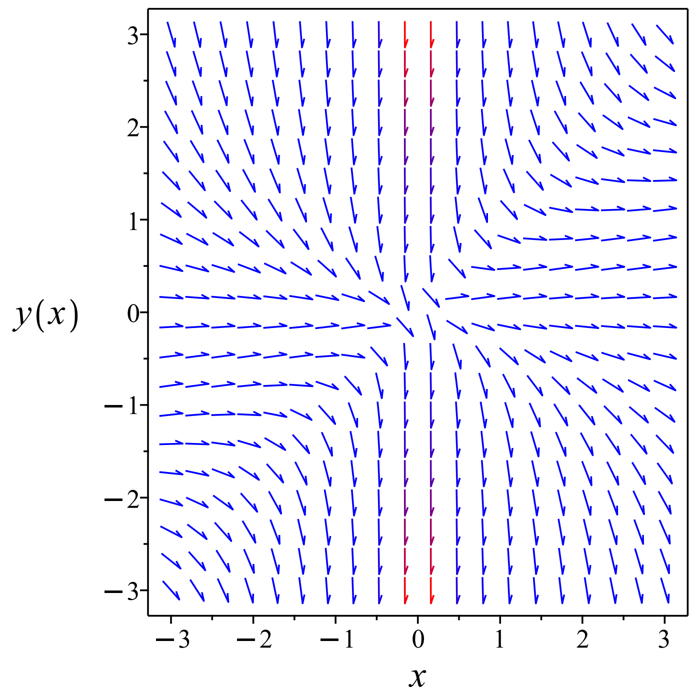


Figure 114: Slope field plot

Verification of solutions

$$y = \frac{x}{2 \ln (x) - c_1}$$

Verified OK.

3.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x + 2y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x} y - \frac{2}{x^2} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{2}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} - \frac{2}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{2}{x^2} \\ w' &= -\frac{w}{x} + \frac{2}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{2}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{2}{x^2}\right) \\
\frac{d}{dx}(xw) &= (x)\left(\frac{2}{x^2}\right) \\
d(xw) &= \left(\frac{2}{x}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
xw &= \int \frac{2}{x} dx \\
xw &= 2\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{2\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{2\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{2\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{2\ln(x) + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{2\ln(x) + c_1} \tag{1}$$

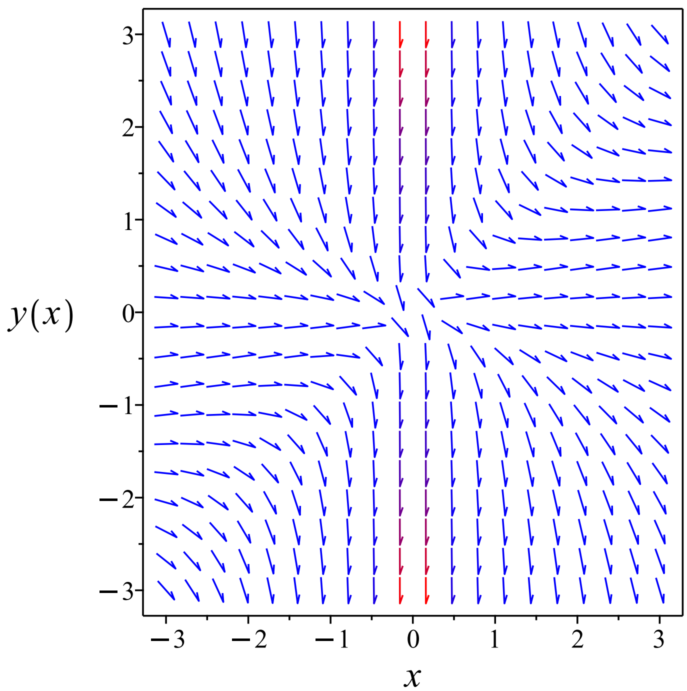


Figure 115: Slope field plot

Verification of solutions

$$y = \frac{x}{2\ln(x) + c_1}$$

Verified OK.

3.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2)\,\mathrm{d}y&=(-y(x-2y))\,\mathrm{d}x\\ (y(x-2y))\,\mathrm{d}x+(-x^2)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y(x-2y)\\ N(x,y)&=-x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(x-2y)) \\ &= x-4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2) \\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y(x-2y)$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{x-2y}{xy} \\ N &= -\frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x}{y^2}\right)\mathrm{d}y = \left(-\frac{x-2y}{xy}\right)\mathrm{d}x$$

$$\left(\frac{x-2y}{xy}\right)\mathrm{d}x + \left(-\frac{x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \frac{x-2y}{xy} \\ N(x, y) &= -\frac{x}{y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x-2y}{xy}\right) \\ &= -\frac{1}{y^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right) \\ &= -\frac{1}{y^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x - 2y}{xy} \, \mathrm{d}x$$
$$\phi = -2 \ln(x) + \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2 \ln(x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln(x) + \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{2\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2\ln(x) + c_1} \tag{1}$$

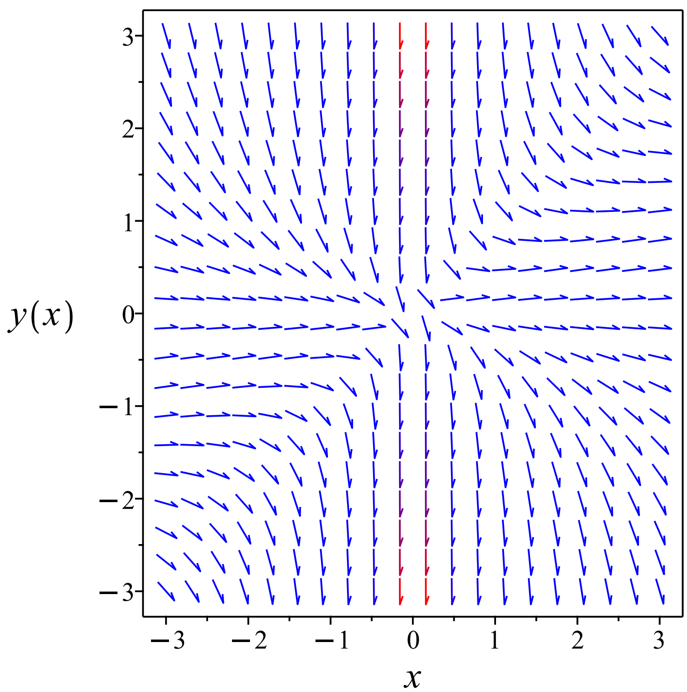


Figure 116: Slope field plot

Verification of solutions

$$y = \frac{x}{2\ln(x) + c_1}$$

Verified OK.

3.11.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{y(-x+2y)}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y}{x} - \frac{2y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{2}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{2u}{x^2}} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= \frac{4}{x^3} \\
f_1 f_2 &= -\frac{2}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{2u''(x)}{x^2} - \frac{2u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 \ln(x) + c_1
$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{2c_2 \ln(x) + 2c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{2\ln(x) + 2c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2\ln(x) + 2c_3} \tag{1}$$

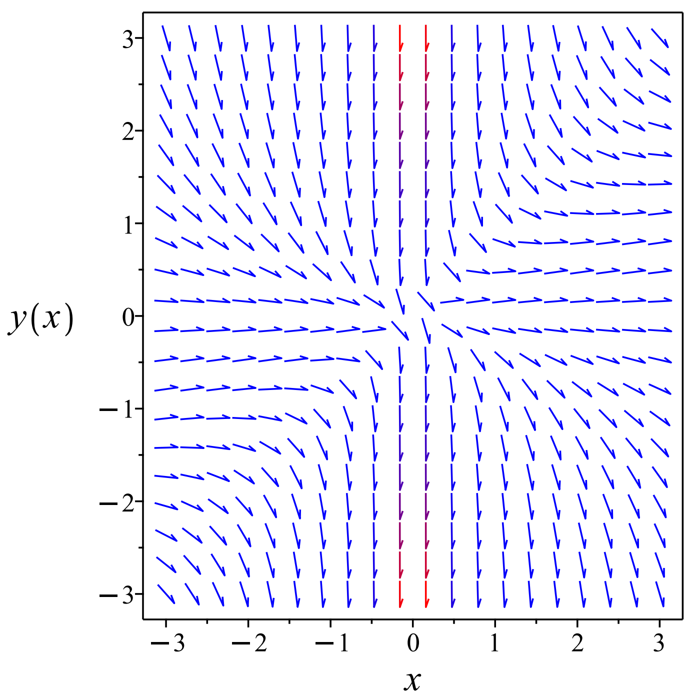


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{x}{2\ln(x) + 2c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)*(x-2*y(x))-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{2\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 21

```
DSolve[y[x]*(x-2*y[x])-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2\log(x) + c_1}$$
$$y(x) \to 0$$

3.12 problem 24 (c)

3.12.1 Solving as homogeneousTypeD2 ode 524
3.12.2 Solving as first order ode lie symmetry lookup ode 526
3.12.3 Solving as bernoulli ode 530
3.12.4 Solving as exact ode 533

Internal problem ID [5273]
Internal file name [`OUTPUT/4764_Friday_February_02_2024_05_11_41_AM_81589728/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 24 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^2 + yy'x = -x^2}$$

3.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + u(x)\,x^2(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2+1}{ux} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+1}{u}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{2u^2+1}{u}}\,du = \int -\frac{1}{x}\,dx$$

$$\frac{\ln\left(2u^2+1\right)}{4} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\left(2u^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\left(2u^2+1\right)^{\frac{1}{4}} = \frac{c_3}{x}$$

Which simplifies to

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

The solution is

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x}$$

<u>Summary</u>

The solution(s) found are the following

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3\mathrm{e}^{c_2}}{x} \tag{1}$$

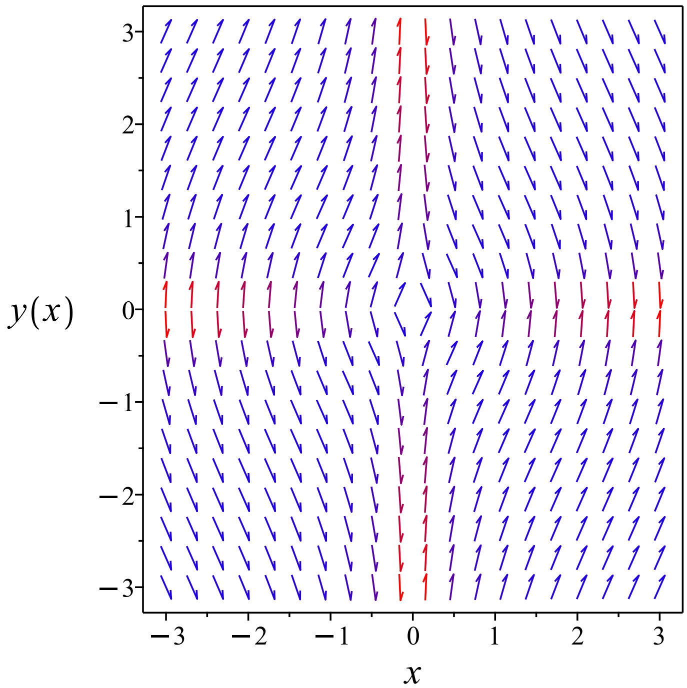


Figure 118: Slope field plot

Verification of solutions

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

3.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x^2+y^2}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 72: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{y\, x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y\,x^2}} dy \end{aligned}$$

Which results in

$$S = \frac{x^2 y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x^2 + y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^2 x \\ S_y &= x^2 y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^4}{4} + c_1 \tag{4}$$

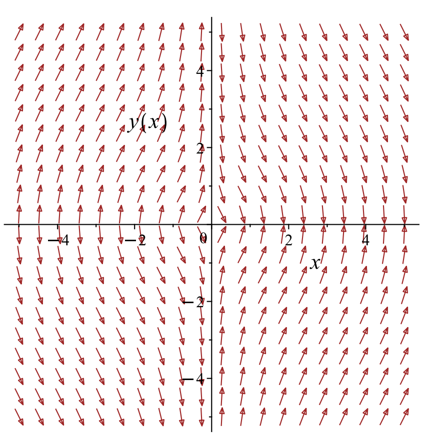
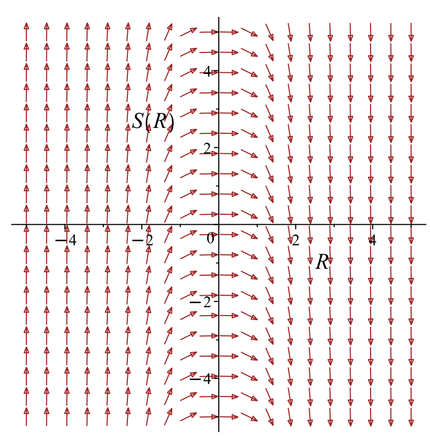
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1$$

Which simplifies to

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{xy}$ 	$R = x$ $S = \frac{x^2y^2}{2}$	$\frac{dS}{dR} = -R^3$ 

Summary

The solution(s) found are the following

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1 \tag{1}$$

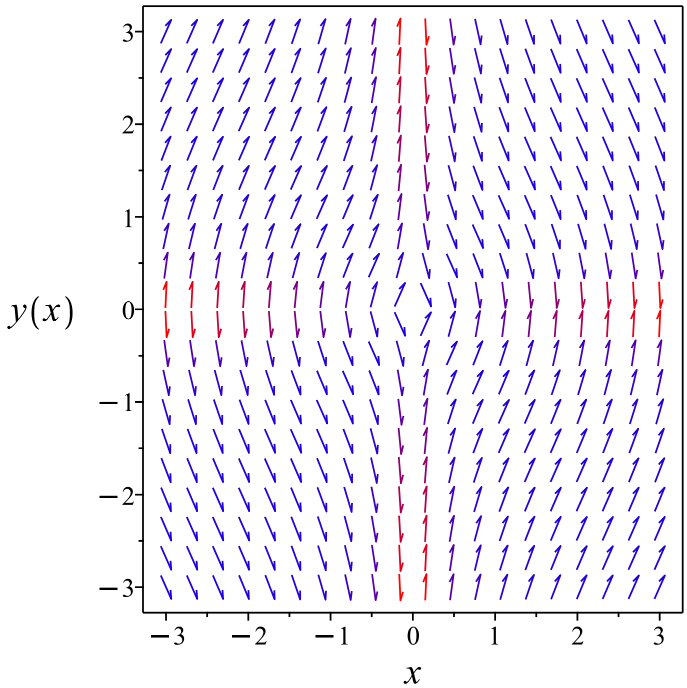


Figure 119: Slope field plot

Verification of solutions

$$\frac{x^2y^2}{2} = -\frac{x^4}{4} + c_1$$

Verified OK.

3.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x^2+y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= -x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{x} - x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{x} - x \\
w' &= -\frac{2w}{x} - 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = -2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-2x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2 w) &= (x^2)(-2x) \\ \mathrm{d}(x^2 w) &= (-2x^3)\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int -2x^3 \,\mathrm{d}x \\ x^2 w &= -\frac{x^4}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{-2x^4 + 4c_1}}{2x} \\ y(x) &= -\frac{\sqrt{-2x^4 + 4c_1}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x} \tag{2}$$

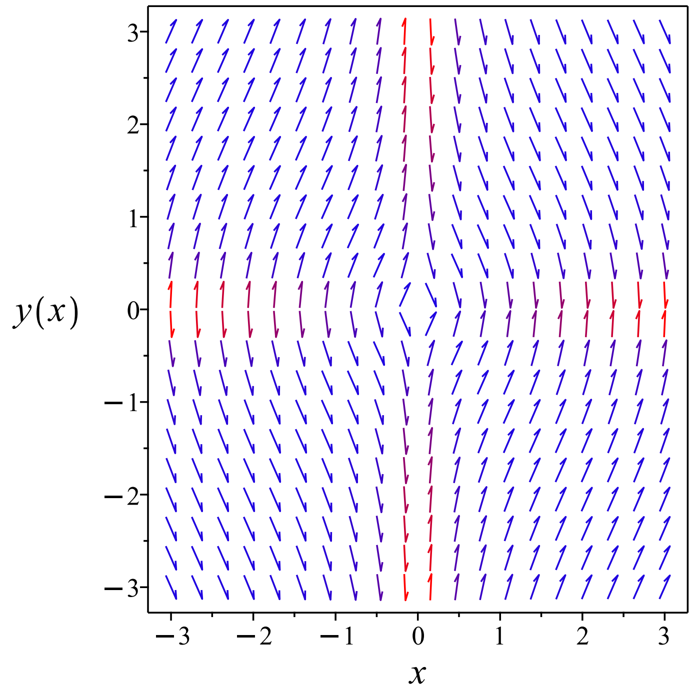


Figure 120: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

Verified OK.

3.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(xy)\,\mathrm{d}y &= \left(-x^2 - y^2\right)\mathrm{d}x \\
\left(x^2 + y^2\right)\mathrm{d}x + (xy)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= x^2 + y^2 \\
N(x,y) &= xy
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^2 + y^2\right) \\
&= 2y
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy) \\
&= y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yx}((2y) - (y)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(x^2 + y^2\right) \\
&= x\left(x^2 + y^2\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(xy) \\
&= x^2 y
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(x\left(x^2 + y^2\right)\right) + \left(x^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int x(x^2 + y^2) \,\mathrm{d}x$$
$$\phi = \frac{(x^2 + y^2)^2}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = (x^2 + y^2)\, y + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2 y$. Therefore equation (4) becomes

$$x^2 y = (x^2 + y^2)\, y + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (-y^3) \,\mathrm{d}y$$
$$f(y) = -\frac{y^4}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4}$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1 \tag{1}$$

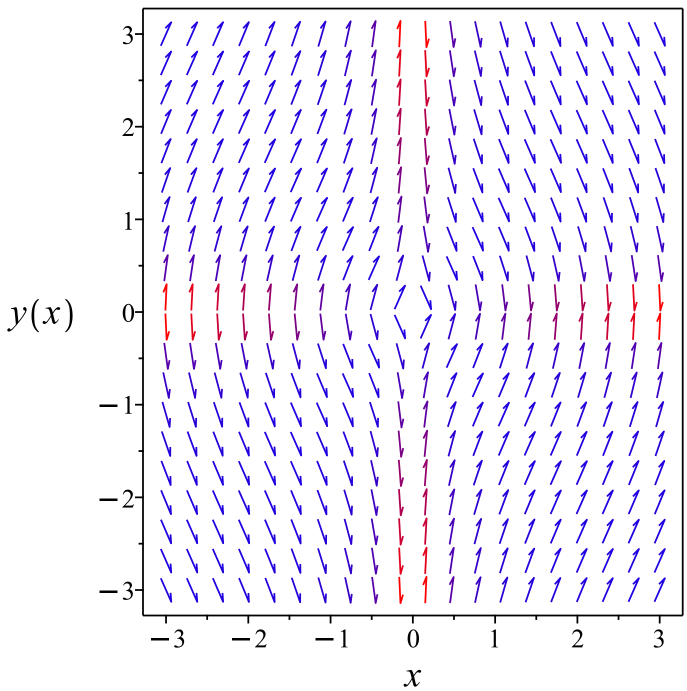


Figure 121: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2)^2}{4} - \frac{y^4}{4} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 39

```
dsolve((x^2+y(x)^2)+x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^4 + 4c_1}}{2x}$$
$$y(x) = \frac{\sqrt{-2x^4 + 4c_1}}{2x}$$

✓ Solution by Mathematica

Time used: 0.198 (sec). Leaf size: 46

```
DSolve[(x^2+y[x]^2)+x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -\frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$
$$y(x) \rightarrow \frac{\sqrt{-\frac{x^4}{2} + c_1}}{x}$$

3.13 problem 24 (d)

3.13.1 Solving as homogeneousTypeD2 ode 539
3.13.2 Solving as first order ode lie symmetry lookup ode 541
3.13.3 Solving as bernoulli ode 545
3.13.4 Solving as exact ode 548
3.13.5 Maple step by step solution 552

Internal problem ID [5274]
Internal file name [`OUTPUT/4765_Friday_February_02_2024_05_11_43_AM_34495175/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 24 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{y^2 + 2yy'x = -x^2}$$

3.13.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + 2u(x)\,x^2(u'(x)\,x + u(x)) = -x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u^2 + 1}{2ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{3u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2+1}{u}}\,du = -\frac{1}{2x}\,dx$$
$$\int \frac{1}{\frac{3u^2+1}{u}}\,du = \int -\frac{1}{2x}\,dx$$
$$\frac{\ln(3u^2+1)}{6} = -\frac{\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\left(3u^2+1\right)^{\frac{1}{6}} = e^{-\frac{\ln(x)}{2}+c_2}$$

Which simplifies to

$$\left(3u^2+1\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}$$

Which simplifies to

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

The solution is

$$\left(3u(x)^2+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{3y^2}{x^2}+1\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$
$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 e^{c_2}}{\sqrt{x}} \tag{1}$$

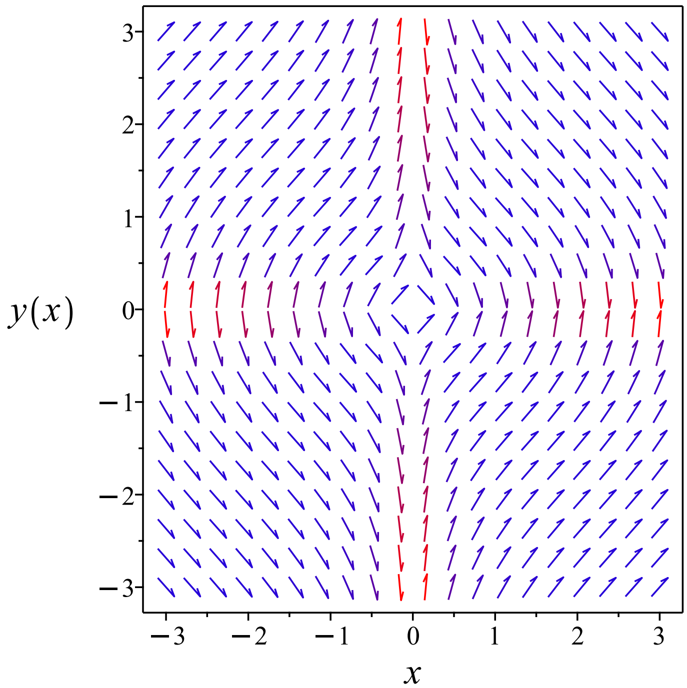


Figure 122: Slope field plot

Verification of solutions

$$\left(\frac{3y^2+x^2}{x^2}\right)^{\frac{1}{6}}=\frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Verified OK.

3.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{x^2+y^2}{2xy}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{yx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= xy \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x^2}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{6} + c_1 \tag{4}$$

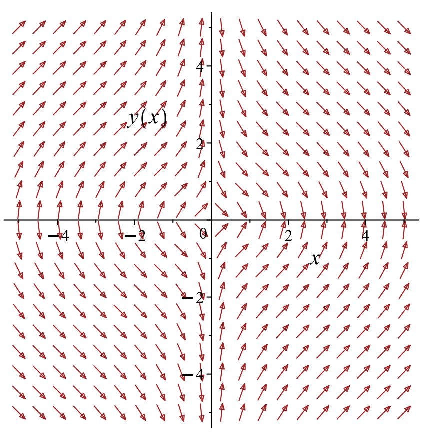
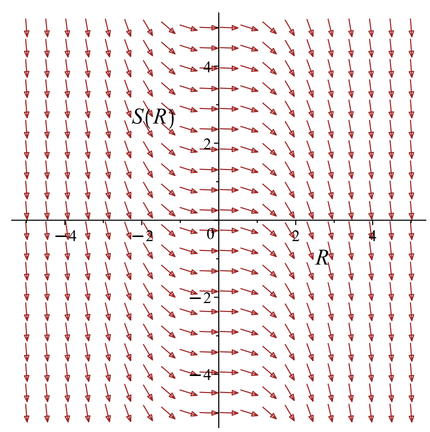
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x y^2}{2} = -\frac{x^3}{6} + c_1$$

Which simplifies to

$$\frac{x y^2}{2} = -\frac{x^3}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2 x}{2}$	$\frac{dS}{dR} = -\frac{R^2}{2}$ 

Summary

The solution(s) found are the following

$$\frac{x y^2}{2} = -\frac{x^3}{6} + c_1 \tag{1}$$

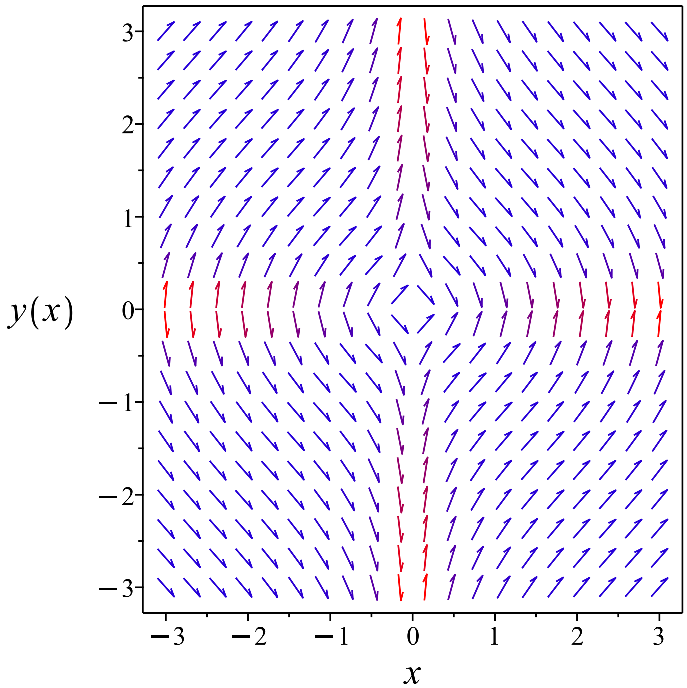


Figure 123: Slope field plot

Verification of solutions

$$\frac{x y^2}{2} = -\frac{x^3}{6} + c_1$$

Verified OK.

3.13.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x} y - \frac{x}{2} \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x) y + f_1(x) y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x) y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= -\frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{2x} - \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{x}{2} \\
w' &= -\frac{w}{x} - x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(-x) \\ \frac{d}{dx}(xw) &= (x)(-x) \\ d(xw) &= \left(-x^2\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -x^2 \, dx \\ xw &= -\frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^2}{3} + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{3} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \\ y(x) &= -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x} \tag{2}$$

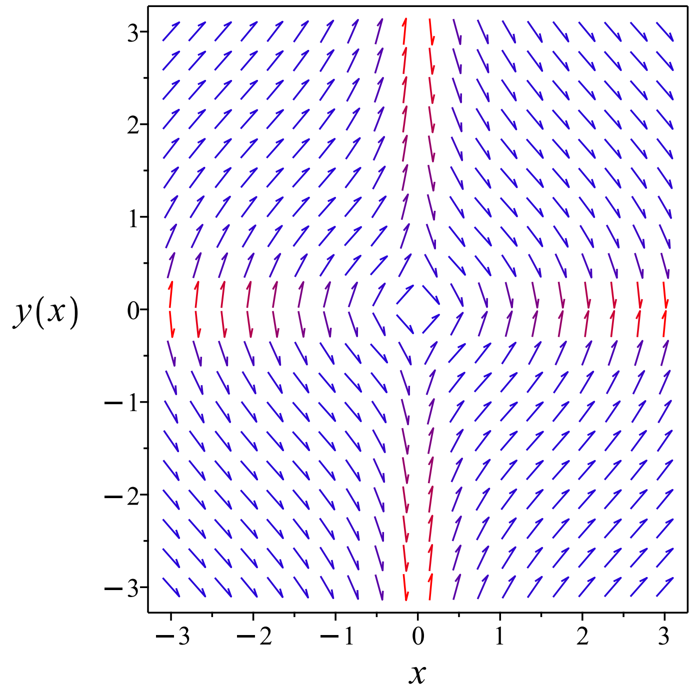


Figure 124: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{3}\,\sqrt{-x\,(x^3 - 3c_1)}}{3x}$$

Verified OK.

3.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(-x^2-y^2\right)\mathrm{d}x\\ \left(x^2+y^2\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x^2+y^2\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x^2+y^2\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy) \\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int x^2 + y^2\, \mathrm{d}x\end{aligned}$$

$$\phi = \frac{1}{3}x^3 + y^2 x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3}x^3 + y^2 x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3}x^3 + y^2 x$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + x y^2 = c_1 \tag{1}$$

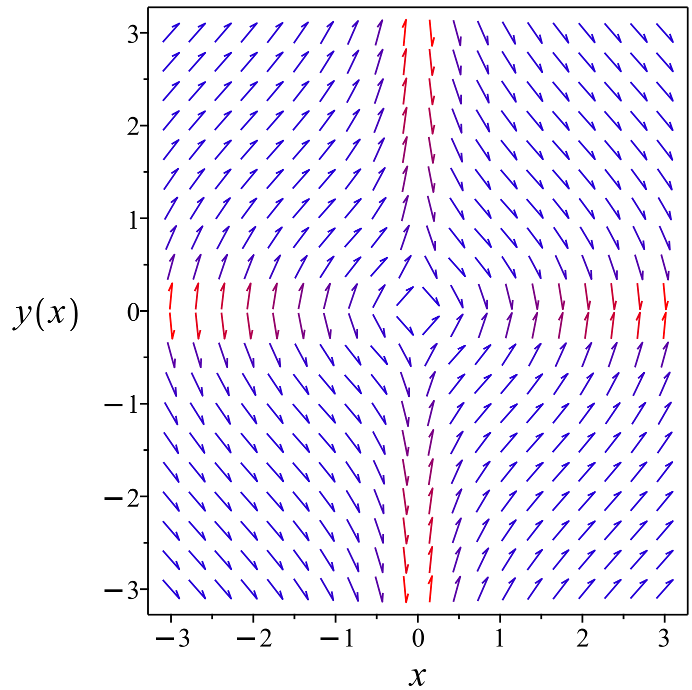


Figure 125: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + x y^2 = c_1$$

Verified OK.

3.13.5 Maple step by step solution

Let's solve

$y^2 + 2yy'x = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  ◦ Evaluate derivatives

    $2y = 2y$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x^2 + y^2)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = \frac{x^3}{3} + y^2x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $2xy = 2xy + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = \frac{1}{3}x^3 + y^2 x$

- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{1}{3}x^3 + y^2 x = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\,\sqrt{x(-x^3+3c_1)}}{3x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve((x^2+y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x} \\
y(x) &= \frac{\sqrt{3}\,\sqrt{-x\,(x^3-3c_1)}}{3x}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.207 (sec). Leaf size: 60

```
DSolve[(x^2+y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}} \\
y(x) &\to \frac{\sqrt{-x^3+3c_1}}{\sqrt{3}\sqrt{x}}
\end{aligned}
$$

3.14 problem 24 (g)

3.14.1 Solving as quadrature ode 554
3.14.2 Maple step by step solution 555

Internal problem ID [5275]
Internal file name [`OUTPUT/4766_Friday_February_02_2024_05_11_44_AM_93727827/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 24 (g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{-\sqrt{a^2-x^2}\, y' = -1}$$

3.14.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{a^2-x^2}}\, \mathrm{d}x \\ &= \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + c_1 \tag{1}$$

Verification of solutions

$$y = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + c_1$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$-\sqrt{a^2-x^2}\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{1}{\sqrt{a^2-x^2}}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{\sqrt{a^2-x^2}} dx + c_1$

- Evaluate integral

  $y = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + c_1$

- Solve for y

  $y = \arctan\left(\frac{x}{\sqrt{a^2-x^2}}\right) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(1-(sqrt(a^2-x^2))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 24

```
DSolve[1-(Sqrt[a^2-x^2])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arctan\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c_1$$

3.15 problem 24 (L)

3.15.1 Solving as homogeneousTypeMapleC ode 557
3.15.2 Solving as first order ode lie symmetry calculated ode 560

Internal problem ID [5276]
Internal file name [`OUTPUT/4767_Friday_February_02_2024_05_11_45_AM_93566786/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 24 (L).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y-3)\,y'=-x-1}$$

3.15.1 Solving as homogeneousTypeMapleC ode

Let $Y=y+y_0$ and $X=x+x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=-\frac{X+x_0+1+Y(X)+y_0}{Y(X)+y_0-X-x_0+3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=1$$
$$y_0=-2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=-\frac{X+Y(X)}{Y(X)-X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= -\frac{X+Y}{Y-X}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M = X + Y$ and $N = -Y + X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-u-1}{u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{-u(X)-1}{u(X)-1} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X u(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 1 = 0
$$

Or

$$
X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{u^2+1}{X(u-1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u^2+1)}{2} - \arctan(u) = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(u(X)^2+1\right)}{2} - \arctan(u(X)) + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2}{X^2}+1\right)}{2} - \arctan\left(\frac{Y(X)}{X}\right) + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 2$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{(y+2)^2}{(x-1)^2}+1\right)}{2} - \arctan\left(\frac{y+2}{x-1}\right) + \ln(x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{(y+2)^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y+2}{x-1}\right)+\ln(x-1)-c_2=0 \tag{1}$$

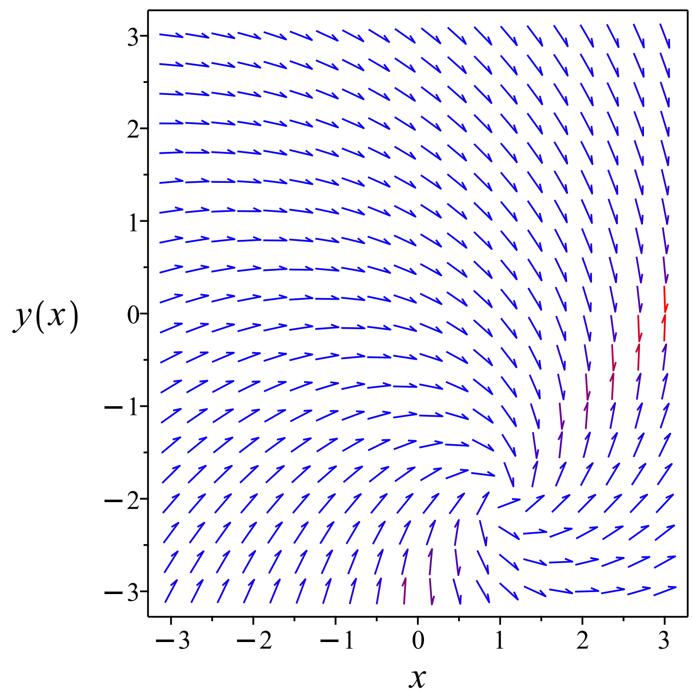


Figure 126: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{(y+2)^2}{(x-1)^2}+1\right)}{2}-\arctan\left(\frac{y+2}{x-1}\right)+\ln(x-1)-c_2=0$$

Verified OK.

3.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{x+y+1}{-x+y+3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(x+y+1)(b_3 - a_2)}{-x+y+3} - \frac{(x+y+1)^2 a_3}{(-x+y+3)^2} \\ &- \left(-\frac{1}{-x+y+3} - \frac{x+y+1}{(-x+y+3)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{-x+y+3} + \frac{x+y+1}{(-x+y+3)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 - 6xa_2 + 2xa}{(x - y -}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 - 2xyb_3 + y^2a_2 \\ &+ y^2a_3 + y^2b_2 - y^2b_3 + 6xa_2 - 2xa_3 - 2xb_1 - 4xb_2 - 2xb_3 + 2ya_1 \\ &+ 4ya_2 + 2ya_3 + 6yb_2 - 2yb_3 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-a_2 v_1^2 + 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 - 2a_3 v_1 v_2 + a_3 v_2^2 - b_2 v_1^2 - 2b_2 v_1 v_2 + b_2 v_2^2 \\
&+ b_3 v_1^2 - 2b_3 v_1 v_2 - b_3 v_2^2 + 2a_1 v_2 + 6a_2 v_1 + 4a_2 v_2 - 2a_3 v_1 + 2a_3 v_2 - 2b_1 v_1 \\
&- 4b_2 v_1 + 6b_2 v_2 - 2b_3 v_1 - 2b_3 v_2 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - a_3 - b_2 + b_3) v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3) v_1 v_2 \\
&+ (6a_2 - 2a_3 - 2b_1 - 4b_2 - 2b_3) v_1 + (a_2 + a_3 + b_2 - b_3) v_2^2 \\
&+ (2a_1 + 4a_2 + 2a_3 + 6b_2 - 2b_3) v_2 + 4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_2 - a_3 - b_2 + b_3 &= 0 \\
a_2 + a_3 + b_2 - b_3 &= 0 \\
2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \\
2a_1 + 4a_2 + 2a_3 + 6b_2 - 2b_3 &= 0 \\
6a_2 - 2a_3 - 2b_1 - 4b_2 - 2b_3 &= 0 \\
4a_1 + 3a_2 - a_3 + 2b_1 + 9b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -2b_2 - b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 + 2b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -2 - y$$
$$\eta = x - 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - 1 - \left(-\frac{x + y + 1}{-x + y + 3}\right)(-2 - y) \\
&= \frac{x^2 + y^2 - 2x + 4y + 5}{x - y - 3} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 + y^2 - 2x + 4y + 5}{x - y - 3}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln\left(x^2 + y^2 - 2x + 4y + 5\right)}{2} + \frac{2(x - 1)\arctan\left(\frac{2y+4}{2x-2}\right)}{2x - 2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y + 1}{-x + y + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-x - 1 - y}{x^2 + y^2 - 2x + 4y + 5} \\ S_y &= \frac{x - y - 3}{x^2 + y^2 - 2x + 4y + 5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

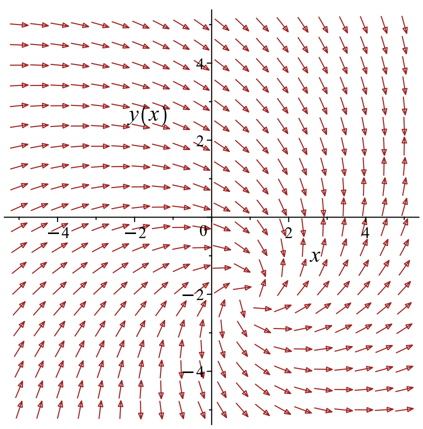
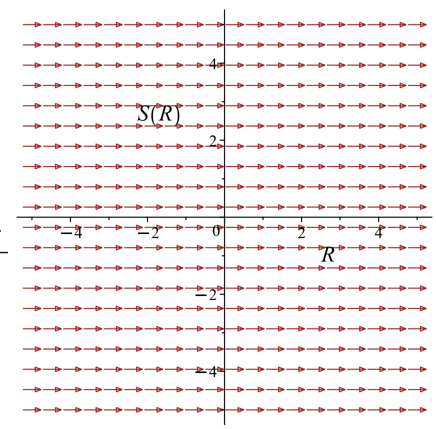
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} + \arctan\left(\frac{y + 2}{x - 1}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y^2 + x^2 + 4y - 2x + 5)}{2} + \arctan\left(\frac{y + 2}{x - 1}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y+1}{-x+y+3}$ 	$R = x$ $S = -\frac{\ln(x^2+y^2-2x+}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\frac{\ln(y^2+x^2+4y-2x+5)}{2} + \arctan\left(\frac{y+2}{x-1}\right) = c_1 \tag{1}$$

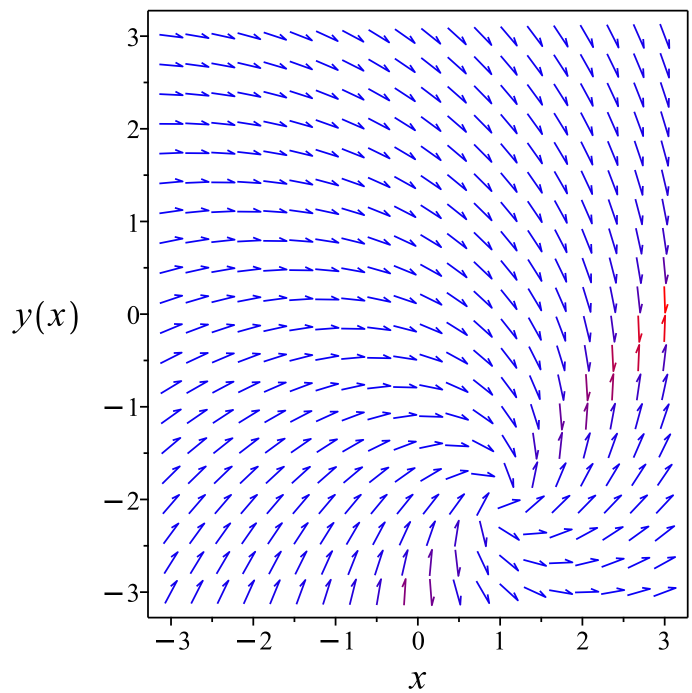


Figure 127: Slope field plot

Verification of solutions

$$-\frac{\ln \left(y^{2}+x^{2}+4y-2x+5\right)}{2}+\arctan\left(\frac{y+2}{x-1}\right)=c_{1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve((x+y(x)+1)-(x-y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -2 - \tan\left(\mathrm{RootOf}\left(2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)(x-1)$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 58

```
DSolve[(x+y[x]+1)-(x-y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[2\arctan\left(\frac{y(x)+x+1}{y(x)-x+3}\right) + \log\left(\frac{x^2+y(x)^2+4y(x)-2x+5}{2(x-1)^2}\right) + 2\log(x-1) + c_1 = 0, y(x)\right]$$

3.16 problem 25 (a)

3.16.1 Solving as exact ode 568

Internal problem ID [5277]
Internal file name [OUTPUT/4768_Friday_February_02_2024_05_11_47_AM_88750982/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Bernoulli]
```

$$\boxed{-y^2 + yy' = x^2 - x}$$

3.16.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y &= \left(x^2+y^2-x\right)\mathrm{d}x\\ \left(-x^2-y^2+x\right)\mathrm{d}x + (y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2-y^2+x\\ N(x,y) &= y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2-y^2+x\right)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{y}((-2y)-(0))\\ &= -2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -2 \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2x}\\ &= \mathrm{e}^{-2x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-2x}\left(-x^2 - y^2 + x\right)\\ &= -\left(x^2 + y^2 - x\right)\mathrm{e}^{-2x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-2x}(y)\\ &= y\,\mathrm{e}^{-2x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\left(x^2 + y^2 - x\right)\mathrm{e}^{-2x}\right) + \left(y\,\mathrm{e}^{-2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\left(x^2 + y^2 - x\right)\mathrm{e}^{-2x}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{\left(x^2 + y^2\right)\mathrm{e}^{-2x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = y\,\mathrm{e}^{-2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y\,\mathrm{e}^{-2x}$. Therefore equation (4) becomes

$$y\,\mathrm{e}^{-2x} = y\,\mathrm{e}^{-2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(x^2 + y^2)\,\mathrm{e}^{-2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(x^2 + y^2)\,\mathrm{e}^{-2x}}{2}$$

Summary

The solution(s) found are the following

$$\frac{(x^2 + y^2)\,\mathrm{e}^{-2x}}{2} = c_1 \tag{1}$$

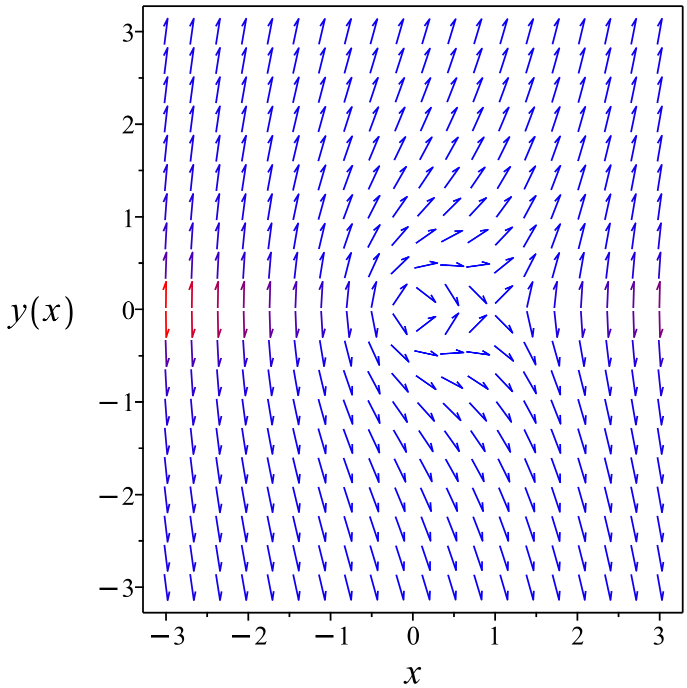


Figure 128: Slope field plot

Verification of solutions

$$\frac{(x^2 + y^2)\,e^{-2x}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve((x-x^2-y(x)^2)+y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \sqrt{\mathrm{e}^{2x} c_1 - x^2} \\
y(x) &= -\sqrt{\mathrm{e}^{2x} c_1 - x^2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 4.613 (sec). Leaf size: 47

```
DSolve[(x-x^2-y[x]^2)+y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{-x^2 + c_1 e^{2x}} \\
y(x) &\to \sqrt{-x^2 + c_1 e^{2x}}
\end{aligned}
$$

3.17 problem 25 (b)

3.17.1 Solving as exact ode 574
3.17.2 Maple step by step solution 579

Internal problem ID [5278]
Internal file name [OUTPUT/4769_Friday_February_02_2024_05_11_47_AM_41865652/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{xy' + 2y = 3x}$$

3.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x)\,\mathrm{d}y &= (-2y + 3x)\,\mathrm{d}x \\
(2y - 3x)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2y - 3x \\
N(x, y) &= x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - 3x) \\
&= 2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((2) - (1)) \\ &= \frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x)} \\ &= x \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x(2y - 3x) \\ &= (2y - 3x)\,x \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x(x) \\ &= x^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((2y - 3x)\,x) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int (2y-3x)\,x\,\mathrm{d}x$$
$$\phi = -x^2(x-y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2(x-y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2(x-y)$$

The solution becomes

$$y = \frac{x^3 + c_1}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{x^3 + c_1}{x^2} \tag{1}$$

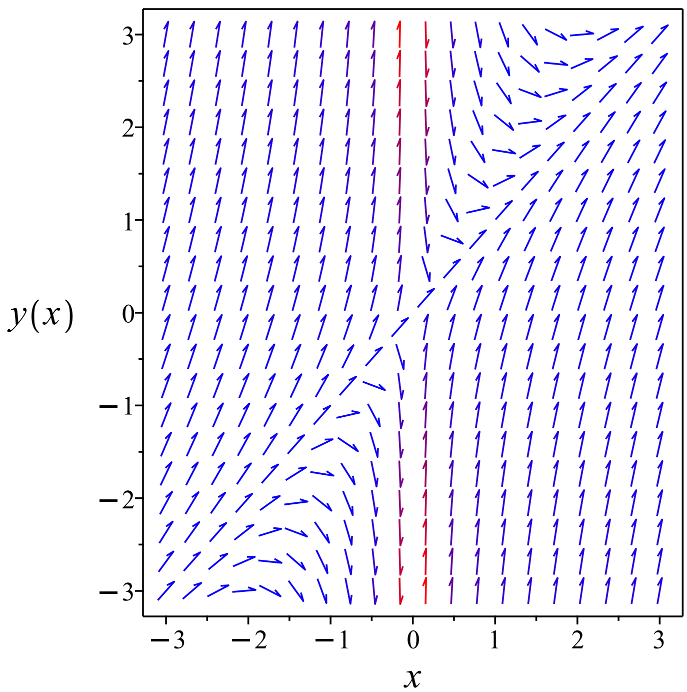


Figure 129: Slope field plot

Verification of solutions

$$y = \frac{x^3 + c_1}{x^2}$$

Verified OK.

3.17.2 Maple step by step solution

Let's solve

$xy' + 2y = 3x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3 - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = 3\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 3\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 3\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int 3\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int 3x^2 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^3 + c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((2*y(x)-3*x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = x + \frac{c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 13

```
DSolve[(2*y[x]-3*x)+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{c_1}{x^2}$$

3.18 problem 25 (c)

3.18.1 Solving as exact ode 581

Internal problem ID [5279]
Internal file name [`OUTPUT/4770_Friday_February_02_2024_05_11_47_AM_45671391/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{-y^2 + 2yy'x = -x}$$

3.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(y^2-x\right)\mathrm{d}x\\ \left(-y^2+x\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^2+x\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-y^2+x\right)\\ &=-2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2xy)\\ &=2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{2yx}((-2y)-(2y))\\ &=-\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(-y^2 + x) \\
&= \frac{-y^2 + x}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(2xy) \\
&= \frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^2 + x}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{-y^2 + x}{x^2} \, \mathrm{d}x \\
\phi &= \frac{y^2}{x} + \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y^2}{x} + \ln(x) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{y^2}{x} + \ln(x)
$$

Summary

The solution(s) found are the following

$$
\frac{y^2}{x} + \ln(x) = c_1 \qquad (1)
$$

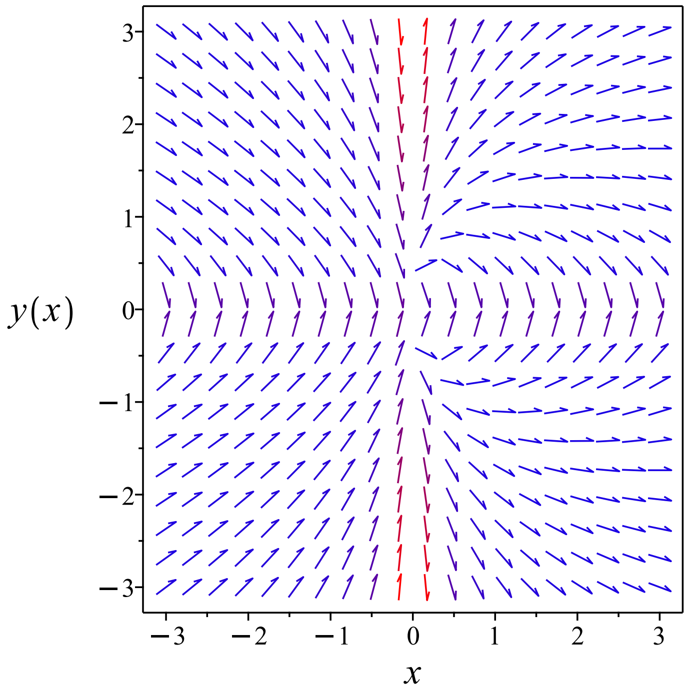


Figure 130: Slope field plot

Verification of solutions

$$\frac{y^2}{x} + \ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((x-y(x)^2)+2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-x\left(\ln\left(x\right)-c_1\right)}$$
$$y(x) = -\sqrt{x\left(-\ln\left(x\right)+c_1\right)}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 44

```
DSolve[(x-y[x]^2)+2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{-\log(x)+c_1}$$
$$y(x) \to \sqrt{x}\sqrt{-\log(x)+c_1}$$

3.19 problem 25 (d)

3.19.1 Solving as exact ode 587

Internal problem ID [5280]
Internal file name [OUTPUT/4771_Friday_February_02_2024_05_11_47_AM_57323726/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{-y - 3x^2(x^2 + y^2) + xy' = 0}$$

3.19.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(y + 3x^2\left(x^2+y^2\right)\right)\mathrm{d}x\\ \left(-y - 3x^2\left(x^2+y^2\right)\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y - 3x^2\left(x^2+y^2\right)\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y - 3x^2\left(x^2+y^2\right)\right)\\ &= -6x^2y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y - 3x^2(x^2+y^2)$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{-y - 3x^2(x^2+y^2)}{x^2+y^2} \\
N &= \frac{x}{x^2+y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{x}{x^2+y^2}\right)\mathrm{d}y &= \left(-\frac{-y - 3x^2(x^2+y^2)}{x^2+y^2}\right)\mathrm{d}x \\
\left(\frac{-y - 3x^2(x^2+y^2)}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{x}{x^2+y^2}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{-y-3x^2(x^2+y^2)}{x^2+y^2} \\
N(x,y) &= \frac{x}{x^2+y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-y-3x^2(x^2+y^2)}{x^2+y^2}\right) \\
&= \frac{-x^2+y^2}{(x^2+y^2)^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \\
&= \frac{-x^2+y^2}{(x^2+y^2)^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int \frac{-y-3x^2(x^2+y^2)}{x^2+y^2}\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^3 - \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \\ &= \frac{x}{x^2+y^2} + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x}{x^2+y^2} = \frac{x}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 - \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 - \arctan\left(\frac{x}{y}\right)$$

The solution becomes

$$y = -\frac{x}{\tan\left(x^3+c_1\right)}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\tan\left(x^3+c_1\right)} \tag{1}$$

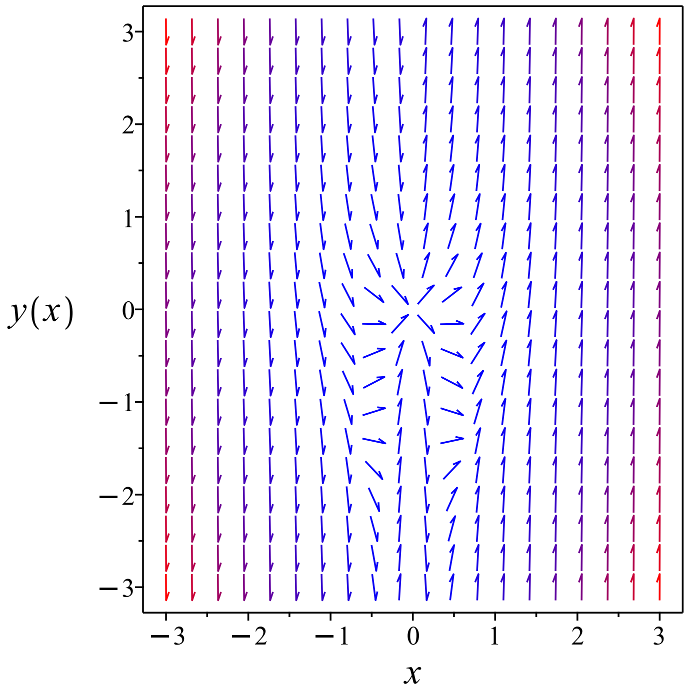


Figure 131: Slope field plot

Verification of solutions

$$y = -\frac{x}{\tan(x^3 + c_1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve((-y(x)-3*x^2*(x^2+y(x)^2))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(x^3 + 3c_1\right) x$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 14

```
DSolve[(-y[x]-3*x^2*(x^2+y[x]^2))+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \tan\left(x^3 + c_1\right)$$

3.20 problem 25 (e)

3.20.1 Solving as exact ode 594
3.20.2 Maple step by step solution 599

Internal problem ID [5281]
Internal file name [`OUTPUT/4772_Friday_February_02_2024_05_11_48_AM_79599042/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-xy' + y = \ln(x)}$$

3.20.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x)\,\mathrm{d}y &= (-y + \ln(x))\,\mathrm{d}x \\
(y - \ln(x))\,\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= y - \ln(x) \\
N(x,y) &= -x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \ln(x)) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x}((1) - (-1)) \\
&= -\frac{2}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(y - \ln(x)) \\
&= \frac{y - \ln(x)}{x^2}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-x) \\
&= -\frac{1}{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y - \ln(x)}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y - \ln(x)}{x^2} \, \mathrm{d}x$$

$$\phi = \frac{-y + \ln(x) + 1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-y + \ln(x) + 1}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-y + \ln(x) + 1}{x}$$

The solution becomes

$$y = -c_1 x + \ln(x) + 1$$

Summary

The solution(s) found are the following

$$y = -c_1 x + \ln(x) + 1 \tag{1}$$

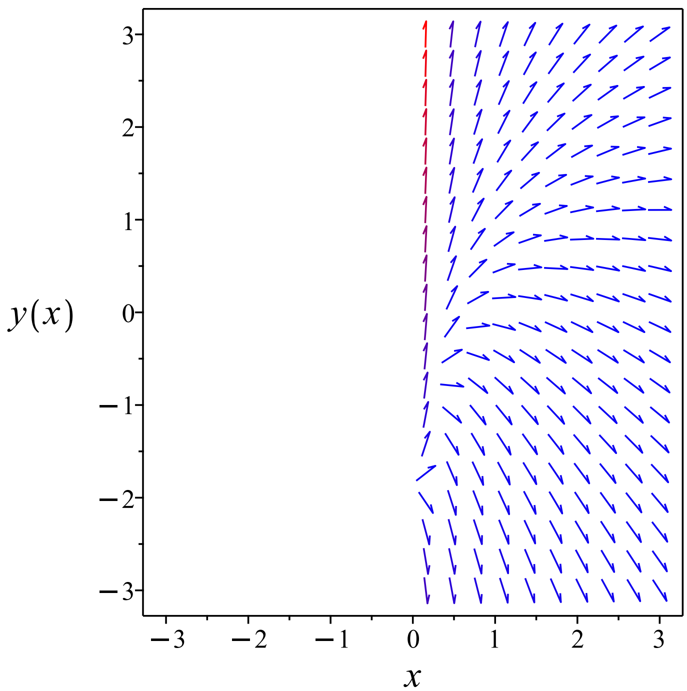


Figure 132: Slope field plot

Verification of solutions

$$y = -c_1 x + \ln(x) + 1$$

Verified OK.

3.20.2 Maple step by step solution

Let's solve

$-xy' + y = \ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} - \frac{\ln(x)}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = -\frac{\ln(x)}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = -\frac{\mu(x)\ln(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)\ln(x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)\ln(x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int -\frac{\ln(x)}{x^2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + c_1\right)$

- Simplify

$$y = c_1 x + \ln(x) + 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((y(x)-ln(x))-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x + \ln(x) + 1$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 13

```
DSolve[(y[x]-Log[x])-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1 x + 1$$

3.21 problem 25 (f)

3.21.1 Solving as exact ode 601

Internal problem ID [5282]
Internal file name [`OUTPUT/4773_Friday_February_02_2024_05_11_48_AM_87394506/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{-2yy'x+y^2=-3x^2}$$

3.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2xy)\,\mathrm{d}y&=\left(-3x^2-y^2\right)\mathrm{d}x\\ \left(3x^2+y^2\right)\mathrm{d}x+(-2xy)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2+y^2\\ N(x,y)&=-2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(3x^2+y^2\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(-2xy)\\ &=-2y\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=-\frac{1}{2yx}((2y)-(-2y))\\ &=-\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(3x^2 + y^2) \\
&= \frac{3x^2 + y^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-2xy) \\
&= -\frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{3x^2 + y^2}{x^2}\right) + \left(-\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{3x^2 + y^2}{x^2} \mathrm{d}x \\
\phi &= 3x - \frac{y^2}{x} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{2y}{x} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{2y}{x}$. Therefore equation (4) becomes

$$
-\frac{2y}{x} = -\frac{2y}{x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = 3x - \frac{y^2}{x} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = 3x - \frac{y^2}{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
3x - \frac{y^2}{x} = c_1 \qquad (1)
$$

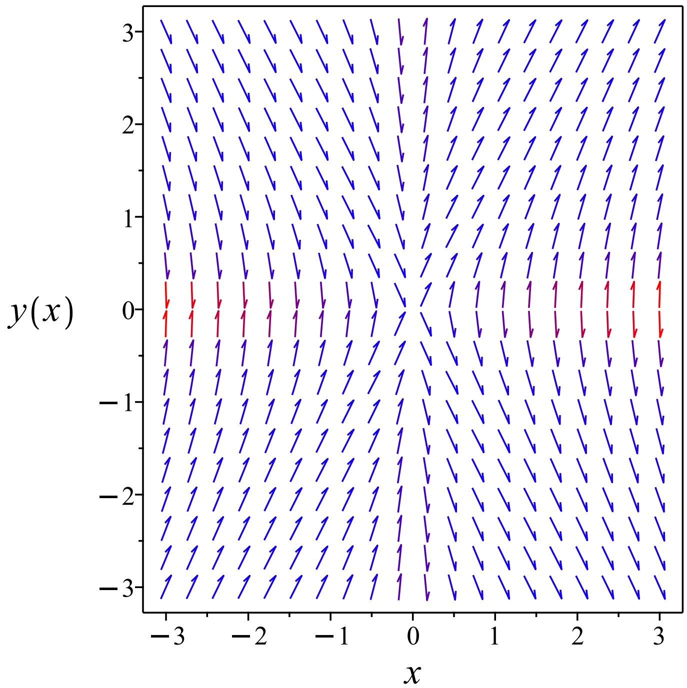


Figure 133: Slope field plot

Verification of solutions

$$3x - \frac{y^2}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 27

```
dsolve((3*x^2+y(x)^2)-2*x*y(x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{(3x + c_1) x}$$
$$y(x) = -\sqrt{(3x + c_1) x}$$

✓ Solution by Mathematica

Time used: 0.182 (sec). Leaf size: 42

```
DSolve[(3*x^2+y[x]^2)-2*x*y[x]*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{x}\sqrt{3x + c_1}$$
$$y(x) \to \sqrt{x}\sqrt{3x + c_1}$$

3.22 problem 25 (g)

3.22.1 Solving as exact ode 607

Internal problem ID [5283]
Internal file name [`OUTPUT/4774_Friday_February_02_2024_05_11_48_AM_23473528/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{yx - 2y^2 - (x^2 - 3yx)\, y' = 0}$$

3.22.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-x^2 + 3xy\right) \mathrm{d}y &= \left(-xy + 2y^2\right) \mathrm{d}x \\
\left(xy - 2y^2\right) \mathrm{d}x + \left(-x^2 + 3xy\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= xy - 2y^2 \\
N(x, y) &= -x^2 + 3xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(xy - 2y^2\right) \\
&= x - 4y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2 + 3xy\right) \\
&= -2x + 3y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = yx - 2y^2$ and $N = -x^2 + 3yx$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{yx - 2y^2}{xy^2}$$

$$N = \frac{-x^2 + 3yx}{xy^2}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x^2+3xy}{x\,y^2}\right)\mathrm{d}y = \left(-\frac{xy-2y^2}{x\,y^2}\right)\mathrm{d}x$$

$$\left(\frac{xy-2y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{-x^2+3xy}{x\,y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{xy-2y^2}{x\,y^2}$$
$$N(x,y) = \frac{-x^2+3xy}{x\,y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{xy-2y^2}{x\,y^2}\right)\\ &= -\frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x^2+3xy}{x\,y^2}\right)\\ &= -\frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{xy-2y^2}{x\,y^2}\,\mathrm{d}x$$
$$\phi = -2\ln(x) + \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+3xy}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{-x^2+3xy}{x\,y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{3}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{3}{y}\right) \mathrm{d}y$$
$$f(y) = 3\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\ln(x) + \frac{x}{y} + 3\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln(x) + \frac{x}{y} + 3\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\mathrm{LambertW}\left(-\frac{x\,\mathrm{e}^{-\frac{2\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)+\frac{2\ln(x)}{3}+\frac{c_1}{3}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}^{-\frac{2\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)+\frac{2\ln(x)}{3}+\frac{c_1}{3}} \tag{1}$$

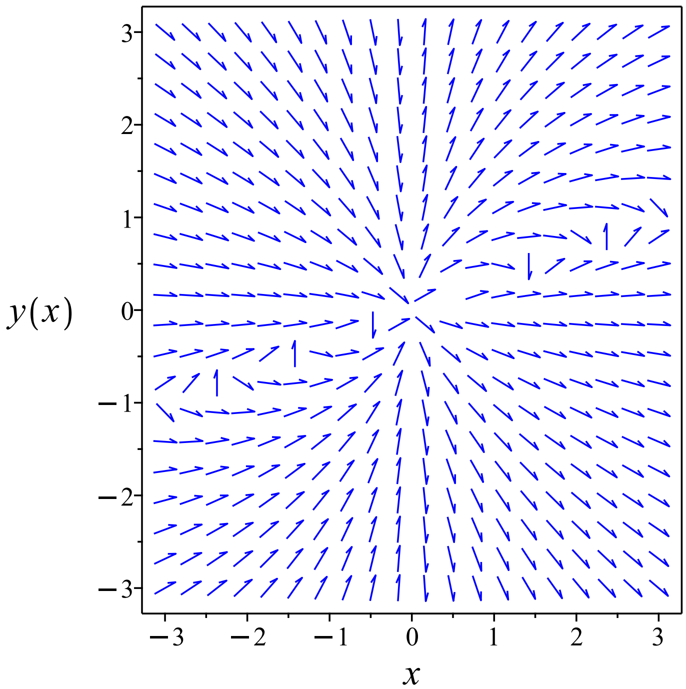


Figure 134: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\operatorname{LambertW}\left(-\frac{x\,\mathrm{e}^{-\frac{2\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)+\frac{2\ln(x)}{3}+\frac{c_1}{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 19

```
dsolve((x*y(x)-2*y(x)^2)-(x^2-3*x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x}{3\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{\frac{c_1}{3}} x^{\frac{1}{3}}}{3}\right)}$$

✓ Solution by Mathematica
Time used: 4.722 (sec). Leaf size: 35

```
DSolve[(x*y[x]-2*y[x]^2)-(x^2-3*x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{3W\left(-\frac{1}{3}e^{-\frac{c_1}{3}}\sqrt[3]{x}\right)}$$
$$y(x) \to 0$$

3.23 problem 25 (h)

3.23.1 Solving as exact ode 614

Internal problem ID [5284]
Internal file name [`OUTPUT/4775_Friday_February_02_2024_05_11_49_AM_67211904/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y-(x-y)\,y'=-x}$$

3.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= M \\
\frac{\partial\phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x+y)\,\mathrm{d}y &= (-x-y)\,\mathrm{d}x \\
(x+y)\,\mathrm{d}x + (-x+y)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= x+y \\
N(x,y) &= -x+y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x+y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x+y) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = x + y$ and $N = -x + y$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x+y}{x^2+y^2}$$

$$N = \frac{-x+y}{x^2+y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{-x+y}{x^2+y^2}\right) \mathrm{d}y = \left(-\frac{x+y}{x^2+y^2}\right) \mathrm{d}x$$

$$\left(\frac{x+y}{x^2+y^2}\right) \mathrm{d}x + \left(\frac{-x+y}{x^2+y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{x+y}{x^2+y^2}$$
$$N(x,y) = \frac{-x+y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-x+y}{x^2+y^2}\right)\\ &= \frac{x^2-2xy-y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{x+y}{x^2+y^2} \mathrm{d}x$$
$$\phi = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{y}{x^2+y^2} - \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \qquad (4)\\
&= \frac{-x+y}{x^2+y^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x+y}{x^2+y^2}$. Therefore equation (4) becomes

$$
\frac{-x+y}{x^2+y^2} = \frac{-x+y}{x^2+y^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right)
$$

Summary

The solution(s) found are the following

$$
\frac{\ln(x^2+y^2)}{2} + \arctan\left(\frac{x}{y}\right) = c_1 \qquad (1)
$$

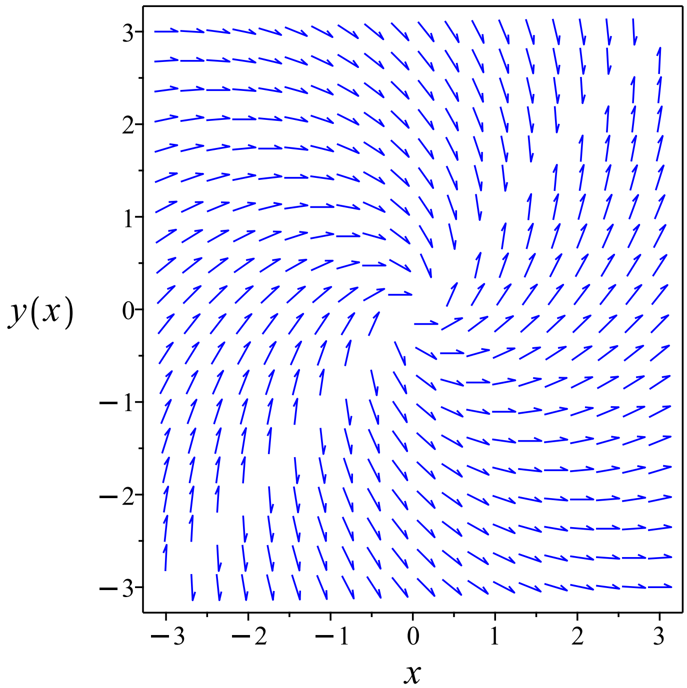


Figure 135: Slope field plot

Verification of solutions

$$\frac{\ln (x^2 + y^2)}{2} + \arctan \left(\frac{x}{y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 24

```
dsolve((x+y(x))-(x-y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 36

```
DSolve[(x+y[x])-(x-y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

3.24 problem 25 (L)

3.24.1 Solving as exact ode 621

Internal problem ID [5285]
Internal file name [OUTPUT/4776_Friday_February_02_2024_05_11_49_AM_91658592/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (L).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2y - 3xy^2 - xy' = 0}$$

3.24.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= (3y^2x - 2y)\,\mathrm{d}x\\ (-3y^2x + 2y)\,\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -3y^2x + 2y\\ N(x,y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3y^2x + 2y)\\ &= -6xy + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x}((-6xy + 2) - (-1))\\ &= \frac{6xy - 3}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-3y^2x + 2y}((-1) - (-6xy + 2)) \\
&= \frac{-6xy + 3}{3y^2x - 2y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(-1) - (-6xy + 2)}{x\left(-3y^2x + 2y\right) - y\left(-x\right)} \\
&= \frac{-2xy + 1}{xy\left(xy - 1\right)}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = \frac{-2t + 1}{t\left(t - 1\right)}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2t+1}{t(t-1)}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(t(t-1))} \\
&= \frac{1}{t\left(t - 1\right)}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{xy\left(xy - 1\right)}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{xy\,(xy-1)}\left(-3y^2x+2y\right) \\ &= \frac{-3xy+2}{x\,(xy-1)}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{xy\,(xy-1)}(-x) \\ &= -\frac{1}{y\,(xy-1)}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-3xy+2}{x\,(xy-1)}\right)+\left(-\frac{1}{y\,(xy-1)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\ \int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{-3xy+2}{x\,(xy-1)}\,\mathrm{d}x\end{aligned}$$

$$\phi = -2\ln(x)-\ln(xy-1)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{x}{xy-1}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y(xy-1)}$. Therefore equation (4) becomes

$$-\frac{1}{y\left(xy-1\right)} = -\frac{x}{xy-1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln\left(y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\ln\left(x\right) - \ln\left(xy-1\right) + \ln\left(y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln\left(x\right) - \ln\left(xy-1\right) + \ln\left(y\right)$$

The solution becomes

$$y = \frac{x^2 \mathrm{e}^{c_1}}{\mathrm{e}^{c_1} x^3 - 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^2 \mathrm{e}^{c_1}}{\mathrm{e}^{c_1} x^3 - 1} \tag{1}$$

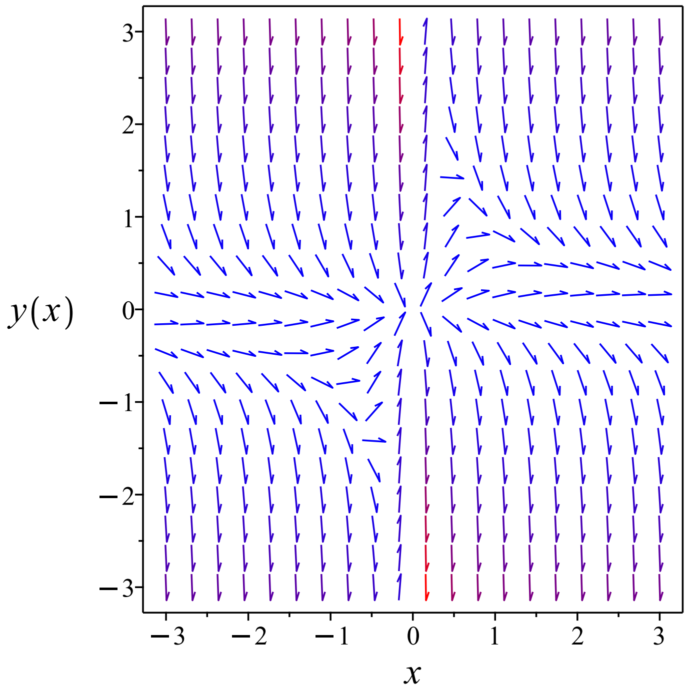


Figure 136: Slope field plot

Verification of solutions

$$y = \frac{x^2 \mathrm{e}^{c_1}}{\mathrm{e}^{c_1} x^3 - 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((2*y(x)-3*x*y(x)^2)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{x^3 + c_1}$$

✓ Solution by Mathematica

Time used: 0.145 (sec). Leaf size: 22

```
DSolve[(2*y[x]-3*x*y[x]^2)-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{x^3 + c_1}$$
$$y(x) \to 0$$

3.25 problem 25 (j)

3.25.1 Solving as exact ode 628

Internal problem ID [5286]
Internal file name [`OUTPUT/4777_Friday_February_02_2024_05_11_50_AM_54804854/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y + x(x^2 y - 1)\, y' = 0}$$

3.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(x^2 y - 1\right)\right) \mathrm{d}y &= (-y)\,\mathrm{d}x\\ (y)\,\mathrm{d}x + \left(x\left(x^2 y - 1\right)\right) \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y\\ N(x, y) &= x\left(x^2 y - 1\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^2 y - 1\right)\right)\\ &= 3x^2 y - 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{y}{x^3}$ is an integrating factor. Therefore by multiplying $M = y$ and $N = x(x^2y - 1)$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{y^2}{x^3} \\
N &= \frac{y(x^2y-1)}{x^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{y(x^2y-1)}{x^2}\right)\mathrm{d}y=\left(-\frac{y^2}{x^3}\right)\mathrm{d}x$$

$$\left(\frac{y^2}{x^3}\right)\mathrm{d}x+\left(\frac{y(x^2y-1)}{x^2}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=\frac{y^2}{x^3}$$
$$N(x,y)=\frac{y(x^2y-1)}{x^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{y^2}{x^3}\right)\\&=\frac{2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{y(x^2y-1)}{x^2}\right)\\&=\frac{2y}{x^3}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$
$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\frac{y^2}{x^3}\,\mathrm{d}x$$
$$\phi=-\frac{y^2}{2x^2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{y}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y(x^2y-1)}{x^2}$. Therefore equation (4) becomes

$$\frac{y(x^2y-1)}{x^2} = -\frac{y}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y^2)\,\mathrm{d}y$$
$$f(y) = \frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y^2}{2x^2} + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y^2}{2x^2} + \frac{y^3}{3}$$

Summary
The solution(s) found are the following

$$-\frac{y^2}{2x^2} + \frac{y^3}{3} = c_1 \tag{1}$$

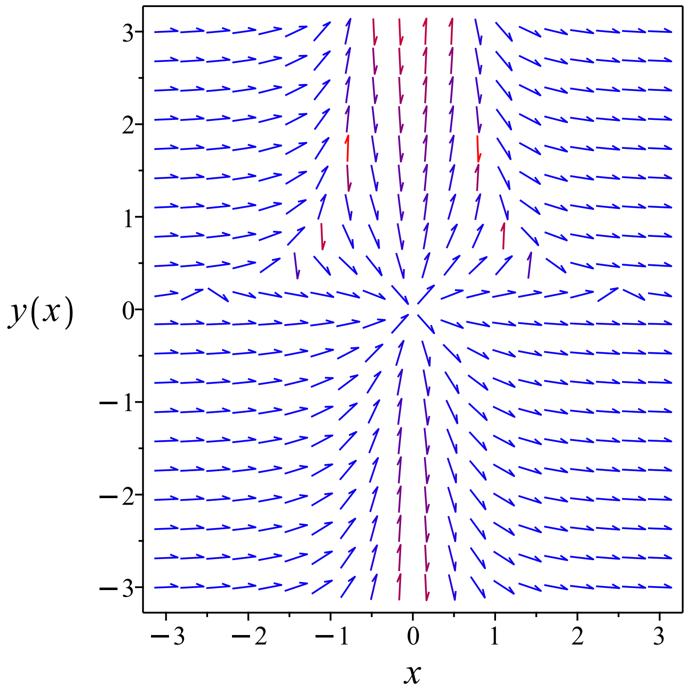


Figure 137: Slope field plot

Verification of solutions

$$-\frac{y^2}{2x^2}+\frac{y^3}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 685

```
dsolve(y(x)+x*(x^2*y(x)-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3 + \frac{\left(\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}}-c_1^2\right)^2}{c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}}}}{2x^2}$$

$$y(x) = \frac{\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}} + c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + c_1^4}{2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2}$$

$$y(x) = \frac{\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}} + c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + c_1^4}{2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2}$$

$$y(x) = \frac{2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(i\sqrt{3}-1\right) - \left(1+i\sqrt{3}\right)c_1^4}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right) - c_1^4\left(i\sqrt{3}-1\right)}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = \frac{2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(i\sqrt{3}-1\right) - \left(1+i\sqrt{3}\right)c_1^4}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right) - c_1^4\left(i\sqrt{3}-1\right)}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = \frac{2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(i\sqrt{3}-1\right) - \left(1+i\sqrt{3}\right)c_1^4}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

$$y(x) = -\frac{-2c_1^2\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} + \left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right) - c_1^4\left(i\sqrt{3}-1\right)}{4\left(x^3+\sqrt{c_1^6+x^6}\right)^{\frac{2}{3}} x^2 c_1^2}$$

✓ Solution by Mathematica

Time used: 56.665 (sec). Leaf size: 452

```
DSolve[y[x]+x*(x^2*y[x]-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{-6c_1}\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-x^6+e^{6c_1}\right)}+e^{18c_1}}+\frac{e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-x^6+e^{6c_1}\right)}+e^{18c}}}}{2x^2}$$

$$y(x) \to \frac{i\left(\sqrt{3}+i\right)e^{-6c_1}\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-x^6+e^{6c_1}\right)}+e^{18c_1}}-\frac{\left(1+i\sqrt{3}\right)e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-x^6+e}}}}{4x^2}$$

$$y(x) \to \frac{-\left(\left(1+i\sqrt{3}\right)e^{-6c_1}\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-x^6+e^{6c_1}\right)}+e^{18c_1}}\right)+\frac{i\left(\sqrt{3}+i\right)e^{6c_1}}{\sqrt[3]{-2e^{12c_1}x^6+2\sqrt{-e^{24c_1}x^6\left(-}}}}{4x^2}$$

$$y(x) \to 0$$

$$y(x) \to \frac{3}{2x^2}$$

3.26 problem 25 (k)

3.26.1 Solving as exact ode 636

Internal problem ID [5287]
Internal file name [OUTPUT/4778_Friday_February_02_2024_05_11_50_AM_68593338/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 25 (k).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_rational]`

$$\boxed{yx^3 + y + (x + 4y^4x + 8y^3)\, y' = -2x^2}$$

3.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(4y^4x+8y^3+x\right)\mathrm{d}y&=\left(-y\,x^3-2x^2-y\right)\mathrm{d}x\\ \left(y\,x^3+2x^2+y\right)\mathrm{d}x+\left(4y^4x+8y^3+x\right)\mathrm{d}y&=0 \qquad (2\text{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y\,x^3+2x^2+y\\ N(x,y)&=4y^4x+8y^3+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y\,x^3+2x^2+y\right)\\ &=x^3+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(4y^4x+8y^3+x\right)\\ &=4y^4+1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{4y^4x+8y^3+x}\left(\left(x^3+1\right)-\left(4y^4+1\right)\right)\\ &=\frac{-4y^4+x^3}{4y^4x+8y^3+x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\,x^3 + 2x^2 + y}\left(\left(4y^4 + 1\right) - \left(x^3 + 1\right)\right) \\
&= \frac{4y^4 - x^3}{y\,x^3 + 2x^2 + y}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(4y^4 + 1) - (x^3 + 1)}{x\,(y\,x^3 + 2x^2 + y) - y\,(4y^4x + 8y^3 + x)} \\
&= -\frac{1}{xy + 2}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{1}{t + 2}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{t+2}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(t+2)} \\
&= \frac{1}{t + 2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{xy + 2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{xy+2}\left(y\,x^{3}+2x^{2}+y\right) \\
&= \frac{y\,x^{3}+2x^{2}+y}{xy+2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{xy+2}\left(4y^{4}x+8y^{3}+x\right) \\
&= \frac{4y^{4}x+8y^{3}+x}{xy+2}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y\,x^{3}+2x^{2}+y}{xy+2}\right)+\left(\frac{4y^{4}x+8y^{3}+x}{xy+2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y\,x^{3}+2x^{2}+y}{xy+2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^{3}}{3}+\ln\left(xy+2\right)+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{xy+2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{4y^4x+8y^3+x}{xy+2}$. Therefore equation (4) becomes

$$\frac{4y^4x + 8y^3 + x}{xy + 2} = \frac{x}{xy + 2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y^3)\,\mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3}{3} + \ln(xy + 2) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3}{3} + \ln(xy + 2) + y^4$$

Summary

The solution(s) found are the following

$$\frac{x^3}{3} + \ln(yx + 2) + y^4 = c_1 \tag{1}$$

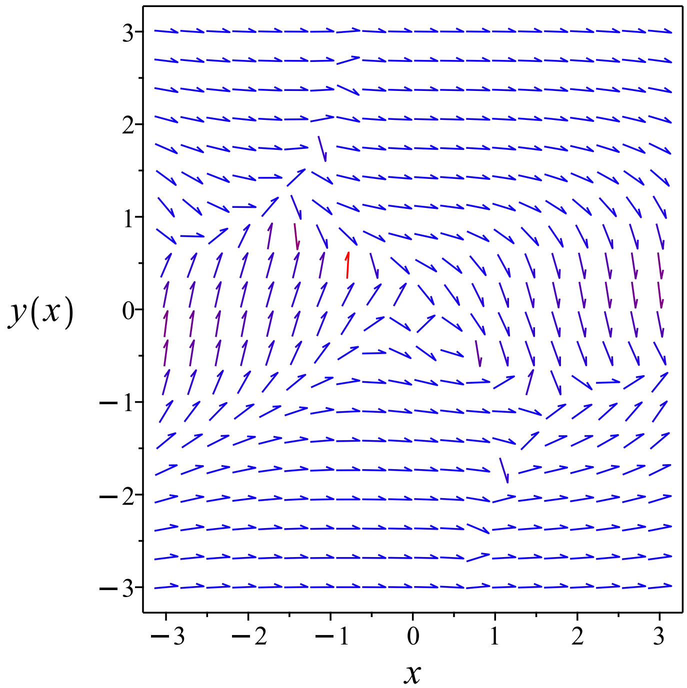


Figure 138: Slope field plot

Verification of solutions

$$\frac{x^3}{3} + \ln(yx + 2) + y^4 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, (x*y+2)/(4*x*y^4+8*y^3+x)]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 25

```
dsolve((y(x)+x^3*y(x)+2*x^2)+(x+4*x*y(x)^4+8*y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$-\frac{x^3}{3} - \ln\left(xy(x)+2\right) - y(x)^4 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.188 (sec). Leaf size: 25

```
DSolve[(y[x]+x^3*y[x]+2*x^2)+(x+4*x*y[x]^4+8*y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolution
```

$$\text{Solve}\left[\frac{x^3}{3} + y(x)^4 + \log(xy(x)+2) = c_1, y(x)\right]$$

3.27 problem 26 (a)

3.27.1 Solving as exact ode 643
3.27.2 Maple step by step solution 648

Internal problem ID [5288]
Internal file name [`OUTPUT/4779_Friday_February_02_2024_05_11_50_AM_53460436/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - y = x^2 e^x}$$

3.27.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x) \,\mathrm{d}y &= \left(x^2 \mathrm{e}^x + y\right) \mathrm{d}x \\
\left(-x^2 \mathrm{e}^x - y\right) \mathrm{d}x + (x) \,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 \mathrm{e}^x - y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 \mathrm{e}^x - y\right) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}\left(-x^2 \mathrm{e}^x - y\right) \\
&= \frac{-x^2 \mathrm{e}^x - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^2 \mathrm{e}^x - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^2 \mathrm{e}^x - y}{x^2} \, \mathrm{d}x$$

$$\phi = \frac{-x \, \mathrm{e}^x + y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x \, \mathrm{e}^x + y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x \, \mathrm{e}^x + y}{x}$$

The solution becomes

$$y = x(\mathrm{e}^x + c_1)$$

Summary

The solution(s) found are the following

$$y = x(\mathrm{e}^x + c_1) \tag{1}$$

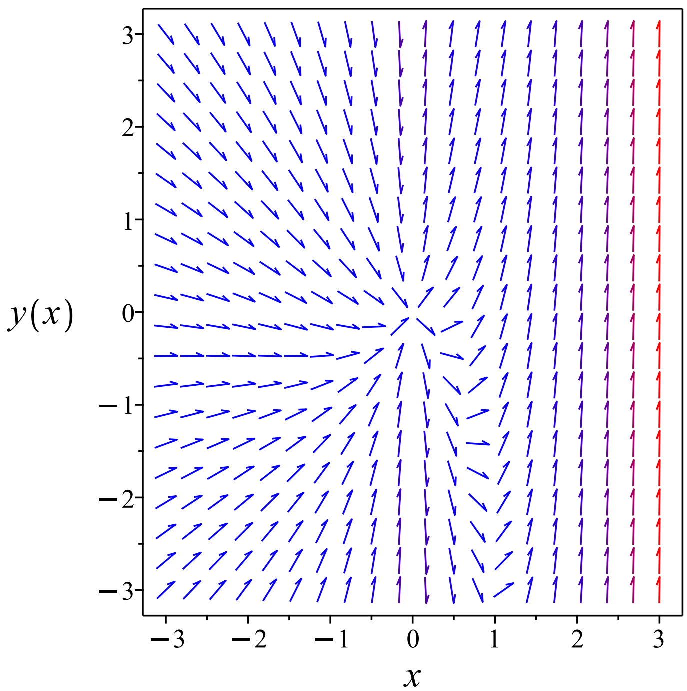


Figure 139: Slope field plot

Verification of solutions

$$y = x(\mathrm{e}^x + c_1)$$

Verified OK.

3.27.2 Maple step by step solution

Let's solve

$xy' - y = x^2 e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x\,e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x\,e^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\,x\,e^x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x\,e^x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x\,e^x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x\,e^x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int e^x dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(e^x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve((-y(x)-x^2*exp(x))+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (e^x + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 13

```
DSolve[(-y[x]-x^2*Exp[x])+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(e^x + c_1)$$

3.28 problem 26 (b)

3.28.1 Solving as exact ode 650
3.28.2 Maple step by step solution 654

Internal problem ID [5289]
Internal file name [`OUTPUT/4780_Friday_February_02_2024_05_11_50_AM_83324793/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y^2 - \left(x^2 + x\right) y' = -1}$$

3.28.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y^2+1}\right) \mathrm{d}y &= \left(\frac{1}{x(x+1)}\right) \mathrm{d}x \\
\left(-\frac{1}{x(x+1)}\right) \mathrm{d}x + \left(\frac{1}{y^2+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x(x+1)} \\
N(x, y) &= \frac{1}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x(x+1)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{1}{x(x+1)} \,\mathrm{d}x \\
\phi &= \ln(x+1) - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x+1) - \ln(x) + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x+1) - \ln(x) + \arctan(y)$$

The solution becomes

$$y = \tan(-\ln(x+1) + \ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = \tan(-\ln(x+1) + \ln(x) + c_1) \tag{1}$$

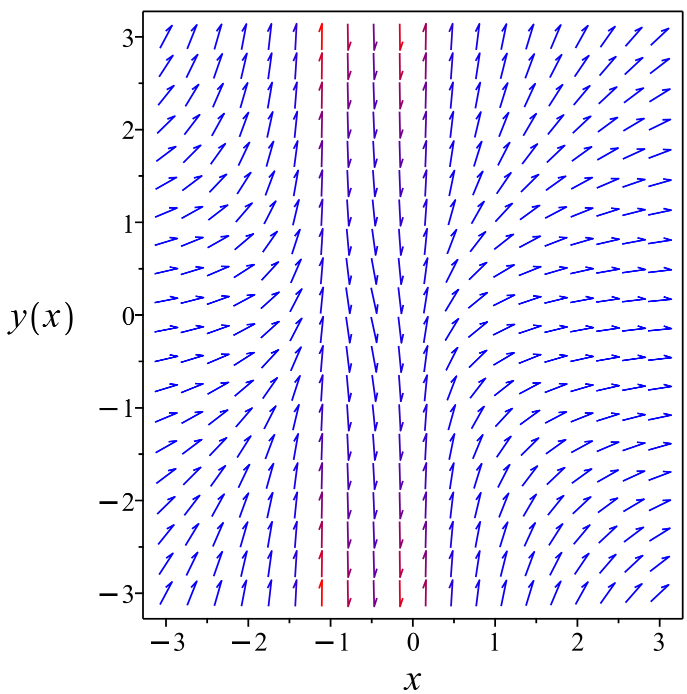


Figure 140: Slope field plot

Verification of solutions

$$y = \tan(-\ln(x+1) + \ln(x) + c_1)$$

Verified OK.

3.28.2 Maple step by step solution

Let's solve

$y^2 - (x^2 + x)\, y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1-y^2} = -\frac{1}{x^2+x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1-y^2} dx = \int -\frac{1}{x^2+x} dx + c_1$

- Evaluate integral

  $-\arctan(y) = \ln(x+1) - \ln(x) + c_1$

- Solve for y

  $y = -\tan(\ln(x+1) - \ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(1+y(x)^2=(x+x^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan(-\ln(x+1) + \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.228 (sec). Leaf size: 31

```
DSolve[1+y[x]^2==(x+x^2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \tan(\log(x) - \log(x+1) + c_1) \\
y(x) &\to -i \\
y(x) &\to i
\end{aligned}
$$

3.29 problem 26 (c)

3.29.1 Solving as exact ode 656
3.29.2 Maple step by step solution 661

Internal problem ID [5290]
Internal file name [`OUTPUT/4781_Friday_February_02_2024_05_11_51_AM_84981706/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + 2y = x^3}$$

3.29.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x)\,\mathrm{d}y &= \left(x^3 - 2y\right)\mathrm{d}x \\
\left(-x^3 + 2y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^3 + 2y \\
N(x, y) &= x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 + 2y\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((2) - (1)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x(-x^3 + 2y) \\
&= -x(x^3 - 2y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-x(x^3 - 2y)) + (x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x(x^3 - 2y) \, \mathrm{d}x$$
$$\phi = -\frac{1}{5}x^5 + x^2 y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{5}x^5 + x^2 y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{5}x^5 + x^2 y$$

The solution becomes

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^5 + 5c_1}{5x^2} \tag{1}$$

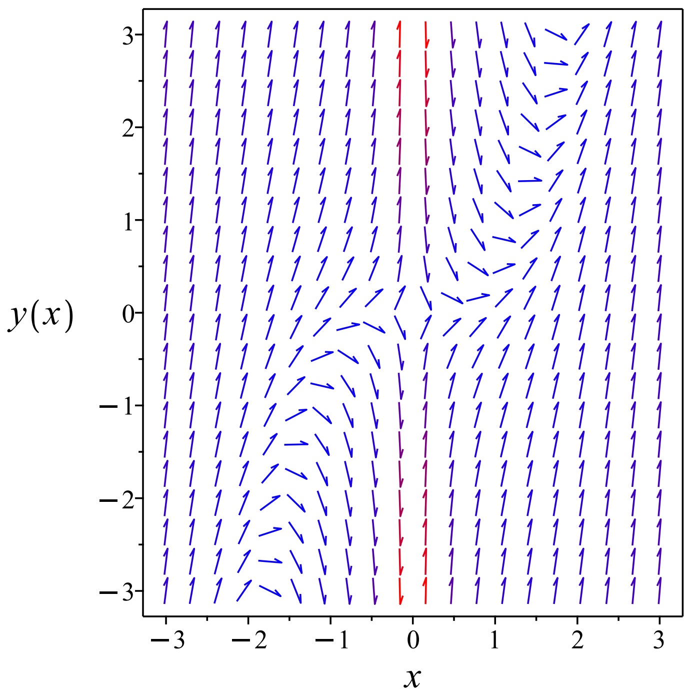


Figure 141: Slope field plot

Verification of solutions

$$y = \frac{x^5 + 5c_1}{5x^2}$$

Verified OK.

3.29.2 Maple step by step solution

Let's solve

$xy' + 2y = x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int x^4 dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{c_1 + \frac{x^5}{5}}{x^2}$

- Simplify

$$y = \frac{x^5+5c_1}{5x^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((2*y(x)-x^3)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^5 + 5c_1}{5x^2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 19

```
DSolve[(2*y[x]-x^3)+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{5} + \frac{c_1}{x^2}$$

3.30 problem 26 (d)

3.30.1 Solving as exact ode 663

Internal problem ID [5291]
Internal file name [`OUTPUT/4782_Friday_February_02_2024_05_11_51_AM_82988294/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y + (-x + y^2)\, y' = 0}$$

3.30.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(y^2 - x)\,\mathrm{d}y &= (-y)\,\mathrm{d}x \\
(y)\,\mathrm{d}x + (y^2 - x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y \\
N(x, y) &= y^2 - x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y^2 - x) \\
&= -1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{y^2 - x}((1) - (-1)) \\
&= -\frac{2}{-y^2 + x}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((-1) - (1)) \\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2}{y} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)} \\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2}(y) \\
&= \frac{1}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(y^2 - x) \\
&= \frac{y^2 - x}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{y}\right) + \left(\frac{y^2 - x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{1}{y} \,\mathrm{d}x$$
$$\phi = \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2-x}{y^2}$. Therefore equation (4) becomes

$$\frac{y^2-x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (1) \,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{y} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{y} + y$$

Summary

The solution(s) found are the following

$$\frac{x}{y} + y = c_1 \tag{1}$$

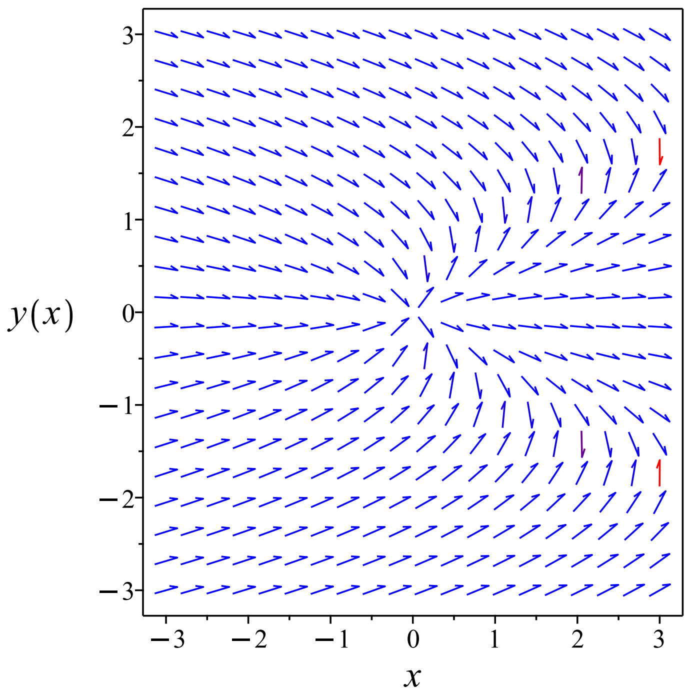


Figure 142: Slope field plot

Verification of solutions

$$\frac{x}{y} + y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(y(x)+(-x+y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= \frac{c_1}{2} - \frac{\sqrt{c_1^2 - 4x}}{2} \\
y(x) &= \frac{c_1}{2} + \frac{\sqrt{c_1^2 - 4x}}{2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.276 (sec). Leaf size: 54

```
DSolve[y[x]+(-x+y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{2}\left(c_1 - \sqrt{-4x + c_1{}^2}\right) \\
y(x) &\to \frac{1}{2}\left(\sqrt{-4x + c_1{}^2} + c_1\right) \\
y(x) &\to 0
\end{aligned}
$$

3.31 problem 26 (e)

3.31.1 Solving as exact ode 669

Internal problem ID [5292]
Internal file name [`OUTPUT/4783_Friday_February_02_2024_05_11_51_AM_10904012/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{3y^3 - yx - (x^2 + 6xy^2)\, y' = 0}$$

3.31.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(-6y^2x - x^2\right)\mathrm{d}y &= \left(-3y^3 + xy\right)\mathrm{d}x \\ \left(3y^3 - xy\right)\mathrm{d}x + \left(-6y^2x - x^2\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= 3y^3 - xy \\ N(x,y) &= -6y^2x - x^2 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y^3 - xy\right) \\ &= 9y^2 - x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-6y^2x - x^2\right) \\ &= -6y^2 - 2x \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2y}$ is an integrating factor. Therefore by multiplying $M = 3y^3 - yx$ and $N = -x^2 - 6xy^2$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{3y^3 - yx}{x^2 y} \\
N &= \frac{-x^2 - 6xy^2}{x^2 y}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{-6y^2 x - x^2}{x^2 y}\right)\mathrm{d}y &= \left(-\frac{3y^3 - xy}{x^2 y}\right)\mathrm{d}x \\
\left(\frac{3y^3 - xy}{x^2 y}\right)\mathrm{d}x + \left(\frac{-6y^2 x - x^2}{x^2 y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{3y^3 - xy}{x^2 y} \\
N(x,y) &= \frac{-6y^2 x - x^2}{x^2 y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3y^3 - xy}{x^2 y}\right) \\
&= \frac{6y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{-6y^2 x - x^2}{x^2 y}\right) \\
&= \frac{6y}{x^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{3y^3 - xy}{x^2 y}\,\mathrm{d}x \\
\phi &= -\frac{3y^2}{x} - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = -\frac{6y}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-6y^2x-x^2}{x^2y}$. Therefore equation (4) becomes

$$\frac{-6y^2x - x^2}{x^2y} = -\frac{6y}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y}\right) \mathrm{d}y \\ f(y) &= -\ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3y^2}{x} - \ln(x) - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3y^2}{x} - \ln(x) - \ln(y)$$

The solution becomes

$$y = \frac{\mathrm{e}^{-\frac{\mathrm{LambertW}\left(\frac{6\,\mathrm{e}^{-2c_1}}{x^3}\right)}{2} - c_1}}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{6\,e^{-2c_1}}{x^3}\right)}{2}-c_1}}{x} \tag{1}$$

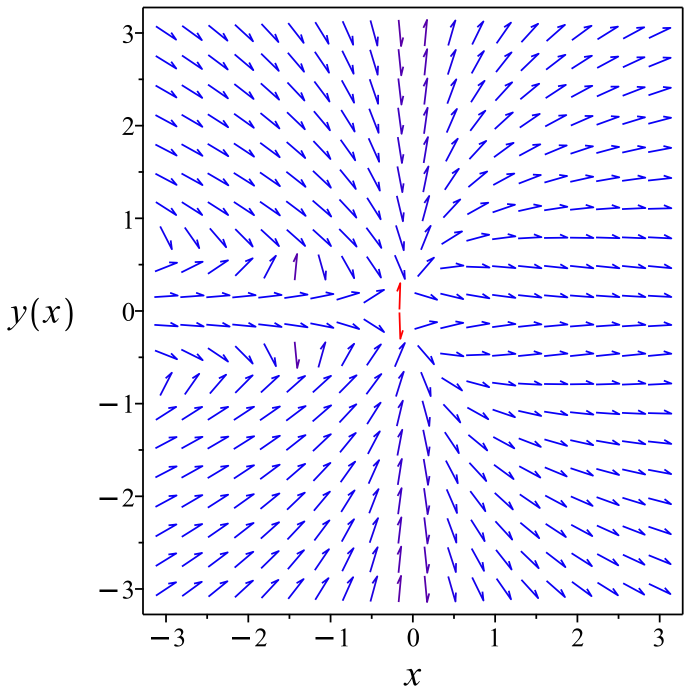


Figure 143: Slope field plot

Verification of solutions

$$y = \frac{e^{-\frac{\operatorname{LambertW}\left(\frac{6\,e^{-2c_1}}{x^3}\right)}{2}-c_1}}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 38

```
dsolve((3*y(x)^3-x*y(x))-(x^2+6*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\frac{3c_1}{2}}\sqrt{6}}{6x\sqrt{\frac{e^{3c_1}}{x^3\,\mathrm{LambertW}\left(\frac{6\,e^{3c_1}}{x^3}\right)}}}$$

✓ Solution by Mathematica

Time used: 3.943 (sec). Leaf size: 69

```
DSolve[(3*y[x]^3-x*y[x])-(x^2+6*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x}\sqrt{W\left(\frac{6e^{3c_1}}{x^3}\right)}}{\sqrt{6}}$$

$$y(x) \to \frac{\sqrt{x}\sqrt{W\left(\frac{6e^{3c_1}}{x^3}\right)}}{\sqrt{6}}$$

$$y(x) \to 0$$

3.32 problem 26 (f)

3.32.1 Solving as exact ode 676

Internal problem ID [5293]
Internal file name [`OUTPUT/4784_Friday_February_02_2024_05_11_52_AM_97656390/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{3x^2y^2 + 4\left(y x^3 - 3\right) y' = 0}$$

3.32.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(4y\,x^3 - 12)\,\mathrm{d}y &= (-3x^2y^2)\,\mathrm{d}x \\
(3x^2y^2)\,\mathrm{d}x + (4y\,x^3 - 12)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= 3x^2y^2 \\
N(x,y) &= 4y\,x^3 - 12
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3x^2y^2) \\
&= 6x^2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4y\,x^3 - 12) \\
&= 12x^2y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{4y\,x^3 - 12}\left(\left(6x^2 y\right) - \left(12x^2 y\right)\right) \\
&= -\frac{3x^2 y}{2y\,x^3 - 6}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{3x^2 y^2}\left(\left(12x^2 y\right) - \left(6x^2 y\right)\right) \\
&= \frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{2}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\ln(y)} \\
&= y^2
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= y^2\left(3x^2 y^2\right) \\
&= 3x^2 y^4
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= y^2\left(4y\,x^3 - 12\right) \\
&= 4\left(y\,x^3 - 3\right) y^2
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3x^2y^4\right) + \left(4\left(y\,x^3-3\right)y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 3x^2y^4\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x^3y^4 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 4x^3y^3 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4\left(y\,x^3-3\right)y^2$. Therefore equation (4) becomes

$$
4\left(y\,x^3-3\right)y^2 = 4x^3y^3 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -12y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-12y^2\right)\mathrm{d}y \\
f(y) &= -4y^3 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3y^4 - 4y^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3y^4 - 4y^3$$

Summary

The solution(s) found are the following

$$y^4x^3 - 4y^3 = c_1 \tag{1}$$

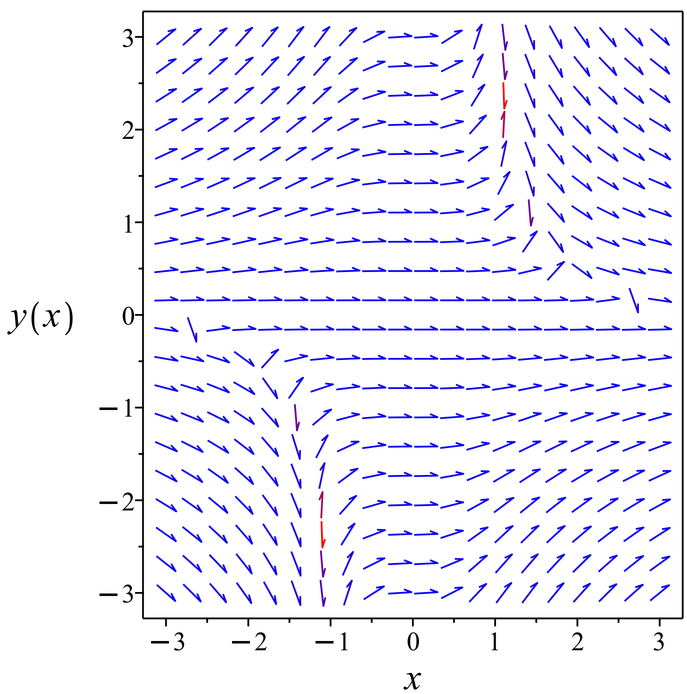


Figure 144: Slope field plot

Verification of solutions

$$y^4x^3 - 4y^3 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 1.297 (sec). Leaf size: 30

```
dsolve((3*x^2*y(x)^2)+4*(x^3*y(x)-3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(\_Z^{12}c_1 + 4\_Z^3c_1 - x^3\right)^9 + 4}{x^3}$$

✓ Solution by Mathematica

Time used: 60.296 (sec). Leaf size: 1175

```
DSolve[(3*x^2*y[x]^2)+4*(x^3*y[x]-3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^3} - \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}+\frac{6}{x^6}+\frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}{x^3}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}+\frac{8}{x^6}-\frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}{3^{2/3}x^3}-\frac{}{x^9\sqrt{-\frac{3^{2/3}c}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27-\right.}}}}}}$$

$$y(x) \to \frac{1}{x^3} - \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}+\frac{6}{x^6}+\frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}{x^3}}}{\sqrt{6}}$$

$$+\frac{1}{2}\sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}+\frac{8}{x^6}-\frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}{3^{2/3}x^3}-\frac{}{x^9\sqrt{-\frac{3^{2/3}c}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27-\right.}}}}}}$$

$$y(x) \to \frac{1}{x^3} + \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}+\frac{6}{x^6}+\frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}{x^3}}}{\sqrt{6}}$$

$$-\frac{1}{2}\sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}+\frac{8}{x^6}-\frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}{3^{2/3}x^3}+\frac{}{x^9\sqrt{-\frac{3^{2/3}c}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27-\right.}}}}}}$$

$$y(x) \to \frac{1}{x^3} + \frac{\sqrt{-\frac{3^{2/3}c_1}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}+\frac{6}{x^6}+\frac{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}{x^3}}}{\sqrt{6}}$$

$$+\frac{1}{2}\sqrt{\frac{2c_1}{\sqrt[3]{3\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-27c_1}}+\frac{8}{x^6}-\frac{2\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27+c_1x^9\right)}-9c_1}}{3^{2/3}x^3}+\frac{}{x^9\sqrt{-\frac{3^{2/3}c}{\sqrt[3]{\sqrt{3}\sqrt{c_1{}^2\left(27-\right.}}}}}}$$

3.33 problem 26 (g)

3.33.1 Solving as exact ode 683

Internal problem ID [5294]
Internal file name [`OUTPUT/4785_Friday_February_02_2024_05_11_52_AM_36040598/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y(x+y)-x^2y'=0}$$

3.33.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2)\,\mathrm{d}y &= (-y(x+y))\,\mathrm{d}x \\ (y(x+y))\,\mathrm{d}x + (-x^2)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y(x+y) \\ N(x,y) &= -x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(x+y)) \\ &= x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2) \\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y(x+y)$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$M = \frac{x+y}{xy}$$
$$N = -\frac{x}{y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{x}{y^2}\right)\mathrm{d}y = \left(-\frac{x+y}{xy}\right)\mathrm{d}x$$
$$\left(\frac{x+y}{xy}\right)\mathrm{d}x + \left(-\frac{x}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= \frac{x+y}{xy} \\
N(x,y) &= -\frac{x}{y^2}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{xy}\right) \\
&= -\frac{1}{y^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right) \\
&= -\frac{1}{y^2}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{x+y}{xy} \mathrm{d}x \\
\phi &= \ln(x) + \frac{x}{y} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) - c_1} \tag{1}$$

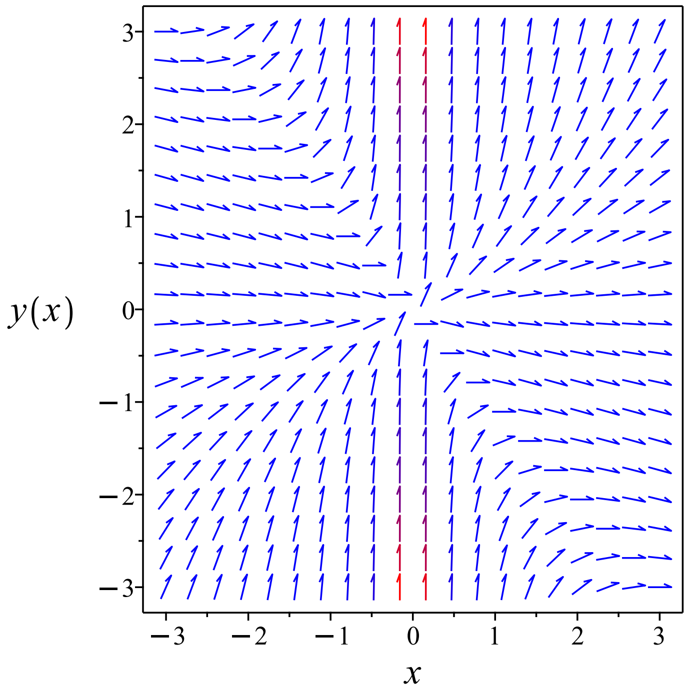


Figure 145: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) - c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)*(x+y(x))-x^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{-\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.14 (sec). Leaf size: 21

```
DSolve[y[x]*(x+y[x])-x^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{-\log(x) + c_1}$$
$$y(x) \to 0$$

3.34 problem 26 (h)

3.34.1 Solving as exact ode 690

Internal problem ID [5295]
Internal file name [`OUTPUT/4786_Friday_February_02_2024_05_11_53_AM_24430299/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{2y+3xy^2+\left(x+2x^2y\right)y'=0}$$

3.34.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(2x^2 y + x\right) \mathrm{d}y &= \left(-3y^2 x - 2y\right) \mathrm{d}x \\ \left(3y^2 x + 2y\right) \mathrm{d}x + \left(2x^2 y + x\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 3y^2 x + 2y \\ N(x, y) &= 2x^2 y + x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y^2 x + 2y\right) \\ &= 6xy + 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2x^2 y + x\right) \\ &= 4xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{2x^2y + x}((6xy + 2) - (4xy + 1)) \\
&= \frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{1}{x} \,\mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= x\left(3y^2x + 2y\right) \\
&= 3x^2y^2 + 2xy
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= x\left(2x^2y + x\right) \\
&= 2y\,x^3 + x^2
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3x^2y^2 + 2xy\right) + \left(2y\,x^3 + x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int 3x^2y^2 + 2xy\,\mathrm{d}x$$
$$\phi = y\,x^2(xy+1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = x^2(xy+1) + y\,x^3 + f'(y) \tag{4}$$
$$= 2y\,x^3 + x^2 + f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = 2y\,x^3 + x^2$. Therefore equation (4) becomes

$$2y\,x^3 + x^2 = 2y\,x^3 + x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2(xy+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2(xy+1)$$

Summary
The solution(s) found are the following

$$yx^2(yx+1) = c_1 \tag{1}$$

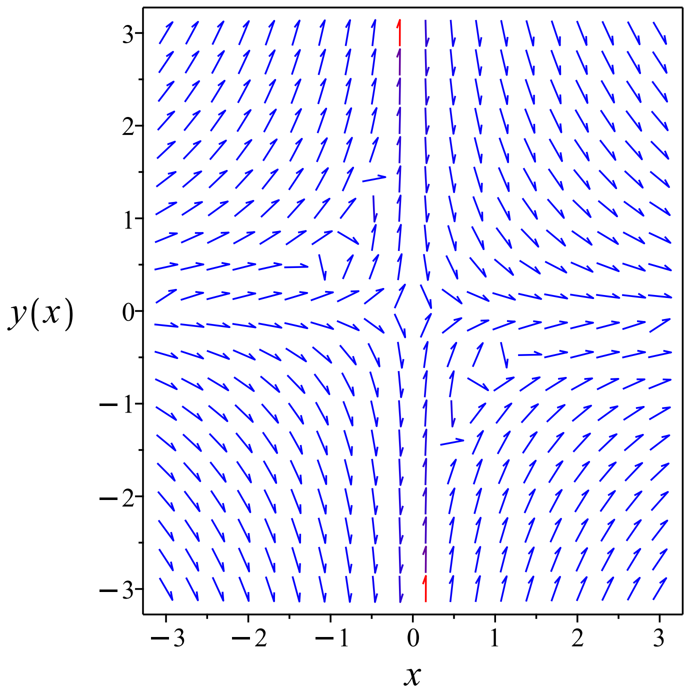


Figure 146: Slope field plot

Verification of solutions

$$yx^2(yx+1) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 45

```
dsolve((2*y(x)+3*x*y(x)^2)+(x+2*x^2*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-x + \sqrt{x(4c_1 + x)}}{2x^2}$$
$$y(x) = \frac{-x - \sqrt{x(4c_1 + x)}}{2x^2}$$

✓ Solution by Mathematica

Time used: 0.526 (sec). Leaf size: 69

```
DSolve[(2*y[x]+3*x*y[x]^2)+(x+2*x^2*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$
$$y(x) \to \frac{-x^{3/2} + \sqrt{x^2(x + 4c_1)}}{2x^{5/2}}$$

3.35 problem 26 (i)

3.35.1 Solving as exact ode 696

Internal problem ID [5296]
Internal file name [OUTPUT/4787_Friday_February_02_2024_05_11_53_AM_15451902/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 26 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y(y^2-2x^2)+x(2y^2-x^2)\,y'=0}$$

3.35.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(-x^2 + 2y^2\right)\right)\mathrm{d}y &= \left(-y\left(-2x^2 + y^2\right)\right)\mathrm{d}x\\ \left(y\left(-2x^2 + y^2\right)\right)\mathrm{d}x + \left(x\left(-x^2 + 2y^2\right)\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y\left(-2x^2 + y^2\right)\\ N(x, y) &= x\left(-x^2 + 2y^2\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\left(-2x^2 + y^2\right)\right)\\ &= -2x^2 + 3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(-x^2 + 2y^2\right)\right)\\ &= -3x^2 + 2y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{(x^2 - 2y^2)\,x}\left(\left(-2x^2 + 3y^2\right) - \left(-3x^2 + 2y^2\right)\right) \\
&= \frac{-x^2 - y^2}{x\,(x^2 - 2y^2)}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\,(-2x^2 + y^2)}\left(\left(-3x^2 + 2y^2\right) - \left(-2x^2 + 3y^2\right)\right) \\
&= \frac{x^2 + y^2}{2x^2y - y^3}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(-3x^2 + 2y^2) - (-2x^2 + 3y^2)}{x\,(y\,(-2x^2 + y^2)) - y\,(x\,(-x^2 + 2y^2))} \\
&= \frac{1}{yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{1}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(\frac{1}{t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(t)} \\
&= t
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = xy
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= xy\left(y\left(-2x^2+y^2\right)\right) \\
&= y^2\left(-2x^2+y^2\right)x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= xy\left(x\left(-x^2+2y^2\right)\right) \\
&= -y\left(x^2-2y^2\right)x^2
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(y^2\left(-2x^2+y^2\right)x\right) + \left(-y\left(x^2-2y^2\right)x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int y^2\left(-2x^2+y^2\right)x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{y^2\left(2x^2-y^2\right)^2}{8} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{y(2x^2 - y^2)^2}{4} + \frac{y^3(2x^2 - y^2)}{2} + f'(y) \\
&= -x^4y + 2x^2y^3 - \frac{3}{4}y^5 + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -y(x^2 - 2y^2)\,x^2$. Therefore equation (4) becomes

$$
-y\left(x^2 - 2y^2\right)x^2 = -x^4y + 2x^2y^3 - \frac{3}{4}y^5 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{3y^5}{4}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{3y^5}{4}\right) \mathrm{d}y \\
f(y) &= \frac{y^6}{8} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8}
$$

Summary

The solution(s) found are the following

$$
-\frac{y^2(2x^2 - y^2)^2}{8} + \frac{y^6}{8} = c_1 \tag{1}
$$

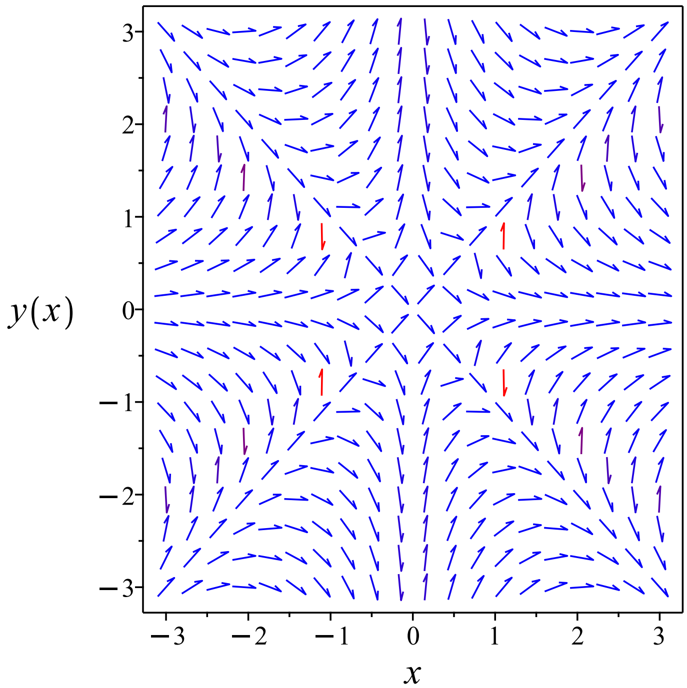


Figure 147: Slope field plot

<u>Verification of solutions</u>

$$-\frac{y^2(2x^2-y^2)^2}{8}+\frac{y^6}{8}=c_1$$

Verified OK.

<u>Maple trace</u>

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.172 (sec). Leaf size: 71

```
dsolve(y(x)*(y(x)^2-2*x^2)+x*(2*y(x)^2-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\frac{2c_1x^3-2\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

$$y(x) = \frac{\sqrt{2}\,\sqrt{\frac{c_1x^3+\sqrt{c_1^2x^6+4}}{c_1x^3}}\, x}{2}$$

✓ Solution by Mathematica

Time used: 11.861 (sec). Leaf size: 277

```
DSolve[y[x]*(y[x]^2-2*x^2)+x*(2*y[x]^2-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -\frac{\sqrt{x^2 - \frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2 - \frac{\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{x^3+\sqrt{x^6-4e^{2c_1}}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{x^2 - \frac{\sqrt{x^6}}{x}}}{\sqrt{2}}$$

$$y(x) \to -\frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

$$y(x) \to \frac{\sqrt{\frac{\sqrt{x^6}+x^3}{x}}}{\sqrt{2}}$$

3.36 problem 27

3.36.1 Solving as exact ode 703
3.36.2 Maple step by step solution 707

Internal problem ID [5297]
Internal file name [`OUTPUT/4788_Friday_February_02_2024_05_11_53_AM_53657820/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 5. Equations of first order and first degree (Exact equations). Supplemetary problems. Page 33
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - y = 0}$$

3.36.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d}y &= \left(\frac{1}{x}\right) \mathrm{d}x \\
\left(-\frac{1}{x}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = x\,\mathrm{e}^{c_1}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{c_1} \tag{1}$$

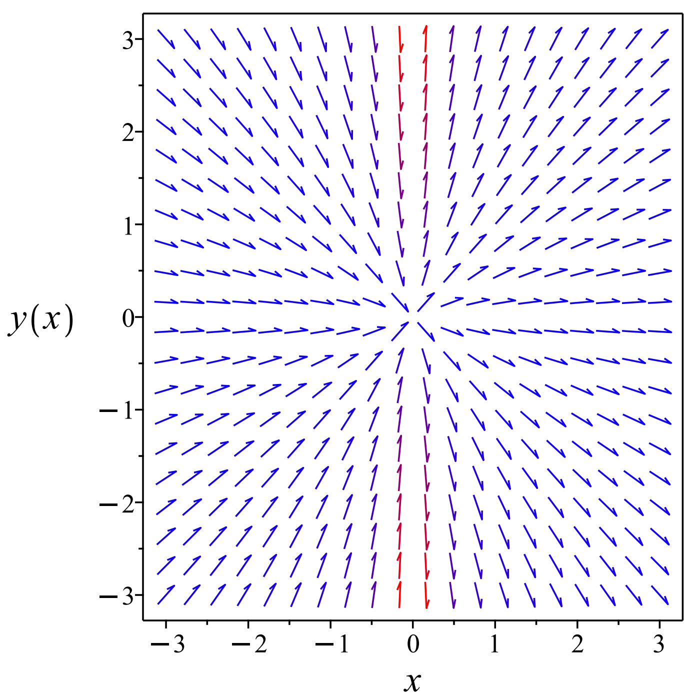


Figure 148: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{c_1}$$

Verified OK.

3.36.2 Maple step by step solution

Let's solve

$xy' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(-y(x)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[-y[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

4 Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39

4.1 problem 19 (a) 710
4.2 problem 19 (c) 725
4.3 problem 19 (d) 740
4.4 problem 19 (e) 753
4.5 problem 19 (f) 766
4.6 problem 19 (g) 777
4.7 problem 19 (h) 783
4.8 problem 19 (i) 790
4.9 problem 19 (j) 807
4.10 problem 19 (k) 820
4.11 problem 19 (L) 826
4.12 problem 19 (m) 841
4.13 problem 19 (o) 843
4.14 problem 19 (p) 857
4.15 problem 19 (q) 870
4.16 problem 19 (r) 884
4.17 problem 19 (s) 898
4.18 problem 19 (t) 908
4.19 problem 22 (a) 914
4.20 problem 22 (b) 928
4.21 problem 23 (a) 941
4.22 problem 23 (b) 947
4.23 problem 23 (c) 950
4.24 problem 23 (d) 964
4.25 problem 23 (e) 971

4.1 problem 19 (a)

4.1.1 Solving as linear ode 710
4.1.2 Solving as homogeneousTypeD2 ode 712
4.1.3 Solving as first order ode lie symmetry lookup ode 714
4.1.4 Solving as exact ode 718
4.1.5 Maple step by step solution 722

Internal problem ID [5298]
Internal file name [`OUTPUT/4789_Friday_February_02_2024_05_11_54_AM_71699313/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y + y' = 2 + 2x}$$

4.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 2 + 2x \end{aligned}$$

Hence the ode is

$$y + y' = 2 + 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 1dx} \\ &= \mathrm{e}^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(2+2x) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\,\mathrm{e}^{x}) &= (\mathrm{e}^{x})\,(2+2x) \\ \mathrm{d}(y\,\mathrm{e}^{x}) &= (2\,\mathrm{e}^{x}(x+1))\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{x} &= \int 2\,\mathrm{e}^{x}(x+1)\,\mathrm{d}x \\ y\,\mathrm{e}^{x} &= 2x\,\mathrm{e}^{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x}$ results in

$$y = 2\,\mathrm{e}^{-x}x\,\mathrm{e}^{x} + c_1\mathrm{e}^{-x}$$

which simplifies to

$$y = 2x + c_1\mathrm{e}^{-x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2x + c_1\mathrm{e}^{-x} \tag{1}$$

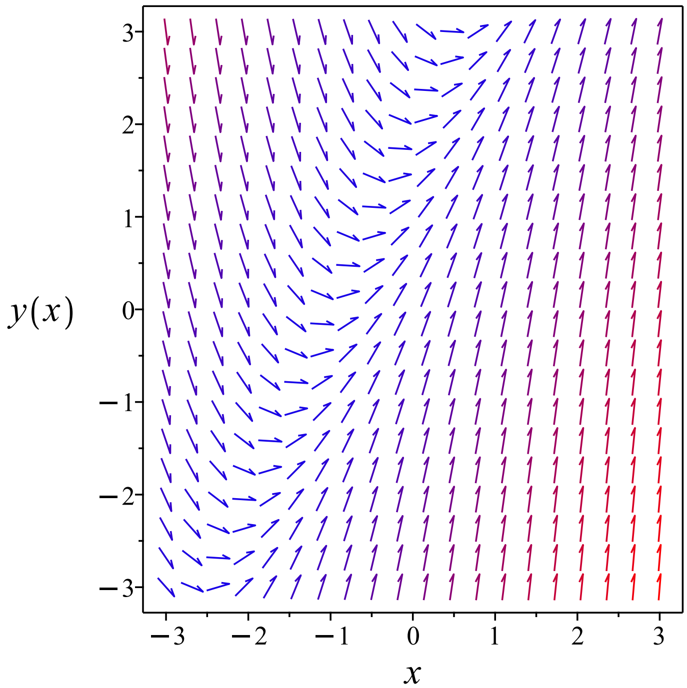


Figure 149: Slope field plot

Verification of solutions

$$y = 2x + c_1 e^{-x}$$

Verified OK.

4.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x + u'(x)\,x + u(x) = 2 + 2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(x+1)\,(-u+2)}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(u) = -u + 2$. Integrating both sides gives

$$\frac{1}{-u+2}\,du = \frac{x+1}{x}\,dx$$

$$
\int \frac{1}{-u+2}du = \int \frac{x+1}{x}dx
$$

$$
-\ln(u-2) = x + \ln(x) + c_2
$$

Raising both side to exponential gives

$$
\frac{1}{u-2} = e^{x+\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{u-2} = c_3 e^{x+\ln(x)}
$$

Which simplifies to

$$
u(x) = \frac{(2c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3 x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{(2c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(2c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3} \tag{1}
$$

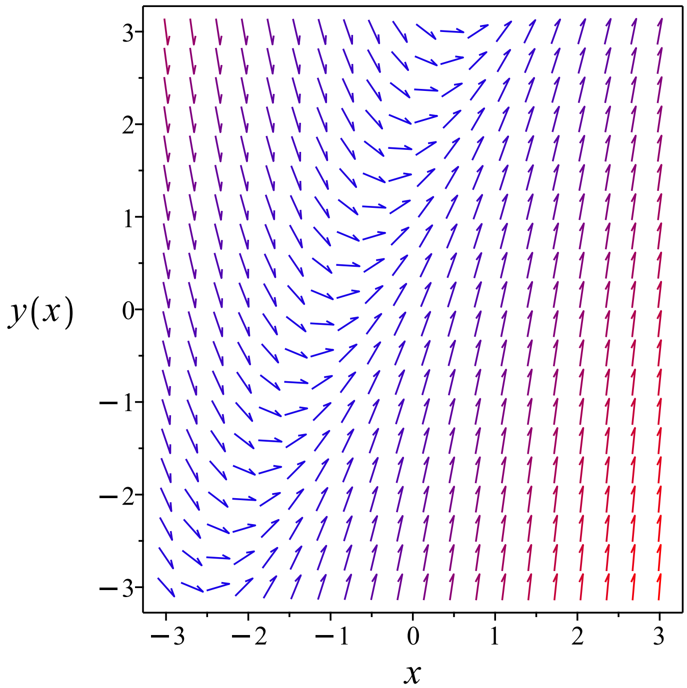


Figure 150: Slope field plot

Verification of solutions

$$y = \frac{(2c_3 \mathrm{e}^{x} x \,\mathrm{e}^{c_2} + 1)\, \mathrm{e}^{-x} \mathrm{e}^{-c_2}}{c_3}$$

Verified OK.

4.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y + 2 + 2x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 84: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-x}} dy
\end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + 2 + 2x$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y\,\mathrm{e}^{x} \\
S_y &= \mathrm{e}^{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,\mathrm{e}^{x}(x + 1) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,\mathrm{e}^{R}(R + 1)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,\mathrm{e}^{R} R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y = 2x\,\mathrm{e}^{x} + c_1$$

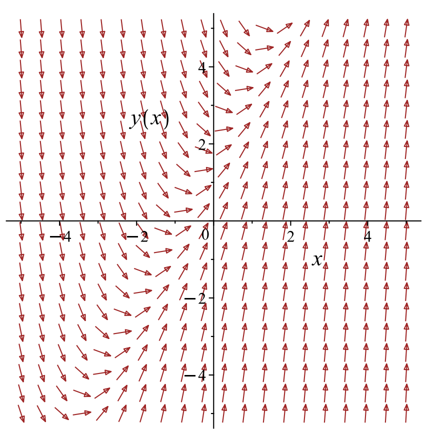
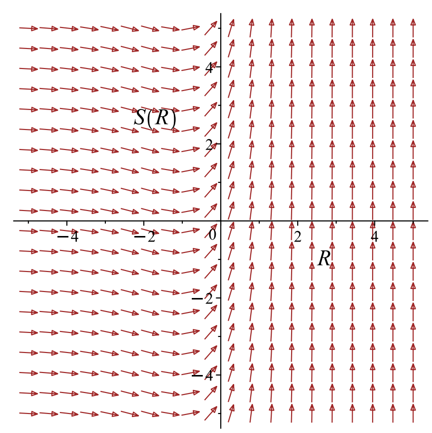
Which simplifies to

$$\mathrm{e}^{x} y = 2x\,\mathrm{e}^{x} + c_1$$

Which gives

$$y = (2x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + 2 + 2x$ 	$R = x$ $S = y\,\mathrm{e}^{x}$	$\frac{dS}{dR} = 2\,\mathrm{e}^{R}(R + 1)$ 

Summary

The solution(s) found are the following

$$y = (2x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x} \tag{1}$$

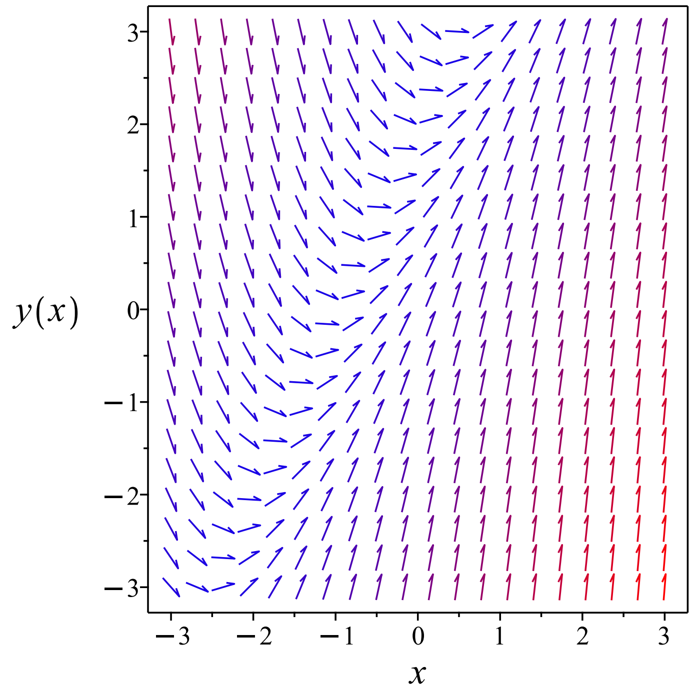


Figure 151: Slope field plot

<u>Verification of solutions</u>

$$y = (2x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Verified OK.

4.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (-y + 2 + 2x)\,\mathrm{d}x \\
(y - 2 - 2x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y - 2 - 2x \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - 2 - 2x) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(y - 2 - 2x) \\
&= (y - 2 - 2x)\,\mathrm{e}^{x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
((y - 2 - 2x)\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int (y-2-2x)\,\mathrm{e}^{x}\,\mathrm{d}x$$

$$\phi = (-2x+y)\,\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-2x+y)\,\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-2x+y)\,\mathrm{e}^{x}$$

The solution becomes

$$y = (2x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = (2x\,\mathrm{e}^{x} + c_1)\,\mathrm{e}^{-x} \tag{1}$$

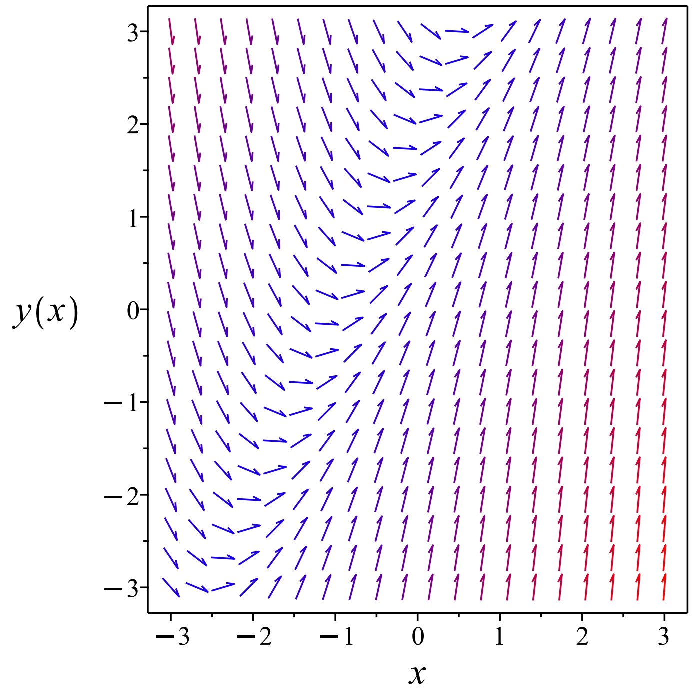


Figure 152: Slope field plot

Verification of solutions

$$y = (2x\,e^{x} + c_1)\,e^{-x}$$

Verified OK.

4.1.5 Maple step by step solution

Let's solve

$y + y' = 2 + 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + 2 + 2x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y + y' = 2 + 2x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y + y') = \mu(x)\,(2 + 2x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y+y') = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\,(2+2x)\, dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\,(2+2x)\, dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(2+2x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int \mathrm{e}^{x}(2+2x)dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{2x\,\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = 2x + c_1 \mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)=2+2*x,y(x), singsol=all)
```

$$y(x) = 2x + c_1 \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 17

```
DSolve[y'[x]+y[x]==2+2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2x + c_1 e^{-x}$$

4.2 problem 19 (c)

4.2.1 Solving as separable ode 725
4.2.2 Solving as linear ode 727
4.2.3 Solving as homogeneousTypeD2 ode 728
4.2.4 Solving as first order ode lie symmetry lookup ode 730
4.2.5 Solving as exact ode 734
4.2.6 Maple step by step solution 738

Internal problem ID [5299]
Internal file name [`OUTPUT/4790_Friday_February_02_2024_05_11_55_AM_66438453/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y - yx = 0}$$

4.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= y(x+1) \end{aligned}$$

Where $f(x) = x + 1$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\, dy &= x + 1\, dx \\
\int \frac{1}{y}\, dy &= \int x + 1\, dx \\
\ln(y) &= \frac{1}{2}x^2 + x + c_1 \\
y &= e^{\frac{1}{2}x^2 + x + c_1} \\
&= c_1 e^{\frac{1}{2}x^2 + x}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{1}{2}x^2 + x} \tag{1}$$

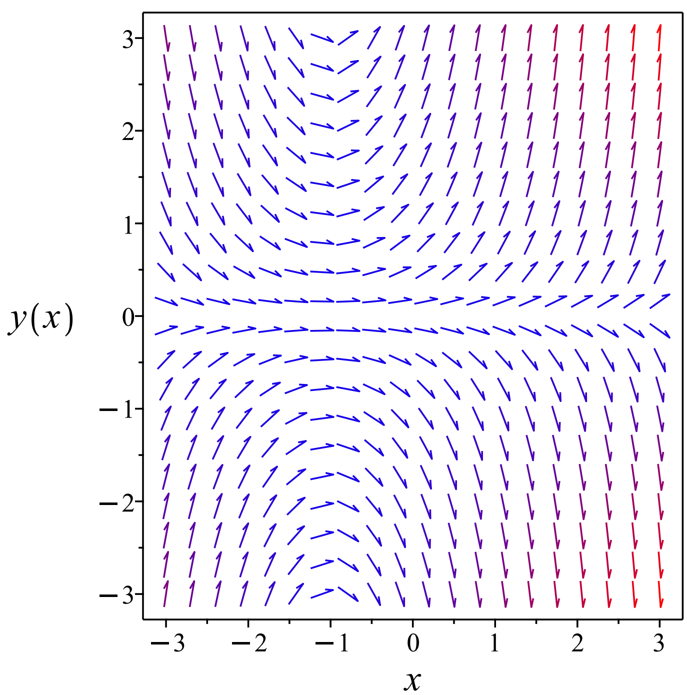


Figure 153: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{1}{2}x^2 + x}$$

Verified OK.

4.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x - 1 \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + (-x - 1)\, y = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-x-1)dx} \\ &= \mathrm{e}^{-\frac{1}{2}x^2 - x} \end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{-\frac{x(2+x)}{2}}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x(2+x)}{2}} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x(2+x)}{2}}$ results in

$$y = c_1 \mathrm{e}^{\frac{x(2+x)}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x(2+x)}{2}} \tag{1}$$

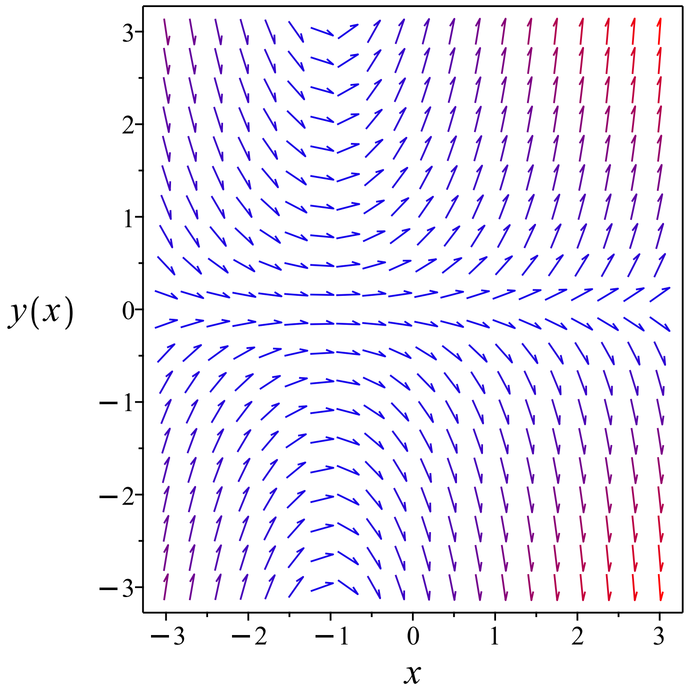


Figure 154: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x(2+x)}{2}}$$

Verified OK.

4.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)\, x - u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x^2 + x - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{x^2+x-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{x^2+x-1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x^2+x-1}{x}\,dx \\
\ln(u) &= \frac{x^2}{2} + x - \ln(x) + c_2 \\
u &= \mathrm{e}^{\frac{x^2}{2}+x-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{\frac{x^2}{2}+x-\ln(x)}
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= x c_2 \mathrm{e}^{\frac{x^2}{2}+x-\ln(x)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x c_2 \mathrm{e}^{\frac{x^2}{2}+x-\ln(x)} \tag{1}
$$

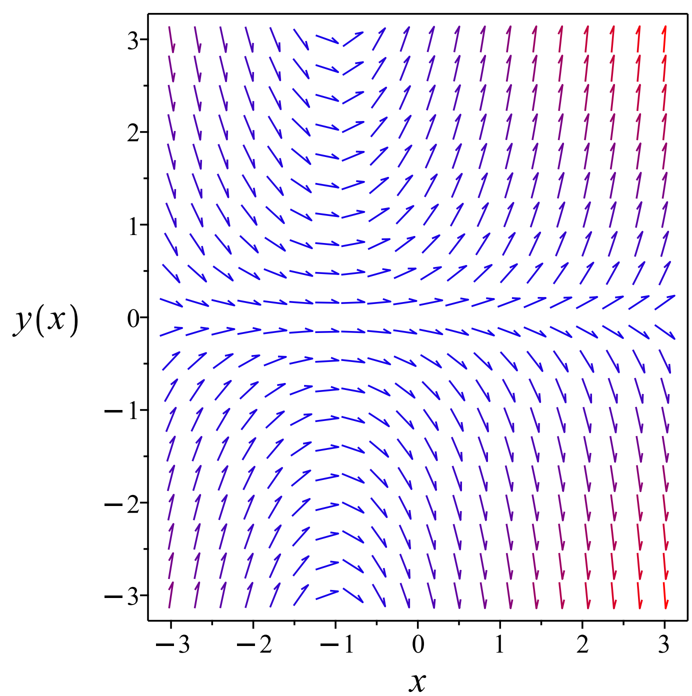


Figure 155: Slope field plot

Verification of solutions

$$y = xc_2 e^{\frac{x^2}{2} + x - \ln(x)}$$

Verified OK.

4.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = xy + y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 87: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\frac{1}{2}x^2 + x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{2}x^2+x}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{1}{2}x^2-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = xy + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -(x+1)\, e^{-\frac{x(2+x)}{2}} y \\ S_y &= e^{-\frac{x(2+x)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = c_1$$

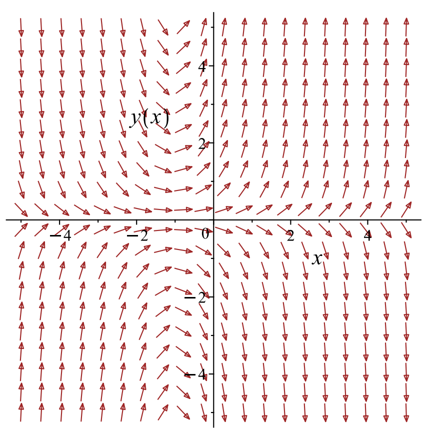
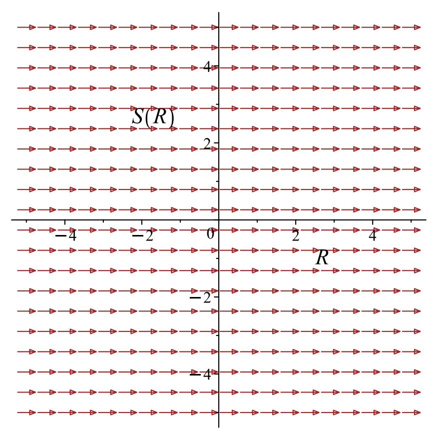
Which simplifies to

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = c_1$$

Which gives

$$y = c_1 \mathrm{e}^{\frac{x(2+x)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = xy + y$ 	$R = x$ $S = \mathrm{e}^{-\frac{x(2+x)}{2}} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x(2+x)}{2}} \tag{1}$$

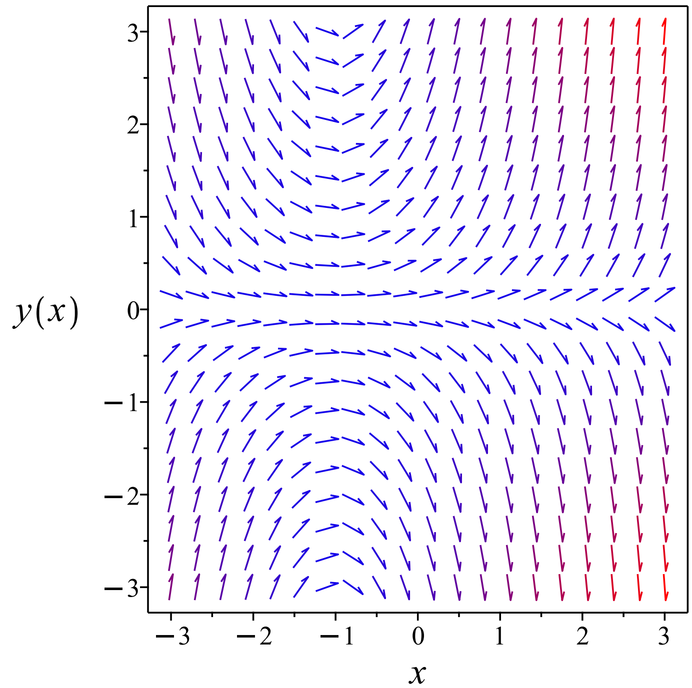


Figure 156: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x(2+x)}{2}}$$

Verified OK.

4.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d}y &= (x + 1) \, \mathrm{d}x \\
(-x - 1) \, \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x - 1 \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x - 1) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x - 1\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{1}{2}x^2 - x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \ln(y)$$

The solution becomes

$$y = e^{\frac{1}{2}x^2 + x + c_1}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{1}{2}x^2 + x + c_1} \tag{1}$$

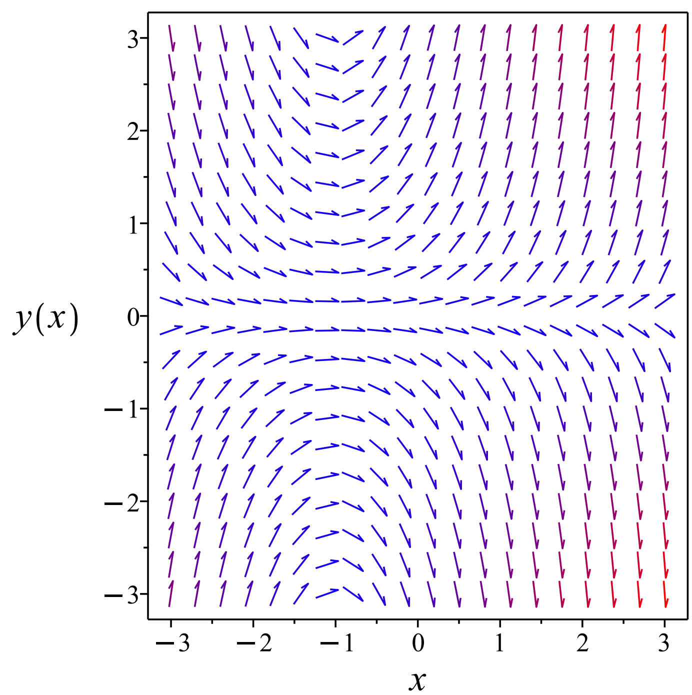


Figure 157: Slope field plot

Verification of solutions

$$y = e^{\frac{1}{2}x^2+x+c_1}$$

Verified OK.

4.2.6 Maple step by step solution

Let's solve

$y' - y - yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x + 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int (x+1)\,dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{1}{2}x^2 + x + c_1$

- Solve for y

  $y = e^{\frac{1}{2}x^2+x+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x)-y(x)=x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x(x+2)}{2}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[y'[x]-y[x]==x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1}{2}x(x+2)}$$
$$y(x) \to 0$$

4.3 problem 19 (d)

4.3.1 Solving as linear ode 740
4.3.2 Solving as first order ode lie symmetry lookup ode 742
4.3.3 Solving as exact ode 746
4.3.4 Maple step by step solution 751

Internal problem ID [5300]
Internal file name [`OUTPUT/4791_Friday_February_02_2024_05_11_55_AM_71430434/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-3y + xy' = (-2 + x)\,\mathrm{e}^{x}}$$

4.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{\mathrm{e}^{x}(-2 + x)}{x} \end{aligned}$$

Hence the ode is

$$y' - \frac{3y}{x} = \frac{\mathrm{e}^{x}(-2 + x)}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{3}{x}dx} \\
&= \frac{1}{x^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\mathrm{e}^x(-2+x)}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(\frac{\mathrm{e}^x(-2+x)}{x}\right) \\
\mathrm{d}\left(\frac{y}{x^3}\right) &= \left(\frac{\mathrm{e}^x(-2+x)}{x^4}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^3} &= \int \frac{\mathrm{e}^x(-2+x)}{x^4}\,\mathrm{d}x \\
\frac{y}{x^3} &= -\frac{\mathrm{e}^x}{6x^2} - \frac{\mathrm{e}^x}{6x} - \frac{\mathrm{expIntegral}_1(-x)}{6} + \frac{2\,\mathrm{e}^x}{3x^3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$
y = x^3\left(-\frac{\mathrm{e}^x}{6x^2} - \frac{\mathrm{e}^x}{6x} - \frac{\mathrm{expIntegral}_1(-x)}{6} + \frac{2\,\mathrm{e}^x}{3x^3}\right) + c_1x^3
$$

which simplifies to

$$
y = -\frac{\mathrm{expIntegral}_1(-x)\,x^3}{6} + \frac{(-x^2-x+4)\,\mathrm{e}^x}{6} + c_1x^3
$$

Summary
The solution(s) found are the following

$$
y = -\frac{\mathrm{expIntegral}_1(-x)\,x^3}{6} + \frac{(-x^2-x+4)\,\mathrm{e}^x}{6} + c_1x^3 \tag{1}
$$

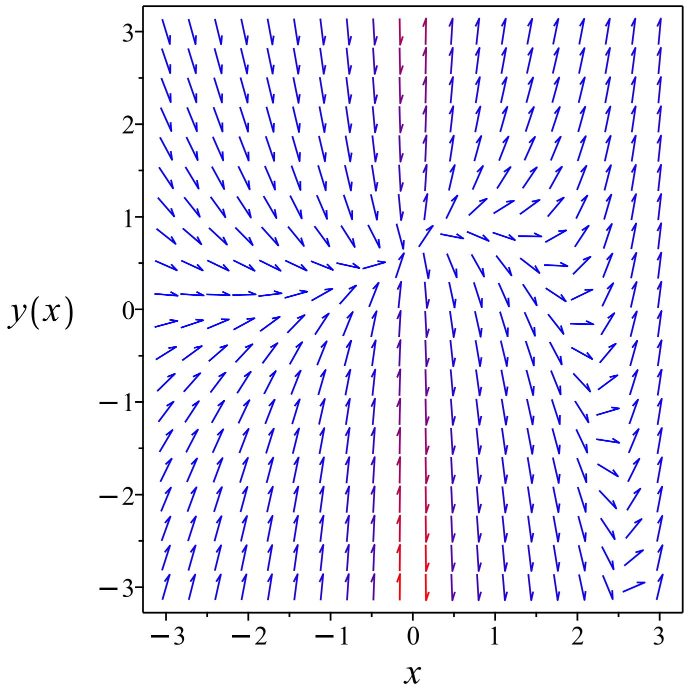


Figure 158: Slope field plot

Verification of solutions

$$
y = -\frac{\text{expIntegral}_1 (-x)\, x^3}{6} + \frac{(-x^2 - x + 4)\, e^x}{6} + c_1 x^3
$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\, e^x - 2\, e^x + 3y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 90: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^3
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x\,\mathrm{e}^x - 2\,\mathrm{e}^x + 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{x^4} \\ S_y &= \frac{1}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^x(-2+x)}{x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\mathrm{e}^R(-2+R)}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{e^{R}}{6R^{2}} - \frac{e^{R}}{6R} - \frac{\operatorname{expIntegral}_{1}(-R)}{6} + \frac{2\,e^{R}}{3R^{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^{3}} = -\frac{e^{x}}{6x^{2}} - \frac{e^{x}}{6x} - \frac{\operatorname{expIntegral}_{1}(-x)}{6} + \frac{2\,e^{x}}{3x^{3}} + c_1$$

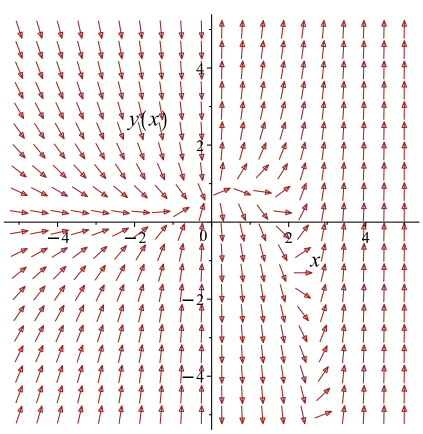
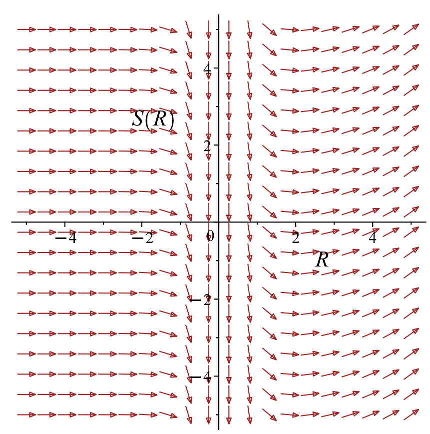
Which simplifies to

$$\frac{\operatorname{expIntegral}_{1}(-x)\,x^{3} + (x^{2}+x-4)\,e^{x} - 6c_1x^{3} + 6y}{6x^{3}} = 0$$

Which gives

$$y = -\frac{\operatorname{expIntegral}_{1}(-x)\,x^{3}}{6} - \frac{x^{2}e^{x}}{6} - \frac{x\,e^{x}}{6} + \frac{2\,e^{x}}{3} + c_1x^{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x\,e^{x}-2\,e^{x}+3y}{x}$ 	$R = x$ $S = \frac{y}{x^{3}}$	$\frac{dS}{dR} = \frac{e^{R}(-2+R)}{R^{4}}$ 

Summary
The solution(s) found are the following

$$y = -\frac{\operatorname{expIntegral}_1(-x)\, x^3}{6} - \frac{x^2 e^x}{6} - \frac{x\, e^x}{6} + \frac{2\, e^x}{3} + c_1 x^3 \tag{1}$$

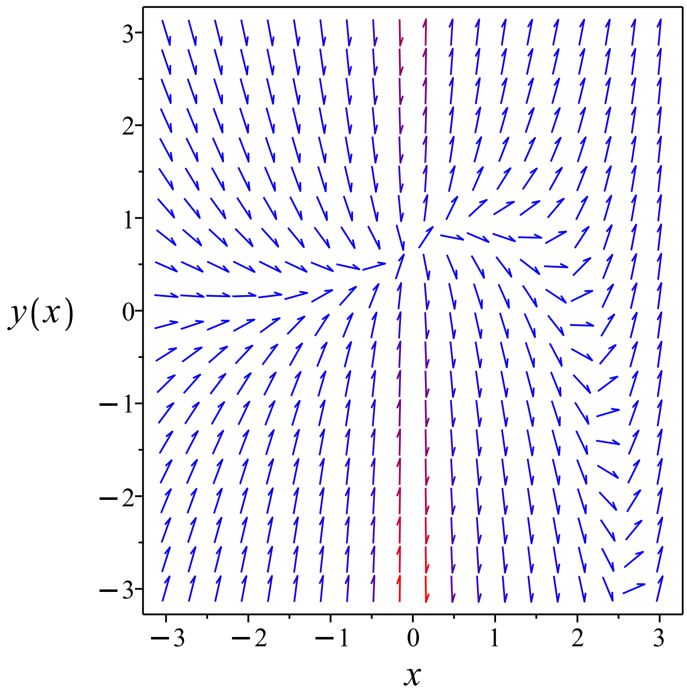


Figure 159: Slope field plot

Verification of solutions

$$y = -\frac{\operatorname{expIntegral}_1(-x)\, x^3}{6} - \frac{x^2 e^x}{6} - \frac{x\, e^x}{6} + \frac{2\, e^x}{3} + c_1 x^3$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(3y+(-2+x)\,\mathrm{e}^{x})\,\mathrm{d}x\\ (-3y-(-2+x)\,\mathrm{e}^{x})\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3y-(-2+x)\,\mathrm{e}^{x}\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-3y-(-2+x)\,\mathrm{e}^{x})\\ &=-3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-3) - (1))\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{4}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}(-3y - (-2 + x)\,\mathrm{e}^x)\\ &= \frac{(-x+2)\,\mathrm{e}^x - 3y}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}(x)\\ &= \frac{1}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{(-x+2)\,\mathrm{e}^{x} - 3y}{x^{4}}\right) + \left(\frac{1}{x^{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{(-x+2)\,\mathrm{e}^{x} - 3y}{x^{4}}\,\mathrm{d}x \\
\phi &= \frac{\mathrm{expIntegral}_{1}(-x)\,x^{3} + (x^{2}+x-4)\,\mathrm{e}^{x} + 6y}{6x^{3}} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^{3}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^{3}}$. Therefore equation (4) becomes

$$
\frac{1}{x^{3}} = \frac{1}{x^{3}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_{1}
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\text{expIntegral}_1(-x)\,x^3 + (x^2+x-4)\,\mathrm{e}^x + 6y}{6x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\text{expIntegral}_1(-x)\,x^3 + (x^2+x-4)\,\mathrm{e}^x + 6y}{6x^3}$$

The solution becomes

$$y = -\frac{\text{expIntegral}_1(-x)\,x^3}{6} - \frac{x^2\mathrm{e}^x}{6} - \frac{x\,\mathrm{e}^x}{6} + \frac{2\,\mathrm{e}^x}{3} + c_1x^3$$

Summary

The solution(s) found are the following

$$y = -\frac{\text{expIntegral}_1(-x)\,x^3}{6} - \frac{x^2\mathrm{e}^x}{6} - \frac{x\,\mathrm{e}^x}{6} + \frac{2\,\mathrm{e}^x}{3} + c_1x^3 \tag{1}$$

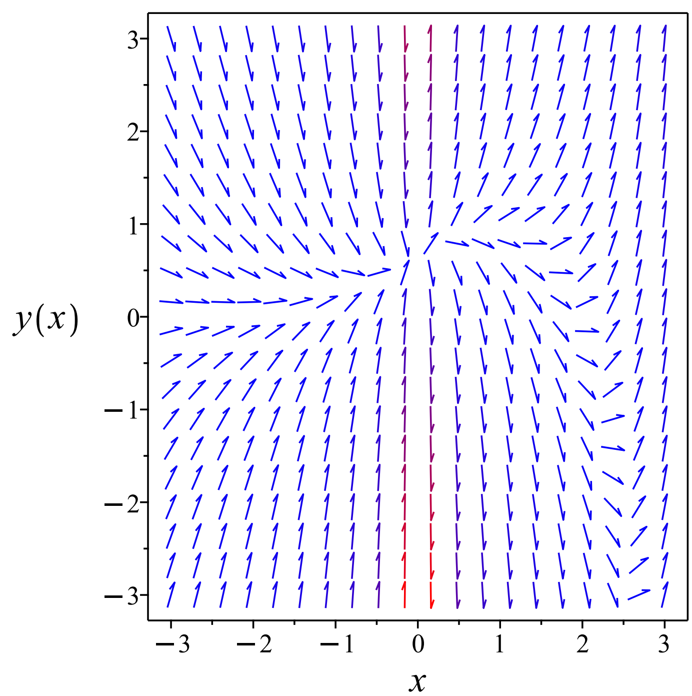


Figure 160: Slope field plot

Verification of solutions

$$y = -\frac{\text{expIntegral}_1\left(-x\right) x^3}{6} - \frac{x^2 \mathrm{e}^x}{6} - \frac{x\,\mathrm{e}^x}{6} + \frac{2\,\mathrm{e}^x}{3} + c_1 x^3$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$-3y + xy' = (-2+x)\,\mathrm{e}^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{x} + \frac{\mathrm{e}^x(-2+x)}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{x} = \frac{\mathrm{e}^x(-2+x)}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \frac{\mu(x)\mathrm{e}^x(-2+x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^3}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)\mathrm{e}^x(-2+x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)\mathrm{e}^x(-2+x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\mathrm{e}^x(-2+x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^3}$

$y = x^3 \left(\int \frac{e^x(-2+x)}{x^4} dx + c_1 \right)$

- Evaluate the integrals on the rhs

  $y = x^3 \left(-\frac{e^x}{6x^2} - \frac{e^x}{6x} - \frac{\mathrm{Ei}_1(-x)}{6} + \frac{2\,e^x}{3x^3} + c_1 \right)$

- Simplify

  $y = -\frac{\mathrm{Ei}_1(-x)x^3}{6} + \frac{(-x^2-x+4)e^x}{6} + c_1 x^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve((-2*y(x)-(x-2)*exp(x))+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{x^3 \operatorname{expIntegral}_1(-x)}{6} + \frac{(-x^2-x+4)\,e^x}{6} + c_1 x^3$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 33

```
DSolve[(-2*y[x]-(x-2)*Exp[x])+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6} x^3 \left(\operatorname{ExpIntegralEi}(x) - \frac{e^x(x^2+x-4)}{x^3} + 6c_1 \right)$$

4.4 problem 19 (e)

4.4.1 Solving as linear ode 753
4.4.2 Solving as first order ode lie symmetry lookup ode 755
4.4.3 Solving as exact ode 759
4.4.4 Maple step by step solution 763

Internal problem ID [5301]
Internal file name [`OUTPUT/4792_Friday_February_02_2024_05_11_56_AM_13116245/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{i' - 6i = 10 \sin(2t)}$$

4.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$i' + p(t)i = q(t)$$

Where here

$$\begin{aligned} p(t) &= -6 \\ q(t) &= 10 \sin(2t) \end{aligned}$$

Hence the ode is

$$i' - 6i = 10 \sin(2t)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-6)dt} \\
&= e^{-6t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu i) &= (\mu)\,(10\sin(2t)) \\
\frac{d}{dt}\left(e^{-6t} i\right) &= \left(e^{-6t}\right)(10\sin(2t)) \\
d\left(e^{-6t} i\right) &= \left(10\,e^{-6t}\sin(2t)\right)\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-6t} i &= \int 10\,e^{-6t}\sin(2t)\,dt \\
e^{-6t} i &= -\frac{e^{-6t}\cos(2t)}{2} - \frac{3\,e^{-6t}\sin(2t)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-6t}$ results in

$$
i = e^{6t}\left(-\frac{e^{-6t}\cos(2t)}{2} - \frac{3\,e^{-6t}\sin(2t)}{2}\right) + c_1 e^{6t}
$$

which simplifies to

$$
i = c_1 e^{6t} - \frac{3\sin(2t)}{2} - \frac{\cos(2t)}{2}
$$

Summary

The solution(s) found are the following

$$
i = c_1 e^{6t} - \frac{3\sin(2t)}{2} - \frac{\cos(2t)}{2} \tag{1}
$$

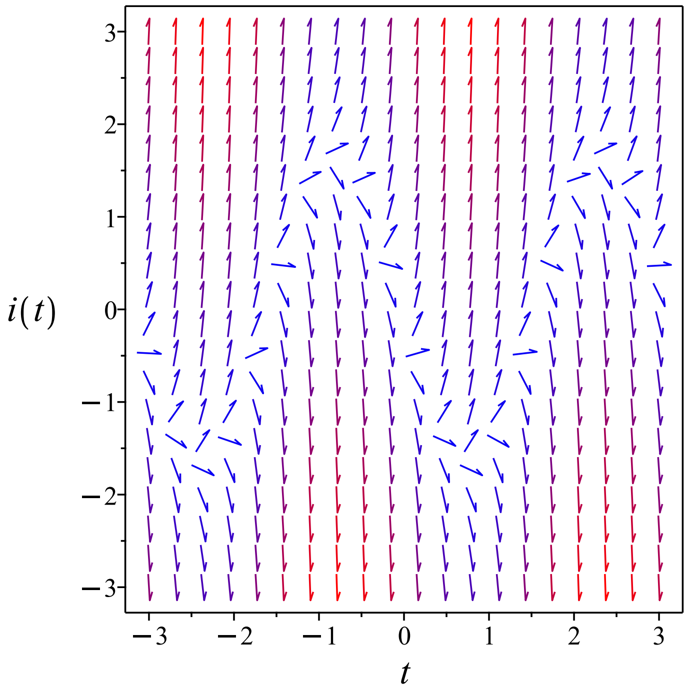


Figure 161: Slope field plot

Verification of solutions

$$i = c_1 e^{6t} - \frac{3\sin(2t)}{2} - \frac{\cos(2t)}{2}$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$i' = 6i + 10\sin(2t)$$
$$i' = \omega(t, i)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_i - \xi_t) - \omega^2 \xi_i - \omega_t \xi - \omega_i \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 93: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(t,i) &= 0 \\
\eta(t,i) &= \mathrm{e}^{6t}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,i) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dt}{\xi} = \frac{di}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial i}\right) S(t,i) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{6t}} dy \end{aligned}$$

Which results in

$$S = e^{-6t} i$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,i) S_i}{R_t + \omega(t,i) R_i} \tag{2}$$

Where in the above R_t, R_i, S_t, S_i are all partial derivatives and $\omega(t,i)$ is the right hand side of the original ode given by

$$\omega(t,i) = 6i + 10 \sin(2t)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_i &= 0 \\ S_t &= -6 e^{-6t} i \\ S_i &= e^{-6t} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 10 e^{-6t} \sin(2t) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, i in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 10 e^{-6R} \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{-6R}(\cos(2R) + 3\sin(2R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, i coordinates. This results in

$$e^{-6t} i = -\frac{(\cos(2t) + 3\sin(2t))\, e^{-6t}}{2} + c_1$$

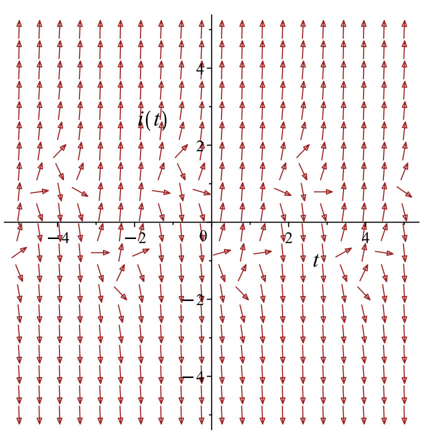
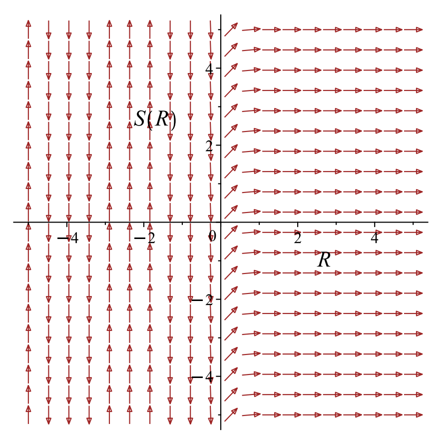
Which simplifies to

$$e^{-6t} i = -\frac{(\cos(2t) + 3\sin(2t))\, e^{-6t}}{2} + c_1$$

Which gives

$$i = -\frac{e^{6t}(3\, e^{-6t} \sin(2t) + e^{-6t} \cos(2t) - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, i coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{di}{dt} = 6i + 10\sin(2t)$ 	$R = t$ $S = e^{-6t} i$	$\frac{dS}{dR} = 10\, e^{-6R} \sin(2R)$ 

Summary

The solution(s) found are the following

$$i = -\frac{e^{6t}(3\, e^{-6t} \sin(2t) + e^{-6t} \cos(2t) - 2c_1)}{2} \tag{1}$$

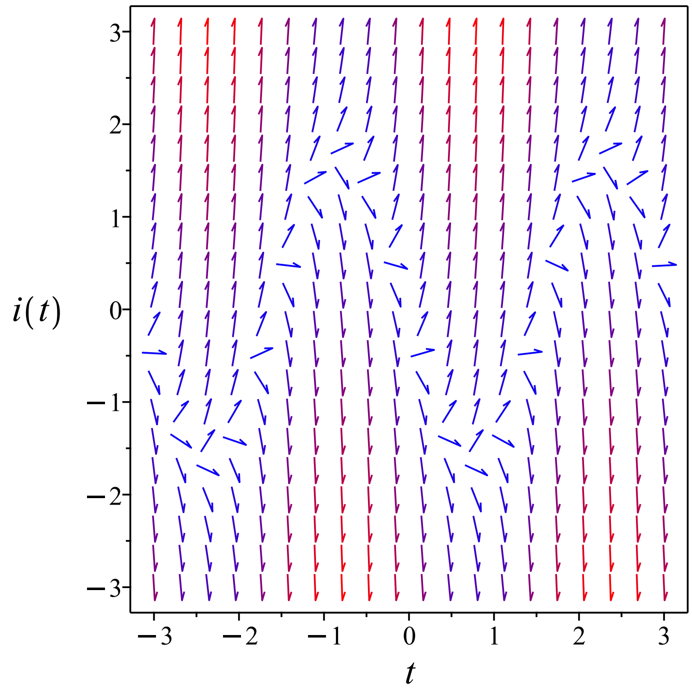


Figure 162: Slope field plot

Verification of solutions

$$
i = -\frac{e^{6t}\left(3\,e^{-6t}\sin\left(2t\right)+e^{-6t}\cos\left(2t\right)-2c_1\right)}{2}
$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, i) \,\mathrm{d}t + N(t, i) \,\mathrm{d}i = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}i &= (6i + 10 \sin(2t)) \,\mathrm{d}t \\
(-6i - 10 \sin(2t)) \,\mathrm{d}t + \mathrm{d}i &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, i) &= -6i - 10 \sin(2t) \\
N(t, i) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial i} &= \frac{\partial}{\partial i}(-6i - 10 \sin(2t)) \\
&= -6
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial i} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial i} - \frac{\partial N}{\partial t}\right) \\
&= 1((-6) - (0)) \\
&= -6
\end{aligned}
$$

Since A does not depend on i, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}t} \\
&= e^{\int -6 \, \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-6t} \\
&= \mathrm{e}^{-6t}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-6t}(-6i - 10\sin(2t)) \\
&= (-6i - 10\sin(2t))\,\mathrm{e}^{-6t}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-6t}(1) \\
&= \mathrm{e}^{-6t}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}i}{\mathrm{d}t} &= 0 \\
\left((-6i - 10\sin(2t))\,\mathrm{e}^{-6t}\right) + \left(\mathrm{e}^{-6t}\right)\frac{\mathrm{d}i}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, i)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial i} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int \overline{M}\,\mathrm{d}t$$
$$\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t = \int (-6i - 10\sin(2t))\,\mathrm{e}^{-6t}\,\mathrm{d}t$$
$$\phi = \frac{(2i + \cos(2t) + 3\sin(2t))\,\mathrm{e}^{-6t}}{2} + f(i) \tag{3}$$

Where $f(i)$ is used for the constant of integration since ϕ is a function of both t and i. Taking derivative of equation (3) w.r.t i gives

$$\frac{\partial \phi}{\partial i} = \mathrm{e}^{-6t} + f'(i) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial i} = \mathrm{e}^{-6t}$. Therefore equation (4) becomes

$$\mathrm{e}^{-6t} = \mathrm{e}^{-6t} + f'(i) \tag{5}$$

Solving equation (5) for $f'(i)$ gives

$$f'(i) = 0$$

Therefore

$$f(i) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(i)$ into equation (3) gives ϕ

$$\phi = \frac{(2i + \cos(2t) + 3\sin(2t))\,\mathrm{e}^{-6t}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2i + \cos(2t) + 3\sin(2t))\,\mathrm{e}^{-6t}}{2}$$

The solution becomes

$$i = -\frac{\mathrm{e}^{6t}(3\,\mathrm{e}^{-6t}\sin(2t) + \mathrm{e}^{-6t}\cos(2t) - 2c_1)}{2}$$

Summary
The solution(s) found are the following

$$
i = -\frac{e^{6t}\left(3\,e^{-6t}\sin\left(2t\right)+e^{-6t}\cos\left(2t\right)-2c_1\right)}{2} \tag{1}
$$

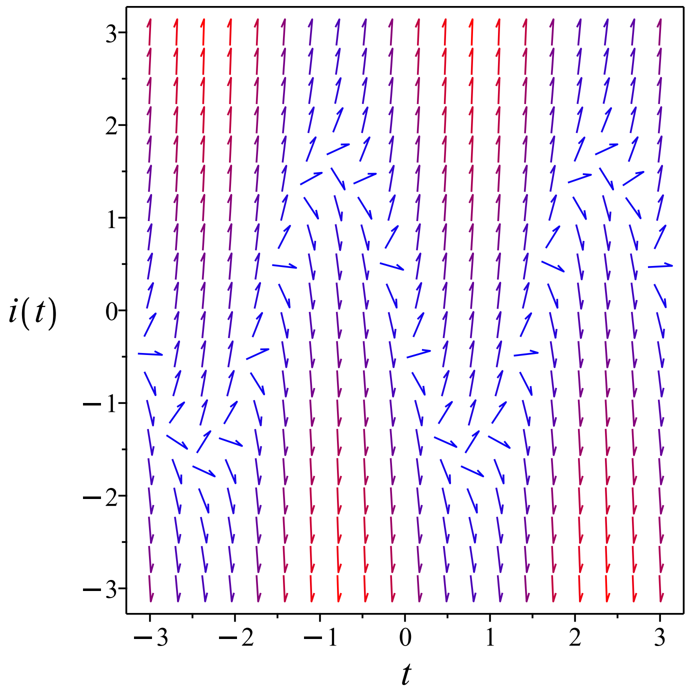


Figure 163: Slope field plot

Verification of solutions

$$
i = -\frac{e^{6t}\left(3\,e^{-6t}\sin\left(2t\right)+e^{-6t}\cos\left(2t\right)-2c_1\right)}{2}
$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$i'-6i=10\sin\left(2t\right)$

- Highest derivative means the order of the ODE is 1

  i'

- Isolate the derivative

$$i' = 6i + 10\sin(2t)$$

- Group terms with i on the lhs of the ODE and the rest on the rhs of the ODE
$$i' - 6i = 10\sin(2t)$$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
$$\mu(t)\,(i' - 6i) = 10\mu(t)\sin(2t)$$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,i)$
$$\mu(t)\,(i' - 6i) = \mu'(t)\,i + \mu(t)\,i'$$
- Isolate $\mu'(t)$
$$\mu'(t) = -6\mu(t)$$
- Solve to find the integrating factor
$$\mu(t) = \mathrm{e}^{-6t}$$
- Integrate both sides with respect to t
$$\int \left(\frac{d}{dt}(\mu(t)\,i)\right) dt = \int 10\mu(t)\sin(2t)\,dt + c_1$$
- Evaluate the integral on the lhs
$$\mu(t)\,i = \int 10\mu(t)\sin(2t)\,dt + c_1$$
- Solve for i
$$i = \frac{\int 10\mu(t)\sin(2t)dt + c_1}{\mu(t)}$$
- Substitute $\mu(t) = \mathrm{e}^{-6t}$
$$i = \frac{\int 10\,\mathrm{e}^{-6t}\sin(2t)dt + c_1}{\mathrm{e}^{-6t}}$$
- Evaluate the integrals on the rhs
$$i = \frac{-\frac{3\,\mathrm{e}^{-6t}\sin(2t)}{2} - \frac{\mathrm{e}^{-6t}\cos(2t)}{2} + c_1}{\mathrm{e}^{-6t}}$$
- Simplify
$$i = c_1\mathrm{e}^{6t} - \frac{3\sin(2t)}{2} - \frac{\cos(2t)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(i(t),t)-6*i(t)=10*sin(2*t),i(t), singsol=all)
```

$$i(t) = -\frac{\cos(2t)}{2} - \frac{3\sin(2t)}{2} + e^{6t}c_1$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 28

```
DSolve[i'[t]-6*i[t]==10*Sin[2*t],i[t],t,IncludeSingularSolutions -> True]
```

$$i(t) \to -\frac{1}{2}\cos(2t) + c_1 e^{6t} - 3\sin(t)\cos(t)$$

4.5 problem 19 (f)

4.5.1 Solving as first order ode lie symmetry lookup ode 766
4.5.2 Solving as bernoulli ode 770
4.5.3 Solving as riccati ode 773

Internal problem ID [5302]
Internal file name [`OUTPUT/4793_Friday_February_02_2024_05_11_57_AM_30202892/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (f).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{y + y' - e^{x} y^{2} = 0}$$

4.5.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= -y + y^{2} e^{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_{x} + \omega(\eta_{y} - \xi_{x}) - \omega^{2} \xi_{y} - \omega_{x} \xi - \omega_{y} \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 96: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= y^2 e^x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 \mathrm{e}^x} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^{-x}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + y^2 \mathrm{e}^x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{-x}}{y} \\ S_y &= \frac{\mathrm{e}^{-x}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{-x}}{y} = x + c_1$$

Which simplifies to

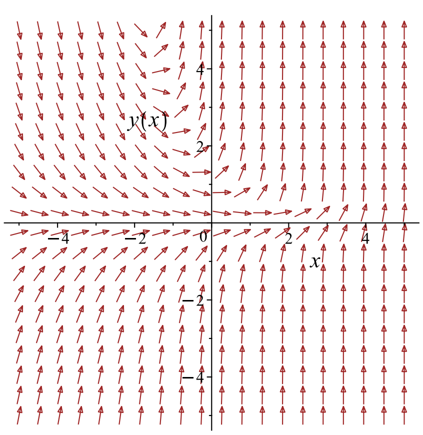
$$-\frac{e^{-x}}{y} = x + c_1$$

Which gives

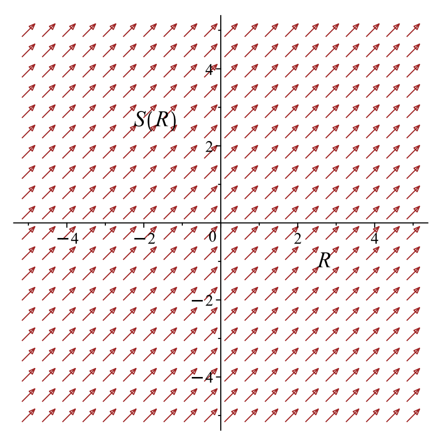
$$y = -\frac{e^{-x}}{x + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -y + y^2 e^x$


</td><td>$R = x$

$S = -\frac{e^{-x}}{y}$</td><td>$\frac{dS}{dR} = 1$


</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\frac{e^{-x}}{x + c_1} \tag{1}$$

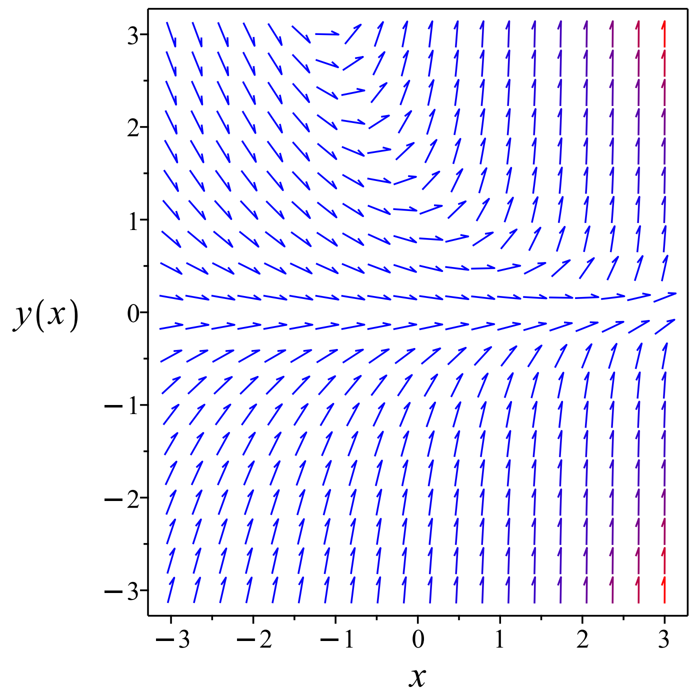


Figure 164: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}}{x + c_1}$$

Verified OK.

4.5.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + y^2 e^x \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + e^x y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -1 \\ f_1(x) &= \mathrm{e}^x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{y} + \mathrm{e}^x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -w(x) + \mathrm{e}^x \\ w' &= w - \mathrm{e}^x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -1 \\
q(x) &= -\mathrm{e}^{x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) - w(x) = -\mathrm{e}^{x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\mathrm{e}^{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x} w\right) &= \left(\mathrm{e}^{-x}\right)\left(-\mathrm{e}^{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{-x} w\right) &= -1 \,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-x} w &= \int -1 \,\mathrm{d}x \\
\mathrm{e}^{-x} w &= -x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x}$ results in

$$
w(x) = -x\,\mathrm{e}^{x} + c_1 \mathrm{e}^{x}
$$

which simplifies to

$$
w(x) = \mathrm{e}^{x}(c_1 - x)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \mathrm{e}^{x}(c_1 - x)
$$

Or

$$
y = \frac{\mathrm{e}^{-x}}{c_1 - x}
$$

Summary

The solution(s) found are the following

$$y = \frac{e^{-x}}{c_1 - x} \tag{1}$$

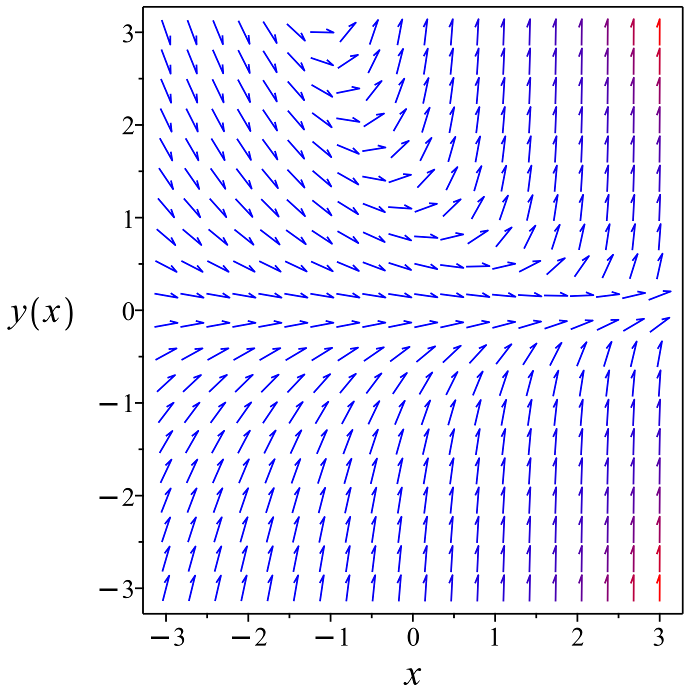


Figure 165: Slope field plot

Verification of solutions

$$y = \frac{e^{-x}}{c_1 - x}$$

Verified OK.

4.5.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + y^2 e^x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y + y^2 e^x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -1$ and $f_2(x) = e^x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^x u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= e^x \\ f_1 f_2 &= -e^x \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^x u''(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 e^{-x}}{c_1 x + c_2}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{c_3 \mathrm{e}^{-x}}{c_3 x + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3 \mathrm{e}^{-x}}{c_3 x + 1} \tag{1}$$

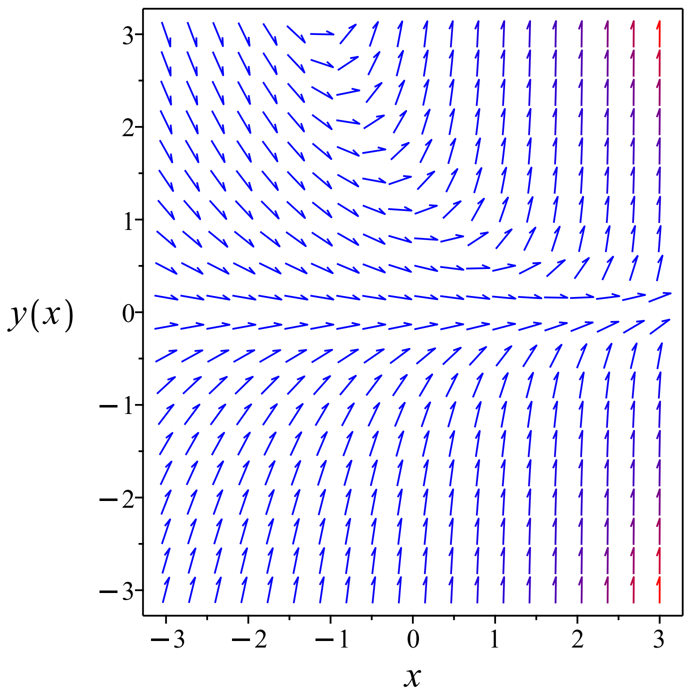


Figure 166: Slope field plot

Verification of solutions

$$y = -\frac{c_3 \mathrm{e}^{-x}}{c_3 x + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)+y(x)=y(x)^2*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{-x}}{-x + c_1}$$

✓ Solution by Mathematica

Time used: 0.204 (sec). Leaf size: 25

```
DSolve[y'[x]+y[x]==y[x]^2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{-x}}{x - c_1}$$
$$y(x) \to 0$$

4.6 problem 19 (g)

4.6.1 Solving as exact ode 777

Internal problem ID [5303]
Internal file name [`OUTPUT/4794_Friday_February_02_2024_05_11_57_AM_30991145/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (g).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [
   _Abel, `2nd type`, `class B`]]
```

$$\boxed{y+(yx+x-3y)\,y'=0}$$

4.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(xy + x - 3y) \, \mathrm{d}y &= (-y) \, \mathrm{d}x \\
(y) \, \mathrm{d}x + (xy + x - 3y) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y \\
N(x, y) &= xy + x - 3y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy + x - 3y) \\
&= y + 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{(x-3)\,y+x}((1)-(y+1)) \\
&= -\frac{y}{(x-3)\,y+x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y}((y+1)-(1)) \\
&= 1
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int 1 \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{y} \\
&= \mathrm{e}^{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{y}(y) \\
&= \mathrm{e}^{y} y
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{y}(xy+x-3y) \\
&= (xy+x-3y)\,\mathrm{e}^{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(\mathrm{e}^{y}y) + ((xy + x - 3y)\,\mathrm{e}^{y})\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \mathrm{e}^{y}y\,\mathrm{d}x$$
$$\phi = \mathrm{e}^{y}xy + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{y}xy + x\,\mathrm{e}^{y} + f'(y) \tag{4}$$
$$= x\,\mathrm{e}^{y}(y+1) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (xy + x - 3y)\,\mathrm{e}^{y}$. Therefore equation (4) becomes

$$(xy + x - 3y)\,\mathrm{e}^{y} = x\,\mathrm{e}^{y}(y+1) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -3\,\mathrm{e}^{y}y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-3\,\mathrm{e}^{y}y)\,\mathrm{d}y$$
$$f(y) = -3(y-1)\,\mathrm{e}^{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^y xy - 3(y-1)e^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^y xy - 3(y-1)e^y$$

The solution becomes

$$y = \frac{\operatorname{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) x - 3 \operatorname{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) - 3}{x-3}$$

Summary
The solution(s) found are the following

$$y = \frac{\operatorname{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) x - 3 \operatorname{LambertW}\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right) - 3}{x-3} \tag{1}$$

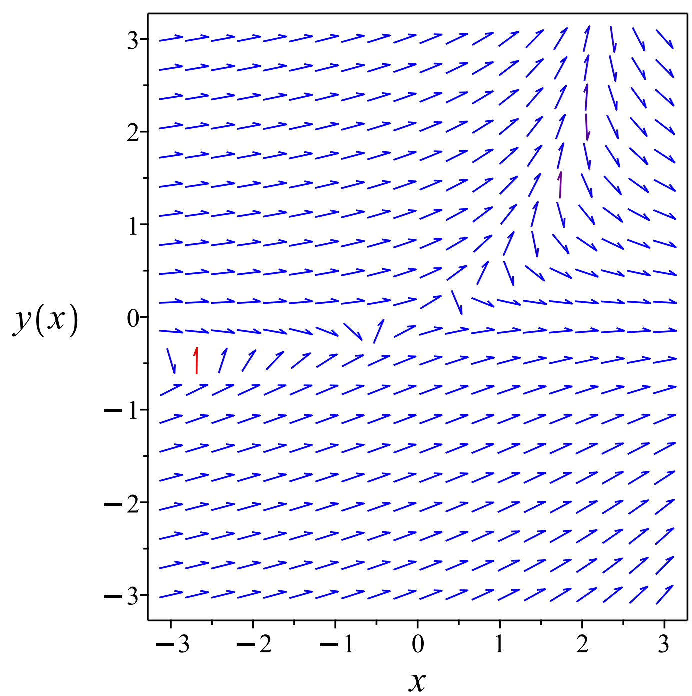


Figure 167: Slope field plot

Verification of solutions

$$y = \frac{\operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{\frac{3}{x-3}}}{x-3}\right) x - 3 \operatorname{LambertW}\left(\frac{c_1 \mathrm{e}^{\frac{3}{x-3}}}{x-3}\right) - 3}{x-3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
      -> Trying a Liouvillian solution using Kovacics algorithm
         A Liouvillian solution exists
         Reducible group (found an exponential solution)
         Reducible group (found another exponential solution)
      <- Kovacics algorithm successful
   <- Abel AIR successful: ODE belongs to the 1F1 2-parameter class
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve(y(x)+(x*y(x)+x-3*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3 + (x-3) \operatorname{LambertW}\left(\frac{\mathrm{e}^{\frac{3}{x-3}}}{c_1 (x-3)}\right)}{x-3}$$

✓ Solution by Mathematica

Time used: 60.04 (sec). Leaf size: 31

```
DSolve[y[x]+(x*y[x]+x-3*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3}{x-3} + W\left(\frac{c_1 e^{\frac{3}{x-3}}}{x-3}\right)$$

4.7 problem 19 (h)

4.7.1 Solving as exact ode 783
4.7.2 Maple step by step solution 787

Internal problem ID [5304]
Internal file name [`OUTPUT/4795_Friday_February_02_2024_05_11_59_AM_65601271/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (h).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel, `2
   nd type`, `class A`]]
```

$$\boxed{\left(2s-\mathrm{e}^{2t}\right)s'-2s\,\mathrm{e}^{2t}=-2\cos\left(2t\right)}$$

4.7.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t, s)\,\mathrm{d}t + N(t, s)\,\mathrm{d}s = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(2s - \mathrm{e}^{2t}\right)\mathrm{d}s &= \left(2s\,\mathrm{e}^{2t} - 2\cos(2t)\right)\mathrm{d}t \\
\left(-2s\,\mathrm{e}^{2t} + 2\cos(2t)\right)\mathrm{d}t + \left(2s - \mathrm{e}^{2t}\right)\mathrm{d}s &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t, s) &= -2s\,\mathrm{e}^{2t} + 2\cos(2t) \\
N(t, s) &= 2s - \mathrm{e}^{2t}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial s} &= \frac{\partial}{\partial s}\left(-2s\,\mathrm{e}^{2t} + 2\cos(2t)\right) \\
&= -2\,\mathrm{e}^{2t}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}\left(2s - \mathrm{e}^{2t}\right) \\
&= -2\,\mathrm{e}^{2t}
\end{aligned}
$$

Since $\frac{\partial M}{\partial s} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,s)$

$$
\frac{\partial \phi}{\partial t} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial s} = N \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int M \, \mathrm{d}t
$$

$$
\int \frac{\partial \phi}{\partial t} \, \mathrm{d}t = \int -2s\,\mathrm{e}^{2t} + 2\cos(2t) \, \mathrm{d}t
$$

$$
\phi = -s\,\mathrm{e}^{2t} + \sin(2t) + f(s) \tag{3}
$$

Where $f(s)$ is used for the constant of integration since ϕ is a function of both t and s. Taking derivative of equation (3) w.r.t s gives

$$
\frac{\partial \phi}{\partial s} = -\mathrm{e}^{2t} + f'(s) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial s} = 2s - \mathrm{e}^{2t}$. Therefore equation (4) becomes

$$
2s - \mathrm{e}^{2t} = -\mathrm{e}^{2t} + f'(s) \tag{5}
$$

Solving equation (5) for $f'(s)$ gives

$$
f'(s) = 2s
$$

Integrating the above w.r.t s gives

$$
\int f'(s) \, \mathrm{d}s = \int (2s) \, \mathrm{d}s
$$

$$
f(s) = s^2 + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(s)$ into equation (3) gives ϕ

$$
\phi = -s\,\mathrm{e}^{2t} + \sin(2t) + s^2 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -s\,e^{2t} + \sin(2t) + s^2$$

Summary

The solution(s) found are the following

$$-s\,e^{2t} + \sin(2t) + s^2 = c_1 \tag{1}$$

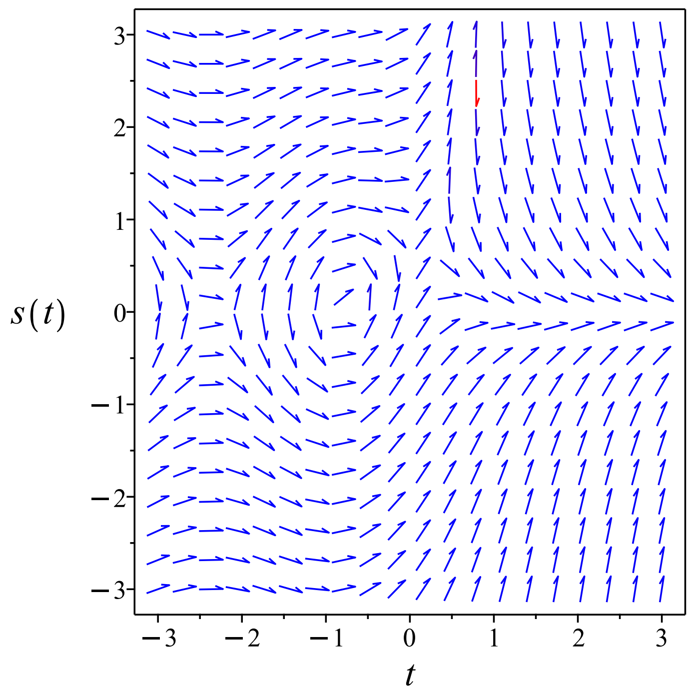


Figure 168: Slope field plot

Verification of solutions

$$-s\,e^{2t} + \sin(2t) + s^2 = c_1$$

Verified OK.

4.7.2 Maple step by step solution

Let's solve

$\left(2s-\mathrm{e}^{2t}\right)s'-2s\,\mathrm{e}^{2t}=-2\cos\left(2t\right)$

- Highest derivative means the order of the ODE is 1

  s'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(t,s)=0$

  ◦ Compute derivative of lhs

    $F'(t,s)+\left(\frac{\partial}{\partial s}F(t,s)\right)s'=0$

  ◦ Evaluate derivatives

    $-2\,\mathrm{e}^{2t}=-2\,\mathrm{e}^{2t}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(t,s)=c_1, M(t,s)=F'(t,s), N(t,s)=\frac{\partial}{\partial s}F(t,s)\right]$

- Solve for $F(t,s)$ by integrating $M(t,s)$ with respect to t

  $F(t,s)=\int\left(-2s\,\mathrm{e}^{2t}+2\cos\left(2t\right)\right)dt+f_1(s)$

- Evaluate integral

  $F(t,s)=-s\,\mathrm{e}^{2t}+\sin\left(2t\right)+f_1(s)$

- Take derivative of $F(t,s)$ with respect to s

  $N(t,s)=\frac{\partial}{\partial s}F(t,s)$

- Compute derivative

  $2s-\mathrm{e}^{2t}=-\mathrm{e}^{2t}+\frac{d}{ds}f_1(s)$

- Isolate for $\frac{d}{ds}f_1(s)$

  $\frac{d}{ds}f_1(s)=2s$

- Solve for $f_1(s)$

  $f_1(s)=s^2$

- Substitute $f_1(s)$ into equation for $F(t,s)$

  $F(t,s)=-s\,\mathrm{e}^{2t}+\sin\left(2t\right)+s^2$

- Substitute $F(t,s)$ into the solution of the ODE

  $$-s\,e^{2t}+\sin(2t)+s^2=c_1$$

- Solve for s

  $$\left\{s=\frac{e^{2t}}{2}-\frac{\sqrt{\left(e^{2t}\right)^2-4\sin(2t)+4c_1}}{2}, s=\frac{e^{2t}}{2}+\frac{\sqrt{\left(e^{2t}\right)^2-4\sin(2t)+4c_1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve((2*s(t)-exp(2*t))*diff(s(t),t)=2*(s(t)*exp(2*t)-cos(2*t)),s(t), singsol=all)
```

$$s(t)=\frac{e^{2t}}{2}-\frac{\sqrt{e^{4t}-4\sin(2t)-4c_1}}{2}$$
$$s(t)=\frac{e^{2t}}{2}+\frac{\sqrt{e^{4t}-4\sin(2t)-4c_1}}{2}$$

✓ Solution by Mathematica

Time used: 15.59 (sec). Leaf size: 81

```
DSolve[(2*s[t]-Exp[2*t])*s'[t]==2*(s[t]*Exp[2*t]-Cos[2*t]),s[t],t,IncludeSingularSolutions -
```

$$s(t) \to \frac{1}{2}\left(e^{2t} - i\sqrt{-e^{4t} + 4\sin(2t) - 4c_1}\right)$$
$$s(t) \to \frac{1}{2}\left(e^{2t} + i\sqrt{-e^{4t} + 4\sin(2t) - 4c_1}\right)$$

4.8 problem 19 (i)

4.8.1 Solving as first order ode lie symmetry lookup ode 790
4.8.2 Solving as bernoulli ode 794
4.8.3 Solving as exact ode 799

Internal problem ID [5305]
Internal file name [`OUTPUT/4796_Friday_February_02_2024_05_12_05_AM_97857565/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (i).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{xy' + y - x^3y^6 = 0}$$

4.8.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x^3y^5 - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 99: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= y^6x^5
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^6 x^5} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{5x^5 y^5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(x^3 y^5 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x^6 y^5} \\ S_y &= \frac{1}{y^6 x^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

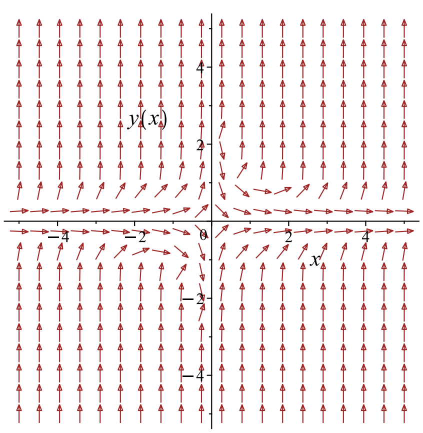
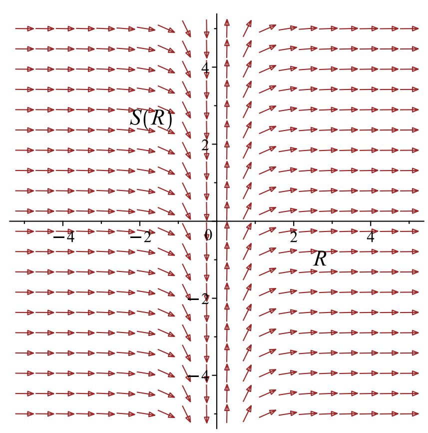
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

Which simplifies to

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x^3y^5-1)}{x}$ 	$R = x$ $S = -\frac{1}{5x^5y^5}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary
The solution(s) found are the following

$$-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1 \tag{1}$$

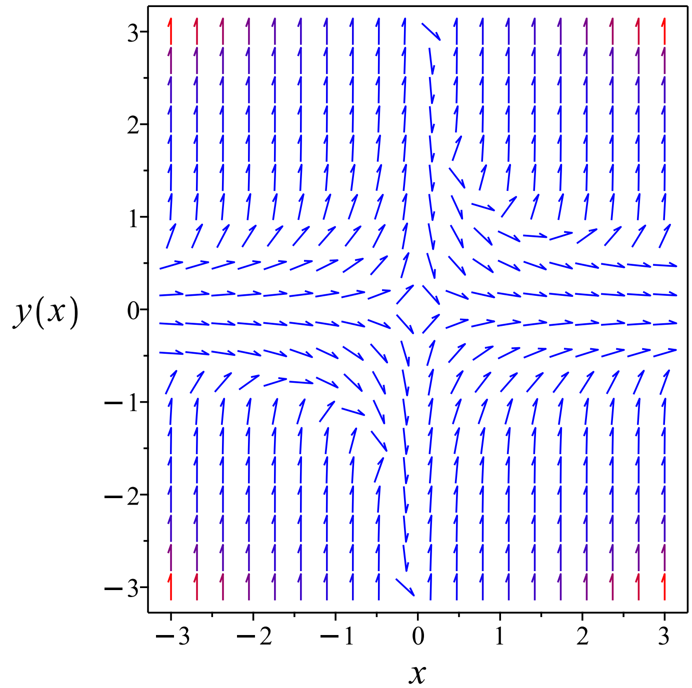


Figure 169: Slope field plot

Verification of solutions

$$
-\frac{1}{5x^5y^5} = -\frac{1}{2x^2} + c_1
$$

Verified OK.

4.8.2 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(x^3y^5 - 1)}{x}
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = -\frac{1}{x}y + x^2y^6 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= x^2 \\
n &= 6
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^6$ gives

$$
y' \frac{1}{y^6} = -\frac{1}{x\, y^5} + x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^5}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{5}{y^6} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{5} &= -\frac{w(x)}{x} + x^2 \\
w' &= \frac{5w}{x} - 5x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{5}{x} \\
q(x) &= -5x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{5w(x)}{x} = -5x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{5}{x}dx} \\ &= \frac{1}{x^5}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-5x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^5}\right) &= \left(\frac{1}{x^5}\right)\left(-5x^2\right) \\ d\left(\frac{w}{x^5}\right) &= \left(-\frac{5}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^5} &= \int -\frac{5}{x^3}\, dx \\ \frac{w}{x^5} &= \frac{5}{2x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^5}$ results in

$$w(x) = \frac{5}{2}x^3 + c_1 x^5$$

Replacing w in the above by $\frac{1}{y^5}$ using equation (5) gives the final solution.

$$\frac{1}{y^5} = \frac{5}{2}x^3 + c_1 x^5$$

Solving for y gives

$$y(x) = \frac{2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{2c_1 x^3 + 5x}$$

$$y(x) = \frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}} + \sqrt{5} - 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = \frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}} - \sqrt{5} - 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = -\frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}} + \sqrt{5} + 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

$$y(x) = -\frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}} - \sqrt{5} + 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{2c_1 x^3 + 5x} \tag{1}$$

$$y = \frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}} + \sqrt{5} - 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x} \tag{2}$$

$$y = \frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}} - \sqrt{5} - 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x} \tag{3}$$

$$y = -\frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}} + \sqrt{5} + 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x} \tag{4}$$

$$y = -\frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}} - \sqrt{5} + 1\right) 2^{\frac{1}{5}} \left(x^2 (2c_1 x^2 + 5)^4\right)^{\frac{1}{5}}}{8c_1 x^3 + 20x} \tag{5}$$

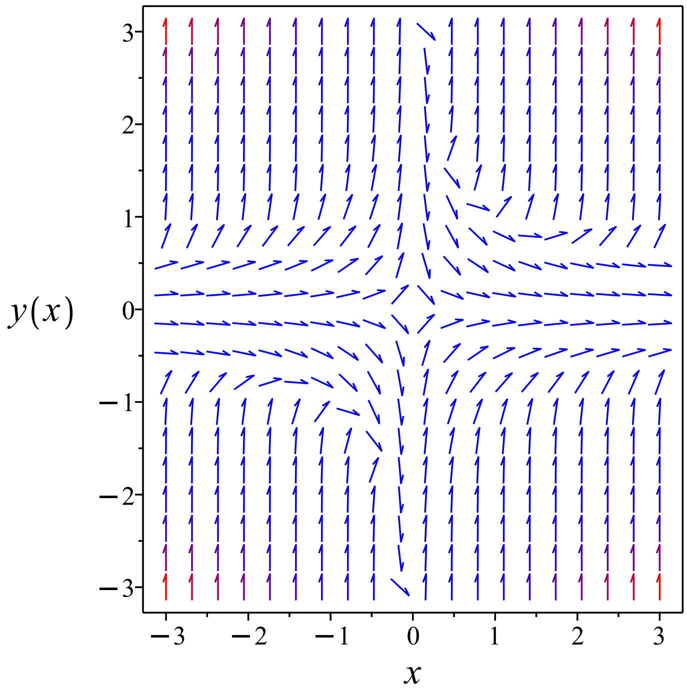


Figure 170: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{5}} \left(x^{2} \left(2 c_{1} x^{2}+5\right)^{4}\right)^{\frac{1}{5}}}{2 c_{1} x^{3}+5 x}$$

Verified OK.

$$y = \frac{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}+\sqrt{5}-1\right) 2^{\frac{1}{5}} \left(x^{2} \left(2 c_{1} x^{2}+5\right)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

Verified OK.

$$y = \frac{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}-\sqrt{5}-1\right) 2^{\frac{1}{5}} \left(x^{2} \left(2 c_{1} x^{2}+5\right)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

Verified OK.

$$y = -\frac{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}+\sqrt{5}+1\right) 2^{\frac{1}{5}} \left(x^{2} \left(2 c_{1} x^{2}+5\right)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

Verified OK.

$$y = -\frac{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}-\sqrt{5}+1\right) 2^{\frac{1}{5}} \left(x^{2} \left(2 c_{1} x^{2}+5\right)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

Verified OK.

4.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^3 y^6 - y\right)\mathrm{d}x \\ \left(-x^3 y^6 + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^3 y^6 + y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 y^6 + y\right) \\ &= -6x^3 y^5 + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left((-6x^3y^5 + 1) - (1)\right) \\
&= -6x^2y^5
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-x^3y^6 + y}\left((1) - \left(-6x^3y^5 + 1\right)\right) \\
&= -\frac{6x^3y^4}{x^3y^5 - 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-6x^3y^5 + 1)}{x(-x^3y^6 + y) - y(x)} \\
&= -\frac{6}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{6}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{6}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-6\ln(t)}\\ &= \frac{1}{t^6}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^6 y^6}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^6 y^6}\left(-x^3 y^6 + y\right)\\ &= \frac{-x^3 y^5 + 1}{y^5 x^6}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^6 y^6}(x)\\ &= \frac{1}{y^6 x^5}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^3 y^5 + 1}{y^5 x^6}\right) + \left(\frac{1}{y^6 x^5}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-x^3y^5+1}{y^5x^6} \mathrm{d}x \\
\phi &= \frac{5x^3y^5-2}{10x^5y^5} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{5}{2y\,x^2} - \frac{5x^3y^5-2}{2x^5y^6} + f'(y) \qquad (4) \\
&= \frac{1}{y^6x^5} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^6x^5}$. Therefore equation (4) becomes

$$
\frac{1}{y^6x^5} = \frac{1}{y^6x^5} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{5x^3y^5-2}{10x^5y^5} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{5x^3y^5-2}{10x^5y^5}
$$

Summary

The solution(s) found are the following

$$\frac{5y^5x^3-2}{10x^5y^5}=c_1 \tag{1}$$

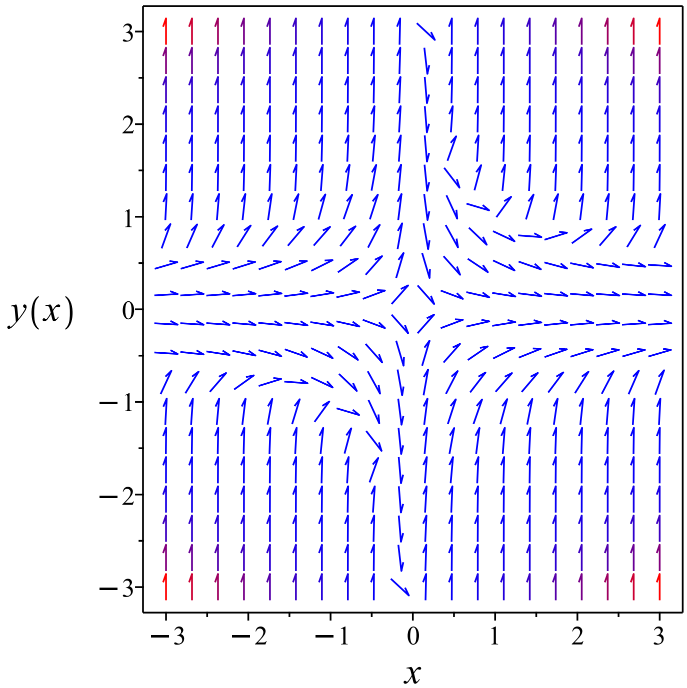


Figure 171: Slope field plot

Verification of solutions

$$\frac{5y^5x^3-2}{10x^5y^5}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 258

```
dsolve(x*diff(y(x),x)+(y(x)-x^3*y(x)^6)=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{1}{5}} \left(x^{2} (2 c_{1} x^{2}+5)^{4}\right)^{\frac{1}{5}}}{2 c_{1} x^{3}+5 x}$$

$$y(x) = -\frac{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}+\sqrt{5}+1\right) 2^{\frac{1}{5}} \left(x^{2} (2 c_{1} x^{2}+5)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

$$y(x) = \frac{\left(i \sqrt{2}\, \sqrt{5-\sqrt{5}}-\sqrt{5}-1\right) 2^{\frac{1}{5}} \left(x^{2} (2 c_{1} x^{2}+5)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

$$y(x) = -\frac{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}-\sqrt{5}+1\right) 2^{\frac{1}{5}} \left(x^{2} (2 c_{1} x^{2}+5)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

$$y(x) = \frac{\left(i \sqrt{2}\, \sqrt{5+\sqrt{5}}+\sqrt{5}-1\right) 2^{\frac{1}{5}} \left(x^{2} (2 c_{1} x^{2}+5)^{4}\right)^{\frac{1}{5}}}{8 c_{1} x^{3}+20 x}$$

✓ Solution by Mathematica

Time used: 0.472 (sec). Leaf size: 141

```
DSolve[x*y'[x]+(y[x]-x^3*y[x]^6)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\frac{\sqrt[5]{-2}}{\sqrt[5]{x^3\left(5+2c_1x^2\right)}} \\
y(x) &\to \frac{1}{\sqrt[5]{\frac{5x^3}{2}+c_1x^5}} \\
y(x) &\to \frac{(-1)^{2/5}}{\sqrt[5]{\frac{5x^3}{2}+c_1x^5}} \\
y(x) &\to -\frac{(-1)^{3/5}}{\sqrt[5]{\frac{5x^3}{2}+c_1x^5}} \\
y(x) &\to \frac{(-1)^{4/5}}{\sqrt[5]{\frac{5x^3}{2}+c_1x^5}} \\
y(x) &\to 0
\end{aligned}
$$

4.9 problem 19 (j)

4.9.1 Solving as linear ode 807
4.9.2 Solving as first order ode lie symmetry lookup ode 809
4.9.3 Solving as exact ode 813
4.9.4 Maple step by step solution 817

Internal problem ID [5306]
Internal file name [`OUTPUT/4797_Friday_February_02_2024_05_12_07_AM_85124973/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (j).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{r' + 2r\cos(\theta) = -\sin(2\theta)}$$

4.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$p(\theta) = 2\cos(\theta)$$
$$q(\theta) = -\sin(2\theta)$$

Hence the ode is

$$r' + 2r\cos(\theta) = -\sin(2\theta)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2\cos(\theta)d\theta} \\
&= \mathrm{e}^{2\sin(\theta)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}(\mu r) &= (\mu)\left(-\sin\left(2\theta\right)\right) \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\mathrm{e}^{2\sin(\theta)} r\right) &= \left(\mathrm{e}^{2\sin(\theta)}\right)\left(-\sin\left(2\theta\right)\right) \\
\mathrm{d}\left(\mathrm{e}^{2\sin(\theta)} r\right) &= \left(-\mathrm{e}^{2\sin(\theta)}\sin\left(2\theta\right)\right)\,\mathrm{d}\theta
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{2\sin(\theta)} r &= \int -\mathrm{e}^{2\sin(\theta)}\sin\left(2\theta\right)\,\mathrm{d}\theta \\
\mathrm{e}^{2\sin(\theta)} r &= -\mathrm{e}^{2\sin(\theta)}\sin\left(\theta\right) + \frac{\mathrm{e}^{2\sin(\theta)}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2\sin(\theta)}$ results in

$$
r = \mathrm{e}^{-2\sin(\theta)}\left(-\mathrm{e}^{2\sin(\theta)}\sin\left(\theta\right) + \frac{\mathrm{e}^{2\sin(\theta)}}{2}\right) + c_1\mathrm{e}^{-2\sin(\theta)}
$$

which simplifies to

$$
r = -\sin\left(\theta\right) + \frac{1}{2} + c_1\mathrm{e}^{-2\sin(\theta)}
$$

Summary

The solution(s) found are the following

$$
r = -\sin\left(\theta\right) + \frac{1}{2} + c_1\mathrm{e}^{-2\sin(\theta)} \tag{1}
$$

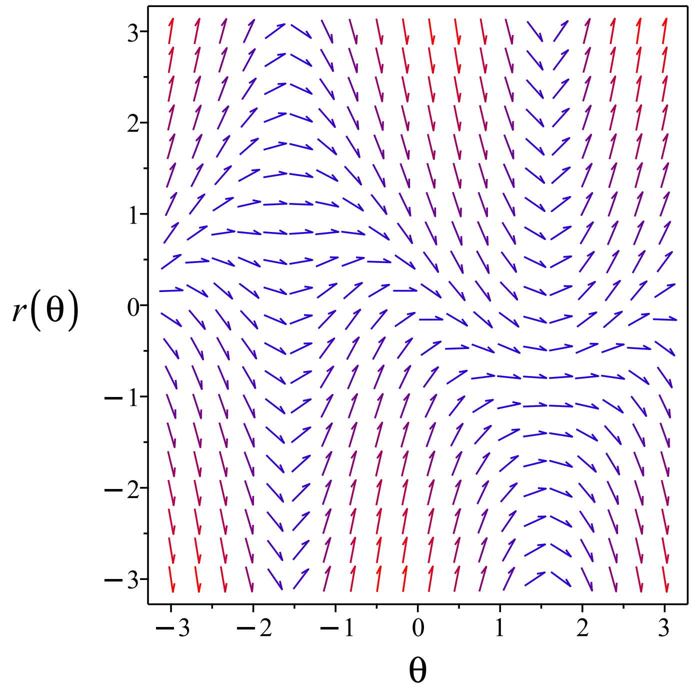


Figure 172: Slope field plot

Verification of solutions

$$r = -\sin(\theta) + \frac{1}{2} + c_1 e^{-2\sin(\theta)}$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = -2\cos(\theta)\, r - \sin(2\theta)$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 101: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \mathrm{e}^{-2\sin(\theta)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2\sin(\theta)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{2\sin(\theta)} r$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = -2\cos(\theta) r - \sin(2\theta)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_\theta &= 1 \\ R_r &= 0 \\ S_\theta &= 2\cos(\theta) \mathrm{e}^{2\sin(\theta)} r \\ S_r &= \mathrm{e}^{2\sin(\theta)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\mathrm{e}^{2\sin(\theta)} \sin(2\theta) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\mathrm{e}^{2\sin(R)} \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 - \frac{e^{2\sin(R)}(-1 + 2\sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$e^{2\sin(\theta)} r = c_1 - \frac{e^{2\sin(\theta)}(-1 + 2\sin(\theta))}{2}$$

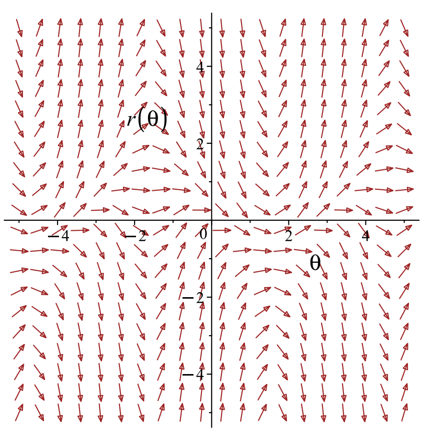
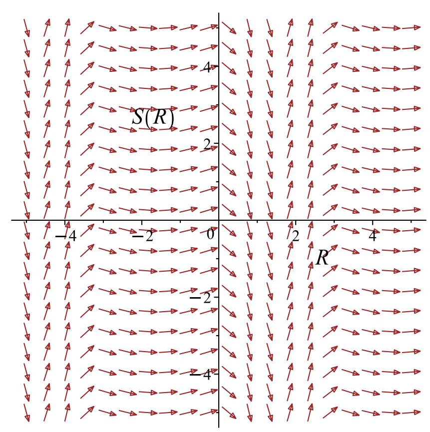
Which simplifies to

$$\frac{(2r + 2\sin(\theta) - 1)\, e^{2\sin(\theta)}}{2} - c_1 = 0$$

Which gives

$$r = -\frac{e^{-2\sin(\theta)}\left(2\, e^{2\sin(\theta)} \sin(\theta) - e^{2\sin(\theta)} - 2c_1\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = -2\cos(\theta) r - \sin(2\theta)$ 	$R = \theta$ $S = e^{2\sin(\theta)} r$	$\frac{dS}{dR} = -e^{2\sin(R)} \sin(2R)$ 

Summary

The solution(s) found are the following

$$r = -\frac{e^{-2\sin(\theta)}\left(2\, e^{2\sin(\theta)} \sin(\theta) - e^{2\sin(\theta)} - 2c_1\right)}{2} \tag{1}$$

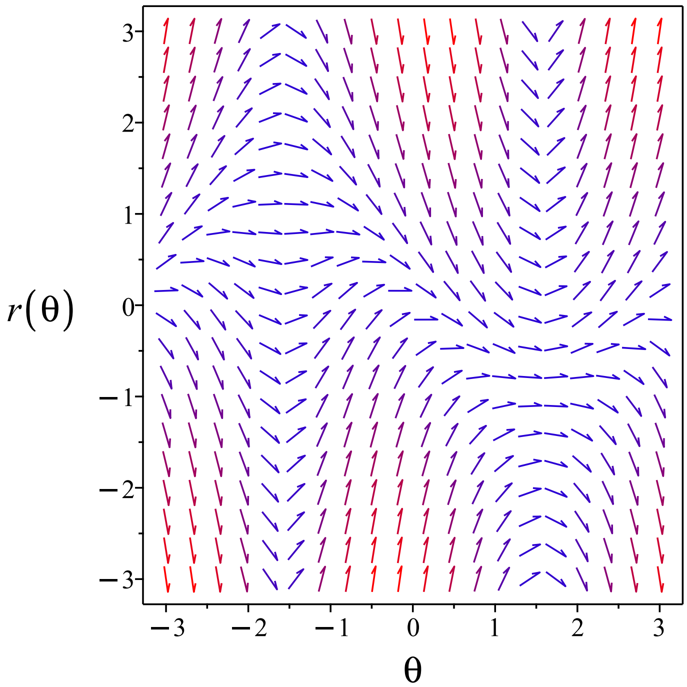


Figure 173: Slope field plot

Verification of solutions

$$r = -\frac{e^{-2\sin(\theta)}\left(2\,e^{2\sin(\theta)}\sin(\theta) - e^{2\sin(\theta)} - 2c_1\right)}{2}$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(\theta, r)\, \mathrm{d}\theta + N(\theta, r)\, \mathrm{d}r = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}r &= (-2\cos(\theta)\, r - \sin(2\theta))\, \mathrm{d}\theta \\
(2\cos(\theta)\, r + \sin(2\theta))\, \mathrm{d}\theta + \mathrm{d}r &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(\theta, r) &= 2\cos(\theta)\, r + \sin(2\theta) \\
N(\theta, r) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}(2\cos(\theta)\, r + \sin(2\theta)) \\
&= 2\cos(\theta)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= 1((2\cos(\theta)) - (0)) \\
&= 2\cos(\theta)
\end{aligned}
$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int 2\cos(\theta) \, \mathrm{d}\theta}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2\sin(\theta)} \\
&= \mathrm{e}^{2\sin(\theta)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2\sin(\theta)}(2\cos(\theta)\, r + \sin(2\theta)) \\
&= 2\cos(\theta)\,(r + \sin(\theta))\,\mathrm{e}^{2\sin(\theta)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2\sin(\theta)}(1) \\
&= \mathrm{e}^{2\sin(\theta)}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
\left(2\cos(\theta)\,(r + \sin(\theta))\,\mathrm{e}^{2\sin(\theta)}\right) + \left(\mathrm{e}^{2\sin(\theta)}\right)\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$
\frac{\partial \phi}{\partial \theta} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial r} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. θ gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial \theta} \, d\theta &= \int \overline{M} \, d\theta \\
\int \frac{\partial \phi}{\partial \theta} \, d\theta &= \int 2\cos(\theta)(r+\sin(\theta))\, e^{2\sin(\theta)} \, d\theta \\
\phi &= \frac{(2r + 2\sin(\theta) - 1)\, e^{2\sin(\theta)}}{2} + f(r) \qquad (3)
\end{aligned}
$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$
\frac{\partial \phi}{\partial r} = e^{2\sin(\theta)} + f'(r) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = e^{2\sin(\theta)}$. Therefore equation (4) becomes

$$
e^{2\sin(\theta)} = e^{2\sin(\theta)} + f'(r) \qquad (5)
$$

Solving equation (5) for $f'(r)$ gives

$$
f'(r) = 0
$$

Therefore

$$
f(r) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$
\phi = \frac{(2r + 2\sin(\theta) - 1)\, e^{2\sin(\theta)}}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{(2r + 2\sin(\theta) - 1)\, e^{2\sin(\theta)}}{2}
$$

The solution becomes

$$
r = -\frac{e^{-2\sin(\theta)} \left(2\, e^{2\sin(\theta)} \sin(\theta) - e^{2\sin(\theta)} - 2c_1\right)}{2}
$$

Summary

The solution(s) found are the following

$$r = -\frac{\mathrm{e}^{-2\sin(\theta)}\left(2\,\mathrm{e}^{2\sin(\theta)}\sin(\theta)-\mathrm{e}^{2\sin(\theta)}-2c_1\right)}{2} \tag{1}$$

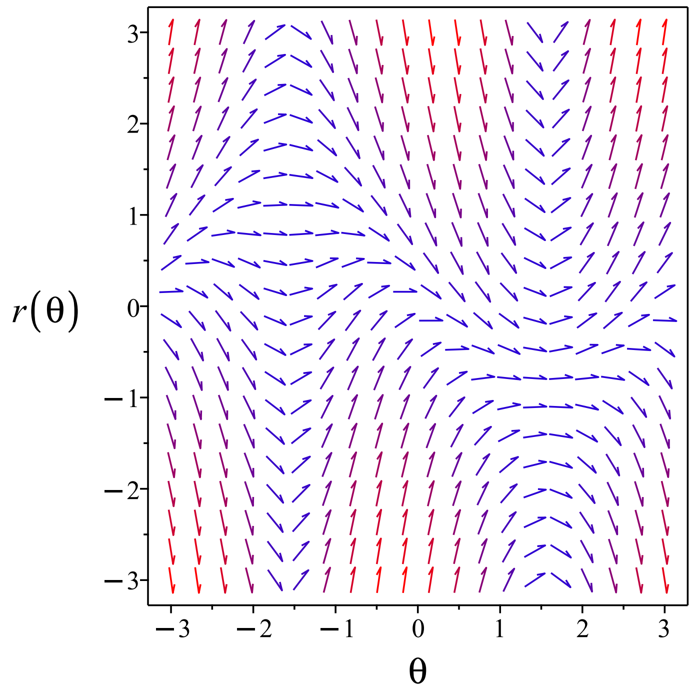


Figure 174: Slope field plot

Verification of solutions

$$r = -\frac{\mathrm{e}^{-2\sin(\theta)}\left(2\,\mathrm{e}^{2\sin(\theta)}\sin(\theta)-\mathrm{e}^{2\sin(\theta)}-2c_1\right)}{2}$$

Verified OK.

4.9.4 Maple step by step solution

Let's solve

$r' + 2r\cos(\theta) = -\sin(2\theta)$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

$$r' = -2r\cos(\theta) - \sin(2\theta)$$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

$$r' + 2r\cos(\theta) = -\sin(2\theta)$$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

$$\mu(\theta)\left(r' + 2r\cos(\theta)\right) = -\mu(\theta)\sin(2\theta)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta)\, r)$

$$\mu(\theta)\left(r' + 2r\cos(\theta)\right) = \mu'(\theta)\, r + \mu(\theta)\, r'$$

- Isolate $\mu'(\theta)$

$$\mu'(\theta) = 2\mu(\theta)\cos(\theta)$$

- Solve to find the integrating factor

$$\mu(\theta) = \mathrm{e}^{2\sin(\theta)}$$

- Integrate both sides with respect to θ

$$\int \left(\frac{d}{d\theta}(\mu(\theta)\, r)\right) d\theta = \int -\mu(\theta)\sin(2\theta)\, d\theta + c_1$$

- Evaluate the integral on the lhs

$$\mu(\theta)\, r = \int -\mu(\theta)\sin(2\theta)\, d\theta + c_1$$

- Solve for r

$$r = \frac{\int -\mu(\theta)\sin(2\theta)d\theta + c_1}{\mu(\theta)}$$

- Substitute $\mu(\theta) = \mathrm{e}^{2\sin(\theta)}$

$$r = \frac{\int -\mathrm{e}^{2\sin(\theta)}\sin(2\theta)d\theta + c_1}{\mathrm{e}^{2\sin(\theta)}}$$

- Evaluate the integrals on the rhs

$$r = \frac{-\mathrm{e}^{2\sin(\theta)}\sin(\theta) + c_1 + \frac{\mathrm{e}^{2\sin(\theta)}}{2}}{\mathrm{e}^{2\sin(\theta)}}$$

- Simplify

$$r = -\sin(\theta) + \frac{1}{2} + c_1\mathrm{e}^{-2\sin(\theta)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(r(theta),theta)+(2*r(theta)*cos(theta)+sin(2*theta))=0,r(theta), singsol=all)
```

$$r(\theta) = -\sin(\theta) + \frac{1}{2} + \mathrm{e}^{-2\sin(\theta)} c_1$$

✓ Solution by Mathematica

Time used: 0.095 (sec). Leaf size: 22

```
DSolve[r'[t]+(2*r[t]*Cos[t]+Sin[2*t])==0,r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \rightarrow -\sin(t) + c_1 e^{-2\sin(t)} + \frac{1}{2}$$

4.10 problem 19 (k)

4.10.1 Solving as exact ode 820

Internal problem ID [5307]
Internal file name [`OUTPUT/4798_Friday_February_02_2024_05_12_08_AM_80125195/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (k).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y(1+y^2)-2(1-2xy^2)\,y'=0}$$

4.10.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4y^2x-2)\,\mathrm{d}y &= (-y(y^2+1))\,\mathrm{d}x\\ (y(y^2+1))\,\mathrm{d}x + (4y^2x-2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y(y^2+1)\\ N(x,y) &= 4y^2x-2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(y^2+1))\\ &= 3y^2+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4y^2x-2)\\ &= 4y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{4y^2x-2}((3y^2+1)-(4y^2))\\ &= \frac{-y^2+1}{4y^2x-2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\left(y^{2}+1\right)}\left(\left(4y^{2}\right) - \left(3y^{2}+1\right)\right) \\
&= \frac{y^{2}-1}{y\left(y^{2}+1\right)}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int \frac{y^{2}-1}{y\left(y^{2}+1\right)} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln\left(y^{2}+1\right)-\ln(y)} \\
&= \frac{y^{2}+1}{y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{y^{2}+1}{y}\left(y\left(y^{2}+1\right)\right) \\
&= \left(y^{2}+1\right)^{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{y^{2}+1}{y}\left(4y^{2}x-2\right) \\
&= \frac{\left(4y^{2}x-2\right)\left(y^{2}+1\right)}{y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(y^{2}+1\right)^{2}\right) + \left(\frac{\left(4y^{2}x-2\right)\left(y^{2}+1\right)}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int (y^2 + 1)^2 \, \mathrm{d}x$$

$$\phi = (y^2 + 1)^2 x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4(y^2 + 1) \, xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(4y^2x - 2)(y^2+1)}{y}$. Therefore equation (4) becomes

$$\frac{(4y^2x - 2)(y^2 + 1)}{y} = 4(y^2 + 1) \, xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{2(y^2 + 1)}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{-2y^2 - 2}{y} \right) \mathrm{d}y$$

$$f(y) = -y^2 - 2 \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = (y^2 + 1)^2 x - y^2 - 2 \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (y^2 + 1)^2 x - y^2 - 2\ln(y)$$

Summary

The solution(s) found are the following

$$(1 + y^2)^2 x - y^2 - 2\ln(y) = c_1 \tag{1}$$

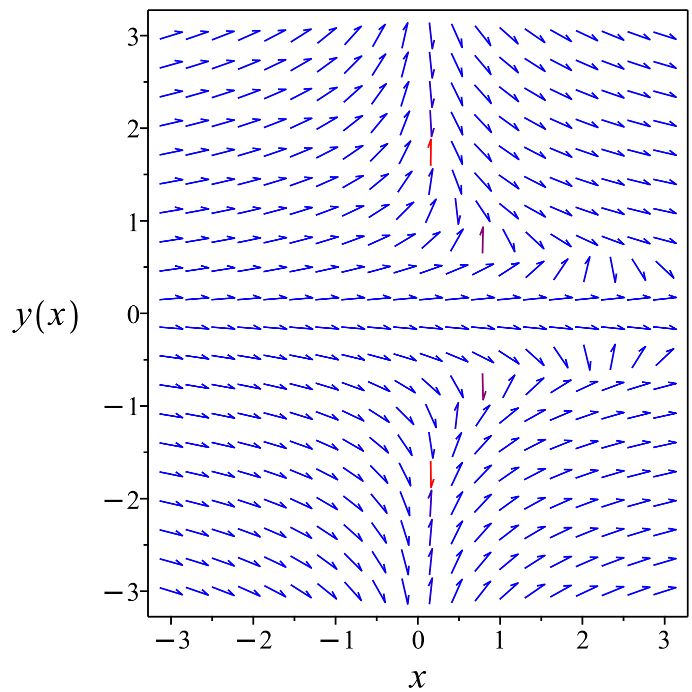


Figure 175: Slope field plot

Verification of solutions

$$(1 + y^2)^2 x - y^2 - 2\ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(y(x)*(1+y(x)^2)=2*(1-2*x*y(x)^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(-x\,\mathrm{e}^{4\_Z}-2x\,\mathrm{e}^{2\_Z}+\mathrm{e}^{2\_Z}+c_1+2\_Z-x\right)}$$

✓ Solution by Mathematica

Time used: 0.176 (sec). Leaf size: 36

```
DSolve[y[x]*(1+y[x]^2)==2*(1-2*x*y[x]^2)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\mathrm{Solve}\left[x = \frac{y(x)^2 + 2\log(y(x))}{\left(y(x)^2+1\right)^2} + \frac{c_1}{\left(y(x)^2+1\right)^2}, y(x)\right]$$

4.11 problem 19 (L)

4.11.1 Solving as separable ode 826
4.11.2 Solving as first order ode lie symmetry lookup ode 828
4.11.3 Solving as bernoulli ode 832
4.11.4 Solving as exact ode 835
4.11.5 Maple step by step solution 839

Internal problem ID [5308]
Internal file name [OUTPUT/4799_Friday_February_02_2024_05_12_09_AM_67397305/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (L).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{yy' - xy^2 = -x}$$

4.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(y^2-1)}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{y^2-1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2-1}{y}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{y^2-1}{y}}\, dy = \int x\, dx$$
$$\frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} = \frac{x^2}{2} + c_1$$

The above can be written as

$$\begin{aligned}\left(\frac{1}{2}\right)(\ln(y-1) + \ln(y+1)) &= \frac{x^2}{2} + 2c_1 \\ \ln(y-1) + \ln(y+1) &= (2)\left(\frac{x^2}{2} + 2c_1\right) \\ &= x^2 + 4c_1\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(y-1)+\ln(y+1)} = \mathrm{e}^{x^2+2c_1}$$

Which simplifies to

$$\begin{aligned}y^2 - 1 &= 2\,\mathrm{e}^{x^2} c_1 \\ &= c_2 \mathrm{e}^{x^2}\end{aligned}$$

The solution is

$$y^2 - 1 = c_2 \mathrm{e}^{x^2}$$

Summary

The solution(s) found are the following

$$y^2 - 1 = c_2 \mathrm{e}^{x^2} \tag{1}$$

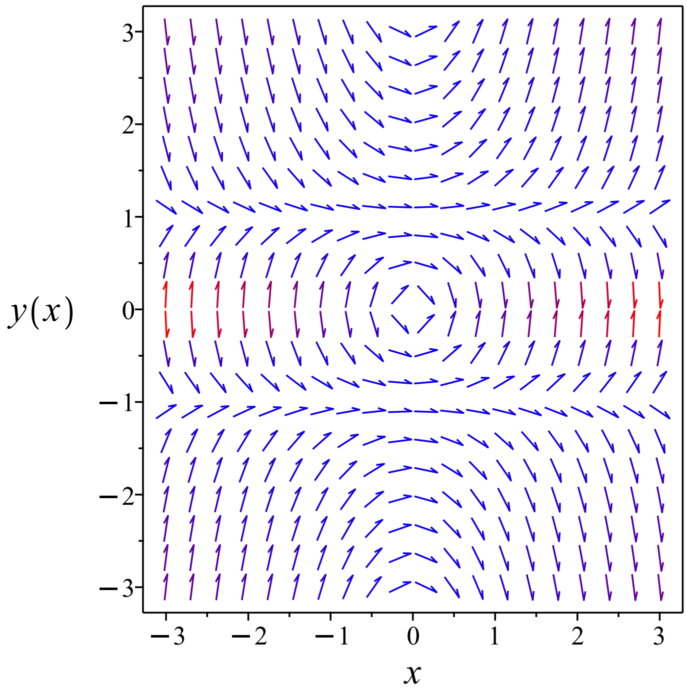


Figure 176: Slope field plot

Verification of solutions

$$y^2 - 1 = c_2 e^{x^2}$$

Verified OK.

4.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x(y^2 - 1)}{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 104: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(y^2 - 1)}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \tag{4}$$

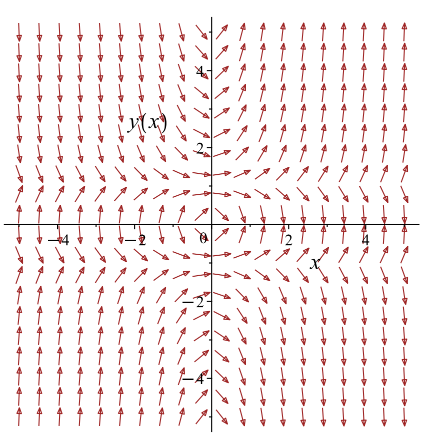
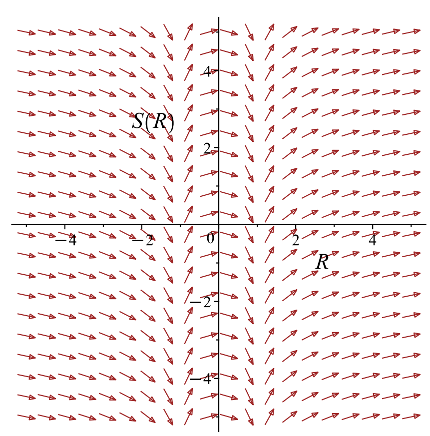
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(y^2-1)}{y}$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

<u>Summary</u>

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1 \tag{1}$$

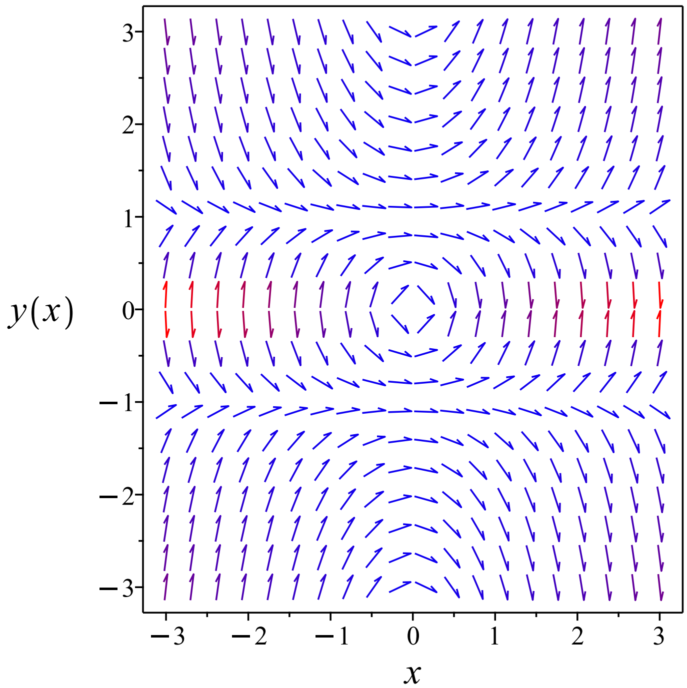


Figure 177: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

Verified OK.

4.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x(y^2-1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy - x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= x \\
f_1(x) &= -x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = y^2x - x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= w(x)\,x - x \\
w' &= 2xw - 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2x \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2w(x)\,x = -2x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2x dx} \\
&= e^{-x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\
\frac{d}{dx}\left(e^{-x^2} w\right) &= \left(e^{-x^2}\right)(-2x) \\
d\left(e^{-x^2} w\right) &= \left(-2x\, e^{-x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x^2} w &= \int -2x\, e^{-x^2} dx \\
e^{-x^2} w &= e^{-x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$
w(x) = e^{x^2} e^{-x^2} + e^{x^2} c_1
$$

which simplifies to

$$
w(x) = 1 + e^{x^2} c_1
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = 1 + e^{x^2} c_1
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \sqrt{1 + e^{x^2} c_1} \\
y(x) &= -\sqrt{1 + e^{x^2} c_1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \sqrt{1 + e^{x^2} c_1} \tag{1}
$$

$$
y = -\sqrt{1 + e^{x^2} c_1} \tag{2}
$$

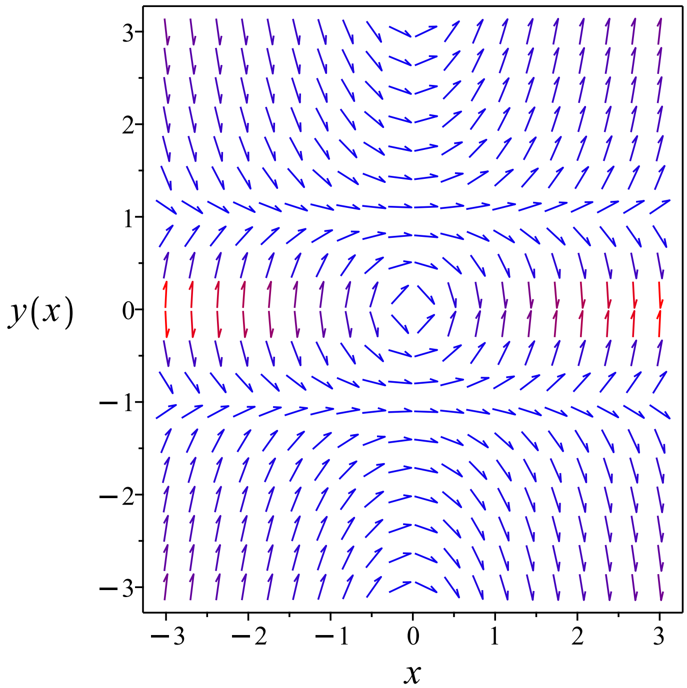


Figure 178: Slope field plot

Verification of solutions

$$y = \sqrt{1 + e^{x^2} c_1}$$

Verified OK.

$$y = -\sqrt{1 + e^{x^2} c_1}$$

Verified OK.

4.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y^2-1}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{y}{y^2-1}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=\frac{y}{y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2-1}$. Therefore equation (4) becomes

$$
\frac{y}{y^2-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{y^2-1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{y}{y^2-1}\right) \mathrm{d}y \\
f(y) &= \frac{\ln (y-1)}{2} + \frac{\ln (y+1)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} = c_1 \tag{1}$$

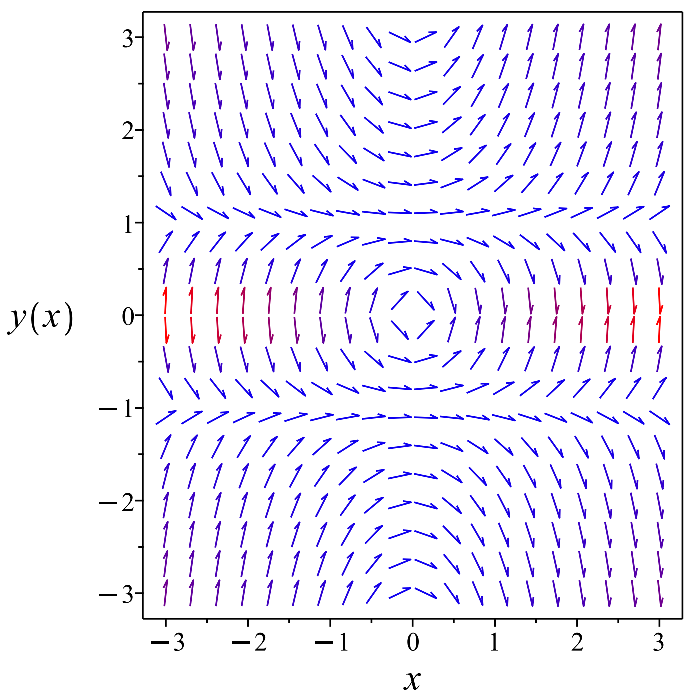


Figure 179: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{\ln(y-1)}{2} + \frac{\ln(y+1)}{2} = c_1$$

Verified OK.

4.11.5 Maple step by step solution

Let's solve

$yy' - xy^2 = -x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{(y-1)(y+1)} = x$

- Integrate both sides with respect to x

  $\int \frac{y'y}{(y-1)(y+1)}dx = \int xdx + c_1$

- Evaluate integral

  $\frac{\ln((y-1)(y+1))}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{1 + e^{x^2+2c_1}}, y = -\sqrt{1 + e^{x^2+2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(y(x)*diff(y(x),x)-x*y(x)^2+x=0,y(x), singsol=all)
```

$$y(x) = \sqrt{e^{x^2}c_1 + 1}$$
$$y(x) = -\sqrt{e^{x^2}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 1.859 (sec). Leaf size: 53

```
DSolve[y[x]*y'[x]-x*y[x]^2+x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{1+e^{x^2+2c_1}} \\
y(x) &\to \sqrt{1+e^{x^2+2c_1}} \\
y(x) &\to -1 \\
y(x) &\to 1
\end{aligned}
$$

4.12 problem 19 (m)

Internal problem ID [5309]
Internal file name [`OUTPUT/4800_Friday_February_02_2024_05_12_10_AM_19531194/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (m).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

Unable to solve or complete the solution.

$$\boxed{\left(x - x\sqrt{x^2 - y^2}\right) y' - y = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5`[0, (x^2-y^2)^(1/2)/((x^2-y^2)^(1/2)-1)]
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 27

```
dsolve((x-x*sqrt(x^2-y(x)^2))*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x)-\arctan\left(\frac{y(x)}{\sqrt{x^2-y(x)^2}}\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.518 (sec). Leaf size: 29

```
DSolve[(x-x*Sqrt[x^2-y[x]^2])*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\arctan\left(\frac{\sqrt{x^2-y(x)^2}}{y(x)}\right)+y(x)=c_1,y(x)\right]$$

4.13 problem 19 (o)

4.13.1 Solving as first order ode lie symmetry lookup ode 843
4.13.2 Solving as bernoulli ode 847
4.13.3 Solving as exact ode 851

Internal problem ID [5310]
Internal file name [`OUTPUT/4801_Friday_February_02_2024_05_12_12_AM_22931473/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (o).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{2x' - \frac{x}{y} + x^3 \cos(y) = 0}$$

4.13.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$x' = -\frac{x(x^2 \cos(y)\, y - 1)}{2y}$$
$$x' = \omega(y, x)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_y + \omega(\eta_x - \xi_y) - \omega^2 \xi_x - \omega_y \xi - \omega_x \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y,x) &= 0 \\
\eta(y,x) &= \frac{x^3}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dx}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial x}\right) S(y,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y}} dy \end{aligned}$$

Which results in

$$S = -\frac{y}{2x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_y + \omega(y, x) S_x}{R_y + \omega(y, x) R_x} \tag{2}$$

Where in the above R_y, R_x, S_y, S_x are all partial derivatives and $\omega(y, x)$ is the right hand side of the original ode given by

$$\omega(y, x) = -\frac{x(x^2 \cos(y) y - 1)}{2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_y &= 1 \\ R_x &= 0 \\ S_y &= -\frac{1}{2x^2} \\ S_x &= \frac{y}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{y \cos(y)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for y, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R \cos(R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(R)}{2} - \frac{R\sin(R)}{2} + c_1 \tag{4}$$

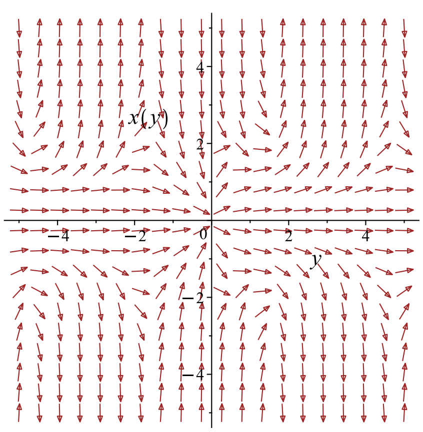
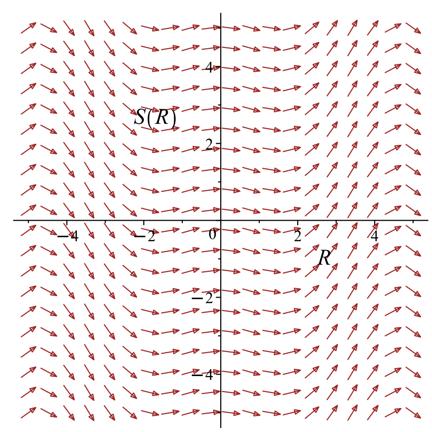
To complete the solution, we just need to transform (4) back to y, x coordinates. This results in

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y\sin(y)}{2} + c_1$$

Which simplifies to

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y\sin(y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in y, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dy} = -\frac{x(x^2\cos(y)y-1)}{2y}$ 	$R = y$ $S = -\frac{y}{2x^2}$	$\frac{dS}{dR} = -\frac{R\cos(R)}{2}$ 

Summary

The solution(s) found are the following

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y\sin(y)}{2} + c_1 \tag{1}$$

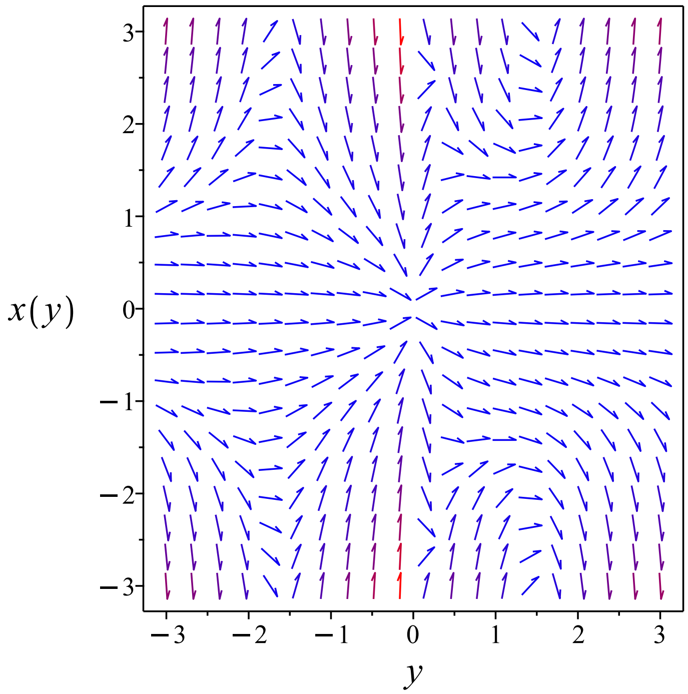


Figure 180: Slope field plot

Verification of solutions

$$-\frac{y}{2x^2} = -\frac{\cos(y)}{2} - \frac{y\sin(y)}{2} + c_1$$

Verified OK.

4.13.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} x' &= F(y,x) \\ &= -\frac{x(x^2\cos(y)\,y-1)}{2y} \end{aligned}$$

This is a Bernoulli ODE.

$$x' = \frac{1}{2y}x - \frac{\cos(y)}{2}x^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$x' = f_0(y)x + f_1(y)x^n \tag{2}$$

The first step is to divide the above equation by x^n which gives

$$\frac{x'}{x^n} = f_0(y)x^{1-n} + f_1(y) \tag{3}$$

The next step is use the substitution $w = x^{1-n}$ in equation (3) which generates a new ODE in $w(y)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $x(y)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(y) &= \frac{1}{2y} \\
f_1(y) &= -\frac{\cos(y)}{2} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $x^n = x^3$ gives

$$
x'\frac{1}{x^3} = \frac{1}{2y\,x^2} - \frac{\cos(y)}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= x^{1-n} \\
&= \frac{1}{x^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t y gives

$$
w' = -\frac{2}{x^3}x' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(y)}{2} &= \frac{w(y)}{2y} - \frac{\cos(y)}{2} \\
w' &= -\frac{w}{y} + \cos(y)
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(y)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(y) + p(y)w(y) = q(y)
$$

Where here

$$
\begin{aligned}
p(y) &= \frac{1}{y} \\
q(y) &= \cos(y)
\end{aligned}
$$

Hence the ode is

$$w'(y) + \frac{w(y)}{y} = \cos(y)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{y}dy} \\ &= y\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dy}(\mu w) &= (\mu)(\cos(y)) \\ \frac{d}{dy}(yw) &= (y)(\cos(y)) \\ d(yw) &= (y\cos(y))\,dy\end{aligned}$$

Integrating gives

$$\begin{aligned}yw &= \int y\cos(y)\,dy \\ yw &= y\sin(y) + \cos(y) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = y$ results in

$$w(y) = \frac{y\sin(y) + \cos(y)}{y} + \frac{c_1}{y}$$

which simplifies to

$$w(y) = \frac{y\sin(y) + \cos(y) + c_1}{y}$$

Replacing w in the above by $\frac{1}{x^2}$ using equation (5) gives the final solution.

$$\frac{1}{x^2} = \frac{y\sin(y) + \cos(y) + c_1}{y}$$

Solving for x gives

$$\begin{aligned}x(y) &= \frac{\sqrt{(y\sin(y) + \cos(y) + c_1)\,y}}{y\sin(y) + \cos(y) + c_1} \\ x(y) &= -\frac{\sqrt{(y\sin(y) + \cos(y) + c_1)\,y}}{y\sin(y) + \cos(y) + c_1}\end{aligned}$$

Summary

The solution(s) found are the following

$$x = \frac{\sqrt{\left(y \sin\left(y\right)+\cos\left(y\right)+c_1\right) y}}{y \sin\left(y\right)+\cos\left(y\right)+c_1} \tag{1}$$

$$x = -\frac{\sqrt{\left(y \sin\left(y\right)+\cos\left(y\right)+c_1\right) y}}{y \sin\left(y\right)+\cos\left(y\right)+c_1} \tag{2}$$

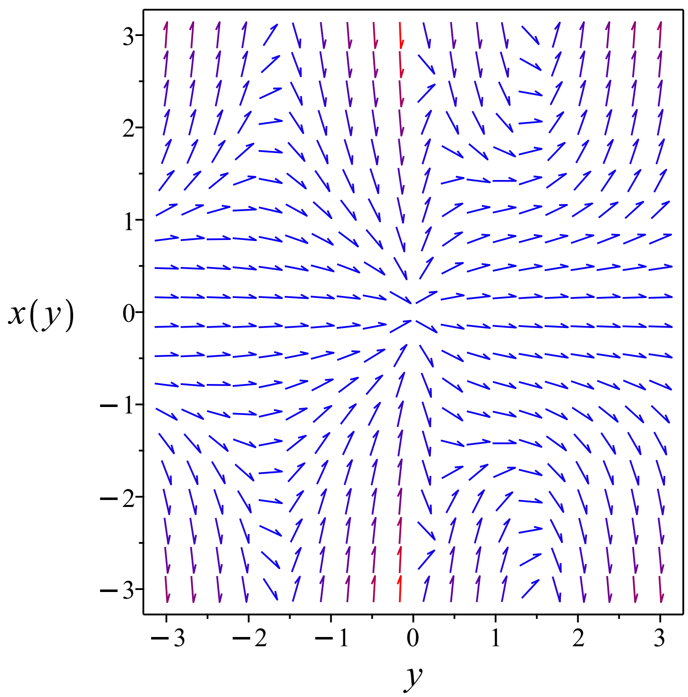


Figure 181: Slope field plot

Verification of solutions

$$x = \frac{\sqrt{\left(y \sin\left(y\right)+\cos\left(y\right)+c_1\right) y}}{y \sin\left(y\right)+\cos\left(y\right)+c_1}$$

Verified OK.

$$x = -\frac{\sqrt{\left(y \sin\left(y\right)+\cos\left(y\right)+c_1\right) y}}{y \sin\left(y\right)+\cos\left(y\right)+c_1}$$

Verified OK.

4.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(y,x)\,\mathrm{d}y + N(y,x)\,\mathrm{d}x = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(2y)\,\mathrm{d}x &= \left(-x\left(x^2\cos(y)\,y - 1\right)\right)\mathrm{d}y \\
\left(x\left(x^2\cos(y)\,y - 1\right)\right)\mathrm{d}y + (2y)\,\mathrm{d}x &= 0
\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(y,x) &= x\left(x^2\cos(y)\,y - 1\right) \\
N(y,x) &= 2y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^2 \cos(y)\, y - 1\right)\right)\\ &= 3x^2 \cos(y)\, y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial y} &= \frac{\partial}{\partial y}(2y)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y}\right)\\ &= \frac{1}{2y}\left(\left(3x^2 \cos(y)\, y - 1\right) - (2)\right)\\ &= \frac{3x^2 \cos(y)\, y - 3}{2y}\end{aligned}$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right)\\ &= \frac{1}{x\left(x^2 \cos(y)\, y - 1\right)}\left((2) - \left(3x^2 \cos(y)\, y - 1\right)\right)\\ &= -\frac{3}{x}\end{aligned}$$

Since B does not depend on y, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}x}\\ &= e^{\int -\frac{3}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(x\left(x^2\cos(y)\,y - 1\right)\right) \\
&= \frac{x^2\cos(y)\,y - 1}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(2y) \\
&= \frac{2y}{x^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}x}{\mathrm{d}y} &= 0 \\
\left(\frac{x^2\cos(y)\,y - 1}{x^2}\right) + \left(\frac{2y}{x^3}\right)\frac{\mathrm{d}x}{\mathrm{d}y} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(y, x)$

$$
\frac{\partial\phi}{\partial y} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial x} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. y gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \overline{M}\,\mathrm{d}y \\
\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int \frac{x^2\cos(y)\,y - 1}{x^2}\,\mathrm{d}y
\end{aligned}
$$

$$
\phi = \frac{x^2\sin(y)\,y + \cos(y)\,x^2 - y}{x^2} + f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both y and x. Taking derivative of equation (3) w.r.t x gives

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= -\frac{2(x^2\sin(y)\,y+\cos(y)\,x^2-y)}{x^3}+\frac{2\sin(y)\,xy+2\cos(y)\,x}{x^2}+f'(x) \\ &= \frac{2y}{x^3}+f'(x)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial x}=\frac{2y}{x^3}$. Therefore equation (4) becomes

$$\frac{2y}{x^3}=\frac{2y}{x^3}+f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x)=0$$

Therefore

$$f(x)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi=\frac{x^2\sin(y)\,y+\cos(y)\,x^2-y}{x^2}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=\frac{x^2\sin(y)\,y+\cos(y)\,x^2-y}{x^2}$$

Summary

The solution(s) found are the following

$$\frac{x^2y\sin(y)+\cos(y)\,x^2-y}{x^2}=c_1 \tag{1}$$

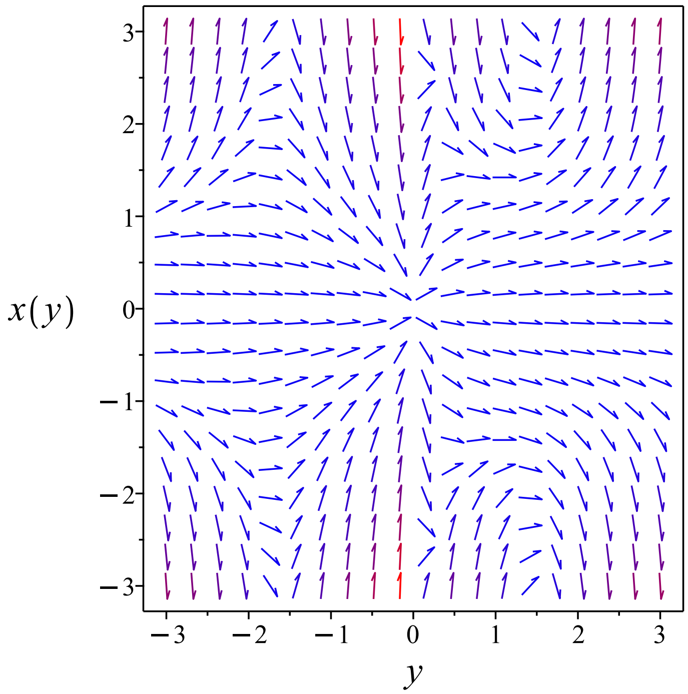


Figure 182: Slope field plot

Verification of solutions

$$
\frac{x^2 y \sin(y) + \cos(y) x^2 - y}{x^2} = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 54

```
dsolve(2*diff(x(y),y)-x(y)/y+x(y)^3*cos(y)=0,x(y), singsol=all)
```

$$x(y) = \frac{\sqrt{(\cos(y) + y\sin(y) + c_1)\, y}}{\cos(y) + y\sin(y) + c_1}$$
$$x(y) = -\frac{\sqrt{(\cos(y) + y\sin(y) + c_1)\, y}}{\cos(y) + y\sin(y) + c_1}$$

✓ Solution by Mathematica

Time used: 0.261 (sec). Leaf size: 53

```
DSolve[2*x'[y]-x[y]/y+x[y]^3*Cos[y]==0,x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to -\frac{\sqrt{y}}{\sqrt{y\sin(y) + \cos(y) + c_1}}$$
$$x(y) \to \frac{\sqrt{y}}{\sqrt{y\sin(y) + \cos(y) + c_1}}$$
$$x(y) \to 0$$

4.14 problem 19 (p)

4.14.1 Solving as linear ode 857
4.14.2 Solving as first order ode lie symmetry lookup ode 859
4.14.3 Solving as exact ode 863
4.14.4 Maple step by step solution 868

Internal problem ID [5311]
Internal file name [`OUTPUT/4802_Friday_February_02_2024_05_12_14_AM_12081669/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (p).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - y(1 - x\tan(x)) = \cos(x)\, x^2}$$

4.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1 - x\tan(x)}{x} \\ q(x) &= \cos(x)\, x \end{aligned}$$

Hence the ode is

$$y' - \frac{(1 - x\tan(x))\, y}{x} = \cos(x)\, x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1-x\tan(x)}{x}dx} \\
&= \mathrm{e}^{-\ln(\cos(x))-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{1}{\cos(x)\, x}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(\cos(x)\, x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\cos(x)\, x}\right) &= \left(\frac{1}{\cos(x)\, x}\right)(\cos(x)\, x) \\
\mathrm{d}\left(\frac{y}{\cos(x)\, x}\right) &= \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\cos(x)\, x} &= \int \mathrm{d}x \\
\frac{y}{\cos(x)\, x} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\cos(x)x}$ results in

$$
y = \cos(x)\, x^2 + \cos(x)\, c_1 x
$$

which simplifies to

$$
y = \cos(x)\, x(x + c_1)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \cos(x)\, x(x + c_1) \tag{1}
$$

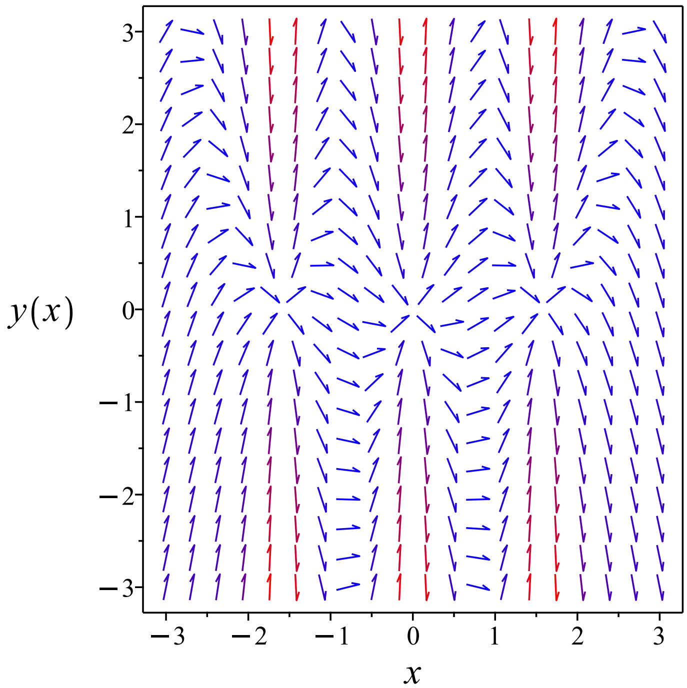


Figure 183: Slope field plot

Verification of solutions

$$y = \cos(x)\, x(x + c_1)$$

Verified OK.

4.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-xy\tan(x) + \cos(x)\, x^2 + y}{x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 109: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{\ln(\cos(x))+\ln(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{\ln(\cos(x))+\ln(x)}} dy
\end{aligned}$$

Which results in

$$S = \frac{y}{\cos(x)\, x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-xy\tan(x) + \cos(x)\, x^2 + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y\sec(x)\,(x\tan(x) - 1)}{x^2} \\
S_y &= \frac{\sec(x)}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\cos(x)\,x} = x + c_1$$

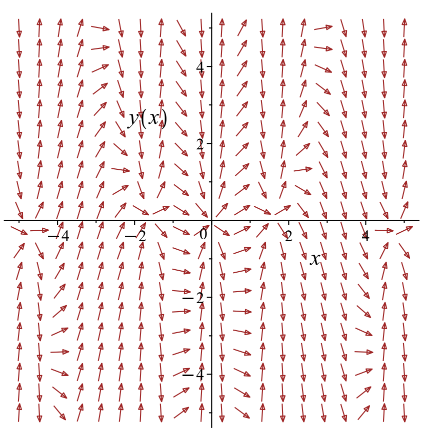
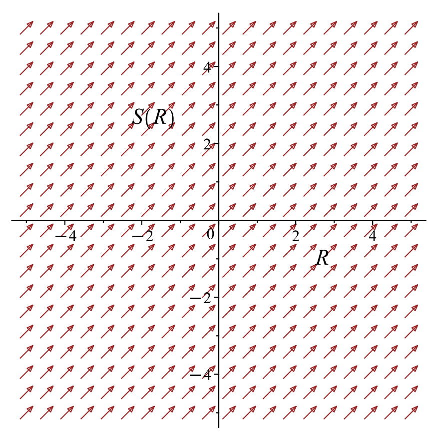
Which simplifies to

$$\frac{y}{\cos(x)\,x} = x + c_1$$

Which gives

$$y = \cos(x)\,x^2 + \cos(x)\,c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-xy\tan(x)+\cos(x)x^2+y}{x}$ 	$R = x$ $S = \frac{y}{\cos(x)\,x}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = \cos(x)\,x^2 + \cos(x)\,c_1 x \tag{1}$$

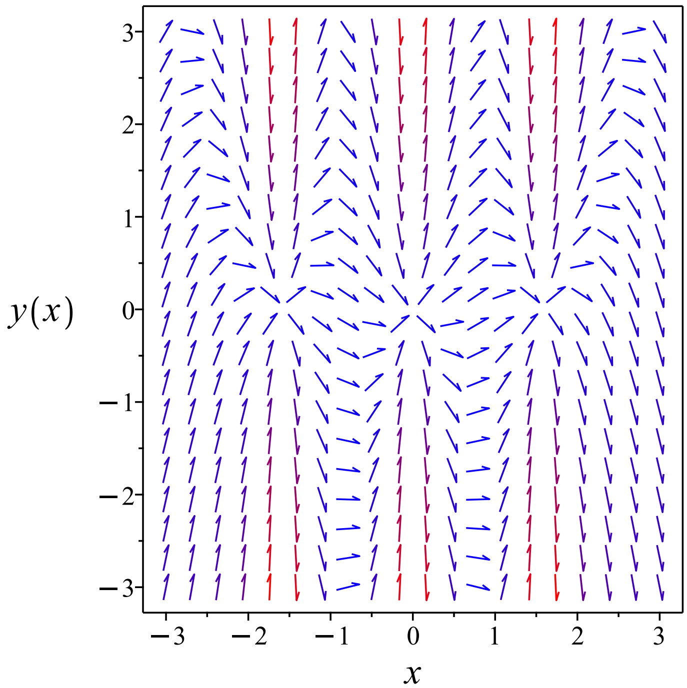


Figure 184: Slope field plot

Verification of solutions

$$y = \cos(x)\, x^2 + \cos(x)\, c_1 x$$

Verified OK.

4.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(y(1 - x\tan(x)) + \cos(x)\,x^2\right)\mathrm{d}x\\ \left(-y(1 - x\tan(x)) - \cos(x)\,x^2\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y(1 - x\tan(x)) - \cos(x)\,x^2\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y(1 - x\tan(x)) - \cos(x)\,x^2\right)\\ &= x\tan(x) - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((x\tan(x) - 1) - (1)) \\
&= \frac{x\tan(x) - 2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{x\tan(x)-2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(x)) - 2\ln(x)} \\
&= \frac{1}{\cos(x)\, x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\cos(x)\, x^2}\left(-y(1 - x\tan(x)) - \cos(x)\, x^2\right) \\
&= \frac{y\sec(x)\,(x\tan(x) - 1) - x^2}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\cos(x)\, x^2}(x) \\
&= \frac{\sec(x)}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y\sec(x)\,(x\tan(x) - 1) - x^2}{x^2}\right) + \left(\frac{\sec(x)}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y \sec(x) (x \tan(x) - 1) - x^2}{x^2} \mathrm{d}x$$
$$\phi = \int^x \frac{y \sec(\_a) (\_a \tan(\_a) - 1) - \_a^2}{\_a^2} d\_a + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2\,\mathrm{e}^{ix}}{x\,(\mathrm{e}^{2ix} + 1)} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sec(x)}{x}$. Therefore equation (4) becomes

$$\frac{\sec(x)}{x} = \frac{2\,\mathrm{e}^{ix}}{x\,(\mathrm{e}^{2ix} + 1)} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \int^x \frac{y \sec(\_a) (\_a \tan(\_a) - 1) - \_a^2}{\_a^2} d\_a + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int^x \frac{y \sec(\_a)(\_a \tan(\_a) - 1) - \_a^2}{\_a^2} d\_a$$

Summary

The solution(s) found are the following

$$\int^x \frac{y \sec(\_a)(\_a \tan(\_a) - 1) - \_a^2}{\_a^2} d\_a = c_1 \tag{1}$$

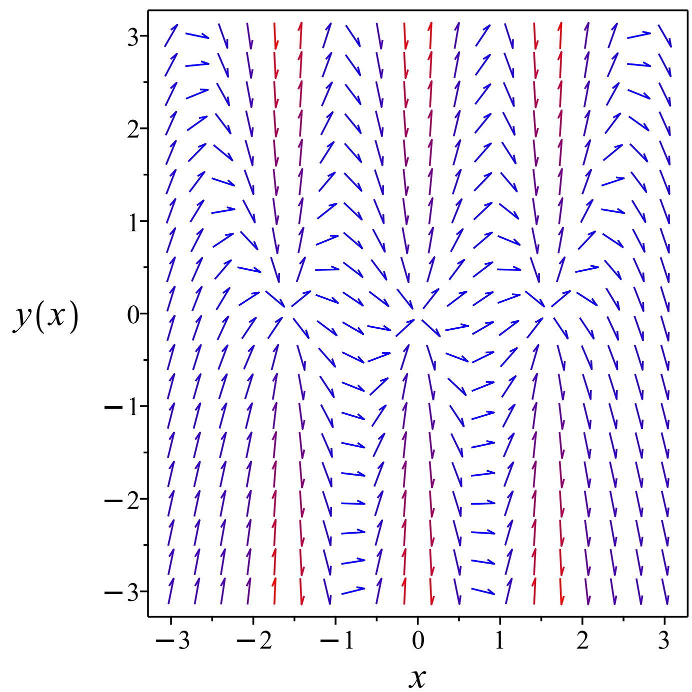


Figure 185: Slope field plot

Verification of solutions

$$\int^x \frac{y \sec(\_a)(\_a \tan(\_a) - 1) - \_a^2}{\_a^2} d\_a = c_1$$

Verified OK.

4.14.4 Maple step by step solution

Let's solve

$xy' - y(1 - x\tan(x)) = \cos(x)\, x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(x\tan(x)-1)y}{x} + \cos(x)\, x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(x\tan(x)-1)y}{x} = \cos(x)\, x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(x\tan(x)-1)y}{x}\right) = \mu(x)\cos(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{(x\tan(x)-1)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(x\tan(x)-1)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{\cos(x)x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\cos(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\cos(x)x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{\cos(x)x}$

  $y = \cos(x)\, x\left(\int 1 dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = \cos(x)\, x(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(x*diff(y(x),x)=y(x)*(1-x*tan(x))+x^2*cos(x),y(x), singsol=all)
```

$$y(x) = (x + c_1) \cos(x) x$$

✓ Solution by Mathematica

Time used: 0.081 (sec). Leaf size: 13

```
DSolve[x*y'[x]==y[x]*(1-x*Tan[x])+x^2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x + c_1) \cos(x)$$

4.15 problem 19 (q)

4.15.1 Solving as first order ode lie symmetry calculated ode 870
4.15.2 Solving as exact ode 877

Internal problem ID [5312]
Internal file name [`OUTPUT/4803_Friday_February_02_2024_05_12_22_AM_50983171/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (q).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first__order__ode__lie__symmetry__calculated"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{y^2 - \left(yx + 2y + y^3\right) y' = -2}$$

4.15.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2 + 2}{y\left(y^2 + x + 2\right)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + xya_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = x^2 b_4 + xyb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 + \frac{(y^2+2)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{y(y^2+x+2)} \\
&- \frac{(y^2+2)^2 (xa_5 + 2ya_6 + a_3)}{y^2 (y^2+x+2)^2} \\
&+ \frac{(y^2+2)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{y(y^2+x+2)^2} - \left(\frac{2}{y^2+x+2}\right. \\
&\left. - \frac{y^2+2}{y^2(y^2+x+2)} - \frac{2(y^2+2)}{(y^2+x+2)^2}\right)(x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x\,y^6b_4 + y^7b_5 + 5x^2y^4b_4 - 2x\,y^5a_4 + 4x\,y^5b_5 - y^6a_5 + y^6b_2 + 3y^6b_6 + x^3y^2b_4 - x^2y^3a_4 + x^2y^3b_5 - x\,y^4a_5 + \cdots}{\cdots}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x\,y^6b_4 + y^7b_5 + 5x^2y^4b_4 - 2x\,y^5a_4 + 4x\,y^5b_5 - y^6a_5 + y^6b_2 + 3y^6b_6 \\
&+ x^3y^2b_4 - x^2y^3a_4 + x^2y^3b_5 - x\,y^4a_5 + 3x\,y^4b_2 + 8x\,y^4b_4 + x\,y^4b_6 - y^5a_2 \\
&- y^5a_6 + 2y^5b_3 + 4y^5b_5 + 12x^2y^2b_4 - 8x\,y^3a_4 + 12x\,y^3b_5 - 4y^4a_5 + y^4b_1 \\
&+ 4y^4b_2 + 12y^4b_6 + 2x^3b_4 - 2x^2ya_4 + 4x^2yb_5 - 4x\,y^2a_5 - x\,y^2b_1 + 8x\,y^2b_2 \\
&+ 8xb_4y^2 + 6x\,y^2b_6 + y^3a_1 - 4y^3a_2 - 6y^3a_6 + 8y^3b_3 + 4y^3b_5 + 2x^2b_2 + 4x^2b_4 \\
&- 8xya_4 + 4xyb_3 + 8xyb_5 - 2y^2a_3 - 4y^2a_5 + 4y^2b_1 + 4b_2y^2 + 12y^2b_6 \\
&- 4xa_5 + 2xb_1 + 4xb_2 + 2ya_1 - 4ya_2 - 8ya_6 + 8yb_3 - 4a_3 + 4b_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2b_4v_1v_2^6 + b_5v_2^7 - 2a_4v_1v_2^5 - a_5v_2^6 + b_2v_2^6 + 5b_4v_1^2v_2^4 + 4b_5v_1v_2^5 + 3b_6v_2^6 \\
&- a_2v_2^5 - a_4v_1^2v_2^3 - a_5v_1v_2^4 - a_6v_2^5 + 3b_2v_1v_2^4 + 2b_3v_2^5 + b_4v_1^3v_2^2 \\
&+ 8b_4v_1v_2^4 + b_5v_1^2v_2^3 + 4b_5v_2^5 + b_6v_1v_2^4 - 8a_4v_1v_2^3 - 4a_5v_2^4 + b_1v_2^4 \\
&+ 4b_2v_2^4 + 12b_4v_1^2v_2^2 + 12b_5v_1v_2^3 + 12b_6v_2^4 + a_1v_2^3 - 4a_2v_2^3 - 2a_4v_1^2v_2 \\
&- 4a_5v_1v_2^2 - 6a_6v_2^3 - b_1v_1v_2^2 + 8b_2v_1v_2^2 + 8b_3v_2^3 + 2b_4v_1^3 + 8b_4v_1v_2^2 \\
&+ 4b_5v_1^2v_2 + 4b_5v_2^3 + 6b_6v_1v_2^2 - 2a_3v_2^2 - 8a_4v_1v_2 - 4a_5v_2^2 + 4b_1v_2^2 \\
&+ 2b_2v_1^2 + 4b_2v_2^2 + 4b_3v_1v_2 + 4b_4v_1^2 + 8b_5v_1v_2 + 12b_6v_2^2 + 2a_1v_2 \\
&- 4a_2v_2 - 4a_5v_1 - 8a_6v_2 + 2b_1v_1 + 4b_2v_1 + 8b_3v_2 - 4a_3 + 4b_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&b_4v_1^3v_2^2 + 2b_4v_1^3 + 5b_4v_1^2v_2^4 + (-a_4 + b_5)\, v_1^2v_2^3 + 12b_4v_1^2v_2^2 \\
&+ (-2a_4 + 4b_5)\, v_1^2v_2 + (2b_2 + 4b_4)\, v_1^2 + 2b_4v_1v_2^6 + (-2a_4 + 4b_5)\, v_1v_2^5 \\
&+ (-a_5 + 3b_2 + 8b_4 + b_6)\, v_1v_2^4 + (-8a_4 + 12b_5)\, v_1v_2^3 \\
&+ (-4a_5 - b_1 + 8b_2 + 8b_4 + 6b_6)\, v_1v_2^2 + (-8a_4 + 4b_3 + 8b_5)\, v_1v_2 \\
&+ (-4a_5 + 2b_1 + 4b_2)\, v_1 + b_5v_2^7 + (-a_5 + b_2 + 3b_6)\, v_2^6 \\
&+ (-a_2 - a_6 + 2b_3 + 4b_5)\, v_2^5 + (-4a_5 + b_1 + 4b_2 + 12b_6)\, v_2^4 \\
&+ (a_1 - 4a_2 - 6a_6 + 8b_3 + 4b_5)\, v_2^3 + (-2a_3 - 4a_5 + 4b_1 + 4b_2 + 12b_6)\, v_2^2 \\
&+ (2a_1 - 4a_2 - 8a_6 + 8b_3)\, v_2 - 4a_3 + 4b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_4 &= 0 \\
b_5 &= 0 \\
2b_4 &= 0 \\
5b_4 &= 0 \\
12b_4 &= 0 \\
-4a_3 + 4b_1 &= 0 \\
-8a_4 + 12b_5 &= 0 \\
-2a_4 + 4b_5 &= 0 \\
-a_4 + b_5 &= 0 \\
2b_2 + 4b_4 &= 0 \\
-8a_4 + 4b_3 + 8b_5 &= 0 \\
-4a_5 + 2b_1 + 4b_2 &= 0 \\
-a_5 + b_2 + 3b_6 &= 0 \\
2a_1 - 4a_2 - 8a_6 + 8b_3 &= 0 \\
-a_2 - a_6 + 2b_3 + 4b_5 &= 0 \\
-4a_5 + b_1 + 4b_2 + 12b_6 &= 0 \\
-a_5 + 3b_2 + 8b_4 + b_6 &= 0 \\
a_1 - 4a_2 - 6a_6 + 8b_3 + 4b_5 &= 0 \\
-2a_3 - 4a_5 + 4b_1 + 4b_2 + 12b_6 &= 0 \\
-4a_5 - b_1 + 8b_2 + 8b_4 + 6b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2a_6 \\
a_2 &= -a_6 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= a_6 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y^2 - x + 2 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - \left(\frac{y^2 + 2}{y\,(y^2 + x + 2)}\right)(y^2 - x + 2) \\
&= \frac{-y^4 + y^2 x - 4y^2 + 2x - 4}{y^3 + xy + 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int\frac{1}{\eta}dy\\&=\int\frac{1}{\frac{-y^4+y^2x-4y^2+2x-4}{y^3+xy+2y}}dy\end{aligned}$$

Which results in

$$S=-\ln\left(y^2-x+2\right)+\frac{\ln\left(y^2+2\right)}{2}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y}\tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y^2+2}{y\left(y^2+x+2\right)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=\frac{1}{y^2-x+2}\\S_y&=-\frac{2y}{y^2-x+2}+\frac{y}{y^2+2}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=0\tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

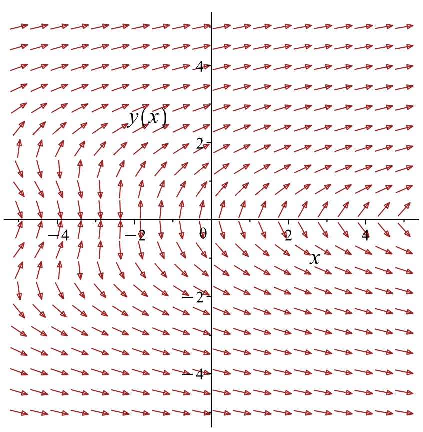
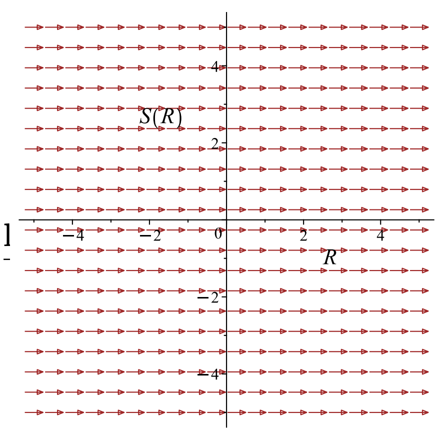
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y^2 - x + 2\right) + \frac{\ln\left(y^2+2\right)}{2} = c_1$$

Which simplifies to

$$-\ln\left(y^2 - x + 2\right) + \frac{\ln\left(y^2+2\right)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+2}{y(y^2+x+2)}$  	$R = x$ $S = -\ln\left(y^2 - x + 2\right) +$	$\frac{dS}{dR} = 0$ 

<u>Summary</u>
The solution(s) found are the following

$$-\ln\left(y^2 - x + 2\right) + \frac{\ln\left(y^2+2\right)}{2} = c_1 \tag{1}$$

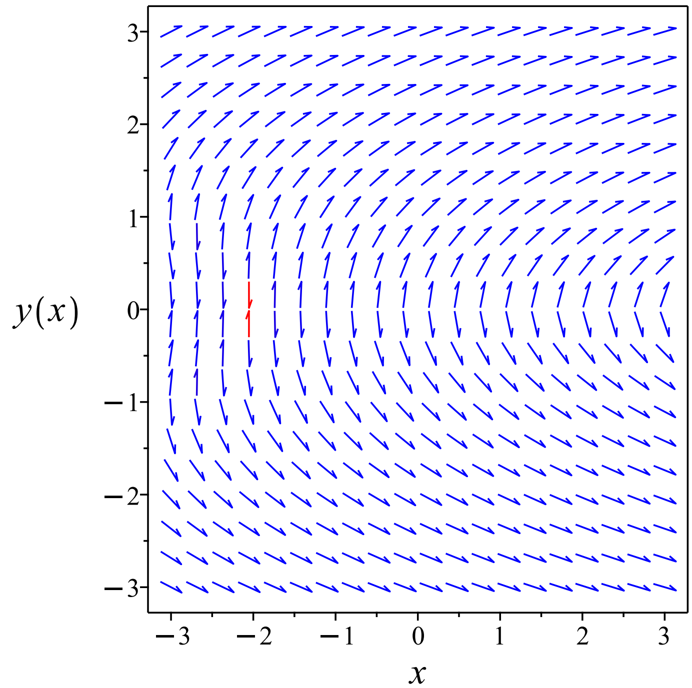


Figure 186: Slope field plot

Verification of solutions

$$-\ln\left(y^2 - x + 2\right) + \frac{\ln\left(y^2 + 2\right)}{2} = c_1$$

Verified OK.

4.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-y^3 - xy - 2y\right)\mathrm{d}y &= \left(-y^2 - 2\right)\mathrm{d}x\\ \left(y^2 + 2\right)\mathrm{d}x + \left(-y^3 - xy - 2y\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2 + 2\\ N(x,y) &= -y^3 - xy - 2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2 + 2\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-y^3 - xy - 2y\right)\\ &= -y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{y\left(y^2 + x + 2\right)}((2y) - (-y)) \\
&= -\frac{3}{y^2 + x + 2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2 + 2}((-y) - (2y)) \\
&= -\frac{3y}{y^2 + 2}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{3y}{y^2+2} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{3 \ln\left(y^2+2\right)}{2}} \\
&= \frac{1}{\left(y^2 + 2\right)^{\frac{3}{2}}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\left(y^2 + 2\right)^{\frac{3}{2}}}\left(y^2 + 2\right) \\
&= \frac{1}{\sqrt{y^2 + 2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(y^{2}+2\right)^{\frac{3}{2}}}\left(-y^{3}-xy-2y\right) \\
&= \frac{\left(-y^{2}-x-2\right) y}{\left(y^{2}+2\right)^{\frac{3}{2}}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{1}{\sqrt{y^{2}+2}}\right) + \left(\frac{\left(-y^{2}-x-2\right) y}{\left(y^{2}+2\right)^{\frac{3}{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\mathrm{d}x &= \int \frac{1}{\sqrt{y^{2}+2}}\mathrm{d}x \\
\phi &= \frac{x}{\sqrt{y^{2}+2}} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\frac{xy}{\left(y^{2}+2\right)^{\frac{3}{2}}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\left(-y^{2}-x-2\right) y}{\left(y^{2}+2\right)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$
\frac{\left(-y^{2}-x-2\right) y}{\left(y^{2}+2\right)^{\frac{3}{2}}} = -\frac{xy}{\left(y^{2}+2\right)^{\frac{3}{2}}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{y}{\sqrt{y^2+2}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{y}{\sqrt{y^2+2}}\right)\mathrm{d}y$$
$$f(y) = -\sqrt{y^2+2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x}{\sqrt{y^2+2}} - \sqrt{y^2+2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x}{\sqrt{y^2+2}} - \sqrt{y^2+2}$$

Summary

The solution(s) found are the following

$$\frac{x}{\sqrt{y^2+2}} - \sqrt{y^2+2} = c_1 \tag{1}$$

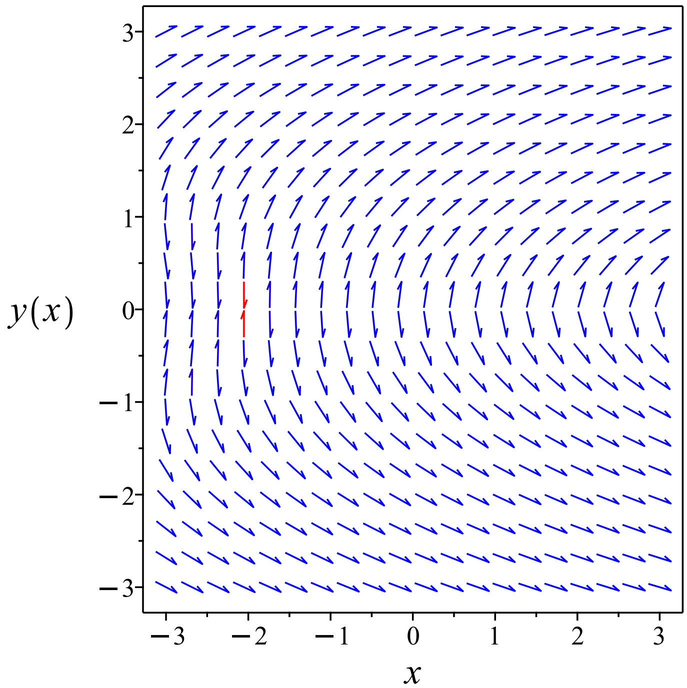


Figure 187: Slope field plot

Verification of solutions

$$\frac{x}{\sqrt{y^2+2}} - \sqrt{y^2+2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve((2+y(x)^2)-(x*y(x)+2*y(x)+y(x)^3)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x - y(x)^2 - 2 - \sqrt{y(x)^2 + 2}\, c_1 = 0$$

✓ Solution by Mathematica

Time used: 5.808 (sec). Leaf size: 189

```
DSolve[(2+y[x]^2)-(x*y[x]+2*y[x]+y[x]^3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{2x - \sqrt{4c_1{}^2x + c_1{}^4} - 4 + c_1{}^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{2x - \sqrt{4c_1{}^2x + c_1{}^4} - 4 + c_1{}^2}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{2x + \sqrt{4c_1{}^2x + c_1{}^4} - 4 + c_1{}^2}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{2x + \sqrt{4c_1{}^2x + c_1{}^4} - 4 + c_1{}^2}}{\sqrt{2}}$$
$$y(x) \to -i\sqrt{2}$$
$$y(x) \to i\sqrt{2}$$

4.16 problem 19 (r)

4.16.1 Solving as first order ode lie symmetry calculated ode 884
4.16.2 Solving as exact ode 891

Internal problem ID [5313]
Internal file name [`OUTPUT/4804_Friday_February_02_2024_05_12_26_AM_45050703/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (r).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(y)]`]]
```

$$\boxed{y^2 - (\arctan(y) - x)\, y' = -1}$$

4.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2+1}{\arctan(y) - x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + xy a_5 + y^2 a_6 + x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x^2 b_4 + xy b_5 + y^2 b_6 + x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 + \frac{(y^2+1)(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{\arctan(y) - x} \\
&- \frac{(y^2+1)^2 (xa_5 + 2ya_6 + a_3)}{(\arctan(y) - x)^2} \\
&- \frac{(y^2+1)(x^2a_4 + xya_5 + y^2a_6 + xa_2 + ya_3 + a_1)}{(\arctan(y) - x)^2} - \left(\frac{2y}{\arctan(y) - x}\right. \\
&\left. - \frac{1}{(\arctan(y) - x)^2}\right)(x^2b_4 + xyb_5 + y^2b_6 + xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{-2\arctan(y)\,x^2yb_4 - 2\arctan(y)\,x\,y^2a_4 - \arctan(y)\,x\,y^2b_5 - 2\arctan(y)\,xyb_5 - 2\arctan(y)\,xyb_2 - ya_3}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-2\arctan(y)\,x^2yb_4 - 2\arctan(y)\,x\,y^2a_4 - \arctan(y)\,x\,y^2b_5 \\
&- 2\arctan(y)\,xyb_5 - 2\arctan(y)\,xyb_2 - ya_3 + xb_2 + yb_3 - a_1 - a_3 \\
&+ b_1 + xyb_5 - 2y^5a_6 - y^4a_6 + 2x^3b_4 - 4y^3a_6 - x^2b_5 - xa_5 - 2ya_6 - y^4a_3 \\
&- y^3a_3 + x^2b_2 - y^2a_1 - 2y^2a_3 - xb_3 + \arctan(y)^2\,b_2 - \arctan(y)\,a_2 \\
&+ \arctan(y)\,b_3 + 2x^2yb_2 + x\,y^2b_3 + 2xyb_1 - \arctan(y)\,y^2a_2 \\
&- \arctan(y)\,y^2b_3 - 2\arctan(y)\,xb_2 - 2\arctan(y)\,yb_1 + x^2a_4 \\
&- y^2a_6 + x^2b_4 + y^2b_6 - x\,y^4a_5 + 2x^3yb_4 + x^2y^2a_4 + x^2y^2b_5 \\
&+ x^2yb_5 - 2x\,y^2a_5 - 2xyb_6 - \arctan(y)\,y^3a_5 + 2\arctan(y)^2\,xb_4 \\
&+ \arctan(y)^2\,yb_5 - 4\arctan(y)\,x^2b_4 - 2\arctan(y)\,xa_4 \\
&+ \arctan(y)\,xb_5 - \arctan(y)\,ya_5 + 2\arctan(y)\,yb_6 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \arctan(y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \arctan(y) = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_1 v_2^4 a_5 - 2v_2^5 a_6 - v_2^4 a_3 + v_1^2 v_2^2 a_4 - 2v_3 v_1 v_2^2 a_4 - v_3 v_2^3 a_5 - v_2^4 a_6 \\
&+ 2v_1^3 v_2 b_4 - 2v_3 v_1^2 v_2 b_4 + v_1^2 v_2^2 b_5 - v_3 v_1 v_2^2 b_5 - v_3 v_2^2 a_2 - v_2^3 a_3 - 2v_1 v_2^2 a_5 \\
&- 4v_2^3 a_6 + 2v_1^2 v_2 b_2 - 2v_3 v_1 v_2 b_2 + v_1 v_2^2 b_3 - v_3 v_2^2 b_3 + 2v_1^3 b_4 - 4v_3 v_1^2 b_4 \\
&+ 2v_3^2 v_1 b_4 + v_1^2 v_2 b_5 - 2v_3 v_1 v_2 b_5 + v_3^2 v_2 b_5 - v_2^2 a_1 - 2v_2^2 a_3 + v_1^2 a_4 \\
&- 2v_3 v_1 a_4 - v_3 v_2 a_5 - v_2^2 a_6 + 2v_1 v_2 b_1 - 2v_3 v_2 b_1 + v_1^2 b_2 - 2v_3 v_1 b_2 + v_3^2 b_2 \\
&+ v_1^2 b_4 - v_1^2 b_5 + v_1 v_2 b_5 + v_3 v_1 b_5 - 2v_1 v_2 b_6 + v_2^2 b_6 + 2v_3 v_2 b_6 - v_3 a_2 \\
&- v_2 a_3 - v_1 a_5 - 2v_2 a_6 + v_1 b_2 - v_1 b_3 + v_2 b_3 + v_3 b_3 - a_1 - a_3 + b_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(a_4 + b_5) v_1^2 v_2^2 + (2b_2 + b_5) v_1^2 v_2 + (-2a_5 + b_3) v_1 v_2^2 \\
&+ (2b_1 + b_5 - 2b_6) v_1 v_2 + (-2a_4 - 2b_2 + b_5) v_1 v_3 + (-a_2 - b_3) v_2^2 v_3 \\
&+ (-a_5 - 2b_1 + 2b_6) v_2 v_3 - 2v_3 v_1^2 v_2 b_4 - a_1 - a_3 + b_1 - 2v_2^5 a_6 + 2v_1^3 b_4 \\
&+ v_3^2 b_2 + (-2a_4 - b_5) v_1 v_2^2 v_3 + (-2b_2 - 2b_5) v_1 v_2 v_3 - v_1 v_2^4 a_5 + 2v_1^3 v_2 b_4 \\
&- v_3 v_2^3 a_5 + 2v_3^2 v_1 b_4 + v_3^2 v_2 b_5 - 4v_3 v_1^2 b_4 + (a_4 + b_2 + b_4 - b_5) v_1^2 \\
&+ (-a_5 + b_2 - b_3) v_1 + (-a_3 - a_6) v_2^4 + (-a_3 - 4a_6) v_2^3 \\
&+ (-a_1 - 2a_3 - a_6 + b_6) v_2^2 + (-a_3 - 2a_6 + b_3) v_2 + (b_3 - a_2) v_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
b_5 &= 0 \\
-a_5 &= 0 \\
-2a_6 &= 0 \\
-4b_4 &= 0 \\
-2b_4 &= 0 \\
2b_4 &= 0 \\
-a_2 - b_3 &= 0 \\
-a_3 - 4a_6 &= 0 \\
-a_3 - a_6 &= 0 \\
-2a_4 - b_5 &= 0 \\
a_4 + b_5 &= 0 \\
-2a_5 + b_3 &= 0 \\
-2b_2 - 2b_5 &= 0 \\
2b_2 + b_5 &= 0 \\
b_3 - a_2 &= 0 \\
-a_1 - a_3 + b_1 &= 0 \\
-a_3 - 2a_6 + b_3 &= 0 \\
-2a_4 - 2b_2 + b_5 &= 0 \\
-a_5 - 2b_1 + 2b_6 &= 0 \\
-a_5 + b_2 - b_3 &= 0 \\
2b_1 + b_5 - 2b_6 &= 0 \\
-a_1 - 2a_3 - a_6 + b_6 &= 0 \\
a_4 + b_2 + b_4 - b_5 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_6 \\
a_2 &= 0 \\
a_3 &= 0 \\
a_4 &= 0 \\
a_5 &= 0 \\
a_6 &= 0 \\
b_1 &= b_6 \\
b_2 &= 0 \\
b_3 &= 0 \\
b_4 &= 0 \\
b_5 &= 0 \\
b_6 &= b_6
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= y^2 + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y^2 + 1 - \left(\frac{y^2 + 1}{\arctan(y) - x}\right)(1) \\
&= \frac{y^2 \arctan(y) - y^2 x - y^2 + \arctan(y) - x - 1}{\arctan(y) - x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2\arctan(y)-y^2x-y^2+\arctan(y)-x-1}{\arctan(y)-x}} dy \end{aligned}$$

Which results in

$$S = \arctan(y) + \ln(-1+\arctan(y)-x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2+1}{\arctan(y)-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{1-\arctan(y)+x} \\ S_y &= \frac{1+\frac{1}{-1+\arctan(y)-x}}{y^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\arctan(y) + \ln(-1 + \arctan(y) - x) = c_1$$

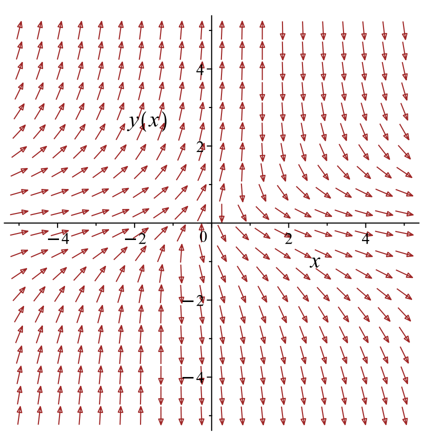
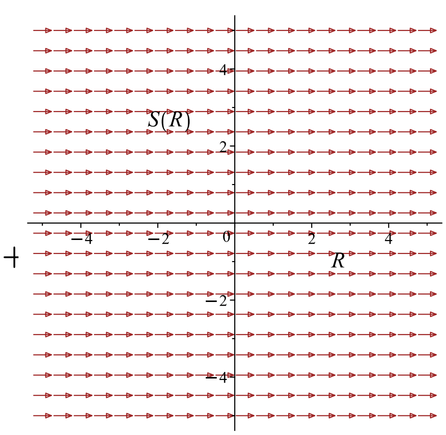
Which simplifies to

$$\arctan(y) + \ln(-1 + \arctan(y) - x) = c_1$$

Which gives

$$y = \tan\left(\mathrm{LambertW}\left(\mathrm{e}^{-1+c_1-x}\right) + 1 + x\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{\arctan(y)-x}$ 	$R = x$ $S = \arctan(y) + \ln(-1 +$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \tan\left(\mathrm{LambertW}\left(\mathrm{e}^{-1+c_1-x}\right)+1+x\right) \tag{1}$$

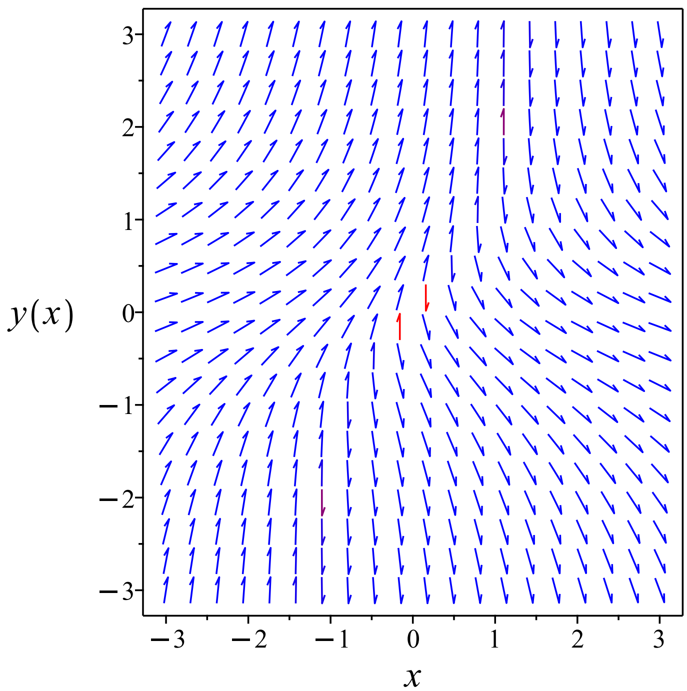


Figure 188: Slope field plot

Verification of solutions

$$y = \tan\left(\mathrm{LambertW}\left(\mathrm{e}^{-1+c_1-x}\right)+1+x\right)$$

Verified OK.

4.16.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-\arctan(y)+x)\,\mathrm{d}y&=\left(-y^2-1\right)\mathrm{d}x\\ \left(y^2+1\right)\mathrm{d}x+(-\arctan(y)+x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2+1\\ N(x,y)&=-\arctan(y)+x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y^2+1\right)\\ &=2y\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-\arctan(y) + x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{-\arctan(y) + x}((2y) - (1)) \\
&= \frac{-2y + 1}{\arctan(y) - x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2 + 1}((1) - (2y)) \\
&= \frac{-2y + 1}{y^2 + 1}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{-2y+1}{y^2+1} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(y^2+1) + \arctan(y)} \\
&= \frac{\mathrm{e}^{\arctan(y)}}{y^2 + 1}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\mathrm{e}^{\arctan(y)}}{y^2 + 1}(y^2 + 1) \\
&= \mathrm{e}^{\arctan(y)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\mathrm{e}^{\arctan(y)}}{y^2+1}(-\arctan(y)+x) \\
&= \frac{(-\arctan(y)+x)\,\mathrm{e}^{\arctan(y)}}{y^2+1}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{\arctan(y)}\right) + \left(\frac{(-\arctan(y)+x)\,\mathrm{e}^{\arctan(y)}}{y^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{\arctan(y)}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{\arctan(y)}x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{\arctan(y)}x}{y^2+1} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{(-\arctan(y)+x)\mathrm{e}^{\arctan(y)}}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{(-\arctan(y)+x)\,\mathrm{e}^{\arctan(y)}}{y^2+1} = \frac{\mathrm{e}^{\arctan(y)}x}{y^2+1} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{e^{\arctan(y)} \arctan(y)}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(-\frac{e^{\arctan(y)} \arctan(y)}{y^2 + 1}\right) dy$$
$$f(y) = -e^{\arctan(y)} \arctan(y) + e^{\arctan(y)} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{\arctan(y)} x - e^{\arctan(y)} \arctan(y) + e^{\arctan(y)} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{\arctan(y)} x - e^{\arctan(y)} \arctan(y) + e^{\arctan(y)}$$

The solution becomes

$$y = \tan\left(\text{LambertW}\left(-c_1 e^{-x-1}\right) + x + 1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\text{LambertW}\left(-c_1 e^{-x-1}\right) + x + 1\right) \tag{1}$$

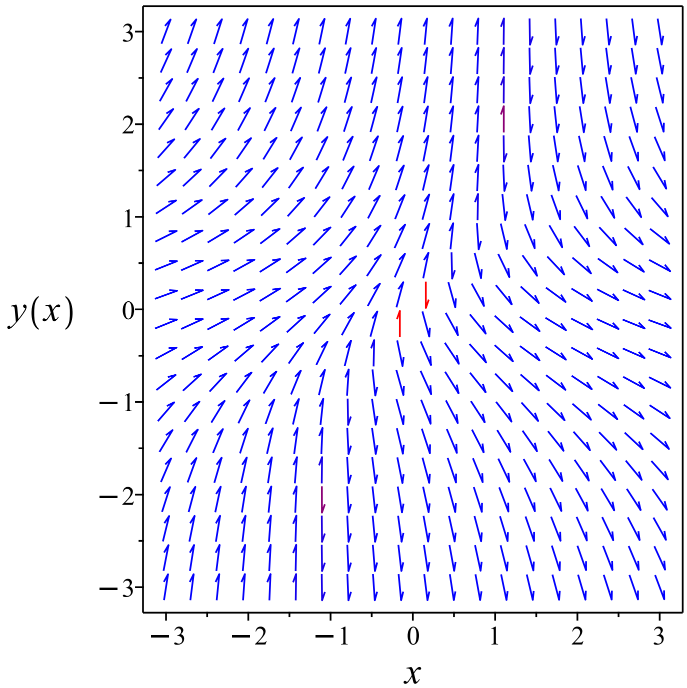


Figure 189: Slope field plot

Verification of solutions

$$y = \tan\left(\text{LambertW}\left(-c_1 e^{-x-1}\right) + x + 1\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve((1+y(x)^2)=(arctan(y(x))-x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{LambertW}\left(-c_1 \mathrm{e}^{-x-1}\right) + x + 1\right)$$

✓ Solution by Mathematica

Time used: 60.157 (sec). Leaf size: 21

```
DSolve[(1+y[x]^2)==(ArcTan[y[x]]-x)*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \tan\left(W\left(c_1\left(-e^{-x-1}\right)\right) + x + 1\right)$$

4.17 problem 19 (s)

4.17.1 Solving as first order ode lie symmetry lookup ode 898
4.17.2 Solving as bernoulli ode 902

Internal problem ID [5314]
Internal file name [`OUTPUT/4805_Friday_February_02_2024_05_12_29_AM_93928993/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (s).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2y^5x - y + 2xy' = 0}$$

4.17.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(2x\,y^4 - 1)}{2x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 112: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^5}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^5}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{4y^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(2x\,y^4 - 1)}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{2y^4} \\ S_y &= \frac{x^2}{y^5} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{3} + c_1 \tag{4}$$

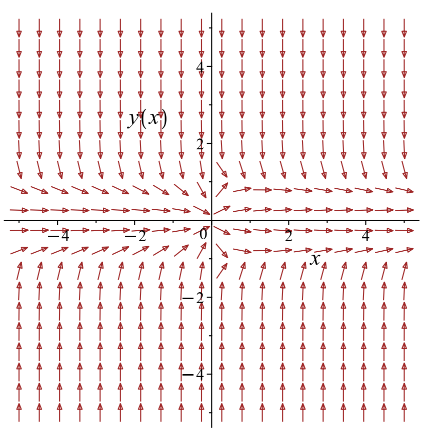
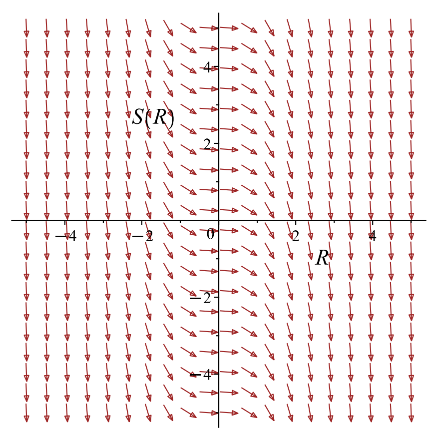
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

Which simplifies to

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(2x\,y^4-1)}{2x}$ 	$R = x$ $S = -\frac{x^2}{4y^4}$	$\frac{dS}{dR} = -R^2$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1 \tag{1}$$

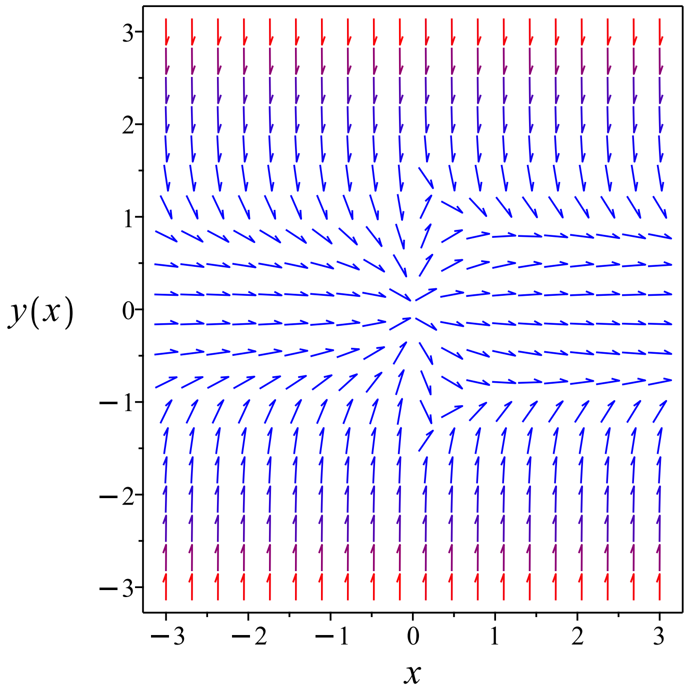


Figure 190: Slope field plot

Verification of solutions

$$-\frac{x^2}{4y^4} = -\frac{x^3}{3} + c_1$$

Verified OK.

4.17.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(2x\,y^4 - 1)}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y - y^5 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= -1 \\
n &= 5
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^5$ gives

$$
y' \frac{1}{y^5} = \frac{1}{2x\, y^4} - 1 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^4}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{4}{y^5} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{4} &= \frac{w(x)}{2x} - 1 \\
w' &= -\frac{2w}{x} + 4
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 4
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = 4$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(4) \\ \frac{d}{dx}(x^2 w) &= (x^2)(4) \\ d(x^2 w) &= (4x^2)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}x^2 w &= \int 4x^2 \, dx \\ x^2 w &= \frac{4x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{4x}{3} + \frac{c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y^4}$ using equation (5) gives the final solution.

$$\frac{1}{y^4} = \frac{4x}{3} + \frac{c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{3^{\frac{1}{4}}\sqrt{x\sqrt{4x^3+3c_1}}}{\sqrt{4x^3+3c_1}} \\ y(x) &= -\frac{3^{\frac{1}{4}}\sqrt{x\sqrt{4x^3+3c_1}}}{\sqrt{4x^3+3c_1}} \\ y(x) &= \frac{3^{\frac{1}{4}}\sqrt{-x\sqrt{4x^3+3c_1}}}{\sqrt{4x^3+3c_1}} \\ y(x) &= -\frac{3^{\frac{1}{4}}\sqrt{-x\sqrt{4x^3+3c_1}}}{\sqrt{4x^3+3c_1}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \tag{1}$$

$$y = -\frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \tag{2}$$

$$y = \frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \tag{3}$$

$$y = -\frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}} \tag{4}$$

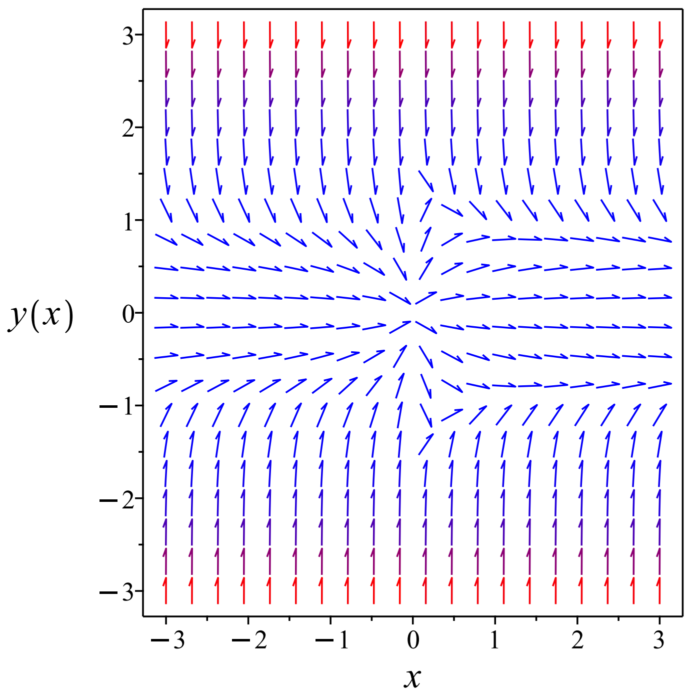


Figure 191: Slope field plot

Verification of solutions

$$y = \frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = -\frac{3^{\frac{1}{4}} \sqrt{x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = \frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

$$y = -\frac{3^{\frac{1}{4}} \sqrt{-x\sqrt{4x^3 + 3c_1}}}{\sqrt{4x^3 + 3c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 137

```
dsolve((2*x*y(x)^5-y(x))+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{1}{4}}\sqrt{-\sqrt{4x^3+3c_1}\,x}}{\sqrt{4x^3+3c_1}}$$
$$y(x) = \frac{3^{\frac{1}{4}}\sqrt{\sqrt{4x^3+3c_1}\,x}}{\sqrt{4x^3+3c_1}}$$
$$y(x) = -\frac{3^{\frac{1}{4}}\sqrt{-\sqrt{4x^3+3c_1}\,x}}{\sqrt{4x^3+3c_1}}$$
$$y(x) = -\frac{3^{\frac{1}{4}}\sqrt{\sqrt{4x^3+3c_1}\,x}}{\sqrt{4x^3+3c_1}}$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 109

```
DSolve[(2*x*y[x]^5-y[x])+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3}+c_1}}$$
$$y(x) \to -\frac{i\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3}+c_1}}$$
$$y(x) \to \frac{i\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3}+c_1}}$$
$$y(x) \to \frac{\sqrt{x}}{\sqrt[4]{\frac{4x^3}{3}+c_1}}$$
$$y(x) \to 0$$

4.18 problem 19 (t)

4.18.1 Solving as exact ode 908

Internal problem ID [5315]
Internal file name [`OUTPUT/4806_Friday_February_02_2024_05_12_30_AM_81058194/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 19 (t).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\sin(y) - (2y\cos(y) - x(\sec(y) + \tan(y)))\, y' = -1}$$

4.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2y\cos(y)+x(\sec(y)+\tan(y)))\,\mathrm{d}y&=(-1-\sin(y))\,\mathrm{d}x\\(1+\sin(y))\,\mathrm{d}x+(-2y\cos(y)+x(\sec(y)+\tan(y)))\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=1+\sin(y)\\N(x,y)&=-2y\cos(y)+x(\sec(y)+\tan(y))\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(1+\sin(y))\\&=\cos(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(-2y\cos(y)+x(\sec(y)+\tan(y)))\\&=\sec(y)+\tan(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\cos(y)}{-2y\cos(y)^2 + \sin(y)x + x}((\cos(y)) - (\sec(y) + \tan(y))) \\
&= -\frac{\sin(y)}{2y\sin(y) + x - 2y}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{1 + \sin(y)}((\sec(y) + \tan(y)) - (\cos(y))) \\
&= \tan(y)
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \tan(y) \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(y))} \\
&= \sec(y)
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sec(y)(1 + \sin(y)) \\
&= \sec(y) + \tan(y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sec(y)(-2y\cos(y) + x(\sec(y) + \tan(y))) \\
&= \frac{-2y\sin(y) - x + 2y}{\sin(y) - 1}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\sec(y) + \tan(y)) + \left(\frac{-2y\sin(y) - x + 2y}{\sin(y) - 1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \sec(y) + \tan(y)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = x(\sec(y) + \tan(y)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = x\left(\sec(y)\tan(y) + 1 + \tan(y)^2\right) + f'(y) \tag{4}
$$

$$
= -\frac{x}{\sin(y) - 1} + f'(y)
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{-2y\sin(y)-x+2y}{\sin(y)-1}$. Therefore equation (4) becomes

$$
\frac{-2y\sin(y) - x + 2y}{\sin(y) - 1} = -\frac{x}{\sin(y) - 1} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -2y
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,dy = \int (-2y)\,dy$$
$$f(y) = -y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(\sec(y) + \tan(y)) - y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(\sec(y) + \tan(y)) - y^2$$

Summary

The solution(s) found are the following

$$x(\sec(y) + \tan(y)) - y^2 = c_1 \tag{1}$$

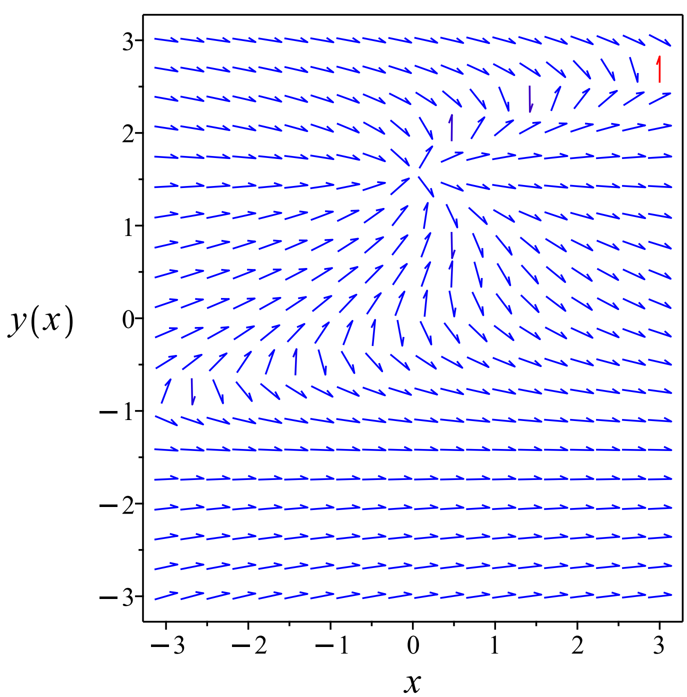


Figure 192: Slope field plot

Verification of solutions

$$x(\sec(y) + \tan(y)) - y^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 25

```
dsolve((1+sin(y(x)))=(2*y(x)*cos(y(x))-x*(sec(y(x))+tan(y(x))) )*diff(y(x),x),y(x), singsol=
```

$$x + \frac{-y(x)^2 - c_1}{\sec(y(x)) + \tan(y(x))} = 0$$

✓ Solution by Mathematica

Time used: 1.489 (sec). Leaf size: 66

```
DSolve[(1+Sin[y[x]])==(2*y[x]*Cos[y[x]]-x*(Sec[y[x]]+Tan[y[x]]) )*y'[x],y[x],x,IncludeSingul
```

$$y(x) \to -\frac{\pi}{2}$$
$$y(x) \to \frac{3\pi}{2}$$
$$\text{Solve}\left[x = y(x)^2 e^{-2\text{arctanh}\left(\tan\left(\frac{y(x)}{2}\right)\right)} + c_1 e^{-2\text{arctanh}\left(\tan\left(\frac{y(x)}{2}\right)\right)}, y(x)\right]$$
$$y(x) \to -\frac{\pi}{2}$$

4.19 problem 22 (a)

4.19.1 Existence and uniqueness analysis . . . 914
4.19.2 Solving as linear ode . . . 915
4.19.3 Solving as first order ode lie symmetry lookup ode . . . 917
4.19.4 Solving as exact ode . . . 921
4.19.5 Maple step by step solution . . . 926

Internal problem ID [5316]
Internal file name [`OUTPUT/4807_Friday_February_02_2024_05_13_49_AM_31079458/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 22 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - 2y = x^3 e^x}$$

With initial conditions

$$[y(1) = 0]$$

4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = x^2 e^x$$

Hence the ode is

$$y' - \frac{2y}{x} = x^2 e^x$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2 e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.19.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{x} dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2 e^x\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(x^2 e^x\right) \\
d\left(\frac{y}{x^2}\right) &= e^x \, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x^2} &= \int e^x \, dx \\
\frac{y}{x^2} &= e^x + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 e^x + c_1 x^2$$

which simplifies to

$$y = x^2(e^x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e} + c_1$$

The solutions are

$$c_1 = -\mathrm{e}$$

Trying the constant

$$c_1 = -\mathrm{e}$$

Substituting this in the general solution gives

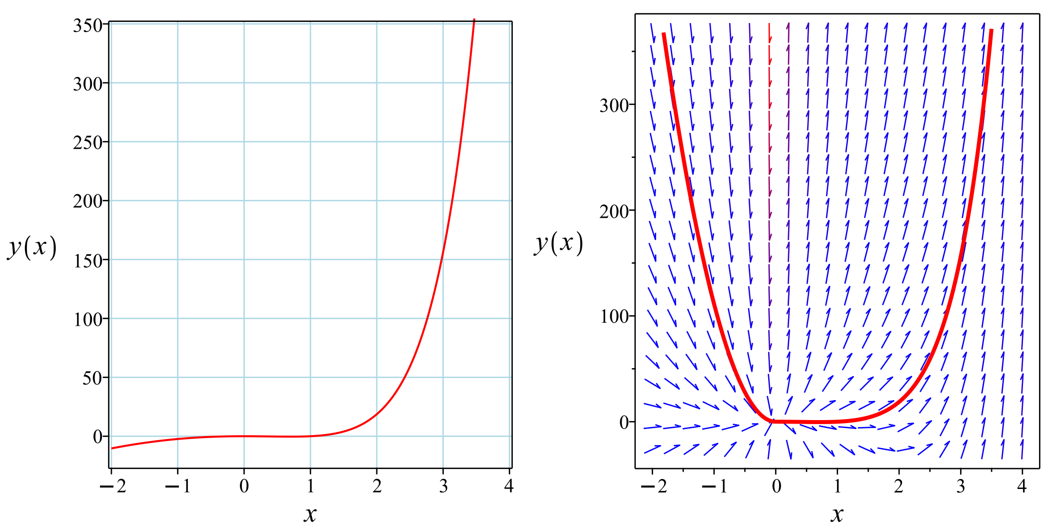
$$y = x^2(\mathrm{e}^x - \mathrm{e})$$

The constant $c_1 = -\mathrm{e}$ gives valid solution.

Summary

The solution(s) found are the following

$$y = x^2(\mathrm{e}^x - \mathrm{e}) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x^2(\mathrm{e}^x - \mathrm{e})$$

Verified OK.

4.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2y + x^3 \mathrm{e}^x}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 114: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{x^2} dy
\end{aligned}
$$

Which results in

$$
S = \frac{y}{x^2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \frac{2y + x^3 \mathrm{e}^x}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2y}{x^3} \\
S_y &= \frac{1}{x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \mathrm{e}^x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \mathrm{e}^R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \mathrm{e}^R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^2} = \mathrm{e}^x + c_1
$$

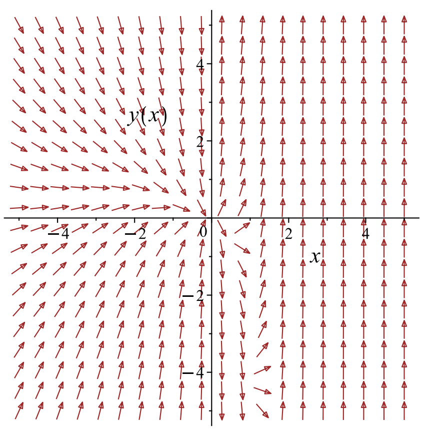
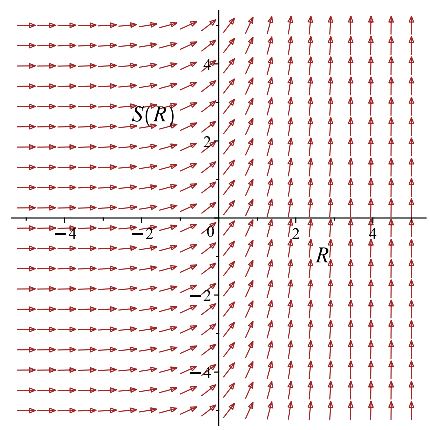
Which simplifies to

$$
\frac{y}{x^2} = \mathrm{e}^x + c_1
$$

Which gives

$$
y = x^2(\mathrm{e}^x + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+x^3 e^x}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = e^R$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e + c_1$$

The solutions are

$$c_1 = -e$$

Trying the constant

$$c_1 = -e$$

Substituting this in the general solution gives

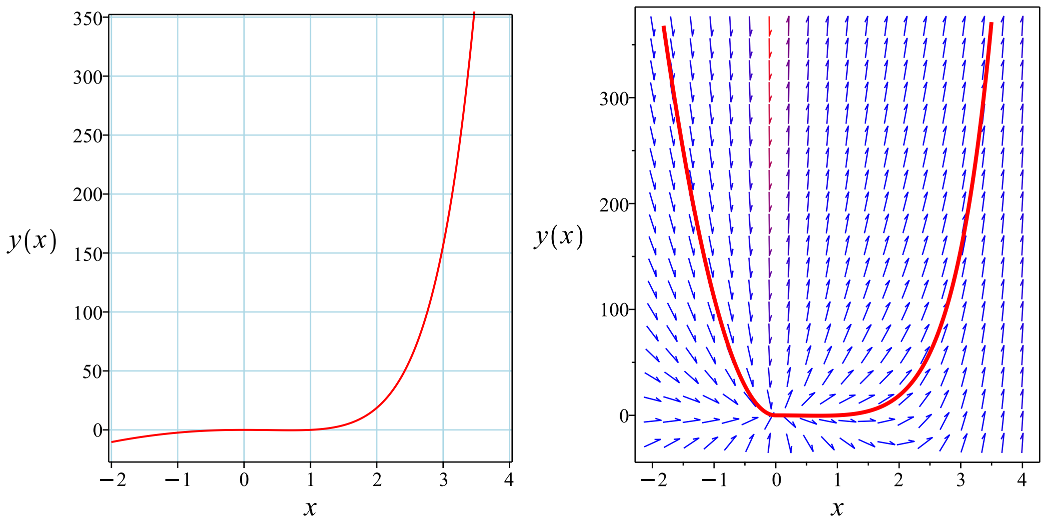
$$y = x^2(e^x - e)$$

The constant $c_1 = -e$ gives valid solution.

Summary

The solution(s) found are the following

$$y = x^2(e^x - e) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = x^2(\mathrm{e}^x - \mathrm{e})$$

Verified OK.

4.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(2y + x^3\mathrm{e}^x\right)\mathrm{d}x\\ \left(-2y - x^3\mathrm{e}^x\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2y - x^3\mathrm{e}^x\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - x^3\mathrm{e}^x\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-2) - (1))\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(x)} \\ &= \frac{1}{x^3}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^3}\left(-2y - x^3 \mathrm{e}^x\right) \\ &= \frac{-2y - x^3 \mathrm{e}^x}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^3}(x) \\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-2y - x^3 \mathrm{e}^x}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-2y - x^3 \mathrm{e}^x}{x^3} \mathrm{d}x \\
\phi &= \frac{-x^2 \mathrm{e}^x + y}{x^2} + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^2 \mathrm{e}^x + y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^2 \mathrm{e}^x + y}{x^2}$$

The solution becomes

$$y = x^2(\mathrm{e}^x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \mathrm{e} + c_1$$

The solutions are

$$c_1 = -\mathrm{e}$$

Trying the constant

$$c_1 = -\mathrm{e}$$

Substituting this in the general solution gives

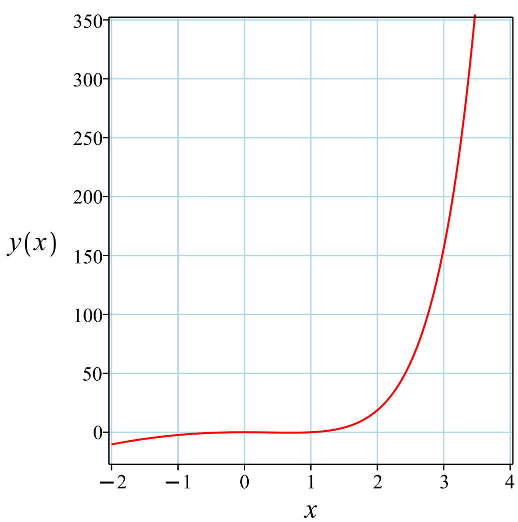
$$y = x^2(\mathrm{e}^x - \mathrm{e})$$

The constant $c_1 = -\mathrm{e}$ gives valid solution.

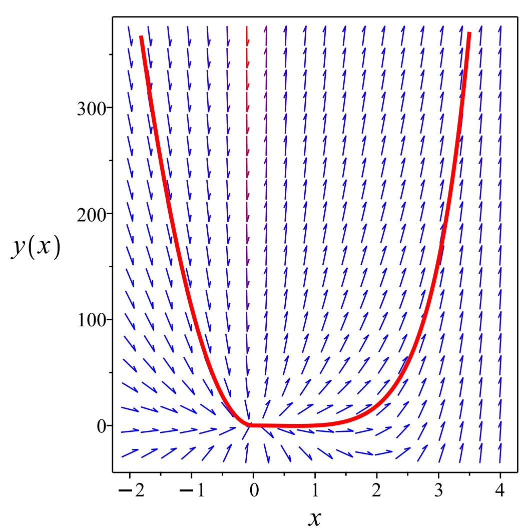
Summary

The solution(s) found are the following

$$y = x^2(\mathrm{e}^x - \mathrm{e}) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^2(\mathrm{e}^x - \mathrm{e})$$

Verified OK.

4.19.5 Maple step by step solution

Let's solve

$[xy' - 2y = x^3 e^x, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + x^2 e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = x^2 e^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu(x)\, x^2 e^x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 e^x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 e^x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 e^x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int e^x dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(e^x + c_1)$

- Use initial condition $y(1) = 0$

  $0 = e + c_1$

- Solve for c_1

  $c_1 = -\mathrm{e}$
- Substitute $c_1 = -\mathrm{e}$ into general solution and simplify

  $y = x^2(\mathrm{e}^x - \mathrm{e})$
- Solution to the IVP

  $y = x^2(\mathrm{e}^x - \mathrm{e})$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.031 (sec). Leaf size: 15

```
dsolve([x*diff(y(x),x)=2*y(x)+x^3*exp(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = -(-\mathrm{e}^x + \mathrm{e})\, x^2$$

✓ Solution by Mathematica
Time used: 0.05 (sec). Leaf size: 16

```
DSolve[{x*y'[x]==2*y[x]+x^3*Exp[x],{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (e^x - e)\, x^2$$

4.20 problem 22 (b)

4.20.1 Existence and uniqueness analysis 928
4.20.2 Solving as linear ode 929
4.20.3 Solving as first order ode lie symmetry lookup ode 930
4.20.4 Solving as exact ode 934
4.20.5 Maple step by step solution 938

Internal problem ID [5317]
Internal file name [`OUTPUT/4808_Friday_February_02_2024_05_13_50_AM_63251945/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 22 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{Li' + Ri = E \sin(2t)}$$

With initial conditions

$$[i(0) = 0]$$

4.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$i' + p(t)i = q(t)$$

Where here

$$p(t) = \frac{R}{L}$$
$$q(t) = \frac{E \sin(2t)}{L}$$

Hence the ode is

$$i' + \frac{Ri}{L} = \frac{E \sin(2t)}{L}$$

The domain of $p(t) = \frac{R}{L}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{E\sin(2t)}{L}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.20.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{R}{L} dt} \\ &= e^{\frac{Rt}{L}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dt}(\mu i) &= (\mu)\left(\frac{E\sin(2t)}{L}\right) \\ \frac{d}{dt}\left(e^{\frac{Rt}{L}} i\right) &= \left(e^{\frac{Rt}{L}}\right)\left(\frac{E\sin(2t)}{L}\right) \\ d\left(e^{\frac{Rt}{L}} i\right) &= \left(\frac{e^{\frac{Rt}{L}} E\sin(2t)}{L}\right) dt\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\frac{Rt}{L}} i &= \int \frac{e^{\frac{Rt}{L}} E\sin(2t)}{L} dt \\ e^{\frac{Rt}{L}} i &= \frac{E\left(-\frac{2\,e^{\frac{Rt}{L}}\cos(2t)}{\frac{R^2}{L^2}+4} + \frac{R\,e^{\frac{Rt}{L}}\sin(2t)}{L\left(\frac{R^2}{L^2}+4\right)}\right)}{L} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{Rt}{L}}$ results in

$$i = \frac{e^{-\frac{Rt}{L}} E\left(-\frac{2\,e^{\frac{Rt}{L}}\cos(2t)}{\frac{R^2}{L^2}+4} + \frac{R\,e^{\frac{Rt}{L}}\sin(2t)}{L\left(\frac{R^2}{L^2}+4\right)}\right)}{L} + c_1 e^{-\frac{Rt}{L}}$$

which simplifies to

$$i = \frac{c_1(4L^2+R^2)\,\mathrm{e}^{-\frac{Rt}{L}} - 2EL\cos(2t) + ER\sin(2t)}{4L^2+R^2}$$

Initial conditions are used to solve for c_1. Substituting $t=0$ and $i=0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4L^2c_1 + R^2c_1 - 2EL}{4L^2+R^2}$$

The solutions are

$$c_1 = \frac{2EL}{4L^2+R^2}$$

Trying the constant

$$c_1 = \frac{2EL}{4L^2+R^2}$$

Substituting this in the general solution gives

$$i = \frac{-2EL\cos(2t) + 2EL\,\mathrm{e}^{-\frac{Rt}{L}} + ER\sin(2t)}{4L^2+R^2}$$

The constant $c_1 = \frac{2EL}{4L^2+R^2}$ gives valid solution.

Summary

The solution(s) found are the following

$$i = \frac{-2EL\cos(2t) + 2EL\,\mathrm{e}^{-\frac{Rt}{L}} + ER\sin(2t)}{4L^2+R^2} \tag{1}$$

Verification of solutions

$$i = \frac{-2EL\cos(2t) + 2EL\,\mathrm{e}^{-\frac{Rt}{L}} + ER\sin(2t)}{4L^2+R^2}$$

Verified OK.

4.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$i' = \frac{-Ri + E\sin(2t)}{L}$$
$$i' = \omega(t,i)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_i - \xi_t) - \omega^2 \xi_i - \omega_t \xi - \omega_i \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 117: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(t,i) &= 0 \\ \eta(t,i) &= \mathrm{e}^{-\frac{Rt}{L}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,i) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{di}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial i}\right) S(t,i) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\frac{Rt}{L}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{Rt}{L}} i$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,i) S_i}{R_t + \omega(t,i) R_i} \tag{2}$$

Where in the above R_t, R_i, S_t, S_i are all partial derivatives and $\omega(t,i)$ is the right hand side of the original ode given by

$$\omega(t,i) = \frac{-Ri + E \sin(2t)}{L}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 1 \\ R_i &= 0 \\ S_t &= \frac{R \,\mathrm{e}^{\frac{Rt}{L}} i}{L} \\ S_i &= \mathrm{e}^{\frac{Rt}{L}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^{\frac{Rt}{L}} E \sin(2t)}{L} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, i in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{e^{\frac{RR}{L}} E \sin(2R)}{L}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{RR}{L}} (2L\cos(2R) - \sin(2R) R)}{4L^2 + R^2} \tag{4}$$

To complete the solution, we just need to transform (4) back to t, i coordinates. This results in

$$e^{\frac{Rt}{L}} i = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{Rt}{L}} (2L\cos(2t) - \sin(2t) R)}{4L^2 + R^2}$$

Which simplifies to

$$e^{\frac{Rt}{L}} i = -\frac{-c_1(4L^2 + R^2) + E e^{\frac{Rt}{L}} (2L\cos(2t) - \sin(2t) R)}{4L^2 + R^2}$$

Which gives

$$i = -\frac{e^{-\frac{Rt}{L}} \left(2 e^{\frac{Rt}{L}} E\cos(2t) L - R e^{\frac{Rt}{L}} E\sin(2t) - 4L^2c_1 - R^2c_1\right)}{4L^2 + R^2}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $i = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{4L^2c_1 + R^2c_1 - 2EL}{4L^2 + R^2}$$

The solutions are

$$c_1 = \frac{2EL}{4L^2 + R^2}$$

Trying the constant

$$c_1 = \frac{2EL}{4L^2 + R^2}$$

Substituting this in the general solution gives

$$i = \frac{-2EL\cos(2t) e^{-\frac{Rt}{L}} e^{\frac{Rt}{L}} + E\sin(2t) e^{-\frac{Rt}{L}} e^{\frac{Rt}{L}} R + 2EL e^{-\frac{Rt}{L}}}{4L^2 + R^2}$$

The constant $c_1 = \frac{2EL}{4L^2+R^2}$ gives valid solution.

Summary

The solution(s) found are the following

$$i = \frac{-2EL\cos(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\mathrm{e}^{\frac{Rt}{L}} + E\sin(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\mathrm{e}^{\frac{Rt}{L}}R + 2EL\,\mathrm{e}^{-\frac{Rt}{L}}}{4L^2+R^2} \tag{1}$$

Verification of solutions

$$i = \frac{-2EL\cos(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\mathrm{e}^{\frac{Rt}{L}} + E\sin(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\mathrm{e}^{\frac{Rt}{L}}R + 2EL\,\mathrm{e}^{-\frac{Rt}{L}}}{4L^2+R^2}$$

Verified OK.

4.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,i)\,\mathrm{d}t + N(t,i)\,\mathrm{d}i = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(L)\,\mathrm{d}i &= (-Ri + E\sin(2t))\,\mathrm{d}t\\ (Ri - E\sin(2t))\,\mathrm{d}t + (L)\,\mathrm{d}i &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,i) &= Ri - E\sin(2t)\\ N(t,i) &= L\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial i} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial i} &= \frac{\partial}{\partial i}(Ri - E\sin(2t))\\ &= R\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(L)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial i} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial i} - \frac{\partial N}{\partial t}\right)\\ &= \frac{1}{L}((R) - (0))\\ &= \frac{R}{L}\end{aligned}$$

Since A does not depend on i, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}t}\\ &= e^{\int \frac{R}{L}\,\mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\frac{Rt}{L}} \\
&= \mathrm{e}^{\frac{Rt}{L}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\frac{Rt}{L}}(Ri - E\sin(2t)) \\
&= -\mathrm{e}^{\frac{Rt}{L}}(-Ri + E\sin(2t))
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\frac{Rt}{L}}(L) \\
&= L\,\mathrm{e}^{\frac{Rt}{L}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}i}{\mathrm{d}t} &= 0 \\
\left(-\mathrm{e}^{\frac{Rt}{L}}(-Ri + E\sin(2t))\right) + \left(L\,\mathrm{e}^{\frac{Rt}{L}}\right)\frac{\mathrm{d}i}{\mathrm{d}t} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(t, i)$

$$
\frac{\partial \phi}{\partial t} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial i} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. t gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int \overline{M}\,\mathrm{d}t \\
\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int -\mathrm{e}^{\frac{Rt}{L}}(-Ri + E\sin(2t))\,\mathrm{d}t
\end{aligned}
$$

$$
\phi = \frac{\mathrm{e}^{\frac{Rt}{L}} L(2EL\cos(2t) - ER\sin(2t) + 4i\,L^2 + i\,R^2)}{4L^2 + R^2} + f(i) \tag{3}
$$

Where $f(i)$ is used for the constant of integration since ϕ is a function of both t and i. Taking derivative of equation (3) w.r.t i gives

$$\frac{\partial \phi}{\partial i} = L\,\mathrm{e}^{\frac{Rt}{L}} + f'(i) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial i} = L\,\mathrm{e}^{\frac{Rt}{L}}$. Therefore equation (4) becomes

$$L\,\mathrm{e}^{\frac{Rt}{L}} = L\,\mathrm{e}^{\frac{Rt}{L}} + f'(i) \tag{5}$$

Solving equation (5) for $f'(i)$ gives

$$f'(i) = 0$$

Therefore

$$f(i) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(i)$ into equation (3) gives ϕ

$$\phi = \frac{\mathrm{e}^{\frac{Rt}{L}} L(2EL\cos(2t) - ER\sin(2t) + 4i\,L^2 + i\,R^2)}{4L^2 + R^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\mathrm{e}^{\frac{Rt}{L}} L(2EL\cos(2t) - ER\sin(2t) + 4i\,L^2 + i\,R^2)}{4L^2 + R^2}$$

The solution becomes

$$i = -\frac{\left(2E\,L^2\cos(2t)\,\mathrm{e}^{\frac{Rt}{L}} - EL\sin(2t)\,\mathrm{e}^{\frac{Rt}{L}} R - 4L^2 c_1 - R^2 c_1\right)\mathrm{e}^{-\frac{Rt}{L}}}{(4L^2 + R^2)\,L}$$

Initial conditions are used to solve for c_1. Substituting $t = 0$ and $i = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-2E\,L^2 + 4L^2 c_1 + R^2 c_1}{4L^3 + L\,R^2}$$

The solutions are

$$c_1 = \frac{2E\,L^2}{4L^2 + R^2}$$

Trying the constant

$$c_1 = \frac{2E\,L^2}{4L^2+R^2}$$

Substituting this in the general solution gives

$$i = \frac{-2EL\cos(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}} + E\sin(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}}\,R + 2EL\,\mathrm{e}^{-\frac{Rt}{L}}}{4L^2+R^2}$$

The constant $c_1 = \frac{2E\,L^2}{4L^2+R^2}$ gives valid solution.

Summary

The solution(s) found are the following

$$i = \frac{-2EL\cos(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}} + E\sin(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}}\,R + 2EL\,\mathrm{e}^{-\frac{Rt}{L}}}{4L^2+R^2} \tag{1}$$

Verification of solutions

$$i = \frac{-2EL\cos(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}} + E\sin(2t)\,\mathrm{e}^{-\frac{Rt}{L}}\,\mathrm{e}^{\frac{Rt}{L}}\,R + 2EL\,\mathrm{e}^{-\frac{Rt}{L}}}{4L^2+R^2}$$

Verified OK.

4.20.5 Maple step by step solution

Let's solve

$[Li' + Ri = E\sin(2t), i(0) = 0]$

- Highest derivative means the order of the ODE is 1

  i'

- Isolate the derivative

  $i' = -\frac{Ri}{L} + \frac{E\sin(2t)}{L}$

- Group terms with i on the lhs of the ODE and the rest on the rhs of the ODE

  $i' + \frac{Ri}{L} = \frac{E\sin(2t)}{L}$

- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(i' + \frac{Ri}{L}\right) = \frac{\mu(t)E\sin(2t)}{L}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(\mu(t)\,i)$

  $\mu(t)\left(i' + \frac{Ri}{L}\right) = \mu'(t)\,i + \mu(t)\,i'$

- Isolate $\mu'(t)$

  $\mu'(t) = \frac{\mu(t) R}{L}$

- Solve to find the integrating factor

  $\mu(t) = \mathrm{e}^{\frac{Rt}{L}}$

- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(\mu(t)\, i)\right) dt = \int \frac{\mu(t) E \sin(2t)}{L} dt + c_1$

- Evaluate the integral on the lhs

  $\mu(t)\, i = \int \frac{\mu(t) E \sin(2t)}{L} dt + c_1$

- Solve for i

  $i = \frac{\int \frac{\mu(t) E \sin(2t)}{L} dt + c_1}{\mu(t)}$

- Substitute $\mu(t) = \mathrm{e}^{\frac{Rt}{L}}$

  $i = \frac{\int \frac{\mathrm{e}^{\frac{Rt}{L}} E \sin(2t)}{L} dt + c_1}{\mathrm{e}^{\frac{Rt}{L}}}$

- Evaluate the integrals on the rhs

  $i = \frac{\frac{E\left(-\frac{2\,\mathrm{e}^{\frac{Rt}{L}} \cos(2t)}{\frac{R^2}{L^2}+4} + \frac{R\,\mathrm{e}^{\frac{Rt}{L}} \sin(2t)}{L\left(\frac{R^2}{L^2}+4\right)}\right)}{L} + c_1}{\mathrm{e}^{\frac{Rt}{L}}}$

- Simplify

  $i = \frac{c_1 (4L^2+R^2) \mathrm{e}^{-\frac{Rt}{L}} - 2EL\cos(2t) + ER\sin(2t)}{4L^2+R^2}$

- Use initial condition $i(0) = 0$

  $0 = \frac{c_1 (4L^2+R^2) - 2EL}{4L^2+R^2}$

- Solve for c_1

  $c_1 = \frac{2EL}{4L^2+R^2}$

- Substitute $c_1 = \frac{2EL}{4L^2+R^2}$ into general solution and simplify

  $i = -\frac{2E\left(L\cos(2t) - L\,\mathrm{e}^{-\frac{Rt}{L}} - \frac{\sin(2t) R}{2}\right)}{4L^2+R^2}$

- Solution to the IVP

  $i = -\frac{2E\left(L\cos(2t) - L\,\mathrm{e}^{-\frac{Rt}{L}} - \frac{\sin(2t) R}{2}\right)}{4L^2+R^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve([L*diff(i(t),t)+R*i(t)=E*sin(2*t),i(0) = 0],i(t), singsol=all)
```

$$i(t) = \frac{E\left(2L\,\mathrm{e}^{-\frac{Rt}{L}} - 2L\cos\left(2t\right) + \sin\left(2t\right)R\right)}{4L^2 + R^2}$$

✓ Solution by Mathematica

Time used: 0.152 (sec). Leaf size: 49

```
DSolve[{L*i'[t]+R*i[t]==e*Sin[2*t],{i[0]==0}},i[t],t,IncludeSingularSolutions -> True]
```

$$i(t) \to \frac{2e\left(L\left(e^{-\frac{Rt}{L}} + \sin^2(t)\right) - L\cos^2(t) + R\sin(t)\cos(t)\right)}{4L^2 + R^2}$$

4.21 problem 23 (a)

4.21.1 Solving as exact ode 941

Internal problem ID [5318]
Internal file name [`OUTPUT/4809_Friday_February_02_2024_05_13_52_AM_30399592/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 23 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{\cos(y)\, y' x^2 - 2x\sin(y) = -1}$$

4.21.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\cos(y)\,x^2\right)\mathrm{d}y &= (2\sin(y)\,x - 1)\,\mathrm{d}x \\
(-2\sin(y)\,x + 1)\,\mathrm{d}x + \left(\cos(y)\,x^2\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -2\sin(y)\,x + 1 \\
N(x,y) &= \cos(y)\,x^2
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2\sin(y)\,x + 1) \\
&= -2\cos(y)\,x
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\cos(y)\,x^2\right) \\
&= 2\cos(y)\,x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{\sec(y)}{x^2}((-2\cos(y)\,x) - (2\cos(y)\,x)) \\
&= -\frac{4}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^4}(-2\sin(y)\,x + 1) \\ &= \frac{-2\sin(y)\,x + 1}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^4}(\cos(y)\,x^2) \\ &= \frac{\cos(y)}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-2\sin(y)\,x + 1}{x^4}\right) + \left(\frac{\cos(y)}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-2\sin(y)\, x + 1}{x^4} \mathrm{d}x \\
\phi &= \frac{3\sin(y)\, x - 1}{3x^3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{x^2}$. Therefore equation (4) becomes

$$
\frac{\cos(y)}{x^2} = \frac{\cos(y)}{x^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{3\sin(y)\, x - 1}{3x^3} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{3\sin(y)\, x - 1}{3x^3}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\frac{3x\sin(y) - 1}{3x^3} = c_1 \qquad (1)
$$

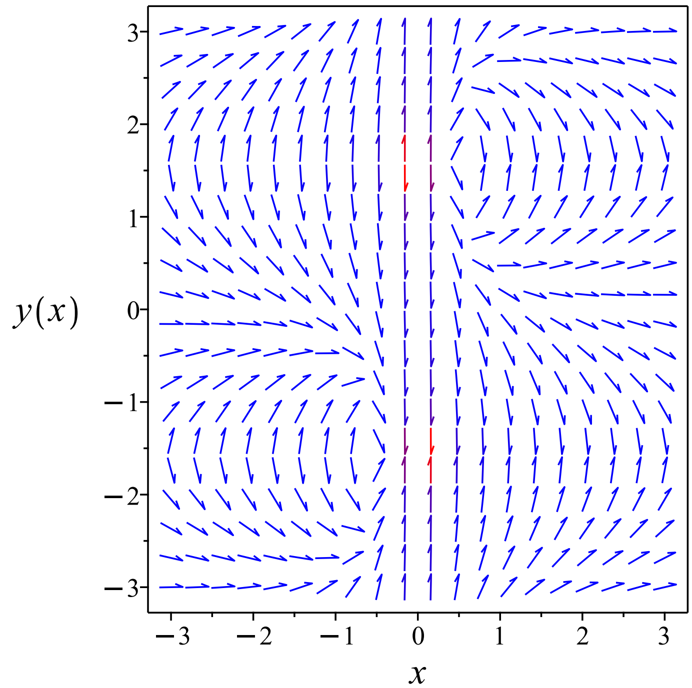


Figure 196: Slope field plot

Verification of solutions

$$\frac{3x\sin(y) - 1}{3x^3} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*cos(y(x))*diff(y(x),x)=2*x*sin(y(x))-1,y(x), singsol=all)
```

$$y(x) = -\arcsin\left(\frac{3c_1x^3 - 1}{3x}\right)$$

✓ Solution by Mathematica

Time used: 10.185 (sec). Leaf size: 21

```
DSolve[x^2*Cos[y[x]]*y'[x]==2*x*Sin[y[x]]-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin\left(\frac{1}{3x} + 2c_1x^2\right)$$

4.22 problem 23 (b)

Internal problem ID [5319]
Internal file name [`OUTPUT/4810_Friday_February_02_2024_05_13_54_AM_49569275/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 23 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

`[_rational]`

Unable to solve or complete the solution.

$$\boxed{4x^2yy' - 3x(3y^2 + 2) - 2(3y^2 + 2)^3 = 0}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 2`[0, (2*x*y^4+x*y^6+8/27*x+8/81*x^2+4/3*x*y^2+4/27
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 175

```
dsolve(4*x^2*y(x)*diff(y(x),x)=3*x*(3*y(x)^2+2)+2*(3*y(x)^2+2)^3,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{6}\,\sqrt{\frac{-3c_1x^8+\sqrt{-3\left(c_1x^8-\frac{1}{3}\right)c_1x^9}+1}{3c_1x^8-1}}}{3}$$

$$y(x) = \frac{\sqrt{6}\,\sqrt{\frac{-3c_1x^8+\sqrt{-3\left(c_1x^8-\frac{1}{3}\right)c_1x^9}+1}{3c_1x^8-1}}}{3}$$

$$y(x) = -\frac{\sqrt{\frac{-18c_1x^8-6\sqrt{-3\left(c_1x^8-\frac{1}{3}\right)c_1x^9}+6}{3c_1x^8-1}}}{3}$$

$$y(x) = \frac{\sqrt{\frac{-18c_1x^8-6\sqrt{-3\left(c_1x^8-\frac{1}{3}\right)c_1x^9}+6}{3c_1x^8-1}}}{3}$$

✓ Solution by Mathematica

Time used: 19.518 (sec). Leaf size: 277

```
DSolve[4*x^2*y[x]*y'[x]==3*x*(3*y[x]^2+2)+2*(3*y[x]^2+2)^3,y[x],x,IncludeSingularSolutions -
```

$$y(x)\to -\frac{1}{3}\sqrt{2}\sqrt{-\frac{3x^8+\sqrt{3}\sqrt{-x^9\left(x^8+72c_1\right)}+216c_1}{x^8+72c_1}}$$
$$y(x)\to \frac{1}{3}\sqrt{2}\sqrt{-\frac{3x^8+\sqrt{3}\sqrt{-x^9\left(x^8+72c_1\right)}+216c_1}{x^8+72c_1}}$$
$$y(x)\to -\frac{1}{3}\sqrt{2}\sqrt{\frac{-3x^8+\sqrt{3}\sqrt{-x^9\left(x^8+72c_1\right)}-216c_1}{x^8+72c_1}}$$
$$y(x)\to \frac{1}{3}\sqrt{2}\sqrt{\frac{-3x^8+\sqrt{3}\sqrt{-x^9\left(x^8+72c_1\right)}-216c_1}{x^8+72c_1}}$$
$$y(x)\to -i\sqrt{\frac{2}{3}}$$
$$y(x)\to i\sqrt{\frac{2}{3}}$$

4.23 problem 23 (c)

4.23.1 Solving as first order ode lie symmetry lookup ode 950
4.23.2 Solving as bernoulli ode 954
4.23.3 Solving as exact ode 958

Internal problem ID [5320]
Internal file name [`OUTPUT/4811_Friday_February_02_2024_05_13_56_AM_14351785/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 23 (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{xy^3 - y^3 + 3y^2y'x = x^2 \mathrm{e}^x}$$

4.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-x\,y^3 + y^3 + x^2 \mathrm{e}^x}{3y^2 x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 120: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\mathrm{e}^{-x+\ln(x)}}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\mathrm{e}^{-x+\ln(x)}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 \mathrm{e}^x}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x\, y^3 + y^3 + x^2 \mathrm{e}^x}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3 \mathrm{e}^x (x-1)}{3x^2} \\ S_y &= \frac{y^2 \mathrm{e}^x}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\mathrm{e}^{2x}}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\mathrm{e}^{2R}}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{6} + c_1 \tag{4}$$

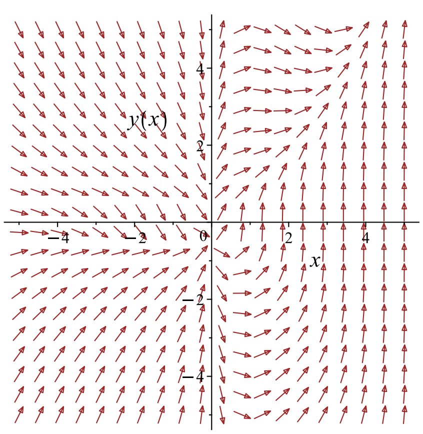
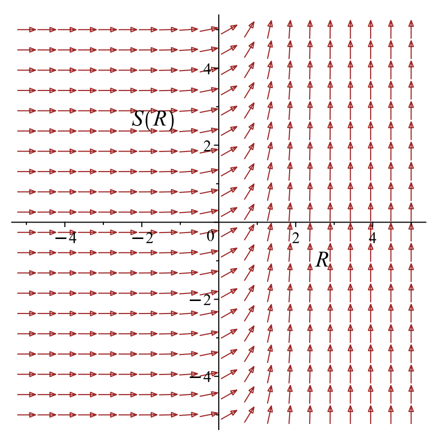
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

Which simplifies to

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x\,y^3+y^3+x^2e^x}{3y^2x}$ 	$R = x$ $S = \frac{y^3 e^x}{3x}$	$\frac{dS}{dR} = \frac{e^{2R}}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1 \tag{1}$$

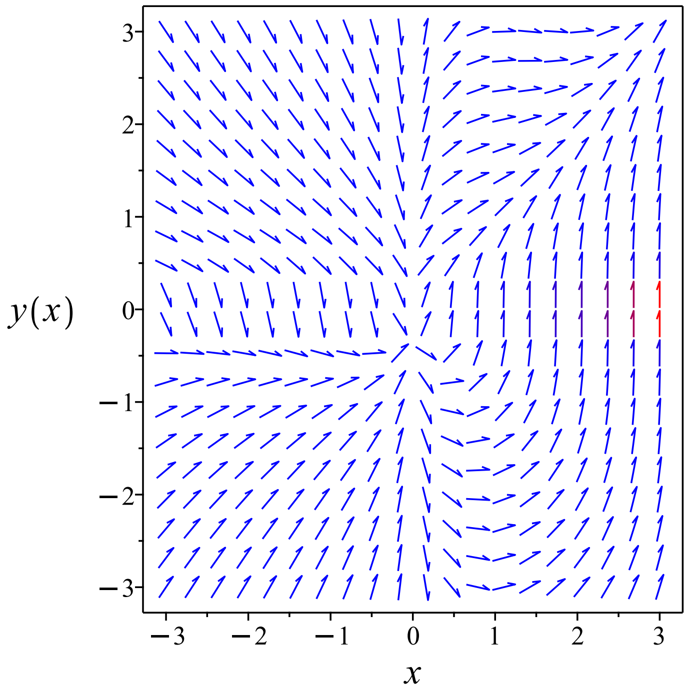


Figure 197: Slope field plot

Verification of solutions

$$\frac{y^3 e^x}{3x} = \frac{e^{2x}}{6} + c_1$$

Verified OK.

4.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{-x\,y^3 + y^3 + x^2 e^x}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1-x}{3x}y + \frac{x\,e^x}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1-x}{3x} \\
f_1(x) &= \frac{x\,\mathrm{e}^x}{3} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y' y^2 = \frac{(1-x)\,y^3}{3x} + \frac{x\,\mathrm{e}^x}{3} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2 y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= \frac{(1-x)\,w(x)}{3x} + \frac{x\,\mathrm{e}^x}{3} \\
w' &= \frac{(1-x)\,w}{x} + x\,\mathrm{e}^x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1-x}{x} \\
q(x) &= x\,\mathrm{e}^x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{(1-x)\,w(x)}{x} = x\,\mathrm{e}^{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1-x}{x}dx}\\ &= \mathrm{e}^{x-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{x}}{x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\,(x\,\mathrm{e}^{x})\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{x} w}{x}\right) &= \left(\frac{\mathrm{e}^{x}}{x}\right)(x\,\mathrm{e}^{x})\\ \mathrm{d}\left(\frac{\mathrm{e}^{x} w}{x}\right) &= \mathrm{e}^{2x}\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{\mathrm{e}^{x} w}{x} &= \int \mathrm{e}^{2x}\,\mathrm{d}x\\ \frac{\mathrm{e}^{x} w}{x} &= \frac{\mathrm{e}^{2x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{x}}{x}$ results in

$$w(x) = \frac{x\,\mathrm{e}^{-x}\mathrm{e}^{2x}}{2} + c_1 x\,\mathrm{e}^{-x}$$

which simplifies to

$$w(x) = \frac{x(2c_1\mathrm{e}^{-x} + \mathrm{e}^{x})}{2}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = \frac{x(2c_1\mathrm{e}^{-x} + \mathrm{e}^{x})}{2}$$

Solving for y gives

$$y(x) = \frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} e^{-x}}{2}$$
$$y(x) = \frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) e^{-x}}{4}$$
$$y(x) = -\frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right) e^{-x}}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} e^{-x}}{2} \tag{1}$$
$$y = \frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right) e^{-x}}{4} \tag{2}$$
$$y = -\frac{2^{\frac{2}{3}} \left(x(e^{2x} + 2c_1)\, e^{2x}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right) e^{-x}}{4} \tag{3}$$

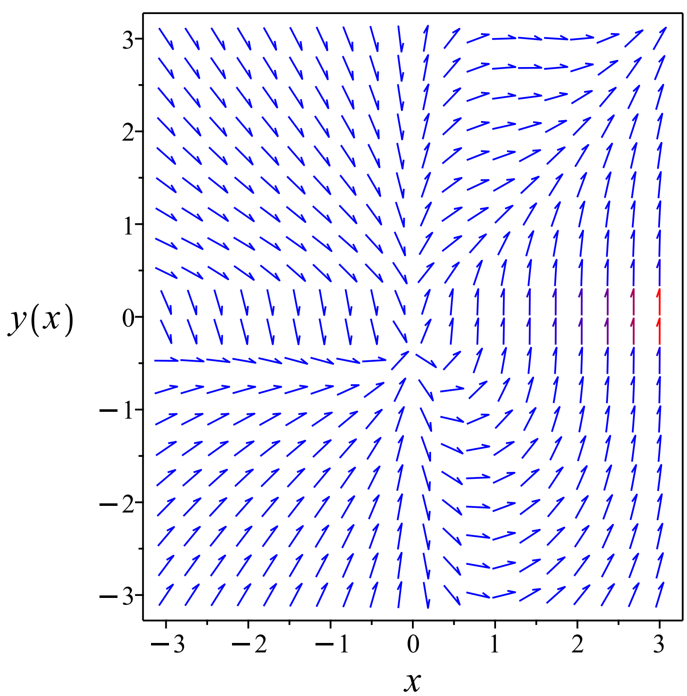


Figure 198: Slope field plot

Verification of solutions

$$y=\frac{2^{\frac{2}{3}}\left(x(\mathrm{e}^{2x}+2c_1)\,\mathrm{e}^{2x}\right)^{\frac{1}{3}}\mathrm{e}^{-x}}{2}$$

Verified OK.

$$y=\frac{2^{\frac{2}{3}}\left(x(\mathrm{e}^{2x}+2c_1)\,\mathrm{e}^{2x}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)\mathrm{e}^{-x}}{4}$$

Verified OK.

$$y=-\frac{2^{\frac{2}{3}}\left(x(\mathrm{e}^{2x}+2c_1)\,\mathrm{e}^{2x}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)\mathrm{e}^{-x}}{4}$$

Verified OK.

4.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3y^2x\right)\mathrm{d}y&=\left(-x\,y^3+y^3+x^2\mathrm{e}^x\right)\mathrm{d}x\\ \left(x\,y^3-y^3-x^2\mathrm{e}^x\right)\mathrm{d}x+\left(3y^2x\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x\,y^3-y^3-x^2\mathrm{e}^x\\ N(x,y)&=3y^2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(x\,y^3-y^3-x^2\mathrm{e}^x\right)\\ &=3y^2(x-1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(3y^2x\right)\\ &=3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{3y^2x}\left(\left(3y^2x-3y^2\right)-\left(3y^2\right)\right)\\ &=\frac{-2+x}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{-2+x}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x-2\ln(x)}\\ &= \frac{\mathrm{e}^{x}}{x^{2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{\mathrm{e}^{x}}{x^{2}}\left(x\,y^{3}-y^{3}-x^{2}\mathrm{e}^{x}\right)\\ &= -\frac{\left(x^{2}\mathrm{e}^{x}-y^{3}(x-1)\right)\mathrm{e}^{x}}{x^{2}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{\mathrm{e}^{x}}{x^{2}}\left(3y^{2}x\right)\\ &= \frac{3y^{2}\mathrm{e}^{x}}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{\left(x^{2}\mathrm{e}^{x}-y^{3}(x-1)\right)\mathrm{e}^{x}}{x^{2}}\right)+\left(\frac{3y^{2}\mathrm{e}^{x}}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{(x^2 \mathrm{e}^x - y^3(x-1)) \mathrm{e}^x}{x^2} \mathrm{d}x$$

$$\phi = -\frac{\mathrm{e}^x(-2y^3 + x \mathrm{e}^x)}{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2 \mathrm{e}^x}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2 \mathrm{e}^x}{x}$. Therefore equation (4) becomes

$$\frac{3y^2 \mathrm{e}^x}{x} = \frac{3y^2 \mathrm{e}^x}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^x(-2y^3 + x \mathrm{e}^x)}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^x(-2y^3 + x \mathrm{e}^x)}{2x}$$

Summary

The solution(s) found are the following

$$-\frac{\mathrm{e}^x(-2y^3 + x \mathrm{e}^x)}{2x} = c_1 \tag{1}$$

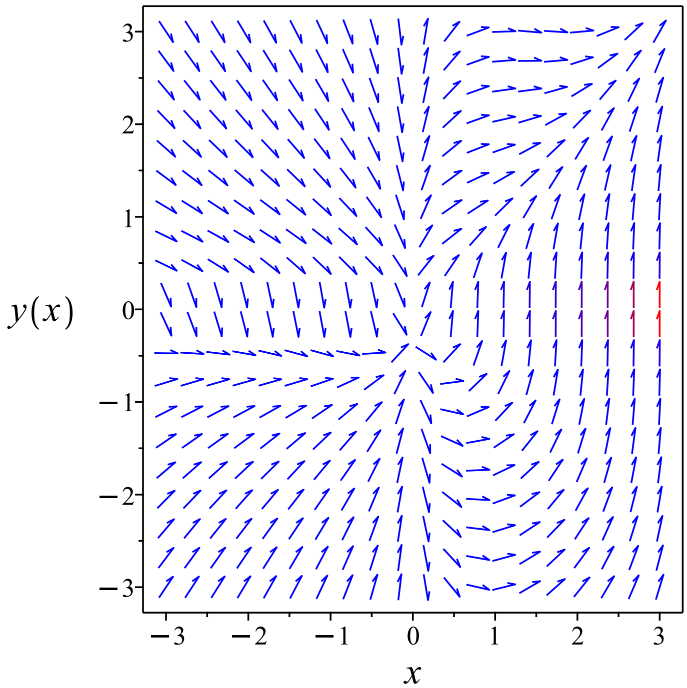


Figure 199: Slope field plot

Verification of solutions

$$-\frac{e^{x}(-2y^{3}+x\,e^{x})}{2x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 101

```
dsolve((x*y(x)^3-y(x)^3-x^2*exp(x))+(3*x*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{2}{3}}\left(\left(e^{2x}+2c_1\right)x\,e^{2x}\right)^{\frac{1}{3}}e^{-x}}{2}$$
$$y(x) = -\frac{2^{\frac{2}{3}}\left(\left(e^{2x}+2c_1\right)x\,e^{2x}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)e^{-x}}{4}$$
$$y(x) = \frac{2^{\frac{2}{3}}\left(\left(e^{2x}+2c_1\right)x\,e^{2x}\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)e^{-x}}{4}$$

✓ Solution by Mathematica

Time used: 0.854 (sec). Leaf size: 117

```
DSolve[(x*y[x]^3-y[x]^3-x^2*Exp[x])+(3*x*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\sqrt[3]{-\frac{1}{2}}e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x}+2c_1}$$
$$y(x) \to \frac{e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x}+2c_1}}{\sqrt[3]{2}}$$
$$y(x) \to \frac{(-1)^{2/3}e^{-x/3}\sqrt[3]{x}\sqrt[3]{e^{2x}+2c_1}}{\sqrt[3]{2}}$$

4.24 problem 23 (d)

4.24.1 Solving as abelFirstKind ode 964

Internal problem ID [5321]
Internal file name [`OUTPUT/4812_Friday_February_02_2024_05_13_57_AM_51034746/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 23 (d).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"abelFirstKind"**

Maple gives the following as the ode type

`[_Abel]`

$$\boxed{y' + x(x+y) - x^3(x+y)^3 = -1}$$

4.24.1 Solving as abelFirstKind ode

This is Abel first kind ODE, it has the form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = x^3y^3 + 3x^4y^2 + (3x^5 - x)\, y + x^6 - x^2 - 1 \tag{1}$$

Therefore

$$\begin{aligned}
f_0(x) &= x^6 - x^2 - 1\\
f_1(x) &= 3x^5 - x\\
f_2(x) &= 3x^4\\
f_3(x) &= x^3
\end{aligned}$$

Since $f_2(x) = 3x^4$ is not zero, then the first step is to apply the following transformation to remove f_2. Let $y = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned} y &= u(x) - \left(\frac{3x^4}{3x^3}\right) \\ &= u(x) - x \end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = x^3 u(x)^3 - x u(x) \tag{2}$$

The above ODE (2) can now be solved as separable.

Writing the ode as

$$\begin{aligned} u'(x) &= x^3 u^3 - x u \\ u'(x) &= \omega(x, u) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_u - \xi_x) - \omega^2 \xi_u - \omega_x \xi - \omega_u \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 122: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, u) &= 0 \\
\eta(x, u) &= u^3 \mathrm{e}^{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, u) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u}\right) S(x, u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{u^3 \mathrm{e}^{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\mathrm{e}^{-x^2}}{2u^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,u) S_u}{R_x + \omega(x,u) R_u} \tag{2}$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x,u)$ is the right hand side of the original ode given by

$$\omega(x,u) = x^3 u^3 - xu$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_u &= 0 \\ S_x &= \frac{x\,\mathrm{e}^{-x^2}}{u^2} \\ S_u &= \frac{\mathrm{e}^{-x^2}}{u^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^3 \mathrm{e}^{-x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^3 \mathrm{e}^{-R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{(R^2+1)\,\mathrm{e}^{-R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, u coordinates. This results in

$$-\frac{\mathrm{e}^{-x^2}}{2u(x)^2} = -\frac{(x^2+1)\,\mathrm{e}^{-x^2}}{2} + c_1$$

Which simplifies to

$$-\frac{\mathrm{e}^{-x^2}}{2u(x)^2} = -\frac{(x^2+1)\,\mathrm{e}^{-x^2}}{2} + c_1$$

Substituting $u = -x + y$ in the above solution gives

$$-\frac{\mathrm{e}^{-x^2}}{2(-x+y)^2} = -\frac{(x^2+1)\,\mathrm{e}^{-x^2}}{2} + c_1$$

Summary

The solution(s) found are the following

$$-\frac{\mathrm{e}^{-x^2}}{2(-x+y)^2} = -\frac{(x^2+1)\,\mathrm{e}^{-x^2}}{2} + c_1 \tag{1}$$

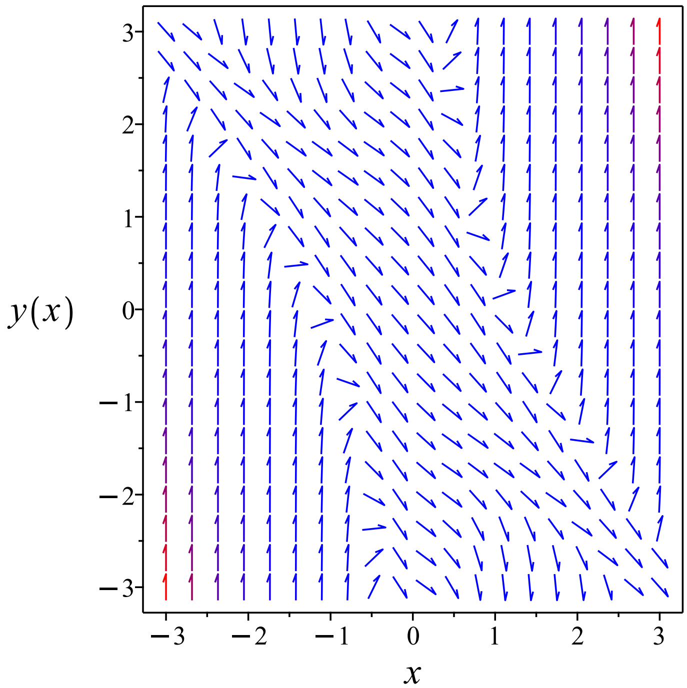


Figure 200: Slope field plot

Verification of solutions

$$-\frac{e^{-x^2}}{2\left(-x+y\right)^2} = -\frac{\left(x^2+1\right)e^{-x^2}}{2} + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
<- Abel successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 109

```
dsolve(diff(y(x),x)+x*(x+y(x))=x^3*(x+y(x))^3-1,y(x), singsol=all)
```

$$y(x) = -\frac{x\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1} + e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1}}$$
$$y(x) = \frac{-x\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1} + e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}x^2 + e^{-x^2} + c_1}}$$

✓ Solution by Mathematica

Time used: 10.062 (sec). Leaf size: 85

```
DSolve[y'[x]+x*(x+y[x])==x^3*(x+y[x])^3-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x - \frac{e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}(x^2+1) + c_1}}$$
$$y(x) \to -x + \frac{e^{-\frac{x^2}{2}}}{\sqrt{e^{-x^2}(x^2+1) + c_1}}$$
$$y(x) \to -x$$

4.25 problem 23 (e)

4.25.1 Solving as exact ode 971

Internal problem ID [5322]
Internal file name [OUTPUT/4813_Friday_February_02_2024_05_13_59_AM_76254145/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 6. Equations of first order and first degree (Linear equations). Supplemetary problems. Page 39
Problem number: 23 (e).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

$$\boxed{y + e^{y} + (1 + e^{y})\, y' = e^{-x}}$$

4.25.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y+1)\,\mathrm{d}y&=\left(-y-\mathrm{e}^y+\mathrm{e}^{-x}\right)\mathrm{d}x\\ \left(y+\mathrm{e}^y-\mathrm{e}^{-x}\right)\mathrm{d}x+(\mathrm{e}^y+1)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y+\mathrm{e}^y-\mathrm{e}^{-x}\\ N(x,y)&=\mathrm{e}^y+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y+\mathrm{e}^y-\mathrm{e}^{-x}\right)\\ &=\mathrm{e}^y+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\mathrm{e}^y+1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{\mathrm{e}^y+1}((\mathrm{e}^y+1)-(0))\\ &=1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x} \left(y + \mathrm{e}^{y} - \mathrm{e}^{-x}\right) \\
&= \mathrm{e}^{x+y} + y\,\mathrm{e}^{x} - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(\mathrm{e}^{y} + 1) \\
&= (\mathrm{e}^{y} + 1)\,\mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{x+y} + y\,\mathrm{e}^{x} - 1\right) + \left((\mathrm{e}^{y} + 1)\,\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \mathrm{e}^{x+y} + y\,\mathrm{e}^{x} - 1 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + y\,\mathrm{e}^{x} + \mathrm{e}^{x+y} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x+y} + \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = (\mathrm{e}^{y} + 1)\,\mathrm{e}^{x}$. Therefore equation (4) becomes

$$(\mathrm{e}^{y} + 1)\,\mathrm{e}^{x} = \mathrm{e}^{x+y} + \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + y\,\mathrm{e}^{x} + \mathrm{e}^{x+y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + y\,\mathrm{e}^{x} + \mathrm{e}^{x+y}$$

The solution becomes

$$y = -\left(\mathrm{LambertW}\left(\mathrm{e}^{-x+(x\,\mathrm{e}^{x}+c_1+x)\mathrm{e}^{-x}}\right)\mathrm{e}^{x} - c_1 - x\right)\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = -\left(\mathrm{LambertW}\left(\mathrm{e}^{-x+(x\,\mathrm{e}^{x}+c_1+x)\mathrm{e}^{-x}}\right)\mathrm{e}^{x} - c_1 - x\right)\mathrm{e}^{-x} \tag{1}$$

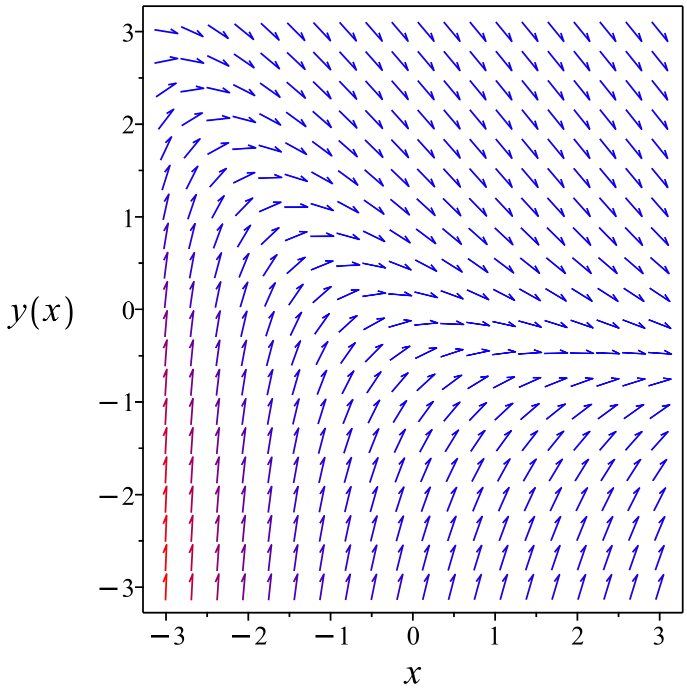


Figure 201: Slope field plot

Verification of solutions

$$y = -\left(\text{LambertW}\left(\mathrm{e}^{-x+(x\,\mathrm{e}^{x}+c_1+x)\mathrm{e}^{-x}}\right)\mathrm{e}^{x} - c_1 - x\right)\mathrm{e}^{-x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 29

```
dsolve((y(x)+exp(y(x))-exp(-x))+(1+exp(y(x)))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(\mathrm{e}^{\mathrm{e}^{-x}(x-c_1)}\right) + \mathrm{e}^{-x}(x - c_1)$$

✓ Solution by Mathematica

Time used: 6.265 (sec). Leaf size: 33

```
DSolve[(y[x]+Exp[y[x]]-Exp[-x])+(1+Exp[y[x]])*y'[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-x}\left(-e^x W\left(e^{e^{-x}(x+c_1)}\right) + x + c_1\right)$$

5 Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65

5.1 problem 17 978
5.2 problem 18 982
5.3 problem 19 986
5.4 problem 20 991
5.5 problem 21 1004
5.6 problem 22 1010
5.7 problem 23 1024
5.8 problem 24 1028
5.9 problem 25 1043
5.10 problem 26 1051
5.11 problem 27 1056
5.12 problem 28 1078
5.13 problem 29 1084
5.14 problem 30 1089
5.15 problem 31 1094

5.1 problem 17

5.1.1 Maple step by step solution 980

Internal problem ID [5323]
Internal file name [OUTPUT/4814_Friday_February_02_2024_05_14_01_AM_61480249/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 17.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{{y'}^2x^2 + yy'x - 6y^2 = 0}$$

The ode

$${y'}^2x^2 + yy'x - 6y^2 = 0$$

is factored to

$$(xy' + 3y)(-xy' + 2y) = 0$$

Which gives the following equations

$$xy' + 3y = 0 \tag{1}$$
$$-xy' + 2y = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{3y}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{3}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{3}{x}\,dx \\
\ln(y) &= -3\ln(x) + c_1 \\
y &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.
Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3}$$

Verified OK.
Solving ODE (2) In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{2y}{x}
\end{aligned}$$

Where $f(x) = \frac{2}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{2}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{2}{x}\,dx \\
\ln(y) &= 2\ln(x) + c_2 \\
y &= \mathrm{e}^{2\ln(x)+c_2} \\
&= c_2 x^2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^2$$

Verified OK.

Summary

The solution(s) found are the following

$$y = c_2 x^2 \tag{1}$$

Verification of solutions

$$y = c_2 x^2$$

Verified OK.

5.1.1 Maple step by step solution

Let's solve

$y'^2 x^2 + y y' x - 6y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{2}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = 2\ln(x) + c_1$
- Solve for y

  $y = x^2 e^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x)^2+x*y(x)*diff(y(x),x)-6*y(x)^2=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2$$
$$y(x) = \frac{c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 26

```
DSolve[x^2*(y'[x])^2+x*y[x]*y'[x]-6*y[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^3}$$
$$y(x) \to c_1 x^2$$
$$y(x) \to 0$$

5.2 problem 18

5.2.1 Maple step by step solution 984

Internal problem ID [5324]
Internal file name [`OUTPUT/4815_Friday_February_02_2024_05_14_02_AM_34816384/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 18.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"exact", "linear", "quadrature", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xy'^2 + (y - 1 - x^2)\, y' - x(-1 + y) = 0}$$

The ode

$$xy'^2 + (y - 1 - x^2)\, y' - x(-1 + y) = 0$$

is factored to

$$(y' - x)\,(y'x + y - 1) = 0$$

Which gives the following equations

$$y' - x = 0 \tag{1}$$
$$y'x + y - 1 = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) Integrating both sides gives

$$
\begin{aligned}
y &= \int x \,\mathrm{d}x \\
&= \frac{x^2}{2} + c_1
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^2}{2} + c_1 \tag{1}
$$

Verification of solutions

$$
y = \frac{x^2}{2} + c_1
$$

Verified OK.

Summary

The solution(s) found are the following

$$
y = \frac{x^2}{2} + c_1 \tag{1}
$$

Verification of solutions

$$
y = \frac{x^2}{2} + c_1
$$

Verified OK.

Solving ODE (2) In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{1-y}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(y) = 1 - y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{1-y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{1-y}\,dy &= \int \frac{1}{x}\,dx \\
-\ln(-1+y) &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$\frac{1}{-1+y} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{-1+y} = c_3 x$$

Which simplifies to

$$y = \frac{(c_3 \mathrm{e}^{c_2} x + 1)\,\mathrm{e}^{-c_2}}{c_3 x}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3 \mathrm{e}^{c_2} x + 1)\,\mathrm{e}^{-c_2}}{c_3 x} \tag{1}$$

Verification of solutions

$$y = \frac{(c_3 \mathrm{e}^{c_2} x + 1)\,\mathrm{e}^{-c_2}}{c_3 x}$$

Verified OK.

Summary

The solution(s) found are the following

$$y = \frac{(c_3 \mathrm{e}^{c_2} x + 1)\,\mathrm{e}^{-c_2}}{c_3 x} \tag{1}$$

Verification of solutions

$$y = \frac{(c_3 \mathrm{e}^{c_2} x + 1)\,\mathrm{e}^{-c_2}}{c_3 x}$$

Verified OK.

5.2.1 Maple step by step solution

Let's solve

$x{y'}^2 + (y - 1 - x^2)\,y' - x(-1+y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int \left(x{y'}^2 + (y - 1 - x^2)\, y' - x(-1 + y)\right) dx = \int 0 dx + c_1$
- Cannot compute integral

  $\int \left(x{y'}^2 + (y - 1 - x^2)\, y' - x(-1 + y)\right) dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)^2+(y(x)-1-x^2)*diff(y(x),x)-x*(y(x)-1)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + c_1$$
$$y(x) = \frac{x + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 32

```
DSolve[x*(y'[x])^2+(y[x]-1-x^2)*y'[x]-x*(y[x]-1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{2} + c_1$$
$$y(x) \to \frac{x + c_1}{x}$$
$$y(x) \to 1$$

5.3 problem 19

5.3.1 Solving as dAlembert ode 986

Internal problem ID [5325]
Internal file name [`OUTPUT/4816_Friday_February_02_2024_05_14_02_AM_44066862/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{{y'}^2 x - 2yy' = -4x}$$

5.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - 2yp = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2+4}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2+4}{2p} = x\left(1 - \frac{p^2+4}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2+4}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 2 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu p &= 0 \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary

The solution(s) found are the following

$$y = -2x \tag{1}$$

$$y = 2x \tag{2}$$

$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.274 (sec). Leaf size: 43

```
DSolve[x*(y'[x])^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x\cosh(-\log(x) + c_1)$$
$$y(x) \to -2x\cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

5.4 problem 20

Internal problem ID [5326]
Internal file name [`OUTPUT/4817_Friday_February_02_2024_05_14_02_AM_30867913/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 20.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{3x^4y'^2 - xy' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 12x^2y}}{6x^3} \tag{1}$$
$$y' = -\frac{-1 + \sqrt{1 + 12x^2y}}{6x^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{1 + \sqrt{12x^2y + 1}}{6x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(1+\sqrt{12x^2y+1}\right)(b_3-a_2)}{6x^3} - \frac{\left(1+\sqrt{12x^2y+1}\right)^2 a_3}{36x^6} \\ &- \left(-\frac{1+\sqrt{12x^2y+1}}{2x^4} + \frac{2y}{x^2\sqrt{12x^2y+1}}\right)(x a_2 + y a_3 + a_1) \\ &- \frac{x b_2 + y b_3 + b_1}{x\sqrt{12x^2y+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36b_2x^6\sqrt{12x^2y+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 + 36x^5b_1 - 144x^4ya_1 - 12\sqrt{12x^2y+1}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &36b_2x^6\sqrt{12x^2y+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 \\ &- 36x^5b_1 + 144x^4ya_1 + 12\sqrt{12x^2y+1}\,x^3a_2 + 6\sqrt{12x^2y+1}\,x^3b_3 \\ &+ 18\sqrt{12x^2y+1}\,x^2ya_3 - \left(12x^2y+1\right)^{\frac{3}{2}} a_3 + 18\sqrt{12x^2y+1}\,x^2a_1 \\ &+ 12x^3a_2 + 6x^3b_3 - 6x^2ya_3 + 18x^2a_1 - a_3\sqrt{12x^2y+1} - 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &36b_2x^6\sqrt{12x^2y+1} - 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 72x^4y^2a_3 \\ &+ 12\left(12x^2y+1\right)x^3a_2 + 6\left(12x^2y+1\right)x^3b_3 + 18\left(12x^2y+1\right)x^2ya_3 \\ &- 36x^5b_1 - 72x^4ya_1 + 18\left(12x^2y+1\right)x^2a_1 + 12\sqrt{12x^2y+1}\,x^3a_2 \\ &+ 6\sqrt{12x^2y+1}\,x^3b_3 + 18\sqrt{12x^2y+1}\,x^2ya_3 - \left(12x^2y+1\right)^{\frac{3}{2}} a_3 \\ &+ 18\sqrt{12x^2y+1}\,x^2a_1 - 2\left(12x^2y+1\right)a_3 - a_3\sqrt{12x^2y+1} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&36 b_{2} x^{6} \sqrt{12 x^{2} y+1}-36 x^{6} b_{2}+72 x^{5} y a_{2}+36 x^{5} y b_{3}+144 x^{4} y^{2} a_{3}-36 x^{5} b_{1}+144 x^{4} y a_{1} \\
&+12 \sqrt{12 x^{2} y+1}\, x^{3} a_{2}+6 \sqrt{12 x^{2} y+1}\, x^{3} b_{3}+6 \sqrt{12 x^{2} y+1}\, x^{2} y a_{3}+12 x^{3} a_{2} \\
&+6 x^{3} b_{3}+18 \sqrt{12 x^{2} y+1}\, x^{2} a_{1}-6 x^{2} y a_{3}+18 x^{2} a_{1}-2 a_{3} \sqrt{12 x^{2} y+1}-2 a_{3}=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{12 x^{2} y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_{1}, y=v_{2}, \sqrt{12 x^{2} y+1}=v_{3}\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36 b_{2} v_{1}^{6} v_{3}+72 v_{1}^{5} v_{2} a_{2}+144 v_{1}^{4} v_{2}^{2} a_{3}-36 v_{1}^{6} b_{2}+36 v_{1}^{5} v_{2} b_{3}+144 v_{1}^{4} v_{2} a_{1} \\
&-36 v_{1}^{5} b_{1}+12 v_{3} v_{1}^{3} a_{2}+6 v_{3} v_{1}^{2} v_{2} a_{3}+6 v_{3} v_{1}^{3} b_{3}+18 v_{3} v_{1}^{2} a_{1} \\
&+12 v_{1}^{3} a_{2}-6 v_{1}^{2} v_{2} a_{3}+6 v_{1}^{3} b_{3}+18 v_{1}^{2} a_{1}-2 a_{3} v_{3}-2 a_{3}=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_{1}, v_{2}, v_{3}\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36 b_{2} v_{1}^{6} v_{3}-36 v_{1}^{6} b_{2}+\left(72 a_{2}+36 b_{3}\right) v_{1}^{5} v_{2}-36 v_{1}^{5} b_{1}+144 v_{1}^{4} v_{2}^{2} a_{3} \\
&+144 v_{1}^{4} v_{2} a_{1}+\left(12 a_{2}+6 b_{3}\right) v_{1}^{3} v_{3}+\left(12 a_{2}+6 b_{3}\right) v_{1}^{3}+6 v_{3} v_{1}^{2} v_{2} a_{3} \\
&-6 v_{1}^{2} v_{2} a_{3}+18 v_{3} v_{1}^{2} a_{1}+18 v_{1}^{2} a_{1}-2 a_{3} v_{3}-2 a_{3}=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
18a_1 &= 0 \\
144a_1 &= 0 \\
-6a_3 &= 0 \\
-2a_3 &= 0 \\
6a_3 &= 0 \\
144a_3 &= 0 \\
-36b_1 &= 0 \\
-36b_2 &= 0 \\
36b_2 &= 0 \\
12a_2 + 6b_3 &= 0 \\
72a_2 + 36b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(\frac{1 + \sqrt{12x^2y + 1}}{6x^3}\right)(x) \\
&= \frac{-12x^2y - \sqrt{12x^2y + 1} - 1}{6x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12x^2y-\sqrt{12x^2y+1}-1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{12x^2y+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{12x^2y+1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{12x^2y+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{12x^2y+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+12x^2y}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= -\frac{-1+\sqrt{12x^2y+1}}{6x^3} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{\left(-1+\sqrt{12x^2y+1}\right)(b_3 - a_2)}{6x^3} - \frac{\left(-1+\sqrt{12x^2y+1}\right)^2 a_3}{36x^6} \\ &- \left(-\frac{2y}{x^2\sqrt{12x^2y+1}} + \frac{-1+\sqrt{12x^2y+1}}{2x^4}\right)(xa_2 + ya_3 + a_1) \\ &+ \frac{xb_2 + yb_3 + b_1}{x\sqrt{12x^2y+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-36b_2x^6\sqrt{12x^2y+1} - 36x^6b_2 + 72x^5ya_2 + 36x^5yb_3 + 144x^4y^2a_3 - 36x^5b_1 + 144x^4ya_1 - 12\sqrt{12x^2y+1}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &36b_2x^6\sqrt{12x^2y+1} + 36x^6b_2 - 72x^5ya_2 - 36x^5yb_3 - 144x^4y^2a_3 \\ &+ 36x^5b_1 - 144x^4ya_1 + 12\sqrt{12x^2y+1}\,x^3a_2 + 6\sqrt{12x^2y+1}\,x^3b_3 \\ &+ 18\sqrt{12x^2y+1}\,x^2ya_3 - \left(12x^2y+1\right)^{\frac{3}{2}}a_3 + 18\sqrt{12x^2y+1}\,x^2a_1 \\ &- 12x^3a_2 - 6x^3b_3 + 6x^2ya_3 - 18x^2a_1 - a_3\sqrt{12x^2y+1} + 2a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12x^2y+1}+36x^6b_2+72x^5ya_2+36x^5yb_3+72x^4y^2a_3\\
&-12\left(12x^2y+1\right)x^3a_2-6\left(12x^2y+1\right)x^3b_3-18\left(12x^2y+1\right)x^2ya_3\\
&+36x^5b_1+72x^4ya_1-18\left(12x^2y+1\right)x^2a_1+12\sqrt{12x^2y+1}\,x^3a_2\\
&+6\sqrt{12x^2y+1}\,x^3b_3+18\sqrt{12x^2y+1}\,x^2ya_3-\left(12x^2y+1\right)^{\frac{3}{2}}a_3\\
&+18\sqrt{12x^2y+1}\,x^2a_1+2\left(12x^2y+1\right)a_3-a_3\sqrt{12x^2y+1}=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&36b_2x^6\sqrt{12x^2y+1}+36x^6b_2-72x^5ya_2-36x^5yb_3-144x^4y^2a_3+36x^5b_1-144x^4ya_1\\
&+12\sqrt{12x^2y+1}\,x^3a_2+6\sqrt{12x^2y+1}\,x^3b_3+6\sqrt{12x^2y+1}\,x^2ya_3-12x^3a_2\\
&-6x^3b_3+18\sqrt{12x^2y+1}\,x^2a_1+6x^2ya_3-18x^2a_1-2a_3\sqrt{12x^2y+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{12x^2y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{12x^2y+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3-72v_1^5v_2a_2-144v_1^4v_2^2a_3+36v_1^6b_2-36v_1^5v_2b_3-144v_1^4v_2a_1\\
&+36v_1^5b_1+12v_3v_1^3a_2+6v_3v_1^2v_2a_3+6v_3v_1^3b_3+18v_3v_1^2a_1\\
&-12v_1^3a_2+6v_1^2v_2a_3-6v_1^3b_3-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36b_2v_1^6v_3+36v_1^6b_2+(-72a_2-36b_3)\,v_1^5v_2+36v_1^5b_1-144v_1^4v_2^2a_3\\
&-144v_1^4v_2a_1+(12a_2+6b_3)\,v_1^3v_3+(-12a_2-6b_3)\,v_1^3\\
&+6v_3v_1^2v_2a_3+6v_1^2v_2a_3+18v_3v_1^2a_1-18v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-144a_1 &= 0 \\
-18a_1 &= 0 \\
18a_1 &= 0 \\
-144a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
36b_1 &= 0 \\
36b_2 &= 0 \\
-72a_2 - 36b_3 &= 0 \\
-12a_2 - 6b_3 &= 0 \\
12a_2 + 6b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= -2y - \left(-\frac{-1 + \sqrt{12x^2y + 1}}{6x^3}\right)(x) \\
&= \frac{-12x^2y + \sqrt{12x^2y + 1} - 1}{6x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-12x^2y+\sqrt{12x^2y+1}-1}{6x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln (y)}{2} + \operatorname{arctanh}\left(\sqrt{12x^2y+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{12x^2y+1}}{6x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{12x^2y+1}} \\ S_y &= \frac{-\frac{1}{\sqrt{12x^2y+1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1 + 12x^2y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 97

```
dsolve(3*x^4*diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{1}{12x^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$
$$y(x) = \frac{-i\sqrt{3}\,c_1 - 3x}{3x\,c_1^2}$$

✓ Solution by Mathematica

Time used: 0.512 (sec). Leaf size: 123

```
DSolve[3*x^4*y'[x]^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$\text{Solve}\left[\frac{x\sqrt{12x^2y(x)+1}\text{arctanh}\left(\sqrt{12x^2y(x)+1}\right)}{\sqrt{12x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right]$$
$$y(x) \to 0$$

5.5 problem 21

5.5.1 Solving as dAlembert ode 1004

Internal problem ID [5327]
Internal file name [`OUTPUT/4818_Friday_February_02_2024_05_14_06_AM_65941968/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 21.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{8y{y'}^2-2xy'+y=0}$$

5.5.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$8y\,p^2-2xp+y=0$$

Solving for y from the above results in

$$y=\frac{2xp}{8p^2+1} \tag{1A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p&=f+(xf'+g')\frac{dp}{dx} \\ p-f&=(xf'+g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{2p}{8p^2 + 1} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{2p}{8p^2 + 1} = x\left(\frac{2}{8p^2 + 1} - \frac{32p^2}{\left(8p^2 + 1\right)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{2p}{8p^2 + 1} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= \frac{\sqrt{2}}{4} \\
p &= -\frac{\sqrt{2}}{4}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -\frac{x\sqrt{2}}{4} \\
y &= \frac{x\sqrt{2}}{4}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{2p(x)}{8p(x)^2+1}}{x\left(\frac{2}{8p(x)^2+1} - \frac{32p(x)^2}{\left(8p(x)^2+1\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2}{8p^2+1} - \frac{32p^2}{\left(8p^2+1\right)^2}\right)}{p - \frac{2p}{8p^2+1}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$\begin{aligned} p(p) &= \frac{2}{8p^3 + p} \\ q(p) &= 0 \end{aligned}$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{8p^3 + p} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{2}{8p^3+p}dp} \\ &= e^{-\ln(8p^2+1)+2\ln(p)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{p^2}{8p^2 + 1}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{p^2 x}{8p^2 + 1}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{p^2 x}{8p^2 + 1} = c_3$$

Dividing both sides by the integrating factor $\mu = \frac{p^2}{8p^2+1}$ results in

$$x(p) = \frac{c_3(8p^2 + 1)}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= \frac{x+\sqrt{x^2-8y^2}}{8y} \\
p &= -\frac{-x+\sqrt{x^2-8y^2}}{8y}
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
\begin{aligned}
x &= \frac{16c_3 x}{x+\sqrt{x^2-8y^2}} \\
x &= -\frac{16c_3 x}{-x+\sqrt{x^2-8y^2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = -\frac{x\sqrt{2}}{4} \tag{2}$$

$$y = \frac{x\sqrt{2}}{4} \tag{3}$$

$$x = \frac{16c_3 x}{x+\sqrt{x^2-8y^2}} \tag{4}$$

$$x = -\frac{16c_3 x}{-x+\sqrt{x^2-8y^2}} \tag{5}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -\frac{x\sqrt{2}}{4}$$

Verified OK.

$$y = \frac{x\sqrt{2}}{4}$$

Verified OK.

$$x = \frac{16c_3x}{x + \sqrt{x^2 - 8y^2}}$$

Verified OK.

$$x = -\frac{16c_3x}{-x + \sqrt{x^2 - 8y^2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.187 (sec). Leaf size: 95

```
dsolve(8*y(x)*diff(y(x),x)^2-2*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\, x}{4}$$
$$y(x) = \frac{\sqrt{2}\, x}{4}$$
$$y(x) = 0$$
$$\ln(x) + \operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{8y(x)^2-x^2}{x^2}}}\right) + \ln\left(\frac{y(x)}{x}\right) - c_1 = 0$$
$$\ln(x) - \operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{8y(x)^2-x^2}{x^2}}}\right) + \ln\left(\frac{y(x)}{x}\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.347 (sec). Leaf size: 174

```
DSolve[8*y[x]*y'[x]^2-2*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{4c_1}\sqrt{e^{8c_1} - 2ix}}{2\sqrt{2}}$$
$$y(x) \to \frac{e^{4c_1}\sqrt{e^{8c_1} - 2ix}}{2\sqrt{2}}$$
$$y(x) \to -\frac{e^{4c_1}\sqrt{2ix + e^{8c_1}}}{2\sqrt{2}}$$
$$y(x) \to \frac{e^{4c_1}\sqrt{2ix + e^{8c_1}}}{2\sqrt{2}}$$
$$y(x) \to 0$$
$$y(x) \to -\frac{x}{2\sqrt{2}}$$
$$y(x) \to \frac{x}{2\sqrt{2}}$$

5.6 problem 22

Internal problem ID [5328]
Internal file name [`OUTPUT/4819_Friday_February_02_2024_05_14_06_AM_46240253/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 22.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{y^2 {y'}^2 + 3xy' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-3x + \sqrt{9x^2 + 4y^3}}{2y^2} \tag{1}$$

$$y' = -\frac{3x + \sqrt{9x^2 + 4y^3}}{2y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{\left(-3x + \sqrt{4y^3 + 9x^2}\right)(b_3 - a_2)}{2y^2} - \frac{\left(-3x + \sqrt{4y^3 + 9x^2}\right)^2 a_3}{4y^4} \\ & - \frac{\left(-3 + \frac{9x}{\sqrt{4y^3+9x^2}}\right)(x a_2 + y a_3 + a_1)}{2y^2} \\ & - \left(-\frac{-3x + \sqrt{4y^3 + 9x^2}}{y^3} + \frac{3}{\sqrt{4y^3 + 9x^2}}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2 y^4 \sqrt{4y^3 + 9x^2} - 4x\, y^4 b_2 + 8y^5 a_2 - 12y^5 b_3 + 12\sqrt{4y^3 + 9x^2}\, x^2 y b_2 - 12\sqrt{4y^3 + 9x^2}\, x\, y^2 a_2 + 18\sqrt{4y^3}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4b_2 y^4 \sqrt{4y^3 + 9x^2} + 4x\, y^4 b_2 - 8y^5 a_2 + 12y^5 b_3 - 12\sqrt{4y^3 + 9x^2}\, x^2 y b_2 \\ & + 12\sqrt{4y^3 + 9x^2}\, x\, y^2 a_2 - 18\sqrt{4y^3 + 9x^2}\, x\, y^2 b_3 + 6\sqrt{4y^3 + 9x^2}\, y^3 a_3 \\ & + 36x^3 y b_2 - 36x^2 y^2 a_2 + 54x^2 y^2 b_3 + 6x\, y^3 a_3 + 4y^4 b_1 \\ & - \left(4y^3 + 9x^2\right)^{\frac{3}{2}} a_3 - 9\sqrt{4y^3 + 9x^2}\, x^2 a_3 - 12\sqrt{4y^3 + 9x^2}\, x y b_1 \\ & + 6\sqrt{4y^3 + 9x^2}\, y^2 a_1 + 54x^3 a_3 + 36x^2 y b_1 - 18x\, y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2y^4\sqrt{4y^3+9x^2}-12x\,y^4b_2-12y^5b_3+4\left(4y^3+9x^2\right)xyb_2\\
&-2\left(4y^3+9x^2\right)y^2a_2+6\left(4y^3+9x^2\right)y^2b_3-12\sqrt{4y^3+9x^2}\,x^2yb_2\\
&+12\sqrt{4y^3+9x^2}\,x\,y^2a_2-18\sqrt{4y^3+9x^2}\,x\,y^2b_3+6\sqrt{4y^3+9x^2}\,y^3a_3\\
&-18x^2y^2a_2-18x\,y^3a_3-12y^4b_1-\left(4y^3+9x^2\right)^{\frac{3}{2}}a_3\\
&+6\left(4y^3+9x^2\right)xa_3+4\left(4y^3+9x^2\right)yb_1-9\sqrt{4y^3+9x^2}\,x^2a_3\\
&-12\sqrt{4y^3+9x^2}\,xyb_1+6\sqrt{4y^3+9x^2}\,y^2a_1-18x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4x\,y^4b_2+4b_2y^4\sqrt{4y^3+9x^2}-8y^5a_2+12y^5b_3+36x^3yb_2-12\sqrt{4y^3+9x^2}\,x^2yb_2\\
&-36x^2y^2a_2+54x^2y^2b_3+12\sqrt{4y^3+9x^2}\,x\,y^2a_2-18\sqrt{4y^3+9x^2}\,x\,y^2b_3\\
&+6x\,y^3a_3+2\sqrt{4y^3+9x^2}\,y^3a_3+4y^4b_1+54x^3a_3-18\sqrt{4y^3+9x^2}\,x^2a_3\\
&+36x^2yb_1-12\sqrt{4y^3+9x^2}\,xyb_1-18x\,y^2a_1+6\sqrt{4y^3+9x^2}\,y^2a_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{4y^3+9x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{4y^3+9x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-8v_2^5a_2+4v_1v_2^4b_2+4b_2v_2^4v_3+12v_2^5b_3-36v_1^2v_2^2a_2+12v_3v_1v_2^2a_2+6v_1v_2^3a_3\\
&+2v_3v_2^3a_3+4v_2^4b_1+36v_1^3v_2b_2-12v_3v_1^2v_2b_2+54v_1^2v_2^2b_3-18v_3v_1v_2^2b_3\\
&-18v_1v_2^2a_1+6v_3v_2^2a_1+54v_1^3a_3-18v_3v_1^2a_3+36v_1^2v_2b_1-12v_3v_1v_2b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36v_1^3v_2b_2 + 54v_1^3a_3 + (-36a_2 + 54b_3)\, v_1^2v_2^2 - 12v_3v_1^2v_2b_2 + 36v_1^2v_2b_1 \\
&- 18v_3v_1^2a_3 + 4v_1v_2^4b_2 + 6v_1v_2^3a_3 + (12a_2 - 18b_3)\, v_1v_2^2v_3 - 18v_1v_2^2a_1 \\
&- 12v_3v_1v_2b_1 + (-8a_2 + 12b_3)\, v_2^5 + 4b_2v_2^4v_3 + 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-18a_1 &= 0 \\
6a_1 &= 0 \\
-18a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
54a_3 &= 0 \\
-12b_1 &= 0 \\
4b_1 &= 0 \\
36b_1 &= 0 \\
-12b_2 &= 0 \\
4b_2 &= 0 \\
36b_2 &= 0 \\
-36a_2 + 54b_3 &= 0 \\
-8a_2 + 12b_3 &= 0 \\
12a_2 - 18b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3x}{2} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}\right)\left(\frac{3x}{2}\right) \\
&= \frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2}\,x}{4y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^3+9x^2-3\sqrt{4y^3+9x^2}\,x}{4y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(y) - \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3+9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{\sqrt{4y^3 + 9x^2}} \\
S_y &= \frac{1 + \frac{3x}{\sqrt{4y^3+9x^2}}}{y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \tag{1}$$

Verification of solutions

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned} y' &= -\frac{\sqrt{4y^3+9x^2}+3x}{2y^2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{\left(\sqrt{4y^3+9x^2}+3x\right)(b_3-a_2)}{2y^2} - \frac{\left(\sqrt{4y^3+9x^2}+3x\right)^2 a_3}{4y^4} \\
& + \frac{\left(\frac{9x}{\sqrt{4y^3+9x^2}}+3\right)(xa_2+ya_3+a_1)}{2y^2} \\
& - \left(-\frac{3}{\sqrt{4y^3+9x^2}} + \frac{\sqrt{4y^3+9x^2}+3x}{y^3}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2\sqrt{4y^3+9x^2}\,y^4 + 4x\,y^4 b_2 - 8y^5 a_2 + 12y^5 b_3 + 12\sqrt{4y^3+9x^2}\,x^2 y b_2 - 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 + 18\sqrt{4y^3}}{}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
& 4b_2\sqrt{4y^3+9x^2}\,y^4 - 4x\,y^4 b_2 + 8y^5 a_2 - 12y^5 b_3 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
& + 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 + 6\sqrt{4y^3+9x^2}\,y^3 a_3 \\
& - 36x^3 y b_2 + 36x^2 y^2 a_2 - 54x^2 y^2 b_3 - 6x\,y^3 a_3 - 4y^4 b_1 \\
& - \left(4y^3+9x^2\right)^{\frac{3}{2}} a_3 - 9\sqrt{4y^3+9x^2}\,x^2 a_3 - 12\sqrt{4y^3+9x^2}\,xyb_1 \\
& + 6\sqrt{4y^3+9x^2}\,y^2 a_1 - 54x^3 a_3 - 36x^2 y b_1 + 18x\,y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& 4b_2\sqrt{4y^3+9x^2}\,y^4 + 12x\,y^4 b_2 + 12y^5 b_3 - 4\left(4y^3+9x^2\right) xyb_2 \\
& + 2\left(4y^3+9x^2\right) y^2 a_2 - 6\left(4y^3+9x^2\right) y^2 b_3 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
& + 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 + 6\sqrt{4y^3+9x^2}\,y^3 a_3 \\
& + 18x^2 y^2 a_2 + 18x\,y^3 a_3 + 12y^4 b_1 - \left(4y^3+9x^2\right)^{\frac{3}{2}} a_3 \\
& - 6\left(4y^3+9x^2\right) xa_3 - 4\left(4y^3+9x^2\right) yb_1 - 9\sqrt{4y^3+9x^2}\,x^2 a_3 \\
& - 12\sqrt{4y^3+9x^2}\,xyb_1 + 6\sqrt{4y^3+9x^2}\,y^2 a_1 + 18x\,y^2 a_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
& -4x\,y^4 b_2 + 4b_2\sqrt{4y^3+9x^2}\,y^4 + 8y^5 a_2 - 12y^5 b_3 - 36x^3 y b_2 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
& + 36x^2 y^2 a_2 - 54x^2 y^2 b_3 + 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 \\
& - 6x\,y^3 a_3 + 2\sqrt{4y^3+9x^2}\,y^3 a_3 - 4y^4 b_1 - 54x^3 a_3 - 18\sqrt{4y^3+9x^2}\,x^2 a_3 \\
& - 36x^2 y b_1 - 12\sqrt{4y^3+9x^2}\,xyb_1 + 18x\,y^2 a_1 + 6\sqrt{4y^3+9x^2}\,y^2 a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4y^3+9x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{4y^3+9x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&8v_2^5a_2 - 4v_1v_2^4b_2 + 4b_2v_3v_2^4 - 12v_2^5b_3 + 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 - 6v_1v_2^3a_3 \\ &+ 2v_3v_2^3a_3 - 4v_2^4b_1 - 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 - 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\ &+ 18v_1v_2^2a_1 + 6v_3v_2^2a_1 - 54v_1^3a_3 - 18v_3v_1^2a_3 - 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&-36v_1^3v_2b_2 - 54v_1^3a_3 + (36a_2 - 54b_3)\, v_1^2v_2^2 - 12v_3v_1^2v_2b_2 - 36v_1^2v_2b_1 \\ &- 18v_3v_1^2a_3 - 4v_1v_2^4b_2 - 6v_1v_2^3a_3 + (12a_2 - 18b_3)\, v_1v_2^2v_3 + 18v_1v_2^2a_1 \\ &- 12v_3v_1v_2b_1 + (8a_2 - 12b_3)\, v_2^5 + 4b_2v_3v_2^4 - 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_1 &= 0 \\
18a_1 &= 0 \\
-54a_3 &= 0 \\
-18a_3 &= 0 \\
-6a_3 &= 0 \\
2a_3 &= 0 \\
-36b_1 &= 0 \\
-12b_1 &= 0 \\
-4b_1 &= 0 \\
-36b_2 &= 0 \\
-12b_2 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
8a_2 - 12b_3 &= 0 \\
12a_2 - 18b_3 &= 0 \\
36a_2 - 54b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3x}{2} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}\right)\left(\frac{3x}{2}\right) \\
&= \frac{4y^3 + 3\sqrt{4y^3 + 9x^2}\,x + 9x^2}{4y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^3 + 3\sqrt{4y^3 + 9x^2}\,x + 9x^2}{4y^2}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) + \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 - \frac{3x}{\sqrt{4y^3+9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$
\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \tag{1}
$$

Verification of solutions

$$
\ln(y) + \frac{2 \operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[3/2*x, y], [2/3*y^3+3*x^2, y*x]
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 126

```
dsolve(y(x)^2*diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{18^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}\left(3i3^{\frac{1}{6}}+3^{\frac{2}{3}}\right)}{4}$$
$$y(x) = \frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}\left(-3^{\frac{2}{3}}+3i3^{\frac{1}{6}}\right)}{4}$$
$$y(x) = 0$$
$$y(x) = \operatorname{RootOf}\left(-2\ln(x)+3\left(\int^{\_Z}-\frac{4\_a^3-3\sqrt{4\_a^3+9}+9}{\_a(4\_a^3+9)}d\_a\right)+2c_1\right)x^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.574 (sec). Leaf size: 239

```
DSolve[y[x]^2*y'[x]^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to -\sqrt[3]{-1}e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to (-1)^{2/3}e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to -\sqrt[3]{-1}e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to (-1)^{2/3}e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -\left(-\frac{3}{2}\right)^{2/3}x^{2/3}$$
$$y(x) \to -\left(\frac{3}{2}\right)^{2/3}x^{2/3}$$
$$y(x) \to \frac{\sqrt[3]{-1}3^{2/3}x^{2/3}}{2^{2/3}}$$

5.7 problem 23

5.7.1 Solving as clairaut ode 1024

Internal problem ID [5329]
Internal file name [`OUTPUT/4820_Friday_February_02_2024_05_14_11_AM_82456155/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 23.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Clairaut]
```

$$\boxed{y'^2 - xy' + y = 0}$$

5.7.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -p^2 + xp \\ &= -p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -p^2$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - 2p \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{2}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{4}$$

Summary

The solution(s) found are the following

$$y = -c_1^2 + c_1 x \tag{1}$$

$$y = \frac{x^2}{4} \tag{2}$$

Verification of solutions

$$y = -c_1^2 + c_1 x$$

Verified OK.

$$y = \frac{x^2}{4}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4}$$
$$y(x) = c_1(x - c_1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[y'[x]^2-x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x - c_1)$$
$$y(x) \to \frac{x^2}{4}$$

5.8 problem 24

Internal problem ID [5330]
Internal file name [`OUTPUT/4821_Friday_February_02_2024_05_14_11_AM_7693094/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 24.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _rational]`

$$\boxed{16y^3 {y'}^2 - 4xy' + y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{x + \sqrt{x^2 - 4y^4}}{8y^3} \tag{1}$$

$$y' = -\frac{-x + \sqrt{x^2 - 4y^4}}{8y^3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$y' = \frac{\sqrt{-(2y^2 - x)(2y^2 + x)} + x}{8y^3}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}+x\right)\left(b_3-a_2\right)}{8y^3} \\ &- \frac{\left(\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}+x\right)^2 a_3}{64y^6} \\ &- \frac{\left(\frac{x}{\sqrt{-(2y^2-x)(2y^2+x)}}+1\right)\left(xa_2+ya_3+a_1\right)}{8y^3} \\ &- \left(\frac{-4y(2y^2+x)-4(2y^2-x)\,y}{16y^3\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}}\right. \\ &\left. - \frac{3\left(\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}+x\right)}{8y^4}\right)\left(xb_2+yb_3+b_1\right) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-64b_2y^6\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}+32x\,y^6b_2-32y^7a_2+64y^7b_3+32y^6b_1-24\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x^2y}{}$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&64 b_2 y^6 \sqrt{(-2y^2+x)(2y^2+x)} - 32 x y^6 b_2 + 32 y^7 a_2 - 64 y^7 b_3 - 32 y^6 b_1 \\
&+ 24\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 y^2 b_2 - 16\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 a_2 \\
&+ 32\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 b_3 - 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^4 a_3 \\
&+ 24 x^3 y^2 b_2 - 16 x^2 y^3 a_2 + 32 x^2 y^3 b_3 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x y^2 b_1 \\
&- 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^3 a_1 + 24 x^2 y^2 b_1 - 8 x y^3 a_1 \\
&- \left((-2y^2+x)(2y^2+x)\right)^{\frac{3}{2}} a_3 - \sqrt{(-2y^2+x)(2y^2+x)}\, x^2 a_3 - 2 x^3 a_3 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&64 b_2 y^6 \sqrt{(-2y^2+x)(2y^2+x)} + 64 x y^6 b_2 + 64 y^7 b_3 + 64 y^6 b_1 \\
&- 24(2y^2-x)(2y^2+x)\, x y^2 b_2 + 8(2y^2-x)(2y^2+x)\, y^3 a_2 \\
&- 32(2y^2-x)(2y^2+x)\, y^3 b_3 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 y^2 b_2 \\
&- 16\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 a_2 \\
&+ 32\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 b_3 - 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^4 a_3 \\
&- 8 x^2 y^3 a_2 - 8 x y^4 a_3 - 24(2y^2-x)(2y^2+x)\, y^2 b_1 \\
&+ 24\sqrt{(-2y^2+x)(2y^2+x)}\, x y^2 b_1 - 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^3 a_1 \\
&- 8 x y^3 a_1 - \left((-2y^2+x)(2y^2+x)\right)^{\frac{3}{2}} a_3 \\
&+ 2(2y^2-x)(2y^2+x)\, x a_3 - \sqrt{(-2y^2+x)(2y^2+x)}\, x^2 a_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-32 x y^6 b_2 + 64 b_2 y^6 \sqrt{(-2y^2+x)(2y^2+x)} + 32 y^7 a_2 - 64 y^7 b_3 \\
&- 32 y^6 b_1 + 24 x^3 y^2 b_2 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 y^2 b_2 \\
&- 16 x^2 y^3 a_2 + 32 x^2 y^3 b_3 - 16\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 a_2 \\
&+ 32\sqrt{(-2y^2+x)(2y^2+x)}\, x y^3 b_3 - 4\sqrt{(-2y^2+x)(2y^2+x)}\, y^4 a_3 \\
&+ 24 x^2 y^2 b_1 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x y^2 b_1 - 8 x y^3 a_1 \\
&- 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^3 a_1 - 2 x^3 a_3 - 2\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 a_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{(-2y^2+x)(2y^2+x)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{(-2y^2+x)(2y^2+x)} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{align*}
&32v_2^7a_2 - 32v_1v_2^6b_2 + 64b_2v_2^6v_3 - 64v_2^7b_3 - 32v_2^6b_1 - 16v_1^2v_2^3a_2 - 16v_3v_1v_2^3a_2 \\
&- 4v_3v_2^4a_3 + 24v_1^3v_2^2b_2 + 24v_3v_1^2v_2^2b_2 + 32v_1^2v_2^3b_3 + 32v_3v_1v_2^3b_3 \\
&- 8v_1v_2^3a_1 - 8v_3v_2^3a_1 + 24v_1^2v_2^2b_1 + 24v_3v_1v_2^2b_1 - 2v_1^3a_3 - 2v_3v_1^2a_3 = 0
\end{align*} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{align*}
&24v_1^3v_2^2b_2 - 2v_1^3a_3 + (-16a_2 + 32b_3)\,v_1^2v_2^3 + 24v_3v_1^2v_2^2b_2 + 24v_1^2v_2^2b_1 \\
&- 2v_3v_1^2a_3 - 32v_1v_2^6b_2 + (-16a_2 + 32b_3)\,v_1v_2^3v_3 - 8v_1v_2^3a_1 + 24v_3v_1v_2^2b_1 \\
&+ (32a_2 - 64b_3)\,v_2^7 + 64b_2v_2^6v_3 - 32v_2^6b_1 - 4v_3v_2^4a_3 - 8v_3v_2^3a_1 = 0
\end{align*} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{align*}
-8a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-32b_1 &= 0 \\
24b_1 &= 0 \\
-32b_2 &= 0 \\
24b_2 &= 0 \\
64b_2 &= 0 \\
-16a_2 + 32b_3 &= 0 \\
32a_2 - 64b_3 &= 0
\end{align*}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{\sqrt{-\left(2y^2 - x\right)\left(2y^2 + x\right)} + x}{8y^3}\right)(2x) \\
&= \frac{4y^4 - \sqrt{-4y^4 + x^2}\,x - x^2}{4y^3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^4 - \sqrt{-4y^4 + x^2}\,x - x^2}{4y^3}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln (4y^4 - x^2)}{4} - \frac{\ln (2y^2 - x)}{4} + \ln (y) - \frac{\ln (2y^2 + x)}{4} + \frac{x \ln \left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{-4y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\sqrt{-\left(2y^2 - x\right)\left(2y^2 + x\right)} + x}{8y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x + \sqrt{-4y^4 + x^2}}{2\sqrt{-4y^4 + x^2}\, x} \\
S_y &= -\frac{4y^3}{\sqrt{-4y^4 + x^2}\,\left(x + \sqrt{-4y^4 + x^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R)}{2} + c_1 \tag{4}$$

Summary

The solution(s) found are the following

$$\frac{\ln(2)}{2}+\frac{\ln(x)}{2}+\frac{\ln\left(x+\sqrt{x^2-4y^4}\right)}{2}=\frac{\ln(x)}{2}+c_1 \tag{1}$$

Verification of solutions

$$\frac{\ln(2)}{2}+\frac{\ln(x)}{2}+\frac{\ln\left(x+\sqrt{x^2-4y^4}\right)}{2}=\frac{\ln(x)}{2}+c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y'=-\frac{\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}-x}{8y^3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2-\frac{\left(\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}-x\right)\left(b_3-a_2\right)}{8y^3} \\
& -\frac{\left(\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}-x\right)^2 a_3}{64y^6} \\
& +\frac{\left(\frac{x}{\sqrt{-(2y^2-x)(2y^2+x)}}-1\right)\left(xa_2+ya_3+a_1\right)}{8y^3} \\
& -\left(-\frac{-4y(2y^2+x)-4(2y^2-x)\,y}{16y^3\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}}\right. \\
& \left.+\frac{\frac{3\sqrt{-(2y^2-x)(2y^2+x)}}{8}-\frac{3x}{8}}{y^4}\right)\left(xb_2+yb_3+b_1\right)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-64b_2y^6\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}-32x\,y^6b_2+32y^7a_2-64y^7b_3-32y^6b_1-24\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x^2y
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 64b_2y^6\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}+32x\,y^6b_2-32y^7a_2+64y^7b_3+32y^6b_1 \\
& +24\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x^2y^2b_2-16\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x\,y^3a_2 \\
& +32\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x\,y^3b_3-8\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,y^4a_3 \\
& -24x^3y^2b_2+16x^2y^3a_2-32x^2y^3b_3+24\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x\,y^2b_1 \\
& -8\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,y^3a_1-24x^2y^2b_1+8x\,y^3a_1 \\
& -\left(\left(-2y^2+x\right)\left(2y^2+x\right)\right)^{\frac{3}{2}}a_3-\sqrt{\left(-2y^2+x\right)\left(2y^2+x\right)}\,x^2a_3+2x^3a_3=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&64 b_2 y^6 \sqrt{(-2y^2+x)(2y^2+x)} - 64 x\, y^6 b_2 - 64 y^7 b_3 - 64 y^6 b_1 \\
&+ 24(2y^2-x)(2y^2+x)\, x\, y^2 b_2 - 8(2y^2-x)(2y^2+x)\, y^3 a_2 \\
&+ 32(2y^2-x)(2y^2+x)\, y^3 b_3 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 y^2 b_2 \\
&- 16\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^3 a_2 \\
&+ 32\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^3 b_3 - 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^4 a_3 \\
&+ 8x^2 y^3 a_2 + 8x\, y^4 a_3 + 24(2y^2-x)(2y^2+x)\, y^2 b_1 \\
&+ 24\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^2 b_1 - 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^3 a_1 \\
&+ 8x\, y^3 a_1 - \left((-2y^2+x)(2y^2+x)\right)^{\frac{3}{2}} a_3 \\
&- 2(2y^2-x)(2y^2+x)\, x a_3 - \sqrt{(-2y^2+x)(2y^2+x)}\, x^2 a_3 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&32x\, y^6 b_2 + 64 b_2 y^6 \sqrt{(-2y^2+x)(2y^2+x)} - 32 y^7 a_2 + 64 y^7 b_3 \\
&+ 32 y^6 b_1 - 24 x^3 y^2 b_2 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 y^2 b_2 \\
&+ 16 x^2 y^3 a_2 - 32 x^2 y^3 b_3 - 16\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^3 a_2 \\
&+ 32\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^3 b_3 - 4\sqrt{(-2y^2+x)(2y^2+x)}\, y^4 a_3 \\
&- 24 x^2 y^2 b_1 + 24\sqrt{(-2y^2+x)(2y^2+x)}\, x\, y^2 b_1 + 8x\, y^3 a_1 \\
&- 8\sqrt{(-2y^2+x)(2y^2+x)}\, y^3 a_1 + 2x^3 a_3 - 2\sqrt{(-2y^2+x)(2y^2+x)}\, x^2 a_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{(-2y^2+x)(2y^2+x)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{(-2y^2+x)(2y^2+x)} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-32 v_2^7 a_2 + 32 v_1 v_2^6 b_2 + 64 b_2 v_2^6 v_3 + 64 v_2^7 b_3 + 32 v_2^6 b_1 + 16 v_1^2 v_2^3 a_2 \\
&- 16 v_3 v_1 v_2^3 a_2 - 4 v_3 v_2^4 a_3 - 24 v_1^3 v_2^2 b_2 + 24 v_3 v_1^2 v_2^2 b_2 - 32 v_1^2 v_2^3 b_3 + 32 v_3 v_1 v_2^3 b_3 \\
&+ 8 v_1 v_2^3 a_1 - 8 v_3 v_2^3 a_1 - 24 v_1^2 v_2^2 b_1 + 24 v_3 v_1 v_2^2 b_1 + 2 v_1^3 a_3 - 2 v_3 v_1^2 a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}
&-24v_1^3v_2^2b_2 + 2v_1^3a_3 + (16a_2 - 32b_3)\, v_1^2v_2^3 + 24v_3v_1^2v_2^2b_2 - 24v_1^2v_2^2b_1 \\
&- 2v_3v_1^2a_3 + 32v_1v_2^6b_2 + (-16a_2 + 32b_3)\, v_1v_2^3v_3 + 8v_1v_2^3a_1 + 24v_3v_1v_2^2b_1 \\
&+ (-32a_2 + 64b_3)\, v_2^7 + 64b_2v_2^6v_3 + 32v_2^6b_1 - 4v_3v_2^4a_3 - 8v_3v_2^3a_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-8a_1 &= 0 \\
8a_1 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
-24b_1 &= 0 \\
24b_1 &= 0 \\
32b_1 &= 0 \\
-24b_2 &= 0 \\
24b_2 &= 0 \\
32b_2 &= 0 \\
64b_2 &= 0 \\
-32a_2 + 64b_3 &= 0 \\
-16a_2 + 32b_3 &= 0 \\
16a_2 - 32b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{\sqrt{-\left(2y^2 - x\right)\left(2y^2 + x\right)} - x}{8y^3}\right)(2x) \\
&= \frac{4y^4 + \sqrt{-4y^4 + x^2}\, x - x^2}{4y^3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^4 + \sqrt{-4y^4 + x^2}\, x - x^2}{4y^3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(4y^4 - x^2\right)}{4} - \frac{\ln\left(2y^2 - x\right)}{4} + \ln\left(y\right) - \frac{\ln\left(2y^2 + x\right)}{4} - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{-4y^4 + x^2}}{y^2}\right)}{2\sqrt{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{-\left(2y^2-x\right)\left(2y^2+x\right)}-x}{8y^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x+\sqrt{-4y^4+x^2}}{2\sqrt{-4y^4+x^2}\, x} \\
S_y &= \frac{-4y^4+2x^2+2\sqrt{-4y^4+x^2}\, x}{y\sqrt{-4y^4+x^2}\, \left(x+\sqrt{-4y^4+x^2}\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{-4y^4+\sqrt{-4y^4+x^2}\, x+x^2}{2\sqrt{-4y^4+x^2}\, x\left(x+\sqrt{-4y^4+x^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

Which gives

$$y = \mathrm{e}^{\frac{\ln(2)}{4}+\frac{\ln\left(-8\,\mathrm{e}^{-2c_1}\mathrm{e}^{4c_1}+2\,\mathrm{e}^{-2c_1}\mathrm{e}^{2c_1}x\right)}{4}+\frac{c_1}{2}}$$

Summary
The solution(s) found are the following

$$y = e^{\frac{\ln(2)}{4}+\frac{\ln\left(-8\,e^{-2c_1}e^{4c_1}+2\,e^{-2c_1}e^{2c_1}x\right)}{4}+\frac{c_1}{2}} \tag{1}$$

Verification of solutions

$$y = e^{\frac{\ln(2)}{4}+\frac{\ln\left(-8\,e^{-2c_1}e^{4c_1}+2\,e^{-2c_1}e^{2c_1}x\right)}{4}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[2*x, y]
```

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 105

```
dsolve(16*y(x)^3*diff(y(x),x)^2-4*x*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\,\sqrt{-x}}{2}$$
$$y(x) = \frac{\sqrt{2}\,\sqrt{-x}}{2}$$
$$y(x) = -\frac{\sqrt{x}\,\sqrt{2}}{2}$$
$$y(x) = \frac{\sqrt{x}\,\sqrt{2}}{2}$$
$$y(x) = 0$$
$$y(x) = \operatorname{RootOf}\left(-\ln(x) + 2\left(\int^{\_Z} -\frac{4\_a^4 - \sqrt{-4\_a^4+1} - 1}{\_a\,(4\_a^4-1)}d\_a\right) + c_1\right)\sqrt{x}$$

✓ Solution by Mathematica

Time used: 0.563 (sec). Leaf size: 303

```
DSolve[16*y[x]^3*y'[x]^2-4*x*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1}-ix} \\
y(x) &\to -ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1}-ix} \\
y(x) &\to ie^{\frac{c_1}{4}}\sqrt[4]{e^{c_1}-ix} \\
y(x) &\to e^{\frac{c_1}{4}}\sqrt[4]{e^{c_1}-ix} \\
y(x) &\to -e^{\frac{c_1}{4}}\sqrt[4]{ix+e^{c_1}} \\
y(x) &\to -ie^{\frac{c_1}{4}}\sqrt[4]{ix+e^{c_1}} \\
y(x) &\to ie^{\frac{c_1}{4}}\sqrt[4]{ix+e^{c_1}} \\
y(x) &\to e^{\frac{c_1}{4}}\sqrt[4]{ix+e^{c_1}} \\
y(x) &\to 0 \\
y(x) &\to -\frac{\sqrt{x}}{\sqrt{2}} \\
y(x) &\to -\frac{i\sqrt{x}}{\sqrt{2}} \\
y(x) &\to \frac{i\sqrt{x}}{\sqrt{2}} \\
y(x) &\to \frac{\sqrt{x}}{\sqrt{2}}
\end{aligned}
$$

5.9 problem 25

Internal problem ID [5331]
Internal file name [OUTPUT/4822_Friday_February_02_2024_05_14_16_AM_58506166/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 25.
ODE order: 1.
ODE degree: 5.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Clairaut]
```

$$\boxed{xy'^5 - yy'^4 + (x^2+1)\,y'^3 - 2xyy'^2 + (y^2+x)\,y' - y = 0}$$

The ode

$$xy'^5 - yy'^4 + (x^2+1)\,y'^3 - 2xyy'^2 + (y^2+x)\,y' - y = 0$$

is factored to

$$\left(-y'^2x + yy' - 1\right)\left(-y'^3 - xy' + y\right) = 0$$

Which gives the following equations

$$-y'^2x + yy' - 1 = 0 \tag{1}$$
$$-y'^3 - xy' + y = 0 \tag{2}$$

Each of the above equations is now solved.

Solving ODE (1) This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^2x + yp = 1$$

Solving for y from the above results in

$$y = \frac{p^2x + 1}{p} \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= px + \frac{1}{p} \\ &= px + \frac{1}{p} \end{aligned}$$

Writing the ode as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Then we see that

$$g = \frac{1}{p}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \frac{1}{c_1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \frac{1}{p}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{1}{p^2} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$\begin{aligned} p_1 &= \frac{1}{\sqrt{x}} \\ p_2 &= -\frac{1}{\sqrt{x}} \end{aligned}$$

Substituting the above back in (1) results in

$$\begin{aligned} y_1 &= 2\sqrt{x} \\ y_2 &= -2\sqrt{x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{1}{c_1} \tag{1}$$

$$y = 2\sqrt{x} \tag{2}$$

$$y = -2\sqrt{x} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{x}$$

Verified OK.

$$y = -2\sqrt{x}$$

Verified OK.

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{1}{c_1} \tag{1}$$
$$y = 2\sqrt{x} \tag{2}$$
$$y = -2\sqrt{x} \tag{3}$$

Verification of solutions

$$y = c_1 x + \frac{1}{c_1}$$

Verified OK.

$$y = 2\sqrt{x}$$

Verified OK.

$$y = -2\sqrt{x}$$

Verified OK.

Solving ODE (2) This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$-p^3 - xp + y = 0$$

Solving for y from the above results in

$$y = p^3 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= p^3 + xp \\ &= p^3 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = p^3$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2^3 + c_2 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^3$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= 3p^2 + x \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$\begin{aligned}
p_1 &= \frac{\sqrt{-3x}}{3} \\
p_2 &= -\frac{\sqrt{-3x}}{3}
\end{aligned}$$

Substituting the above back in (1) results in

$$\begin{aligned}
y_1 &= -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \\
y_2 &= \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2^3 + c_2 x \tag{1}$$

$$y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{2}$$

$$y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{3}$$

Verification of solutions

$$y = c_2^3 + c_2 x$$

Verified OK.

$$y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

$$y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$

Verified OK.

Summary

The solution(s) found are the following

$$y = c_2^3 + c_2 x \tag{1}$$

$$y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{2}$$

$$y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9} \tag{3}$$

Verification of solutions

$$
y = c_2^3 + c_2 x
$$

Verified OK.

$$
y = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}
$$

Verified OK.

$$
y = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 63

```
dsolve(x*diff(y(x),x)^5-y(x)*diff(y(x),x)^4+(1+x^2)*diff(y(x),x)^3-2*x*y(x)*diff(y(x),x)^2+(
```

$$y(x) = \frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$
$$y(x) = -\frac{2\sqrt{3}\,(-x)^{\frac{3}{2}}}{9}$$
$$y(x) = c_1(c_1^2 + x)$$
$$y(x) = -2\sqrt{x}$$
$$y(x) = 2\sqrt{x}$$
$$y(x) = c_1 x + \frac{1}{c_1}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 142

```
DSolve[x*y'[x]^5-y[x]*y'[x]^4+(1+x^2)*y'[x]^3-2*x*y[x]*y'[x]^2+(x+y[x]^2)*y'[x]-y[x]==0,y[x]
```

$$y(x) \to c_1 x + \frac{1}{c_1}$$
$$y(x) \to c_1\left(x + c_1{}^2\right)$$
$$y(x) \to \text{Indeterminate}$$
$$y(x) \to -x - 1$$
$$y(x) \to -2\sqrt{x}$$
$$y(x) \to 2\sqrt{x}$$
$$y(x) \to -\frac{2ix^{3/2}}{3\sqrt{3}}$$
$$y(x) \to \frac{2ix^{3/2}}{3\sqrt{3}}$$
$$y(x) \to x + 1$$
$$y(x) \to -\sqrt{-(x-1)^2}$$
$$y(x) \to \sqrt{-(x-1)^2}$$

5.10 problem 26

5.10.1 Solving as dAlembert ode 1051

Internal problem ID [5332]
Internal file name [`OUTPUT/4823_Friday_February_02_2024_05_14_16_AM_5154403/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 26.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{{y'}^2 x - y y' - y = 0}$$

5.10.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - y p - y = 0$$

Solving for y from the above results in

$$y = \frac{p^2 x}{p+1} \tag{1A}$$

This has the form

$$y = x f(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (x f' + g') \frac{dp}{dx}$$
$$p - f = (x f' + g') \frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{p^2}{p+1} \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$p - \frac{p^2}{p+1} = x\left(\frac{2p}{p+1} - \frac{p^2}{(p+1)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p^2}{p+1} = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)^2}{p(x)+1}}{x\left(\frac{2p(x)}{p(x)+1} - \frac{p(x)^2}{(p(x)+1)^2}\right)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(\frac{2p}{p+1} - \frac{p^2}{(p+1)^2}\right)}{p - \frac{p^2}{p+1}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$
\begin{aligned}
p(p) &= -\frac{p+2}{p+1} \\
q(p) &= 0
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) - \frac{x(p)\,(p+2)}{p+1} = 0
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{p+2}{p+1}dp} \\
&= \mathrm{e}^{-p-\ln(p+1)}
\end{aligned}
$$

Which simplifies to

$$
\mu = \frac{\mathrm{e}^{-p}}{p+1}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p}\mu x &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{\mathrm{e}^{-p}x}{p+1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\frac{\mathrm{e}^{-p}x}{p+1} = c_2
$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{-p}}{p+1}$ results in

$$
x(p) = c_2(p+1)\,\mathrm{e}^{p}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= \frac{y+\sqrt{y^2+4yx}}{2x} \\
p &= -\frac{-y+\sqrt{y^2+4yx}}{2x}
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
x = \frac{c_2 \left(y + \sqrt{y (y+4x)} + 2x\right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x}
$$
$$
x = -\frac{c_2 \left(-y + \sqrt{y (y+4x)} - 2x\right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x}
$$

Summary

The solution(s) found are the following

$$
y = 0 \tag{1}
$$
$$
x = \frac{c_2 \left(y + \sqrt{y (y+4x)} + 2x\right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x} \tag{2}
$$
$$
x = -\frac{c_2 \left(-y + \sqrt{y (y+4x)} - 2x\right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x} \tag{3}
$$

Verification of solutions

$$
y = 0
$$

Verified OK.

$$
x = \frac{c_2 \left(y + \sqrt{y (y+4x)} + 2x\right) e^{\frac{y+\sqrt{y(y+4x)}}{2x}}}{2x}
$$

Verified OK.

$$
x = -\frac{c_2 \left(-y + \sqrt{y (y+4x)} - 2x\right) e^{-\frac{-y+\sqrt{y(y+4x)}}{2x}}}{2x}
$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 32

```
dsolve(x*diff(y(x),x)^2-y(x)*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\left(\operatorname{LambertW}\left(\frac{x\,\mathrm{e}}{c_1}\right) - 1\right)^2 x}{\operatorname{LambertW}\left(\frac{x\,\mathrm{e}}{c_1}\right)}$$

✓ Solution by Mathematica

Time used: 2.255 (sec). Leaf size: 158

```
DSolve[x*y'[x]^2-y[x]*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\frac{y(x)}{4x}+\frac{1}{4}\sqrt{\frac{y(x)}{x}}\sqrt{\frac{y(x)}{x}+4}-\log\left(\sqrt{\frac{y(x)}{x}+4}-\sqrt{\frac{y(x)}{x}}\right)=-\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$\text{Solve}\left[\frac{1}{4}\left(\frac{y(x)}{x}+\sqrt{\frac{y(x)}{x}}\sqrt{\frac{y(x)}{x}+4}-4\log\left(\sqrt{\frac{y(x)}{x}+4}-\sqrt{\frac{y(x)}{x}}\right)\right)=\frac{\log(x)}{2}+c_1,y(x)\right]$$

$$y(x) \to 0$$

5.11 problem 27

Internal problem ID [5333]
Internal file name [`OUTPUT/4824_Friday_February_02_2024_05_14_17_AM_71139267/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 27.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y - 2xy' - y^2 {y'}^3 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{6y} - \frac{4x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} \tag{1}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}{6y} \cdots\right)}{\cdots} \tag{2}$$

$$y' = -\frac{\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{12y} + \frac{2x}{y\left(108y^2 + 12\sqrt{3}\,\sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(108y^2+12\sqrt{3}\,\sqrt{27y^4+32x^3}\right)^{\frac{1}{3}}}{6y} \cdots\right)}{\cdots} \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\left(2\,12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}}{6y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(2\,12^{\frac{1}{3}} x - \left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}} (b_3 - a_2)}{6y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}} \\
&- \frac{\left(2\,12^{\frac{1}{3}} x - \left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)^2 12^{\frac{2}{3}} a_3}{36y^2\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}} \\
&- \left(-\frac{\left(2\,12^{\frac{1}{3}} - \frac{32\sqrt{3}\,x^2}{\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}} \sqrt{27y^4+32x^3}}\right) 12^{\frac{1}{3}}}{6y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}} \right. \\
&\left. + \frac{8\left(2\,12^{\frac{1}{3}} x - \left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}} \sqrt{3}\,x^2}{3y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}} \sqrt{27y^4+32x^3}} \right) (x a_2 + y a_3 + a_1) \\
&- \left(\frac{\sqrt{3}\left(6\sqrt{3}\,y + \frac{54y^3}{\sqrt{27y^4+32x^3}}\right) 12^{\frac{1}{3}}}{9\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}} y} \right. \\
&+ \frac{\left(2\,12^{\frac{1}{3}} x - \left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}}{6y^2\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}} \\
&\left. + \frac{\left(2\,12^{\frac{1}{3}} x - \left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}} \sqrt{3}\left(6\sqrt{3}\,y + \frac{54y^3}{\sqrt{27y^4+32x^3}}\right)}{18y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}}} \right) (x b_2 \\
&+ y b_3 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}} = v_3, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}} = v_4, \sqrt{27y^4 +\right.$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-54\,12^{\frac{2}{3}} v_5 v_2^4 a_3 + 108\,12^{\frac{2}{3}} v_5 v_2^3 a_1 + 972\sqrt{3}\, v_3 v_2^6 b_2 + 1536\sqrt{3}\, v_1^4 v_3 a_3 \\
&- 162\sqrt{3}\,12^{\frac{2}{3}} v_2^6 a_3 - 384\sqrt{3}\,12^{\frac{2}{3}} v_1^5 b_2 + 324\sqrt{3}\,12^{\frac{2}{3}} v_2^5 a_1 \\
&- 384\sqrt{3}\,12^{\frac{2}{3}} v_1^4 b_1 + 324 v_3 v_5 v_2^4 b_2 - 96\,12^{\frac{2}{3}} v_1^3 v_5 a_3 \\
&+ 384\sqrt{3}\,12^{\frac{1}{3}} v_1^3 v_4 v_2 b_3 - 96\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1^2 v_2 a_1 + 18 v_4 12^{\frac{1}{3}} v_5 v_1 v_2^2 b_2 \\
&+ 54\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1 v_2^4 b_2 - 288\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1^3 v_2 a_2 - 96\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1^2 v_2^2 a_3 \\
&+ 648\sqrt{3}\,12^{\frac{2}{3}} v_1 v_2^5 a_2 - 768\sqrt{3}\,12^{\frac{2}{3}} v_1^4 v_2 b_3 + 432 v_1 v_3 v_5 v_2^2 a_3 \\
&+ 72 v_4 12^{\frac{1}{3}} v_5 v_2^3 b_3 - 180\,12^{\frac{2}{3}} v_5 v_1^2 v_2^2 b_2 - 288\,12^{\frac{2}{3}} v_5 v_1 v_2^3 b_3 \\
&+ 216\sqrt{3}\, v_4 12^{\frac{1}{3}} v_2^5 b_3 - 540\sqrt{3}\,12^{\frac{2}{3}} v_1^2 v_2^4 b_2 - 864\sqrt{3}\,12^{\frac{2}{3}} v_1 v_2^5 b_3 \\
&- 48 v_4 12^{\frac{1}{3}} v_5 v_1^2 a_3 + 18 v_4 12^{\frac{1}{3}} v_5 v_2^2 b_1 - 180\,12^{\frac{2}{3}} v_5 v_1 v_2^2 b_1 \\
&+ 54\sqrt{3}\, v_4 12^{\frac{1}{3}} v_2^4 b_1 + 576\sqrt{3}\,12^{\frac{2}{3}} v_1^4 v_2 a_2 - 384\sqrt{3}\,12^{\frac{2}{3}} v_1^3 v_2^2 a_3 \\
&- 540\sqrt{3}\,12^{\frac{2}{3}} v_1 v_2^4 b_1 + 192\sqrt{3}\,12^{\frac{2}{3}} v_1^3 v_2 a_1 - 54\,12^{\frac{1}{3}} v_4 v_5 v_2^3 a_2 \\
&+ 216\,12^{\frac{2}{3}} v_5 v_1 v_2^3 a_2 - 162\sqrt{3}\,12^{\frac{1}{3}} v_4 v_2^5 a_2 + 192\sqrt{3}\,12^{\frac{1}{3}} v_1^4 v_4 b_2 \\
&+ 192\sqrt{3}\,12^{\frac{1}{3}} v_1^3 v_4 b_1 + 1152\sqrt{3}\, v_1^3 v_3 v_2^2 b_2 + 1296\sqrt{3}\, v_1 v_3 v_2^4 a_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-54\,12^{\frac{2}{3}} v_5 v_2^4 a_3 + 108\,12^{\frac{2}{3}} v_5 v_2^3 a_1 + 972\sqrt{3}\, v_3 v_2^6 b_2 \\
&+ 1536\sqrt{3}\, v_1^4 v_3 a_3 - 162\sqrt{3}\, 12^{\frac{2}{3}} v_2^6 a_3 - 384\sqrt{3}\, 12^{\frac{2}{3}} v_1^5 b_2 \\
&+ 324\sqrt{3}\, 12^{\frac{2}{3}} v_2^5 a_1 - 384\sqrt{3}\, 12^{\frac{2}{3}} v_1^4 b_1 + 324 v_3 v_5 v_2^4 b_2 \\
&- 96\,12^{\frac{2}{3}} v_1^3 v_5 a_3 + \left(648\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 864\sqrt{3}\, 12^{\frac{2}{3}} b_3\right) v_1 v_2^5 \\
&+ \left(-162\sqrt{3}\, 12^{\frac{1}{3}} a_2 + 216\sqrt{3}\, 12^{\frac{1}{3}} b_3\right) v_2^5 v_4 \\
&- 96\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1^2 v_2 a_1 + 18 v_4 12^{\frac{1}{3}} v_5 v_1 v_2^2 b_2 + 54\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1 v_2^4 b_2 \\
&- 96\sqrt{3}\, v_4 12^{\frac{1}{3}} v_1^2 v_2^2 a_3 + \left(576\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 768\sqrt{3}\, 12^{\frac{2}{3}} b_3\right) v_1^4 v_2 \\
&+ \left(-288\sqrt{3}\, 12^{\frac{1}{3}} a_2 + 384\sqrt{3}\, 12^{\frac{1}{3}} b_3\right) v_1^3 v_2 v_4 \\
&+ \left(216\,12^{\frac{2}{3}} a_2 - 288\,12^{\frac{2}{3}} b_3\right) v_1 v_2^3 v_5 + \left(-54\,12^{\frac{1}{3}} a_2 + 72\,12^{\frac{1}{3}} b_3\right) v_2^3 v_4 v_5 \\
&+ 432 v_1 v_3 v_5 v_2^2 a_3 - 180\,12^{\frac{2}{3}} v_5 v_1^2 v_2^2 b_2 - 540\sqrt{3}\, 12^{\frac{2}{3}} v_1^2 v_2^4 b_2 \\
&- 48 v_4 12^{\frac{1}{3}} v_5 v_1^2 a_3 + 18 v_4 12^{\frac{1}{3}} v_5 v_2^2 b_1 - 180\,12^{\frac{2}{3}} v_5 v_1 v_2^2 b_1 \\
&+ 54\sqrt{3}\, v_4 12^{\frac{1}{3}} v_2^4 b_1 - 384\sqrt{3}\, 12^{\frac{2}{3}} v_1^3 v_2^2 a_3 - 540\sqrt{3}\, 12^{\frac{2}{3}} v_1 v_2^4 b_1 \\
&+ 192\sqrt{3}\, 12^{\frac{2}{3}} v_1^3 v_2 a_1 + 192\sqrt{3}\, 12^{\frac{1}{3}} v_1^4 v_4 b_2 + 192\sqrt{3}\, 12^{\frac{1}{3}} v_1^3 v_4 b_1 \\
&+ 1152\sqrt{3}\, v_1^3 v_3 v_2^2 b_2 + 1296\sqrt{3}\, v_1 v_3 v_2^4 a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
432a_3 &= 0\\
324b_2 &= 0\\
1296\sqrt{3}\,a_3 &= 0\\
1536\sqrt{3}\,a_3 &= 0\\
972\sqrt{3}\,b_2 &= 0\\
1152\sqrt{3}\,b_2 &= 0\\
-48\,12^{\frac{1}{3}}a_3 &= 0\\
18\,12^{\frac{1}{3}}b_1 &= 0\\
18\,12^{\frac{1}{3}}b_2 &= 0\\
108\,12^{\frac{2}{3}}a_1 &= 0\\
-96\,12^{\frac{2}{3}}a_3 &= 0\\
-54\,12^{\frac{2}{3}}a_3 &= 0\\
-180\,12^{\frac{2}{3}}b_1 &= 0\\
-180\,12^{\frac{2}{3}}b_2 &= 0\\
-96\sqrt{3}\,12^{\frac{1}{3}}a_1 &= 0\\
-96\sqrt{3}\,12^{\frac{1}{3}}a_3 &= 0\\
54\sqrt{3}\,12^{\frac{1}{3}}b_1 &= 0\\
192\sqrt{3}\,12^{\frac{1}{3}}b_1 &= 0\\
54\sqrt{3}\,12^{\frac{1}{3}}b_2 &= 0\\
192\sqrt{3}\,12^{\frac{1}{3}}b_2 &= 0\\
192\sqrt{3}\,12^{\frac{2}{3}}a_1 &= 0\\
324\sqrt{3}\,12^{\frac{2}{3}}a_1 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}}a_3 &= 0\\
-162\sqrt{3}\,12^{\frac{2}{3}}a_3 &= 0\\
-540\sqrt{3}\,12^{\frac{2}{3}}b_1 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}}b_1 &= 0\\
-540\sqrt{3}\,12^{\frac{2}{3}}b_2 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}}b_2 &= 0\\
-54\,12^{\frac{1}{3}}a_2 + 72\,12^{\frac{1}{3}}b_3 &= 0\\
216\,12^{\frac{2}{3}}a_2 - 288\,12^{\frac{2}{3}}b_3 &= 0\\
-288\sqrt{3}\,12^{\frac{1}{3}}a_2 + 384\sqrt{3}\,12^{\frac{1}{3}}b_3 &= 0\\
-162\sqrt{3}\,12^{\frac{1}{3}}a_2 + 216\sqrt{3}\,12^{\frac{1}{3}}b_3 &= 0\\
576\sqrt{3}\,12^{\frac{2}{3}}a_2 - 768\sqrt{3}\,12^{\frac{2}{3}}b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{\frac{4x}{3}} \\
&= \frac{3y}{4x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^{\frac{3}{4}}
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^{\frac{3}{4}}}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{\frac{4x}{3}} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \frac{3\ln(x)}{4} \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\left(2\, 12^{\frac{1}{3}} x - \left(\sqrt{3}\left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}}{6y\left(\sqrt{3}\, \left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)\right)^{\frac{1}{3}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{3y}{4x^{\frac{7}{4}}} \\ R_y &= \frac{1}{x^{\frac{3}{4}}} \\ S_x &= \frac{3}{4x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9x^{\frac{3}{4}} y 3^{\frac{1}{6}} \left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}}}{2\left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)^{\frac{2}{3}} x 6^{\frac{2}{3}} - 8\, 3^{\frac{2}{3}} 2^{\frac{1}{3}} x^2 - 9\, 3^{\frac{1}{6}} \left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)^{\frac{1}{3}} y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9R 3^{\frac{1}{6}} \left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^{\frac{1}{3}}}{2 \left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^{\frac{2}{3}} 6^{\frac{2}{3}} - 9 \left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}} R^2 - 8\, 2^{\frac{1}{3}} 3^{\frac{2}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{9R\left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}}{2\, 36^{\frac{1}{3}} \left(\left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^2\right)^{\frac{1}{3}} - 9 \left(3\sqrt{3}\, R^2 + \sqrt{27R^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}} R^2 - 8\, 18^{\frac{1}{3}}} dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}}{2\, 36^{\frac{1}{3}} \left(\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^2\right)^{\frac{1}{3}} - 9 \left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}\_a^2 - 8\, 18^{\frac{1}{3}}} d\_$$

Which simplifies to

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}}{2\, 36^{\frac{1}{3}} \left(\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^2\right)^{\frac{1}{3}} - 9 \left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}\_a^2 - 8\, 18^{\frac{1}{3}}} d\_$$

Summary

The solution(s) found are the following

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}}{2\, 36^{\frac{1}{3}} \left(\left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^2\right)^{\frac{1}{3}} - 9 \left(3\sqrt{3}\_a^2 + \sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}\_a^2 - 8\, 18^{\frac{1}{3}}} d\_a + c_1 \tag{1}$$

Verification of solutions

$$\frac{3\ln(x)}{4} = \int^{\frac{y}{x^{\frac{3}{4}}}} \frac{9\_a\left(3\sqrt{3}\,\_a^2+\sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}}{2\,36^{\frac{1}{3}}\left(\left(3\sqrt{3}\,\_a^2+\sqrt{27\_a^4+32}\right)^2\right)^{\frac{1}{3}} - 9\left(3\sqrt{3}\,\_a^2+\sqrt{27\_a^4+32}\right)^{\frac{1}{3}} 3^{\frac{1}{6}}\_a^2 - 8\,18^{\frac{1}{3}}} d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = \frac{\left(i\sqrt{3}\,x12^{\frac{1}{3}} - 12^{\frac{1}{3}}x - \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}}{3y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{\left(i\sqrt{3}\,x12^{\frac{1}{3}}-12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}\left(b_3-a_2\right)}{3y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)} \\
&-\frac{\left(i\sqrt{3}\,x12^{\frac{1}{3}}-12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)^2 12^{\frac{2}{3}}a_3}{9y^2\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2} \\
&-\left(\frac{\left(i\sqrt{3}\,12^{\frac{1}{3}}-12^{\frac{1}{3}}-\frac{32\sqrt{3}\,x^2}{\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}\right)12^{\frac{1}{3}}}{3y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}\right. \\
&\left.-\frac{16\left(i\sqrt{3}\,x12^{\frac{1}{3}}-12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}\sqrt{3}\,x^2}{3y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)\sqrt{27y^4+32x^3}}\right)(xa_2 \\
&+ya_3+a_1)-\left(-\frac{2\sqrt{3}\left(6\sqrt{3}\,y+\frac{54y^3}{\sqrt{27y^4+32x^3}}\right)12^{\frac{1}{3}}}{9\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}y\left(1+i\sqrt{3}\right)}\right. \\
&-\frac{\left(i\sqrt{3}\,x12^{\frac{1}{3}}-12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}}{3y^2\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)} \\
&\left.-\frac{\left(i\sqrt{3}\,x12^{\frac{1}{3}}-12^{\frac{1}{3}}x-\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)12^{\frac{1}{3}}\sqrt{3}\left(6\sqrt{3}\,y+\frac{54y^3}{\sqrt{27y^4+32x^3}}\right)}{9y\left(\sqrt{3}\ \left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)}\right)(xb_2 \\
&+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}=v_3, \left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}=v_4, \sqrt{27y^4+\right.$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(216i\sqrt{3}\,a_3-216a_3\right)v_1v_2^2v_3v_5+\left(-9i\sqrt{3}\,12^{\frac{1}{3}}b_2-9\,12^{\frac{1}{3}}b_2\right)v_1v_2^2v_4v_5\\
&+\left(216\,12^{\frac{2}{3}}a_2-288\,12^{\frac{2}{3}}b_3\right)v_1v_2^3v_5+\left(2304ia_3-768\sqrt{3}\,a_3\right)v_1^4v_3\\
&+\left(-288i12^{\frac{1}{3}}b_2-96\sqrt{3}\,12^{\frac{1}{3}}b_2\right)v_1^4v_4+\left(-288i12^{\frac{1}{3}}b_1-96\sqrt{3}\,12^{\frac{1}{3}}b_1\right)v_1^3v_4\\
&+\left(1458ib_2-486\sqrt{3}\,b_2\right)v_2^6v_3+\left(243i12^{\frac{1}{3}}a_2-324i12^{\frac{1}{3}}b_3+81\sqrt{3}\,12^{\frac{1}{3}}a_2-108\sqrt{3}\,12^{\frac{1}{3}}b_3\right)v_2^5v_4\\
&+\left(-81i12^{\frac{1}{3}}b_1-27\sqrt{3}\,12^{\frac{1}{3}}b_1\right)v_2^4v_4+\left(576\sqrt{3}\,12^{\frac{2}{3}}a_2-768\sqrt{3}\,12^{\frac{2}{3}}b_3\right)v_1^4v_2\\
&+\left(648\sqrt{3}\,12^{\frac{2}{3}}a_2-864\sqrt{3}\,12^{\frac{2}{3}}b_3\right)v_1v_2^5+\left(1728ib_2-576\sqrt{3}\,b_2\right)v_1^3v_2^2v_3\\
&+\left(432i12^{\frac{1}{3}}a_2-576i12^{\frac{1}{3}}b_3+144\sqrt{3}\,12^{\frac{1}{3}}a_2-192\sqrt{3}\,12^{\frac{1}{3}}b_3\right)v_1^3v_2v_4\\
&+\left(144i12^{\frac{1}{3}}a_3+48\sqrt{3}\,12^{\frac{1}{3}}a_3\right)v_1^2v_2^2v_4+\left(144i12^{\frac{1}{3}}a_1+48\sqrt{3}\,12^{\frac{1}{3}}a_1\right)v_1^2v_2v_4\\
&+\left(24i\sqrt{3}\,12^{\frac{1}{3}}a_3+24\,12^{\frac{1}{3}}a_3\right)v_1^2v_4v_5+\left(1944ia_3-648\sqrt{3}\,a_3\right)v_1v_2^4v_3\\
&+\left(-81i12^{\frac{1}{3}}b_2-27\sqrt{3}\,12^{\frac{1}{3}}b_2\right)v_1v_2^4v_4+\left(162i\sqrt{3}\,b_2-162b_2\right)v_2^4v_3v_5\\
&+\left(27i\sqrt{3}\,12^{\frac{1}{3}}a_2-36i\sqrt{3}\,12^{\frac{1}{3}}b_3+27\,12^{\frac{1}{3}}a_2-36\,12^{\frac{1}{3}}b_3\right)v_2^3v_4v_5\\
&+\left(-9i\sqrt{3}\,12^{\frac{1}{3}}b_1-9\,12^{\frac{1}{3}}b_1\right)v_2^2v_4v_5-180v_5 12^{\frac{2}{3}}v_1^2v_2^2b_2-540\sqrt{3}\,12^{\frac{2}{3}}v_1^2v_2^4b_2\\
&-180v_5 12^{\frac{2}{3}}v_1v_2^2b_1-384\sqrt{3}\,12^{\frac{2}{3}}v_1^3v_2^2a_3-540\sqrt{3}\,12^{\frac{2}{3}}v_1v_2^4b_1\\
&+192\sqrt{3}\,12^{\frac{2}{3}}v_1^3v_2a_1+108v_5 12^{\frac{2}{3}}v_2^3a_1-162\sqrt{3}\,12^{\frac{2}{3}}v_2^6a_3-384\sqrt{3}\,12^{\frac{2}{3}}v_1^5b_2\\
&+324\sqrt{3}\,12^{\frac{2}{3}}v_2^5a_1-384\sqrt{3}\,12^{\frac{2}{3}}v_1^4b_1-96\,12^{\frac{2}{3}}v_1^3v_5a_3-54v_5 12^{\frac{2}{3}}v_2^4a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
108\, 12^{\frac{2}{3}} a_1 &= 0\\
-96\, 12^{\frac{2}{3}} a_3 &= 0\\
-54\, 12^{\frac{2}{3}} a_3 &= 0\\
-180\, 12^{\frac{2}{3}} b_1 &= 0\\
-180\, 12^{\frac{2}{3}} b_2 &= 0\\
192\sqrt{3}\, 12^{\frac{2}{3}} a_1 &= 0\\
324\sqrt{3}\, 12^{\frac{2}{3}} a_1 &= 0\\
-384\sqrt{3}\, 12^{\frac{2}{3}} a_3 &= 0\\
-162\sqrt{3}\, 12^{\frac{2}{3}} a_3 &= 0\\
-540\sqrt{3}\, 12^{\frac{2}{3}} b_1 &= 0\\
-384\sqrt{3}\, 12^{\frac{2}{3}} b_1 &= 0\\
-540\sqrt{3}\, 12^{\frac{2}{3}} b_2 &= 0\\
-384\sqrt{3}\, 12^{\frac{2}{3}} b_2 &= 0\\
216\, 12^{\frac{2}{3}} a_2 - 288\, 12^{\frac{2}{3}} b_3 &= 0\\
1458 i b_2 - 486\sqrt{3}\, b_2 &= 0\\
1728 i b_2 - 576\sqrt{3}\, b_2 &= 0\\
1944 i a_3 - 648\sqrt{3}\, a_3 &= 0\\
2304 i a_3 - 768\sqrt{3}\, a_3 &= 0\\
576\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 768\sqrt{3}\, 12^{\frac{2}{3}} b_3 &= 0\\
648\sqrt{3}\, 12^{\frac{2}{3}} a_2 - 864\sqrt{3}\, 12^{\frac{2}{3}} b_3 &= 0\\
-288 i 12^{\frac{1}{3}} b_1 - 96\sqrt{3}\, 12^{\frac{1}{3}} b_1 &= 0\\
-288 i 12^{\frac{1}{3}} b_2 - 96\sqrt{3}\, 12^{\frac{1}{3}} b_2 &= 0\\
-81 i 12^{\frac{1}{3}} b_1 - 27\sqrt{3}\, 12^{\frac{1}{3}} b_1 &= 0\\
-81 i 12^{\frac{1}{3}} b_2 - 27\sqrt{3}\, 12^{\frac{1}{3}} b_2 &= 0\\
144 i 12^{\frac{1}{3}} a_1 + 48\sqrt{3}\, 12^{\frac{1}{3}} a_1 &= 0\\
144 i 12^{\frac{1}{3}} a_3 + 48\sqrt{3}\, 12^{\frac{1}{3}} a_3 &= 0\\
162 i\sqrt{3}\, b_2 - 162 b_2 &= 0\\
216 i\sqrt{3}\, a_3 - 216 a_3 &= 0\\
-9 i\sqrt{3}\, 12^{\frac{1}{3}} b_1 - 9\, 12^{\frac{1}{3}} b_1 &= 0\\
-9 i\sqrt{3}\, 12^{\frac{1}{3}} b_2 - 9\, 12^{\frac{1}{3}} b_2 &= 0\\
24 i\sqrt{3}\, 12^{\frac{1}{3}} a_3 + 24\, 12^{\frac{1}{3}} a_3 &= 0\\
243 i 12^{\frac{1}{3}} a_2 - 324 i 12^{\frac{1}{3}} b_3 + 81\sqrt{3}\, 12^{\frac{1}{3}} a_2 - 108\sqrt{3}\, 12^{\frac{1}{3}} b_3 &= 0\\
432 i 12^{\frac{1}{3}} a_2 - 576 i 12^{\frac{1}{3}} b_3 + 144\sqrt{3}\, 12^{\frac{1}{3}} a_2 - 192\sqrt{3}\, 12^{\frac{1}{3}} b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\left(i\sqrt{3}\, x 12^{\frac{1}{3}} + 12^{\frac{1}{3}} x + \left(\sqrt{3}\left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}}{3y\left(\sqrt{3}\ \left(3\sqrt{3}\, y^2 + \sqrt{27y^4 + 32x^3}\right)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{\left(i\sqrt{3}\,x 12^{\frac{1}{3}} + 12^{\frac{1}{3}}x + \left(\sqrt{3}\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}(b_3 - a_2)}{3y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)} \\
&- \frac{\left(i\sqrt{3}\,x 12^{\frac{1}{3}} + 12^{\frac{1}{3}}x + \left(\sqrt{3}\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right)^2 12^{\frac{2}{3}} a_3}{9y^2\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\left(i\sqrt{3}-1\right)^2} \\
&- \left(\frac{\left(i\sqrt{3}\,12^{\frac{1}{3}} + 12^{\frac{1}{3}} + \frac{32\sqrt{3}\,x^2}{\left(\sqrt{3}\left(3\sqrt{3}\,y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\sqrt{27y^4+32x^3}}\right)12^{\frac{1}{3}}}{3y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}\right. \\
&\left. - \frac{16\left(i\sqrt{3}\,x 12^{\frac{1}{3}} + 12^{\frac{1}{3}}x + \left(\sqrt{3}\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}\sqrt{3}\,x^2}{3y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}}\left(i\sqrt{3}-1\right)\sqrt{27y^4+32x^3}}\right)(xa_2 \\
&+ ya_3 + a_1) - \left(\frac{2\sqrt{3}\left(6\sqrt{3}\,y + \frac{54y^3}{\sqrt{27y^4+32x^3}}\right)12^{\frac{1}{3}}}{9\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}} y\left(i\sqrt{3}-1\right)}\right. \\
&- \frac{\left(i\sqrt{3}\,x 12^{\frac{1}{3}} + 12^{\frac{1}{3}}x + \left(\sqrt{3}\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}}{3y^2\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)} \\
&\left. - \frac{\left(i\sqrt{3}\,x 12^{\frac{1}{3}} + 12^{\frac{1}{3}}x + \left(\sqrt{3}\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}\right) 12^{\frac{1}{3}}\sqrt{3}\left(6\sqrt{3}\,y + \frac{54y^3}{\sqrt{27y^4+32x^3}}\right)}{9y\left(\sqrt{3}\,\left(3\sqrt{3}\,y^2 + \sqrt{27y^4+32x^3}\right)\right)^{\frac{4}{3}}\left(i\sqrt{3}-1\right)}\right)(xb_2 \\
&+ yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

Expression too large to display (6E)

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(\sqrt{3}\left(3\sqrt{3}\, y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}, \left(\sqrt{3}\left(3\sqrt{3}\, y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}, \sqrt{27y^4+32x^3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x=v_1, y=v_2, \left(\sqrt{3}\left(3\sqrt{3}\, y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{1}{3}}=v_3, \left(\sqrt{3}\left(3\sqrt{3}\, y^2+\sqrt{27y^4+32x^3}\right)\right)^{\frac{2}{3}}=v_4, \sqrt{27y^4+\right.$$

The above PDE (6E) now becomes

Expression too large to display (7E)

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(576\sqrt{3}\,12^{\frac{2}{3}}a_2-768\sqrt{3}\,12^{\frac{2}{3}}b_3\right)v_1^4v_2\\
&+\left(648\sqrt{3}\,12^{\frac{2}{3}}a_2-864\sqrt{3}\,12^{\frac{2}{3}}b_3\right)v_1v_2^5-540\sqrt{3}\,12^{\frac{2}{3}}v_1v_2^4b_1\\
&+192\sqrt{3}\,12^{\frac{2}{3}}v_1^3v_2a_1-180v_5 12^{\frac{2}{3}}v_1^2v_2^2b_2-540\sqrt{3}\,12^{\frac{2}{3}}v_1^2v_2^4b_2\\
&+\left(216\,12^{\frac{2}{3}}a_2-288\,12^{\frac{2}{3}}b_3\right)v_1v_2^3v_5+\left(-432i12^{\frac{1}{3}}a_2\right.\\
&\left.+576i12^{\frac{1}{3}}b_3+144\sqrt{3}\,12^{\frac{1}{3}}a_2-192\sqrt{3}\,12^{\frac{1}{3}}b_3\right)v_1^3v_2v_4\\
&+\left(-144i12^{\frac{1}{3}}a_3+48\sqrt{3}\,12^{\frac{1}{3}}a_3\right)v_1^2v_2^2v_4\\
&+\left(-144i12^{\frac{1}{3}}a_1+48\sqrt{3}\,12^{\frac{1}{3}}a_1\right)v_1^2v_2v_4\\
&+\left(-24i\sqrt{3}\,12^{\frac{1}{3}}a_3+24\,12^{\frac{1}{3}}a_3\right)v_1^2v_4v_5\\
&+\left(-1944ia_3-648\sqrt{3}\,a_3\right)v_1v_2^4v_3\\
&+\left(81i12^{\frac{1}{3}}b_2-27\sqrt{3}\,12^{\frac{1}{3}}b_2\right)v_1v_2^4v_4\\
&+\left(-162i\sqrt{3}\,b_2-162b_2\right)v_2^4v_3v_5+\left(-27i\sqrt{3}\,12^{\frac{1}{3}}a_2\right.\\
&\left.+36i\sqrt{3}\,12^{\frac{1}{3}}b_3+27\,12^{\frac{1}{3}}a_2-36\,12^{\frac{1}{3}}b_3\right)v_2^3v_4v_5\\
&+\left(9i\sqrt{3}\,12^{\frac{1}{3}}b_1-9\,12^{\frac{1}{3}}b_1\right)v_2^2v_4v_5\\
&-384\sqrt{3}\,12^{\frac{2}{3}}v_1^5b_2+324\sqrt{3}\,12^{\frac{2}{3}}v_2^5a_1\\
&-384\sqrt{3}\,12^{\frac{2}{3}}v_1^4b_1-96\,12^{\frac{2}{3}}v_1^3v_5a_3-54v_5 12^{\frac{2}{3}}v_2^4a_3\\
&+108v_5 12^{\frac{2}{3}}v_2^3a_1+\left(-1728ib_2-576\sqrt{3}\,b_2\right)v_1^3v_2^2v_3\\
&-162\sqrt{3}\,12^{\frac{2}{3}}v_2^6a_3+\left(-243i12^{\frac{1}{3}}a_2+324i12^{\frac{1}{3}}b_3\right.\\
&\left.+81\sqrt{3}\,12^{\frac{1}{3}}a_2-108\sqrt{3}\,12^{\frac{1}{3}}b_3\right)v_2^5v_4\\
&+\left(81i12^{\frac{1}{3}}b_1-27\sqrt{3}\,12^{\frac{1}{3}}b_1\right)v_2^4v_4-384\sqrt{3}\,12^{\frac{2}{3}}v_1^3v_2^2a_3\\
&-180v_5 12^{\frac{2}{3}}v_1v_2^2b_1+\left(-2304ia_3-768\sqrt{3}\,a_3\right)v_1^4v_3\\
&+\left(288i12^{\frac{1}{3}}b_2-96\sqrt{3}\,12^{\frac{1}{3}}b_2\right)v_1^4v_4\\
&+\left(288i12^{\frac{1}{3}}b_1-96\sqrt{3}\,12^{\frac{1}{3}}b_1\right)v_1^3v_4\\
&+\left(-1458ib_2-486\sqrt{3}\,b_2\right)v_2^6v_3\\
&+\left(-216i\sqrt{3}\,a_3-216a_3\right)v_1v_2^2v_3v_5\\
&+\left(9i\sqrt{3}\,12^{\frac{1}{3}}b_2-9\,12^{\frac{1}{3}}b_2\right)v_1v_2^2v_4v_5=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
108\,12^{\frac{2}{3}} a_1 &= 0\\
-96\,12^{\frac{2}{3}} a_3 &= 0\\
-54\,12^{\frac{2}{3}} a_3 &= 0\\
-180\,12^{\frac{2}{3}} b_1 &= 0\\
-180\,12^{\frac{2}{3}} b_2 &= 0\\
192\sqrt{3}\,12^{\frac{2}{3}} a_1 &= 0\\
324\sqrt{3}\,12^{\frac{2}{3}} a_1 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}} a_3 &= 0\\
-162\sqrt{3}\,12^{\frac{2}{3}} a_3 &= 0\\
-540\sqrt{3}\,12^{\frac{2}{3}} b_1 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}} b_1 &= 0\\
-540\sqrt{3}\,12^{\frac{2}{3}} b_2 &= 0\\
-384\sqrt{3}\,12^{\frac{2}{3}} b_2 &= 0\\
216\,12^{\frac{2}{3}} a_2 - 288\,12^{\frac{2}{3}} b_3 &= 0\\
-2304 i a_3 - 768\sqrt{3}\, a_3 &= 0\\
-1944 i a_3 - 648\sqrt{3}\, a_3 &= 0\\
-1728 i b_2 - 576\sqrt{3}\, b_2 &= 0\\
-1458 i b_2 - 486\sqrt{3}\, b_2 &= 0\\
576\sqrt{3}\,12^{\frac{2}{3}} a_2 - 768\sqrt{3}\,12^{\frac{2}{3}} b_3 &= 0\\
648\sqrt{3}\,12^{\frac{2}{3}} a_2 - 864\sqrt{3}\,12^{\frac{2}{3}} b_3 &= 0\\
-216 i\sqrt{3}\, a_3 - 216 a_3 &= 0\\
-162 i\sqrt{3}\, b_2 - 162 b_2 &= 0\\
-144 i 12^{\frac{1}{3}} a_1 + 48\sqrt{3}\,12^{\frac{1}{3}} a_1 &= 0\\
-144 i 12^{\frac{1}{3}} a_3 + 48\sqrt{3}\,12^{\frac{1}{3}} a_3 &= 0\\
81 i 12^{\frac{1}{3}} b_1 - 27\sqrt{3}\,12^{\frac{1}{3}} b_1 &= 0\\
81 i 12^{\frac{1}{3}} b_2 - 27\sqrt{3}\,12^{\frac{1}{3}} b_2 &= 0\\
288 i 12^{\frac{1}{3}} b_1 - 96\sqrt{3}\,12^{\frac{1}{3}} b_1 &= 0\\
288 i 12^{\frac{1}{3}} b_2 - 96\sqrt{3}\,12^{\frac{1}{3}} b_2 &= 0\\
-24 i\sqrt{3}\,12^{\frac{1}{3}} a_3 + 24\,12^{\frac{1}{3}} a_3 &= 0\\
9 i\sqrt{3}\,12^{\frac{1}{3}} b_1 - 9\,12^{\frac{1}{3}} b_1 &= 0\\
9 i\sqrt{3}\,12^{\frac{1}{3}} b_2 - 9\,12^{\frac{1}{3}} b_2 &= 0\\
-432 i 12^{\frac{1}{3}} a_2 + 576 i 12^{\frac{1}{3}} b_3 + 144\sqrt{3}\,12^{\frac{1}{3}} a_2 - 192\sqrt{3}\,12^{\frac{1}{3}} b_3 &= 0\\
-243 i 12^{\frac{1}{3}} a_2 + 324 i 12^{\frac{1}{3}} b_3 + 81\sqrt{3}\,12^{\frac{1}{3}} a_2 - 108\sqrt{3}\,12^{\frac{1}{3}} b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{4b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{4x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   trying dAlembert
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-2*y(x)^2*x^3-y(x))/(2*y(x)*x^4+x), y(
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.14 (sec). Leaf size: 97

```
dsolve(y(x)=2*x*diff(y(x),x)+y(x)^2*diff(y(x),x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = -\frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = \frac{2i(-x^3)^{\frac{1}{4}} 6^{\frac{1}{4}}}{3}$$
$$y(x) = 0$$
$$y(x) = \sqrt{c_1 (c_1^2 + 2x)}$$
$$y(x) = -\sqrt{c_1 (c_1^2 + 2x)}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 119

```
DSolve[y[x]==2*x*y'[x]+y[x]^2*y'[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{2c_1 x + c_1{}^3}$$
$$y(x) \to \sqrt{2c_1 x + c_1{}^3}$$
$$y(x) \to (-1-i)\left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1-i)\left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (-1+i)\left(\frac{2}{3}\right)^{3/4} x^{3/4}$$
$$y(x) \to (1+i)\left(\frac{2}{3}\right)^{3/4} x^{3/4}$$

5.12 problem 28

5.12.1 Solving as dAlembert ode 1078

Internal problem ID [5334]
Internal file name [`OUTPUT/4825_Friday_February_02_2024_05_14_23_AM_52910488/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 28.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y'^2 - xy' - y = 0}$$

5.12.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 - xp - y = 0$$

Solving for y from the above results in

$$y = p^2 - xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -p$$
$$g = p^2$$

Hence (2) becomes

$$2p = (-x + 2p)\, p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$2p = 0$$

Solving for p from the above gives

$$p = 0$$

Substituting these in (1A) gives

$$y = 0$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{2p(x)}{-x + 2p\,(x)} \tag{3}$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-x(p) + 2p}{2p} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = \frac{1}{2p}$$
$$q(p) = 1$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{x(p)}{2p} = 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{2p}dp} \\ &= \sqrt{p}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= \mu \\ \frac{\mathrm{d}}{\mathrm{d}p}(\sqrt{p}\,x) &= \sqrt{p} \\ \mathrm{d}(\sqrt{p}\,x) &= \sqrt{p}\mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}\sqrt{p}\,x &= \int \sqrt{p}\,\mathrm{d}p \\ \sqrt{p}\,x &= \frac{2p^{\frac{3}{2}}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{p}$ results in

$$x(p) = \frac{2p}{3} + \frac{c_1}{\sqrt{p}}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= \frac{x}{2} + \frac{\sqrt{x^2+4y}}{2} \\ p &= \frac{x}{2} - \frac{\sqrt{x^2+4y}}{2}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{x}{3} + \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x+2\sqrt{x^2+4y}}} \\ x &= \frac{x}{3} - \frac{\sqrt{x^2+4y}}{3} + \frac{2c_1}{\sqrt{2x-2\sqrt{x^2+4y}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}} \tag{2}$$

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{x}{3} + \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x + 2\sqrt{x^2 + 4y}}}$$

Verified OK.

$$x = \frac{x}{3} - \frac{\sqrt{x^2 + 4y}}{3} + \frac{2c_1}{\sqrt{2x - 2\sqrt{x^2 + 4y}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 77

```
dsolve(diff(y(x),x)^2-x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$\frac{c_1}{\sqrt{2x-2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}+\frac{\sqrt{x^2+4y(x)}}{3}=0$$
$$\frac{c_1}{\sqrt{2x+2\sqrt{x^2+4y(x)}}}+\frac{2x}{3}-\frac{\sqrt{x^2+4y(x)}}{3}=0$$

✓ Solution by Mathematica

Time used: 60.129 (sec). Leaf size: 1003

```
DSolve[y'[x]^2-x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(x^2+\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)^2+8e^{3c_1}x}{4\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}$$

$$y(x) \to \frac{1}{8}\left(4x^2-\frac{i\left(\sqrt{3}-i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}+i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{8}\left(4x^2+\frac{i\left(\sqrt{3}+i\right)x\left(x^3+8e^{3c_1}\right)}{\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}}-\left(1+i\sqrt{3}\right)\sqrt[3]{-x^6+20e^{3c_1}x^3+8\sqrt{e^{3c_1}\left(-x^3+e^{3c_1}\right)^3}+8e^{6c_1}}\right)$$

$$y(x) \to \frac{2\sqrt[3]{2}x^4+2^{2/3}\left(-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}\right)^{2/3}+4x^2\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x\right.}}}{8\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}$$

$$y(x) \to \frac{1}{16}\left(8x^2+\frac{2\sqrt[3]{2}\left(1+i\sqrt{3}\right)x\left(-x^3+2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}+i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{16}\left(8x^2+\frac{2i\sqrt[3]{2}\left(\sqrt{3}+i\right)x\left(x^3-2e^{3c_1}\right)}{\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}}-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{-2x^6-10e^{3c_1}x^3+\sqrt{e^{3c_1}\left(4x^3+e^{3c_1}\right)^3}+e^{6c_1}}\right)$$

5.13 problem 29

5.13.1 Solving as dAlembert ode 1084

Internal problem ID [5335]
Internal file name [`OUTPUT/4826_Friday_February_02_2024_05_14_23_AM_73749985/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 29.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{y - (y'+1)\,x - {y'}^2 = 0}$$

5.13.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y - (p+1)\,x - p^2 = 0$$

Solving for y from the above results in

$$y = (p+1)\,x + p^2 \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p + 1 \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
-1 = (x + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-1 = 0
$$

No singular solution are found

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{1}{x + 2p\,(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = -x(p) - 2p \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp}x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= 1 \\
q(p) &= -2p
\end{aligned}
$$

Hence the ode is

$$
\frac{d}{dp}x(p) + x(p) = -2p
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 1 dp} \\
&= \mathrm{e}^{p}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dp}(\mu x) &= (\mu)(-2p) \\
\frac{d}{dp}(e^p x) &= (e^p)(-2p) \\
d(e^p x) &= (-2\,e^p p)\;dp
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^p x &= \int -2\,e^p p\,dp \\
e^p x &= -2(p-1)\,e^p + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^p$ results in

$$
x(p) = -2\,e^{-p}(p-1)\,e^p + c_1 e^{-p}
$$

which simplifies to

$$
x(p) = -2p + 2 + c_1 e^{-p}
$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$
\begin{aligned}
p &= -\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2} \\
p &= -\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}
\end{aligned}
$$

Substituting the above in the solution for x found above gives

$$
\begin{aligned}
x &= x - \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2-4x+4y}}{2}} \\
x &= x + \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2-4x+4y}}{2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = x - \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2-4x+4y}}{2}} \tag{1}
$$

$$
x = x + \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2-4x+4y}}{2}} \tag{2}
$$

Verification of solutions

$$x = x - \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} - \frac{\sqrt{x^2 - 4x + 4y}}{2}}$$

Verified OK.

$$x = x + \sqrt{x^2 - 4x + 4y} + 2 + c_1 e^{\frac{x}{2} + \frac{\sqrt{x^2 - 4x + 4y}}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 36

```
dsolve(y(x)=(1+diff(y(x),x))*x+diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = x - \frac{x^2}{4} + \text{LambertW}\left(\frac{c_1 e^{-1+\frac{x}{2}}}{2}\right)^2 + 2\,\text{LambertW}\left(\frac{c_1 e^{-1+\frac{x}{2}}}{2}\right) + 1$$

✓ Solution by Mathematica

Time used: 1.048 (sec). Leaf size: 177

```
DSolve[y[x]==(1+y'[x])*x+y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[-\sqrt{x^2+4y(x)-4x}+2\log\left(\sqrt{x^2+4y(x)-4x}-x+2\right)-2\log\left(-x\sqrt{x^2+4y(x)-4x}+x^2+4y(x)-2x-4\right)+x=c_1,y(x)\right]$$

$$\text{Solve}\left[-4\text{arctanh}\left(\frac{(x-5)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+7x-6}{(x-3)\sqrt{x^2+4y(x)-4x}-x^2-4y(x)+5x-2}\right)+\sqrt{x^2+4y(x)-4x}+x=c_1,y(x)\right]$$

5.14 problem 30

5.14.1 Maple step by step solution 1092

Internal problem ID [5336]
Internal file name [`OUTPUT/4827_Friday_February_02_2024_05_14_23_AM_68613889/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 30.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y - 2y' - \sqrt{1 + y'^2} = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2y}{3} + \frac{\sqrt{y^2+3}}{3} \tag{1}$$

$$y' = \frac{2y}{3} - \frac{\sqrt{y^2+3}}{3} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} dy = \int dx$$

$$
-\frac{\sqrt{(-1+y)^2+2+2y}}{2}-\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)
+\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)+\frac{\sqrt{(y+1)^2-2y+2}}{2}
-\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_1
$$

Summary

The solution(s) found are the following

$$
-\frac{\sqrt{(-1+y)^2+2+2y}}{2}-\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)
+\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)+\frac{\sqrt{(y+1)^2-2y+2}}{2}
-\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_1 \tag{1}
$$

Verification of solutions

$$
-\frac{\sqrt{(-1+y)^2+2+2y}}{2}-\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)
+\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)+\frac{\sqrt{(y+1)^2-2y+2}}{2}
-\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_1
$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$
\int \frac{1}{\frac{2y}{3}-\frac{\sqrt{y^2+3}}{3}}dy=\int dx
$$

$$
\frac{\sqrt{(-1+y)^2+2+2y}}{2}+\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)-\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)-\frac{\sqrt{(y+1)^2-2y+2}}{2}+\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_2
$$

Summary

The solution(s) found are the following

$$
\frac{\sqrt{(-1+y)^2+2+2y}}{2}+\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)-\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)-\frac{\sqrt{(y+1)^2-2y+2}}{2}+\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_2 \tag{1}
$$

Verification of solutions

$$
\frac{\sqrt{(-1+y)^2+2+2y}}{2}+\operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right)-\operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right)-\frac{\sqrt{(y+1)^2-2y+2}}{2}+\operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2-2y+2}}\right)+\ln(-1+y)+\ln(y+1)=x+c_2
$$

Verified OK.

5.14.1 Maple step by step solution

Let's solve

$y - 2y' - \sqrt{1 + y'^2} = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} = 1$

- Integrate both sides with respect to x

$\int \frac{y'}{\frac{2y}{3} + \frac{\sqrt{y^2+3}}{3}} dx = \int 1dx + c_1$

- Evaluate integral

$-\frac{\sqrt{(-1+y)^2+2+2y}}{2} - \operatorname{arcsinh}\left(\frac{\sqrt{3}\,y}{3}\right) + \operatorname{arctanh}\left(\frac{6+2y}{4\sqrt{(-1+y)^2+2+2y}}\right) + \frac{\sqrt{(y+1)^2-2y+2}}{2} - \operatorname{arctanh}\left(\frac{6-2y}{4\sqrt{(y+1)^2\ldots}}\right.$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 125

```
dsolve(y(x)=2*diff(y(x),x)+sqrt(1+diff(y(x),x)^2),y(x), singsol=all)
```

$$x - \operatorname{arcsinh}\left(\frac{\sqrt{3}\, y(x)}{3}\right) + \operatorname{arctanh}\left(\frac{3 + y(x)}{2\sqrt{y(x)^{2} + 3}}\right) + \operatorname{arctanh}\left(\frac{-3 + y(x)}{2\sqrt{y(x)^{2} + 3}}\right) - \ln\left(y(x) - 1\right) - \ln\left(y(x) + 1\right) - c_1 = 0$$

$$x + \operatorname{arcsinh}\left(\frac{\sqrt{3}\, y(x)}{3}\right) - \operatorname{arctanh}\left(\frac{3 + y(x)}{2\sqrt{y(x)^{2} + 3}}\right) - \operatorname{arctanh}\left(\frac{-3 + y(x)}{2\sqrt{y(x)^{2} + 3}}\right) - \ln\left(y(x) - 1\right) - \ln\left(y(x) + 1\right) - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.301 (sec). Leaf size: 4821

```
DSolve[y[x]==2*y'[x]+Sqrt[1+y'[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

5.15 problem 31

5.15.1 Solving as dAlembert ode 1094

Internal problem ID [5337]
Internal file name [`OUTPUT/4828_Friday_February_02_2024_05_14_27_AM_37782308/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 9. Equations of first order and higher degree. Supplemetary problems. Page 65
Problem number: 31.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_homogeneous, `class A`], _rational, _dAlembert]`

$$\boxed{yy'^2 - xy' + 3y = 0}$$

5.15.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$y\,p^2 - xp + 3y = 0$$

Solving for y from the above results in

$$y = \frac{xp}{p^2 + 3} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{p^2+3} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p}{p^2+3} = x\left(\frac{1}{p^2+3} - \frac{2p^2}{(p^2+3)^2}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p}{p^2+3} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= i\sqrt{2} \\
p &= -i\sqrt{2}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -i\sqrt{2}\,x \\
y &= i\sqrt{2}\,x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)}{p(x)^2+3}}{x\left(\frac{1}{p(x)^2+3} - \frac{2p(x)^2}{\left(p(x)^2+3\right)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(x,p) \\
&= f(x)g(p) \\
&= -\frac{(p^2+2)\,p(p^2+3)}{x\,(p^2-3)}
\end{aligned}
$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = \frac{(p^2+2)p(p^2+3)}{p^2-3}$. Integrating both sides gives

$$\frac{1}{\frac{(p^2+2)p(p^2+3)}{p^2-3}}\,dp = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{(p^2+2)p(p^2+3)}{p^2-3}}\,dp = \int -\frac{1}{x}\,dx$$

$$-\ln\left(p^2+3\right) + \frac{5\ln\left(p^2+2\right)}{4} - \frac{\ln(p)}{2} = -\ln(x) + c_1$$

Raising both side to exponential gives

$$e^{-\ln(p^2+3)+\frac{5\ln\left(p^2+2\right)}{4}-\frac{\ln(p)}{2}} = e^{-\ln(x)+c_1}$$

Which simplifies to

$$\frac{(p^2+2)^{\frac{5}{4}}}{(p^2+3)\sqrt{p}} = \frac{c_2}{x}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x^3\mathrm{RootOf}\left((c_2^4-x^4)\_Z^{20}+2c_2^4\_Z^{16}-2c_2^4\_Z^{12}-8c_2^4\_Z^{8}-\cdots\right.}{c_2^2\left(\mathrm{RootOf}\left((c_2^4-x^4)\_Z^{20}+2c_2^4\_Z^{16}-2c_2^4\_Z^{12}-8c_2^4\_Z^{8}-7c_2^4\_Z^{4}-2c_2^4\right)^4+1\right)^2\left(\frac{\mathrm{RootOf}\left((c_2^4-x^4\cdots\right.}{c_2^4\left(\mathrm{RootOf}\left((c_2^4-x\cdots\right.\right.}\right.}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$

$$y = -i\sqrt{2}\,x \tag{2}$$

$$y = i\sqrt{2}\,x \tag{3}$$

$$y = \frac{x^3\mathrm{RootOf}\left((c_2^4-x^4)\_Z^{20}+2c_2^4\_Z^{16}-2c_2^4\_Z^{12}-8c_2^4\_Z^{8}-7c_2^4\cdots\right.}{c_2^2\left(\mathrm{RootOf}\left((c_2^4-x^4)\_Z^{20}+2c_2^4\_Z^{16}-2c_2^4\_Z^{12}-8c_2^4\_Z^{8}-7c_2^4\_Z^{4}-2c_2^4\right)^4+1\right)^2\left(\frac{\mathrm{RootOf}\left((c_2^4-x^4)\_\cdots\right.}{c_2^4\left(\mathrm{RootOf}\left((c_2^4-x^4)\_\cdots\right.\right.}\right.} \tag{4}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$y = -i\sqrt{2}\, x$$

Verified OK.

$$y = i\sqrt{2}\, x$$

Verified OK.

$$y = \frac{x^3 \mathrm{RootOf}\left(\left(c_2^4 - x^4\right)\_Z^{20} + 2c_2^4\_Z^{16} - 2c_2^4\_Z^{12} - 8c_2^4\_Z^8 - 7c_2^4\right.}{c_2^2\left(\mathrm{RootOf}\left(\left(c_2^4 - x^4\right)\_Z^{20} + 2c_2^4\_Z^{16} - 2c_2^4\_Z^{12} - 8c_2^4\_Z^8 - 7c_2^4\_Z^4 - 2c_2^4\right)^4 + 1\right)^2 \left(\frac{\mathrm{RootOf}\left(\left(c_2^4 - x^4\right)\_\right.}{c_2^4\left(\mathrm{RootOf}\left(\left(c_2^4 - x^4\right)\_\right.\right.}\right.}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 5.141 (sec). Leaf size: 159

```
dsolve(y(x)*diff(y(x),x)^2-x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = 0$$

$$\ln(x) - \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{-x^2+12y(x)^2}{x^2}}}\right)}{4} + \frac{5\operatorname{arctanh}\left(\frac{\sqrt{\frac{x^2-12y(x)^2}{x^2}}}{5}\right)}{4} + \frac{5\ln\left(\frac{2x^2+y(x)^2}{x^2}\right)}{8} - \frac{\ln\left(\frac{y(x)}{x}\right)}{4} - c_1 = 0$$

$$\ln(x) + \frac{\operatorname{arctanh}\left(\frac{1}{\sqrt{-\frac{-x^2+12y(x)^2}{x^2}}}\right)}{4} - \frac{5\operatorname{arctanh}\left(\frac{\sqrt{\frac{x^2-12y(x)^2}{x^2}}}{5}\right)}{4} + \frac{5\ln\left(\frac{2x^2+y(x)^2}{x^2}\right)}{8} - \frac{\ln\left(\frac{y(x)}{x}\right)}{4} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.281 (sec). Leaf size: 1131

```
DSolve[y[x]*y'[x]^2-x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to \sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to -\sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to \sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to -\sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to \sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to -\sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to \sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to -\sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

$$y(x) \to \sqrt{\text{Root}\left[62208\#1^5 + 622080\#1^4x^2 + \#1^3\left(2488320x^4 - 864e^{8c_1}\right) + \#1^2\left(4976640x^6 + 16416e^{8c_1}x^2\right) +\right.}$$

6 Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74

6.1 problem 10 1101
6.2 problem 11 1105
6.3 problem 12 1119
6.4 problem 13 1124
6.5 problem 14 1129
6.6 problem 15 1134
6.7 problem 16 1147
6.8 problem 17 1153
6.9 problem 18 1156
6.10 problem 19 1183

6.1 problem 10

6.1.1 Solving as clairaut ode 1101

Internal problem ID [5338]
Internal file name [`OUTPUT/4829_Sunday_February_04_2024_12_46_05_AM_16279652/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 10.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"clairaut"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _Clairaut]`

$$\boxed{y - xy' + 2y'^2 = 0}$$

6.1.1 Solving as clairaut ode

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$2p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -2p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved. We start by replacing y' by p which gives

$$\begin{aligned} y &= -2p^2 + xp \\ &= -2p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -2p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p. The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = -2c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -2p^2$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - 4p \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = \frac{x}{4}$$

Substituting the above back in (1) results in

$$y_1 = \frac{x^2}{8}$$

Summary

The solution(s) found are the following

$$y = -2c_1^2 + c_1 x \tag{1}$$

$$y = \frac{x^2}{8} \tag{2}$$

Verification of solutions

$$y = -2c_1^2 + c_1 x$$

Verified OK.

$$y = \frac{x^2}{8}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(y(x)=diff(y(x),x)*x-2*diff(y(x),x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{8}$$
$$y(x) = c_1(x - 2c_1)$$

✓ Solution by Mathematica

Time used: 0.006 (sec). Leaf size: 25

```
DSolve[y[x]==y'[x]*x-2*y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x - 2c_1)$$
$$y(x) \to \frac{x^2}{8}$$

6.2 problem 11

Internal problem ID [5339]
Internal file name [`OUTPUT/4830_Sunday_February_04_2024_12_46_05_AM_48988675/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 11.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _rational]
```

$$\boxed{y^2 {y'}^2 + 3xy' - y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{-3x + \sqrt{9x^2 + 4y^3}}{2y^2} \tag{1}$$
$$y' = -\frac{3x + \sqrt{9x^2 + 4y^3}}{2y^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$\begin{aligned} y' &= \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{\left(-3x + \sqrt{4y^3 + 9x^2}\right)(b_3 - a_2)}{2y^2} - \frac{\left(-3x + \sqrt{4y^3 + 9x^2}\right)^2 a_3}{4y^4} \\
&- \frac{\left(-3 + \frac{9x}{\sqrt{4y^3+9x^2}}\right)(xa_2 + ya_3 + a_1)}{2y^2} \\
&- \left(-\frac{-3x + \sqrt{4y^3 + 9x^2}}{y^3} + \frac{3}{\sqrt{4y^3 + 9x^2}}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2y^4\sqrt{4y^3 + 9x^2} - 4x\,y^4b_2 + 8y^5a_2 - 12y^5b_3 + 12\sqrt{4y^3 + 9x^2}\,x^2yb_2 - 12\sqrt{4y^3 + 9x^2}\,x\,y^2a_2 + 18\sqrt{4y^3}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&4b_2y^4\sqrt{4y^3 + 9x^2} + 4x\,y^4b_2 - 8y^5a_2 + 12y^5b_3 - 12\sqrt{4y^3 + 9x^2}\,x^2yb_2 \\
&+ 12\sqrt{4y^3 + 9x^2}\,x\,y^2a_2 - 18\sqrt{4y^3 + 9x^2}\,x\,y^2b_3 + 6\sqrt{4y^3 + 9x^2}\,y^3a_3 \\
&+ 36x^3yb_2 - 36x^2y^2a_2 + 54x^2y^2b_3 + 6x\,y^3a_3 + 4y^4b_1 \\
&- \left(4y^3 + 9x^2\right)^{\frac{3}{2}} a_3 - 9\sqrt{4y^3 + 9x^2}\,x^2a_3 - 12\sqrt{4y^3 + 9x^2}\,xyb_1 \\
&+ 6\sqrt{4y^3 + 9x^2}\,y^2a_1 + 54x^3a_3 + 36x^2yb_1 - 18x\,y^2a_1 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2y^4\sqrt{4y^3+9x^2}-12x\,y^4b_2-12y^5b_3+4\left(4y^3+9x^2\right)xyb_2 \\
&-2\left(4y^3+9x^2\right)y^2a_2+6\left(4y^3+9x^2\right)y^2b_3-12\sqrt{4y^3+9x^2}\,x^2yb_2 \\
&+12\sqrt{4y^3+9x^2}\,x\,y^2a_2-18\sqrt{4y^3+9x^2}\,x\,y^2b_3+6\sqrt{4y^3+9x^2}\,y^3a_3 \\
&-18x^2y^2a_2-18x\,y^3a_3-12y^4b_1-\left(4y^3+9x^2\right)^{\frac{3}{2}}a_3 \\
&+6\left(4y^3+9x^2\right)xa_3+4\left(4y^3+9x^2\right)yb_1-9\sqrt{4y^3+9x^2}\,x^2a_3 \\
&-12\sqrt{4y^3+9x^2}\,xyb_1+6\sqrt{4y^3+9x^2}\,y^2a_1-18x\,y^2a_1=0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4x\,y^4b_2+4b_2y^4\sqrt{4y^3+9x^2}-8y^5a_2+12y^5b_3+36x^3yb_2-12\sqrt{4y^3+9x^2}\,x^2yb_2 \\
&-36x^2y^2a_2+54x^2y^2b_3+12\sqrt{4y^3+9x^2}\,x\,y^2a_2-18\sqrt{4y^3+9x^2}\,x\,y^2b_3 \\
&+6x\,y^3a_3+2\sqrt{4y^3+9x^2}\,y^3a_3+4y^4b_1+54x^3a_3-18\sqrt{4y^3+9x^2}\,x^2a_3 \\
&+36x^2yb_1-12\sqrt{4y^3+9x^2}\,xyb_1-18x\,y^2a_1+6\sqrt{4y^3+9x^2}\,y^2a_1=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x,y,\sqrt{4y^3+9x^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1,y=v_2,\sqrt{4y^3+9x^2}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-8v_2^5a_2+4v_1v_2^4b_2+4b_2v_2^4v_3+12v_2^5b_3-36v_1^2v_2^2a_2+12v_3v_1v_2^2a_2+6v_1v_2^3a_3 \\
&+2v_3v_2^3a_3+4v_2^4b_1+36v_1^3v_2b_2-12v_3v_1^2v_2b_2+54v_1^2v_2^2b_3-18v_3v_1v_2^2b_3 \\
&-18v_1v_2^2a_1+6v_3v_2^2a_1+54v_1^3a_3-18v_3v_1^2a_3+36v_1^2v_2b_1-12v_3v_1v_2b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&36v_1^3v_2b_2+54v_1^3a_3+\left(-36a_2+54b_3\right)v_1^2v_2^2-12v_3v_1^2v_2b_2+36v_1^2v_2b_1 \\
&-18v_3v_1^2a_3+4v_1v_2^4b_2+6v_1v_2^3a_3+\left(12a_2-18b_3\right)v_1v_2^2v_3-18v_1v_2^2a_1 \\
&-12v_3v_1v_2b_1+\left(-8a_2+12b_3\right)v_2^5+4b_2v_2^4v_3+4v_2^4b_1+2v_3v_2^3a_3+6v_3v_2^2a_1=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-18a_1 &= 0 \\
6a_1 &= 0 \\
-18a_3 &= 0 \\
2a_3 &= 0 \\
6a_3 &= 0 \\
54a_3 &= 0 \\
-12b_1 &= 0 \\
4b_1 &= 0 \\
36b_1 &= 0 \\
-12b_2 &= 0 \\
4b_2 &= 0 \\
36b_2 &= 0 \\
-36a_2+54b_3 &= 0 \\
-8a_2+12b_3 &= 0 \\
12a_2-18b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = \frac{3x}{2}$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}\right)\left(\frac{3x}{2}\right) \\
&= \frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2}\,x}{4y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^3 + 9x^2 - 3\sqrt{4y^3 + 9x^2}\,x}{4y^2}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) - \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3 + 9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-3x + \sqrt{4y^3 + 9x^2}}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2}{\sqrt{4y^3 + 9x^2}} \\
S_y &= \frac{1 + \frac{3x}{\sqrt{4y^3+9x^2}}}{y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary

The solution(s) found are the following

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1 \tag{1}$$

Verification of solutions

$$\ln(y) - \frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$\begin{aligned}
y' &= -\frac{\sqrt{4y^3+9x^2}+3x}{2y^2} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(\sqrt{4y^3+9x^2}+3x\right)(b_3-a_2)}{2y^2} - \frac{\left(\sqrt{4y^3+9x^2}+3x\right)^2 a_3}{4y^4} \\
&+ \frac{\left(\frac{9x}{\sqrt{4y^3+9x^2}}+3\right)(xa_2+ya_3+a_1)}{2y^2} \\
&- \left(-\frac{3}{\sqrt{4y^3+9x^2}}+\frac{\sqrt{4y^3+9x^2}+3x}{y^3}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{-4b_2\sqrt{4y^3+9x^2}\,y^4 + 4x\,y^4 b_2 - 8y^5 a_2 + 12y^5 b_3 + 12\sqrt{4y^3+9x^2}\,x^2 y b_2 - 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 + 18\sqrt{4y^3}}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&4b_2\sqrt{4y^3+9x^2}\,y^4 - 4x\,y^4 b_2 + 8y^5 a_2 - 12y^5 b_3 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
&+ 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 + 6\sqrt{4y^3+9x^2}\,y^3 a_3 \\
&- 36x^3 y b_2 + 36x^2 y^2 a_2 - 54x^2 y^2 b_3 - 6x\,y^3 a_3 - 4y^4 b_1 \\
&- \left(4y^3+9x^2\right)^{\frac{3}{2}} a_3 - 9\sqrt{4y^3+9x^2}\,x^2 a_3 - 12\sqrt{4y^3+9x^2}\,xyb_1 \\
&+ 6\sqrt{4y^3+9x^2}\,y^2 a_1 - 54x^3 a_3 - 36x^2 y b_1 + 18x\,y^2 a_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2\sqrt{4y^3+9x^2}\,y^4 + 12x\,y^4 b_2 + 12y^5 b_3 - 4\left(4y^3+9x^2\right)xyb_2 \\
&+ 2\left(4y^3+9x^2\right)y^2 a_2 - 6\left(4y^3+9x^2\right)y^2 b_3 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
&+ 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 + 6\sqrt{4y^3+9x^2}\,y^3 a_3 \\
&+ 18x^2 y^2 a_2 + 18x\,y^3 a_3 + 12y^4 b_1 - \left(4y^3+9x^2\right)^{\frac{3}{2}} a_3 \\
&- 6\left(4y^3+9x^2\right)xa_3 - 4\left(4y^3+9x^2\right)yb_1 - 9\sqrt{4y^3+9x^2}\,x^2 a_3 \\
&- 12\sqrt{4y^3+9x^2}\,xyb_1 + 6\sqrt{4y^3+9x^2}\,y^2 a_1 + 18x\,y^2 a_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-4x\,y^4 b_2 + 4b_2\sqrt{4y^3+9x^2}\,y^4 + 8y^5 a_2 - 12y^5 b_3 - 36x^3 y b_2 - 12\sqrt{4y^3+9x^2}\,x^2 y b_2 \\
&+ 36x^2 y^2 a_2 - 54x^2 y^2 b_3 + 12\sqrt{4y^3+9x^2}\,x\,y^2 a_2 - 18\sqrt{4y^3+9x^2}\,x\,y^2 b_3 \\
&- 6x\,y^3 a_3 + 2\sqrt{4y^3+9x^2}\,y^3 a_3 - 4y^4 b_1 - 54x^3 a_3 - 18\sqrt{4y^3+9x^2}\,x^2 a_3 \\
&- 36x^2 y b_1 - 12\sqrt{4y^3+9x^2}\,xyb_1 + 18x\,y^2 a_1 + 6\sqrt{4y^3+9x^2}\,y^2 a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{4y^3+9x^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{4y^3+9x^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &8v_2^5a_2 - 4v_1v_2^4b_2 + 4b_2v_3v_2^4 - 12v_2^5b_3 + 36v_1^2v_2^2a_2 + 12v_3v_1v_2^2a_2 - 6v_1v_2^3a_3 \\ &+ 2v_3v_2^3a_3 - 4v_2^4b_1 - 36v_1^3v_2b_2 - 12v_3v_1^2v_2b_2 - 54v_1^2v_2^2b_3 - 18v_3v_1v_2^2b_3 \\ &+ 18v_1v_2^2a_1 + 6v_3v_2^2a_1 - 54v_1^3a_3 - 18v_3v_1^2a_3 - 36v_1^2v_2b_1 - 12v_3v_1v_2b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-36v_1^3v_2b_2 - 54v_1^3a_3 + (36a_2 - 54b_3)\, v_1^2v_2^2 - 12v_3v_1^2v_2b_2 - 36v_1^2v_2b_1 \\ &- 18v_3v_1^2a_3 - 4v_1v_2^4b_2 - 6v_1v_2^3a_3 + (12a_2 - 18b_3)\, v_1v_2^2v_3 + 18v_1v_2^2a_1 \\ &- 12v_3v_1v_2b_1 + (8a_2 - 12b_3)\, v_2^5 + 4b_2v_3v_2^4 - 4v_2^4b_1 + 2v_3v_2^3a_3 + 6v_3v_2^2a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
6a_1 &= 0 \\
18a_1 &= 0 \\
-54a_3 &= 0 \\
-18a_3 &= 0 \\
-6a_3 &= 0 \\
2a_3 &= 0 \\
-36b_1 &= 0 \\
-12b_1 &= 0 \\
-4b_1 &= 0 \\
-36b_2 &= 0 \\
-12b_2 &= 0 \\
-4b_2 &= 0 \\
4b_2 &= 0 \\
8a_2 - 12b_3 &= 0 \\
12a_2 - 18b_3 &= 0 \\
36a_2 - 54b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{3b_3}{2} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{3x}{2} \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}\right)\left(\frac{3x}{2}\right) \\
&= \frac{4y^3 + 3\sqrt{4y^3 + 9x^2}\,x + 9x^2}{4y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4y^3+3\sqrt{4y^3+9x^2}\,x+9x^2}{4y^2}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(y) + \frac{2x \operatorname{arctanh}\left(\frac{\sqrt{4y^3+9x^2}}{3\sqrt{x^2}}\right)}{3\sqrt{x^2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\sqrt{4y^3 + 9x^2} + 3x}{2y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{\sqrt{4y^3 + 9x^2}} \\ S_y &= \frac{1 - \frac{3x}{\sqrt{4y^3+9x^2}}}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(y) + \frac{2\operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Which simplifies to

$$\ln(y) + \frac{2\operatorname{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3} = c_1$$

Summary
The solution(s) found are the following

$$\ln(y)+\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3}=c_1 \tag{1}$$

Verification of solutions

$$\ln(y)+\frac{2\,\mathrm{arctanh}\left(\frac{\sqrt{9x^2+4y^3}}{3x}\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 3`[3/2*x, y], [2/3*y^3+3*x^2, y*x]
```

✓ Solution by Maple

Time used: 0.141 (sec). Leaf size: 123

```
dsolve(y(x)^2*diff(y(x),x)^2+3*x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{18^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}}{2}$$
$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}\left(3i3^{\frac{1}{6}}+3^{\frac{2}{3}}\right)}{4}$$
$$y(x) = -\frac{2^{\frac{1}{3}}(-x^2)^{\frac{1}{3}}\left(3^{\frac{2}{3}}-3i3^{\frac{1}{6}}\right)}{4}$$
$$y(x) = 0$$
$$y(x) = \mathrm{RootOf}\left(-2\ln(x)-3\left(\int^{\_Z}\frac{4\_a^3+3\sqrt{4\_a^3+9}+9}{\_a\,(4\_a^3+9)}d\_a\right)+2c_1\right)x^{\frac{2}{3}}$$

✓ Solution by Mathematica

Time used: 0.597 (sec). Leaf size: 239

```
DSolve[y[x]^2*y'[x]^2+3*x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to -\sqrt[3]{-1}e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to (-1)^{2/3}e^{\frac{c_1}{3}}\sqrt[3]{-3x+e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to -\sqrt[3]{-1}e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to (-1)^{2/3}e^{\frac{c_1}{3}}\sqrt[3]{3x+e^{c_1}}$$
$$y(x) \to 0$$
$$y(x) \to -\left(-\frac{3}{2}\right)^{2/3}x^{2/3}$$
$$y(x) \to -\left(\frac{3}{2}\right)^{2/3}x^{2/3}$$
$$y(x) \to \frac{\sqrt[3]{-1}3^{2/3}x^{2/3}}{2^{2/3}}$$

6.3 problem 12

6.3.1 Solving as dAlembert ode 1119

Internal problem ID [5340]
Internal file name [`OUTPUT/4831_Sunday_February_04_2024_12_46_09_AM_99128311/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 12.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{{y'}^2x - 2yy' = -4x}$$

6.3.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2x - 2yp = -4x$$

Solving for y from the above results in

$$y = \frac{x(p^2+4)}{2p} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2+4}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2+4}{2p} = x\left(1 - \frac{p^2+4}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2+4}{2p} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 2 \\
p &= -2
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -2x \\
y &= 2x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2+4}{2p(x)}}{x\left(1 - \frac{p(x)^2+4}{2p(x)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= 0
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{x}\right) &= 0\end{aligned}$$

Integrating gives

$$\frac{p}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Summary
The solution(s) found are the following

$$y = -2x \tag{1}$$
$$y = 2x \tag{2}$$
$$y = \frac{c_1^2 x^2 + 4}{2c_1} \tag{3}$$

Verification of solutions

$$y = -2x$$

Verified OK.

$$y = 2x$$

Verified OK.

$$y = \frac{c_1^2 x^2 + 4}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 30

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+4*x=0,y(x), singsol=all)
```

$$y(x) = -2x$$
$$y(x) = 2x$$
$$y(x) = \frac{4c_1^2 + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.271 (sec). Leaf size: 43

```
DSolve[x*y'[x]^2-2*y[x]*y'[x]+4*x==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x\cosh(-\log(x) + c_1)$$
$$y(x) \to -2x\cosh(\log(x) + c_1)$$
$$y(x) \to -2x$$
$$y(x) \to 2x$$

6.4 problem 13

6.4.1 Solving as dAlembert ode 1124

Internal problem ID [5341]
Internal file name [`OUTPUT/4832_Sunday_February_04_2024_12_46_10_AM_46687559/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 13.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{{y'}^2 x - 2yy' + 2y = -x}$$

6.4.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$p^2 x - 2yp + 2y = -x$$

Solving for y from the above results in

$$y = \frac{x(p^2 + 1)}{2p - 2} \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 + 1}{2p - 2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 + 1}{2p - 2} = x\left(\frac{2p}{2p - 2} - \frac{2(p^2 + 1)}{(2p - 2)^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 + 1}{2p - 2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 1 + \sqrt{2} \\
p &= 1 - \sqrt{2}
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= x - x\sqrt{2} \\
y &= x + x\sqrt{2}
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 + 1}{2p(x) - 2}}{x\left(\frac{2p(x)}{2p(x) - 2} - \frac{2\left(p(x)^2 + 1\right)}{(2p(x) - 2)^2}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -\frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = -\frac{1}{x}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)\left(-\frac{1}{x}\right)$$
$$\frac{d}{dx}\left(\frac{p}{x}\right) = \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right)$$
$$d\left(\frac{p}{x}\right) = \left(-\frac{1}{x^2}\right) dx$$

Integrating gives

$$\frac{p}{x} = \int -\frac{1}{x^2} dx$$
$$\frac{p}{x} = \frac{1}{x} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = c_1 x + 1$$

Substituing the above solution for p in (2A) gives

$$y = \frac{(c_1 x + 1)^2 + 1}{2c_1}$$

Summary

The solution(s) found are the following

$$y = x - x\sqrt{2} \tag{1}$$
$$y = x + x\sqrt{2} \tag{2}$$
$$y = \frac{(c_1 x + 1)^2 + 1}{2c_1} \tag{3}$$

Verification of solutions

$$y = x - x\sqrt{2}$$

Verified OK.

$$y = x + x\sqrt{2}$$

Verified OK.

$$y = \frac{(c_1 x + 1)^2 + 1}{2c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 44

```
dsolve(x*diff(y(x),x)^2-2*y(x)*diff(y(x),x)+x+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(1 - \sqrt{2}\right) x$$
$$y(x) = \left(1 + \sqrt{2}\right) x$$
$$y(x) = \frac{2c_1^2 + 2c_1 x + x^2}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.153 (sec). Leaf size: 78

```
DSolve[x*y'[x]^2-2*y[x]*y'[x]+x+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}e^{-c_1}x^2 + x - e^{c_1}$$
$$y(x) \to -e^{c_1}x^2 + x - \frac{e^{-c_1}}{2}$$
$$y(x) \to x - \sqrt{2}x$$
$$y(x) \to \left(1 + \sqrt{2}\right)x$$

6.5 problem 14

6.5.1 Maple step by step solution 1130

Internal problem ID [5342]
Internal file name [`OUTPUT/4833_Sunday_February_04_2024_12_46_10_AM_26499802/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 14.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(3y-1)^2 {y'}^2 - 4y = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2\sqrt{y}}{3y-1} \tag{1}$$

$$y' = -\frac{2\sqrt{y}}{3y-1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{3y-1}{2\sqrt{y}} dy = \int dx$$

$$\sqrt{y}\,(-1+y) = x + c_1$$

Summary
The solution(s) found are the following

$$\sqrt{y}\,(-1+y) = x + c_1 \tag{1}$$

Verification of solutions

$$\sqrt{y}\,(-1+y) = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{3y-1}{2\sqrt{y}}dy = \int dx$$
$$-\sqrt{y}\,(-1+y) = x + c_2$$

Summary

The solution(s) found are the following

$$-\sqrt{y}\,(-1+y) = x + c_2 \tag{1}$$

Verification of solutions

$$-\sqrt{y}\,(-1+y) = x + c_2$$

Verified OK.

6.5.1 Maple step by step solution

Let's solve

$(3y-1)^2\,y'^2 - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'
- Separate variables

  $\frac{y'(3y-1)}{\sqrt{y}} = 2$
- Integrate both sides with respect to x

  $\int \frac{y'(3y-1)}{\sqrt{y}}dx = \int 2dx + c_1$
- Evaluate integral

  $2y^{\frac{3}{2}} - 2\sqrt{y} = 2x + c_1$
- Solve for y

$$y=\left(\frac{\left(54c_1+108x+6\sqrt{81c_1^2+324c_1x+324x^2-48}\right)^{\frac{1}{3}}}{6}+\frac{2}{\left(54c_1+108x+6\sqrt{81c_1^2+324c_1x+324x^2-48}\right)^{\frac{1}{3}}}\right)^2$$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.046 (sec). Leaf size: 519

```
dsolve((3*y(x)-1)^2*diff(y(x),x)^2=4*y(x),y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \frac{\left(\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}+12\right)^2}{36\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3}\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}-12i\sqrt{3}+\left(-108x+108c_1+12\sqrt{81c_1^2-\right.\right.}{144\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(\left(i\sqrt{3}-1\right)\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}-12i\sqrt{3}-12\right)^2}{144\left(-108x+108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}+12\right)^2}{36\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(i\sqrt{3}\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}-12i\sqrt{3}+\left(108x-108c_1+12\sqrt{81c_1^2-162\right.\right.}{144\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

$$y(x) = \frac{\left(\left(i\sqrt{3}-1\right)\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}-12i\sqrt{3}-12\right)^2}{144\left(108x-108c_1+12\sqrt{81c_1^2-162c_1x+81x^2-12}\right)^{\frac{2}{3}}}$$

✓ Solution by Mathematica

Time used: 4.472 (sec). Leaf size: 892

```
DSolve[(3*y[x]-1)^2*y'[x]^2==4*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\left(2+\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}\right)^2}{6\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}}$$

$$y(x) \to \frac{1}{24}\left(2i\left(\sqrt{3}+i\right)\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}+\frac{-8-8i\sqrt{3}}{\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}}+16\right)$$

$$y(x) \to \frac{1}{24}\left(-2\left(1+i\sqrt{3}\right)\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}+\frac{-8+8i\sqrt{3}}{\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(-2x+c_1)^2(108x^2-108c_1x-16+27c_1{}^2)}-108c_1x-8+27c_1{}^2}}+16\right)$$

$$y(x) \to \frac{\left(2+\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(2x+c_1)^2(108x^2+108c_1x-16+27c_1{}^2)}+108c_1x-8+27c_1{}^2}\right)^2}{6\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(2x+c_1)^2(108x^2+108c_1x-16+27c_1{}^2)}+108c_1x-8+27c_1{}^2}}$$

$$y(x) \to \frac{1}{24}\left(2i\left(\sqrt{3}+i\right)\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(2x+c_1)^2(108x^2+108c_1x-16+27c_1{}^2)}+108c_1x-8+27c_1{}^2}-\frac{8\left(1+i\sqrt{3}\right)}{\sqrt[3]{108x^2+3\sqrt{3}\sqrt{(2x+c_1)^2(108x^2+108c_1x-16+27c_1{}^2)}+108c_1x-8+27c_1{}^2}}+16\right)$$

6.6 problem 15

Internal problem ID [5343]
Internal file name [`OUTPUT/4834_Sunday_February_04_2024_12_46_11_AM_27770854/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 15.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y + xy' - x^4 {y'}^2 = 0}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{1 + \sqrt{1 + 4x^2 y}}{2x^3} \tag{1}$$

$$y' = -\frac{-1 + \sqrt{1 + 4x^2 y}}{2x^3} \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Writing the ode as

$$\begin{aligned} y' &= \frac{1 + \sqrt{4x^2 y + 1}}{2x^3} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{\left(1+\sqrt{4x^2y+1}\right)(b_3-a_2)}{2x^3} - \frac{\left(1+\sqrt{4x^2y+1}\right)^2 a_3}{4x^6} \\ &- \left(-\frac{3\left(1+\sqrt{4x^2y+1}\right)}{2x^4} + \frac{2y}{x^2\sqrt{4x^2y+1}}\right)(xa_2+ya_3+a_1) \\ &- \frac{xb_2+yb_3+b_1}{x\sqrt{4x^2y+1}} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 - 4\sqrt{4x^2y+1}\,x^3a_2 - 2\sqrt{4}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 \\ &+4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}} a_3 \\ &+6\sqrt{4x^2y+1}\,x^2a_1 + 4x^3a_2 + 2x^3b_3 - 2x^2ya_3 + 6x^2a_1 - a_3\sqrt{4x^2y+1} - 2a_3 \\ &= 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 8x^4y^2a_3 + 4\left(4x^2y+1\right)x^3a_2 \\ &+2\left(4x^2y+1\right)x^3b_3 + 6\left(4x^2y+1\right)x^2ya_3 - 4x^5b_1 - 8x^4ya_1 + 6\left(4x^2y+1\right)x^2a_1 \\ &+4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 \\ &-\left(4x^2y+1\right)^{\frac{3}{2}} a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 - 2\left(4x^2y+1\right)a_3 - a_3\sqrt{4x^2y+1} = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^6\sqrt{4x^2y+1}-4x^6b_2+8x^5ya_2+4x^5yb_3+16x^4y^2a_3-4x^5b_1+16x^4ya_1\\
&+4\sqrt{4x^2y+1}\,x^3a_2+2\sqrt{4x^2y+1}\,x^3b_3+2\sqrt{4x^2y+1}\,x^2ya_3+4x^3a_2\\
&+2x^3b_3+6\sqrt{4x^2y+1}\,x^2a_1-2x^2ya_3+6x^2a_1-2a_3\sqrt{4x^2y+1}-2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{4x^2y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{4x^2y+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^6v_3+8v_1^5v_2a_2+16v_1^4v_2^2a_3-4v_1^6b_2+4v_1^5v_2b_3+16v_1^4v_2a_1\\
&-4v_1^5b_1+4v_3v_1^3a_2+2v_3v_1^2v_2a_3+2v_3v_1^3b_3+6v_3v_1^2a_1\\
&+4v_1^3a_2-2v_1^2v_2a_3+2v_1^3b_3+6v_1^2a_1-2a_3v_3-2a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^6v_3-4v_1^6b_2+(8a_2+4b_3)\,v_1^5v_2-4v_1^5b_1+16v_1^4v_2^2a_3\\
&+16v_1^4v_2a_1+(4a_2+2b_3)\,v_1^3v_3+(4a_2+2b_3)\,v_1^3+2v_3v_1^2v_2a_3\\
&-2v_1^2v_2a_3+6v_3v_1^2a_1+6v_1^2a_1-2a_3v_3-2a_3=0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
6a_1 &= 0\\
16a_1 &= 0\\
-2a_3 &= 0\\
2a_3 &= 0\\
16a_3 &= 0\\
-4b_1 &= 0\\
-4b_2 &= 0\\
4b_2 &= 0\\
4a_2 + 2b_3 &= 0\\
8a_2 + 4b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= -2a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= -2y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi\\
&= -2y - \left(\frac{1 + \sqrt{4x^2y + 1}}{2x^3}\right)(x)\\
&= \frac{-4x^2y - \sqrt{4x^2y + 1} - 1}{2x^2}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4x^2y - \sqrt{4x^2y+1} - 1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{4x^2y+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{1 + \sqrt{4x^2y+1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x\sqrt{4x^2y+1}} \\ S_y &= \frac{-1 + \frac{1}{\sqrt{4x^2y+1}}}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Summary

The solution(s) found are the following

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} - \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$y' = -\frac{-1+\sqrt{4x^2y+1}}{2x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{\left(-1+\sqrt{4x^2y+1}\right)(b_3-a_2)}{2x^3} - \frac{\left(-1+\sqrt{4x^2y+1}\right)^2 a_3}{4x^6} - \left(-\frac{2y}{x^2\sqrt{4x^2y+1}} + \frac{-\frac{3}{2}+\frac{3\sqrt{4x^2y+1}}{2}}{x^4}\right)(xa_2+ya_3+a_1) + \frac{xb_2+yb_3+b_1}{x\sqrt{4x^2y+1}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2x^6\sqrt{4x^2y+1} - 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 16x^4y^2a_3 - 4x^5b_1 + 16x^4ya_1 - 4\sqrt{4x^2y+1}\,x^3a_2 - 2\sqrt{4}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{gathered}4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 - 8x^5ya_2 - 4x^5yb_3 - 16x^4y^2a_3 + 4x^5b_1 - 16x^4ya_1 \\ +4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 - \left(4x^2y+1\right)^{\frac{3}{2}}a_3 \\ +6\sqrt{4x^2y+1}\,x^2a_1 - 4x^3a_2 - 2x^3b_3 + 2x^2ya_3 - 6x^2a_1 - a_3\sqrt{4x^2y+1} + 2a_3 \\ = 0\end{gathered} \tag{6E}$$

Simplifying the above gives

$$\begin{gathered}4b_2x^6\sqrt{4x^2y+1} + 4x^6b_2 + 8x^5ya_2 + 4x^5yb_3 + 8x^4y^2a_3 - 4\left(4x^2y+1\right)x^3a_2 \\ -2\left(4x^2y+1\right)x^3b_3 - 6\left(4x^2y+1\right)x^2ya_3 + 4x^5b_1 + 8x^4ya_1 - 6\left(4x^2y+1\right)x^2a_1 \\ +4\sqrt{4x^2y+1}\,x^3a_2 + 2\sqrt{4x^2y+1}\,x^3b_3 + 6\sqrt{4x^2y+1}\,x^2ya_3 \\ -\left(4x^2y+1\right)^{\frac{3}{2}}a_3 + 6\sqrt{4x^2y+1}\,x^2a_1 + 2\left(4x^2y+1\right)a_3 - a_3\sqrt{4x^2y+1} = 0\end{gathered} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&4b_2x^6\sqrt{4x^2y+1}+4x^6b_2-8x^5ya_2-4x^5yb_3-16x^4y^2a_3+4x^5b_1-16x^4ya_1 \\
&+4\sqrt{4x^2y+1}\,x^3a_2+2\sqrt{4x^2y+1}\,x^3b_3+2\sqrt{4x^2y+1}\,x^2ya_3-4x^3a_2 \\
&-2x^3b_3+6\sqrt{4x^2y+1}\,x^2a_1+2x^2ya_3-6x^2a_1-2a_3\sqrt{4x^2y+1}+2a_3=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\left\{x, y, \sqrt{4x^2y+1}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\left\{x=v_1, y=v_2, \sqrt{4x^2y+1}=v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&4b_2v_1^6v_3-8v_1^5v_2a_2-16v_1^4v_2^2a_3+4v_1^6b_2-4v_1^5v_2b_3-16v_1^4v_2a_1 \\
&+4v_1^5b_1+4v_3v_1^3a_2+2v_3v_1^2v_2a_3+2v_3v_1^3b_3+6v_3v_1^2a_1 \\
&-4v_1^3a_2+2v_1^2v_2a_3-2v_1^3b_3-6v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&4b_2v_1^6v_3+4v_1^6b_2+(-8a_2-4b_3)\,v_1^5v_2+4v_1^5b_1-16v_1^4v_2^2a_3 \\
&-16v_1^4v_2a_1+(4a_2+2b_3)\,v_1^3v_3+(-4a_2-2b_3)\,v_1^3 \\
&+2v_3v_1^2v_2a_3+2v_1^2v_2a_3+6v_3v_1^2a_1-6v_1^2a_1-2a_3v_3+2a_3=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-16a_1 &= 0 \\
-6a_1 &= 0 \\
6a_1 &= 0 \\
-16a_3 &= 0 \\
-2a_3 &= 0 \\
2a_3 &= 0 \\
4b_1 &= 0 \\
4b_2 &= 0 \\
-8a_2 - 4b_3 &= 0 \\
-4a_2 - 2b_3 &= 0 \\
4a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= -2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= -2y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= -2y - \left(-\frac{-1 + \sqrt{4x^2y + 1}}{2x^3}\right)(x) \\
&= \frac{-4x^2y + \sqrt{4x^2y + 1} - 1}{2x^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-4x^2y+\sqrt{4x^2y+1}-1}{2x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{4x^2y+1}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-1 + \sqrt{4x^2y+1}}{2x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{x\sqrt{4x^2y+1}} \\ S_y &= \frac{-\frac{1}{\sqrt{4x^2y+1}} - 1}{2y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Which simplifies to

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Summary
The solution(s) found are the following

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1 \tag{1}$$

Verification of solutions

$$-\frac{\ln(y)}{2} + \operatorname{arctanh}\left(\sqrt{1+4x^2y}\right) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      <- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 81

```
dsolve(y(x)=-x*diff(y(x),x)+x^4*diff(y(x),x)^2,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= -\frac{1}{4x^2} \\
y(x) &= \frac{-c_1 i - x}{x\, c_1^2} \\
y(x) &= \frac{c_1 i - x}{x\, c_1^2} \\
y(x) &= \frac{c_1 i - x}{x\, c_1^2} \\
y(x) &= \frac{-c_1 i - x}{x\, c_1^2}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.498 (sec). Leaf size: 123

```
DSolve[y[x]==-x*y'[x]+x^4*y'[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
&\text{Solve}\left[-\frac{x\sqrt{4x^2y(x)+1}\text{arctanh}\left(\sqrt{4x^2y(x)+1}\right)}{\sqrt{4x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right] \\
&\text{Solve}\left[\frac{x\sqrt{4x^2y(x)+1}\text{arctanh}\left(\sqrt{4x^2y(x)+1}\right)}{\sqrt{4x^4y(x)+x^2}} - \frac{1}{2}\log(y(x)) = c_1, y(x)\right] \\
&y(x) \to 0
\end{aligned}
$$

6.7 problem 16

6.7.1 Solving as dAlembert ode 1147

Internal problem ID [5344]
Internal file name [`OUTPUT/4835_Sunday_February_04_2024_12_46_13_AM_25059485/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 16.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _dAlembert]`

$$\boxed{2y - {y'}^2 - 4xy' = 0}$$

6.7.1 Solving as dAlembert ode

Let $p = y'$ the ode becomes

$$-p^2 - 4xp + 2y = 0$$

Solving for y from the above results in

$$y = \frac{1}{2}p^2 + 2xp \tag{1A}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= \frac{p^2}{2}
\end{aligned}
$$

Hence (2) becomes

$$
-p = (2x + p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving for p from the above gives

$$
p = 0
$$

Substituting these in (1A) gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = -\frac{p(x)}{2x + p(x)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp} x(p) = -\frac{2x(p) + p}{p} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
\frac{d}{dp} x(p) + p(p)x(p) = q(p)
$$

Where here

$$
\begin{aligned}
p(p) &= \frac{2}{p} \\
q(p) &= -1
\end{aligned}
$$

Hence the ode is

$$\frac{d}{dp}x(p) + \frac{2x(p)}{p} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{p}dp} \\ &= p^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}(\mu x) &= (\mu)(-1) \\ \frac{\mathrm{d}}{\mathrm{d}p}(p^2 x) &= (p^2)(-1) \\ \mathrm{d}(p^2 x) &= (-p^2)\, \mathrm{d}p\end{aligned}$$

Integrating gives

$$\begin{aligned}p^2 x &= \int -p^2\,\mathrm{d}p \\ p^2 x &= -\frac{p^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = p^2$ results in

$$x(p) = -\frac{p}{3} + \frac{c_1}{p^2}$$

Now we need to eliminate p between the above and (1A). One way to do this is by solving (1) for p. This results in

$$\begin{aligned}p &= -2x + \sqrt{4x^2 + 2y} \\ p &= -2x - \sqrt{4x^2 + 2y}\end{aligned}$$

Substituting the above in the solution for x found above gives

$$\begin{aligned}x &= \frac{(-16x^2 - 2y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x - \sqrt{4x^2 + 2y}\right)^2} \\ x &= \frac{2(8x^2 + y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x + \sqrt{4x^2 + 2y}\right)^2}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$

$$x = \frac{(-16x^2 - 2y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x - \sqrt{4x^2 + 2y}\right)^2} \tag{2}$$

$$x = \frac{2(8x^2 + y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x + \sqrt{4x^2 + 2y}\right)^2} \tag{3}$$

Verification of solutions

$$y = 0$$

Verified OK.

$$x = \frac{(-16x^2 - 2y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x - \sqrt{4x^2 + 2y}\right)^2}$$

Verified OK.

$$x = \frac{2(8x^2 + y)\sqrt{4x^2 + 2y} + 32x^3 + 12yx + 3c_1}{3\left(2x + \sqrt{4x^2 + 2y}\right)^2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 700

```
dsolve(2*y(x)=diff(y(x),x)^2+4*x*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{\left(\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{2}{3}} - 2x\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{1}{3}} + 4x^2\right)\left(\left(12c_1 - 8\right.\right.}{8\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3\right.}$$

$$y(x) = \frac{\left(i\sqrt{3}\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{2}{3}} - 4i\sqrt{3}\,x^2 + \left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{2}{3}} + 4x\left(\right.\right.}{}$$

$$y(x) = \frac{\left(4i\sqrt{3}\,x^2 - i\sqrt{3}\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)^{\frac{2}{3}} + 4x^2 + 4x\left(12c_1 - 8x^3 + 4\sqrt{3}\,\sqrt{-4c_1x^3 + 3c_1^2}\right)\right.}{}$$

✓ Solution by Mathematica

Time used: 60.241 (sec). Leaf size: 1344

```
DSolve[2*y[x]==y'[x]^2+4*x*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-x^2 + \frac{x\left(x^3+2\sqrt{2}e^{3c_1}\right)}{\sqrt[3]{-x^6+5\sqrt{2}e^{3c_1}x^3+\sqrt{e^{3c_1}\left(-16\sqrt{2}x^9+24e^{3c_1}x^6-6\sqrt{2}e^{6c_1}x^3+e^{9c_1}\right)}+e^{6c_1}}} + \sqrt[3]{-x^6+5\sqrt{2}e^{3c_1}x^3+\sqrt{e^{3c_1}\left(-16\sqrt{2}x^9+24e^{3c_1}x^6-6\sqrt{2}e^{6c_1}x^3+e^{9c_1}\right)}+e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{4}\left(-2x^2 - \frac{\left(1+i\sqrt{3}\right)x\left(x^3+2\sqrt{2}e^{3c_1}\right)}{\sqrt[3]{-x^6+5\sqrt{2}e^{3c_1}x^3+\sqrt{e^{3c_1}\left(-16\sqrt{2}x^9+24e^{3c_1}x^6-6\sqrt{2}e^{6c_1}x^3+e^{9c_1}\right)}+e^{6c_1}}} + i\left(\sqrt{3}+i\right)\sqrt[3]{-x^6+5\sqrt{2}e^{3c_1}x^3+\sqrt{e^{3c_1}\left(-16\sqrt{2}x^9+24e^{3c_1}x^6-6\sqrt{2}e^{6c_1}x^3+e^{9c_1}\right)}+e^{6c_1}}\right)$$

$$y(x) \to \frac{1}{4}\left(-2x^2 + \frac{i\left(\sqrt{3}+i\right)x\left(x^3+2\sqrt{2}e^{3c_1}\right)}{\sqrt[3]{-x^6+5\sqrt{2}e^{3c_1}x^3+\sqrt{e^{3c_1}\left(-16\sqrt{2}x^9+24e^{3c_1}x^6-6\sqrt{2}e^{6c_1}x^3+e^{9c_1}\right)}+e^{6c_1}}} - \left(1\right.\right.$$

6.8 problem 17

6.8.1 Maple step by step solution 1154

Internal problem ID [5345]
Internal file name [`OUTPUT/4836_Sunday_February_04_2024_12_46_13_AM_76649283/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 17.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y(3-4y)^2 {y'}^2+4y=4}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \frac{2\sqrt{-y\,(-1+y)}}{y\,(4y-3)} \tag{1}$$

$$y' = -\frac{2\sqrt{-y\,(-1+y)}}{y\,(4y-3)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y(4y-3)}{2\sqrt{-y\,(-1+y)}}dy = \int dx$$

$$\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x+c_1$$

Summary
The solution(s) found are the following

$$\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x + c_1 \tag{1}$$

Verification of solutions

$$\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x + c_1$$

Verified OK.

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y(4y-3)}{2\sqrt{-y\,(-1+y)}}dy = \int dx$$

$$-\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x + c_2$$

Summary
The solution(s) found are the following

$$-\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x + c_2 \tag{1}$$

Verification of solutions

$$-\frac{y^2(-1+y)}{\sqrt{-y\,(-1+y)}} = x + c_2$$

Verified OK.

6.8.1 Maple step by step solution

Let's solve

$y(3-4y)^2\,y'^2 + 4y = 4$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'y(4y-3)}{\sqrt{-y(-1+y)}} = 2$

- Integrate both sides with respect to x

  $\int \frac{y'y(4y-3)}{\sqrt{-y(-1+y)}}dx = \int 2dx + c_1$

- Evaluate integral

  $-2y\sqrt{-y^2+y} = 2x + c_1$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   <- differential order: 1; missing  x  successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 58

```
dsolve(y(x)*(3-4*y(x))^2*diff(y(x),x)^2=4*(1-y(x)),y(x), singsol=all)
```

$$y(x) = 1$$
$$x + \frac{y(x)^2 (y(x)-1)}{\sqrt{-y(x)(y(x)-1)}} - c_1 = 0$$
$$x - \frac{y(x)^2 (y(x)-1)}{\sqrt{-y(x)(y(x)-1)}} - c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.264 (sec). Leaf size: 3751

```
DSolve[y[x]*(3-4*y[x])^2*y'[x]^2==4*(1-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

6.9 problem 18

Internal problem ID [5346]
Internal file name [`OUTPUT/4837_Sunday_February_04_2024_12_46_14_AM_38484777/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 18.
ODE order: 1.
ODE degree: 3.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{y'^3 - 4x^4 y' + 8yx^3 = 0}$$

Solving the given ode for y' results in 3 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = 2\left(\frac{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^2}{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}\right)x \tag{1}$$

$$y' = 2\left(-\frac{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}{12} - \frac{x^2}{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}} + \frac{i\sqrt{3}\left(\frac{\left(-108y+12\sqrt{-12x^6+81y}\right.}{6}\right.}{} \right. \tag{2}$$

$$y' = 2\left(-\frac{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}{12} - \frac{x^2}{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}} - \frac{i\sqrt{3}\left(\frac{\left(-108y+12\sqrt{-12x^6+81y}\right.}{6}\right.}{} \right. \tag{3}$$

Now each one of the above ODE is solved.

Solving equation (1)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x\left(\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2\right)}{3\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{x\left(\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2\right)(b_3-a_2)}{3\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}} \\
&-\frac{x^2\left(\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2\right)^2 a_3}{9\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}} \\
&-\left(\frac{\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2}{3\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}}\right. \\
&+\frac{x\left(-\frac{288x^5}{\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\sqrt{-12x^6+81y^2}}+24x\right)}{3\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}} \\
&\left.+\frac{48x^6\left(\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2\right)}{\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}\sqrt{-12x^6+81y^2}}\right)(xa_2+ya_3+a_1) \\
&-\left(\frac{2\left(-108+\frac{972y}{\sqrt{-12x^6+81y^2}}\right)x}{9\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}}\right. \\
&\left.-\frac{x\left(\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}+12x^2\right)\left(-108+\frac{972y}{\sqrt{-12x^6+81y^2}}\right)}{9\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}}\right)(xb_2+yb_3+b_1) \\
&=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{1728x^9b_3-5184x^9a_2-3456x^8a_1+46656x^3y^2a_2-34992x^2y^3a_3+34992x^2y^2a_1-36\left(-108y+12\sqrt{-12\right.}}{}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-1728x^9b_3+5184x^9a_2+3456x^8a_1\\
&-46656x^3y^2a_2+34992x^2y^3a_3-34992x^2y^2a_1\\
&+36\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,xyb_3\\
&-6912x^8ya_3+3888x^4yb_2+15552x^3y^2b_3+3888x^3yb_1\\
&+144\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^7a_2\\
&+144\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^6a_1\\
&-\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,a_1\\
&+3b_2\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}\sqrt{-12x^6+81y^2}\\
&-432\sqrt{-12x^6+81y^2}\,x^4b_2-432\sqrt{-12x^6+81y^2}\,x^3b_1\\
&-48\left(-12x^6+81y^2\right)^{\frac{3}{2}}x^2a_3\\
&-8\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}\sqrt{-12x^6+81y^2}\,x^4a_3\\
&-48\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x^6a_3\\
&+144\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^6ya_3\\
&-2\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,xa_2\\
&+\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,xb_3\\
&-\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,ya_3\\
&+36\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x^2b_2\\
&-324\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^2yb_2\\
&-324\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x\,y^2b_3\\
&-1728\sqrt{-12x^6+81y^2}\,x^3yb_3\\
&+36\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,xb_1\\
&-324\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}xyb_1\\
&+5184\sqrt{-12x^6+81y^2}\,x^3a_2y\\
&+3888\sqrt{-12x^6+81y^2}\,x^2a_1y=0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-5184x^9a_2-5184x^8a_1-108\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)\sqrt{-12x^6+81y^2}\,x^2ya_3\\
&+108\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,xyb_3\\
&-5184x^8ya_3+11664x^4yb_2+11664x^3y^2b_3+11664x^3yb_1\\
&+432\Big(-108y+12\sqrt{-12x^6+81y^2}\Big)^{\frac{2}{3}}x^7a_2\\
&+432\Big(-108y+12\sqrt{-12x^6+81y^2}\Big)^{\frac{2}{3}}x^6a_1\\
&-3\Big(-108y+12\sqrt{-12x^6+81y^2}\Big)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,a_1\\
&+9b_2\Big(-108y+12\sqrt{-12x^6+81y^2}\Big)^{\frac{4}{3}}\sqrt{-12x^6+81y^2}\\
&-1296\sqrt{-12x^6+81y^2}\,x^4b_2\\
&-1296\sqrt{-12x^6+81y^2}\,x^3b_1-\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{2}\sqrt{-12x^6+81y^2}\,x^2a_3\\
&-24\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{4}{3}}\sqrt{-12x^6+81y^2}\,x^4a_3\\
&-144\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x^6a_3\\
&+432\Big(-108y+12\sqrt{-12x^6+81y^2}\Big)^{\frac{2}{3}}x^6ya_3-6\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,xa_2\\
&+3\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,xb_3\\
&-3\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)^{\frac{5}{3}}\sqrt{-12x^6+81y^2}\,ya_3\\
&-144\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)\sqrt{-12x^6+81y^2}\,x^3a_2\\
&+36\Big(-108y\\
&+12\sqrt{-12x^6+81y^2}\Big)\sqrt{-12x^6+81y^2}\,x^3b_3
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&648\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x a_2 y \\
&-11664x^2\sqrt{-12x^6+81y^2}\,y^2 a_3 \\
&+324\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,y^2 a_3 \\
&+324\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,a_1 y \\
&-972b_2\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\sqrt{-12x^6+81y^2}\,y-5184x^9 b_3 \\
&+15552x^9 a_2+10368x^8 a_1-139968x^3y^2a_2+104976x^2y^3a_3-104976x^2y^2a_1 \\
&-216\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,xyb_3 \\
&-23328x^4\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}y^2a_3 \\
&-5832x\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}y^2a_2-20736x^8ya_3+11664x^4yb_2 \\
&+46656x^3y^2b_3+11664x^3yb_1+1296\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^7a_2 \\
&+864\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^6a_1-1296\sqrt{-12x^6+81y^2}\,x^4b_2 \\
&-1296\sqrt{-12x^6+81y^2}\,x^3b_1-144\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x^6a_3 \\
&+864\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^6ya_3 \\
&+108\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,x^2b_2 \\
&+2592\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\sqrt{-12x^6+81y^2}\,x^4a_3y \\
&-972\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x^2yb_2+1944\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}x\,y^2b_3 \\
&-5184\sqrt{-12x^6+81y^2}\,x^3yb_3+108\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\sqrt{-12x^6+81y^2}\,xb_1 \\
&-972\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}xyb_1+3456x^{10}\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}a_3 \\
&-432x^7\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}b_3-1296x^6\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}b_2 \\
&+1728x^8\sqrt{-12x^6+81y^2}\,a_3-2916\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}y^3a_3 \\
&-2916\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}y^2a_1+8748\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}y^2b_2 \\
&+15552\sqrt{-12x^6+81y^2}\,x^3a_2y+11664\sqrt{-12x^6+81y^2}\,x^2a_1y=0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6+81y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6+81y^2} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&3456v_1^{10}v_3a_3 + 15552v_1^9a_2 - 20736v_1^8v_2a_3 + 1728v_1^8v_5a_3 \\
&- 5184v_1^9b_3 + 10368v_1^8a_1 + 1296v_4v_1^7a_2 + 864v_4v_1^6v_2a_3 \\
&- 144v_4v_5v_1^6a_3 - 432v_1^7v_4b_3 + 864v_4v_1^6a_1 - 23328v_1^4v_3v_2^2a_3 \\
&+ 2592v_3v_5v_1^4a_3v_2 - 1296v_1^6v_3b_2 - 139968v_1^3v_2^2a_2 + 15552v_5v_1^3a_2v_2 \\
&+ 104976v_1^2v_2^3a_3 - 11664v_1^2v_5v_2^2a_3 + 11664v_1^4v_2b_2 - 1296v_5v_1^4b_2 \\
&+ 46656v_1^3v_2^2b_3 - 5184v_5v_1^3v_2b_3 - 104976v_1^2v_2^2a_1 + 11664v_5v_1^2a_1v_2 \\
&- 5832v_1v_4v_2^2a_2 + 648v_4v_5v_1a_2v_2 - 2916v_4v_2^3a_3 + 324v_4v_5v_2^2a_3 \\
&+ 11664v_1^3v_2b_1 - 1296v_5v_1^3b_1 - 972v_4v_1^2v_2b_2 + 108v_4v_5v_1^2b_2 \\
&+ 1944v_4v_1v_2^2b_3 - 216v_4v_5v_1v_2b_3 - 2916v_4v_2^2a_1 + 324v_4v_5a_1v_2 \\
&- 972v_4v_1v_2b_1 + 108v_4v_5v_1b_1 + 8748v_3v_2^2b_2 - 972b_2v_3v_5v_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&10368v_1^8a_1+(648a_2-216b_3)\,v_1v_2v_4v_5+(15552a_2-5184b_3)\,v_1^9\\
&-11664v_1^2v_5v_2^2a_3+(15552a_2-5184b_3)\,v_1^3v_2v_5\\
&+(-5832a_2+1944b_3)\,v_1v_2^2v_4+104976v_1^2v_2^3a_3-104976v_1^2v_2^2a_1\\
&-20736v_1^8v_2a_3+11664v_1^4v_2b_2+11664v_1^3v_2b_1+864v_4v_1^6a_1\\
&-1296v_5v_1^4b_2-1296v_5v_1^3b_1+3456v_1^{10}v_3a_3-1296v_1^6v_3b_2+1728v_1^8v_5a_3\\
&-2916v_4v_2^3a_3-2916v_4v_2^2a_1+8748v_3v_2^2b_2+324v_4v_5v_2^2a_3\\
&+324v_4v_5a_1v_2-972b_2v_3v_5v_2-23328v_1^4v_3v_2^2a_3-144v_4v_5v_1^6a_3\\
&+864v_4v_1^6v_2a_3+108v_4v_5v_1^2b_2-972v_4v_1^2v_2b_2+108v_4v_5v_1b_1\\
&-972v_4v_1v_2b_1+11664v_5v_1^2a_1v_2+(1296a_2-432b_3)\,v_1^7v_4\\
&+(-139968a_2+46656b_3)\,v_1^3v_2^2+2592v_3v_5v_1^4a_3v_2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-104976a_1 &= 0 \\
-2916a_1 &= 0 \\
324a_1 &= 0 \\
864a_1 &= 0 \\
10368a_1 &= 0 \\
11664a_1 &= 0 \\
-23328a_3 &= 0 \\
-20736a_3 &= 0 \\
-11664a_3 &= 0 \\
-2916a_3 &= 0 \\
-144a_3 &= 0 \\
324a_3 &= 0 \\
864a_3 &= 0 \\
1728a_3 &= 0 \\
2592a_3 &= 0 \\
3456a_3 &= 0 \\
104976a_3 &= 0 \\
-1296b_1 &= 0 \\
-972b_1 &= 0 \\
108b_1 &= 0 \\
11664b_1 &= 0 \\
-1296b_2 &= 0 \\
-972b_2 &= 0 \\
108b_2 &= 0 \\
8748b_2 &= 0 \\
11664b_2 &= 0 \\
-139968a_2 + 46656b_3 &= 0 \\
-5832a_2 + 1944b_3 &= 0 \\
648a_2 - 216b_3 &= 0 \\
1296a_2 - 432b_3 &= 0 \\
15552a_2 - 5184b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 3a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 3y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{3y}{x} \\
&= \frac{3y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x^3
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x^3}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{x}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= \ln(x)
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x\left(\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 12x^2\right)}{3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= -\frac{3y}{x^4} \\
R_y &= \frac{1}{x^3} \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{3x^3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}}{x^2\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 12x^4 - 9\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}} y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{3\, 12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^2 - 4} - 9R\right)^{\frac{1}{3}}}{12^{\frac{2}{3}}\left(\sqrt{3}\,\sqrt{27R^2 - 4} - 9R\right)^{\frac{2}{3}} - 9\, 12^{\frac{1}{3}}\left(\sqrt{3}\,\sqrt{27R^2 - 4} - 9R\right)^{\frac{1}{3}} R + 12}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{3\left(12\sqrt{81R^2-12}-108R\right)^{\frac{1}{3}}}{2\,18^{\frac{1}{3}}\left(\left(\sqrt{81R^2-12}-9R\right)^2\right)^{\frac{1}{3}}-9R\left(12\sqrt{81R^2-12}-108R\right)^{\frac{1}{3}}+12}dR + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{3\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}}{2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}-9\_a\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}+12}d\_a + c_1$$

Which simplifies to

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{3\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}}{2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}-9\_a\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}+12}d\_a + c_1$$

Summary

The solution(s) found are the following

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{3\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}}{2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}-9\_a\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}+12}d\_a + c_1 \tag{1}$$

Verification of solutions

$$\ln(x) = \int^{\frac{y}{x^3}} \frac{3\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}}{2\,18^{\frac{1}{3}}\left(\left(\sqrt{81\_a^2-12}-9\_a\right)^2\right)^{\frac{1}{3}}-9\_a\left(12\sqrt{81\_a^2-12}-108\_a\right)^{\frac{1}{3}}+12}d\_a + c_1$$

Verified OK.

Solving equation (2)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x\left(6i\sqrt{3}\,x^2+\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}-6x^2\right)}{3\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{2x\left(6i\sqrt{3}\,x^2 + \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} - 6x^2\right)(b_3 - a_2)}{3\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}$$
$$- \frac{4x^2\left(6i\sqrt{3}\,x^2 + \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} - 6x^2\right)^2 a_3}{9\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)^2}$$
$$- \left(-\frac{2\left(6i\sqrt{3}\,x^2 + \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} - 6x^2\right)}{3\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}\right.$$
$$- \frac{2x\left(12i\sqrt{3}\,x - \frac{288x^5}{\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\sqrt{-12x^6+81y^2}} - 12x\right)}{3\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}$$
$$\left.- \frac{96x^6\left(6i\sqrt{3}\,x^2 + \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} - 6x^2\right)}{\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)\sqrt{-12x^6+81y^2}}\right)(xa_2 + ya_3 + a_1)$$
$$- \left(-\frac{4x\left(-108 + \frac{972y}{\sqrt{-12x^6+81y^2}}\right)}{9\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}\left(1+i\sqrt{3}\right)}\right.$$
$$\left.+ \frac{2x\left(6i\sqrt{3}\,x^2 + \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} - 6x^2\right)\left(-108 + \frac{972y}{\sqrt{-12x^6+81y^2}}\right)}{9\left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{4}{3}}\left(1+i\sqrt{3}\right)}\right)(xb_2$$
$$+ yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\text{Expression too large to display}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6+81y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6+81y^2} = v_5\right\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-3888 v_4 v_1 v_2^2 b_3-216 v_4 v_5 v_1 b_1+1944 v_4 v_1 v_2 b_1 \\
&+62208 v_5 v_1^3 a_2 v_2+46656 v_1^4 v_3 v_2^2 a_3+11664 v_1 v_4 v_2^2 a_2 \\
&+46656 v_5 v_1^2 a_1 v_2+1944 v_3 v_5 b_2 v_2-648 v_4 v_5 a_1 v_2 \\
&-46656 v_1^2 v_5 v_2^2 a_3-648 v_4 v_5 v_2^2 a_3-20736 v_5 v_1^3 v_2 b_3 \\
&-1728 v_4 v_1^6 v_2 a_3-216 v_4 v_5 v_1^2 b_2+1944 v_4 v_1^2 v_2 b_2 \\
&+288 v_4 v_5 v_1^6 a_3-46656 i \sqrt{3}\, v_1^4 v_3 v_2^2 a_3+11664 i \sqrt{3}\, v_1 v_4 v_2^2 a_2 \\
&-3888 i v_4 \sqrt{3}\, v_1 v_2^2 b_3-648 i v_4 v_5 \sqrt{3}\, a_1 v_2-1944 i v_3 v_5 \sqrt{3}\, b_2 v_2 \\
&-648 i v_4 v_5 \sqrt{3}\, v_2^2 a_3-1728 i v_4 \sqrt{3}\, v_1^6 v_2 a_3+288 i v_4 v_5 \sqrt{3}\, v_1^6 a_3 \\
&-216 i v_4 v_5 \sqrt{3}\, v_1^2 b_2+186624 v_1^3 v_2^2 b_3+46656 v_1^3 v_2 b_1 \\
&-5184 v_5 v_1^4 b_2-2592 v_4 v_1^7 a_2-5184 v_5 v_1^3 b_1-1728 v_4 v_1^6 a_1 \\
&+6912 v_1^8 v_5 a_3-559872 v_1^3 v_2^2 a_2+419904 v_1^2 v_2^3 a_3-6912 v_1^{10} v_3 a_3 \\
&+864 v_1^7 v_4 b_3+2592 v_1^6 v_3 b_2+5832 v_4 v_2^3 a_3+5832 v_4 v_2^2 a_1 \\
&-17496 v_3 v_2^2 b_2-82944 v_1^8 v_2 a_3+46656 v_1^4 v_2 b_2-419904 v_1^2 v_2^2 a_1 \\
&+5184 i v_3 v_5 \sqrt{3}\, v_1^4 a_3 v_2-1296 i v_4 v_5 \sqrt{3}\, v_1 a_2 v_2 \\
&+432 i v_4 v_5 \sqrt{3}\, v_1 v_2 b_3+1944 i v_4 \sqrt{3}\, v_1^2 v_2 b_2-216 i v_4 v_5 \sqrt{3}\, v_1 b_1 \\
&+1944 i v_4 \sqrt{3}\, v_1 v_2 b_1+432 v_4 v_5 v_1 v_2 b_3-5184 v_3 v_5 v_1^4 a_3 v_2 \\
&-1296 v_4 v_5 v_1 a_2 v_2+6912 i \sqrt{3}\, v_1^{10} v_3 a_3-2592 i v_4 \sqrt{3}\, v_1^7 a_2 \\
&+864 i \sqrt{3}\, v_1^7 v_4 b_3-1728 i v_4 \sqrt{3}\, v_1^6 a_1-2592 i \sqrt{3}\, v_1^6 v_3 b_2 \\
&+5832 i \sqrt{3}\, v_4 v_2^3 a_3+5832 i \sqrt{3}\, v_4 v_2^2 a_1+17496 i \sqrt{3}\, v_3 v_2^2 b_2 \\
&-20736 v_1^9 b_3+62208 v_1^9 a_2+41472 v_1^8 a_1=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&\left(-1944i\sqrt{3}\,b_2+1944b_2\right)v_2v_3v_5+\left(-46656i\sqrt{3}\,a_3+46656a_3\right)v_1^4v_2^2v_3\\
&+46656v_5v_1^2a_1v_2-46656v_1^2v_5v_2^2a_3+\left(1944i\sqrt{3}\,b_2+1944b_2\right)v_1^2v_2v_4\\
&+\left(-216i\sqrt{3}\,b_2-216b_2\right)v_1^2v_4v_5+\left(1944i\sqrt{3}\,b_1+1944b_1\right)v_1v_2v_4\\
&+\left(-216i\sqrt{3}\,b_1-216b_1\right)v_1v_4v_5+\left(-1728i\sqrt{3}\,a_3-1728a_3\right)v_1^6v_2v_4\\
&+\left(11664i\sqrt{3}\,a_2-3888i\sqrt{3}\,b_3+11664a_2-3888b_3\right)v_1v_2^2v_4+\left(288i\sqrt{3}\,a_3+288a_3\right)v_1^6v_4v_5\\
&+\left(-648i\sqrt{3}\,a_3-648a_3\right)v_2^2v_4v_5+(-559872a_2+186624b_3)\,v_1^3v_2^2\\
&+\left(-648i\sqrt{3}\,a_1-648a_1\right)v_2v_4v_5+(62208a_2-20736b_3)\,v_1^3v_2v_5+(62208a_2-20736b_3)\,v_1^9\\
&+46656v_1^3v_2b_1-5184v_5v_1^4b_2-5184v_5v_1^3b_1+6912v_1^8v_5a_3+419904v_1^2v_2^3a_3\\
&-82944v_1^8v_2a_3+46656v_1^4v_2b_2-419904v_1^2v_2^2a_1+\left(-1728i\sqrt{3}\,a_1-1728a_1\right)v_1^6v_4\\
&+\left(-2592i\sqrt{3}\,a_2+864i\sqrt{3}\,b_3-2592a_2+864b_3\right)v_1^7v_4+\left(6912i\sqrt{3}\,a_3-6912a_3\right)v_1^{10}v_3\\
&+\left(17496i\sqrt{3}\,b_2-17496b_2\right)v_2^2v_3+\left(5832i\sqrt{3}\,a_1+5832a_1\right)v_2^2v_4\\
&+\left(5832i\sqrt{3}\,a_3+5832a_3\right)v_2^3v_4+\left(-2592i\sqrt{3}\,b_2+2592b_2\right)v_1^6v_3\\
&+41472v_1^8a_1+\left(-1296i\sqrt{3}\,a_2+432i\sqrt{3}\,b_3-1296a_2+432b_3\right)v_1v_2v_4v_5\\
&+\left(5184i\sqrt{3}\,a_3-5184a_3\right)v_1^4v_2v_3v_5=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-419904a_1 &= 0 \\
41472a_1 &= 0 \\
46656a_1 &= 0 \\
-82944a_3 &= 0 \\
-46656a_3 &= 0 \\
6912a_3 &= 0 \\
419904a_3 &= 0 \\
-5184b_1 &= 0 \\
46656b_1 &= 0 \\
-5184b_2 &= 0 \\
46656b_2 &= 0 \\
-559872a_2 + 186624b_3 &= 0 \\
62208a_2 - 20736b_3 &= 0 \\
-46656i\sqrt{3}\, a_3 + 46656a_3 &= 0 \\
-2592i\sqrt{3}\, b_2 + 2592b_2 &= 0 \\
-1944i\sqrt{3}\, b_2 + 1944b_2 &= 0 \\
-1728i\sqrt{3}\, a_1 - 1728a_1 &= 0 \\
-1728i\sqrt{3}\, a_3 - 1728a_3 &= 0 \\
-648i\sqrt{3}\, a_1 - 648a_1 &= 0 \\
-648i\sqrt{3}\, a_3 - 648a_3 &= 0 \\
-216i\sqrt{3}\, b_1 - 216b_1 &= 0 \\
-216i\sqrt{3}\, b_2 - 216b_2 &= 0 \\
288i\sqrt{3}\, a_3 + 288a_3 &= 0 \\
1944i\sqrt{3}\, b_1 + 1944b_1 &= 0 \\
1944i\sqrt{3}\, b_2 + 1944b_2 &= 0 \\
5184i\sqrt{3}\, a_3 - 5184a_3 &= 0 \\
5832i\sqrt{3}\, a_1 + 5832a_1 &= 0 \\
5832i\sqrt{3}\, a_3 + 5832a_3 &= 0 \\
6912i\sqrt{3}\, a_3 - 6912a_3 &= 0 \\
17496i\sqrt{3}\, b_2 - 17496b_2 &= 0 \\
-2592i\sqrt{3}\, a_2 + 864i\sqrt{3}\, b_3 - 2592a_2 + 864b_3 &= 0 \\
-1296i\sqrt{3}\, a_2 + 432i\sqrt{3}\, b_3 - 1296a_2 + 432b_3 &= 0 \\
11664i\sqrt{3}\, a_2 - 3888i\sqrt{3}\, b_3 + 11664a_2 - 3888b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Solving equation (3)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)}{3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{2x\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)(b_3 - a_2)}{3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)} \\
&- \frac{4x^2\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)^2 a_3}{9\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}}\left(i\sqrt{3} - 1\right)^2} \\
&- \left(-\frac{2\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)}{3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}\right. \\
&- \frac{2x\left(12i\sqrt{3}\,x + \frac{288x^5}{\left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}\sqrt{-12x^6+81y^2}} + 12x\right)}{3\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)} \\
&\left.- \frac{96x^6\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)}{\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{4}{3}}\left(i\sqrt{3} - 1\right)\sqrt{-12x^6 + 81y^2}}\right)(xa_2 + ya_3 + a_1) \\
&- \left(\frac{4x\left(-108 + \frac{972y}{\sqrt{-12x^6+81y^2}}\right)}{9\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}}\left(i\sqrt{3} - 1\right)}\right. \\
&\left.+ \frac{2x\left(6i\sqrt{3}\,x^2 - \left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{2}{3}} + 6x^2\right)\left(-108 + \frac{972y}{\sqrt{-12x^6+81y^2}}\right)}{9\left(-108y + 12\sqrt{-12x^6 + 81y^2}\right)^{\frac{4}{3}}\left(i\sqrt{3} - 1\right)}\right)(xb_2 \\
&+ yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

$$\text{Expression too large to display} \tag{6E}$$

Simplifying the above gives

$$\text{Expression too large to display} \tag{6E}$$

Since the PDE has radicals, simplifying gives

Expression too large to display

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}}, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}}, \sqrt{-12x^6+81y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \left(-108y+12\sqrt{-12x^6+81y^2}\right)^{\frac{1}{3}} = v_3, \left(-108y\right.\right.$$
$$\left.\left. + 12\sqrt{-12x^6+81y^2}\right)^{\frac{2}{3}} = v_4, \sqrt{-12x^6+81y^2} = v_5\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-5184v_5v_1^3b_1 - 1728v_4v_1^6a_1 + 6912v_1^8v_5a_3 - 6912v_1^{10}v_3a_3 \\
&+ 864v_1^7v_4b_3 + 2592v_1^6v_3b_2 + 5832v_4v_2^3a_3 + 5832v_4v_2^2a_1 \\
&- 17496v_3v_2^2b_2 - 82944v_1^8v_2a_3 + 46656v_1^4v_2b_2 \\
&+ 186624v_1^3v_2^2b_3 + 46656v_1^3v_2b_1 - 5184v_5v_1^4b_2 - 559872v_1^3v_2^2a_2 \\
&+ 419904v_1^2v_2^3a_3 - 419904v_1^2v_2^2a_1 - 2592v_4v_1^7a_2 \\
&- 288i\sqrt{3}\,v_4v_5v_1^6a_3 + 216i\sqrt{3}\,v_4v_5v_1^2b_2 - 1944i\sqrt{3}\,v_4v_1^2v_2b_2 \\
&+ 216i\sqrt{3}\,v_4v_5v_1b_1 - 1944i\sqrt{3}\,v_4v_1v_2b_1 + 1728i\sqrt{3}\,v_4v_1^6v_2a_3 \\
&+ 46656i\sqrt{3}\,v_1^4v_3v_2^2a_3 - 11664i\sqrt{3}\,v_1v_4v_2^2a_2 \\
&+ 3888i\sqrt{3}\,v_4v_1v_2^2b_3 + 648i\sqrt{3}\,v_4v_5a_1v_2 + 1944i\sqrt{3}\,v_3v_5b_2v_2 \\
&+ 648i\sqrt{3}\,v_4v_5v_2^2a_3 - 6912i\sqrt{3}\,v_1^{10}v_3a_3 - 5832i\sqrt{3}\,v_4v_2^3a_3 \\
&- 5832i\sqrt{3}\,v_4v_2^2a_1 - 17496i\sqrt{3}\,v_3v_2^2b_2 - 5184v_3v_5v_1^4a_3v_2 \\
&- 1296v_4v_5v_1a_2v_2 + 432v_4v_5v_1v_2b_3 + 2592i\sqrt{3}\,v_4v_1^7a_2 \\
&- 864i\sqrt{3}\,v_1^7v_4b_3 + 1728i\sqrt{3}\,v_4v_1^6a_1 + 2592i\sqrt{3}\,v_1^6v_3b_2 \\
&- 5184i\sqrt{3}\,v_3v_5v_1^4a_3v_2 + 1296i\sqrt{3}\,v_4v_5v_1a_2v_2 \\
&- 432i\sqrt{3}\,v_4v_5v_1v_2b_3 - 20736v_1^9b_3 + 62208v_1^9a_2 + 41472v_1^8a_1 \\
&- 648v_4v_5a_1v_2 - 46656v_1^2v_5v_2^2a_3 - 1728v_4v_1^6v_2a_3 \\
&- 216v_4v_5v_1^2b_2 + 1944v_4v_1^2v_2b_2 - 3888v_4v_1v_2^2b_3 - 216v_4v_5v_1b_1 \\
&+ 1944v_4v_1v_2b_1 + 62208v_5v_1^3a_2v_2 + 46656v_5v_1^2a_1v_2 \\
&+ 288v_4v_5v_1^6a_3 + 46656v_1^4v_3v_2^2a_3 + 11664v_1v_4v_2^2a_2 \\
&- 648v_4v_5v_2^2a_3 + 1944v_3v_5b_2v_2 - 20736v_5v_1^3v_2b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-5184v_5v_1^3b_1+6912v_1^8v_5a_3-82944v_1^8v_2a_3+46656v_1^4v_2b_2\\
&+46656v_1^3v_2b_1-5184v_5v_1^4b_2+419904v_1^2v_2^3a_3\\
&-419904v_1^2v_2^2a_1+\left(-6912i\sqrt{3}\,a_3-6912a_3\right)v_1^{10}v_3\\
&+\left(2592i\sqrt{3}\,a_2-864i\sqrt{3}\,b_3-2592a_2+864b_3\right)v_1^7v_4\\
&+(62208a_2-20736b_3)\,v_1^9+\left(-5184i\sqrt{3}\,a_3-5184a_3\right)v_1^4v_2v_3v_5\\
&+\left(1296i\sqrt{3}\,a_2-432i\sqrt{3}\,b_3-1296a_2+432b_3\right)v_1v_2v_4v_5\\
&+(62208a_2-20736b_3)\,v_1^3v_2v_5+\left(2592i\sqrt{3}\,b_2+2592b_2\right)v_1^6v_3\\
&+\left(1728i\sqrt{3}\,a_1-1728a_1\right)v_1^6v_4\\
&+\left(-5832i\sqrt{3}\,a_3+5832a_3\right)v_2^3v_4\\
&+\left(-17496i\sqrt{3}\,b_2-17496b_2\right)v_2^2v_3\\
&+\left(-5832i\sqrt{3}\,a_1+5832a_1\right)v_2^2v_4+41472v_1^8a_1-46656v_1^2v_5v_2^2a_3\\
&+46656v_5v_1^2a_1v_2+\left(1728i\sqrt{3}\,a_3-1728a_3\right)v_1^6v_2v_4\\
&+\left(-288i\sqrt{3}\,a_3+288a_3\right)v_1^6v_4v_5\\
&+\left(46656i\sqrt{3}\,a_3+46656a_3\right)v_1^4v_2^2v_3\\
&+\left(-1944i\sqrt{3}\,b_2+1944b_2\right)v_1^2v_2v_4\\
&+\left(216i\sqrt{3}\,b_2-216b_2\right)v_1^2v_4v_5\\
&+\left(-11664i\sqrt{3}\,a_2+3888i\sqrt{3}\,b_3+11664a_2-3888b_3\right)v_1v_2^2v_4\\
&+\left(-1944i\sqrt{3}\,b_1+1944b_1\right)v_1v_2v_4\\
&+\left(216i\sqrt{3}\,b_1-216b_1\right)v_1v_4v_5+\left(648i\sqrt{3}\,a_3-648a_3\right)v_2^2v_4v_5\\
&+\left(1944i\sqrt{3}\,b_2+1944b_2\right)v_2v_3v_5+\left(648i\sqrt{3}\,a_1-648a_1\right)v_2v_4v_5\\
&+(-559872a_2+186624b_3)\,v_1^3v_2^2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-419904a_1 &= 0\\
41472a_1 &= 0\\
46656a_1 &= 0\\
-82944a_3 &= 0\\
-46656a_3 &= 0\\
6912a_3 &= 0\\
419904a_3 &= 0\\
-5184b_1 &= 0\\
46656b_1 &= 0\\
-5184b_2 &= 0\\
46656b_2 &= 0\\
-559872a_2 + 186624b_3 &= 0\\
62208a_2 - 20736b_3 &= 0\\
-17496i\sqrt{3}\,b_2 - 17496b_2 &= 0\\
-6912i\sqrt{3}\,a_3 - 6912a_3 &= 0\\
-5832i\sqrt{3}\,a_1 + 5832a_1 &= 0\\
-5832i\sqrt{3}\,a_3 + 5832a_3 &= 0\\
-5184i\sqrt{3}\,a_3 - 5184a_3 &= 0\\
-1944i\sqrt{3}\,b_1 + 1944b_1 &= 0\\
-1944i\sqrt{3}\,b_2 + 1944b_2 &= 0\\
-288i\sqrt{3}\,a_3 + 288a_3 &= 0\\
216i\sqrt{3}\,b_1 - 216b_1 &= 0\\
216i\sqrt{3}\,b_2 - 216b_2 &= 0\\
648i\sqrt{3}\,a_1 - 648a_1 &= 0\\
648i\sqrt{3}\,a_3 - 648a_3 &= 0\\
1728i\sqrt{3}\,a_1 - 1728a_1 &= 0\\
1728i\sqrt{3}\,a_3 - 1728a_3 &= 0\\
1944i\sqrt{3}\,b_2 + 1944b_2 &= 0\\
2592i\sqrt{3}\,b_2 + 2592b_2 &= 0\\
46656i\sqrt{3}\,a_3 + 46656a_3 &= 0\\
-11664i\sqrt{3}\,a_2 + 3888i\sqrt{3}\,b_3 + 11664a_2 - 3888b_3 &= 0\\
1296i\sqrt{3}\,a_2 - 432i\sqrt{3}\,b_3 - 1296a_2 + 432b_3 &= 0\\
2592i\sqrt{3}\,a_2 - 864i\sqrt{3}\,b_3 - 2592a_2 + 864b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_3}{3} \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= \frac{x}{3} \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Unable to determine R. Terminating

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting ODE
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying exact
      Looking for potential symmetries
      trying an equivalence to an Abel ODE
      trying 1st order ODE linearizable_by_differentiation
   -> Solving 1st order ODE of high degree, Lie methods, 1st trial
   `, `-> Computing symmetries using: way = 2
   `, `-> Computing symmetries using: way = 2
   -> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
   -> Calling odsolve with the ODE`, diff(y(x), x) = (-4*y(x)^5+3*y(x)*x^2)/(-4*y(x)^4*x+3*x
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
   <- 1st order, parametric methods successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 40

```
dsolve(diff(y(x),x)^3-4*x^4*diff(y(x),x)+8*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{2\sqrt{3}\,x^3}{9}$$
$$y(x) = \frac{2\sqrt{3}\,x^3}{9}$$
$$y(x) = \frac{x^2}{2c_1} - \frac{1}{8c_1^3}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x]^3-4*x^4*y'[x]+8*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Timed out

6.10 problem 19

6.10.1 Solving as dAlembert ode 1183

Internal problem ID [5347]
Internal file name [`OUTPUT/4838_Sunday_February_04_2024_12_46_20_AM_38250850/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 10. Singular solutions, Extraneous loci. Supplemetary problems. Page 74
Problem number: 19.
ODE order: 1.
ODE degree: 2.

The type(s) of ODE detected by this program : **"dAlembert"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{\left(1+{y'}^{2}\right)(x-y)^{2}-(x+yy')^{2}=0}$$

6.10.1 Solving as dAlembert ode

Let $p=y'$ the ode becomes

$$\left(p^{2}+1\right)(x-y)^{2}-(yp+x)^{2}=0$$

Solving for y from the above results in

$$y=\left(p^{2}+p+1+\sqrt{p^{4}+2p^{3}+2p^{2}+2p+1}\right)x \tag{1A}$$
$$y=\left(p^{2}+p+1-\sqrt{p^{4}+2p^{3}+2p^{2}+2p+1}\right)x \tag{2A}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. Each of the above ode's is dAlembert ode which is now solved. Solving ode 1A Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 + p + 1 + \sqrt{(p^2+1)(p+1)^2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 - 1 - \sqrt{(p^2+1)(p+1)^2} = x\left(2p + 1 + \frac{2p(p+1)^2 + 2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}}\right)p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 - 1 - \sqrt{(p^2+1)(p+1)^2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 - 1 - \sqrt{(p(x)^2+1)(p(x)+1)^2}}{x\left(2p(x) + 1 + \frac{2p(x)(p(x)+1)^2 + 2\left(p(x)^2+1\right)(p(x)+1)}{2\sqrt{\left(p(x)^2+1\right)(p(x)+1)^2}}\right)} \tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$
\frac{d}{dp}x(p) = \frac{x(p)\left(2p + 1 + \frac{2p(p+1)^2 + 2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}}\right)}{-p^2 - 1 - \sqrt{(p^2+1)(p+1)^2}} \tag{4}
$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p) + p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p) - \frac{\left((-2p-1)\sqrt{(p^2+1)(p+1)^2} - 2p^3 - 3p^2 - 2p - 1\right)x(p)}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}dp} x\right) = 0$$

Integrating gives

$$e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}dp} x = c_2$$

Dividing both sides by the integrating factor $\mu = e^{\int -\frac{(-2p-1)\sqrt{(p^2+1)(p+1)^2}-2p^3-3p^2-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}$ results in

$$x(p) = c_2 e^{-\left(\int \frac{(2p+1)\sqrt{(p^2+1)(p+1)^2}+2p^3+3p^2+2p+1}{\sqrt{(p^2+1)(p+1)^2}\left(p^2+1+\sqrt{(p^2+1)(p+1)^2}\right)}dp\right)}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Solving ode 2A Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}
\tag{2}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 + p + 1 - \sqrt{(p^2+1)(p+1)^2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} = x\left(2p + 1 - \frac{2p(p+1)^2 + 2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}}\right)p'(x)
\tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 - 1 + \sqrt{(p^2+1)(p+1)^2} = 0
$$

Solving for p from the above gives

$$
\begin{aligned}
p &= 0 \\
p &= i \\
p &= -i
\end{aligned}
$$

Substituting these in (1A) gives

$$
\begin{aligned}
y &= 0 \\
y &= -ix \\
y &= ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 - 1 + \sqrt{(p(x)^2+1)(p(x)+1)^2}}{x\left(2p(x) + 1 - \frac{2p(x)(p(x)+1)^2 + 2\left(p(x)^2+1\right)(p(x)+1)}{2\sqrt{\left(p(x)^2+1\right)(p(x)+1)^2}}\right)}
\tag{3}
$$

This ODE is now solved for $p(x)$.

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{x(p)\left(2p+1-\frac{2p(p+1)^2+2(p^2+1)(p+1)}{2\sqrt{(p^2+1)(p+1)^2}}\right)}{-p^2-1+\sqrt{(p^2+1)(p+1)^2}} \tag{4}$$

This ODE is now solved for $x(p)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$\frac{d}{dp}x(p)+p(p)x(p) = q(p)$$

Where here

$$p(p) = -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}$$

$$q(p) = 0$$

Hence the ode is

$$\frac{d}{dp}x(p)-\frac{x(p)\left(-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1\right)}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}p}\mu x = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}p}\left(e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}x\right) = 0$$

Integrating gives

$$e^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}x = c_4$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\int -\frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}$ results in

$$x(p) = c_4 \mathrm{e}^{\int \frac{-2p^3+2p\sqrt{(p^2+1)(p+1)^2}-3p^2+\sqrt{(p^2+1)(p+1)^2}-2p-1}{\sqrt{(p^2+1)(p+1)^2}\left(-p^2-1+\sqrt{(p^2+1)(p+1)^2}\right)}dp}$$

Since the solution $x(p)$ has unresolved integral, unable to continue.

Summary

The solution(s) found are the following

$$y = -ix \tag{1}$$

$$y = ix \tag{2}$$

$$y = 0 \tag{3}$$

$$y = -ix \tag{4}$$

$$y = ix \tag{5}$$

Verification of solutions

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

$$y = 0$$

Verified OK.

$$y = -ix$$

Verified OK.

$$y = ix$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   <- symmetries for implicit equations successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 103

```
dsolve((diff(y(x),x)^2+1)*(x-y(x))^2=(x+y(x)*diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = 0$$

$$y(x) = \operatorname{RootOf}\left(-2\ln(x) - \left(\int^{\_Z} \frac{2\_a^2 + \sqrt{2}\,\sqrt{\_a\,(\_a-1)^2}}{\_a\,(\_a^2+1)} d\_a\right) + 2c_1\right) x$$

$$y(x) = \operatorname{RootOf}\left(-2\ln(x) + \int^{\_Z} \frac{\sqrt{2}\,\sqrt{\_a\,(\_a-1)^2} - 2\_a^2}{\_a\,(\_a^2+1)} d\_a + 2c_1\right) x$$

✓ Solution by Mathematica

Time used: 4.36 (sec). Leaf size: 167

```
DSolve[(y'[x]^2+1)*(x-y[x])^2==(x+y[x]*y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)}-e^{\frac{c_1}{2}}$$
$$y(x) \to \sqrt{-x\left(x+2e^{\frac{c_1}{2}}\right)}-e^{\frac{c_1}{2}}$$
$$y(x) \to e^{\frac{c_1}{2}}-\sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)}$$
$$y(x) \to \sqrt{x\left(-x+2e^{\frac{c_1}{2}}\right)}+e^{\frac{c_1}{2}}$$
$$y(x) \to -\sqrt{-x^2}$$
$$y(x) \to \sqrt{-x^2}$$

7 Chapter 12. Linear equations of order n. Supplemetary problems. Page 81

7.1 problem 10 1192
7.2 problem 11 1200
7.3 problem 12 1202
7.4 problem 13 1213
7.5 problem 14 1224
7.6 problem 15 1240
7.7 problem 16 1247
7.8 problem 17 1263
7.9 problem 18 1268
7.10 problem 19 1278

7.1 problem 10

7.1.1 Solving as second order linear constant coeff ode 1192
7.1.2 Solving using Kovacic algorithm 1194
7.1.3 Maple step by step solution 1198

Internal problem ID [5348]
Internal file name [`OUTPUT/4839_Sunday_February_04_2024_12_46_22_AM_33866243/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 0}$$

7.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{1^2 - (4)(1)(-6)} \\
&= -\frac{1}{2} \pm \frac{5}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{5}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -3
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-3)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x} \tag{1}
$$

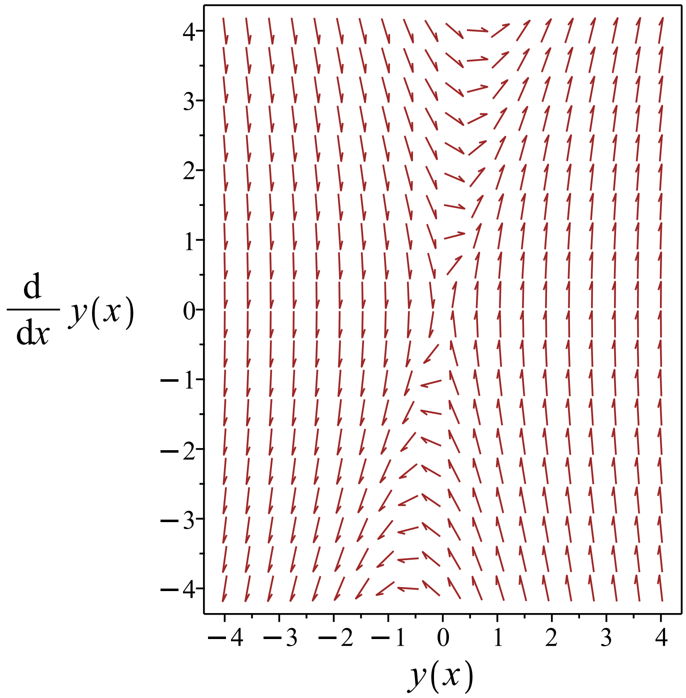


Figure 202: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Verified OK.

7.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-3x} + \frac{c_2\mathrm{e}^{2x}}{5} \tag{1}
$$

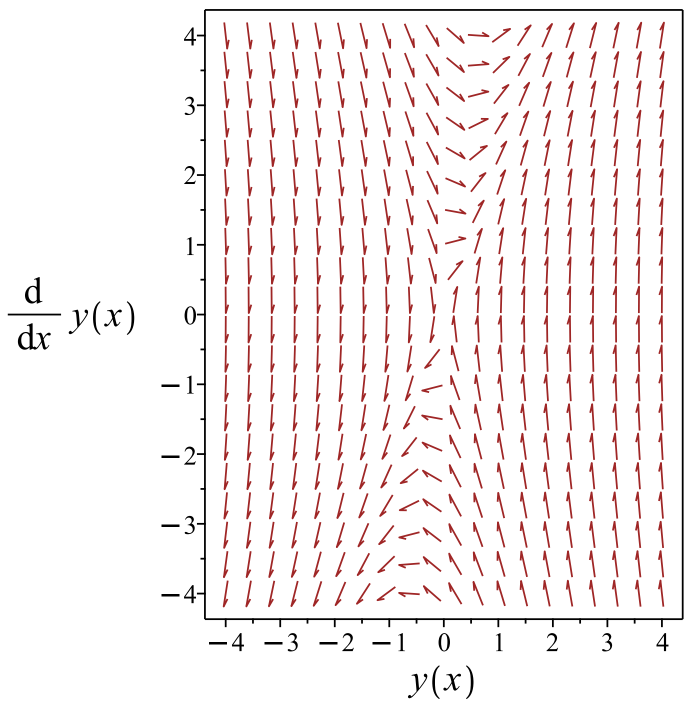


Figure 203: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1\mathrm{e}^{-3x} + \frac{c_2\mathrm{e}^{2x}}{5}
$$

Verified OK.

7.1.3 Maple step by step solution

Let's solve

$y'' + y' - 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-6*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_1 \mathrm{e}^{5x} + c_2)\, \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+y'[x]-6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_2 e^{5x} + c_1\right)$$

7.2 problem 11

Internal problem ID [5349]
Internal file name [`OUTPUT/4840_Sunday_February_04_2024_12_46_23_AM_24262917/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 11.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 6y'' + 12y' - 8y = 0}$$

The characteristic equation is

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above equation are

$$\lambda_1 = 2$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{2x} + x\, e^{2x} c_2 + x^2 e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{2x}$$
$$y_2 = e^{2x} x$$
$$y_3 = x^2 e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{2x} + x e^{2x} c_2 + x^2 e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{2x} + x e^{2x} c_2 + x^2 e^{2x} c_3$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$3)-6*diff(y(x),x$2)+12*diff(y(x),x)-8*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_3 x^2 + c_2 x + c_1)$$

✓ Solution by Mathematica
Time used: 0.003 (sec). Leaf size: 23

```
DSolve[y'''[x]-6*y''[x]+12*y'[x]-8*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(x(c_3 x + c_2) + c_1)$$

7.3 problem 12

7.3.1 Solving as second order linear constant coeff ode 1202
7.3.2 Solving using Kovacic algorithm 1205
7.3.3 Maple step by step solution 1210

Internal problem ID [5350]
Internal file name [`OUTPUT/4841_Sunday_February_04_2024_12_46_23_AM_84963062/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 3y' + 2y = e^{5x}}$$

7.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = e^{5x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{5x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{5x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{5x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^{5x} = e^{5x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{5x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^x\right) + \left(\frac{e^{5x}}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^x + \frac{e^{5x}}{12} \tag{1}$$

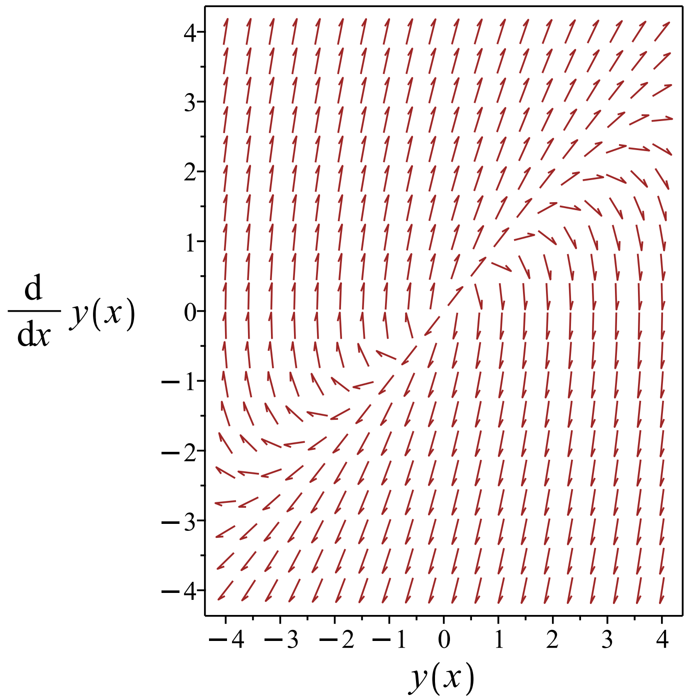


Figure 204: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} + c_2 e^{x} + \frac{e^{5x}}{12}
$$

Verified OK.

7.3.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 3y' + 2y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -3 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{5x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{5x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^x, \mathrm{e}^{2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{5x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
12A_1 \mathrm{e}^{5x} = \mathrm{e}^{5x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{5x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{2x}\right) + \left(\frac{e^{5x}}{12}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{2x} + \frac{e^{5x}}{12} \tag{1}$$

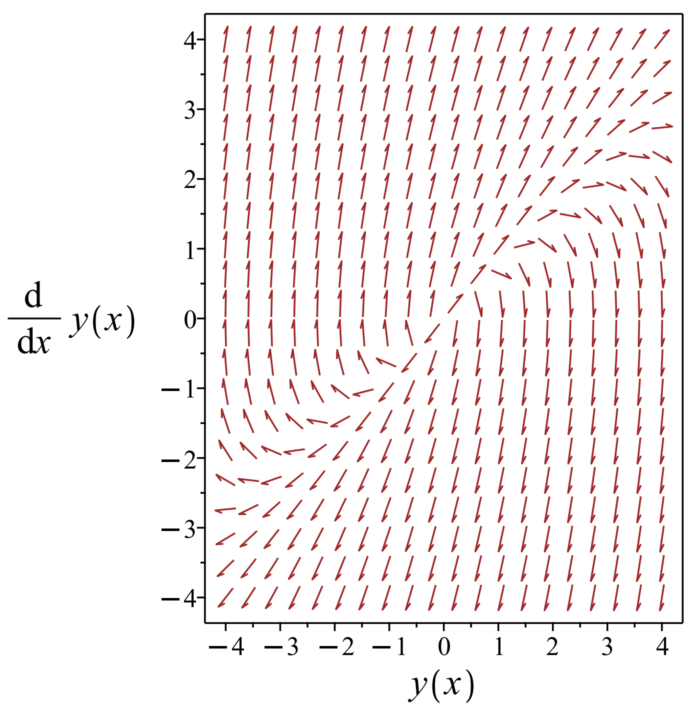


Figure 205: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + \frac{\mathrm{e}^{5x}}{12}$$

Verified OK.

7.3.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \mathrm{e}^{5x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{5x}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{3x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^x\left(\int e^{4x}dx\right) + e^{2x}\left(\int e^{3x}dx\right)$
- Compute integrals

  $y_p(x) = \frac{e^{5x}}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1e^x + c_2e^{2x} + \frac{e^{5x}}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=exp(5*x),y(x), singsol=all)
```

$$y(x) = \frac{(e^{4x} + 12e^x c_1 + 12c_2)e^x}{12}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Exp[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{5x}}{12} + c_1 e^x + c_2 e^{2x}$$

7.4 problem 13

7.4.1 Solving as second order linear constant coeff ode 1213
7.4.2 Solving using Kovacic algorithm 1216
7.4.3 Maple step by step solution 1221

Internal problem ID [5351]
Internal file name [`OUTPUT/4842_Sunday_February_04_2024_12_46_23_AM_16762942/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + 9y = \cos(x)\, x}$$

7.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \cos(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_1 \cos(x)\, x - 2A_1 \sin(x) + 8A_2 \sin(x)\, x + 2A_2 \cos(x) + 8A_3 \cos(x) + 8A_4 \sin(x)\\ &\quad = \cos(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = 0, A_4 = \frac{1}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32} \tag{1}$$

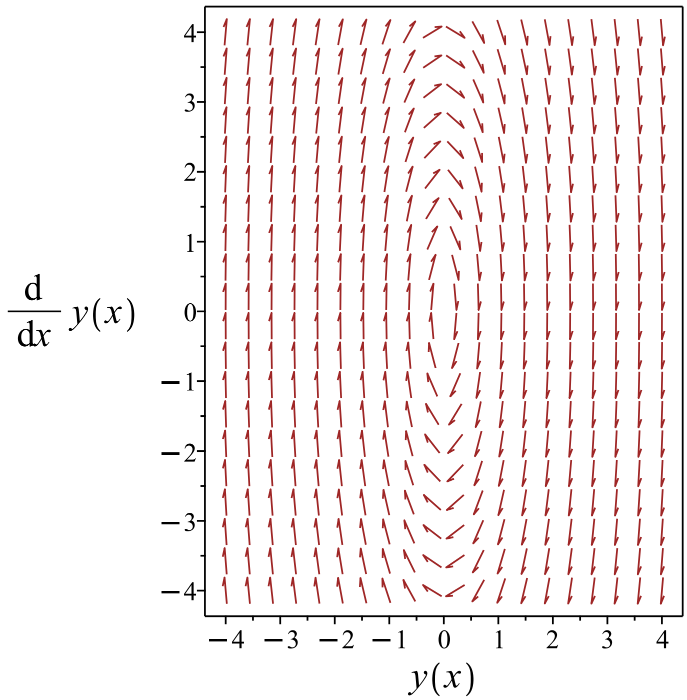


Figure 206: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}$$

Verified OK.

7.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\ &= \cos(3x) \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x + A_3 \cos(x) + A_4 \sin(x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_1 \cos(x)\, x - 2A_1 \sin(x) + 8A_2 \sin(x)\, x + 2A_2 \cos(x) + 8A_3 \cos(x) + 8A_4 \sin(x)\\&= \cos(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = 0, A_4 = \frac{1}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32} \tag{1}$$

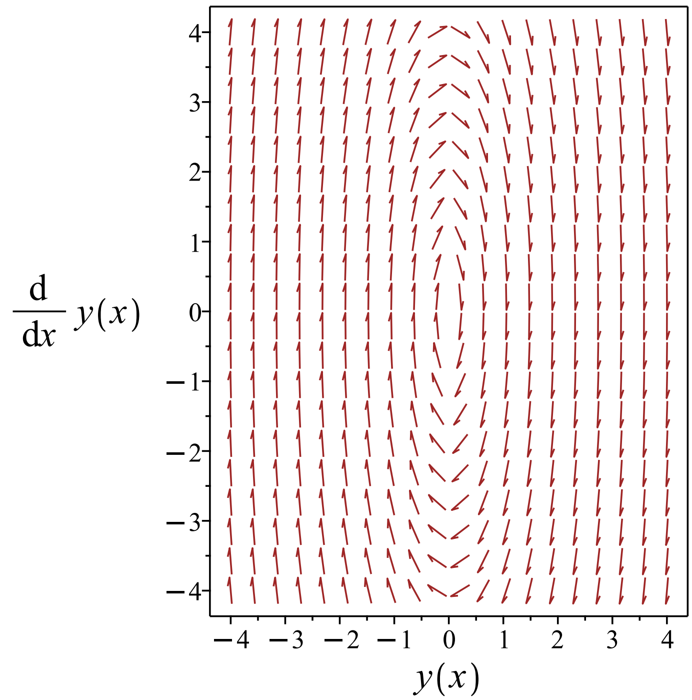


Figure 207: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\cos(x)\, x}{8} + \frac{\sin(x)}{32}$$

Verified OK.

7.4.3 Maple step by step solution

Let's solve

$y'' + 9y = \cos(x)\, x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)\, x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)\cos(x) x dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)\cos(x) x dx\right)}{3}$

  - Compute integrals

    $y_p(x) = \frac{\cos(x) x}{8} + \frac{\sin(x)}{32}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\cos(x) x}{8} + \frac{\sin(x)}{32}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+9*y(x)=x*cos(x),y(x), singsol=all)
```

$$y(x) = \sin(3x)\, c_2 + \cos(3x)\, c_1 + \frac{\sin(x)}{32} + \frac{\cos(x)\, x}{8}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 32

```
DSolve[y''[x]+9*y[x]==x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32}(\sin(x) + 4x\cos(x)) + c_1\cos(3x) + c_2\sin(3x)$$

7.5 problem 14

7.5.1 Solving as second order euler ode ode 1224
7.5.2 Solving as second order change of variable on x method 2 ode . 1225
7.5.3 Solving as second order change of variable on x method 1 ode . 1228
7.5.4 Solving as second order change of variable on y method 2 ode . 1230
7.5.5 Solving using Kovacic algorithm 1232
7.5.6 Maple step by step solution 1237

Internal problem ID [5352]
Internal file name [`OUTPUT/4843_Sunday_February_04_2024_12_46_24_AM_71040505/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 0}$$

7.5.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary
The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

7.5.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)} dx \\
&= \int e^{3\ln(x)}\, dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_2\ln(x^4) - 2c_2\ln(2) + c_1\right)\sqrt{x^4}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

Verified OK.

7.5.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\,\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

7.5.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln (u) &= -\ln (x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln (x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= (c_1 \ln (x) + c_2)\, x^2 \\ &= (c_1 \ln (x) + c_2)\, x^2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1 \ln (x) + c_2)\, x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \ln (x) + c_2)\, x^2$$

Verified OK.

7.5.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 + c_2 x^2 \ln(x)
$$

Verified OK.

7.5.6 Maple step by step solution

Let's solve

$x^2y'' - 3xy' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 3xy' + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{2t} + c_2 t\,\mathrm{e}^{2t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 + c_2 x^2 \ln(x)$
- Simplify

  $y = x^2(c_1 + c_2 \ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2 \log(x) + c_1)$$

7.6 problem 15

7.6.1 Maple step by step solution 1245

Internal problem ID [5353]
Internal file name [`OUTPUT/4844_Sunday_February_04_2024_12_46_25_AM_48686744/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' + xy' - y = 3x^4}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + xy' - y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + xy' - y = 3x^4$$

gives

$$
x\lambda\,x^{\lambda-1} + x^3\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda-3} - x^\lambda = 0
$$

Which simplifies to

$$
\lambda\,x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\,x^\lambda - x^\lambda = 0
$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$
\lambda + \lambda(\lambda-1)\,(\lambda-2) - 1 = 0
$$

Simplifying gives the characteristic equation as

$$
(\lambda-1)^3 = 0
$$

Solving the above gives the following roots

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= 1 \\
\lambda_3 &= 1
\end{aligned}
$$

This table summarises the result

root	multiplicity	type of root
1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\,x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^\alpha(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$
y = c_1x + c_2\ln(x)\,x + c_3\ln(x)^2\,x
$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x\ln(x) \\
y_3 &= x\ln(x)^2
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$x^3 y''' + x y' - y = 3x^4$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$\begin{aligned} W &= \begin{bmatrix} x & x \ln(x) & x \ln(x)^2 \\ 1 & \ln(x) + 1 & \ln(x) (\ln(x) + 2) \\ 0 & \frac{1}{x} & \frac{2 \ln(x) + 2}{x} \end{bmatrix} \\ |W| &= 2 \end{aligned}$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} x \ln(x) & x \ln(x)^2 \\ \ln(x) + 1 & \ln(x) (\ln(x) + 2) \end{bmatrix} \\ &= x \ln(x)^2 \end{aligned}$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} x & x\ln(x)^2 \\ 1 & \ln(x)(\ln(x)+2) \end{bmatrix} \\
&= 2x\ln(x)
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\
&= x
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2 \int \frac{(3x^4)\left(x\ln(x)^2\right)}{(x^3)(2)}\,dx \\
&= \int \frac{3x^5\ln(x)^2}{2x^3}\,dx \\
&= \int \left(\frac{3x^2\ln(x)^2}{2}\right)\,dx \\
&= \frac{x^3\ln(x)^2}{2} - \frac{x^3\ln(x)}{3} + \frac{x^3}{9}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(3x^4)(2x\ln(x))}{(x^3)(2)}\,dx \\
&= -\int \frac{6x^5\ln(x)}{2x^3}\,dx \\
&= -\int \left(3x^2\ln(x)\right)\,dx \\
&= -x^3\ln(x) + \frac{x^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^0 \int \frac{(3x^4)(x)}{(x^3)(2)}\, dx \\
&= \int \frac{3x^5}{2x^3}\, dx \\
&= \int \left(\frac{3x^2}{2}\right) dx \\
&= \frac{x^3}{2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9}\right)(x) \\
&+ \left(-x^3 \ln(x) + \frac{x^3}{3}\right)(x \ln(x)) \\
&+ \left(\frac{x^3}{2}\right)\left(x \ln(x)^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^4}{9}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x + c_2 \ln(x)\, x + c_3 \ln(x)^2 x\right) + \left(\frac{x^4}{9}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9}
$$

Summary

The solution(s) found are the following

$$
y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9} \tag{1}
$$

Verification of solutions

$$y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9}$$

Verified OK.

7.6.1 Maple step by step solution

Let's solve

$x^3 y''' + x y' - y = 3x^4$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)+x*diff(y(x),x)-y(x)=3*x^4,y(x), singsol=all)
```

$$y(x) = \frac{x\left(9c_2 \ln(x)^2 + x^3 + 9c_3 \ln(x) + 9c_1\right)}{9}$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 31

```
DSolve[x^3*y'''[x]+x*y'[x]-y[x]==3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{9} + c_1 x + c_3 x \log^2(x) + c_2 x \log(x)$$

7.7 problem 16

7.7.1 Solving as second order change of variable on x method 2 ode . 1247
7.7.2 Solving as second order change of variable on x method 1 ode . 1250
7.7.3 Solving as second order bessel ode ode 1252
7.7.4 Solving using Kovacic algorithm 1253
7.7.5 Maple step by step solution 1259

Internal problem ID [5354]
Internal file name [`OUTPUT/4845_Sunday_February_04_2024_12_46_25_AM_91415232/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy'' - y' + 4yx^3 = 0}$$

7.7.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = 4x^2$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{4x^2}{x^2} \\
&= 4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

7.7.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{x^2}}{c} \\ \tau'' &= \frac{2x}{c\sqrt{x^2}}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{x^2}dx}{c} \\ &= \frac{x\sqrt{x^2}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

Verified OK.

7.7.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - x y' + 4 y x^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= 1 \\ n &= \frac{1}{2} \\ \gamma &= 2 \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}$$

Verified OK.

7.7.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
xy'' - y' + 4yx^3 &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x \\
B &= -1 \tag{3}\\
C &= 4x^3
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -16x^4 + 3 \\
t &= 4x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty)=-2$ then

$$
v=\frac{-O_r(\infty)}{2}=\frac{2}{2}=1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v=x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r}\approx 2ix-\frac{3i}{16x^3}-\frac{9i}{1024x^7}-\frac{27i}{32768x^{11}}-\frac{405i}{4194304x^{15}}-\frac{1701i}{134217728x^{19}}-\frac{15309i}{8589934592x^{23}}-\frac{72171i}{274877906944x^{27}}+ \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a=2i
$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2=-4x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$
r=Q+\frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &= -\frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\left(2ix\right) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-i x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-i x^2}\right) + c_2\left(\mathrm{e}^{-i x^2}\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-i x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4} \tag{1}
$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-i x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4}$$

Verified OK.

7.7.5 Maple step by step solution

Let's solve

$y'' x - y' + 4 y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + 4x^2 y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x - y' + 4 y x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - Shift index using $k- >k-3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + a_1(1+r)\,(-1+r)\, x^r + a_2(2+r)\, r\, x^{1+r} + a_3(3+r)\,(1+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r = 0, a_3(3+r)\,(1+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + 4a_{k-3} = 0$

- Shift index using $k->k+3$

  $$a_{k+4}(k+4+r)(k+2+r)+4a_k=0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+4}=-\frac{4a_k}{(k+4+r)(k+2+r)}$$

- Recursion relation for $r=0$

  $$a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}$$

- Solution for $r=0$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0\right]$$

- Recursion relation for $r=2$

  $$a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}$$

- Solution for $r=2$

  $$\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}, a_1=0, a_2=0, a_3=0\right]$$

- Combine solutions and rename parameters

  $$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+6)(k+4)}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\sin\left(x^2\right)+c_2\cos\left(x^2\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 20

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

7.8 problem 17

7.8.1 Solving as second order ode missing y ode 1263
7.8.2 Solving as second order ode missing x ode 1264
7.8.3 Maple step by step solution 1266

Internal problem ID [5355]
Internal file name [`OUTPUT/4846_Sunday_February_04_2024_12_46_26_AM_79790319/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + {y'}^2 = -1}$$

7.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2 - 1} dp = x + c_1$$
$$-\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(x + c_1)\, \mathrm{d}x \\ &= -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

7.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2+1}dp = \int dy$$
$$-\frac{\ln (p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p^2+1}} = \mathrm{e}^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p^2+1}} = c_2\mathrm{e}^{y}$$

Solving for $p(y)$ gives

$$p(y) = \mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2y} + c_2^2 \mathrm{e}^{2y} - 1\right)$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2y} + c_2^2 \mathrm{e}^{2y} - 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2y} + c_2^2 \mathrm{e}^{2y} - 1\right)}dy = \int dx$$
$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2\_a} + c_2^2 \mathrm{e}^{2\_a} - 1\right)}d\_a = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2\_a} + c_2^2 \mathrm{e}^{2\_a} - 1\right)}d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^y \frac{1}{\mathrm{RootOf}\left(\_Z^2 c_2^2 \mathrm{e}^{2\_a} + c_2^2 \mathrm{e}^{2\_a} - 1\right)}d\_a = c_3 + x$$

Verified OK.

7.8.3 Maple step by step solution

Let's solve

$y'' + y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)^2 = -1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1} dx = \int 1 dx + c_1$

- Evaluate integral

  $-\arctan\left(u(x)\right) = x + c_1$

- Solve for $u(x)$

  $u(x) = -\tan\left(x + c_1\right)$

- Solve 1st ODE for $u(x)$

  $u(x) = -\tan\left(x + c_1\right)$

- Make substitution $u = y'$

  $y' = -\tan\left(x + c_1\right)$

- Integrate both sides to solve for y

  $\int y' dx = \int -\tan\left(x + c_1\right) dx + c_2$

- Compute integrals

  $y = -\frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \ln\left(\cos\left(x\right) c_2 - c_1 \sin\left(x\right)\right)$$

✓ Solution by Mathematica

Time used: 1.861 (sec). Leaf size: 16

```
DSolve[y''[x]+y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(\cos(x - c_1)) + c_2$$

7.9 problem 18

7.9.1 Solving as second order integrable as is ode 1268
7.9.2 Solving as second order ode missing x ode 1269
7.9.3 Solving as type second_order_integrable_as_is (not using ABC version) 1272
7.9.4 Solving as exact nonlinear second order ode ode 1273
7.9.5 Maple step by step solution 1274

Internal problem ID [5356]
Internal file name [`OUTPUT/4847_Sunday_February_04_2024_12_46_27_AM_79210250/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 = 2}$$

7.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + {y'}^2\right) dx = \int 2dx$$
$$yy' = 2x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{c_1+2x}{y}
\end{aligned}
$$

Where $f(x)=c_1+2x$ and $g(y)=\frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\,dy &= c_1+2x\,dx \\
\int \frac{1}{\frac{1}{y}}\,dy &= \int c_1+2x\,dx \\
\frac{y^2}{2} &= c_1x+x^2+c_2
\end{aligned}
$$

The solution is

$$
\frac{y^2}{2}-c_1x-x^2-c_2=0
$$

Summary

The solution(s) found are the following

$$
\frac{y^2}{2}-c_1x-x^2-c_2=0 \tag{1}
$$

Verification of solutions

$$
\frac{y^2}{2}-c_1x-x^2-c_2=0
$$

Verified OK.

7.9.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y'=p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
yp(y)\left(\frac{d}{dy}p(y)\right)+p(y)^2=2
$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p^2-2}{yp}
\end{aligned}
$$

Where $f(y)=-\frac{1}{y}$ and $g(p)=\frac{p^2-2}{p}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{p^2-2}{p}}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{\frac{p^2-2}{p}}\,dp &= \int -\frac{1}{y}\,dy \\
\frac{\ln(p^2-2)}{2} &= -\ln(y)+c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{p^2-2}=\mathrm{e}^{-\ln(y)+c_1}
$$

Which simplifies to

$$
\sqrt{p^2-2}=\frac{c_2}{y}
$$

Which simplifies to

$$
\sqrt{p(y)^2-2}=\frac{c_2\mathrm{e}^{c_1}}{y}
$$

The solution is

$$\sqrt{p\left(y\right)^{2}-2}=\frac{c_{2}\mathrm{e}^{c_{1}}}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\sqrt{y'^{2}-2}=\frac{c_{2}\mathrm{e}^{c_{1}}}{y}$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y'=\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{y} \tag{1}$$

$$y'=-\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{y} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{y}{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}dy=\int dx$$

$$\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{2}=c_{3}+x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{y}{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}dy=\int dx$$

$$-\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{2}=x+c_{4}$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{2}=c_{3}+x \tag{1}$$

$$-\frac{\sqrt{c_{2}^{2}\mathrm{e}^{2c_{1}}+2y^{2}}}{2}=x+c_{4} \tag{2}$$

Verification of solutions

$$\frac{\sqrt{c_2^2 e^{2c_1} + 2y^2}}{2} = c_3 + x$$

Verified OK.

$$-\frac{\sqrt{c_2^2 e^{2c_1} + 2y^2}}{2} = x + c_4$$

Verified OK.

7.9.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 2$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = \int 2 dx$$
$$yy' = 2x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{c_1 + 2x}{y} \end{aligned}$$

Where $f(x) = c_1 + 2x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = c_1 + 2x\, dx$$
$$\int \frac{1}{\frac{1}{y}}\, dy = \int c_1 + 2x\, dx$$
$$\frac{y^2}{2} = c_1 x + x^2 + c_2$$

The solution is

$$\frac{y^2}{2} - c_1 x - x^2 - c_2 = 0$$

Summary
The solution(s) found are the following

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - c_1x - x^2 - c_2 = 0$$

Verified OK.

7.9.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}
a_2 &= y \\
a_1 &= y' \\
a_0 &= -2
\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}
\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\
\int y\, dy' + \int y'\, dy + \int -2\, dx &= c_1
\end{aligned}$$

Which results in

$$2yy' - 2x = c_1$$

Which is now solved In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{\frac{c_1}{2}+x}{y}
\end{aligned}
$$

Where $f(x) = \frac{c_1}{2} + x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\,dy &= \frac{c_1}{2} + x\,dx \\
\int \frac{1}{\frac{1}{y}}\,dy &= \int \frac{c_1}{2} + x\,dx \\
\frac{y^2}{2} &= \frac{1}{2}c_1 x + \frac{1}{2}x^2 + c_2
\end{aligned}
$$

The solution is

$$
\frac{y^2}{2} - \frac{c_1 x}{2} - \frac{x^2}{2} - c_2 = 0
$$

<u>Summary</u>

The solution(s) found are the following

$$
\frac{y^2}{2} - \frac{c_1 x}{2} - \frac{x^2}{2} - c_2 = 0 \tag{1}
$$

<u>Verification of solutions</u>

$$
\frac{y^2}{2} - \frac{c_1 x}{2} - \frac{x^2}{2} - c_2 = 0
$$

Warning, solution could not be verified

7.9.5 Maple step by step solution

Let's solve

$yy'' + y'^2 = 2$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$
- Compute y''

  $u'(x) = y''$
- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$
- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + u(y)^2 = 2$
- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2+2} = \frac{1}{y}$
- Integrate both sides with respect to y

  $\int \frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2+2}dy = \int \frac{1}{y}dy + c_1$
- Evaluate integral

  $-\frac{\ln\left(u(y)^2-2\right)}{2} = \ln(y) + c_1$
- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{2(\mathrm{e}^{c_1})^2y^2+1}}{\mathrm{e}^{c_1}y}, u(y) = -\frac{\sqrt{2(\mathrm{e}^{c_1})^2y^2+1}}{\mathrm{e}^{c_1}y}\right\}$
- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{2(\mathrm{e}^{c_1})^2y^2+1}}{\mathrm{e}^{c_1}y}$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{\sqrt{2(\mathrm{e}^{c_1})^2y^2+1}}{\mathrm{e}^{c_1}y}$
- Separate variables

  $\frac{y'y}{\sqrt{2(\mathrm{e}^{c_1})^2y^2+1}} = \frac{1}{\mathrm{e}^{c_1}}$
- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}} dx = \int \frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}}{2 (\mathrm{e}^{c_1})^2} = \frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{-2+8 (\mathrm{e}^{c_1})^4 c_2^2+16 (\mathrm{e}^{c_1})^3 c_2 x+8 x^2 (\mathrm{e}^{c_1})^2}}{2 \,\mathrm{e}^{c_1}}, y = \frac{\sqrt{-2+8 (\mathrm{e}^{c_1})^4 c_2^2+16 (\mathrm{e}^{c_1})^3 c_2 x+8 x^2 (\mathrm{e}^{c_1})^2}}{2 \,\mathrm{e}^{c_1}}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}}{\mathrm{e}^{c_1} y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}}{\mathrm{e}^{c_1} y}$$

- Separate variables

$$\frac{y' y}{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}} = -\frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}} dx = \int -\frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$\frac{\sqrt{2 (\mathrm{e}^{c_1})^2 y^2 + 1}}{2 (\mathrm{e}^{c_1})^2} = -\frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{-2+8 (\mathrm{e}^{c_1})^4 c_2^2-16 (\mathrm{e}^{c_1})^3 c_2 x+8 x^2 (\mathrm{e}^{c_1})^2}}{2 \,\mathrm{e}^{c_1}}, y = \frac{\sqrt{-2+8 (\mathrm{e}^{c_1})^4 c_2^2-16 (\mathrm{e}^{c_1})^3 c_2 x+8 x^2 (\mathrm{e}^{c_1})^2}}{2 \,\mathrm{e}^{c_1}}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=2,y(x), singsol=all)
```

$$y(x) = \sqrt{-2c_1x + 2x^2 + 2c_2}$$
$$y(x) = -\sqrt{-2c_1x + 2x^2 + 2c_2}$$

✓ Solution by Mathematica

Time used: 6.295 (sec). Leaf size: 101

```
DSolve[y[x]*y''[x]+y'[x]^2==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{4(x+c_2)^2 - e^{2c_1}}}{\sqrt{2}}$$
$$y(x) \to \sqrt{2(x+c_2)^2 - \frac{e^{2c_1}}{2}}$$
$$y(x) \to -\sqrt{2}\sqrt{(x+c_2)^2}$$
$$y(x) \to \sqrt{2}\sqrt{(x+c_2)^2}$$

7.10 problem 19

7.10.1 Solving as second order ode missing x ode 1278
7.10.2 Maple step by step solution 1280

Internal problem ID [5357]
Internal file name [`OUTPUT/4848_Sunday_February_04_2024_12_46_29_AM_29877559/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 12. Linear equations of order n. Supplemetary problems. Page 81
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
  [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{y y'' + {y'}^{3} = 0}$$

7.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y p(y) \left(\frac{d}{dy} p(y)\right) + p(y)^{3} = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p^2}{y}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p^2}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p^2}\,dp &= \int -\frac{1}{y}\,dy \\
-\frac{1}{p} &= -\ln(y) + c_1
\end{aligned}
$$

The solution is

$$
-\frac{1}{p(y)} + \ln(y) - c_1 = 0
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
-\frac{1}{y'} + \ln(y) - c_1 = 0
$$

Integrating both sides gives

$$
\begin{aligned}
\int (\ln(y) - c_1)\,dy &= x + c_2 \\
-c_1 y + \ln(y)\,y - y &= x + c_2
\end{aligned}
$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$
y = e^{\mathrm{LambertW}\left((x+c_2)e^{-1-c_1}\right)+1+c_1} \tag{1}
$$

Verification of solutions

$$
y = e^{\mathrm{LambertW}\left((x+c_2)e^{-1-c_1}\right)+1+c_1}
$$

Verified OK.

7.10.2 Maple step by step solution

Let's solve

$yy'' + y'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y' \left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y) \left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y) \left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y) \left(\frac{d}{dy}u(y)\right) + u(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = -\frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{1}{\ln(y)-c_1}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{1}{\ln(y)-c_1}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{1}{\ln(y)-c_1}$

- Separate variables

  $y'(\ln(y) - c_1) = 1$
- Integrate both sides with respect to x

  $\int y'(\ln(y) - c_1)\, dx = \int 1 dx + c_2$
- Evaluate integral

  $-yc_1 + \ln(y)\, y - y = x + c_2$
- Solve for y

  $y = e^{LambertW\left((x+c_2)e^{-1-c_1}\right)+1+c_1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^3/_a = 0, _b(_a)`   *** S
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= c_1 \\
y(x) &= \frac{x + c_2}{\operatorname{LambertW}\left((x + c_2)\, \mathrm{e}^{c_1 - 1}\right)}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 60.091 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{x + c_2}{W\left(e^{-1-c_1}(x + c_2)\right)}
$$

8 Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86

8.1 problem 16 . . . 1284
8.2 problem 17 . . . 1292
8.3 problem 18 . . . 1297
8.4 problem 19 . . . 1306
8.5 problem 20 . . . 1308
8.6 problem 21 . . . 1316
8.7 problem 22 . . . 1326
8.8 problem 23 . . . 1331
8.9 problem 24 . . . 1337
8.10 problem 25 . . . 1344

8.1 problem 16

8.1.1 Solving as second order linear constant coeff ode 1284
8.1.2 Solving using Kovacic algorithm 1286
8.1.3 Maple step by step solution 1290

Internal problem ID [5358]
Internal file name [`OUTPUT/4849_Sunday_February_04_2024_12_46_29_AM_58753252/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' - 15y = 0}$$

8.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -15$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} - 15\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 15 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -15$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(-15)} \\
&= -1 \pm 4
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= -1 + 4 \\
\lambda_2 &= -1 - 4
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -5
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(-5)x}
\end{aligned}
$$

Or

$$
y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-5x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-5x} \tag{1}
$$

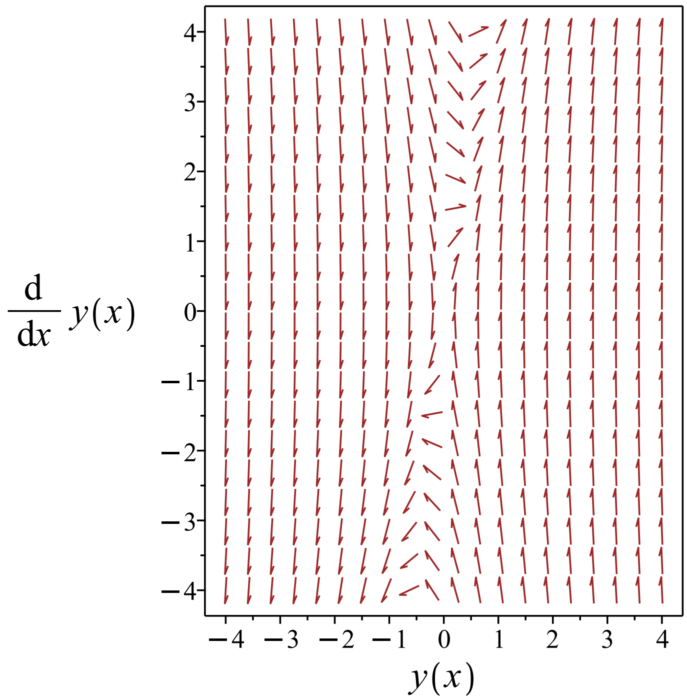


Figure 208: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2 e^{-5x}$$

Verified OK.

8.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 15y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{8x}}{8}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-5x}\right) + c_2\left(\mathrm{e}^{-5x}\left(\frac{\mathrm{e}^{8x}}{8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{3x}}{8} \tag{1}
$$

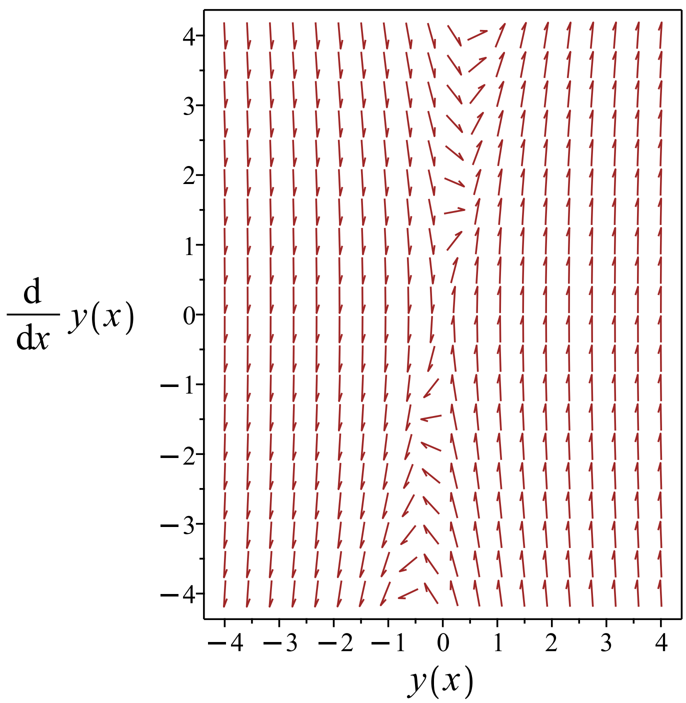


Figure 209: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{3x}}{8}
$$

Verified OK.

8.1.3 Maple step by step solution

Let's solve

$y'' + 2y' - 15y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 15 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-5x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-15*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(\mathrm{e}^{8x} c_2 + c_1\right) \mathrm{e}^{-5x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]-15*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x}\left(c_2 e^{8x} + c_1\right)$$

8.2 problem 17

8.2.1 Maple step by step solution 1293

Internal problem ID [5359]
Internal file name [`OUTPUT/4850_Sunday_February_04_2024_12_46_30_AM_68380213/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 17.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - 2y' = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 1$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{-2x} + c_3 \mathrm{e}^{x}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{-2x}$$
$$y_3 = \mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{-2x} + c_3 e^{x}$$

Verified OK.

8.2.1 Maple step by step solution

Let's solve

$y''' + y'' - 2y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ∘ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ∘ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ∘ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ∘ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + 2y_2(x)]$

- Define vector

  $\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix} + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(4c_3 \mathrm{e}^{3x} + 4c_2 \mathrm{e}^{2x} + c_1) \mathrm{e}^{-2x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_3 \mathrm{e}^{3x} + \mathrm{e}^{2x} c_1 + c_2\right) \mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[y'''[x]+y''[x]-2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2} c_1 e^{-2x} + c_2 e^x + c_3$$

8.3 problem 18

8.3.1 Solving as second order linear constant coeff ode 1297
8.3.2 Solving as linear second order ode solved by an integrating factor ode 1299
8.3.3 Solving using Kovacic algorithm 1300
8.3.4 Maple step by step solution 1304

Internal problem ID [5360]
Internal file name [`OUTPUT/4851_Sunday_February_04_2024_12_46_30_AM_10925479/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 9y = 0}$$

8.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)}\\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-3x} + c_2 x\,\mathrm{e}^{-3x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2 \tag{1}$$

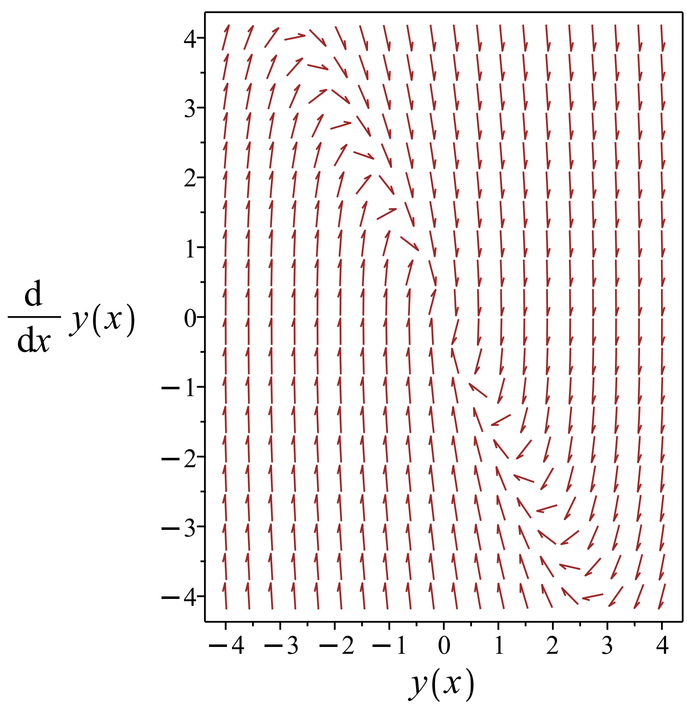


Figure 210: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2$$

Verified OK.

8.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\, y' + \frac{\left(p(x)^{2}+p'(x)\right) y}{2} = f(x)
$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int 6\, dx} \\
&= \mathrm{e}^{3x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{3x} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{3x} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{3x} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{3x}}
$$

Or

$$
y = c_1 x\, \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\, \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x} \tag{1}
$$

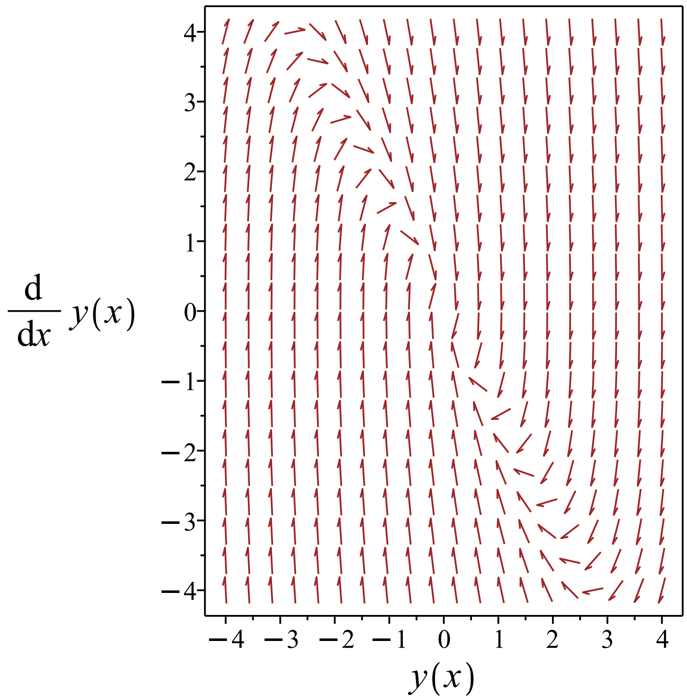


Figure 211: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x \, \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-3x}$$

Verified OK.

8.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 146: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx}\\ &= z_1 e^{-3x}\\ &= z_1\left(\mathrm{e}^{-3x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-3x}\right) + c_2\left(e^{-3x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-3x} + x\,e^{-3x} c_2 \tag{1}
$$

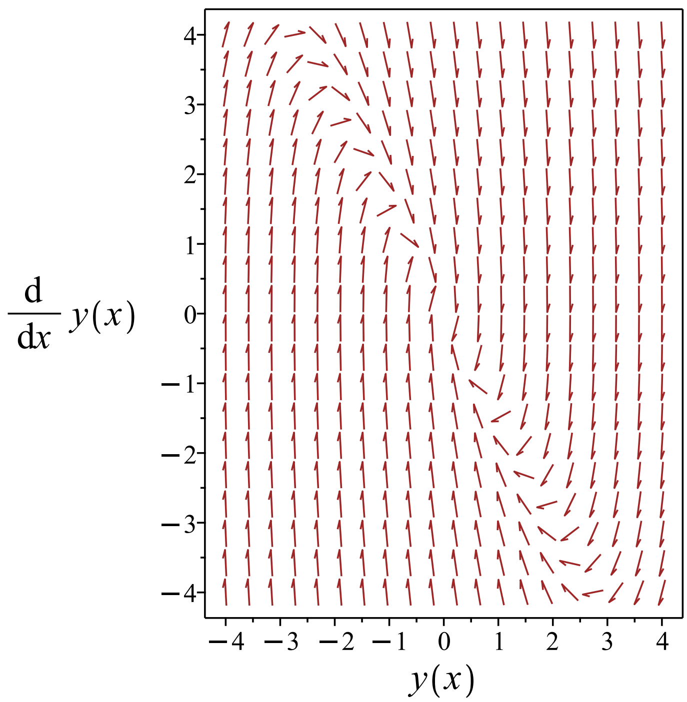


Figure 212: Slope field plot

Verification of solutions

$$
y = c_1 e^{-3x} + x\,e^{-3x} c_2
$$

Verified OK.

8.3.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-3x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[y''[x]+6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(c_2x + c_1)$$

8.4 problem 19

Internal problem ID [5361]
Internal file name [`OUTPUT/4852_Sunday_February_04_2024_12_46_30_AM_84496447/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 19.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 6y''' + 12y'' - 8y' = 0}$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 12\lambda^2 - 8\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{2x} + x\, e^{2x} c_3 + x^2 e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{2x}$$
$$y_3 = e^{2x} x$$
$$y_4 = x^2 e^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{2x} + x\,e^{2x} c_3 + x^2 e^{2x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{2x} + x\,e^{2x} c_3 + x^2 e^{2x} c_4$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+12*diff(y(x),x$2)-8*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_4 x^2 + c_3 x + c_2\right) e^{2x} + c_1$$

✓ Solution by Mathematica
Time used: 0.03 (sec). Leaf size: 43

```
DSolve[y''''[x]-6*y'''[x]+12*y''[x]-8*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4} e^{2x}\left(c_3\left(2x^2 - 2x + 1\right) + c_2(2x - 1) + 2c_1\right) + c_4$$

8.5 problem 20

8.5.1 Solving as second order linear constant coeff ode 1308
8.5.2 Solving using Kovacic algorithm 1310
8.5.3 Maple step by step solution 1314

Internal problem ID [5362]
Internal file name [`OUTPUT/4853_Sunday_February_04_2024_12_46_30_AM_31817868/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 13y = 0}$$

8.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 13\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-4^2 - (4)(1)(13)} \\
&= 2 \pm 3i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= 2 + 3i \\
\lambda_2 &= 2 - 3i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 + 3i \\
\lambda_2 &= 2 - 3i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$
y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{2x}(c_1 \cos(3x) + c_2 \sin(3x)) \tag{1}
$$

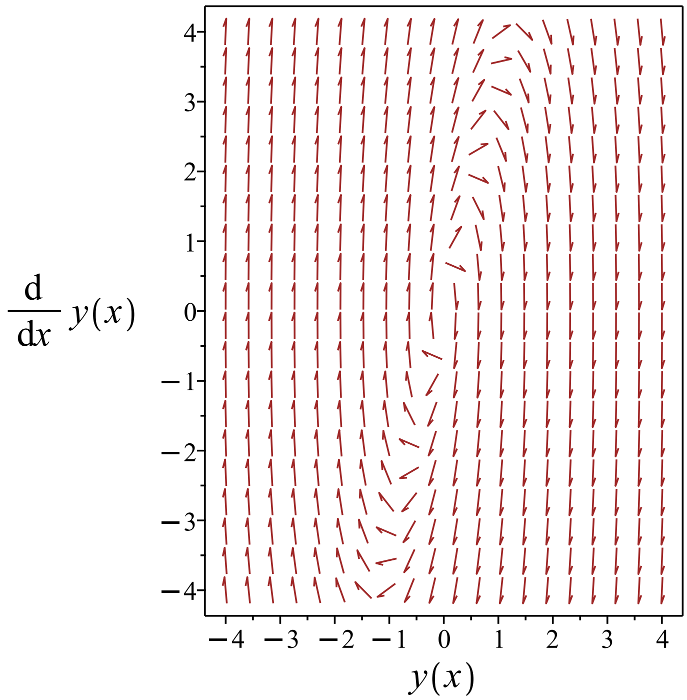


Figure 213: Slope field plot

<u>Verification of solutions</u>

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Verified OK.

8.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 148: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\, dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x} \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x} \cos(3x)\right) + c_2 \left(e^{2x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3} \tag{1}
$$

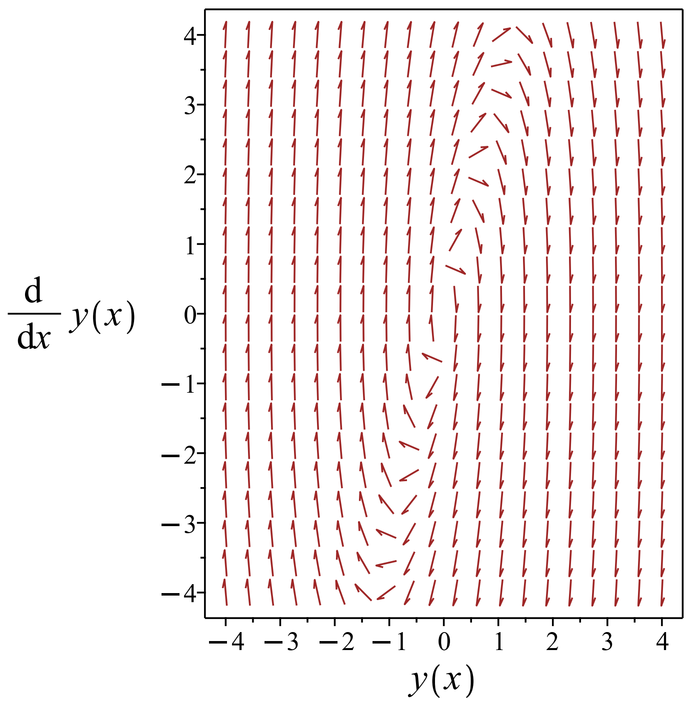


Figure 214: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} \cos(3x) + \frac{c_2 e^{2x} \sin(3x)}{3}
$$

Verified OK.

8.5.3 Maple step by step solution

Let's solve

$y'' - 4y' + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{2x} \cos(3x) + c_2 \mathrm{e}^{2x} \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 26

```
DSolve[y''[x]-4*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 \cos(3x) + c_1 \sin(3x))$$

8.6 problem 21

8.6.1 Solving as second order linear constant coeff ode 1316
8.6.2 Solving as second order ode can be made integrable ode 1318
8.6.3 Solving using Kovacic algorithm 1320
8.6.4 Maple step by step solution 1324

Internal problem ID [5363]
Internal file name [`OUTPUT/4854_Sunday_February_04_2024_12_46_31_AM_33768958/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 25y = 0}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(25)} \\ &= \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5i \\ \lambda_2 &= -5i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5i \\ \lambda_2 &= -5i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(5x) + c_2 \sin(5x))$$

Or

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(5x) + c_2 \sin(5x) \tag{1}$$

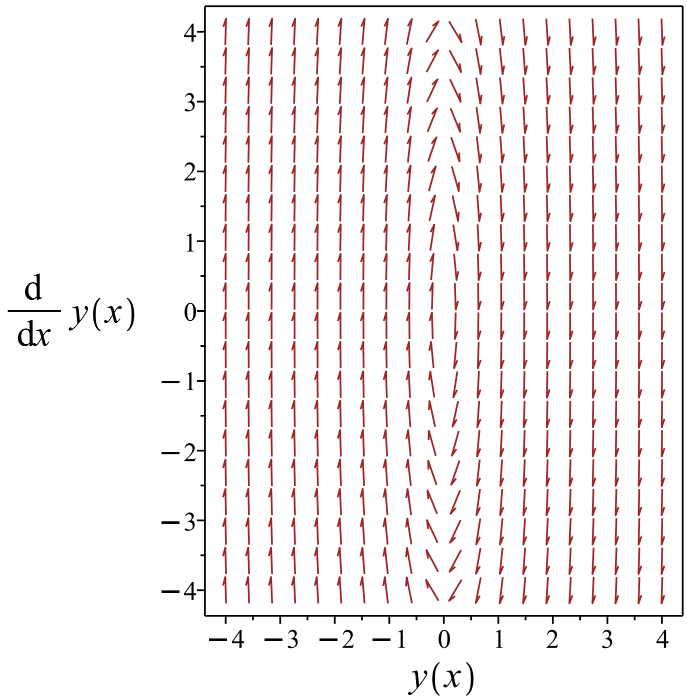


Figure 215: Slope field plot

Verification of solutions

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Verified OK.

8.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 25yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 25yy')\,dx = 0$$
$$\frac{y'^2}{2} + \frac{25y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-25y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-25y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-25y^2 + 2c_1}}dy = \int dx$$

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-25y^2 + 2c_1}}dy = \int dx$$

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_3 + x$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_2 \tag{1}$$

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_3 + x \tag{2}$$

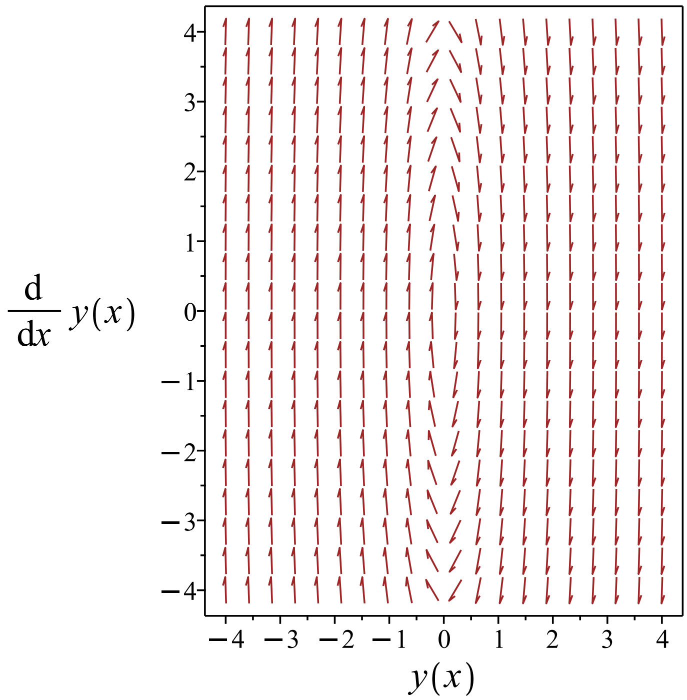


Figure 216: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_2$$

Verified OK.

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_3 + x$$

Verified OK.

8.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 150: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(5x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(5x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(5x) \int \frac{1}{\cos(5x)^2}\, dx \\
&= \cos(5x) \left(\frac{\tan(5x)}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(5x)) + c_2\left(\cos(5x)\left(\frac{\tan(5x)}{5}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} \tag{1}$$

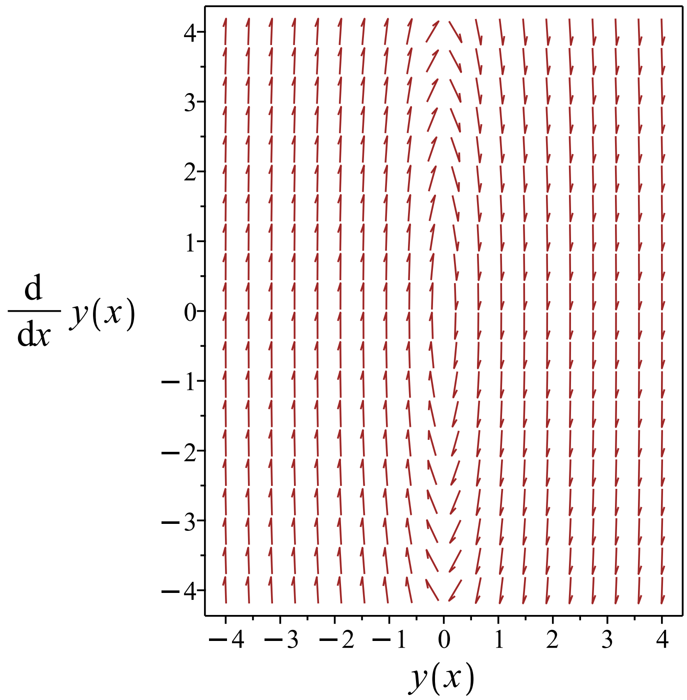


Figure 217: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5}$$

Verified OK.

8.6.4 Maple step by step solution

Let's solve

$y'' + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

$r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(5x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(5x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(5x) + c_2 \sin(5x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(5x) + c_2 \cos(5x)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[y''[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow c_1 \cos(5x) + c_2 \sin(5x)$$

8.7 problem 22

8.7.1 Maple step by step solution 1327

Internal problem ID [5364]
Internal file name [`OUTPUT/4855_Sunday_February_04_2024_12_46_31_AM_84621573/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 22.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - y'' + 9y' - 9y = 0}$$

The characteristic equation is

$$\lambda^3 - \lambda^2 + 9\lambda - 9 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 3i$$
$$\lambda_3 = -3i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^x$$
$$y_2 = e^{-3ix}$$
$$y_3 = e^{3ix}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^x + e^{-3ix} c_2 + e^{3ix} c_3$$

Verified OK.

8.7.1 Maple step by step solution

Let's solve

$y''' - y'' + 9y' - 9y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = y_3(x) - 9y_2(x) + 9y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = y_3(x) - 9y_2(x) + 9y_1(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -9 & 1 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -9 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ -\frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-3\,\mathrm{I}x}\cdot\begin{bmatrix}-\frac{1}{9}\\ \frac{\mathrm{I}}{3}\\ 1\end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\right)\cdot\begin{bmatrix}-\frac{1}{9}\\ \frac{\mathrm{I}}{3}\\ 1\end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix}-\frac{\cos(3x)}{9}+\frac{\mathrm{I}\sin(3x)}{9}\\ \frac{\mathrm{I}}{3}\left(\cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\right)\\ \cos\left(3x\right)-\mathrm{I}\sin\left(3x\right)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=\begin{bmatrix}-\frac{\cos(3x)}{9}\\ \frac{\sin(3x)}{3}\\ \cos\left(3x\right)\end{bmatrix},\vec{y}_{3}(x)=\begin{bmatrix}\frac{\sin(3x)}{9}\\ \frac{\cos(3x)}{3}\\ -\sin\left(3x\right)\end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y}=c_{1}e^{x}\cdot\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}-\frac{c_{2}\cos(3x)}{9}+\frac{c_{3}\sin(3x)}{9}\\ \frac{c_{2}\sin(3x)}{3}+\frac{c_{3}\cos(3x)}{3}\\ c_{2}\cos\left(3x\right)-c_{3}\sin\left(3x\right)\end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y=c_{1}e^{x}+\frac{c_{3}\sin(3x)}{9}-\frac{c_{2}\cos(3x)}{9}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)+9*diff(y(x),x)-9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x} c_1 + \sin(3x)\, c_2 + c_3 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 26

```
DSolve[y'''[x]-y''[x]+9*y'[x]-9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_3 e^{x} + c_1 \cos(3x) + c_2 \sin(3x)$$

8.8 problem 23

8.8.1 Maple step by step solution 1332

Internal problem ID [5365]
Internal file name [`OUTPUT/4856_Sunday_February_04_2024_12_46_32_AM_25985800/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 23.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y'''' + 4y'' = 0}$$

The characteristic equation is

$$\lambda^4 + 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$
$$\lambda_3 = 2i$$
$$\lambda_4 = -2i$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + \mathrm{e}^{2ix} c_3 + \mathrm{e}^{-2ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = x$$
$$y_3 = \mathrm{e}^{2ix}$$
$$y_4 = \mathrm{e}^{-2ix}$$

Summary
The solution(s) found are the following

$$y = c_2 x + c_1 + e^{2ix} c_3 + e^{-2ix} c_4 \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{2ix} c_3 + e^{-2ix} c_4$$

Verified OK.

8.8.1 Maple step by step solution

Let's solve

$y'''' + 4y'' = 0$

- Highest derivative means the order of the ODE is 4

  y''''

□ Convert linear ODE into a system of first order ODEs

  - ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  - ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  - ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  - ◦ Define new variable $y_4(x)$

    $y_4(x) = y'''$

  - ◦ Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_3(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_3(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}^{\prime}(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{y}^{\prime}(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[0, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[-2\,\mathrm{I}, \left[\begin{array}{c} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[2\,\mathrm{I}, \left[\begin{array}{c} \frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ -\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[0, \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-2\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-2\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of sin and cos

$$(\cos(2x) - \mathrm{I}\sin(2x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{8} \\ -\frac{1}{4} \\ \frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{8}(\cos(2x) - \mathrm{I}\sin(2x)) \\ -\frac{\cos(2x)}{4} + \frac{\mathrm{I}\sin(2x)}{4} \\ \frac{\mathrm{I}}{2}(\cos(2x) - \mathrm{I}\sin(2x)) \\ \cos(2x) - \mathrm{I}\sin(2x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\frac{\sin(2x)}{8} \\ -\frac{\cos(2x)}{4} \\ \frac{\sin(2x)}{2} \\ \cos(2x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\frac{\cos(2x)}{8} \\ \frac{\sin(2x)}{4} \\ \frac{\cos(2x)}{2} \\ -\sin(2x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = \begin{bmatrix} -\frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8} + c_1 \\ \frac{c_4\sin(2x)}{4} - \frac{c_3\cos(2x)}{4} \\ \frac{c_4\cos(2x)}{2} + \frac{c_3\sin(2x)}{2} \\ -c_4\sin(2x) + c_3\cos(2x) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = -\frac{c_4\cos(2x)}{8} - \frac{c_3\sin(2x)}{8} + c_1$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$4)+4*diff(y(x),x$2)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2x + c_3 \sin(2x) + c_4 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 32

```
DSolve[y''''[x]+4*y''[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_4x - \frac{1}{4}c_1 \cos(2x) - \frac{1}{4}c_2 \sin(2x) + c_3$$

8.9 problem 24

8.9.1 Maple step by step solution 1338

Internal problem ID [5366]
Internal file name [`OUTPUT/4857_Sunday_February_04_2024_12_46_32_AM_35836327/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 24.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 6y''' + 13y'' - 12y' + 4y = 0}$$

The characteristic equation is

$$\lambda^4 - 6\lambda^3 + 13\lambda^2 - 12\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 2$$
$$\lambda_4 = 2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^x + c_2 x\, e^x + e^{2x} c_3 + x\, e^{2x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= x\,\mathrm{e}^{x} \\
y_3 &= \mathrm{e}^{2x} \\
y_4 &= \mathrm{e}^{2x} x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{2x} c_3 + x\,\mathrm{e}^{2x} c_4 \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 x\,\mathrm{e}^{x} + \mathrm{e}^{2x} c_3 + x\,\mathrm{e}^{2x} c_4
$$

Verified OK.

8.9.1 Maple step by step solution

Let's solve

$y'''' - 6y''' + 13y'' - 12y' + 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = 6y_4(x) - 13y_3(x) + 12y_2(x) - 4y_1(x)$

  Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 6y_4(x) - 13y_3(x) + 12y_2(x) - 4y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right], \left[2, \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]\right], \left[2, \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 1

$$\vec{y}_1(x) = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 1$ is the eigenvalue, an

$$\vec{y}_2(x) = \mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_2(x)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \left(\mathrm{e}^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda x}\left(x\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda x}\vec{v} = \mathrm{e}^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{y}_2(x)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 1

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 1

$$\vec{y}_2(x) = e^{x} \cdot \left(x \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[2, \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue 2

$$\vec{y}_3(x) = e^{2x} \cdot \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, an

$$\vec{y}_4(x) = e^{\lambda x}\left(x\vec{v} + \vec{p}\right)$$

- Note that the x multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obt
- Substitute $\vec{y}_4(x)$ into the homogeneous system

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = \left(e^{\lambda x} A\right) \cdot \left(x\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda x}\left(x\vec{v} + \vec{p}\right) + e^{\lambda x}\vec{v} = e^{\lambda x}\left(\lambda x\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$\lambda \vec{p} + \vec{v} = A \cdot \vec{p}$

- Make use of the identity matrix I

  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$

- Condition $\vec{p}$ must meet for $\vec{y}_4(x)$ to be a solution to the homogeneous system

  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2

$$\left(\left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 12 & -13 & 6 \end{array}\right] - 2 \cdot \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 2

$$\vec{y}_4(x) = \mathrm{e}^{2x} \cdot \left(x \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- General solution to the system of ODEs

  $\vec{y} = c_1 \vec{y}_1(x) + c_2 \vec{y}_2(x) + c_3 \vec{y}_3(x) + c_4 \vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + c_2 \mathrm{e}^{x} \cdot \left(x \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} -1 \\ 0 \\ 0 \\ 0 \end{array}\right]\right) + \mathrm{e}^{2x} c_3 \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \mathrm{e}^{2x} c_4 \cdot \left(x \cdot \left[\begin{array}{c} \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{array}\right] + \left[\begin{array}{c} -\frac{1}{16} \\ 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- First component of the vector is the solution to the ODE

$$y = \frac{((2x-1) c_4 + 2c_3) e^{2x}}{16} + ((x-1) c_2 + c_1) e^x$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$4)-6*diff(y(x),x$3)+13*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=0,y(x), sing
```

$$y(x) = (c_2 x + c_1) e^{2x} + e^x (c_4 x + c_3)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 29

```
DSolve[y''''[x]-6*y'''[x]+13*y''[x]-12*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^x (c_3 e^x + x(c_4 e^x + c_2) + c_1)$$

8.10 problem 25

Internal problem ID [5367]
Internal file name [`OUTPUT/4858_Sunday_February_04_2024_12_46_32_AM_13284602/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 13. Homogeneous Linear equations with constant coefficients. Supplemetary problems. Page 86
Problem number: 25.
ODE order: 6.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

`[[_high_order, _missing_x]]`

$$\boxed{y^{(6)}+9y''''+24y''+16y=0}$$

The characteristic equation is

$$\lambda^6+9\lambda^4+24\lambda^2+16=0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i \\
\lambda_3 &= 2i \\
\lambda_4 &= -2i \\
\lambda_5 &= 2i \\
\lambda_6 &= -2i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x)=c_1\mathrm{e}^{-ix}+\mathrm{e}^{ix}c_2+\mathrm{e}^{2ix}c_3+x\,\mathrm{e}^{2ix}c_4+\mathrm{e}^{-2ix}c_5+x\,\mathrm{e}^{-2ix}c_6$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= e^{-ix} \\
y_2 &= e^{ix} \\
y_3 &= e^{2ix} \\
y_4 &= x\, e^{2ix} \\
y_5 &= e^{-2ix} \\
y_6 &= x\, e^{-2ix}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-ix} + e^{ix} c_2 + e^{2ix} c_3 + x\, e^{2ix} c_4 + e^{-2ix} c_5 + x\, e^{-2ix} c_6 \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-ix} + e^{ix} c_2 + e^{2ix} c_3 + x\, e^{2ix} c_4 + e^{-2ix} c_5 + x\, e^{-2ix} c_6
$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$6)+9*diff(y(x),x$4)+24*diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$
y(x) = (c_6 x + c_4) \cos(2x) + (c_5 x + c_3) \sin(2x) + c_1 \sin(x) + \cos(x) c_2
$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 40

```
DSolve[y''''''[x]+9*y''''[x]+24*y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to (c_2 x + c_1) \cos(2x) + c_6 \sin(x) + \cos(x)(2(c_4 x + c_3) \sin(x) + c_5)
$$

9 Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92

9.1 problem 11 1347
9.2 problem 12 1358
9.3 problem 13 1378
9.4 problem 14 1382
9.5 problem 15 1386
9.6 problem 16 1394
9.7 problem 17 1406
9.8 problem 18 1417
9.9 problem 19 1429
9.10 problem 20 1440
9.11 problem 21 1454
9.12 problem 22 1467

9.1 problem 11

9.1.1 Solving as second order linear constant coeff ode 1347
9.1.2 Solving using Kovacic algorithm 1350
9.1.3 Maple step by step solution 1355

Internal problem ID [5368]
Internal file name [`OUTPUT/4859_Sunday_February_04_2024_12_46_32_AM_63793941/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 3y = 1}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)}\\ &= 2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$
$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x}c_1 + c_2 e^{x}\right) + \left(\frac{1}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2 e^{x} + \frac{1}{3} \tag{1}$$

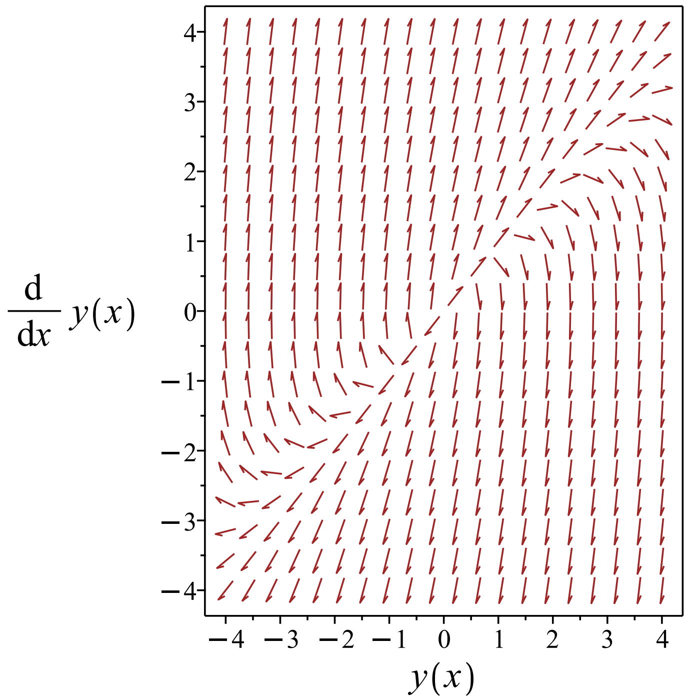


Figure 218: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2 e^x + \frac{1}{3}$$

Verified OK.

9.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + \frac{c_2 \mathrm{e}^{3x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3x}}{2}, \mathrm{e}^x\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2}\right) + \left(\frac{1}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2} + \frac{1}{3} \tag{1}$$

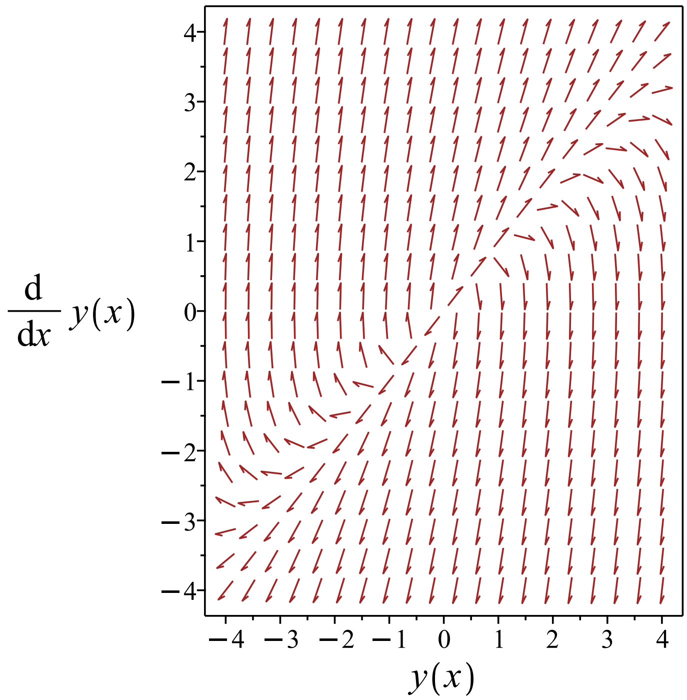


Figure 219: Slope field plot

Verification of solutions

$$
y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{1}{3}
$$

Verified OK.

9.1.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{x} & \mathrm{e}^{3x} \\ \mathrm{e}^{x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{-x} dx\right)}{2} + \frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{1}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{3x} + \frac{1}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=1,y(x), singsol=all)
```

$$y(x) = e^x c_2 + e^{3x} c_1 + \frac{1}{3}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+3*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2 e^{3x} + \frac{1}{3}$$

9.2 problem 12

9.2.1 Solving as second order linear constant coeff ode 1358
9.2.2 Solving as second order integrable as is ode 1362
9.2.3 Solving as second order ode missing y ode 1364
9.2.4 Solving as type second_order_integrable_as_is (not using ABC version) 1365
9.2.5 Solving using Kovacic algorithm 1368
9.2.6 Solving as exact linear second order ode ode 1372
9.2.7 Maple step by step solution 1375

Internal problem ID [5369]
Internal file name [`OUTPUT/4860_Sunday_February_04_2024_12_46_33_AM_16089114/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' = 5}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 0, f(x) = 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-4y'=0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=0$$

Where in the above $A=1, B=-4, C=0$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}-4\lambda\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2-4\lambda=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=-4, C=0$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{4}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{-4^2-(4)\,(1)\,(0)}\\&=2\pm 2\end{aligned}$$

Hence

$$\lambda_1=2+2$$
$$\lambda_2=2-2$$

Which simplifies to

$$\lambda_1=4$$
$$\lambda_2=0$$

Since roots are real and distinct, then the solution is

$$y=c_1e^{\lambda_1 x}+c_2e^{\lambda_2 x}$$
$$y=c_1e^{(4)x}+c_2e^{(0)x}$$

Or

$$y = c_1 e^{4x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{4x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{4x} + c_2\right) + \left(-\frac{5x}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{4x} + c_2 - \frac{5x}{4} \tag{1}
$$

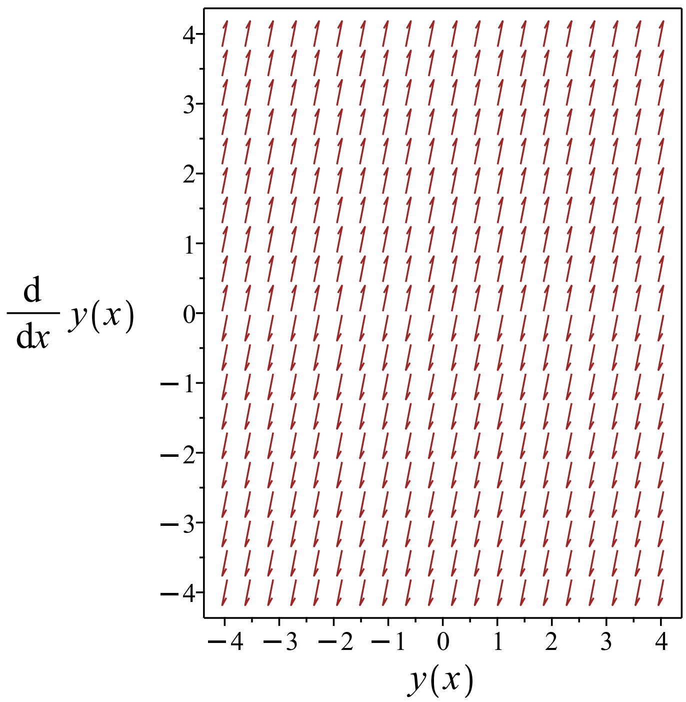


Figure 220: Slope field plot

Verification of solutions

$$
y = c_1 e^{4x} + c_2 - \frac{5x}{4}
$$

Verified OK.

9.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' - 4y')\, dx &= \int 5dx \\
-4y + y' &= 5x + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= 5x + c_1
\end{aligned}
$$

Hence the ode is

$$
-4y + y' = 5x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-4)dx} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(5x + c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-4x} y\right) &= \left(\mathrm{e}^{-4x}\right)(5x + c_1) \\
\mathrm{d}\left(\mathrm{e}^{-4x} y\right) &= \left(\mathrm{e}^{-4x}(5x + c_1)\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-4x} y &= \int \mathrm{e}^{-4x}(5x + c_1)\, \mathrm{d}x \\
\mathrm{e}^{-4x} y &= -\frac{(20x + 4c_1 + 5)\, \mathrm{e}^{-4x}}{16} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-4x}$ results in

$$
y = -\frac{\mathrm{e}^{4x}(20x + 4c_1 + 5)\, \mathrm{e}^{-4x}}{16} + c_2 \mathrm{e}^{4x}
$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x} \tag{1}$$

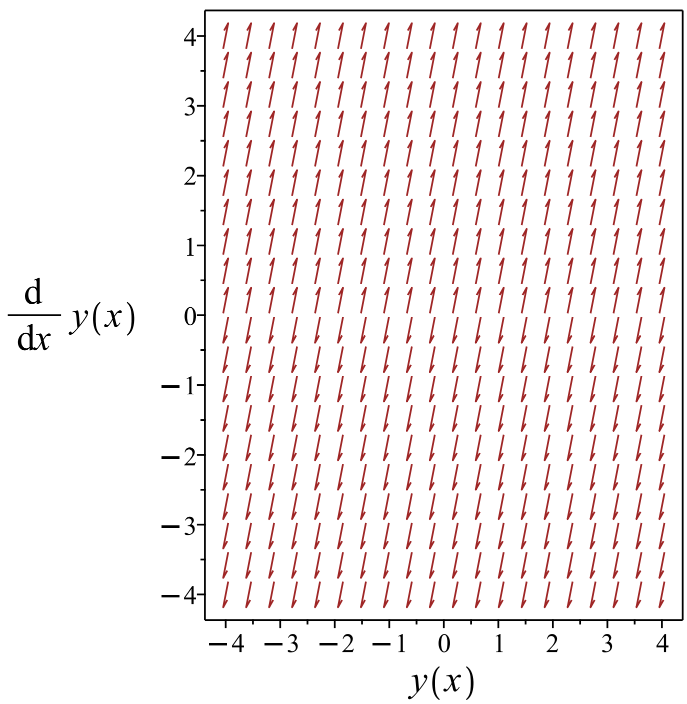


Figure 221: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4p(x) - 5 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{4p+5} dp = \int dx$$
$$\frac{\ln(4p+5)}{4} = x + c_1$$

Raising both side to exponential gives

$$(4p+5)^{\frac{1}{4}} = e^{x+c_1}$$

Which simplifies to

$$(4p+5)^{\frac{1}{4}} = c_2 e^{x}$$

Solving for $p(x)$ gives

$$p(x) = \frac{c_2^4 e^{4x}}{4} - \frac{5}{4}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_2^4 e^{4x}}{4} - \frac{5}{4}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_2^4 e^{4x}}{4} - \frac{5}{4} \, \mathrm{d}x \\ &= -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3 \tag{1}$$

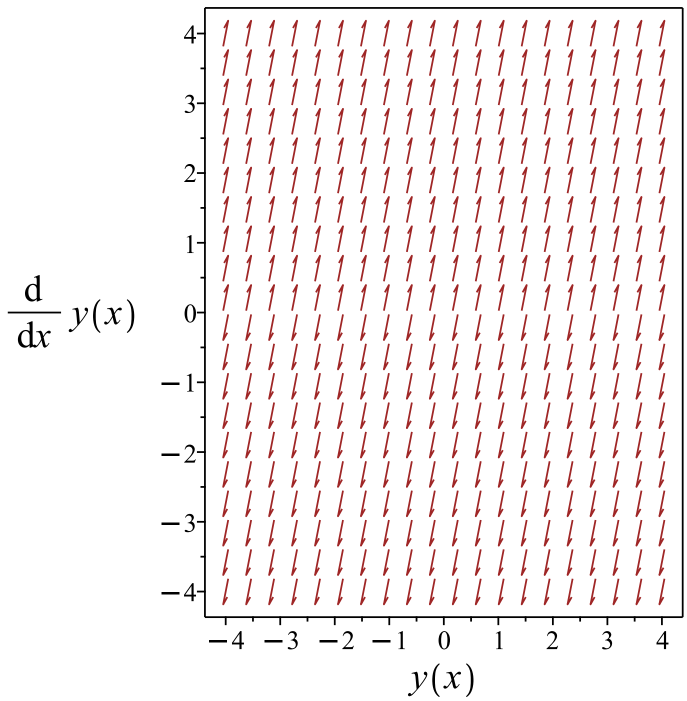


Figure 222: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} + \frac{c_2^4 e^{4x}}{16} + c_3$$

Verified OK.

9.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = 5$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y')\, dx = \int 5 dx$$
$$-4y + y' = 5x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4$$
$$q(x) = 5x + c_1$$

Hence the ode is

$$-4y + y' = 5x + c_1$$

The integrating factor μ is

$$\mu = e^{\int (-4)dx}$$
$$= e^{-4x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(5x + c_1)$$
$$\frac{d}{dx}(e^{-4x}y) = (e^{-4x})(5x + c_1)$$
$$d(e^{-4x}y) = (e^{-4x}(5x + c_1))\, dx$$

Integrating gives

$$e^{-4x}y = \int e^{-4x}(5x + c_1)\, dx$$
$$e^{-4x}y = -\frac{(20x + 4c_1 + 5)\, e^{-4x}}{16} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 4c_1 + 5)\, e^{-4x}}{16} + c_2 e^{4x}$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x} \tag{1}$$

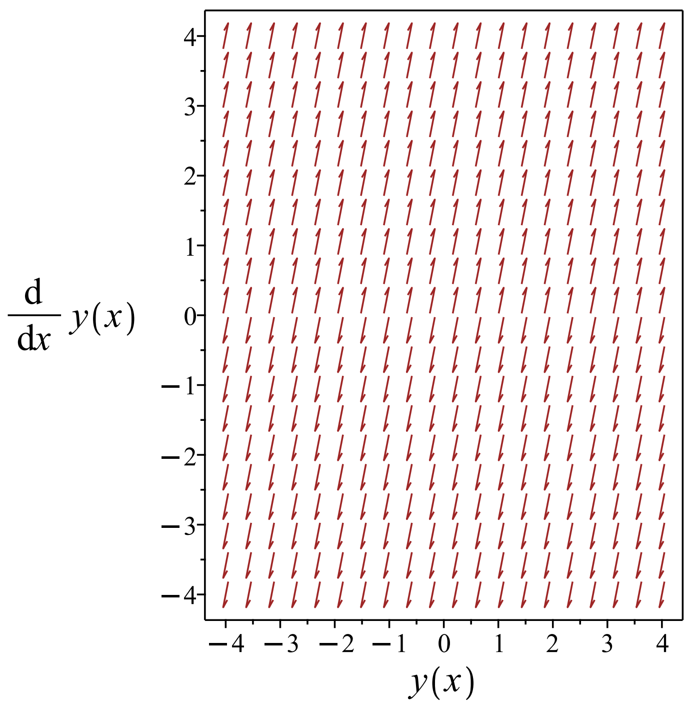


Figure 223: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' - 4y' &= 0 &(1)\\
Ay'' + By' + Cy &= 0 &(2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= -4 &(3)\\
C &= 0
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} &(5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (1) + c_2 \left(1 \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 \mathrm{e}^{4x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \frac{e^{4x}}{4}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 + \frac{c_2 e^{4x}}{4}\right) + \left(-\frac{5x}{4}\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{4} \tag{1}$$

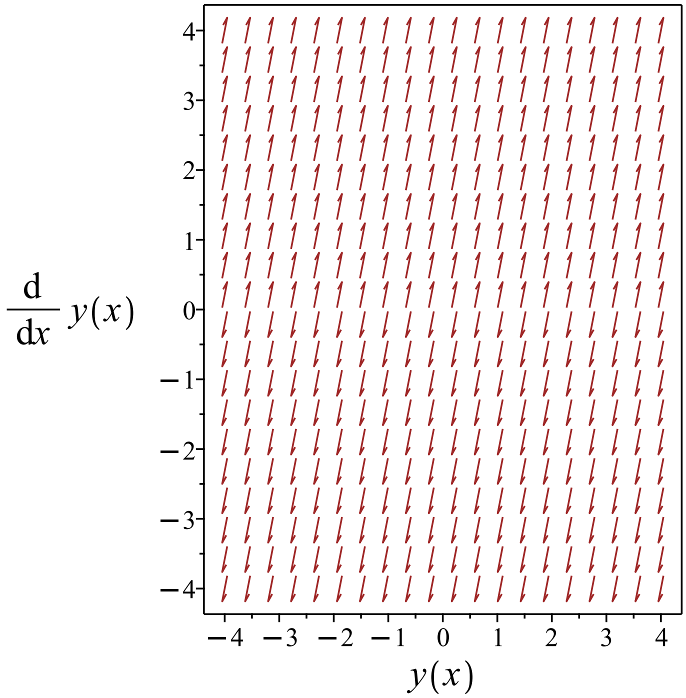


Figure 224: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{4}$$

Verified OK.

9.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= -4 \\
r(x) &= 0 \\
s(x) &= 5
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx
$$

Substituting the above values for p,q,r,s gives

$$
-4y+y'=\int 5\,dx
$$

We now have a first order ode to solve which is

$$
-4y+y'=5x+c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= 5x+c_1
\end{aligned}
$$

Hence the ode is

$$-4y + y' = 5x + c_1$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-4)dx} \\ &= e^{-4x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(5x + c_1) \\ \frac{d}{dx}(e^{-4x} y) &= (e^{-4x})(5x + c_1) \\ d(e^{-4x} y) &= (e^{-4x}(5x + c_1))\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-4x} y &= \int e^{-4x}(5x + c_1)\ dx \\ e^{-4x} y &= -\frac{(20x + 4c_1 + 5)\, e^{-4x}}{16} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 4c_1 + 5)\, e^{-4x}}{16} + c_2 e^{4x}$$

which simplifies to

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Summary
The solution(s) found are the following

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x} \tag{1}$$

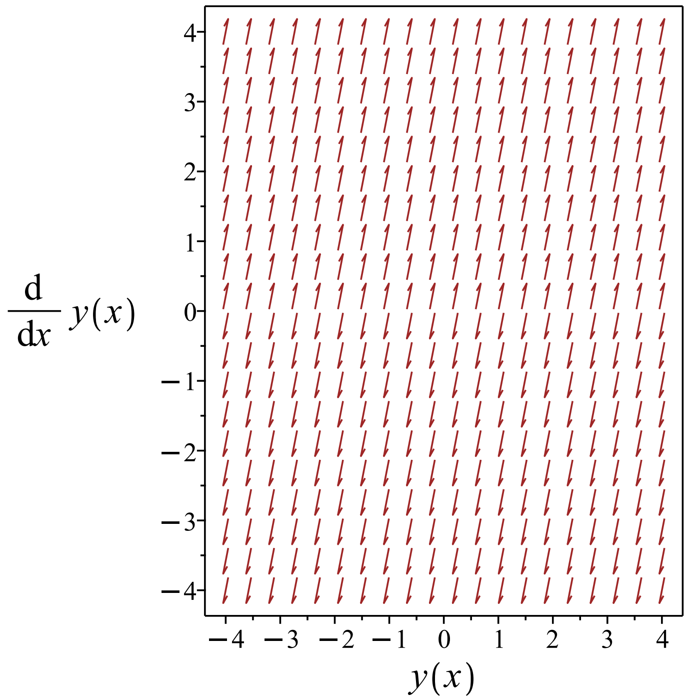


Figure 225: Slope field plot

Verification of solutions

$$y = -\frac{5x}{4} - \frac{c_1}{4} - \frac{5}{16} + c_2 e^{4x}$$

Verified OK.

9.2.7 Maple step by step solution

Let's solve

$y'' - 4y' = 5$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r - 4) = 0$

- Roots of the characteristic polynomial

$r = (0, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{4x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{4x} \\ 0 & 4\,e^{4x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5(\int 1dx)}{4} + \frac{5\,e^{4x}(\int e^{-4x}dx)}{4}$

- ○ Compute integrals

  $y_p(x) = -\frac{5x}{4} - \frac{5}{16}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{4x} - \frac{5x}{4} - \frac{5}{16}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*_b(_a)+5, _b(_a)`   *** Sublevel 2 **
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)=5,y(x), singsol=all)
```

$$y(x) = \frac{e^{4x}c_1}{4} - \frac{5x}{4} + c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 24

```
DSolve[y''[x]-4*y'[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5x}{4} + \frac{1}{4}c_1e^{4x} + c_2$$

9.3 problem 13

Internal problem ID [5370]
Internal file name [`OUTPUT/4861_Sunday_February_04_2024_12_46_34_AM_82222025/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 13.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' - 4y'' = 5}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y'' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = 4$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_2 x + c_1 + e^{4x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= \mathrm{e}^{4x}
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' - 4y'' = 5
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x, \mathrm{e}^{4x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\}]
$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x^2
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-8A_1 = 5
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x^2}{8}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_2 x + c_1 + e^{4x} c_3\right) + \left(-\frac{5x^2}{8}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2 x + c_1 + e^{4x} c_3 - \frac{5x^2}{8} \tag{1}$$

Verification of solutions

$$y = c_2 x + c_1 + e^{4x} c_3 - \frac{5x^2}{8}$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x$2)=5,y(x), singsol=all)
```

$$y(x) = \frac{e^{4x} c_1}{16} - \frac{5x^2}{8} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 30

```
DSolve[y'''[x]-4*y''[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5x^2}{8} + c_3 x + \frac{1}{16} c_1 e^{4x} + c_2$$

9.4 problem 14

Internal problem ID [5371]
Internal file name [`OUTPUT/4862_Sunday_February_04_2024_12_46_34_AM_81323681/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 14.
ODE order: 5.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y^{(5)} - 4y''' = 5}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y^{(5)} - 4y''' = 0$$

The characteristic equation is

$$\lambda^5 - 4\lambda^3 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 0\\ \lambda_3 &= 0\\ \lambda_4 &= 2\\ \lambda_5 &= -2\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_3x^2 + c_2x + c_1 + e^{-2x}c_4 + e^{2x}c_5$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned}
y_1 &= 1 \\
y_2 &= x \\
y_3 &= x^2 \\
y_4 &= e^{-2x} \\
y_5 &= e^{2x}
\end{aligned}$$

Now the particular solution to the given ODE is found

$$y^{(5)} - 4y''' = 5$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x, x^2, e^{-2x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}]$$

Since x^2 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^3$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-24A_1 = 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{24}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x^3}{24}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_3x^2 + c_2x + c_1 + \mathrm{e}^{-2x}c_4 + \mathrm{e}^{2x}c_5\right) + \left(-\frac{5x^3}{24}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{-2x}c_4 + \mathrm{e}^{2x}c_5 - \frac{5x^3}{24} \tag{1}$$

Verification of solutions

$$y = c_3x^2 + c_2x + c_1 + \mathrm{e}^{-2x}c_4 + \mathrm{e}^{2x}c_5 - \frac{5x^3}{24}$$

Verified OK.

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 5; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 4*_b(_a)+5, _b(_a)`   *** Sub
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 5; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$5)-4*diff(y(x),x$3)=5,y(x), singsol=all)
```

$$y(x) = -\frac{5x^3}{24} + \frac{c_2 e^{2x}}{8} + \frac{c_3 x^2}{2} - \frac{e^{-2x} c_1}{8} + c_4 x + c_5$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 47

```
DSolve[y'''''[x]-4*y'''[x]==5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5x^3}{24} + c_5 x^2 + c_4 x + \frac{1}{8} c_1 e^{2x} - \frac{1}{8} c_2 e^{-2x} + c_3$$

9.5 problem 15

9.5.1 Maple step by step solution 1388

Internal problem ID [5372]
Internal file name [`OUTPUT/4863_Sunday_February_04_2024_12_46_34_AM_22823233/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 15.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' - 4y' = x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - 4y' = 0$$

The characteristic equation is

$$\lambda^3 - 4\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 e^{-2x} + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = e^{-2x}$$
$$y_3 = e^{2x}$$

Now the particular solution to the given ODE is found

$$y''' - 4y' = x$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}, e^{2x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8xA_2 - 4A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x^2}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 \mathrm{e}^{-2x} + \mathrm{e}^{2x} c_3\right) + \left(-\frac{x^2}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-2x} + \mathrm{e}^{2x} c_3 - \frac{x^2}{8} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \mathrm{e}^{-2x} + \mathrm{e}^{2x} c_3 - \frac{x^2}{8}$$

Verified OK.

9.5.1 Maple step by step solution

Let's solve

$y''' - 4y' = x$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$

- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x + 4y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x + 4y_2(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & 1 & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & 0 & \frac{e^{2x}}{2} \\ e^{-2x} & 0 & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} 1 & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \\ 0 & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} \\ 0 & -e^{-2x} + e^{2x} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

$\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} -\frac{1}{16} + \frac{e^{-2x}}{32} + \frac{e^{2x}}{32} - \frac{x^2}{8} \\ -\frac{x}{4} - \frac{e^{-2x}}{16} + \frac{e^{2x}}{16} \\ \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3 + \left[\begin{array}{c} -\frac{1}{16} + \frac{e^{-2x}}{32} + \frac{e^{2x}}{32} - \frac{x^2}{8} \\ -\frac{x}{4} - \frac{e^{-2x}}{16} + \frac{e^{2x}}{16} \\ \frac{e^{-2x}}{8} - \frac{1}{4} + \frac{e^{2x}}{8} \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = \frac{(8c_1+1)e^{-2x}}{32} + \frac{(8c_3+1)e^{2x}}{32} - \frac{x^2}{8} + c_2 - \frac{1}{16}$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = 4*_b(_a)+_a, _b(_a)`   *** Su
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   -> Try solving first the homogeneous part of the ODE
      checking if the LODE has constant coefficients
      <- constant coefficients successful
   <- solving first the homogeneous part of the ODE successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$3)-4*diff(y(x),x)=x,y(x), singsol=all)
```

$$y(x) = -\frac{x^2}{8} + \frac{c_2 e^{2x}}{2} - \frac{e^{-2x} c_1}{2} + c_3$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[y'''[x]-4*y'[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{8} + \frac{1}{2} c_1 e^{2x} - \frac{1}{2} c_2 e^{-2x} + c_3$$

9.6 problem 16

9.6.1 Solving as second order linear constant coeff ode 1394
9.6.2 Solving as linear second order ode solved by an integrating factor ode 1397
9.6.3 Solving using Kovacic algorithm 1399
9.6.4 Maple step by step solution 1404

Internal problem ID [5373]
Internal file name [`OUTPUT/4864_Sunday_February_04_2024_12_46_35_AM_19377599/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = e^{2x}}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-6)^2 - (4)\,(1)\,(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 x\, e^{3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{3x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 x\, e^{3x}\right) + \left(e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + e^{2x} \tag{1}$$

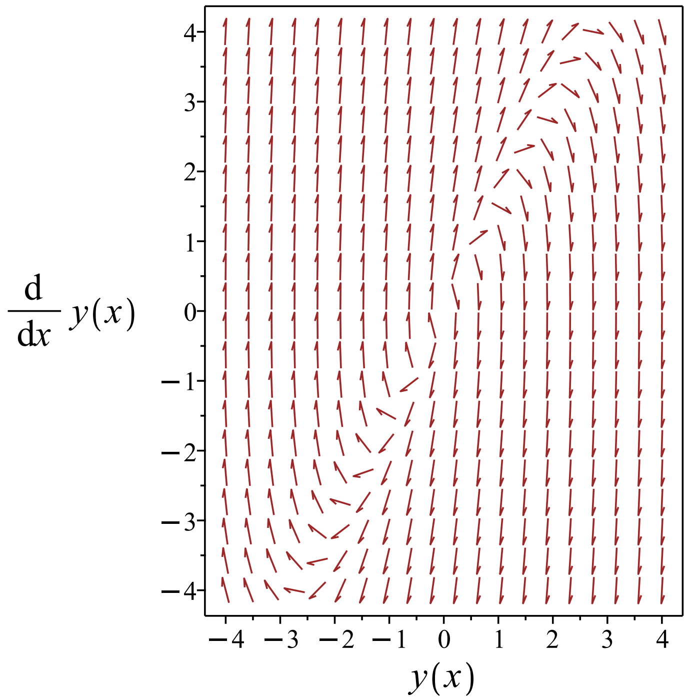


Figure 226: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2 x + c_1) + e^{2x}$$

Verified OK.

9.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -6\,dx} \\ &= e^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{-3x}e^{2x}$$
$$\left(e^{-3x}y\right)'' = e^{-3x}e^{2x}$$

Integrating once gives

$$\left(e^{-3x}y\right)' = -e^{-x} + c_1$$

Integrating again gives

$$\left(e^{-3x}y\right) = c_1x + e^{-x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + e^{-x} + c_2}{e^{-3x}}$$

Or

$$y = c_1x\,e^{3x} + c_2e^{3x} + e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1x\,e^{3x} + c_2e^{3x} + e^{2x} \tag{1}$$

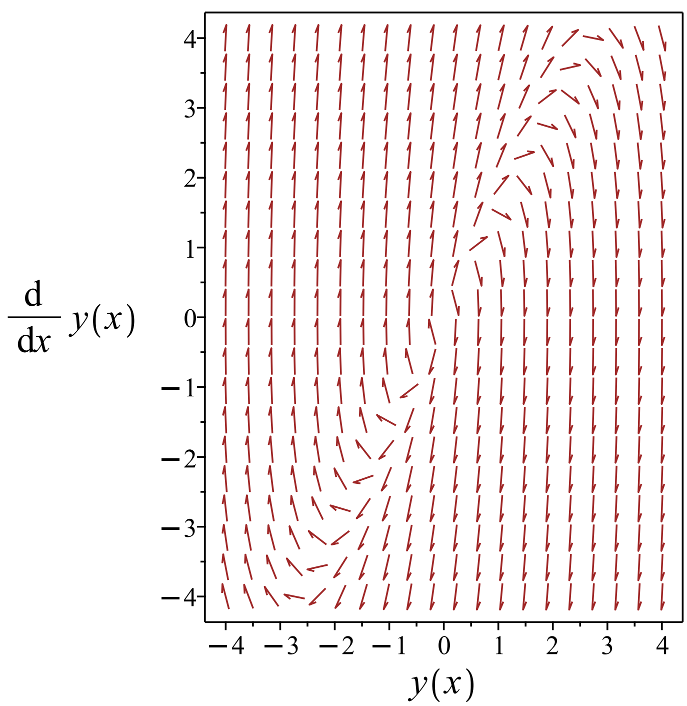


Figure 227: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} + \mathrm{e}^{2x}$$

Verified OK.

9.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 160: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1\left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{3x} c_1 + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{3x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 x\, e^{3x}\right) + \left(e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2x + c_1) + e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2x + c_1) + e^{2x} \tag{1}$$

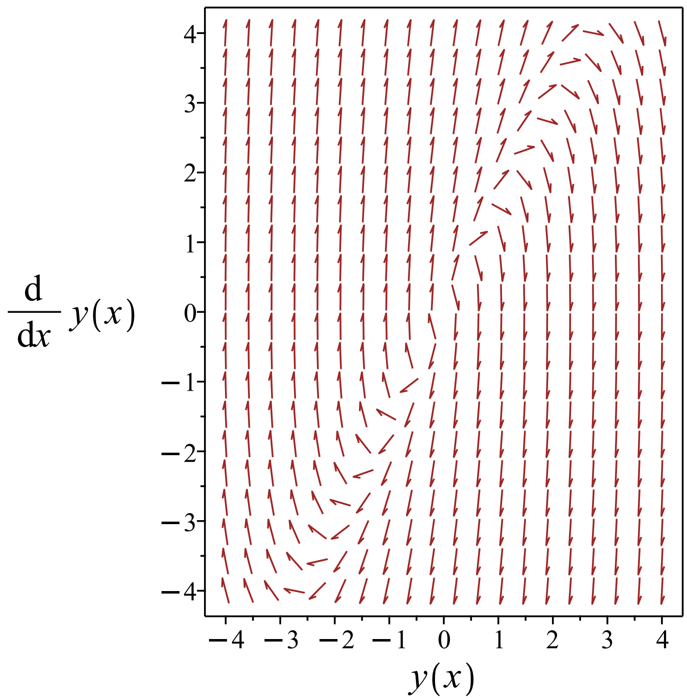


Figure 228: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + e^{2x}$$

Verified OK.

9.6.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{3x} c_1 + c_2 x\,e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{2x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & x\,e^{3x} \\ 3\,e^{3x} & e^{3x} + 3x\,e^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{3x}\left(-\left(\int x\,e^{-x} dx\right) + \left(\int e^{-x} dx\right) x\right)$

- Compute integrals

  $y_p(x) = e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{3x} + e^{3x} c_1 + e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\, e^{3x} + e^{2x}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 24

```
DSolve[y''[x]-6*y'[x]+9*y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(1 + e^x(c_2 x + c_1))$$

9.7 problem 17

9.7.1 Solving as second order linear constant coeff ode 1406
9.7.2 Solving using Kovacic algorithm 1409
9.7.3 Maple step by step solution 1414

Internal problem ID [5374]
Internal file name [`OUTPUT/4865_Sunday_February_04_2024_12_46_35_AM_92023248/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = -2x^2 + 2x + 2}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = -2x^2 + 2x + 2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda\, e^{\lambda x} - 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = -2x^2 + 2x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^x + c_2e^{-2x}\right) + \left(x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1e^x + c_2e^{-2x} + x^2 \tag{1}$$

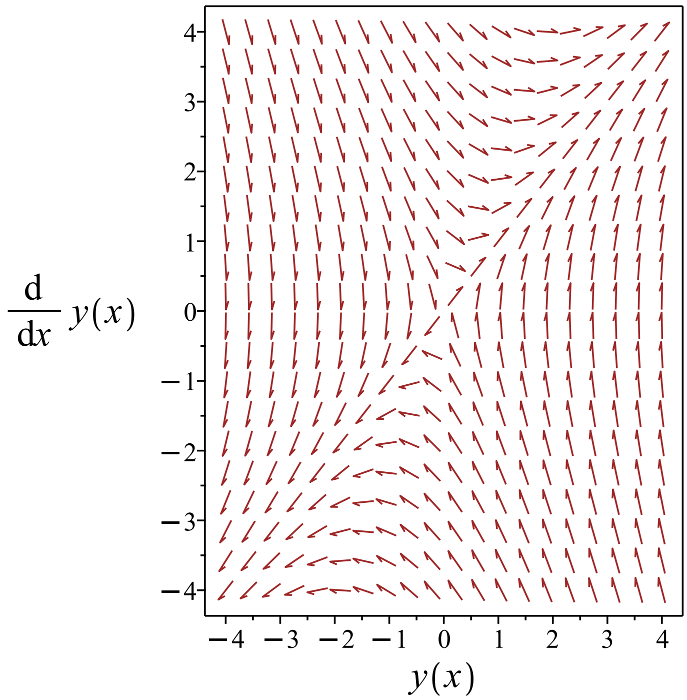


Figure 229: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-2x} + x^2$$

Verified OK.

9.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 162: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{1}{1} \, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{3}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_3x^2 - 2A_2x + 2xA_3 - 2A_1 + A_2 + 2A_3 = -2x^2 + 2x + 2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^x}{3}\right) + \left(x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^x}{3} + x^2 \tag{1}$$

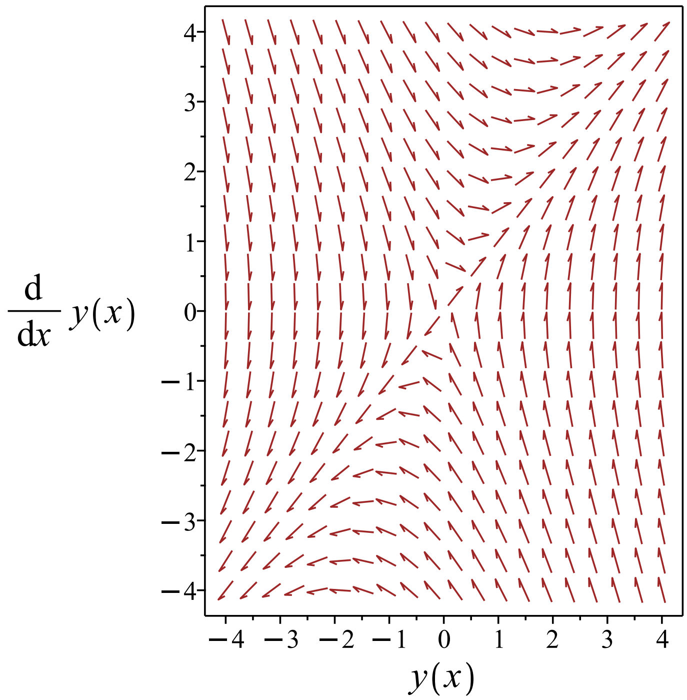


Figure 230: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^x}{3} + x^2$$

Verified OK.

9.7.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = -2x^2 + 2x + 2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = -2x^2 + 2x + 2\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-x}$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{2\left(\mathrm{e}^{3x}\left(\int \mathrm{e}^{-x}(x^2-x-1)dx\right) - \left(\int \mathrm{e}^{2x}(x^2-x-1)dx\right)\right)\mathrm{e}^{-2x}}{3}$
  - Compute integrals

    $y_p(x) = x^2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=2*(1+x-x^2),y(x), singsol=all)
```

$$y(x) = \left(e^{3x}c_1 + e^{2x}x^2 + c_2\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 23

```
DSolve[y''[x]+y'[x]-2*y[x]==2*(1+x-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 e^{-2x} + c_2 e^x$$

9.8 problem 18

9.8.1 Solving as second order linear constant coeff ode 1417
9.8.2 Solving using Kovacic algorithm 1420
9.8.3 Maple step by step solution 1426

Internal problem ID [5375]
Internal file name [`OUTPUT/4866_Sunday_February_04_2024_12_46_36_AM_94845678/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = 4x\,e^{x}}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = 4x\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{x}, x^{2}\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{x} + A_2 x^{2}\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{x} + 2A_2\mathrm{e}^{x} + 4A_2 x\,\mathrm{e}^{x} = 4x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{x} + x^{2}\mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x}\right) + \left(-x\,\mathrm{e}^{x} + x^{2}\mathrm{e}^{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - x e^x + x^2 e^x \tag{1}$$

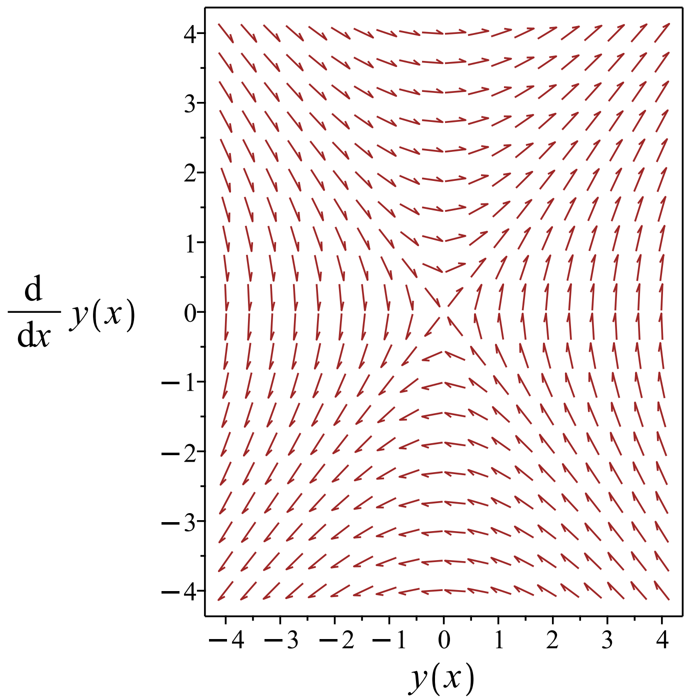


Figure 231: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - x e^x + x^2 e^x$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 164: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\ &= e^{-x} \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\,\mathrm{e}^{2x} x}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 2\,\mathrm{e}^{2x} x dx$$

Hence

$$u_1 = -\frac{(2x-1)\,\mathrm{e}^{2x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{-x} x\,\mathrm{e}^{x}}{1}\,dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(2x-1)\,\mathrm{e}^{2x}\mathrm{e}^{-x}}{2} + x^2\mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{x}\left(\frac{1}{2} + x^2 - x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2}\right) + \left(\mathrm{e}^{x}\left(\frac{1}{2} + x^2 - x\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + e^{x}\left(\frac{1}{2} + x^2 - x\right) \tag{1}$$

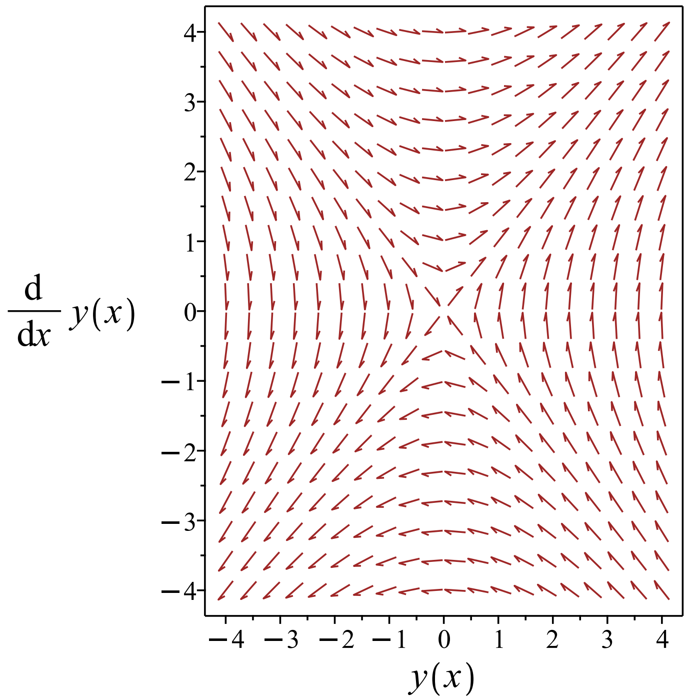


Figure 232: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + e^{x}\left(\frac{1}{2} + x^2 - x\right)$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$y'' - y = 4x\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4x\,\mathrm{e}^{x}\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x} x dx\right) + 2\,\mathrm{e}^{x}\left(\int x dx\right)$
  - Compute integrals

    $y_p(x) = \mathrm{e}^{x}\left(\frac{1}{2} + x^2 - x\right)$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{x}\left(\frac{1}{2} + x^2 - x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=4*x*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 \mathrm{e}^{-x} + \mathrm{e}^{x} \left(x^2 + c_1 - x\right)$$

✓ Solution by Mathematica
Time used: 0.019 (sec). Leaf size: 30

```
DSolve[y''[x]-y[x]==4*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(x^2 - x + \frac{1}{2} + c_1\right) + c_2 e^{-x}$$

9.9 problem 19

9.9.1 Solving as second order linear constant coeff ode 1429
9.9.2 Solving using Kovacic algorithm 1432
9.9.3 Maple step by step solution 1437

Internal problem ID [5376]
Internal file name [`OUTPUT/4867_Sunday_February_04_2024_12_46_36_AM_65151594/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \sin(x)^2}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 e^{-x}) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10} \tag{1}$$

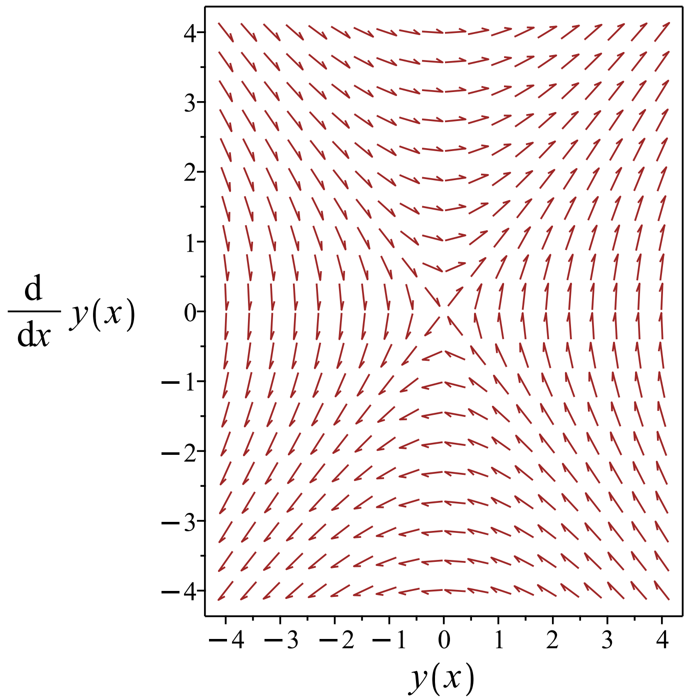


Figure 233: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\,e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 166: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^x}{2}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_2 \cos(2x) - 5A_3 \sin(2x) - A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{10}, A_3 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{2} + \frac{\cos(2x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-\frac{1}{2} + \frac{\cos(2x)}{10}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} - \frac{1}{2} + \frac{\cos(2x)}{10} \tag{1}$$

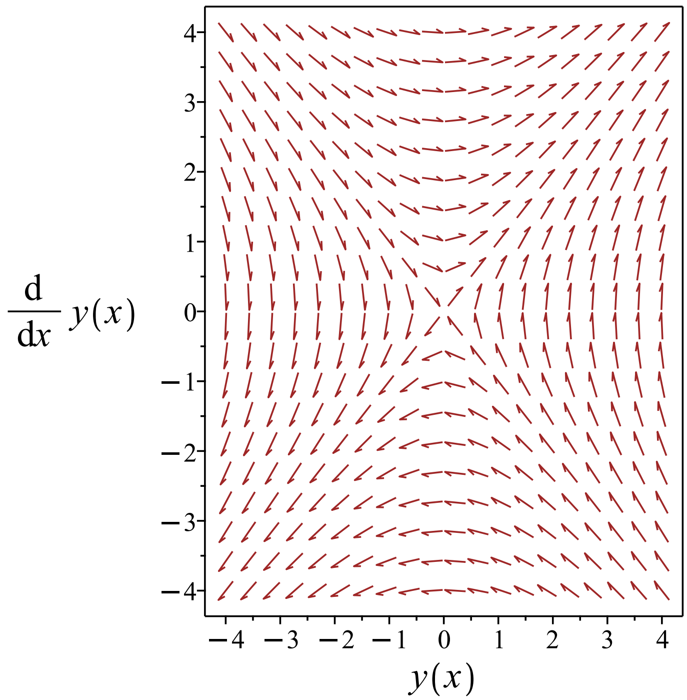


Figure 234: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} - \frac{1}{2} + \frac{\cos(2x)}{10}$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$y'' - y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \sin(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{array}\right]$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} \sin(x)^2 dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \sin(x)^2 \mathrm{e}^{-x} dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = -\frac{1}{2} + \frac{\cos(2x)}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{1}{2} + \frac{\cos(2x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = c_2 \mathrm{e}^{-x} + \mathrm{e}^{x} c_1 + \frac{\cos(x)^2}{5} - \frac{3}{5}$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 30

```
DSolve[y''[x]-y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10}(\cos(2x) - 5) + c_1 e^{x} + c_2 e^{-x}$$

9.10 problem 20

9.10.1 Solving as second order linear constant coeff ode 1440
9.10.2 Solving using Kovacic algorithm 1445
9.10.3 Maple step by step solution 1451

Internal problem ID [5377]
Internal file name [`OUTPUT/4868_Sunday_February_04_2024_12_46_37_AM_60385736/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \frac{1}{(1+e^{-x})^2}}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \frac{1}{(1+e^{-x})^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= \mathrm{e}^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \mathrm{e}^{x} & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})\left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right)(\mathrm{e}^{x})$$

Which simplifies to

$$W = -2\,\mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{-2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x}}{2\,(1+e^{-x})^2}dx$$

Hence

$$u_1 = \frac{1}{2+2\,e^{-x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{x}}{(1+e^{-x})^2}}{-2}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{3x}}{2\,(1+e^{x})^2}dx$$

Hence

$$u_2 = -\frac{e^{x}}{2} + \frac{1}{2+2\,e^{x}} + \ln\left(1+e^{x}\right)$$

Which simplifies to

$$u_1 = \frac{1}{2+2\,e^{-x}}$$

$$u_2 = \frac{(2+2\,e^{x})\ln\left(1+e^{x}\right) - e^{x} - e^{2x} + 1}{2+2\,e^{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{x}}{2+2\,e^{-x}} + \frac{\left((2+2\,e^{x})\ln\left(1+e^{x}\right) - e^{x} - e^{2x} + 1\right)e^{-x}}{2+2\,e^{x}}$$

Which simplifies to

$$
y_p(x) = \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x}\right) + \left(\frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} + \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}} \tag{1}
$$

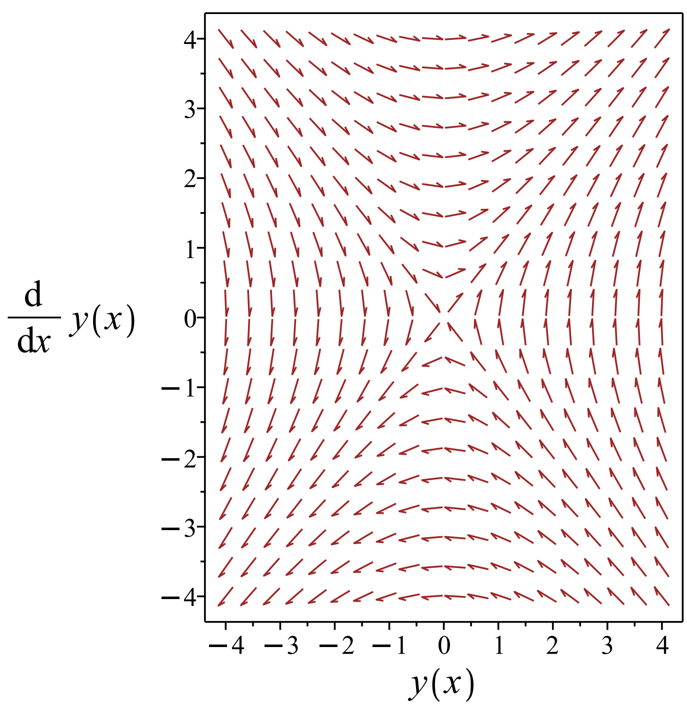


Figure 235: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-x} + \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}
$$

Verified OK.

9.10.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 168: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-x}$$
$$y_2 = \frac{e^x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(\frac{e^x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^x}{2} \\ -e^{-x} & \frac{e^x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{x}}{2(1+\mathrm{e}^{-x})^2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3x}}{2\left(1+\mathrm{e}^{x}\right)^2}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{x}}{2} + \frac{1}{2+2\,\mathrm{e}^{x}} + \ln\left(1+\mathrm{e}^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}}{1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{-x}}{\left(1+\mathrm{e}^{-x}\right)^2}dx$$

Hence

$$u_2 = \frac{1}{1+\mathrm{e}^{-x}}$$

Which simplifies to

$$u_1 = \frac{(2+2\,\mathrm{e}^{x})\ln\left(1+\mathrm{e}^{x}\right) - \mathrm{e}^{x} - \mathrm{e}^{2x} + 1}{2+2\,\mathrm{e}^{x}}$$

$$u_2 = \frac{1}{1+\mathrm{e}^{-x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2+2\,e^{-x}} + \frac{\left((2+2\,e^x)\ln(1+e^x) - e^x - e^{2x} + 1\right)e^{-x}}{2+2\,e^x}$$

Which simplifies to

$$y_p(x) = \frac{(2+2\,e^{-x})\ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(\frac{(2+2\,e^{-x})\ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2+2\,e^{-x})\ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x} \tag{1}$$

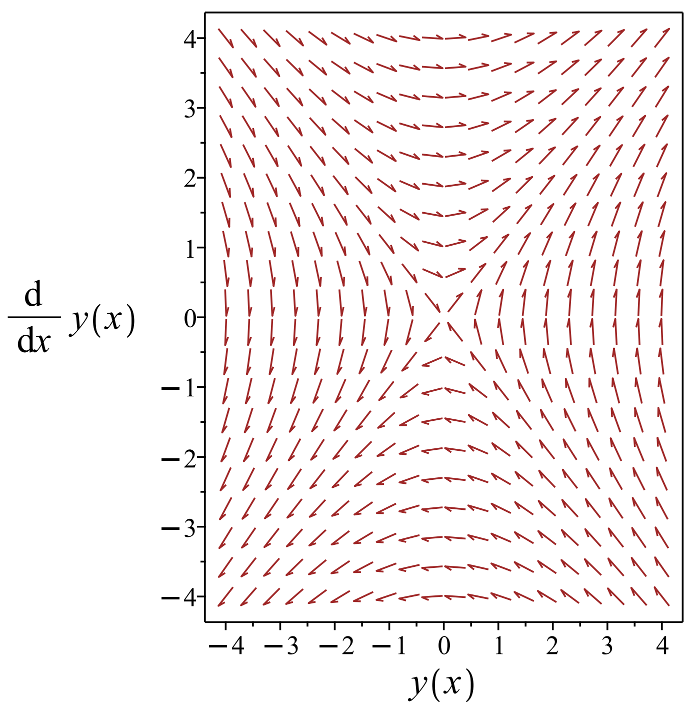


Figure 236: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}$$

Verified OK.

9.10.3 Maple step by step solution

Let's solve

$y'' - y = \frac{1}{(1+\mathrm{e}^{-x})^2}$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{1}{(1+\mathrm{e}^{-x})^2}\right]$
- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^{-x}\left(\int \frac{e^{3x}}{(1+e^{x})^2}dx\right)}{2} + \frac{e^{x}\left(\int \frac{e^{-x}}{\left(1+e^{-x}\right)^2}dx\right)}{2}$$

- Compute integrals

  $$y_p(x) = \frac{(2+2\,e^{-x})\ln(1+e^{x})-e^{x}+e^{-x}+e^{2x}-1}{2+2\,e^{x}}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 e^{-x} + c_2 e^{x} + \frac{(2+2\,e^{-x})\ln(1+e^{x})-e^{x}+e^{-x}+e^{2x}-1}{2+2\,e^{x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-y(x)=(1+exp(-x))^(-2),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{x} c_1 + \frac{e^{x}}{2} - 1 + e^{-x}\ln\left(e^{x}+1\right) + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 42

```
DSolve[y''[x]-y[x]==(1+Exp[-x])^(-2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{2}e^{-x}\left(-2e^{x}+2\log\left(e^{x}+1\right)+2c_1e^{2x}+1+2c_2\right)$$

9.11 problem 21

9.11.1 Solving as second order linear constant coeff ode 1454
9.11.2 Solving using Kovacic algorithm 1458
9.11.3 Maple step by step solution 1464

Internal problem ID [5378]
Internal file name [`OUTPUT/4869_Sunday_February_04_2024_12_46_38_AM_25923686/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \csc(x)}$$

9.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x) dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x)c_1 + c_2\sin(x)) + (-\cos(x)x + \ln(\sin(x))\sin(x))
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x) \tag{1}$$

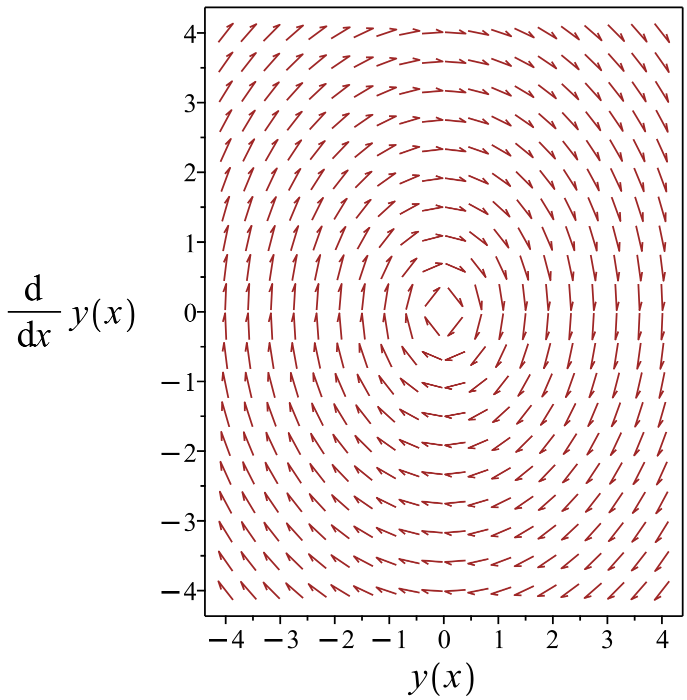


Figure 237: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(x) \\ y_2 &= \sin(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1}\, dx$$

Which simplifies to

$$u_2 = \int \cot(x)\, dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\, x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2 \sin(x)) + (-\cos(x)\, x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\, c_1 + c_2 \sin(x) - \cos(x)\, x + \ln(\sin(x))\sin(x) \tag{1}$$

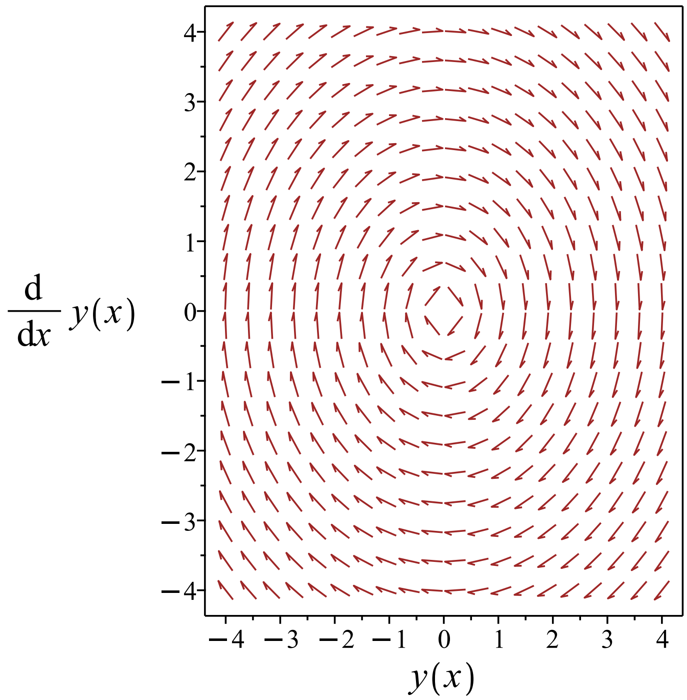


Figure 238: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

9.11.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int 1 dx\right) + \sin(x)\left(\int \cot(x) dx\right)$
- ○ Compute integrals

  $y_p(x) = -\cos(x) x + \ln(\sin(x)) \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=csc(x),y(x), singsol=all)
```

$$y(x) = -\ln\left(\csc\left(x\right)\right)\sin\left(x\right) + \left(-x + c_1\right)\cos\left(x\right) + \sin\left(x\right)c_2$$

✓ Solution by Mathematica

Time used: 0.082 (sec). Leaf size: 86

```
DSolve[y''[x]-y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{2} + \frac{i}{2}\right)e^{ix}\left(\text{Hypergeometric2F1}\left(\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, e^{2ix}\right) + i\,\text{Hypergeometric2F1}\left(\frac{1}{2} + \frac{i}{2}, 1, \frac{3}{2} + \frac{i}{2}, e^{2ix}\right)\right) + c_1e^x + c_2e^{-x}$$

9.12 problem 22

9.12.1 Solving as second order linear constant coeff ode 1467
9.12.2 Solving using Kovacic algorithm 1472
9.12.3 Maple step by step solution 1478

Internal problem ID [5379]
Internal file name [`OUTPUT/4870_Sunday_February_04_2024_12_46_38_AM_28670253/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 14. Linear equations with constant coefficients. Supplemetary problems. Page 92
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 3y' + 2y = \sin\left(\mathrm{e}^{-x}\right)}$$

9.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -3, C = 2, f(x) = \sin\left(\mathrm{e}^{-x}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 3\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\
&= \frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(1)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(e^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{x} \\ 2\,e^{2x} & e^{x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(e^{x}\right) - \left(e^{x}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -e^{2x}e^{x}$$

Which simplifies to

$$W = -e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{x}\sin\left(e^{-x}\right)}{-e^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\sin\left(e^{-x}\right)e^{-2x}dx$$

Hence

$$u_1 = -\frac{e^{-x}\tan\left(\frac{e^{-x}}{2}\right)^2 - e^{-x} + 2\tan\left(\frac{e^{-x}}{2}\right)}{1+\tan\left(\frac{e^{-x}}{2}\right)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x}\sin\left(e^{-x}\right)}{-e^{3x}}\,dx$$

Which simplifies to

$$u_2 = \int -e^{-x}\sin\left(e^{-x}\right)dx$$

Hence

$$u_2 = -\cos\left(e^{-x}\right)$$

Which simplifies to

$$u_1 = \cos\left(e^{-x}\right)e^{-x} - \sin\left(e^{-x}\right)$$
$$u_2 = -\cos\left(e^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(e^{-x}\right)e^{-x} - \sin\left(e^{-x}\right)\right)e^{2x} - \cos\left(e^{-x}\right)e^{x}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}\right) + \left(-\mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} - \mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right) \tag{1}$$

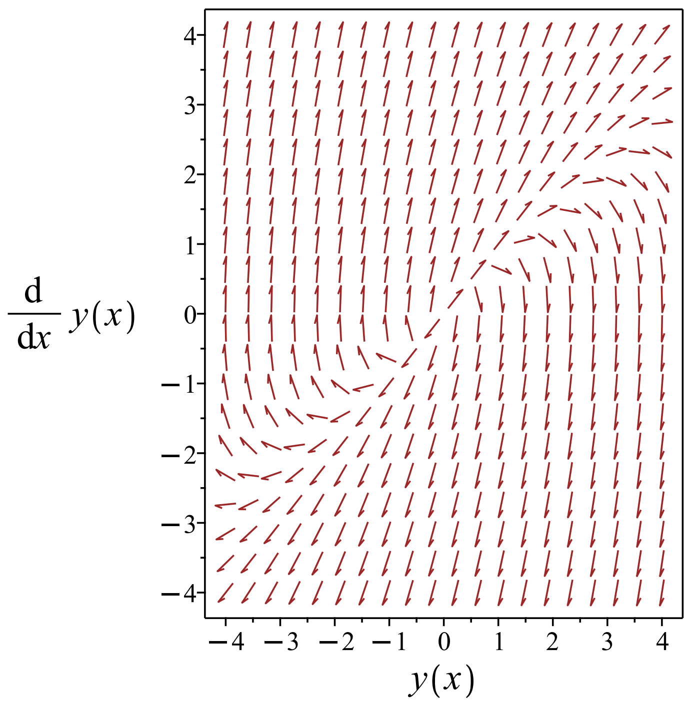


Figure 239: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} - \mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)$$

Verified OK.

9.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 172: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\, dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\, dx \\
&= y_1(\mathrm{e}^x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^x + c_2\mathrm{e}^{2x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{x}$$
$$y_2 = \mathrm{e}^{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{2x} \\ \frac{d}{dx}\left(\mathrm{e}^{x}\right) & \frac{d}{dx}\left(\mathrm{e}^{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{2x} \\ \mathrm{e}^{x} & 2\,\mathrm{e}^{2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{x}\right)\left(2\,\mathrm{e}^{2x}\right) - \left(\mathrm{e}^{2x}\right)\left(\mathrm{e}^{x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{2x}\mathrm{e}^{x}$$

Which simplifies to

$$W = \mathrm{e}^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)}{\mathrm{e}^{3x}} dx$$

Which simplifies to

$$u_1 = -\int \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) dx$$

Hence

$$u_1 = -\cos\left(\mathrm{e}^{-x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{x} \sin\left(\mathrm{e}^{-x}\right)}{\mathrm{e}^{3x}} dx$$

Which simplifies to

$$u_2 = \int \sin\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-2x} dx$$

Hence

$$u_2 = \frac{-\mathrm{e}^{-x} \tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2 + \mathrm{e}^{-x} - 2\tan\left(\frac{\mathrm{e}^{-x}}{2}\right)}{1 + \tan\left(\frac{\mathrm{e}^{-x}}{2}\right)^2}$$

Which simplifies to

$$u_1 = -\cos\left(\mathrm{e}^{-x}\right)$$
$$u_2 = \cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-x} - \sin\left(\mathrm{e}^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-x} - \sin\left(\mathrm{e}^{-x}\right)\right) \mathrm{e}^{2x} - \cos\left(\mathrm{e}^{-x}\right) \mathrm{e}^{x}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x}\right) + \left(-\mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} - \mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right) \tag{1}
$$

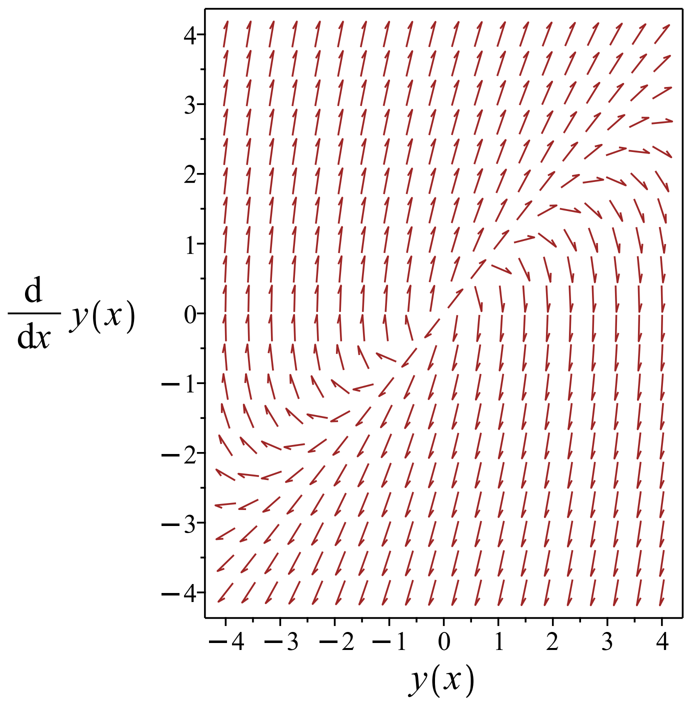


Figure 240: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{2x} - \mathrm{e}^{2x} \sin\left(\mathrm{e}^{-x}\right)
$$

Verified OK.

9.12.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = \sin\left(\mathrm{e}^{-x}\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin\left(\mathrm{e}^{-x}\right)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & \mathrm{e}^{2x} \\ \mathrm{e}^x & 2\,\mathrm{e}^{2x} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{3x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\mathrm{e}^x \left(\int \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) dx\right) + \mathrm{e}^{2x}\left(\int \sin\left(\mathrm{e}^{-x}\right) \mathrm{e}^{-2x} dx\right)$

- Compute integrals

  $y_p(x) = -e^{2x} \sin(e^{-x})$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{2x} - e^{2x} \sin(e^{-x})$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=sin(exp(-x)),y(x), singsol=all)
```

$$y(x) = \left(e^x c_1 - e^x \sin\left(e^{-x}\right) + c_2\right) e^x$$

✓ Solution by Mathematica

Time used: 0.085 (sec). Leaf size: 29

```
DSolve[y''[x]-3*y'[x]+2*y[x]==Sin[Exp[-x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(-e^x \sin\left(e^{-x}\right) + c_2 e^x + c_1\right)$$

10 Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98

10.1 problem 10 1481
10.2 problem 11 1494
10.3 problem 12 1507
10.4 problem 13 1519
10.5 problem 14 1532
10.6 problem 15 1546
10.7 problem 16 1557
10.8 problem 17 1568
10.9 problem 18 1579
10.10 problem 19 1590
10.11 problem 20 1598
10.12 problem 21 1609

10.1 problem 10

10.1.1 Solving as second order linear constant coeff ode 1481
10.1.2 Solving using Kovacic algorithm 1485
10.1.3 Maple step by step solution 1491

Internal problem ID [5380]
Internal file name [`OUTPUT/4871_Sunday_February_04_2024_12_46_39_AM_21402703/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \csc(x)}$$

10.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1} dx$$

Which simplifies to

$$u_2 = \int \cot(x)\, dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\, x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\, c_1 + c_2\sin(x)) + (-\cos(x)\, x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x) \tag{1}$$

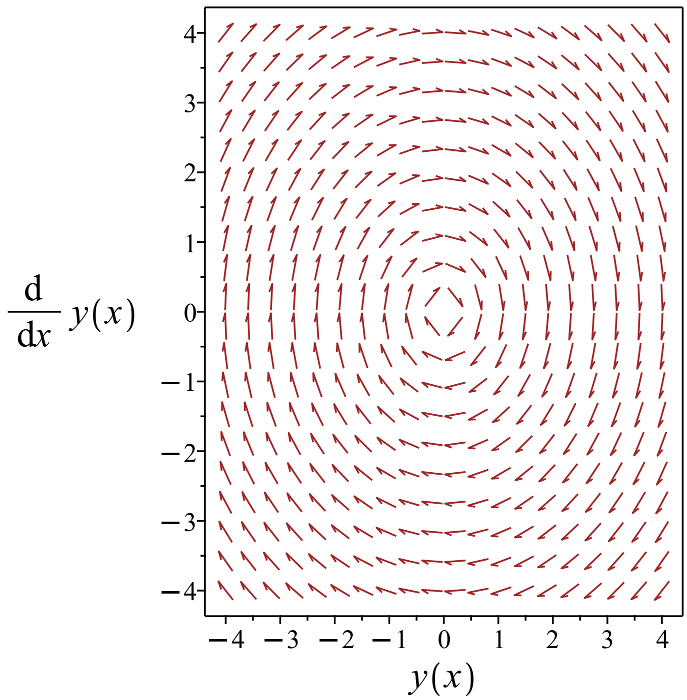


Figure 241: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

10.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$A = 1$$
$$B = 0 \tag{3}$$
$$C = 1$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 174: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \cos(x)^2 + \sin(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int \cot(x)\,dx$$

Hence

$$u_2 = \ln(\sin(x))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\cos(x)\,x + \ln(\sin(x))\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + (-\cos(x)\,x + \ln(\sin(x))\sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2\sin(x) - \cos(x)\,x + \ln(\sin(x))\sin(x) \tag{1}$$

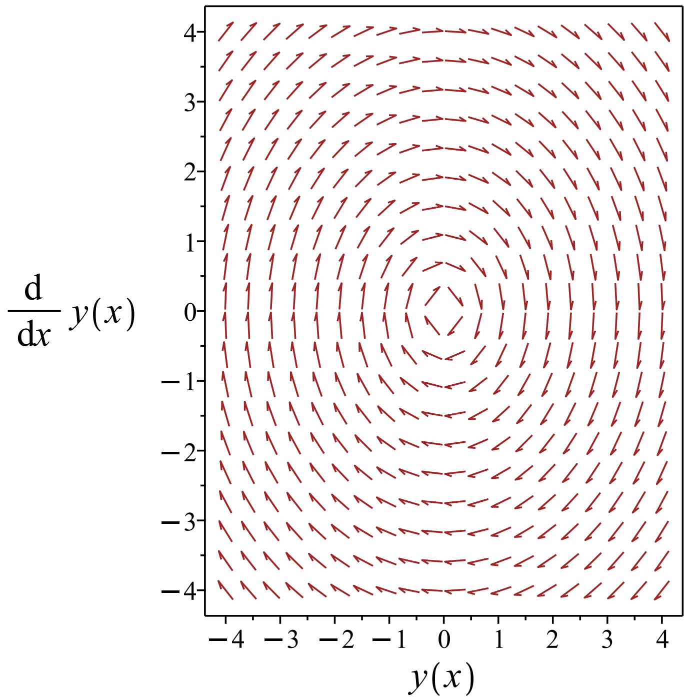


Figure 242: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$$

Verified OK.

10.1.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int 1 dx\right) + \sin(x)\left(\int \cot(x) dx\right)$

  - Compute integrals

    $y_p(x) = -\cos(x) x + \ln(\sin(x)) \sin(x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) - \cos(x) x + \ln(\sin(x)) \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=csc(x),y(x), singsol=all)
```

$$y(x) = -\ln(\csc(x))\sin(x) + (-x + c_1)\cos(x) + \sin(x) c_2$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 24

```
DSolve[y''[x]+y[x]==Csc[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-x + c_1)\cos(x) + \sin(x)(\log(\sin(x)) + c_2)$$

10.2 problem 11

10.2.1 Solving as second order linear constant coeff ode 1494
10.2.2 Solving using Kovacic algorithm 1499
10.2.3 Maple step by step solution 1505

Internal problem ID [5381]
Internal file name [`OUTPUT/4872_Sunday_February_04_2024_12_46_40_AM_16041237/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 4\sec(x)^2}$$

10.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 4\sec(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$
y = c_1 \cos(2x) + c_2 \sin(2x)
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1 \cos(2x) + c_2 \sin(2x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos(2x) \\
y_2 &= \sin(2x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}(\sin(2x)) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}
$$

Therefore

$$W = (\cos(2x))(2\cos(2x)) - (\sin(2x))(-2\sin(2x))$$

Which simplifies to

$$W = 2\cos(2x)^2 + 2\sin(2x)^2$$

Which simplifies to

$$W = 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\sin(2x)\sec(x)^2}{2}\,dx$$

Which simplifies to

$$u_1 = -\int 4\tan(x)\,dx$$

Hence

$$u_1 = 4\ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\cos(2x)\sec(x)^2}{2}\,dx$$

Which simplifies to

$$u_2 = \int 2\cos(2x)\sec(x)^2\,dx$$

Hence

$$u_2 = 4x - 2\tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 4\ln(\cos(x))\cos(2x) + (4x - 2\tan(x))\sin(2x)$$

Which simplifies to

$$y_p(x) = \left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2 \tag{1}$$

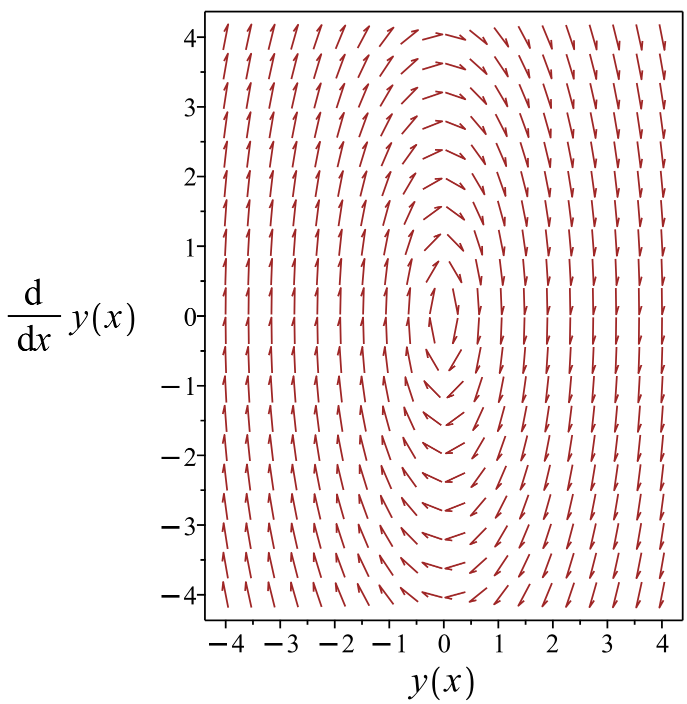


Figure 243: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2$$

Verified OK.

10.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 176: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B=0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B=0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\
&= \cos(2x)\left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(2x)$$
$$y_2 = \frac{\sin(2x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ \frac{d}{dx}(\cos(2x)) & \frac{d}{dx}\left(\frac{\sin(2x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(2x) & \frac{\sin(2x)}{2} \\ -2\sin(2x) & \cos(2x) \end{vmatrix}$$

Therefore

$$W = (\cos(2x))(\cos(2x)) - \left(\frac{\sin(2x)}{2}\right)(-2\sin(2x))$$

Which simplifies to

$$W = \cos(2x)^2 + \sin(2x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(2x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 4\tan(x)\,dx$$

Hence

$$u_1 = 4\ln(\cos(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\cos(2x)\sec(x)^2}{1}\,dx$$

Which simplifies to

$$u_2 = \int 4\cos(2x)\sec(x)^2\,dx$$

Hence

$$u_2 = 8x - 4\tan(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 4\ln(\cos(x))\cos(2x) + \frac{(8x - 4\tan(x))\sin(2x)}{2}$$

Which simplifies to

$$y_p(x) = \left(8 \cos \left(x\right)^2 - 4\right) \ln \left(\cos \left(x\right)\right) + 8 \sin \left(x\right) x \cos \left(x\right) - 4 \sin \left(x\right)^2$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 \cos \left(2x\right) + \frac{c_2 \sin \left(2x\right)}{2}\right) + \left(\left(8 \cos \left(x\right)^2 - 4\right) \ln \left(\cos \left(x\right)\right) + 8 \sin \left(x\right) x \cos \left(x\right) - 4 \sin \left(x\right)^2\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos \left(2x\right) + \frac{c_2 \sin \left(2x\right)}{2} + \left(8 \cos \left(x\right)^2 - 4\right) \ln \left(\cos \left(x\right)\right) + 8 \sin \left(x\right) x \cos \left(x\right) - 4 \sin \left(x\right)^2 \tag{1}$$

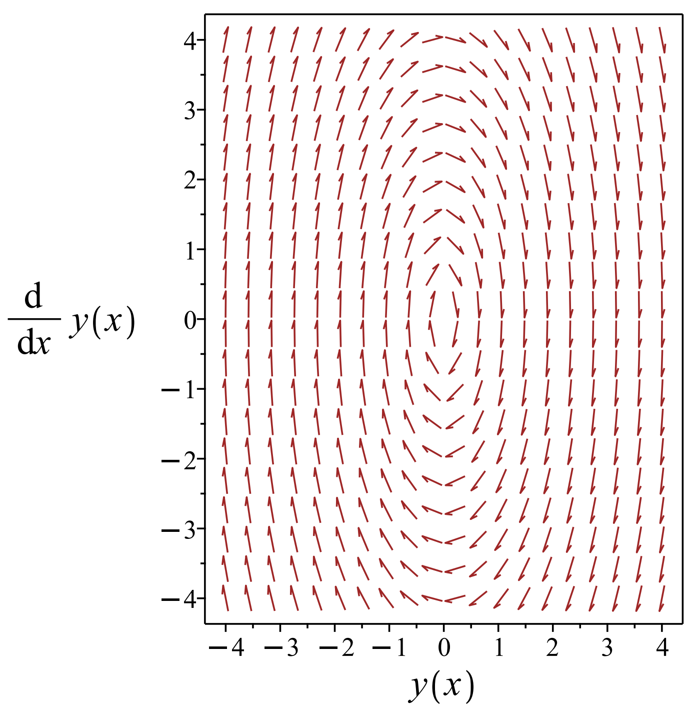


Figure 244: Slope field plot

Verification of solutions

$$y = c_1 \cos \left(2x\right) + \frac{c_2 \sin \left(2x\right)}{2} + \left(8 \cos \left(x\right)^2 - 4\right) \ln \left(\cos \left(x\right)\right) + 8 \sin \left(x\right) x \cos \left(x\right) - 4 \sin \left(x\right)^2$$

Verified OK.

10.2.3 Maple step by step solution

Let's solve

$y'' + 4y = 4\sec(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\sec(x)^2\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -4\cos(2x)\left(\int \tan(x)\,dx\right) + 2\sin(2x)\left(\int \cos(2x)\sec(x)^2\,dx\right)$$

- Compute integrals

$$y_p(x) = \left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(2x) + c_2\sin(2x) + \left(8\cos(x)^2 - 4\right)\ln(\cos(x)) + 8\sin(x)\,x\cos(x) - 4\sin(x)^2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$2)+4*y(x)=4*sec(x)^2,y(x), singsol=all)
```

$$y(x) = \left(-8\cos(x)^2 + 4\right)\ln(\sec(x)) + 2c_1\cos(x)^2 + 8\left(x + \frac{c_2}{4}\right)\sin(x)\cos(x) - 4\sin(x)^2 - c_1$$

✓ Solution by Mathematica

Time used: 0.075 (sec). Leaf size: 44

```
DSolve[y''[x]+4*y[x]==4*Sec[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin(2x)\arctan(\tan(x)) + 2x\sin(2x) + c_2\sin(2x) + \cos(2x)(4\log(\cos(x)) + 2 + c_1) - 2$$

10.3 problem 12

10.3.1 Solving as second order linear constant coeff ode 1507
10.3.2 Solving using Kovacic algorithm 1511
10.3.3 Maple step by step solution 1517

Internal problem ID [5382]
Internal file name [`OUTPUT/4873_Sunday_February_04_2024_12_46_40_AM_96331104/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y' + 3y = \frac{1}{1 + e^{-x}}}$$

10.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = \frac{1}{1+e^{-x}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)}\\ &= 2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$
$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{3x} \\ y_2 &= \mathrm{e}^{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{3x} & \mathrm{e}^{x} \\ \frac{d}{dx}\left(\mathrm{e}^{3x}\right) & \frac{d}{dx}\left(\mathrm{e}^{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{3x} & \mathrm{e}^{x} \\ 3\,\mathrm{e}^{3x} & \mathrm{e}^{x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{3x}\right)\left(\mathrm{e}^{x}\right) - \left(\mathrm{e}^{x}\right)\left(3\,\mathrm{e}^{3x}\right)$$

Which simplifies to

$$W = -2\,\mathrm{e}^{3x}\mathrm{e}^{x}$$

Which simplifies to

$$W = -2\,\mathrm{e}^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{x}}{1+\mathrm{e}^{-x}}}{-2\,\mathrm{e}^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-3x}}{2+2\,\mathrm{e}^{-x}}\,dx$$

Hence

$$u_1 = -\frac{\ln\left(1+\mathrm{e}^{x}\right)}{2} - \frac{\mathrm{e}^{-2x}}{4} + \frac{\ln\left(\mathrm{e}^{x}\right)}{2} + \frac{\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{3x}}{1+\mathrm{e}^{-x}}}{-2\,\mathrm{e}^{4x}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\mathrm{e}^{-x}}{2+2\,\mathrm{e}^{-x}}\,dx$$

Hence

$$u_2 = \frac{\ln\left(1+\mathrm{e}^{-x}\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln\left(1+\mathrm{e}^{x}\right)}{2} - \frac{\mathrm{e}^{-2x}}{4} + \frac{\ln\left(\mathrm{e}^{x}\right)}{2} + \frac{\mathrm{e}^{-x}}{2}\right)\mathrm{e}^{3x} + \frac{\mathrm{e}^{x}\ln\left(1+\mathrm{e}^{-x}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\mathrm{e}^{x}\left(-2\ln\left(1+\mathrm{e}^{x}\right)\mathrm{e}^{2x} + 2\ln\left(\mathrm{e}^{x}\right)\mathrm{e}^{2x} + 2\,\mathrm{e}^{x} + 2\ln\left(1+\mathrm{e}^{-x}\right) - 1\right)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{3x}c_1 + c_2 e^{x}\right) + \left(\frac{e^{x}\left(-2\ln\left(1+e^{x}\right)e^{2x} + 2\ln\left(e^{x}\right)e^{2x} + 2\,e^{x} + 2\ln\left(1+e^{-x}\right) - 1\right)}{4}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = e^{3x}c_1 + c_2 e^{x} + \frac{e^{x}\left(-2\ln\left(1+e^{x}\right)e^{2x} + 2\ln\left(e^{x}\right)e^{2x} + 2\,e^{x} + 2\ln\left(1+e^{-x}\right) - 1\right)}{4} \tag{1}
$$

Verification of solutions

$$
y = e^{3x}c_1 + c_2 e^{x} + \frac{e^{x}\left(-2\ln\left(1+e^{x}\right)e^{2x} + 2\ln\left(e^{x}\right)e^{2x} + 2\,e^{x} + 2\ln\left(1+e^{-x}\right) - 1\right)}{4}
$$

Verified OK.

10.3.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 4y' + 3y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -4 \\
C &= 3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 178: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2 \left(\mathrm{e}^{x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{x} \\
y_2 &= \frac{\mathrm{e}^{3x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{x} & \frac{e^{3x}}{2} \\ \frac{d}{dx}(e^{x}) & \frac{d}{dx}\left(\frac{e^{3x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{x} & \frac{e^{3x}}{2} \\ e^{x} & \frac{3\,e^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = (e^{x})\left(\frac{3\,e^{3x}}{2}\right) - \left(\frac{e^{3x}}{2}\right)(e^{x})$$

Which simplifies to

$$W = e^{3x} e^{x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{3x}}{2+2\,e^{-x}}}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{-x}}{2 + 2\,e^{-x}} dx$$

Hence

$$u_1 = \frac{\ln\left(1 + e^{-x}\right)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{x}}{1+e^{-x}}}{e^{4x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-3x}}{1 + e^{-x}} dx$$

Hence

$$u_2 = -\ln(1 + e^x) - \frac{e^{-2x}}{2} + \ln(e^x) + e^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x \ln(1 + e^{-x})}{2} + \frac{\left(-\ln(1 + e^x) - \frac{e^{-2x}}{2} + \ln(e^x) + e^{-x}\right) e^{3x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{e^x(-2\ln(1 + e^x) e^{2x} + 2\ln(e^x) e^{2x} + 2 e^x + 2\ln(1 + e^{-x}) - 1)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^{3x}}{2}\right) + \left(\frac{e^x(-2\ln(1 + e^x) e^{2x} + 2\ln(e^x) e^{2x} + 2 e^x + 2\ln(1 + e^{-x}) - 1)}{4}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{e^x(-2\ln(1 + e^x) e^{2x} + 2\ln(e^x) e^{2x} + 2 e^x + 2\ln(1 + e^{-x}) - 1)}{4} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} + \frac{e^x(-2\ln(1 + e^x) e^{2x} + 2\ln(e^x) e^{2x} + 2 e^x + 2\ln(1 + e^{-x}) - 1)}{4}$$

Verified OK.

10.3.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = \frac{1}{1+\mathrm{e}^{-x}}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{1+\mathrm{e}^{-x}}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & \mathrm{e}^{3x} \\ \mathrm{e}^x & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{4x}$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^x\left(\int \frac{e^{-x}}{1+e^{-x}}dx\right)}{2} + \frac{e^{3x}\left(\int \frac{e^{-3x}}{1+e^{-x}}dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{e^x\left(-2\ln(1+e^x)e^{2x}+2\ln(e^x)e^{2x}+2\,e^x+2\ln(1+e^{-x})-1\right)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^x + c_2 e^{3x} + \frac{e^x\left(-2\ln(1+e^x)e^{2x}+2\ln(e^x)e^{2x}+2\,e^x+2\ln(1+e^{-x})-1\right)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 57

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=1/(1+exp(-x)),y(x), singsol=all)
```

$$y(x) = \frac{\ln(1+e^{-x})\,e^x}{2} - \frac{e^{3x}\ln(e^x+1)}{2} + \frac{(4c_1+2\ln(e^x))\,e^{3x}}{4} + \frac{e^{2x}}{2} + \frac{(4c_2-1)\,e^x}{4}$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 49

```
DSolve[y''[x]-4*y'[x]+3*y[x]==1/(1+Exp[-x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^x\left(-4\left(e^{2x}-1\right)\operatorname{arctanh}(2e^x+1)+2e^x+4c_2e^{2x}-1+4c_1\right)$$

10.4 problem 13

10.4.1 Solving as second order linear constant coeff ode 1519
10.4.2 Solving using Kovacic algorithm 1524
10.4.3 Maple step by step solution 1530

Internal problem ID [5383]
Internal file name [`OUTPUT/4874_Sunday_February_04_2024_12_46_41_AM_90539611/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \mathrm{e}^{-x} \sin\left(\mathrm{e}^{-x}\right) + \cos\left(\mathrm{e}^{-x}\right)}$$

10.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \mathrm{e}^{-x} \sin(\mathrm{e}^{-x}) + \cos(\mathrm{e}^{-x})$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = e^{-x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & e^{-x} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}(e^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(-e^{-x}\right) - \left(e^{-x}\right)(e^x)$$

Which simplifies to

$$W = -2\, e^x e^{-x}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{-x}(\mathrm{e}^{-x}\sin(\mathrm{e}^{-x}) + \cos(\mathrm{e}^{-x}))}{-2} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-x}(\mathrm{e}^{-x}\sin(\mathrm{e}^{-x}) + \cos(\mathrm{e}^{-x}))}{2} dx$$

Hence

$$u_1 = -\sin\left(\mathrm{e}^{-x}\right) + \frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{-x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{x}(\mathrm{e}^{-x}\sin(\mathrm{e}^{-x}) + \cos(\mathrm{e}^{-x}))}{-2} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}}{2} - \frac{\sin(\mathrm{e}^{-x})}{2}\right) dx$$

Hence

$$u_2 = -\frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\sin\left(\mathrm{e}^{-x}\right) + \frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{-x}}{2}\right)\mathrm{e}^{x} - \frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}\mathrm{e}^{-x}}{2}$$

Which simplifies to

$$y_p(x) = -\mathrm{e}^{x}\sin\left(\mathrm{e}^{-x}\right)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x}\right) + \left(-\mathrm{e}^x \sin\left(\mathrm{e}^{-x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x} - \mathrm{e}^x \sin\left(\mathrm{e}^{-x}\right) \tag{1}
$$

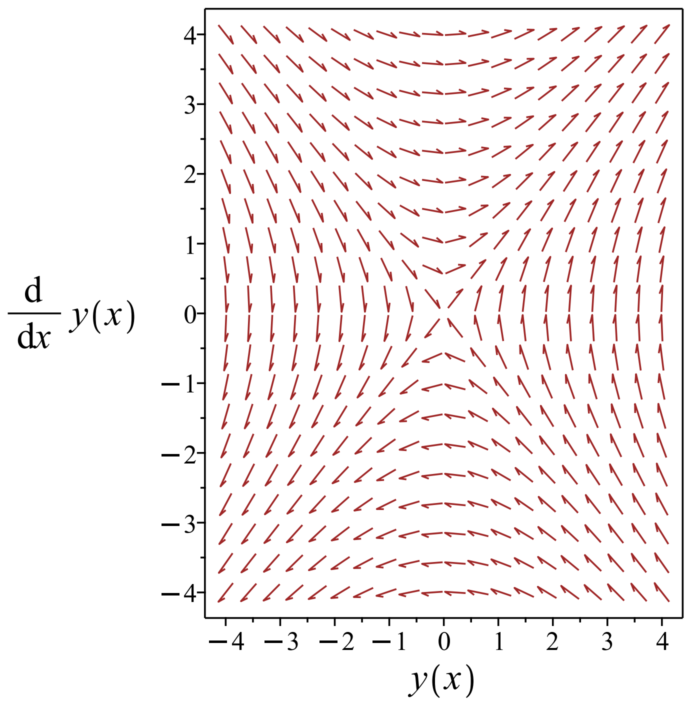


Figure 245: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-x} - \mathrm{e}^x \sin\left(\mathrm{e}^{-x}\right)
$$

Verified OK.

10.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 180: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\
&= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \frac{\mathrm{e}^{x}}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-x})\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)(-\mathrm{e}^{-x})$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{x}(\mathrm{e}^{-x}\sin(\mathrm{e}^{-x})+\cos(\mathrm{e}^{-x}))}{2}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}}{2} + \frac{\sin(\mathrm{e}^{-x})}{2}\right) dx$$

Hence

$$u_1 = -\frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-x}(\mathrm{e}^{-x}\sin(\mathrm{e}^{-x}) + \cos(\mathrm{e}^{-x}))}{1}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-x}\left(\mathrm{e}^{-x}\sin\left(\mathrm{e}^{-x}\right) + \cos\left(\mathrm{e}^{-x}\right)\right) dx$$

Hence

$$u_2 = \cos\left(\mathrm{e}^{-x}\right)\mathrm{e}^{-x} - 2\sin\left(\mathrm{e}^{-x}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\cos(\mathrm{e}^{-x})\,\mathrm{e}^{x}\mathrm{e}^{-x}}{2} + \frac{(\cos(\mathrm{e}^{-x})\,\mathrm{e}^{-x} - 2\sin(\mathrm{e}^{-x}))\,\mathrm{e}^{x}}{2}$$

Which simplifies to

$$y_p(x) = -e^x \sin(e^{-x})$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-e^x \sin(e^{-x})\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - e^x \sin(e^{-x}) \tag{1}$$

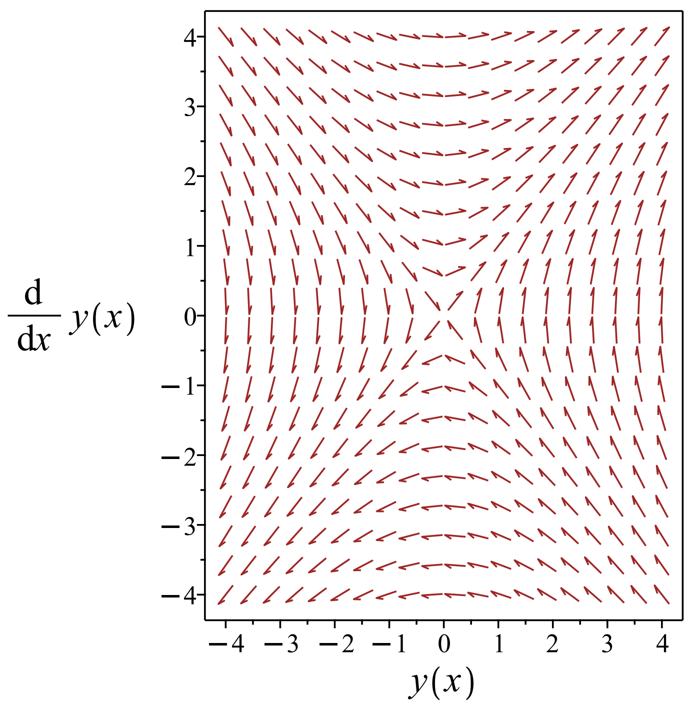


Figure 246: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - e^x \sin(e^{-x})$$

Verified OK.

10.4.3 Maple step by step solution

Let's solve

$y'' - y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^{-x}\sin(e^{-x}) + \cos(e^{-x}\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-x}\left(\int \left(\cos\left(e^{-x}\right)e^{x}+\sin\left(e^{-x}\right)\right)dx\right)}{2} + \frac{e^{x}\left(\int e^{-x}\left(e^{-x}\sin\left(e^{-x}\right)+\cos\left(e^{-x}\right)\right)dx\right)}{2}$$

- Compute integrals

$$y_p(x) = -e^{x}\sin\left(e^{-x}\right)$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-x} + c_2 e^{x} - e^{x}\sin\left(e^{-x}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-y(x)=exp(-x)*sin(exp(-x))+cos(exp(-x)),y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{x}c_1 - e^{x}\sin\left(e^{-x}\right)$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 31

```
DSolve[y''[x]-y[x]==Exp[-x]*Sin[Exp[-x]]+Cos[Exp[-x]],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to -e^{x}\sin\left(e^{-x}\right) + c_1 e^{x} + c_2 e^{-x}$$

10.5 problem 14

10.5.1 Solving as second order linear constant coeff ode 1532
10.5.2 Solving using Kovacic algorithm 1537
10.5.3 Maple step by step solution 1543

Internal problem ID [5384]
Internal file name [`OUTPUT/4875_Sunday_February_04_2024_12_46_42_AM_81101878/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \frac{1}{(1+\mathrm{e}^{-x})^2}}$$

10.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \frac{1}{(1+\mathrm{e}^{-x})^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= \mathrm{e}^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \mathrm{e}^{x} & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})\left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right)(\mathrm{e}^{x})$$

Which simplifies to

$$W = -2\,\mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{-x}}{(1+e^{-x})^2}}{-2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x}}{2\left(1+e^{-x}\right)^2} dx$$

Hence

$$u_1 = \frac{1}{2+2\,e^{-x}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{x}}{(1+e^{-x})^2}}{-2}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{3x}}{2\left(1+e^{x}\right)^2} dx$$

Hence

$$u_2 = -\frac{e^{x}}{2} + \frac{1}{2+2\,e^{x}} + \ln\left(1+e^{x}\right)$$

Which simplifies to

$$u_1 = \frac{1}{2+2\,e^{-x}}$$

$$u_2 = \frac{\left(2+2\,e^{x}\right)\ln\left(1+e^{x}\right) - e^{x} - e^{2x} + 1}{2+2\,e^{x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^{x}}{2+2\,e^{-x}} + \frac{\left(\left(2+2\,e^{x}\right)\ln\left(1+e^{x}\right) - e^{x} - e^{2x} + 1\right)e^{-x}}{2+2\,e^{x}}$$

Which simplifies to

$$y_p(x) = \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}} \tag{1}$$

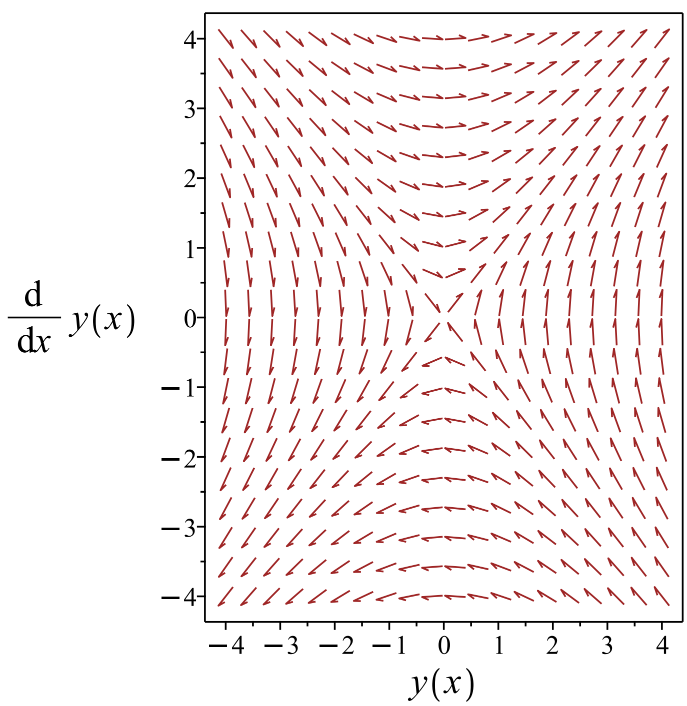


Figure 247: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{(2+2\,\mathrm{e}^{-x})\ln(1+\mathrm{e}^{x}) - \mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{2x} - 1}{2+2\,\mathrm{e}^{x}}$$

Verified OK.

10.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 182: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \frac{\mathrm{e}^{x}}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{x}}{2(1+\mathrm{e}^{-x})^2}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\mathrm{e}^{3x}}{2\left(1+\mathrm{e}^{x}\right)^2} dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{x}}{2} + \frac{1}{2+2\,\mathrm{e}^{x}} + \ln\left(1+\mathrm{e}^{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}}{1}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{-x}}{\left(1+\mathrm{e}^{-x}\right)^2} dx$$

Hence

$$u_2 = \frac{1}{1+\mathrm{e}^{-x}}$$

Which simplifies to

$$u_1 = \frac{\left(2+2\,\mathrm{e}^{x}\right)\ln\left(1+\mathrm{e}^{x}\right) - \mathrm{e}^{x} - \mathrm{e}^{2x} + 1}{2+2\,\mathrm{e}^{x}}$$

$$u_2 = \frac{1}{1+\mathrm{e}^{-x}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{e^x}{2+2\,e^{-x}} + \frac{\left(\left(2+2\,e^x\right)\ln\left(1+e^x\right) - e^x - e^{2x} + 1\right)e^{-x}}{2+2\,e^x}$$

Which simplifies to

$$y_p(x) = \frac{\left(2+2\,e^{-x}\right)\ln\left(1+e^x\right) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(\frac{\left(2+2\,e^{-x}\right)\ln\left(1+e^x\right) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{\left(2+2\,e^{-x}\right)\ln\left(1+e^x\right) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x} \tag{1}$$

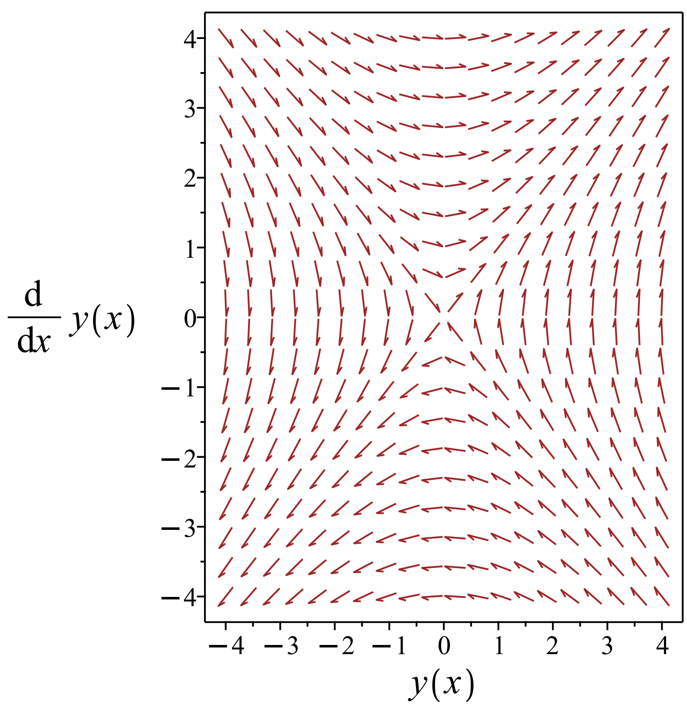


Figure 248: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} + \frac{(2+2\,e^{-x})\ln(1+e^x) - e^x + e^{-x} + e^{2x} - 1}{2+2\,e^x}$$

Verified OK.

10.5.3 Maple step by step solution

Let's solve

$y'' - y = \frac{1}{(1+e^{-x})^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{(1+e^{-x})^2}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = 2$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^{-x}\left(\int \frac{e^{3x}}{(1+e^x)^2}dx\right)}{2} + \frac{e^{x}\left(\int \frac{e^{-x}}{\left(1+e^{-x}\right)^2}dx\right)}{2}$$

- Compute integrals

  $$y_p(x) = \frac{(2+2\,e^{-x})\ln(1+e^x)-e^x+e^{-x}+e^{2x}-1}{2+2\,e^x}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1e^{-x} + c_2e^{x} + \frac{(2+2\,e^{-x})\ln(1+e^x)-e^x+e^{-x}+e^{2x}-1}{2+2\,e^x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-y(x)=1/(1+exp(-x))^2,y(x), singsol=all)
```

$$y(x) = c_2e^{-x} + e^{x}c_1 + \frac{e^{x}}{2} - 1 + e^{-x}\ln\left(e^{x}+1\right) + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 42

```
DSolve[y''[x]-y[x]==1/(1+Exp[-x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-2e^{x}+2\log\left(e^{x}+1\right)+2c_1e^{2x}+1+2c_2\right)$$

10.6 problem 15

10.6.1 Solving as second order linear constant coeff ode 1546
10.6.2 Solving using Kovacic algorithm 1549
10.6.3 Maple step by step solution 1554

Internal problem ID [5385]
Internal file name [`OUTPUT/4876_Sunday_February_04_2024_12_46_42_AM_79064302/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y = 2 + \mathrm{e}^{x}}$$

10.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = 2 + \mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)}\\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)$$

Or

$$y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 + e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(x\sqrt{2}\right), \sin\left(x\sqrt{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2 e^x + 2A_1 = 2 + e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + \frac{e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right) + \left(1 + \frac{e^x}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) + 1 + \frac{e^x}{3} \tag{1}$$

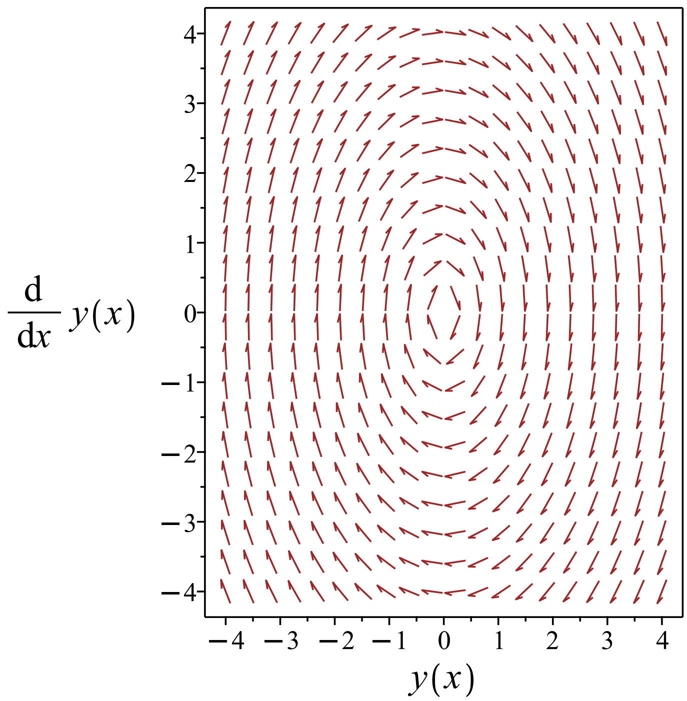


Figure 249: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) + 1 + \frac{e^x}{3}$$

Verified OK.

10.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 184: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(x\sqrt{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(x\sqrt{2}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos\left(x\sqrt{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(x\sqrt{2}\right) \int \frac{1}{\cos\left(x\sqrt{2}\right)^2}\, dx \\
&= \cos\left(x\sqrt{2}\right) \left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\cos\left(x\sqrt{2}\right)\right) + c_2\left(\cos\left(x\sqrt{2}\right)\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 + e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}, \cos\left(x\sqrt{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_2 e^x + 2A_1 = 2 + e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 1 + \frac{e^x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}\right) + \left(1 + \frac{e^x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2} + 1 + \frac{e^x}{3} \tag{1}$$

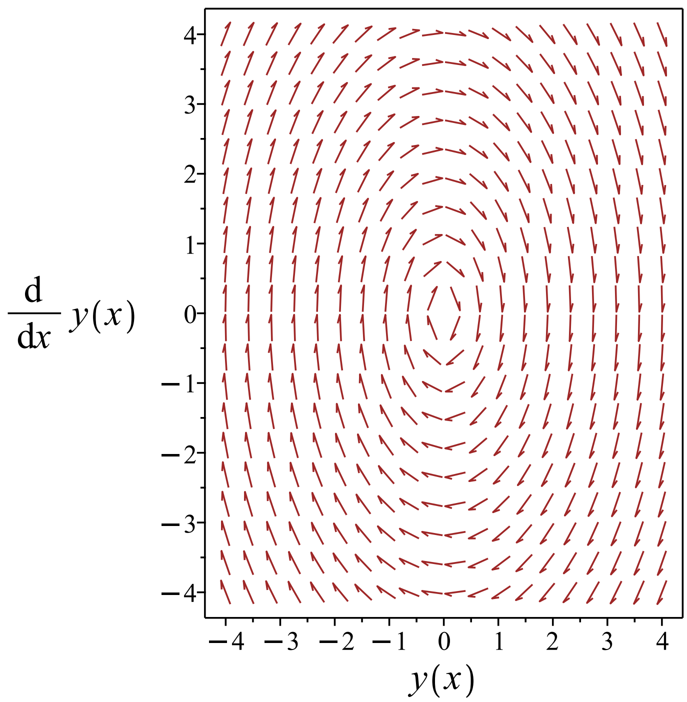


Figure 250: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2} + 1 + \frac{e^x}{3}$$

Verified OK.

10.6.3 Maple step by step solution

Let's solve

$y'' + 2y = 2 + e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2+2=0$

- Use quadratic formula to solve for r

$r=\frac{0\pm(\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r=\left(-\mathrm{I}\sqrt{2},\mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

$y_1(x)=\cos\left(x\sqrt{2}\right)$

- 2nd solution of the homogeneous ODE

$y_2(x)=\sin\left(x\sqrt{2}\right)$

- General solution of the ODE

$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y=c_1\cos\left(x\sqrt{2}\right)+c_2\sin\left(x\sqrt{2}\right)+y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=2+\mathrm{e}^x\right]$

- ○ Wronskian of solutions of the homogeneous equation

$W(y_1(x),y_2(x))=\left[\begin{array}{cc}\cos\left(x\sqrt{2}\right) & \sin\left(x\sqrt{2}\right)\\ -\sqrt{2}\,\sin\left(x\sqrt{2}\right) & \sqrt{2}\,\cos\left(x\sqrt{2}\right)\end{array}\right]$

- ○ Compute Wronskian

$W(y_1(x),y_2(x))=\sqrt{2}$

- ○ Substitute functions into equation for $y_p(x)$

$y_p(x)=-\frac{\sqrt{2}\left(\cos\left(x\sqrt{2}\right)\left(\int(2+\mathrm{e}^x)\sin\left(x\sqrt{2}\right)dx\right)-\sin\left(x\sqrt{2}\right)\left(\int(2+\mathrm{e}^x)\cos\left(x\sqrt{2}\right)dx\right)\right)}{2}$

- ○ Compute integrals

$y_p(x)=1+\frac{\mathrm{e}^x}{3}$

- Substitute particular solution into general solution to ODE

$y=c_1\cos\left(x\sqrt{2}\right)+c_2\sin\left(x\sqrt{2}\right)+1+\frac{\mathrm{e}^x}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+2*y(x)=2+exp(x),y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{2}\, x\right) c_2 + \cos\left(\sqrt{2}\, x\right) c_1 + 1 + \frac{e^x}{3}$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 36

```
DSolve[y''[x]+2*y[x]==2+Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{3} + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + 1$$

10.7 problem 16

10.7.1 Solving as second order linear constant coeff ode 1557
10.7.2 Solving using Kovacic algorithm 1560
10.7.3 Maple step by step solution 1565

Internal problem ID [5386]
Internal file name [`OUTPUT/4877_Sunday_February_04_2024_12_46_43_AM_3374812/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = \mathrm{e}^{x} \sin(2x)}$$

10.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \mathrm{e}^{x} \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\
&= \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x \cos(2x) + A_2 e^x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^x \cos(2x) - 4A_1 e^x \sin(2x) - 4A_2 e^x \sin(2x) + 4A_2 e^x \cos(2x) = e^x \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{8}, A_2 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-x}\right) + \left(-\frac{e^x \cos(2x)}{8} - \frac{e^x \sin(2x)}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
y = c_1 e^{x} + c_2 e^{-x} - \frac{e^{x} \cos(2x)}{8} - \frac{e^{x} \sin(2x)}{8} \tag{1}
$$

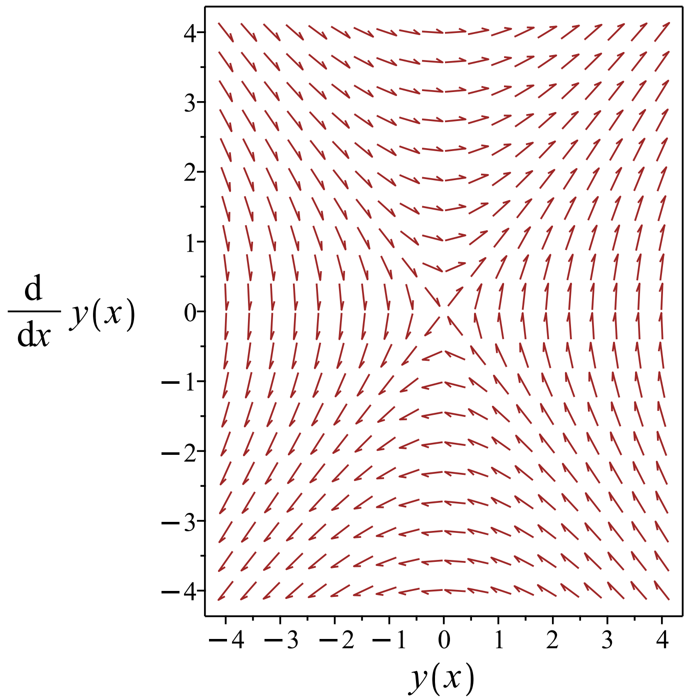


Figure 251: Slope field plot

Verification of solutions

$$
y = c_1 e^{x} + c_2 e^{-x} - \frac{e^{x} \cos(2x)}{8} - \frac{e^{x} \sin(2x)}{8}
$$

Verified OK.

10.7.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\ &= e^{-x} \left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{x} \sin(2x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x} \cos(2x), \mathrm{e}^{x} \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x} \cos(2x) + A_2 \mathrm{e}^{x} \sin(2x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-4A_1 \mathrm{e}^{x} \cos(2x) - 4A_1 \mathrm{e}^{x} \sin(2x) - 4A_2 \mathrm{e}^{x} \sin(2x) + 4A_2 \mathrm{e}^{x} \cos(2x) = \mathrm{e}^{x} \sin(2x)
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{8}, A_2 = -\frac{1}{8}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -\frac{\mathrm{e}^{x} \cos(2x)}{8} - \frac{\mathrm{e}^{x} \sin(2x)}{8}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}\right) + \left(-\frac{\mathrm{e}^{x} \cos(2x)}{8} - \frac{\mathrm{e}^{x} \sin(2x)}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - \frac{e^{x} \cos(2x)}{8} - \frac{e^{x} \sin(2x)}{8} \tag{1}$$

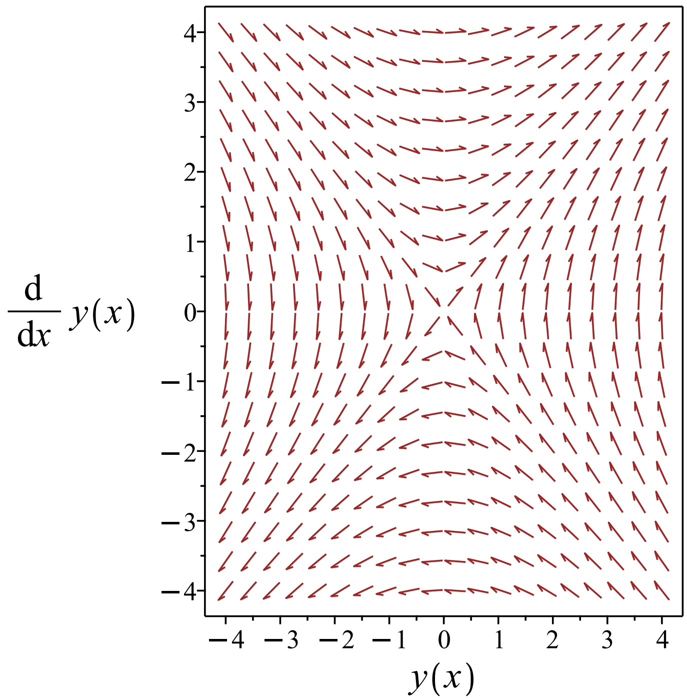


Figure 252: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - \frac{e^{x} \cos(2x)}{8} - \frac{e^{x} \sin(2x)}{8}$$

Verified OK.

10.7.3 Maple step by step solution

Let's solve

$y'' - y = e^{x} \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{x} \sin(2x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x} \sin(2x) dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \sin(2x) dx\right)}{2}$
  - Compute integrals

    $y_p(x) = -\frac{\mathrm{e}^{x}(\sin(2x) + \cos(2x))}{8}$
- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} - \frac{\mathrm{e}^{x}(\sin(2x) + \cos(2x))}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-y(x)=exp(x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{e^{x} \cos(2x)}{8} + c_2 e^{-x} + e^{x}\left(c_1 - \frac{\sin(2x)}{8}\right)$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 37

```
DSolve[y''[x]-y[x]==Exp[x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2 e^{-x} - \frac{1}{8} e^x (\sin(2x) + \cos(2x) + 2)$$

10.8 problem 17

10.8.1 Solving as second order linear constant coeff ode 1568
10.8.2 Solving using Kovacic algorithm 1571
10.8.3 Maple step by step solution 1576

Internal problem ID [5387]
Internal file name [`OUTPUT/4878_Sunday_February_04_2024_12_46_43_AM_88184614/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = x^2 + \sin(x)}$$

10.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = x^2 + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)}\\
&= -1 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -1 + i\\
\lambda_2 &= -1 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 + i\\
\lambda_2 &= -1 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(\cos(x)\,c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-x}(\cos(x)\,c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(x), e^{-x} \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}A_1 \cos(x) + A_2 \sin(x) + 2A_5 - 2A_1 \sin(x) + 2A_2 \cos(x)\\ + 2A_4 + 4A_5 x + 2A_3 + 2A_4 x + 2A_5 x^2 = x^2 + \sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}, A_3 = \frac{1}{2}, A_4 = -1, A_5 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(e^{-x}(\cos(x) c_1 + c_2 \sin(x))\right) + \left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \tag{1}$$

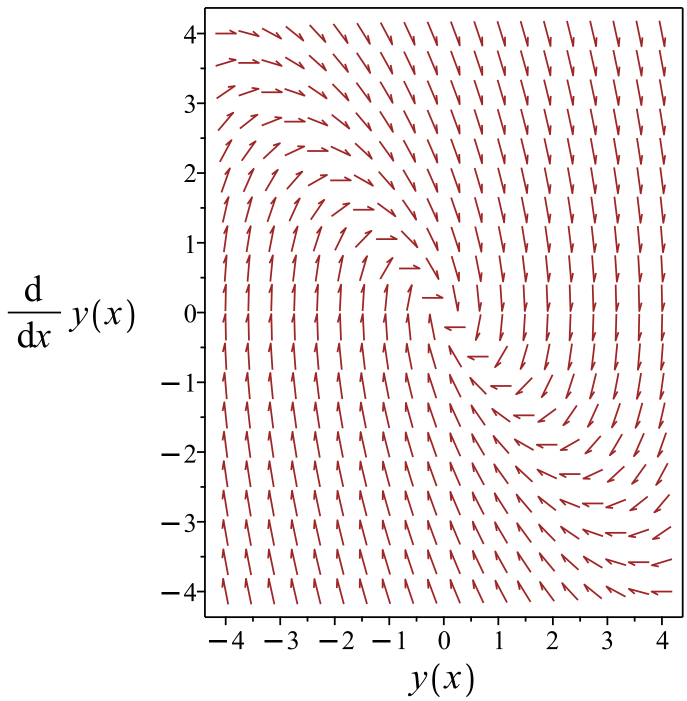


Figure 253: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Verified OK.

10.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{-x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{-x} \cos(x)\right) + c_2\left(e^{-x} \cos(x) (\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} \cos(x) + c_2 e^{-x} \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(x), e^{-x}\sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(x) + A_2\sin(x) + A_3 + A_4 x + A_5 x^2$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{gathered} A_1\cos(x) + A_2\sin(x) + 2A_5 - 2A_1\sin(x) + 2A_2\cos(x) \\ + 2A_4 + 4A_5 x + 2A_3 + 2A_4 x + 2A_5 x^2 = x^2 + \sin(x) \end{gathered}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{5}, A_2 = \frac{1}{5}, A_3 = \frac{1}{2}, A_4 = -1, A_5 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)\right) + \left(-\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(\cos(x) c_1 + c_2\sin(x)) - \frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2} \tag{1}$$

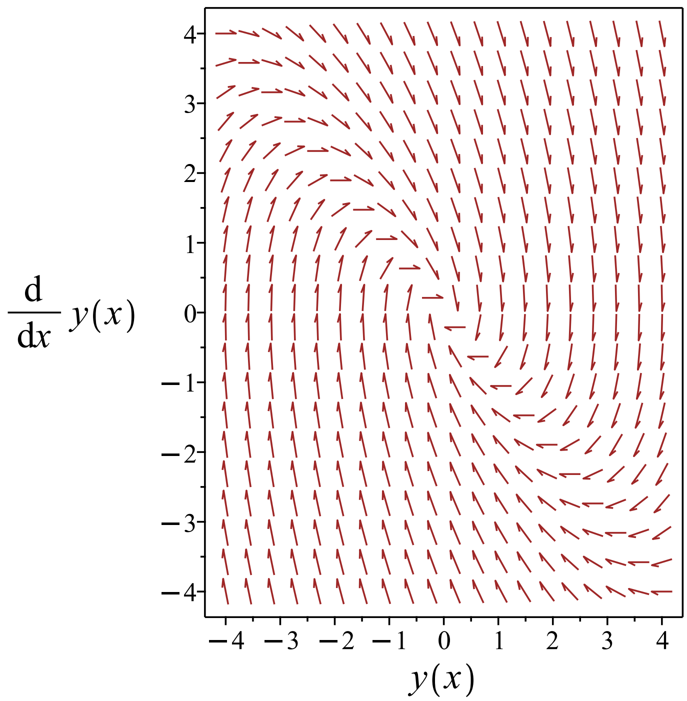


Figure 254: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) - \frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$$

Verified OK.

10.8.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = x^2 + \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} \cos(x) + c_2 \mathrm{e}^{-x} \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = x^2 + \sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(x) & \mathrm{e}^{-x} \sin(x) \\ -\mathrm{e}^{-x} \cos(x) - \mathrm{e}^{-x} \sin(x) & -\mathrm{e}^{-x} \sin(x) + \mathrm{e}^{-x} \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x} \left(-\cos(x) \left(\int \sin(x) \mathrm{e}^{x} (x^2 + \sin(x)) dx\right) + \sin(x) \left(\int \cos(x) \mathrm{e}^{x} (x^2 + \sin(x)) dx\right)\right)$

  - Compute integrals

    $y_p(x) = -\frac{2\cos(x)}{5} + \frac{\sin(x)}{5} + \frac{1}{2} - x + \frac{x^2}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_2 \mathrm{e}^{-x} \sin(x) + c_1 \mathrm{e}^{-x} \cos(x) + \frac{x^2}{2} + \frac{\sin(x)}{5} - \frac{2\cos(x)}{5} - x + \frac{1}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=x^2+sin(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} \sin(x)\, c_2 + \cos(x)\, \mathrm{e}^{-x} c_1 + \frac{x^2}{2} - x + \frac{1}{2} + \frac{\sin(x)}{5} - \frac{2\cos(x)}{5}$$

✓ Solution by Mathematica

Time used: 0.308 (sec). Leaf size: 50

```
DSolve[y''[x]+2*y'[x]+2*y[x]==x^2+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{10} e^{-x} \left(5e^x(x-1)^2 + (-4e^x + 10c_2)\cos(x) + 2(e^x + 5c_1)\sin(x)\right)$$

10.9 problem 18

10.9.1 Solving as second order linear constant coeff ode 1579
10.9.2 Solving using Kovacic algorithm 1582
10.9.3 Maple step by step solution 1587

Internal problem ID [5388]
Internal file name [`OUTPUT/4879_Sunday_February_04_2024_12_46_44_AM_76522474/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 9y = x + e^{2x} - \sin(2x)}$$

10.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -9, f(x) = x + e^{2x} - \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)}\\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3\\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x} - \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{2x} - 13A_4 \cos(2x) - 13A_5 \sin(2x) - 9A_2 - 9A_3 x = x + e^{2x} - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0, A_3 = -\frac{1}{9}, A_4 = 0, A_5 = \frac{1}{13}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{-3x}\right) + \left(-\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2e^{-3x} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \tag{1}$$

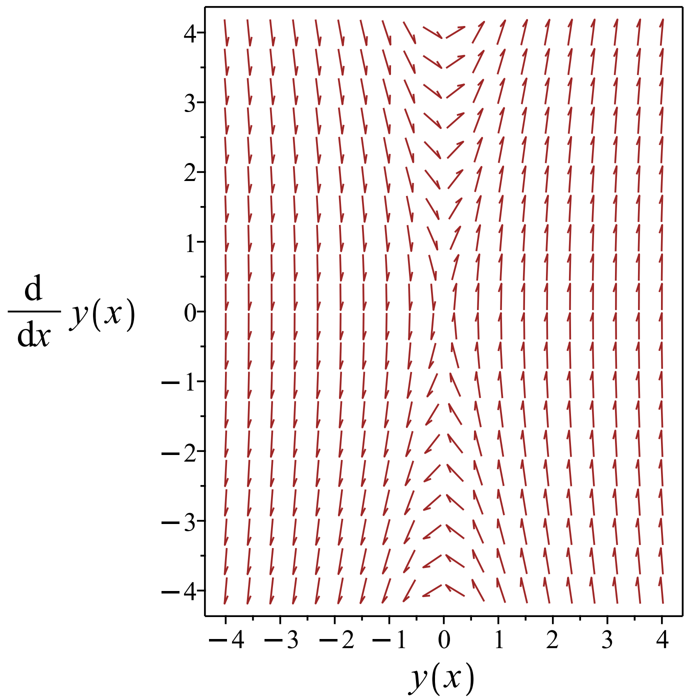


Figure 255: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2e^{-3x} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Verified OK.

10.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-3x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-3x}
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= e^{-3x} \int \frac{1}{e^{-6x}}\, dx \\
&= e^{-3x} \left(\frac{e^{6x}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-3x}\right) + c_2 \left(e^{-3x} \left(\frac{e^{6x}}{6}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x} - \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{6}, e^{-3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x + A_4 \cos(2x) + A_5 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 e^{2x} - 13A_4 \cos(2x) - 13A_5 \sin(2x) - 9A_2 - 9A_3 x = x + e^{2x} - \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0, A_3 = -\frac{1}{9}, A_4 = 0, A_5 = \frac{1}{13}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}\right) + \left(-\frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13} \tag{1}$$

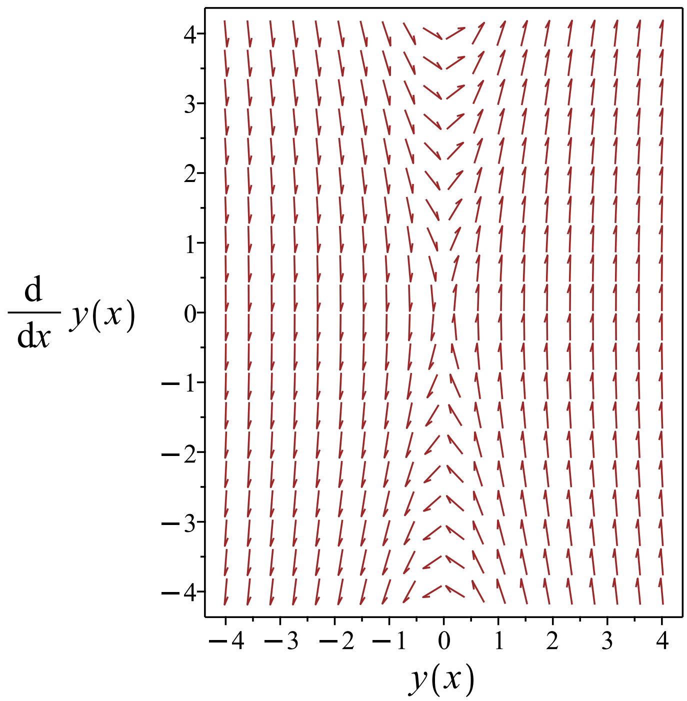


Figure 256: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} - \frac{e^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$$

Verified OK.

10.9.3 Maple step by step solution

Let's solve

$y'' - 9y = x + e^{2x} - \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{3x} + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + \mathrm{e}^{2x} - \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{3x} \\ -3\,\mathrm{e}^{-3x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 6$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\mathrm{e}^{-3x}\left(\int \mathrm{e}^{3x}\left(x+\mathrm{e}^{2x}-\sin(2x)\right)dx\right)}{6} + \frac{\mathrm{e}^{3x}\left(\int \mathrm{e}^{-3x}\left(x+\mathrm{e}^{2x}-\sin(2x)\right)dx\right)}{6}$

  - Compute integrals

    $y_p(x) = -\frac{\mathrm{e}^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{3x} - \frac{\mathrm{e}^{2x}}{5} - \frac{x}{9} + \frac{\sin(2x)}{13}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 40

```
dsolve(diff(y(x),x$2)-9*y(x)=x+exp(2*x)-sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{\mathrm{e}^{-3x}\left(\left(x-\frac{9\sin(2x)}{13}\right)\mathrm{e}^{3x}-9\,\mathrm{e}^{6x}c_1-9c_2+\frac{9\,\mathrm{e}^{5x}}{5}\right)}{9}$$

✓ Solution by Mathematica

Time used: 0.846 (sec). Leaf size: 44

```
DSolve[y''[x]-9*y[x]==x+Exp[2*x]-Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{9}-\frac{e^{2x}}{5}+\frac{1}{13}\sin(2x)+c_1e^{3x}+c_2e^{-3x}$$

10.10 problem 19

10.10.1 Maple step by step solution 1592

Internal problem ID [5389]
Internal file name [`OUTPUT/4880_Sunday_February_04_2024_12_46_45_AM_62370958/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 19.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + 3y'' + 2y' = x^2 + 4x + 8}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + 3y'' + 2y' = 0$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + e^{-2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = 1$$
$$y_3 = e^{-2x}$$

Now the particular solution to the given ODE is found

$$y''' + 3y'' + 2y' = x^2 + 4x + 8$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, e^{-2x}, e^{-x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^3 + A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6x^2 A_3 + 4x A_2 + 18x A_3 + 2A_1 + 6A_2 + 6A_3 = x^2 + 4x + 8$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{11}{4}, A_2 = \frac{1}{4}, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{-2x} c_3\right) + \left(\frac{1}{6}x^3 + \frac{1}{4}x^2 + \frac{11}{4}x\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{-2x} c_3 + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 + \mathrm{e}^{-2x} c_3 + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4}$$

Verified OK.

10.10.1 Maple step by step solution

Let's solve

$y''' + 3y'' + 2y' = x^2 + 4x + 8$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = x^2 - 3y_3(x) - 2y_2(x) + 4x + 8$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = x^2 - 3y_3(x) - 2y_2(x) + 4x + 8]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

  $$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ x^2 + 4x + 8 \end{bmatrix}$$

- Define the forcing function

  $$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ x^2 + 4x + 8 \end{bmatrix}$$

- Define the coefficient matrix

  $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

  $\overrightarrow{y}(x) = c_1 \overrightarrow{y}_1 + c_2 \overrightarrow{y}_2 + c_3 \overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

    $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

    $$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{-x} & 1 \\ -\frac{e^{-2x}}{2} & -e^{-x} & 0 \\ e^{-2x} & e^{-x} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & 1 \\ -\frac{1}{2} & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

    $$\Phi(x) = \begin{bmatrix} 1 & \frac{e^{-2x}}{2} - 2\,e^{-x} + \frac{3}{2} & \frac{e^{-2x}}{2} - e^{-x} + \frac{1}{2} \\ 0 & -e^{-2x} + 2\,e^{-x} & -e^{-2x} + e^{-x} \\ 0 & 2\,e^{-2x} - 2\,e^{-x} & 2\,e^{-2x} - e^{-x} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

    $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

    $\overrightarrow{y}'_p(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$
- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds$
- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

  $$\overrightarrow{y}_p(x) = \left[\begin{array}{c} \frac{11x}{4} + \frac{x^2}{4} + \frac{x^3}{6} - \frac{35}{8} + 6\,\mathrm{e}^{-x} - \frac{13\,\mathrm{e}^{-2x}}{8} \\ \frac{11}{4} - 6\,\mathrm{e}^{-x} + \frac{13\,\mathrm{e}^{-2x}}{4} + \frac{x^2}{2} + \frac{x}{2} \\ \frac{1}{2} + 6\,\mathrm{e}^{-x} - \frac{13\,\mathrm{e}^{-2x}}{2} + x \end{array}\right]$$

- Plug particular solution back into general solution

  $$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \left[\begin{array}{c} \frac{11x}{4} + \frac{x^2}{4} + \frac{x^3}{6} - \frac{35}{8} + 6\,\mathrm{e}^{-x} - \frac{13\,\mathrm{e}^{-2x}}{8} \\ \frac{11}{4} - 6\,\mathrm{e}^{-x} + \frac{13\,\mathrm{e}^{-2x}}{4} + \frac{x^2}{2} + \frac{x}{2} \\ \frac{1}{2} + 6\,\mathrm{e}^{-x} - \frac{13\,\mathrm{e}^{-2x}}{2} + x \end{array}\right]$$

- First component of the vector is the solution to the ODE

  $y = -\frac{35}{8} + \frac{(-13+2c_1)\mathrm{e}^{-2x}}{8} + (c_2+6)\,\mathrm{e}^{-x} + \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} + c_3$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
-> Calling odsolve with the ODE`, diff(diff(_b(_a), _a), _a) = _a^2-3*(diff(_b(_a), _a))-2*_
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying high order exact linear fully integrable
   trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
   trying a double symmetry of the form [xi=0, eta=F(x)]
   <- double symmetry of the form [xi=0, eta=F(x)] successful
<- differential order: 3; linear nonhomogeneous with symmetry [0,1] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)+2*diff(y(x),x)=x^2+4*x+8,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{4} + \frac{x^3}{6} - c_2 e^{-x} + \frac{e^{-2x} c_1}{2} + \frac{11x}{4} + c_3$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 47

```
DSolve[y'''[x]+3*y''[x]+2*y'[x]==x^2+4*x+8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3}{6} + \frac{x^2}{4} + \frac{11x}{4} - \frac{1}{2} c_1 e^{-2x} - c_2 e^{-x} + c_3$$

10.11 problem 20

10.11.1 Solving as second order linear constant coeff ode 1598
10.11.2 Solving using Kovacic algorithm 1602
10.11.3 Maple step by step solution 1606

Internal problem ID [5390]
Internal file name [`OUTPUT/4881_Sunday_February_04_2024_12_46_45_AM_57829874/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 4\cos(x)\,x - 2\sin(x)}$$

10.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 4\cos(x)\,x - 2\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x)\, c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4 \cos(x) x - 2 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ &= 4 \cos(x) x - 2 \sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 2, A_3 = 0, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \sin(x) + 2 \cos(x) x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (\cos(x) c_1 + c_2 \sin(x)) + \left(x^2 \sin(x) + 2\cos(x) x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2\cos(x) x \tag{1}
$$

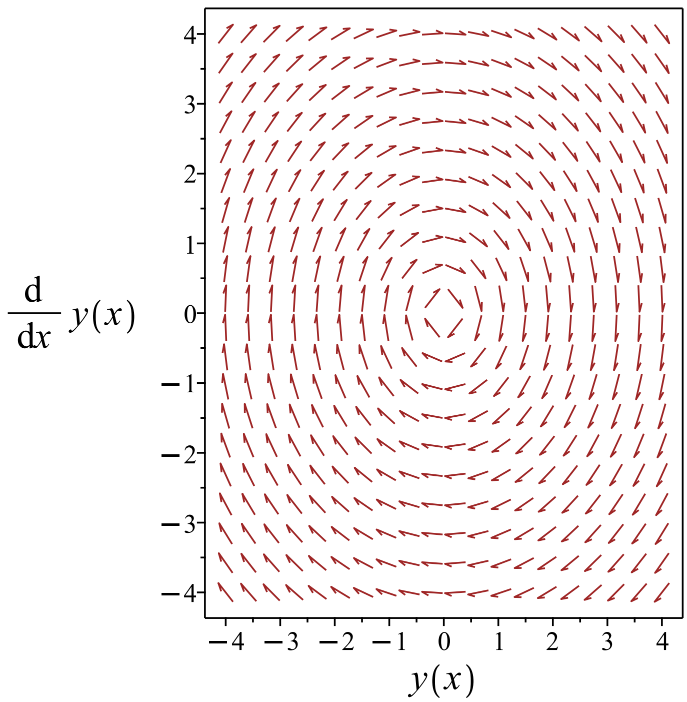


Figure 257: Slope field plot

Verification of solutions

$$
y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2\cos(x) x
$$

Verified OK.

10.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 193: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2} dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(x) c_1 + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\cos(x)\,x-2\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,x,\sin(x)\,x,\cos(x)\,,\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,,\sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\sin(x)\,,\cos(x)\,x,\cos(x)\,x^2,\sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p=A_1x^2\sin(x)+A_2\cos(x)\,x+A_3\cos(x)\,x^2+A_4\sin(x)\,x$$

The unknowns $\{A_1,A_2,A_3,A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\sin(x)+4A_1x\cos(x)-2A_2\sin(x)-4A_3\sin(x)\,x+2A_3\cos(x)+2A_4\cos(x)\\&=4\cos(x)\,x-2\sin(x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=1,A_2=2,A_3=0,A_4=0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=x^2\sin(x)+2\cos(x)\,x$$

Therefore the general solution is

$$\begin{aligned}y&=y_h+y_p\\&=(\cos(x)\,c_1+c_2\sin(x))+\left(x^2\sin(x)+2\cos(x)\,x\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2\cos(x) x \tag{1}$$

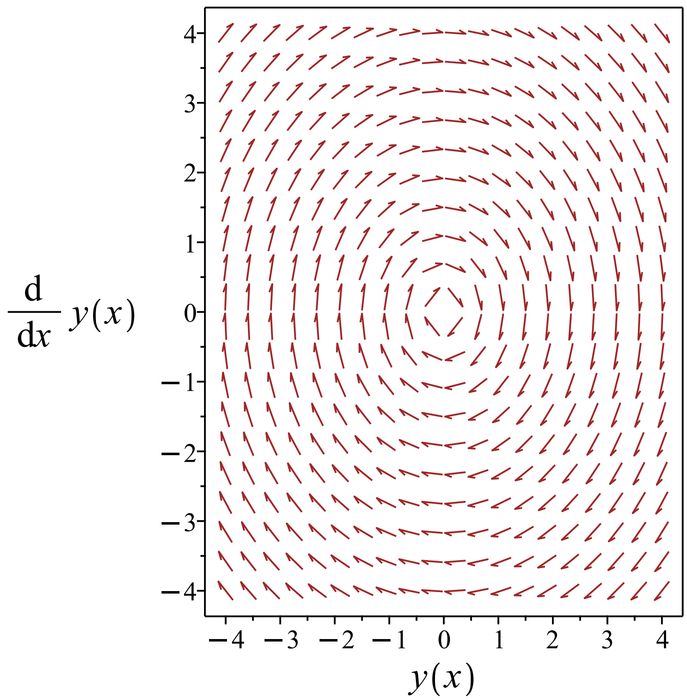


Figure 258: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2\cos(x) x$$

Verified OK.

10.11.3 Maple step by step solution

Let's solve

$y'' + y = 4\cos(x) x - 2\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\cos(x) x - 2\sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x) \left(\int (2x \sin(2x) - 1 + \cos(2x)) dx\right) + \sin(x) \left(\int (2x \cos(2x) + 2x - \sin(2x)) dx\right)$
  - Compute integrals

    $y_p(x) = x^2 \sin(x) + 2\cos(x) x - \sin(x)$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) + 2\cos(x) x - \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+y(x)=-2*sin(x)+4*x*cos(x),y(x), singsol=all)
```

$$y(x) = (x^2 + c_2 - 1)\sin(x) + 2\cos(x)\left(x + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 32

```
DSolve[y''[x]+y[x]==-2*Sin[x]+4*x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(2x^2 - 1 + 2c_2)\sin(x) + (2x + c_1)\cos(x)$$

10.12 problem 21

10.12.1 Maple step by step solution 1611

Internal problem ID [5391]
Internal file name [`OUTPUT/4882_Sunday_February_04_2024_12_46_46_AM_39250907/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 15. Linear equations with constant coefficients (Variation of parameters). Supplemetary problems. Page 98
Problem number: 21.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\, e^{2x} x + e^{2x}}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' - y'' - 4y' + 4y = 0$$

The characteristic equation is

$$\lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

The roots of the above equation are

$$\lambda_1 = 1$$
$$\lambda_2 = 2$$
$$\lambda_3 = -2$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x + e^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^x \\ y_3 &= e^{2x} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\, e^{2x} x + e^{2x}$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\, e^{2x} x + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}, \{x^2 e^{2x}, e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1, x, x^2\}, \{x^2 e^{2x}, e^{2x} x, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1 + A_4 x^2 e^{2x} + A_5 e^{2x} x + A_6 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &4A_1 - 4A_2 - 2A_3 - 8A_3 x + 10A_4 e^{2x} + 6A_6 e^{2x} + 8A_4 x\, e^{2x} + 12A_6 e^{2x} x^2 \\ &\quad + 30A_6 e^{2x} x + 4A_2 x + 4A_3 x^2 + 4A_5 e^{2x} = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\, e^{2x} x + e^{2x} \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = 0, A_3 = \frac{1}{2}, A_4 = 0, A_5 = 0, A_6 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2} + \frac{e^{2x} x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{x} + e^{2x} c_3\right) + \left(\frac{x^2}{2} + \frac{e^{2x} x^3}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{2x} c_3 + \frac{x^2}{2} + \frac{e^{2x} x^3}{6} \tag{1}$$

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{x} + e^{2x} c_3 + \frac{x^2}{2} + \frac{e^{2x} x^3}{6}$$

Verified OK.

10.12.1 Maple step by step solution

Let's solve

$y''' - y'' - 4y' + 4y = 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\, e^{2x} x + e^{2x}$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$

- Define new variable $y_3(x)$

  $y_3(x) = y''$
- Isolate for $y_3'(x)$ using original ODE

  $y_3'(x) = 2x^2 e^{2x} + 5\,e^{2x}x + 2x^2 + y_3(x) + 4y_2(x) - 4y_1(x) + e^{2x} - 4x - 1$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x)\,, y_3(x) = y_2'(x)\,, y_3'(x) = 2x^2 e^{2x} + 5\,e^{2x}x + 2x^2 + y_3(x) + 4y_2(x) - 4y_1(x) + e^{2x} -$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\,e^{2x}x + e^{2x} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 2x^2 - 4x - 1 + 2x^2 e^{2x} + 5\,e^{2x}x + e^{2x} \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-2x} \cdot \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{y}_p($

$\overrightarrow{y}(x) = c_1\overrightarrow{y}_1 + c_2\overrightarrow{y}_2 + c_3\overrightarrow{y}_3 + \overrightarrow{y}_p(x)$

- ☐ Fundamental matrix
  - Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{x} & e^{2x} \end{bmatrix}$$

  - The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

$\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} \frac{e^{-2x}}{4} & e^{x} & \frac{e^{2x}}{4} \\ -\frac{e^{-2x}}{2} & e^{x} & \frac{e^{2x}}{2} \\ e^{-2x} & e^{x} & e^{2x} \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} -\frac{(3 e^{4x}-8 e^{3x}-1) e^{-2x}}{6} & -\frac{e^{-2x}}{4} + \frac{e^{2x}}{4} & \frac{(3 e^{4x}-4 e^{3x}+1) e^{-2x}}{12} \\ -\frac{(3 e^{4x}-4 e^{3x}+1) e^{-2x}}{3} & \frac{e^{-2x}}{2} + \frac{e^{2x}}{2} & \frac{(3 e^{4x}-2 e^{3x}-1) e^{-2x}}{6} \\ -\frac{2(3 e^{4x}-2 e^{3x}-1) e^{-2x}}{3} & -e^{-2x} + e^{2x} & -\frac{(-3 e^{4x}+e^{3x}-1) e^{-2x}}{3} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

$\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

  - Take the derivative of the particular solution

$\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

  - Substitute particular solution and its derivative into the system of ODEs

$\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x)\cdot \overrightarrow{v}'(x)=\overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x)=\frac{1}{\Phi(x)}\cdot \overrightarrow{f}(x)$

- Integrate to solve for $\overrightarrow{v}(x)$

  $\overrightarrow{v}(x)=\int_0^x \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds$

- Plug $\overrightarrow{v}(x)$ into the equation for the particular solution

  $\overrightarrow{y}_p(x)=\Phi(x)\cdot\left(\int_0^x \frac{1}{\Phi(s)}\cdot \overrightarrow{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{y}_p(x)=\left[\begin{array}{c}\frac{\left(2\,\mathrm{e}^{4x}x^3-3\,\mathrm{e}^{4x}+6x^2\mathrm{e}^{2x}+4\,\mathrm{e}^{3x}-1\right)\mathrm{e}^{-2x}}{12}\\ \frac{\left(2\,\mathrm{e}^{4x}x^3+3\,\mathrm{e}^{4x}x^2-3\,\mathrm{e}^{4x}+2\,\mathrm{e}^{3x}+6\,\mathrm{e}^{2x}x+1\right)\mathrm{e}^{-2x}}{6}\\ \frac{\mathrm{e}^{-2x}\left(\left(2x^3+6x^2+3x-3\right)\mathrm{e}^{4x}+3\,\mathrm{e}^{2x}+\mathrm{e}^{3x}-1\right)}{3}\end{array}\right]$$

- Plug particular solution back into general solution

$$\overrightarrow{y}(x)=c_1\overrightarrow{y}_1+c_2\overrightarrow{y}_2+c_3\overrightarrow{y}_3+\left[\begin{array}{c}\frac{\left(2\,\mathrm{e}^{4x}x^3-3\,\mathrm{e}^{4x}+6x^2\mathrm{e}^{2x}+4\,\mathrm{e}^{3x}-1\right)\mathrm{e}^{-2x}}{12}\\ \frac{\left(2\,\mathrm{e}^{4x}x^3+3\,\mathrm{e}^{4x}x^2-3\,\mathrm{e}^{4x}+2\,\mathrm{e}^{3x}+6\,\mathrm{e}^{2x}x+1\right)\mathrm{e}^{-2x}}{6}\\ \frac{\mathrm{e}^{-2x}\left(\left(2x^3+6x^2+3x-3\right)\mathrm{e}^{4x}+3\,\mathrm{e}^{2x}+\mathrm{e}^{3x}-1\right)}{3}\end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y=\frac{\mathrm{e}^{-2x}\left(\left(2x^3+3c_3-3\right)\mathrm{e}^{4x}+6x^2\mathrm{e}^{2x}+12c_2\mathrm{e}^{3x}+3c_1+4\,\mathrm{e}^{3x}-1\right)}{12}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 42

```
dsolve(diff(y(x),x$3)-diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=2*x^2-4*x-1+2*x^2*exp(2*x)+5*x*ex
```

$$y(x) = \frac{((x^3 + 6c_3)\,\mathrm{e}^{4x} + 3\,\mathrm{e}^{2x}x^2 + 6\,\mathrm{e}^{3x}c_1 + 6c_2)\,\mathrm{e}^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 0.519 (sec). Leaf size: 44

```
DSolve[y'''[x]-y''[x]-4*y'[x]+4*y[x]==2*x^2-4*x-1+2*x^2*Exp[2*x]+5*x*Exp[2*x]+Exp[2*x],y[x],
```

$$y(x) \to \frac{1}{6}\left(e^{2x}x + 3\right)x^2 + c_1e^{-2x} + c_2e^x + c_3e^{2x}$$

11 Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107

11.1 problem 26 . . . 1618
11.2 problem 27 . . . 1634
11.3 problem 28 . . . 1646
11.4 problem 29 . . . 1659
11.5 problem 30 . . . 1668
11.6 problem 31 . . . 1676
11.7 problem 32 . . . 1687
11.8 problem 33 . . . 1698
11.9 problem 34 . . . 1708
11.10 problem 36 . . . 1719
11.11 problem 37 . . . 1730
11.12 problem 38 . . . 1741
11.13 problem 39 . . . 1756
11.14 problem 40 . . . 1767

11.1 problem 26

11.1.1 Solving as second order linear constant coeff ode 1618
11.1.2 Solving using Kovacic algorithm 1624
11.1.3 Maple step by step solution 1631

Internal problem ID [5392]
Internal file name [`OUTPUT/4883_Sunday_February_04_2024_12_46_46_AM_17206234/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = e^{3x} + 6\,e^{x} - 3\,e^{-2x} + 5}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = (e^{5x} + 6\,e^{3x} + 5\,e^{2x} - 3)\,e^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

$$y_2 = \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} & \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) \sqrt{3}\, e^{-\frac{x}{2}}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) \left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right) \sqrt{3}\, e^{-\frac{x}{2}}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}}}{2}\right) - \left(\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}}\right) \left(-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)$$

Which simplifies to

$$W = \frac{e^{-x} \cos\left(\frac{\sqrt{3}\,x}{2}\right)^2 \sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)^2 e^{-x} \sqrt{3}}{2}$$

Which simplifies to

$$W = \frac{\sqrt{3}\, e^{-x}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \left(e^{5x} + 6\, e^{3x} + 5\, e^{2x} - 3\right) e^{-2x}}{\frac{\sqrt{3}\, e^{-x}}{2}} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{2\sqrt{3}\, e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \left(e^{5x} + 6\, e^{3x} + 5\, e^{2x} - 3\right)}{3} dx$$

Hence

$u_1 =$

$$-\frac{2\sqrt{3}\left(\frac{\sqrt{3}\, e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{3\, e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sqrt{3}\, e^{\frac{7x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{26} + \frac{7 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{\frac{7x}{2}}}{26} - \sqrt{3}\, e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3 \sin\left(\right.\right.}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(e^{5x} + 6\,e^{3x} + 5\,e^{2x} - 3\right) e^{-2x}}{\frac{\sqrt{3}\,e^{-x}}{2}} dx$$

Which simplifies to

$$u_2 = \int \frac{2\sqrt{3}\,e^{-\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(e^{5x} + 6\,e^{3x} + 5\,e^{2x} - 3\right)}{3} dx$$

Hence

$$u_2 = \frac{2\sqrt{3}\left(\frac{3\,e^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sqrt{3}\,e^{-\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{7\cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{\frac{7x}{2}}}{26} + \frac{\sqrt{3}\,e^{\frac{7x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{26} + 3\,e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sqrt{3}\,e^{\frac{3x}{2}}\sin\left(\right.\right.}{3}$$

Which simplifies to

$$u_1 = -\frac{5\,e^{-\frac{3x}{2}}\left(\left(-3\,e^{2x} - \frac{6\,e^{3x}}{5} - \frac{3\,e^{5x}}{65} + \frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(e^{2x} + \frac{6\,e^{3x}}{5} + \frac{7\,e^{5x}}{65} + \frac{3}{5}\right)\right)}{3}$$

$$u_2 = \frac{5\,e^{-\frac{3x}{2}}\left(\sqrt{3}\left(e^{2x} + \frac{6\,e^{3x}}{5} + \frac{7\,e^{5x}}{65} + \frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(-\frac{1}{5} + e^{2x} + \frac{2\,e^{3x}}{5} + \frac{e^{5x}}{65}\right)\right)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,e^{-\frac{3x}{2}}\left(\left(-3\,e^{2x} - \frac{6\,e^{3x}}{5} - \frac{3\,e^{5x}}{65} + \frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(e^{2x} + \frac{6\,e^{3x}}{5} + \frac{7\,e^{5x}}{65} + \frac{3}{5}\right)\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3} + \frac{5\,e^{-\frac{3x}{2}}\left(\sqrt{3}\left(e^{2x} + \frac{6\,e^{3x}}{5} + \frac{7\,e^{5x}}{65} + \frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right) + 3\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(-\frac{1}{5} + e^{2x} + \frac{2\,e^{3x}}{5} + \frac{e^{5x}}{65}\right)\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-}}{3}$$

Which simplifies to

$$y_p(x) = \frac{e^{-2x}\left(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13\right)}{13}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) + \left(\frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + \frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13} \tag{1}
$$

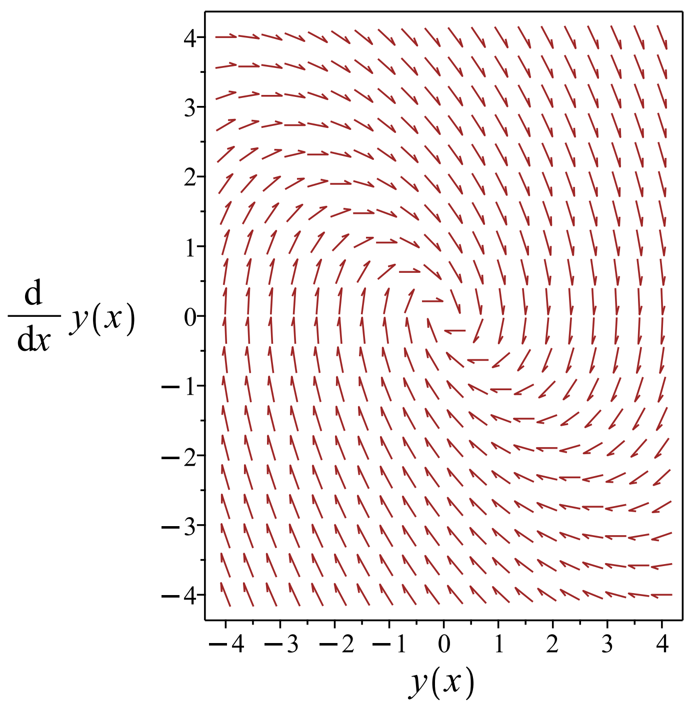


Figure 259: Slope field plot

Verification of solutions

$$
y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) + \frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13}
$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 196: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \frac{2 c_2 \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

$$y_2 = \frac{2\,\mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ \frac{d}{dx}\left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) & \frac{d}{dx}\left(\frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} & -\frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right)\left(-\frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) \\ & - \left(\frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\left(-\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2}\right) \end{aligned}$$

Which simplifies to

$$W = e^{-x} \cos\left(\frac{\sqrt{3}\, x}{2}\right)^2 + e^{-x} \sin\left(\frac{\sqrt{3}\, x}{2}\right)^2$$

Which simplifies to

$$W = e^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \left(e^{5x} + 6\, e^{3x} + 5\, e^{2x} - 3\right) e^{-2x} \sqrt{3}}{3}}{e^{-x}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2\sqrt{3}\, e^{-\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \left(e^{5x} + 6\, e^{3x} + 5\, e^{2x} - 3\right)}{3} dx$$

Hence

$$u_1 = -\frac{2\sqrt{3}\left(\frac{\sqrt{3}\,\mathrm{e}^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{3\,\mathrm{e}^{-\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sqrt{3}\,\mathrm{e}^{\frac{7x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{26}+\frac{7\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{\frac{7x}{2}}}{26}-\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+3\sin\left(\right.\right.}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\mathrm{e}^{5x}+6\,\mathrm{e}^{3x}+5\,\mathrm{e}^{2x}-3\right)\mathrm{e}^{-2x}}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int \mathrm{e}^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\mathrm{e}^{5x}+6\,\mathrm{e}^{3x}+5\,\mathrm{e}^{2x}-3\right)dx$$

Hence

$$u_2 = \frac{3\,\mathrm{e}^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{\sqrt{3}\,\mathrm{e}^{-\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{7\cos\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{\frac{7x}{2}}}{26}+\frac{\sqrt{3}\,\mathrm{e}^{\frac{7x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{26}$$
$$+3\,\mathrm{e}^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)+\frac{5\,\mathrm{e}^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{5\,\mathrm{e}^{\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}$$

Which simplifies to

$$u_1 = -\frac{5\,\mathrm{e}^{-\frac{3x}{2}}\left(\left(-3\,\mathrm{e}^{2x}-\frac{6\,\mathrm{e}^{3x}}{5}-\frac{3\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(\mathrm{e}^{2x}+\frac{6\,\mathrm{e}^{3x}}{5}+\frac{7\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\right)}{3}$$

$$u_2 = \frac{5\left(\left(\mathrm{e}^{2x}+\frac{6\,\mathrm{e}^{3x}}{5}+\frac{7\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(-\frac{1}{5}+\mathrm{e}^{2x}+\frac{2\,\mathrm{e}^{3x}}{5}+\frac{\mathrm{e}^{5x}}{65}\right)\right)\mathrm{e}^{-\frac{3x}{2}}}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,\mathrm{e}^{-\frac{3x}{2}}\left(\left(-3\,\mathrm{e}^{2x}-\frac{6\,\mathrm{e}^{3x}}{5}-\frac{3\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(\mathrm{e}^{2x}+\frac{6\,\mathrm{e}^{3x}}{5}+\frac{7\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\right)\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}}{2}\right.}{3}$$
$$+\frac{5\left(\left(\mathrm{e}^{2x}+\frac{6\,\mathrm{e}^{3x}}{5}+\frac{7\,\mathrm{e}^{5x}}{65}+\frac{3}{5}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\left(-\frac{1}{5}+\mathrm{e}^{2x}+\frac{2\,\mathrm{e}^{3x}}{5}+\frac{\mathrm{e}^{5x}}{65}\right)\right)\mathrm{e}^{-\frac{3x}{2}}\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right.}{3}$$

Which simplifies to

$$y_p(x) = \frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right) + \left(\frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13}\right)$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} + \frac{e^{-2x}(e^{5x} + 65\,e^{2x} + 26\,e^{3x} - 13)}{13} \tag{1}$$

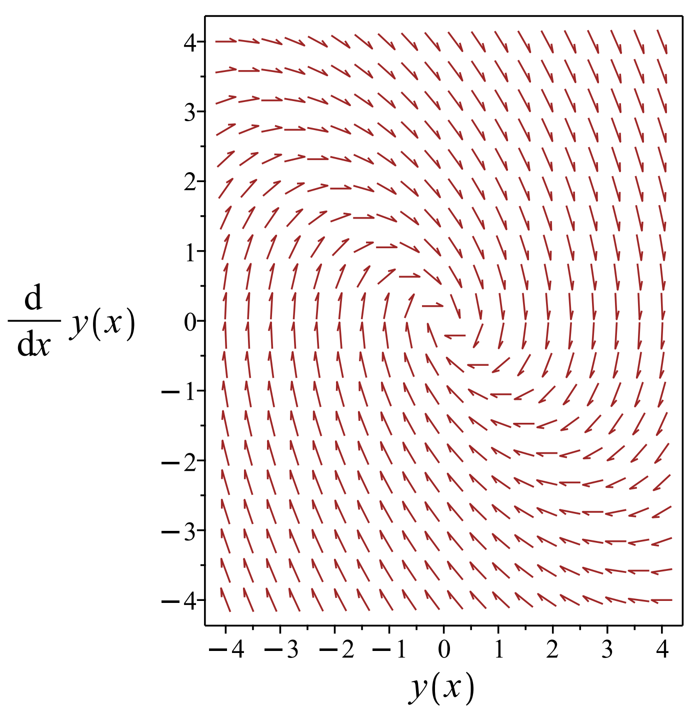


Figure 260: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \frac{2c_2 \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} + \frac{\mathrm{e}^{-2x}(\mathrm{e}^{5x} + 65\, \mathrm{e}^{2x} + 26\, \mathrm{e}^{3x} - 13)}{13}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$y'' + y' + y = (\mathrm{e}^{5x} + 6\, \mathrm{e}^{3x} + 5\, \mathrm{e}^{2x} - 3)\, \mathrm{e}^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\frac{\sqrt{3}\, x}{2}\right) \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 + \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = (\mathrm{e}^{5x} + 6\, \mathrm{e}^{3x} + 5\, \mathrm{e}^{2x} - 3\right.$

  ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & \sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}} \\ -\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}-\frac{e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} & \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}\,e^{-\frac{x}{2}}}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x)\,,y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{-\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)(e^{5x}+6\,e^{3x}+5\,e^{2x}-3)\,dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{-\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)(e^{5x}+6\,e^{3x}+5\,e^{2x}\cdots\right)\right)}{3}$$

- Compute integrals

  $$y_p(x) = \frac{e^{-2x}\left(e^{5x}+65\,e^{2x}+26\,e^{3x}-13\right)}{13}$$

- Substitute particular solution into general solution to ODE

  $$y = e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1+e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_2+\frac{e^{-2x}\left(e^{5x}+65\,e^{2x}+26\,e^{3x}-13\right)}{13}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 55

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=exp(3*x)+6*exp(x)-3*exp(-2*x)+5,y(x), singsol=all)
```

$$y(x) = \left(\frac{e^{5x}}{13}+2\,e^{3x}+\sin\left(\frac{\sqrt{3}\,x}{2}\right)c_2e^{\frac{3x}{2}}+\cos\left(\frac{\sqrt{3}\,x}{2}\right)c_1e^{\frac{3x}{2}}+5\,e^{2x}-1\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 6.996 (sec). Leaf size: 70

```
DSolve[y''[x]+y'[x]+y[x]==Exp[3*x]+6*Exp[x]-3*Exp[-2*x]+5,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -e^{-2x} + 2e^x + \frac{e^{3x}}{13} + c_2 e^{-x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + 5$$

11.2 problem 27

11.2.1 Solving as second order linear constant coeff ode 1634
11.2.2 Solving using Kovacic algorithm 1637
11.2.3 Maple step by step solution 1643

Internal problem ID [5393]
Internal file name [`OUTPUT/4884_Sunday_February_04_2024_12_46_47_AM_72162890/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = \mathrm{e}^{x}}$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since e^x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-x}\right) + \left(\frac{x\,e^x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} + \frac{x\,e^x}{2} \tag{1}$$

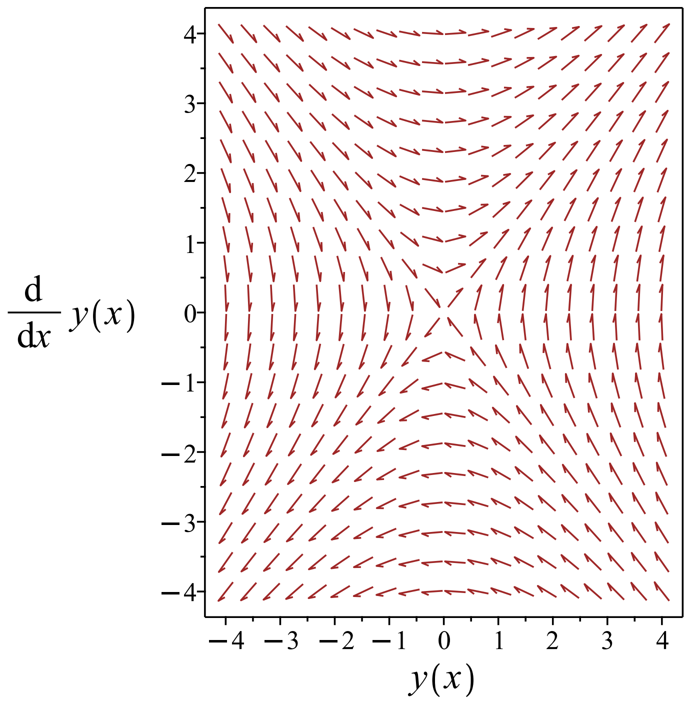


Figure 261: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} + \frac{x\,e^x}{2}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 198: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-x} \int \frac{1}{e^{-2x}}\, dx \\ &= e^{-x}\left(\frac{e^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{-x}\right) + c_2\left(e^{-x}\left(\frac{e^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ -\mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{x}}{2}\right) - \left(\frac{\mathrm{e}^{x}}{2}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}}{2}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}}{2} dx$$

Hence

$$u_1 = -\frac{e^{2x}}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} e^{-x}}{1}\, dx$$

Which simplifies to

$$u_2 = \int 1 dx$$

Hence

$$u_2 = x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-x} e^{2x}}{4} + \frac{x\, e^{x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{(2x-1)\, e^{x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{x}}{2}\right) + \left(\frac{(2x-1)\, e^{x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \frac{(2x-1)\,\mathrm{e}^{x}}{4} \tag{1}$$

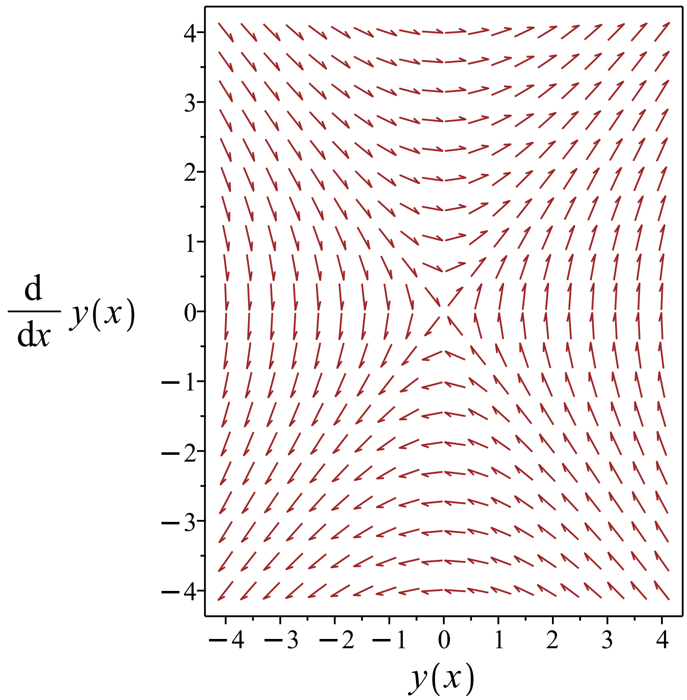


Figure 262: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \frac{(2x-1)\,\mathrm{e}^{x}}{4}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$y'' - y = \mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \mathrm{e}^{x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int \mathrm{e}^{2x}dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int 1dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = \frac{(2x-1)\mathrm{e}^{x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{(2x-1)\mathrm{e}^{x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = c_2 \mathrm{e}^{-x} + \frac{\mathrm{e}^{x}(x + 2c_1)}{2}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 29

```
DSolve[y''[x]-y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}\left(\frac{x}{2} - \frac{1}{4} + c_1\right) + c_2 e^{-x}$$

11.3 problem 28

11.3.1 Solving as second order linear constant coeff ode 1646
11.3.2 Solving as linear second order ode solved by an integrating factor ode 1649
11.3.3 Solving using Kovacic algorithm 1651
11.3.4 Maple step by step solution 1656

Internal problem ID [5394]
Internal file name [`OUTPUT/4885_Sunday_February_04_2024_12_46_48_AM_44622789/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 4y = e^{x} + e^{2x}x}$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = e^{x} + e^{2x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x + e^{2x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x^2 e^{2x}, e^{2x} x\}]$$

Since $e^{2x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{x} + A_2 x^2 e^{2x} + A_3 e^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{x} + 2A_2 e^{2x} + 6A_3 e^{2x} x = e^{x} + e^{2x} x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{x} + \frac{e^{2x} x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + x\, e^{2x} c_2\right) + \left(e^{x} + \frac{e^{2x} x^3}{6}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^{x} + \frac{e^{2x} x^3}{6}$$

Summary
The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^x + \frac{e^{2x} x^3}{6} \tag{1}$$

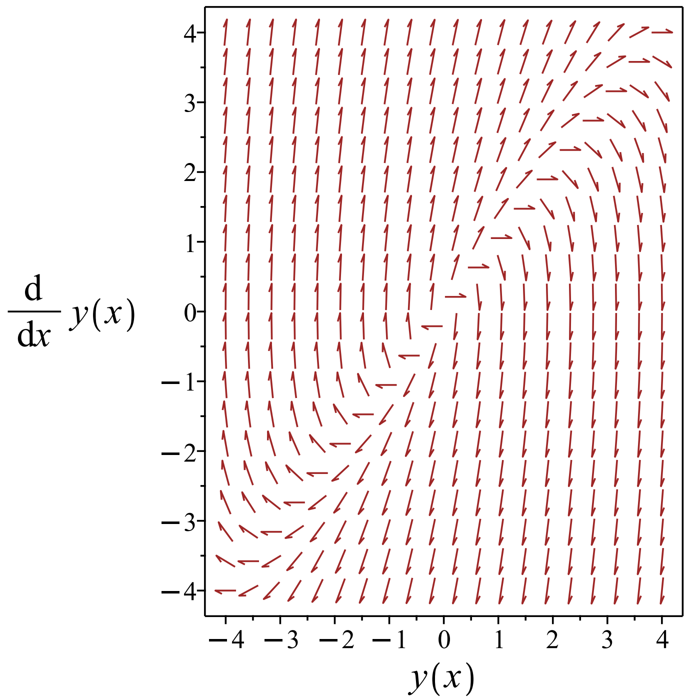


Figure 263: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + e^x + \frac{e^{2x} x^3}{6}$$

Verified OK.

11.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -4\,dx} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-2x}\left(\mathrm{e}^{x} + \mathrm{e}^{2x}x\right) \\
\left(y\,\mathrm{e}^{-2x}\right)'' &= \mathrm{e}^{-2x}\left(\mathrm{e}^{x} + \mathrm{e}^{2x}x\right)
\end{aligned}
$$

Integrating once gives

$$
\left(y\,\mathrm{e}^{-2x}\right)' = \frac{x^2}{2} - \mathrm{e}^{-x} + c_1
$$

Integrating again gives

$$
\left(y\,\mathrm{e}^{-2x}\right) = \frac{x^3}{6} + c_1 x + \mathrm{e}^{-x} + c_2
$$

Hence the solution is

$$
y = \frac{\frac{x^3}{6} + c_1 x + \mathrm{e}^{-x} + c_2}{\mathrm{e}^{-2x}}
$$

Or

$$
y = \frac{\mathrm{e}^{2x}x^3}{6} + c_1 x\,\mathrm{e}^{2x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\mathrm{e}^{2x}x^3}{6} + c_1 x\,\mathrm{e}^{2x} + c_2\mathrm{e}^{2x} + \mathrm{e}^{x} \tag{1}
$$

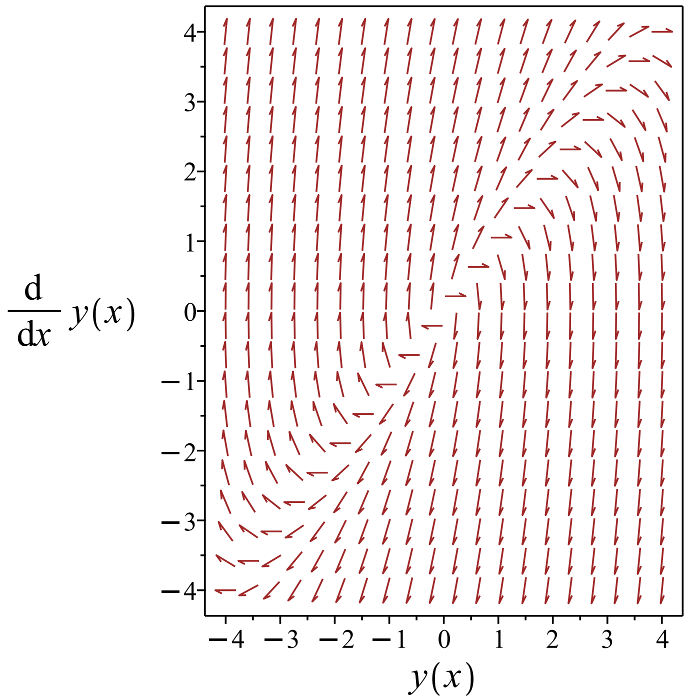


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{e^{2x}x^3}{6} + c_1 x\, e^{2x} + c_2 e^{2x} + e^x$$

Verified OK.

11.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{2x}\right) + c_2\left(e^{2x}(x)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x} + e^{2x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{e^{2x} x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x^2 e^{2x}, e^{2x} x\}]$$

Since $e^{2x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{x}\}, \{x^2 e^{2x}, e^{2x} x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^{x} + A_2 x^2 \mathrm{e}^{2x} + A_3 \mathrm{e}^{2x} x^3$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{x} + 2A_2 \mathrm{e}^{2x} + 6A_3 \mathrm{e}^{2x} x = \mathrm{e}^{x} + \mathrm{e}^{2x} x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = 0, A_3 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x} + \frac{\mathrm{e}^{2x} x^3}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2\right) + \left(\mathrm{e}^{x} + \frac{\mathrm{e}^{2x} x^3}{6}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + \mathrm{e}^{x} + \frac{\mathrm{e}^{2x} x^3}{6}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_2 x + c_1) + \mathrm{e}^{x} + \frac{\mathrm{e}^{2x} x^3}{6} \tag{1}$$

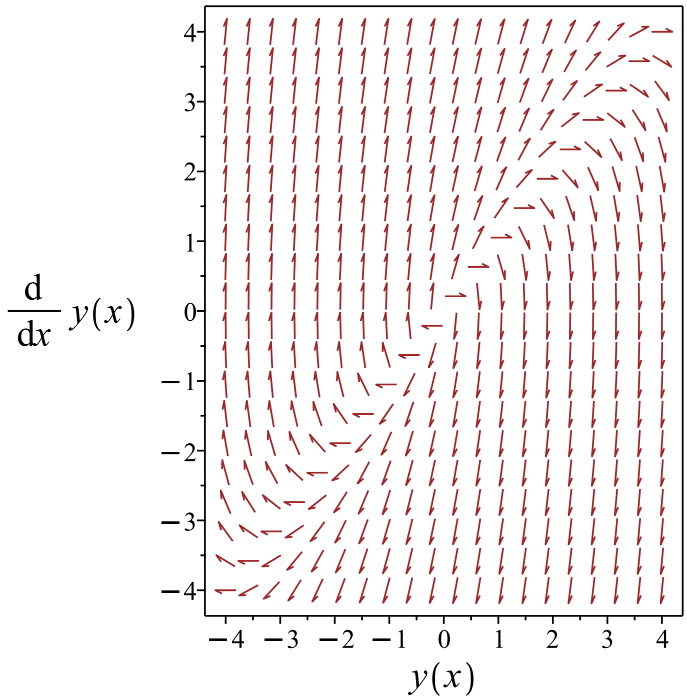


Figure 265: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^x + \frac{e^{2x}x^3}{6}$$

Verified OK.

11.3.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = e^x + e^{2x}x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)^2 = 0$

- Root of the characteristic polynomial

$r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{2x}x$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1e^{2x} + x\,e^{2x}c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = e^x + e^{2x}x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x}x \\ 2\,e^{2x} & 2\,e^{2x}x + e^{2x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = e^{2x}\left(-\left(\int x(x + e^{-x})\,dx\right) + \left(\int (x + e^{-x})\,dx\right)x\right)$

  ◦ Compute integrals

  $y_p(x) = e^x + \frac{e^{2x}x^3}{6}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{2x}c_2 + c_1e^{2x} + e^x + \frac{e^{2x}x^3}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(x)+x*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(x^3 + 6c_1x + 6c_2)\,\mathrm{e}^{2x}}{6} + \mathrm{e}^x$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 31

```
DSolve[y''[x]-4*y'[x]+4*y[x]==Exp[x]+x*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^x\left(6 + e^x\left(x^3 + 6c_2x + 6c_1\right)\right)$$

11.4 problem 29

11.4.1 Maple step by step solution 1661

Internal problem ID [5395]
Internal file name [`OUTPUT/4886_Sunday_February_04_2024_12_46_49_AM_13873342/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 29.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'''' - y = \sin(2x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y'''' - y = 0$$

The characteristic equation is

$$\lambda^4 - 1 = 0$$

The roots of the above equation are

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\\ \lambda_3 &= i\\ \lambda_4 &= -i\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{x} \\ y_3 &= \mathrm{e}^{-ix} \\ y_4 &= \mathrm{e}^{ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y'''' - y = \sin(2x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{ix}, \mathrm{e}^{-x}, \mathrm{e}^{-ix}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$15A_1 \cos(2x) + 15A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{15}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(2x)}{15}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4\right) + \left(\frac{\sin(2x)}{15}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4 + \frac{\sin(2x)}{15} \tag{1}$$

Verification of solutions

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \mathrm{e}^{-ix} c_3 + \mathrm{e}^{ix} c_4 + \frac{\sin(2x)}{15}$$

Verified OK.

11.4.1 Maple step by step solution

Let's solve

$y'''' - y = \sin(2x)$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

$y_4(x) = y'''$

- Isolate for $y_4'(x)$ using original ODE

  $y_4'(x) = \sin(2x) + y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = \sin(2x) + y_1(x)]$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(2x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin(2x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A

- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right], \left[-\mathrm{I}, \left[\begin{array}{c} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{1} = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} -1 \\ 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_{2} = \mathrm{e}^{x} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-\mathrm{I}x} \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \begin{bmatrix} -\sin(x) \\ -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_4(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \vec{y}_p(x)$$

□ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} -e^{-x} & e^{x} & -\sin(x) & -\cos(x) \\ e^{-x} & e^{x} & -\cos(x) & \sin(x) \\ -e^{-x} & e^{x} & \sin(x) & \cos(x) \\ e^{-x} & e^{x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^{x}}{4} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^{x}}{4} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} \\ \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^{x}}{4} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{4} + \frac{e^{x}}{4} + \frac{\sin(x)}{2} & \frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\cos(x)}{2} & -\frac{e^{-x}}{4} + \frac{e^{x}}{4} - \frac{\sin(x)}{2} & \frac{\cos(x)}{2} + \frac{e^{-x}}{4} + \frac{e^{x}}{4} \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

  $\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$
- Take the derivative of the particular solution

  $\vec{y}'_p(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$
- Cancel like terms

$\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \begin{bmatrix} \frac{\sin(2x)}{15} - \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{\sin(x)}{3} \\ \frac{4\cos(x)^2}{15} - \frac{2}{15} + \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{\cos(x)}{3} \\ -\frac{4\sin(2x)}{15} - \frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{\sin(x)}{3} \\ -\frac{16\cos(x)^2}{15} + \frac{8}{15} + \frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{\cos(x)}{3} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2 + c_3\vec{y}_3(x) + c_4\vec{y}_4(x) + \begin{bmatrix} \frac{\sin(2x)}{15} - \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{\sin(x)}{3} \\ \frac{4\cos(x)^2}{15} - \frac{2}{15} + \frac{e^{-x}}{10} + \frac{e^{x}}{10} - \frac{\cos(x)}{3} \\ -\frac{4\sin(2x)}{15} - \frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{\sin(x)}{3} \\ -\frac{16\cos(x)^2}{15} + \frac{8}{15} + \frac{e^{-x}}{10} + \frac{e^{x}}{10} + \frac{\cos(x)}{3} \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = \frac{(-1-10c_1)e^{-x}}{10} + \frac{(-5-15c_3+2\cos(x))\sin(x)}{15} - c_4\cos(x) + \frac{e^{x}(10c_2+1)}{10}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 4; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 4; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$4)-y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)}{15} + \cos(x) c_1 + e^x c_2 + c_3 \sin(x) + c_4 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 37

```
DSolve[y''''[x]-y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_3 e^{-x} + c_4 \sin(x) + \cos(x) \left(\frac{2\sin(x)}{15} + c_2 \right)$$

11.5 problem 30

11.5.1 Maple step by step solution 1670

Internal problem ID [5396]
Internal file name [`OUTPUT/4887_Sunday_February_04_2024_12_46_49_AM_92657304/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y = \cos(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y = 0$$

The characteristic equation is

$$\lambda^3 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$

Now the particular solution to the given ODE is found

$$y''' + y = \cos(x)$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}, e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \sin(x) - A_2 \cos(x) + A_1 \cos(x) + A_2 \sin(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_3\right) + \left(\frac{\cos(x)}{2} - \frac{\sin(x)}{2}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)x} c_3 + \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Verified OK.

11.5.1 Maple step by step solution

Let's solve
$y''' + y = \cos(x)$

- Highest derivative means the order of the ODE is 3
  y'''
- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$
    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$
    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$
    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

$y_3'(x) = \cos(x) - y_1(x)$

Convert linear ODE into a system of first order ODEs

$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \cos(x) - y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right] \cdot \vec{y}(x) + \left[\begin{array}{c} 0 \\ 0 \\ \cos(x) \end{array}\right]$$

- Define the forcing function

$$\vec{f}(x) = \left[\begin{array}{c} 0 \\ 0 \\ \cos(x) \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) - \mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x)=e^{\frac{x}{2}}\cdot\begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)\end{bmatrix}, \vec{y}_3(x)=e^{\frac{x}{2}}\cdot\begin{bmatrix}\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($
  $\vec{y}(x)=c_1\vec{y}_1+c_2\vec{y}_2(x)+c_3\vec{y}_3(x)+\vec{y}_p(x)$

☐ Fundamental matrix

- Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x)=\begin{bmatrix} e^{-x} & e^{\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ -e^{-x} & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{-x} & e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0)=I$ where I is t
  $\Phi(x)=\phi(x)\cdot\frac{1}{\phi(0)}$
- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x)=\begin{bmatrix} e^{-x} & e^{\frac{x}{2}}\left(-\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ -e^{-x} & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}\right) & e^{\frac{x}{2}}\left(\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right) \\ e^{-x} & e^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & -e^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix} 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -1 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x)=\begin{bmatrix} \frac{2\cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{-x}e^{\frac{3x}{2}}}{3}+\frac{e^{-x}}{3} & \frac{\left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)+e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)e^{-x}}{3} & \left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\right. \\ -\frac{\left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)-e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+1\right)e^{-x}}{3} & \frac{2\cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{-x}e^{\frac{3x}{2}}}{3}+\frac{e^{-x}}{3} & \left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\right. \\ -\frac{\left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)+e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)e^{-x}}{3} & -\frac{\left(\sqrt{3}\,e^{\frac{3x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)-e^{\frac{3x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+1\right)e^{-x}}{3} & 2\cos\left(\right.\end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(x)$ and solve for $\vec{v}(x)$

$\vec{y}_p(x) = \Phi(x) \cdot \vec{v}(x)$

- Take the derivative of the particular solution

  $\vec{y}_p'(x) = \Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \vec{v}(x) + \Phi(x) \cdot \vec{v}'(x) = A \cdot \Phi(x) \cdot \vec{v}(x) + \vec{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \vec{v}'(x) = \vec{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(x) = \frac{1}{\Phi(x)} \cdot \vec{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{\left(2\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\cos(x)\mathrm{e}^{x}-3\sin(x)\mathrm{e}^{x}-2\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)\mathrm{e}^{-x}}{6} \\ -\frac{\left(-2\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\cos(x)\mathrm{e}^{x}+3\sin(x)\mathrm{e}^{x}-2\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)\mathrm{e}^{-x}}{6} \\ -\frac{\left(-\frac{4\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{1}{3}+(-\sin(x)+\cos(x))\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{2} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{\left(2\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\cos(x)\mathrm{e}^{x}-3\sin(x)\mathrm{e}^{x}-2\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)\mathrm{e}^{-x}}{6} \\ -\frac{\left(-2\sqrt{3}\,\mathrm{e}^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)+3\cos(x)\mathrm{e}^{x}+3\sin(x)\mathrm{e}^{x}-2\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)-1\right)\mathrm{e}^{-x}}{6} \\ -\frac{\left(-\frac{4\,\mathrm{e}^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+\frac{1}{3}+(-\sin(x)+\cos(x))\mathrm{e}^{x}\right)\mathrm{e}^{-x}}{2} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = -\frac{e^{-x} \left(e^{\frac{3x}{2}} \left(-\sqrt{3}\, c_3+c_2+\frac{2}{3}\right) \cos\left(\frac{\sqrt{3}\, x}{2}\right)-\left(\left(c_2+\frac{2}{3}\right)\sqrt{3}+c_3\right) e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)+(-\cos(x)+\sin(x))e^{x}-2c_1+\frac{1}{3}\right)}{2}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 48

```
dsolve(diff(y(x),x$3)+y(x)=cos(x),y(x), singsol=all)
```

$$y(x) = \left(\frac{(\cos(x)-\sin(x))\,e^{x}}{2}+c_3 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{\frac{3x}{2}}+c_2 \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{\frac{3x}{2}}+c_1\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.515 (sec). Leaf size: 68

```
DSolve[y'''[x]+y[x]==Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sin(x)}{2}+\frac{\cos(x)}{2}+c_1 e^{-x}+c_3 e^{x/2} \cos\left(\frac{\sqrt{3}x}{2}\right)+c_2 e^{x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

11.6 problem 31

11.6.1 Solving as second order linear constant coeff ode 1676
11.6.2 Solving using Kovacic algorithm 1680
11.6.3 Maple step by step solution 1685

Internal problem ID [5397]
Internal file name [`OUTPUT/4888_Sunday_February_04_2024_12_46_49_AM_1617631/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(2x)}$$

11.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(2x)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(-\frac{x\cos(2x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{x\cos(2x)}{4} \tag{1}
$$

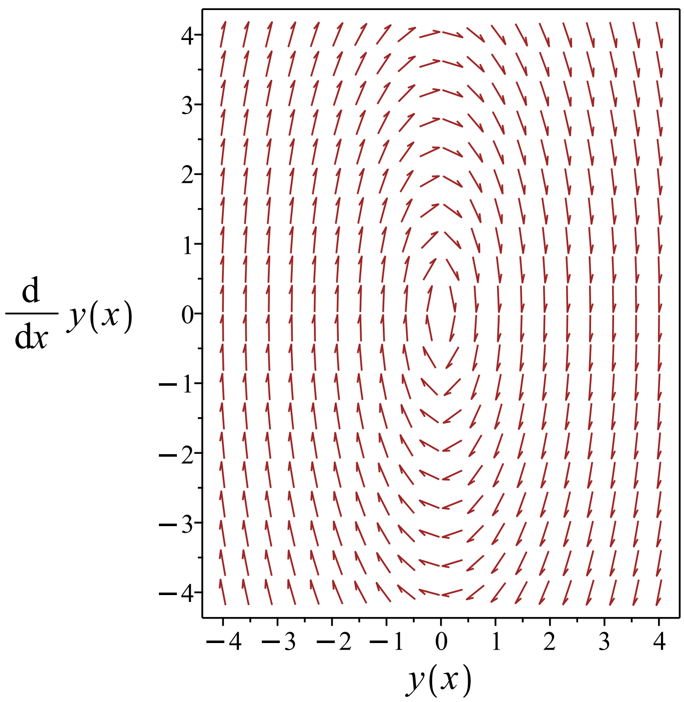


Figure 266: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{x\cos(2x)}{4}
$$

Verified OK.

11.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 204: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 \sin(2x) + 4A_2 \cos(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(2x)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(-\frac{x \cos(2x)}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{x \cos(2x)}{4} \tag{1}
$$

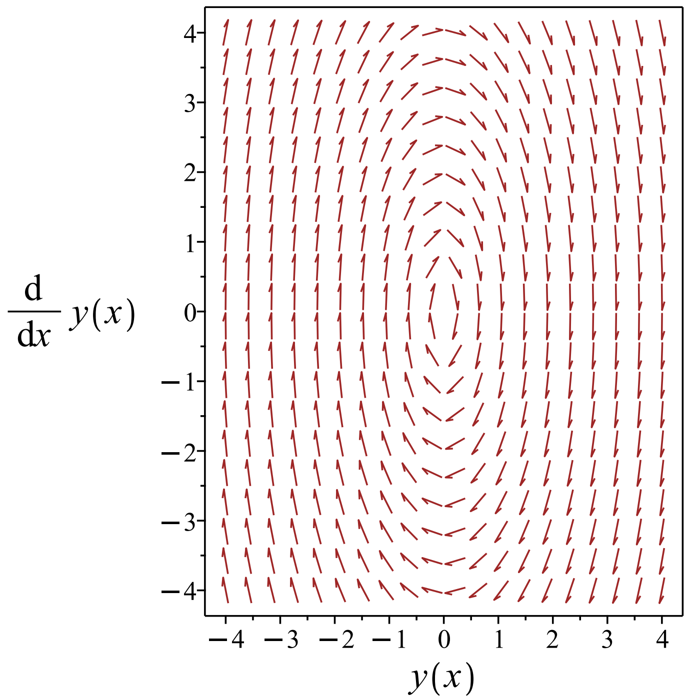


Figure 267: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{x \cos(2x)}{4}
$$

Verified OK.

11.6.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \sin(4x) dx\right)}{4}$$

- Compute integrals

$$y_p(x) = \frac{\sin(2x)}{16} - \frac{x\cos(2x)}{4}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin(2x)}{16} - \frac{x\cos(2x)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-x + 4c_1)\cos(2x)}{4} + \sin(2x) c_2$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 33

```
DSolve[y''[x]+4*y[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x}{4} + c_1\right)\cos(2x) + \frac{1}{8}(1 + 16c_2)\sin(x)\cos(x)$$

11.7 problem 32

11.7.1 Solving as second order linear constant coeff ode 1687
11.7.2 Solving using Kovacic algorithm 1691
11.7.3 Maple step by step solution 1696

Internal problem ID [5398]
Internal file name [`OUTPUT/4889_Sunday_February_04_2024_12_46_50_AM_80123366/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y = \cos\left(x\sqrt{5}\right)}$$

11.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 5, f(x) = \cos\left(x\sqrt{5}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(5)}\\ &= \pm i\sqrt{5}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{5}\\ \lambda_2 &= -i\sqrt{5}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{5}\\ \lambda_2 &= -i\sqrt{5}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right)$$

Or

$$y = c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos\left(x\sqrt{5}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(x\sqrt{5}\right), \sin\left(x\sqrt{5}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(x\sqrt{5}\right), \sin\left(x\sqrt{5}\right)\right\}$$

Since $\cos\left(x\sqrt{5}\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x\cos\left(x\sqrt{5}\right), x\sin\left(x\sqrt{5}\right)\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos\left(x\sqrt{5}\right) + A_2 x\sin\left(x\sqrt{5}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{5}\,\sin\left(x\sqrt{5}\right) + 2A_2\sqrt{5}\,\cos\left(x\sqrt{5}\right) = \cos\left(x\sqrt{5}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{5}}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sqrt{5}\, x \sin\left(x\sqrt{5}\right)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right) + \left(\frac{\sqrt{5}\, x \sin\left(x\sqrt{5}\right)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right) + \frac{\sqrt{5}\, x \sin\left(x\sqrt{5}\right)}{10} \tag{1}$$

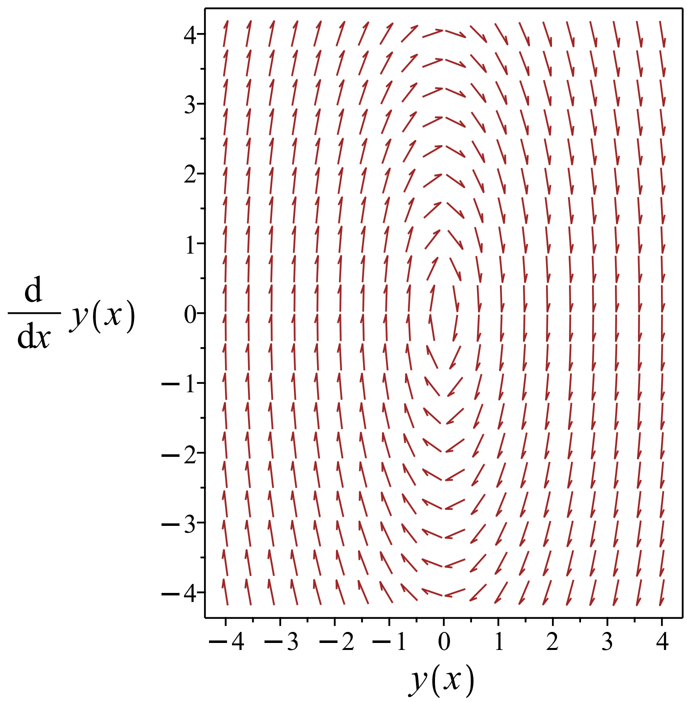


Figure 268: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right) + \frac{\sqrt{5}\, x \sin\left(x\sqrt{5}\right)}{10}$$

Verified OK.

11.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -5z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 206: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -5$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(x\sqrt{5}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos\left(x\sqrt{5}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(x\sqrt{5}\right) \int \frac{1}{\cos\left(x\sqrt{5}\right)^2}\, dx \\
&= \cos\left(x\sqrt{5}\right) \left(\frac{\sqrt{5}\, \tan\left(x\sqrt{5}\right)}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(x\sqrt{5}\right)\right) + c_2 \left(\cos\left(x\sqrt{5}\right) \left(\frac{\sqrt{5}\, \tan\left(x\sqrt{5}\right)}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(x\sqrt{5}\right) + \frac{c_2\sqrt{5}\,\sin\left(x\sqrt{5}\right)}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos\left(x\sqrt{5}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\cos\left(x\sqrt{5}\right), \sin\left(x\sqrt{5}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{5}\,\sin\left(x\sqrt{5}\right)}{5}, \cos\left(x\sqrt{5}\right)\right\}$$

Since $\cos\left(x\sqrt{5}\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$\left[\left\{x\cos\left(x\sqrt{5}\right), x\sin\left(x\sqrt{5}\right)\right\}\right]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos\left(x\sqrt{5}\right) + A_2 x\sin\left(x\sqrt{5}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sqrt{5}\,\sin\left(x\sqrt{5}\right) + 2A_2\sqrt{5}\,\cos\left(x\sqrt{5}\right) = \cos\left(x\sqrt{5}\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{\sqrt{5}}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sqrt{5}\, x \sin \left(x \sqrt{5}\right)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos \left(x \sqrt{5}\right) + \frac{c_2 \sqrt{5}\, \sin \left(x \sqrt{5}\right)}{5}\right) + \left(\frac{\sqrt{5}\, x \sin \left(x \sqrt{5}\right)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos \left(x \sqrt{5}\right) + \frac{c_2 \sqrt{5}\, \sin \left(x \sqrt{5}\right)}{5} + \frac{\sqrt{5}\, x \sin \left(x \sqrt{5}\right)}{10} \tag{1}$$

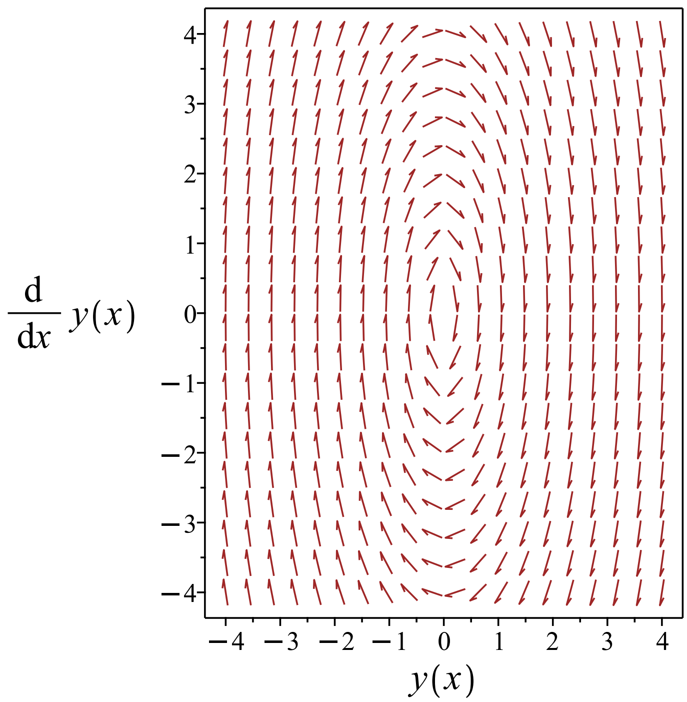


Figure 269: Slope field plot

Verification of solutions

$$y = c_1 \cos \left(x \sqrt{5}\right) + \frac{c_2 \sqrt{5}\, \sin \left(x \sqrt{5}\right)}{5} + \frac{\sqrt{5}\, x \sin \left(x \sqrt{5}\right)}{10}$$

Verified OK.

11.7.3 Maple step by step solution

Let's solve

$y'' + 5y = \cos\left(x\sqrt{5}\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\mathrm{I}\sqrt{5}, \mathrm{I}\sqrt{5}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(x\sqrt{5}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(x\sqrt{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \cos\left(x\sqrt{5}\right)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(x\sqrt{5}\right) & \sin\left(x\sqrt{5}\right) \\ -\sqrt{5}\sin\left(x\sqrt{5}\right) & \sqrt{5}\cos\left(x\sqrt{5}\right) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \sqrt{5}$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\sqrt{5}\left(\cos\left(x\sqrt{5}\right)\left(\int \sin\left(2x\sqrt{5}\right)dx\right)-2\sin\left(x\sqrt{5}\right)\left(\int \cos\left(x\sqrt{5}\right)^2 dx\right)\right)}{10}$$

- Compute integrals

$$y_p(x) = \frac{\sqrt{5}\,x\sin\left(x\sqrt{5}\right)}{10} + \frac{\cos\left(x\sqrt{5}\right)}{20}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos\left(x\sqrt{5}\right) + c_2\sin\left(x\sqrt{5}\right) + \frac{\sqrt{5}\,x\sin\left(x\sqrt{5}\right)}{10} + \frac{\cos\left(x\sqrt{5}\right)}{20}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+5*y(x)=cos(sqrt(5)*x),y(x), singsol=all)
```

$$y(x) = \frac{(10c_1+1)\cos\left(\sqrt{5}\,x\right)}{10} + \frac{\sin\left(\sqrt{5}\,x\right)\left(\sqrt{5}\,x+10c_2\right)}{10}$$

✓ Solution by Mathematica

Time used: 0.16 (sec). Leaf size: 45

```
DSolve[y''[x]+5*y[x]==Cos[Sqrt[5]*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{20}+c_1\right)\cos\left(\sqrt{5}x\right) + \frac{1}{10}\left(\sqrt{5}x+10c_2\right)\sin\left(\sqrt{5}x\right)$$

11.8 problem 33

11.8.1 Maple step by step solution 1702

Internal problem ID [5399]
Internal file name [`OUTPUT/4890_Sunday_February_04_2024_12_46_50_AM_73408246/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 33.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y''' + y'' + y' + y = \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x)}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' + y' + y = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 + \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = i$$
$$\lambda_3 = -i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-ix} c_2 + \mathrm{e}^{ix} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \mathrm{e}^{-ix} \\ y_3 &= \mathrm{e}^{ix} \end{aligned}$$

Now the particular solution to the given ODE is found

$$y''' + y'' + y' + y = \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x)$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x) W_i(x)}{a W(x)} \, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ -\mathrm{e}^{-x} & -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \\ \mathrm{e}^{-x} & -\mathrm{e}^{-ix} & -\mathrm{e}^{ix} \end{bmatrix}$$

$$|W| = 4i \mathrm{e}^{-x} \mathrm{e}^{-ix} \mathrm{e}^{ix}$$

The determinant simplifies to

$$|W| = 4i\mathrm{e}^{-x}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \mathrm{e}^{-ix} & \mathrm{e}^{ix} \\ -i\mathrm{e}^{-ix} & i\mathrm{e}^{ix} \end{bmatrix} \\ &= 2i \end{aligned}$$

$$\begin{aligned} W_2(x) &= \det \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{ix} \\ -\mathrm{e}^{-x} & i\mathrm{e}^{ix} \end{bmatrix} \\ &= (1+i)\,\mathrm{e}^{(-1+i)x} \end{aligned}$$

$$\begin{aligned} W_3(x) &= \det \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{-ix} \\ -\mathrm{e}^{-x} & -i\mathrm{e}^{-ix} \end{bmatrix} \\ &= (1-i)\,\mathrm{e}^{(-1-i)x} \end{aligned}$$

Now we are ready to evaluate each $U_i(x)$.

$$\begin{aligned} U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\,dx \\ &= (-1)^2 \int \frac{(\mathrm{e}^x + \mathrm{e}^{-x} + \sin(x))\,(2i)}{(1)\,(4i\mathrm{e}^{-x})}\,dx \\ &= \int \frac{2i(\mathrm{e}^x + \mathrm{e}^{-x} + \sin(x))}{4i\mathrm{e}^{-x}}\,dx \\ &= \int \left(\frac{\mathrm{e}^{2x}}{2} + \frac{1}{2} + \frac{\sin(x)\,\mathrm{e}^x}{2}\right)\,dx \\ &= \frac{x}{2} - \frac{\cos(x)\,\mathrm{e}^x}{4} + \frac{\sin(x)\,\mathrm{e}^x}{4} + \frac{\mathrm{e}^{2x}}{4} \\ &= \frac{x}{2} - \frac{\cos(x)\,\mathrm{e}^x}{4} + \frac{\sin(x)\,\mathrm{e}^x}{4} + \frac{\mathrm{e}^{2x}}{4} \end{aligned}$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1 \int \frac{(\mathrm{e}^x+\mathrm{e}^{-x}+\sin(x))\left((1+i)\,\mathrm{e}^{(-1+i)x}\right)}{(1)\,(4i\mathrm{e}^{-x})}\,dx \\
&= -\int \frac{(1+i)\,(\mathrm{e}^x+\mathrm{e}^{-x}+\sin(x))\,\mathrm{e}^{(-1+i)x}}{4i\mathrm{e}^{-x}}\,dx \\
&= -\int \left(\left(\frac{1}{4}-\frac{i}{4}\right)\mathrm{e}^{(-1+i)x}\left(1+\mathrm{e}^{2x}+\sin(x)\,\mathrm{e}^x\right)\right)dx \\
&= \frac{\mathrm{e}^{(-1+i)x}}{4}+\frac{i\mathrm{e}^{(1+i)x}}{4}+\frac{\mathrm{e}^{2ix}}{16}-\frac{i\mathrm{e}^{2ix}}{16}-\frac{x}{8}-\frac{ix}{8} \\
&= \frac{\mathrm{e}^{(-1+i)x}}{4}+\frac{i\mathrm{e}^{(1+i)x}}{4}+\frac{\mathrm{e}^{2ix}}{16}-\frac{i\mathrm{e}^{2ix}}{16}-\frac{x}{8}-\frac{ix}{8}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx \\
&= (-1)^0 \int \frac{(\mathrm{e}^x+\mathrm{e}^{-x}+\sin(x))\left((1-i)\,\mathrm{e}^{(-1-i)x}\right)}{(1)\,(4i\mathrm{e}^{-x})}\,dx \\
&= \int \frac{(1-i)\,(\mathrm{e}^x+\mathrm{e}^{-x}+\sin(x))\,\mathrm{e}^{(-1-i)x}}{4i\mathrm{e}^{-x}}\,dx \\
&= \int \left(\left(-\frac{1}{4}-\frac{i}{4}\right)\mathrm{e}^{(-1-i)x}\left(1+\mathrm{e}^{2x}+\sin(x)\,\mathrm{e}^x\right)\right)dx \\
&= \frac{\mathrm{e}^{(-1-i)x}}{4}-\frac{i\mathrm{e}^{(1-i)x}}{4}-\frac{x}{8}+\frac{ix}{8}+\frac{\mathrm{e}^{-2ix}}{16}+\frac{i\mathrm{e}^{-2ix}}{16} \\
&= \frac{\mathrm{e}^{(-1-i)x}}{4}-\frac{i\mathrm{e}^{(1-i)x}}{4}-\frac{x}{8}+\frac{ix}{8}+\frac{\mathrm{e}^{-2ix}}{16}+\frac{i\mathrm{e}^{-2ix}}{16}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x}{2}-\frac{\cos(x)\,\mathrm{e}^x}{4}+\frac{\sin(x)\,\mathrm{e}^x}{4}+\frac{\mathrm{e}^{2x}}{4}\right)(\mathrm{e}^{-x}) \\
&+\left(\frac{\mathrm{e}^{(-1+i)x}}{4}+\frac{i\mathrm{e}^{(1+i)x}}{4}+\frac{\mathrm{e}^{2ix}}{16}-\frac{i\mathrm{e}^{2ix}}{16}-\frac{x}{8}-\frac{ix}{8}\right)(\mathrm{e}^{-ix}) \\
&+\left(\frac{\mathrm{e}^{(-1-i)x}}{4}-\frac{i\mathrm{e}^{(1-i)x}}{4}-\frac{x}{8}+\frac{ix}{8}+\frac{\mathrm{e}^{-2ix}}{16}+\frac{i\mathrm{e}^{-2ix}}{16}\right)(\mathrm{e}^{ix})
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{(4x+4)\,\mathrm{e}^{-x}}{8}+\frac{(-1-2x)\cos(x)}{8}+\frac{(-2x+3)\sin(x)}{8}+\frac{\mathrm{e}^x}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + \mathrm{e}^{-ix} c_2 + \mathrm{e}^{ix} c_3\right) + \left(\frac{(4x+4)\,\mathrm{e}^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{\mathrm{e}^{x}}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-ix} c_2 + \mathrm{e}^{ix} c_3 + \frac{(4x+4)\,\mathrm{e}^{-x}}{8} \\
+ \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{\mathrm{e}^{x}}{4}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \mathrm{e}^{-ix} c_2 + \mathrm{e}^{ix} c_3 + \frac{(4x+4)\,\mathrm{e}^{-x}}{8} + \frac{(-1-2x)\cos(x)}{8} + \frac{(-2x+3)\sin(x)}{8} + \frac{\mathrm{e}^{x}}{4}
$$

Verified OK.

11.8.1 Maple step by step solution

Let's solve

$y''' + y'' + y' + y = \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x)$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x) - y_3(x) - y_2(x) - y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x) - y_3(x) - y_2(x) - y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} \cdot \vec{y}(x) + \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x) \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(x) = \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{x} + \mathrm{e}^{-x} + \sin(x) \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}, \begin{bmatrix} -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-\mathrm{I}, \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-\mathrm{I}x} \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(x) - \mathrm{I}\sin(x)) \cdot \begin{bmatrix} -1 \\ \mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\cos(x) + \mathrm{I}\sin(x) \\ \mathrm{I}(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = \begin{bmatrix} -\cos(x) \\ \sin(x) \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = \begin{bmatrix} \sin(x) \\ \cos(x) \\ -\sin(x) \end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{y}_p($

$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \vec{y}_p(x)$

☐ Fundamental matrix

  ○ Let $\phi(x)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix}$$

- The fundamental matrix, $\Phi(x)$ is a normalized version of $\phi(x)$ satisfying $\Phi(0) = I$ where I is t

  $\Phi(x) = \phi(x) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(x)$ and $\phi(0)$

$$\Phi(x) = \begin{bmatrix} e^{-x} & -\cos(x) & \sin(x) \\ -e^{-x} & \sin(x) & \cos(x) \\ e^{-x} & \cos(x) & -\sin(x) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(x) = \begin{bmatrix} \frac{e^{-x}}{2} + \frac{\cos(x)}{2} + \frac{\sin(x)}{2} & \sin(x) & \frac{e^{-x}}{2} - \frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ -\frac{e^{-x}}{2} - \frac{\sin(x)}{2} + \frac{\cos(x)}{2} & \cos(x) & -\frac{e^{-x}}{2} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ \frac{e^{-x}}{2} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2} & -\sin(x) & \frac{e^{-x}}{2} + \frac{\cos(x)}{2} - \frac{\sin(x)}{2} \end{bmatrix}$$

□ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(x)$ and solve for $\overrightarrow{v}(x)$

  $\overrightarrow{y}_p(x) = \Phi(x) \cdot \overrightarrow{v}(x)$

- Take the derivative of the particular solution

  $\overrightarrow{y}_p'(x) = \Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x)$

- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \Phi(x) \cdot \overrightarrow{v}'(x) = A \cdot \Phi(x) \cdot \overrightarrow{v}(x) + \overrightarrow{f}(x)$

- Cancel like terms

  $\Phi(x) \cdot \overrightarrow{v}'(x) = \overrightarrow{f}(x)$

- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}'(x) = \frac{1}{\Phi(x)} \cdot \overrightarrow{f}(x)$

- Integrate to solve for $\vec{v}(x)$

  $\vec{v}(x) = \int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(x)$ into the equation for the particular solution

  $\vec{y}_p(x) = \Phi(x) \cdot \left(\int_0^x \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{y}_p(x) = \left[\begin{array}{c} \frac{(2+2x)\,\mathrm{e}^{-x}}{4} + \frac{(-x-3)\cos(x)}{4} - \frac{\sin(x)x}{4} + \frac{\mathrm{e}^{x}}{4} \\ -\frac{x\,\mathrm{e}^{-x}}{2} + \frac{(-x-1)\cos(x)}{4} + \frac{(2+x)\sin(x)}{4} + \frac{\mathrm{e}^{x}}{4} \\ \frac{(2x-2)\,\mathrm{e}^{-x}}{4} + \frac{(x+1)\cos(x)}{4} + \frac{(2+x)\sin(x)}{4} + \frac{\mathrm{e}^{x}}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{y}(x) = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + \left[\begin{array}{c} \frac{(2+2x)\,\mathrm{e}^{-x}}{4} + \frac{(-x-3)\cos(x)}{4} - \frac{\sin(x)x}{4} + \frac{\mathrm{e}^{x}}{4} \\ -\frac{x\,\mathrm{e}^{-x}}{2} + \frac{(-x-1)\cos(x)}{4} + \frac{(2+x)\sin(x)}{4} + \frac{\mathrm{e}^{x}}{4} \\ \frac{(2x-2)\,\mathrm{e}^{-x}}{4} + \frac{(x+1)\cos(x)}{4} + \frac{(2+x)\sin(x)}{4} + \frac{\mathrm{e}^{x}}{4} \end{array}\right]$$

- First component of the vector is the solution to the ODE

$$y = \frac{(2x+4c_1+2)\,\mathrm{e}^{-x}}{4} + \frac{(-x-4c_2-3)\cos(x)}{4} + \frac{(-x+4c_3)\sin(x)}{4} + \frac{\mathrm{e}^{x}}{4}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 46

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)+diff(y(x),x)+y(x)=exp(x)+exp(-x)+sin(x),y(x), singsol=a
```

$$y(x) = \frac{(2x + 4c_3 + 2)\,\mathrm{e}^{-x}}{4} + \frac{(-x + 4c_2 + 1)\sin(x)}{4} + \frac{(-x + 4c_1)\cos(x)}{4} + \frac{\mathrm{e}^{x}}{4}$$

✓ Solution by Mathematica

Time used: 0.265 (sec). Leaf size: 55

```
DSolve[y'''[x]+y''[x]+y'[x]+y[x]==Exp[x]+Exp[-x]+Sin[x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{8}\left(2e^{-x}\left(2x + e^{2x} + 2 + 4c_3\right) + (-2x - 1 + 8c_1)\cos(x) + (-2x + 3 + 8c_2)\sin(x)\right)$$

11.9 problem 34

11.9.1 Solving as second order linear constant coeff ode 1708
11.9.2 Solving using Kovacic algorithm 1711
11.9.3 Maple step by step solution 1716

Internal problem ID [5400]
Internal file name [`OUTPUT/4891_Sunday_February_04_2024_12_46_51_AM_49659287/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y = x^2}$$

11.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-x}\right) + \left(-x^2 - 2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-x} - x^2 - 2 \tag{1}$$

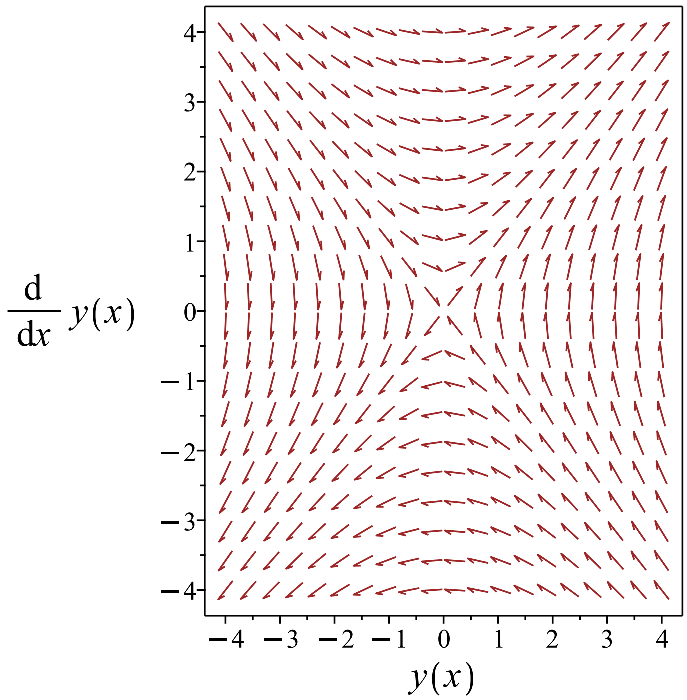


Figure 270: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 e^{-x} - x^2 - 2$$

Verified OK.

11.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 209: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_3x^2 - A_2x - A_1 + 2A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 - 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^x}{2}\right) + \left(-x^2 - 2\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2} - x^2 - 2 \tag{1}$$

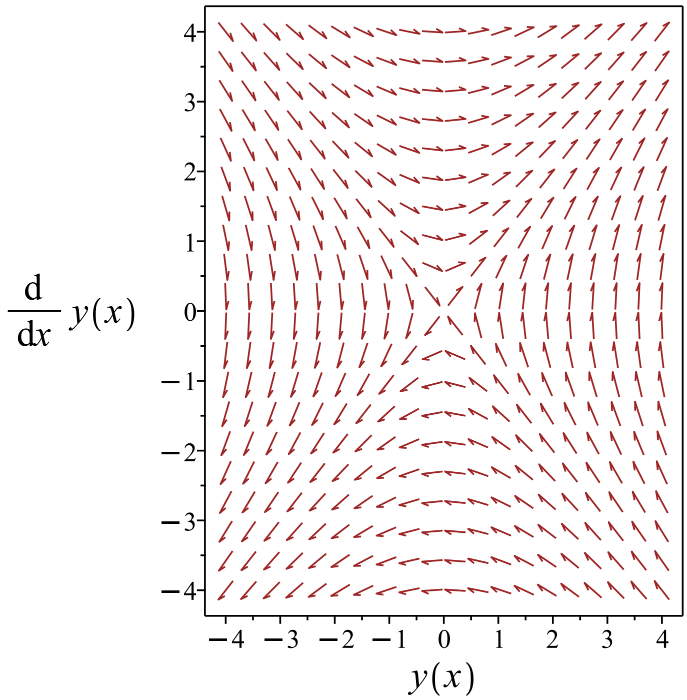


Figure 271: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} - x^2 - 2$$

Verified OK.

11.9.3 Maple step by step solution

Let's solve

$y'' - y = x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^x + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = x^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^x \\ -e^{-x} & e^x \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-x}\left(\int x^2 e^x dx\right)}{2} + \frac{e^x\left(\int x^2 e^{-x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -x^2 - 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^x - x^2 - 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-y(x)=x^2,y(x), singsol=all)
```

$$y(x) = c_2 \mathrm{e}^{-x} + \mathrm{e}^{x} c_1 - x^2 - 2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[y''[x]-y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x^2 + c_1 e^x + c_2 e^{-x} - 2$$

11.10 problem 36

11.10.1 Solving as second order linear constant coeff ode 1719
11.10.2 Solving using Kovacic algorithm 1723
11.10.3 Maple step by step solution 1728

Internal problem ID [5401]
Internal file name [`OUTPUT/4892_Sunday_February_04_2024_12_46_52_AM_3321707/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y = x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)}$$

11.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\lambda_1 = +i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Which simplifies to

$$\lambda_1 = i\sqrt{2}$$
$$\lambda_2 = -i\sqrt{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)$$

Or

$$y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}, \{\cos(3x), \sin(3x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(x\sqrt{2}\right), \sin\left(x\sqrt{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-2x} + A_2 \cos(3x) + A_3 \sin(3x) + A_4 + A_5 x + A_6 x^2 + A_7 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&6A_1\mathrm{e}^{-2x} - 7A_2 \cos(3x) - 7A_3 \sin(3x) + 2A_6 + 6A_7 x + 2A_4 + 2A_5 x + 2A_6 x^2 + 2A_7 x^3 \\ &\quad = x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = -\frac{1}{7}, A_3 = 0, A_4 = -\frac{1}{2}, A_5 = -\frac{3}{2}, A_6 = \frac{1}{2}, A_7 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right) + \left(\frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \tag{1}
$$

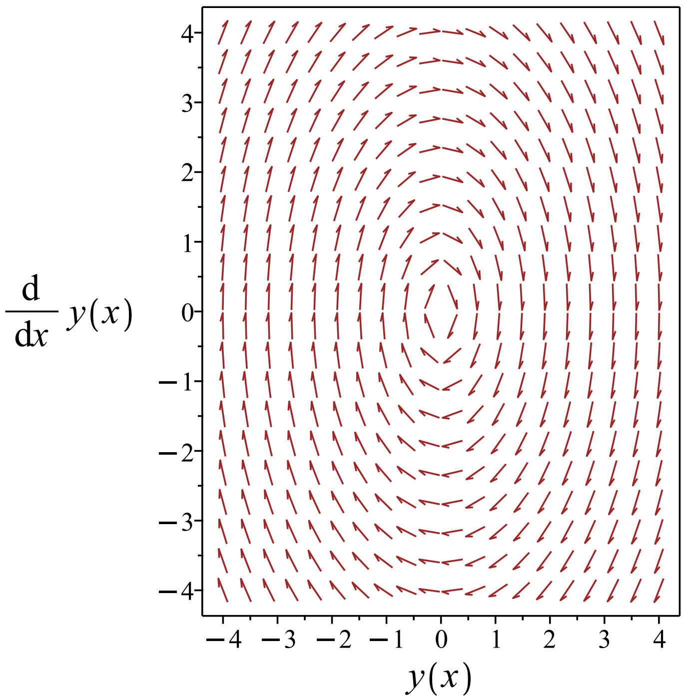


Figure 272: Slope field plot

Verification of solutions

$$
y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}
$$

Verified OK.

11.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 211: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B=0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1\\
&= \cos\left(x\sqrt{2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(x\sqrt{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B=0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx\\
&= \cos\left(x\sqrt{2}\right) \int \frac{1}{\cos\left(x\sqrt{2}\right)^2}\, dx\\
&= \cos\left(x\sqrt{2}\right)\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1\left(\cos\left(x\sqrt{2}\right)\right) + c_2\left(\cos\left(x\sqrt{2}\right)\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}, \{\cos(3x), \sin(3x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}, \cos\left(x\sqrt{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-2x} + A_2\cos(3x) + A_3\sin(3x) + A_4 + A_5x + A_6x^2 + A_7x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&6A_1\mathrm{e}^{-2x} - 7A_2\cos(3x) - 7A_3\sin(3x) + 2A_6 + 6A_7x + 2A_4 + 2A_5x + 2A_6x^2 + 2A_7x^3\\&\quad= x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = -\frac{1}{7}, A_3 = 0, A_4 = -\frac{1}{2}, A_5 = -\frac{3}{2}, A_6 = \frac{1}{2}, A_7 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\mathrm{e}^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}\right) + \left(\frac{\mathrm{e}^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2} + \frac{\mathrm{e}^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2} \tag{1}
$$

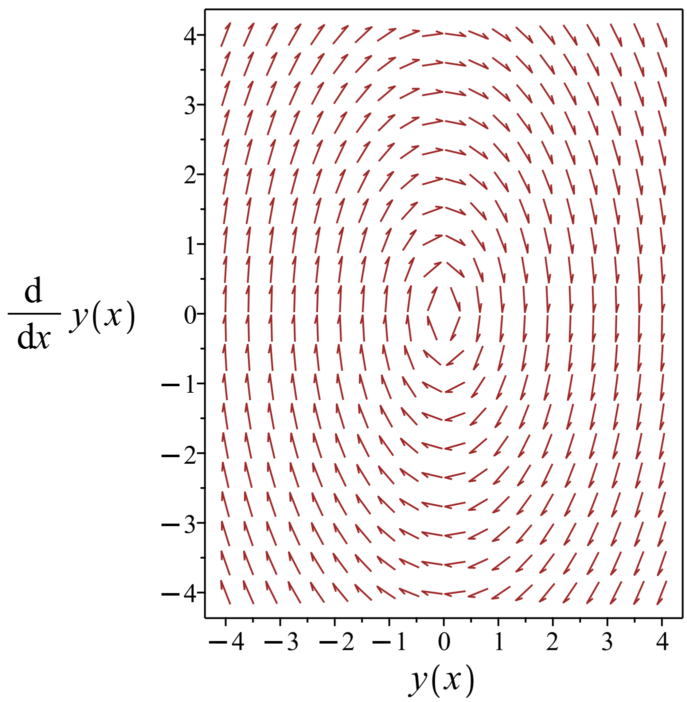


Figure 273: Slope field plot

Verification of solutions

$$
y = c_1 \cos\left(x\sqrt{2}\right) + \frac{c_2\sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2} + \frac{\mathrm{e}^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}
$$

Verified OK.

11.10.3 Maple step by step solution

Let's solve

$y'' + 2y = x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x\sqrt{2})$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x\sqrt{2})$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x^3 + x^2 + \mathrm{e}^{-2x} + \cos(3\right.$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x\sqrt{2}) & \sin(x\sqrt{2}) \\ -\sqrt{2}\sin(x\sqrt{2}) & \sqrt{2}\cos(x\sqrt{2}) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \sqrt{2}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\sqrt{2}\left(\cos\left(x\sqrt{2}\right)\left(\int \sin\left(x\sqrt{2}\right)(x^3+x^2+e^{-2x}+\cos(3x))dx\right)-\sin\left(x\sqrt{2}\right)\left(\int \cos\left(x\sqrt{2}\right)(x^3+x^2+e^{-2x}+\cos(3x))dx\right)\right)}{2}$$

- Compute integrals

  $y_p(x) = \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) + \frac{e^{-2x}}{6} - \frac{\cos(3x)}{7} - \frac{1}{2} - \frac{3x}{2} + \frac{x^2}{2} + \frac{x^3}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x$2)+2*y(x)=x^3+x^2+exp(-2*x)+cos(3*x),y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{2}\,x\right)c_2 + \cos\left(\sqrt{2}\,x\right)c_1 + \frac{x^3}{2} + \frac{x^2}{2} - \frac{3x}{2} - \frac{\cos(3x)}{7} - \frac{1}{2} + \frac{e^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 4.775 (sec). Leaf size: 69

```
DSolve[y''[x]+2*y[x]==x^3+x^2+Exp[-2*x]+Cos[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{42}\Big(21x^3 + 21x^2 - 63x + 7e^{-2x} + 9\sin(x)\sin(2x) - 6\cos^3(x) + 42c_1\cos\left(\sqrt{2}x\right) + 42c_2\sin\left(\sqrt{2}x\right) - 21\Big)$$

11.11 problem 37

11.11.1 Solving as second order linear constant coeff ode 1730
11.11.2 Solving using Kovacic algorithm 1734
11.11.3 Maple step by step solution 1739

Internal problem ID [5402]
Internal file name [`OUTPUT/4893_Sunday_February_04_2024_12_46_52_AM_84754092/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' - y = \cos(x)\,\mathrm{e}^{x}}$$

11.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -1, f(x) = \cos(x)\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-1)}\\ &= 1 \pm \sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + \sqrt{2}\\ \lambda_2 &= 1 - \sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + \sqrt{2}\\ \lambda_2 &= 1 - \sqrt{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(1+\sqrt{2}\right)x} + c_2 e^{\left(1-\sqrt{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\left(1+\sqrt{2}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{2}\right)x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{\left(1+\sqrt{2}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{2}\right)x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(x)\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x)\,\mathrm{e}^{x}, \sin(x)\,\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{\left(1-\sqrt{2}\right)x}, \mathrm{e}^{\left(1+\sqrt{2}\right)x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x)\,\mathrm{e}^{x} + A_2 \sin(x)\,\mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-3A_1 \cos(x)\,\mathrm{e}^{x} - 3A_2 \sin(x)\,\mathrm{e}^{x} = \cos(x)\,\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{3}, A_2 = 0\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -\frac{\cos(x)\,\mathrm{e}^{x}}{3}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{\left(1+\sqrt{2}\right)x} + c_2 \mathrm{e}^{\left(1-\sqrt{2}\right)x}\right) + \left(-\frac{\cos(x)\,\mathrm{e}^{x}}{3}\right)
\end{aligned}
$$

Which simplifies to

$$y = c_1 e^{\left(1+\sqrt{2}\right)x} + c_2 e^{-\left(\sqrt{2}-1\right)x} - \frac{\cos\left(x\right) e^{x}}{3}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\left(1+\sqrt{2}\right)x} + c_2 e^{-\left(\sqrt{2}-1\right)x} - \frac{\cos\left(x\right) e^{x}}{3} \tag{1}$$

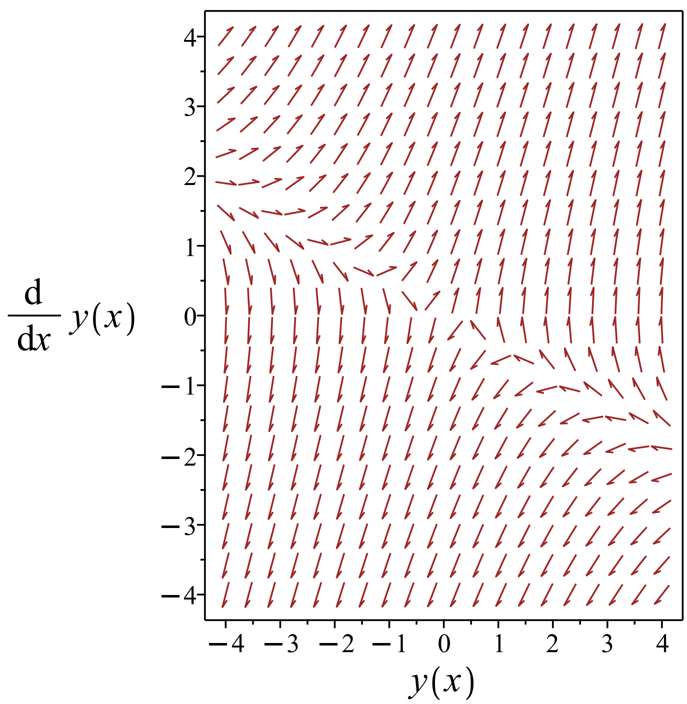


Figure 274: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(1+\sqrt{2}\right)x} + c_2 e^{-\left(\sqrt{2}-1\right)x} - \frac{\cos\left(x\right) e^{x}}{3}$$

Verified OK.

11.11.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' - 2y' - y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= -2 && (3)\\
C &= -1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} && (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 2z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 213: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x\sqrt{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\left(\sqrt{2}-1\right)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\left(\sqrt{2}-1\right)x}\right) + c_2\left(\mathrm{e}^{-\left(\sqrt{2}-1\right)x}\left(\frac{\sqrt{2}\,\mathrm{e}^{2x\sqrt{2}}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\left(\sqrt{2}-1\right)x} + \frac{c_2\sqrt{2}\,\mathrm{e}^{\left(1+\sqrt{2}\right)x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos\left(x\right)\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(x\right)\mathrm{e}^{x}, \sin\left(x\right)\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\sqrt{2}\,\mathrm{e}^{\left(1+\sqrt{2}\right)x}}{4}, \mathrm{e}^{-\left(\sqrt{2}-1\right)x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos\left(x\right)\mathrm{e}^{x} + A_2\sin\left(x\right)\mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-3A_1\cos\left(x\right)\mathrm{e}^{x} - 3A_2\sin\left(x\right)\mathrm{e}^{x} = \cos\left(x\right)\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{1}{3}, A_2 = 0\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -\frac{\cos\left(x\right)\mathrm{e}^{x}}{3}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-\left(\sqrt{2}-1\right)x} + \frac{c_2\sqrt{2}\,\mathrm{e}^{\left(1+\sqrt{2}\right)x}}{4}\right) + \left(-\frac{\cos(x)\,\mathrm{e}^{x}}{3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{2}-1\right)x} + \frac{c_2\sqrt{2}\,\mathrm{e}^{\left(1+\sqrt{2}\right)x}}{4} - \frac{\cos(x)\,\mathrm{e}^{x}}{3} \tag{1}
$$

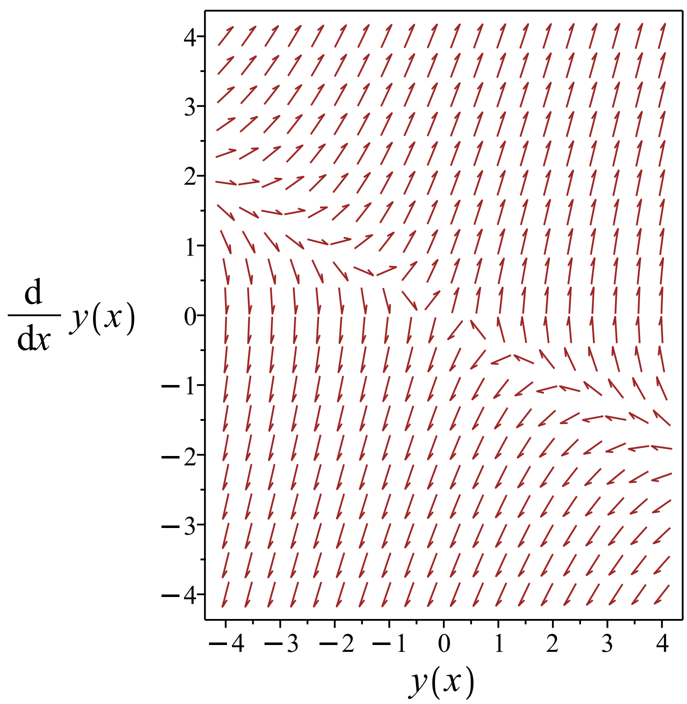


Figure 275: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\left(\sqrt{2}-1\right)x} + \frac{c_2\sqrt{2}\,\mathrm{e}^{\left(1+\sqrt{2}\right)x}}{4} - \frac{\cos(x)\,\mathrm{e}^{x}}{3}
$$

Verified OK.

11.11.3 Maple step by step solution

Let's solve

$y'' - 2y' - y = \cos(x)\, e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm \left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \sqrt{2}, 1 + \sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{\left(1-\sqrt{2}\right)x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{\left(1+\sqrt{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{\left(1-\sqrt{2}\right)x} + c_2 e^{\left(1+\sqrt{2}\right)x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(x)\, e^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\left(1-\sqrt{2}\right)x} & e^{\left(1+\sqrt{2}\right)x} \\ \left(1-\sqrt{2}\right) e^{\left(1-\sqrt{2}\right)x} & \left(1+\sqrt{2}\right) e^{\left(1+\sqrt{2}\right)x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\sqrt{2}\, e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\sqrt{2}\left(\mathrm{e}^{\left(1+\sqrt{2}\right)x}\left(\int \cos(x)\mathrm{e}^{-x\sqrt{2}}dx\right)-\mathrm{e}^{-\left(\sqrt{2}-1\right)x}\left(\int \cos(x)\mathrm{e}^{x\sqrt{2}}dx\right)\right)}{4}$

- Compute integrals

  $y_p(x) = -\frac{\cos(x)\mathrm{e}^{x}}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{\left(1-\sqrt{2}\right)x} + c_2\mathrm{e}^{\left(1+\sqrt{2}\right)x} - \frac{\cos(x)\mathrm{e}^{x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-y(x)=exp(x)*cos(x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\left(1+\sqrt{2}\right)x}c_2 + \mathrm{e}^{-\left(\sqrt{2}-1\right)x}c_1 - \frac{\mathrm{e}^{x}\cos(x)}{3}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 56

```
DSolve[y''[x]-2*y'[x]-y[x]==Exp[x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{-\sqrt{2}x}\left(-e^{\left(1+\sqrt{2}\right)x}\cos(x) + 3e^{x}\left(c_2e^{2\sqrt{2}x} + c_1\right)\right)$$

11.12 problem 38

11.12.1 Solving as second order linear constant coeff ode 1741
11.12.2 Solving as linear second order ode solved by an integrating factor ode 1745
11.12.3 Solving using Kovacic algorithm 1747
11.12.4 Maple step by step solution 1753

Internal problem ID [5403]
Internal file name [`OUTPUT/4894_Sunday_February_04_2024_12_46_53_AM_6482310/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - 4y' + 4y = \frac{e^{2x}}{x^2}}$$

11.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = \frac{e^{2x}}{x^2}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + x\, \mathrm{e}^{2x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^{2x}\\ y_2 &= \mathrm{e}^{2x} x\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{2x}x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ 2\,e^{2x} & 2\,e^{2x}x + e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})\,(2\,e^{2x}x + e^{2x}) - (e^{2x}x)\,(2\,e^{2x})$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{4x}}{x}}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x} dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{4x}}{x^2}}{e^{4x}} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x) e^{2x} - e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + x e^{2x} c_2\right) + \left(e^{2x}(-1 - \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x))$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x)) \tag{1}$$

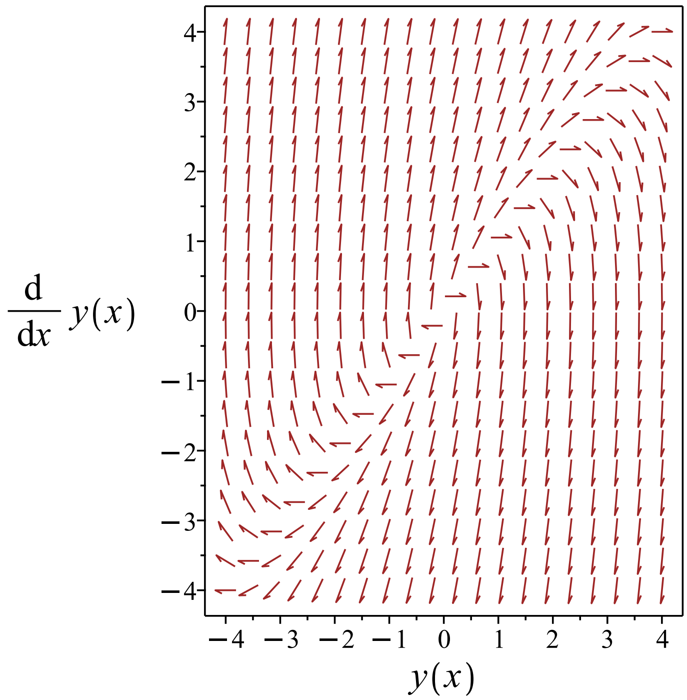


Figure 276: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + e^{2x}(-1 - \ln(x))$$

Verified OK.

11.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \frac{\mathrm{e}^{-2x}\mathrm{e}^{2x}}{x^2}$$
$$\left(y\,\mathrm{e}^{-2x}\right)'' = \frac{\mathrm{e}^{-2x}\mathrm{e}^{2x}}{x^2}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{-2x}\right)' = c_1 - \frac{1}{x}$$

Integrating again gives

$$\left(y\,\mathrm{e}^{-2x}\right) = c_1 x - \ln(x) + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \ln(x) + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} - \ln(x)\,\mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} - \ln(x)\,\mathrm{e}^{2x} \tag{1}$$

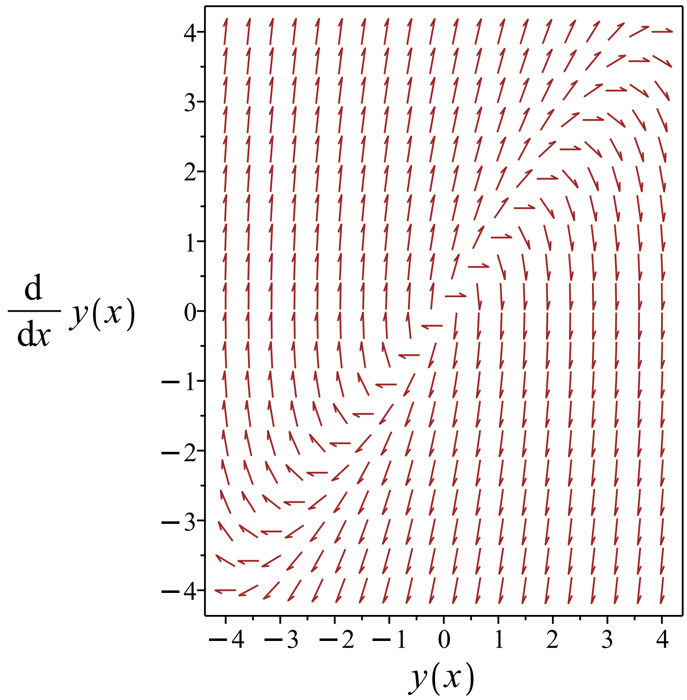


Figure 277: Slope field plot

Verification of solutions

$$
y = c_1 x \,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} - \ln(x)\,\mathrm{e}^{2x}
$$

Verified OK.

11.12.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - 4y' + 4y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= -4 \tag{3} \\
C &= 4
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1\left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{2x} \\
y_2 &= \mathrm{e}^{2x} x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{2x}x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{2x}x \\ 2\,e^{2x} & 2\,e^{2x}x + e^{2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})\left(2\,e^{2x}x + e^{2x}\right) - (e^{2x}x)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{4x}$$

Which simplifies to

$$W = e^{4x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{4x}}{x}}{e^{4x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{x}dx$$

Hence

$$u_1 = -\ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{4x}}{x^2}}{e^{4x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2}dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\ln(x)\, e^{2x} - e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x}(-1 - \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + x\, e^{2x} c_2\right) + \left(e^{2x}(-1 - \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x))$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x)) \tag{1}$$

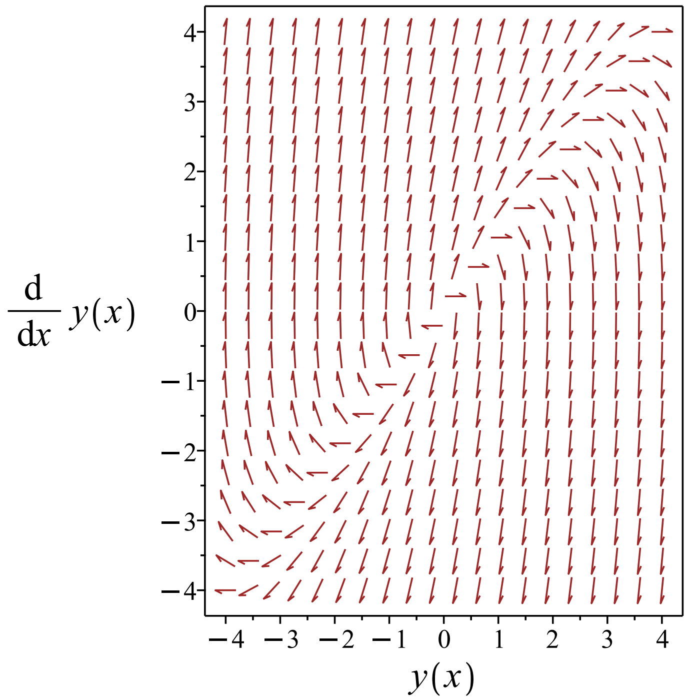


Figure 278: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + e^{2x}(-1 - \ln(x))$$

Verified OK.

11.12.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

$r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + x\, e^{2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{e^{2x}}{x^2}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x}x \\ 2\,e^{2x} & 2\,e^{2x}x + e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = e^{2x}\left(-\left(\int \frac{1}{x}dx\right) + \left(\int \frac{1}{x^2}dx\right)x\right)$

  - Compute integrals

    $y_p(x) = -e^{2x}(\ln(x) + 1)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + x\, e^{2x} c_2 - e^{2x}(\ln(x) + 1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=exp(2*x)/x^2,y(x), singsol=all)
```

$$y(x) = e^{2x}(-1 + c_1x - \ln(x) + c_2)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+4*y[x]==Exp[2*x]/x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(-\log(x) + c_2x - 1 + c_1)$$

11.13 problem 39

11.13.1 Solving as second order linear constant coeff ode 1756
11.13.2 Solving using Kovacic algorithm 1759
11.13.3 Maple step by step solution 1764

Internal problem ID [5404]
Internal file name [`OUTPUT/4895_Sunday_February_04_2024_12_46_54_AM_80399894/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y = x\,\mathrm{e}^{3x}}$$

11.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = x\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{3x}, \mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,\mathrm{e}^{3x} + A_2 \mathrm{e}^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{3x} + 8A_1 x\,\mathrm{e}^{3x} + 8A_2 \mathrm{e}^{3x} = x\,\mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = -\frac{3}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32} \tag{1}$$

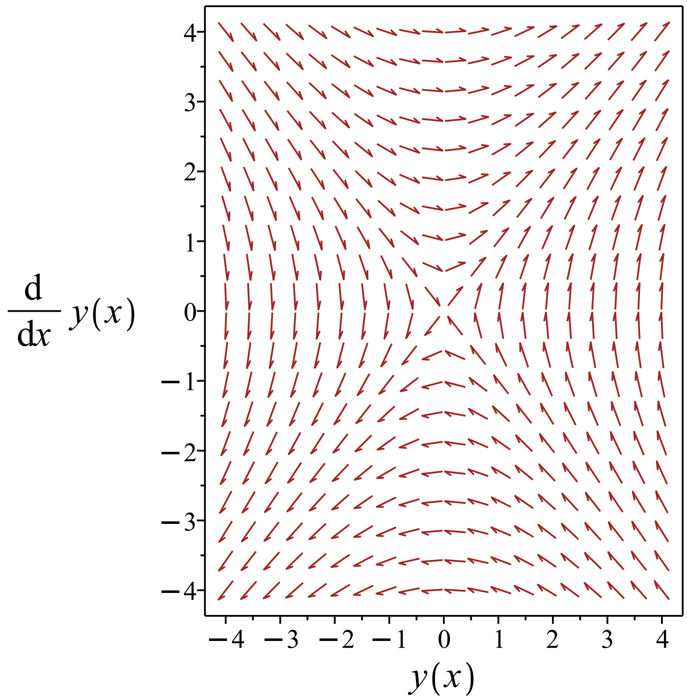


Figure 279: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{x\,e^{3x}}{8} - \frac{3\,e^{3x}}{32}$$

Verified OK.

11.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 217: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\
&= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x \, \mathrm{e}^{3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x \, \mathrm{e}^{3x}, \mathrm{e}^{3x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{x}}{2}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x \, \mathrm{e}^{3x} + A_2 \mathrm{e}^{3x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1\mathrm{e}^{3x} + 8A_1x\,\mathrm{e}^{3x} + 8A_2\mathrm{e}^{3x} = x\,\mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = -\frac{3}{32}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2}\right) + \left(\frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{x}}{2} + \frac{x\,\mathrm{e}^{3x}}{8} - \frac{3\,\mathrm{e}^{3x}}{32} \tag{1}$$

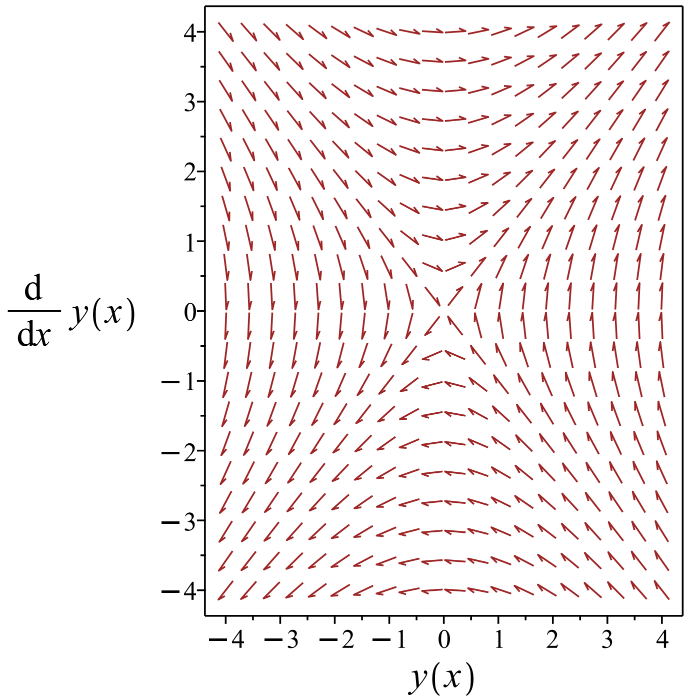


Figure 280: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \frac{x\, e^{3x}}{8} - \frac{3\, e^{3x}}{32}$$

Verified OK.

11.13.3 Maple step by step solution

Let's solve

$y'' - y = x\, e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\,\mathrm{e}^{3x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & \mathrm{e}^{x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{x} \end{array}\right]$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\mathrm{e}^{-x}\left(\int x\,\mathrm{e}^{4x}dx\right)}{2} + \frac{\mathrm{e}^{x}\left(\int \mathrm{e}^{2x}x dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = \frac{(-3+4x)\mathrm{e}^{3x}}{32}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x} + \frac{(-3+4x)\mathrm{e}^{3x}}{32}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)-y(x)=x*exp(3*x),y(x), singsol=all)
```

$$y(x) = c_2 \mathrm{e}^{-x} + \mathrm{e}^{x} c_1 + \frac{(4x-3)\,\mathrm{e}^{3x}}{32}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 34

```
DSolve[y''[x]-y[x]==x*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32} e^{3x}(4x-3) + c_1 e^{x} + c_2 e^{-x}$$

11.14 problem 40

11.14.1 Solving as second order linear constant coeff ode 1767
11.14.2 Solving using Kovacic algorithm 1772
11.14.3 Maple step by step solution 1779

Internal problem ID [5405]
Internal file name [`OUTPUT/4896_Sunday_February_04_2024_12_46_54_AM_96116373/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 16. Linear equations with constant coefficients (Short methods). Supplemetary problems. Page 107
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 5y' + 6y = \mathrm{e}^{-2x} \sec\left(x\right)^{2} \left(1 + 2\tan\left(x\right)\right)}$$

11.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 5, C = 6, f(x) = \mathrm{e}^{-2x} \sec\left(x\right)^{2} \left(1 + 2\tan\left(x\right)\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(6)}\\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-2)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 e^{-2x} + c_2 e^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-3x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \mathrm{e}^{-3x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-3x} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-3x} \\ -2\,\mathrm{e}^{-2x} & -3\,\mathrm{e}^{-3x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(-3\,\mathrm{e}^{-3x}\right) - \left(\mathrm{e}^{-3x}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = -e^{-2x} e^{-3x}$$

Which simplifies to

$$W = -e^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-3x} e^{-2x} \sec(x)^2 (1 + 2\tan(x))}{-e^{-5x}} dx$$

Which simplifies to

$$u_1 = -\int (-1 - 2\tan(x)) \sec(x)^2 \, dx$$

Hence

$$u_1 = \tan(x)^2 + \tan(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-4x} \sec(x)^2 (1 + 2\tan(x))}{-e^{-5x}} dx$$

Which simplifies to

$$u_2 = \int (-1 - 2\tan(x)) \sec(x)^2 e^x dx$$

Hence

$$u_2 = \frac{e^{4ix} + 2 e^{2ix} - 4 e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$

Which simplifies to

$$u_1 = \tan(x)(\tan(x) + 1)$$

$$u_2 = \frac{e^{4ix} + 2 e^{2ix} - 4 e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \tan(x)(\tan(x)+1)\,\mathrm{e}^{-2x} + \frac{\left(\mathrm{e}^{4ix} + 2\,\mathrm{e}^{2ix} - 4\,\mathrm{e}^{(1+2i)x} + 1\right)\mathrm{e}^{-3x}}{\left(\mathrm{e}^{2ix}+1\right)^2}$$

Which simplifies to

$$y_p(x) = \frac{\left(2\tan(x)^2 + 2\tan(x) - 4\right)\mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right)\mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2x}\left(\tan\right.}{\left(\mathrm{e}^{2ix}+1\right)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-3x}\right) \\ &\quad + \left(\frac{\left(2\tan(x)^2 + 2\tan(x) - 4\right)\mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right)\mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2}}{\left(\mathrm{e}^{2ix}+1\right)^2}\right. \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y &= c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-3x} \\ &\quad + \frac{\left(2\tan(x)^2 + 2\tan(x) - 4\right)\mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right)\mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2x}\left(\mathrm{t}\right.}{\left(\mathrm{e}^{2ix}+1\right)^2} \end{aligned} \tag{1}$$

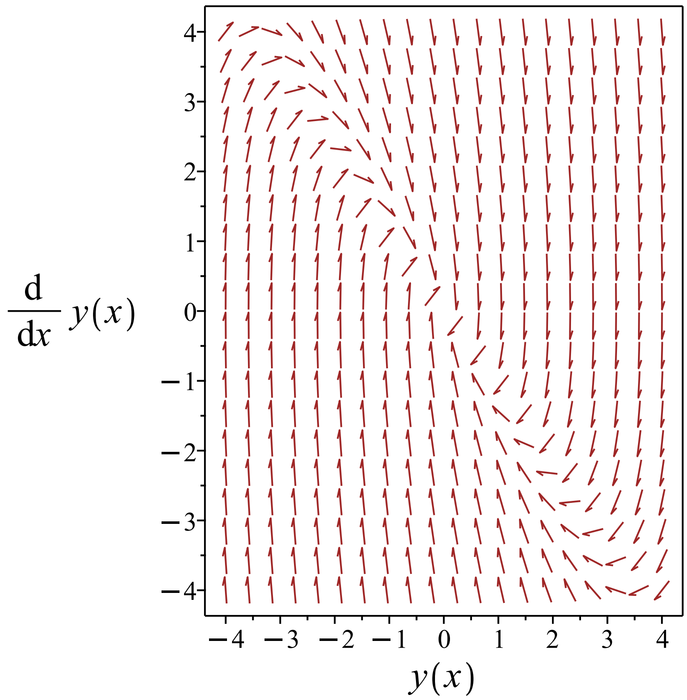


Figure 281: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{\left(2\tan(x)^2 + 2\tan(x) - 4\right) e^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right) e^{(-2+4i)x} + 2\,e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x}(\mathrm{t}}{\left(e^{2ix} + 1\right)^2}$$

Verified OK.

11.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 219: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\
&= z_1 e^{-\frac{5x}{2}} \\
&= z_1 \left(e^{-\frac{5x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(\mathrm{e}^{x})\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}(\mathrm{e}^{-3x}) & \frac{d}{dx}(\mathrm{e}^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-2x} \\ -3\,\mathrm{e}^{-3x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-2x}\mathrm{e}^{-3x}$$

Which simplifies to

$$W = \mathrm{e}^{-5x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-4x} \sec(x)^2 (1 + 2\tan(x))}{e^{-5x}} dx$$

Which simplifies to

$$u_1 = -\int \sec(x)^2 e^{x} (1 + 2\tan(x)) dx$$

Hence

$$u_1 = -\frac{-e^{4ix} - 2e^{2ix} + 4e^{(1+2i)x} - 1}{(e^{2ix} + 1)^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x} e^{-2x} \sec(x)^2 (1 + 2\tan(x))}{e^{-5x}} dx$$

Which simplifies to

$$u_2 = \int \sec(x)^2 (1 + 2\tan(x)) dx$$

Hence

$$u_2 = \tan(x)^2 + \tan(x)$$

Which simplifies to

$$u_1 = \frac{e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1}{(e^{2ix} + 1)^2}$$
$$u_2 = \tan(x)(\tan(x) + 1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \tan(x)(\tan(x) + 1) e^{-2x} + \frac{(e^{4ix} + 2e^{2ix} - 4e^{(1+2i)x} + 1) e^{-3x}}{(e^{2ix} + 1)^2}$$

Which simplifies to

$$y_p(x) = \frac{(2\tan(x)^2 + 2\tan(x) - 4) e^{(-2+2i)x} + (\tan(x)^2 + \tan(x)) e^{(-2+4i)x} + 2e^{(-3+2i)x} + e^{(-3+4i)x} + e^{-2x} (\tan}{(e^{2ix} + 1)^2}$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x}\right)$$

$$+ \left(\frac{\left(2 \tan(x)^2 + 2 \tan(x) - 4\right) \mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right) \mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2}}{\left(\mathrm{e}^{2ix} + 1\right)^2} \right.$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x} + \frac{\left(2 \tan(x)^2 + 2 \tan(x) - 4\right) \mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right) \mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2x}(\mathrm{t}}{\left(\mathrm{e}^{2ix} + 1\right)^2} \tag{1}$$

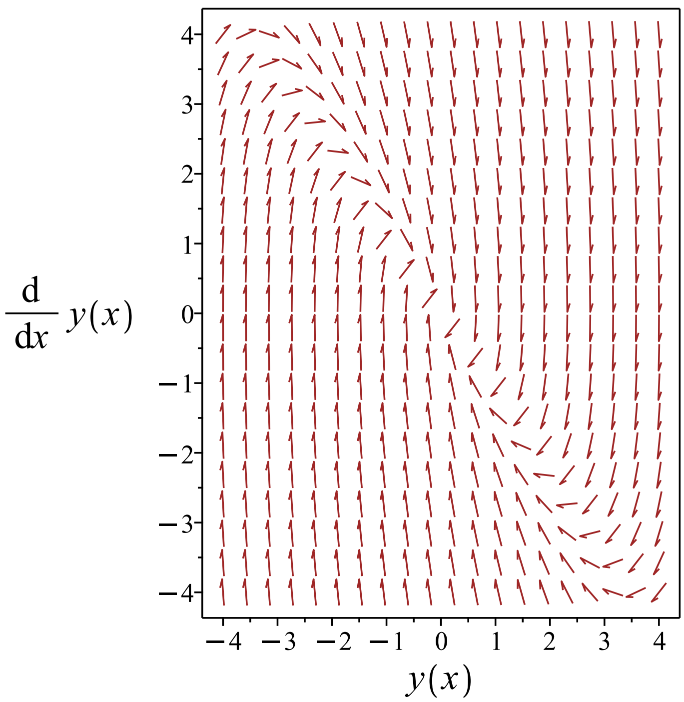


Figure 282: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x} + \frac{\left(2 \tan(x)^2 + 2 \tan(x) - 4\right) \mathrm{e}^{(-2+2i)x} + \left(\tan(x)^2 + \tan(x)\right) \mathrm{e}^{(-2+4i)x} + 2\,\mathrm{e}^{(-3+2i)x} + \mathrm{e}^{(-3+4i)x} + \mathrm{e}^{-2x}(\mathrm{t}}{\left(\mathrm{e}^{2ix} + 1\right)^2}$$

Verified OK.

11.14.3 Maple step by step solution

Let's solve

$y'' + 5y' + 6y = \mathrm{e}^{-2x} \sec(x)^2 (1 + 2\tan(x))$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r + 3)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{-2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{-2x} \sec(x)^2 (1 + 2\tan\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{-2x} \\ -3\,\mathrm{e}^{-3x} & -2\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-5x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-3x} \left(\int \sec(x)^2 \mathrm{e}^{x} (1 + 2\tan(x))\, dx\right) + \mathrm{e}^{-2x} \left(\int \sec(x)^2 (1 + 2\tan(x))\, dx\right)$

- Compute integrals

$$y_p(x) = \frac{\left(2\tan(x)^2+2\tan(x)-4\right)\mathrm{e}^{(-2+2\,\mathrm{I})x}+\tan(x)\left(\mathrm{e}^{-2x}+\mathrm{e}^{(-2+4\,\mathrm{I})x}\right)(\tan(x)+1)}{\left(\mathrm{e}^{2\,\mathrm{I}x}+1\right)^2}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{-2x} + \frac{\left(2\tan(x)^2+2\tan(x)-4\right)\mathrm{e}^{(-2+2\,\mathrm{I})x}+\tan(x)\left(\mathrm{e}^{-2x}+\mathrm{e}^{(-2+4\,\mathrm{I})x}\right)(\tan(x)+1)}{\left(\mathrm{e}^{2\,\mathrm{I}x}+1\right)^2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 95

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)+6*y(x)=exp(-2*x)*sec(x)^2*(1+2*tan(x)),y(x), singsol=al
```

$$y(x) = \frac{\left(2\sec(x)^2+2c_2+2\tan(x)-6\right)\mathrm{e}^{(-2+2i)x}+\left(\tan(x)^2+c_2+\tan(x)\right)\mathrm{e}^{(-2+4i)x}+2c_1\mathrm{e}^{(-3+2i)x}+c_1\mathrm{e}^{(-3+4}}{\mathrm{e}^{4ix}+2\,\mathrm{e}^{2ix}+1}$$

✓ Solution by Mathematica

Time used: 0.147 (sec). Leaf size: 26

```
DSolve[y''[x]+5*y'[x]+6*y[x]==Exp[-2*x]*Sec[x]^2*(1+2*Tan[x]),y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-3x}\left(e^x\tan(x)+c_2e^x+c_1\right)$$

12 Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110

12.1 problem 6 1782
12.2 problem 7 1810
12.3 problem 8 1848
12.4 problem 9 1855
12.5 problem 10 1862
12.6 problem 11 1893

12.1 problem 6

12.1.1 Solving as second order euler ode ode 1782
12.1.2 Solving as second order change of variable on x method 2 ode . 1786
12.1.3 Solving as second order change of variable on x method 1 ode . 1791
12.1.4 Solving as second order change of variable on y method 2 ode . 1796
12.1.5 Solving using Kovacic algorithm 1801

Internal problem ID [5406]
Internal file name [`OUTPUT/4897_Sunday_February_04_2024_12_46_55_AM_34394589/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 3xy' + 4y = x + x^2 \ln(x)}$$

12.1.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x(x \ln(x) + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 4y = x(x\ln(x) + 1)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^2 \ln(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^2 \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x \ln(x) + x \end{vmatrix}$$

Therefore

$$W = (x^2)(2x \ln(x) + x) - (x^2 \ln(x))(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \ln(x)(x \ln(x) + 1)}{x^5} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)(x\ln(x)+1)}{x^2}dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3(x\ln(x)+1)}{x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x\ln(x)+1}{x^2}dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}\right)x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x}\right)x^2\ln(x)$$

Which simplifies to

$$y_p(x) = \frac{x(\ln(x)^3 x + 6)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^2\ln(x)^3}{6} + c_2x^2\ln(x) + c_1x^2 + x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2\ln(x)^3}{6} + c_2x^2\ln(x) + c_1x^2 + x \tag{1}$$

Verification of solutions

$$
y = \frac{x^2 \ln(x)^3}{6} + c_2 x^2 \ln(x) + c_1 x^2 + x
$$

Verified OK.

12.1.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 3xy' + 4y = 0
$$

In normal form the ode

$$
x^2 y'' - 3xy' + 4y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}} \end{vmatrix}
$$

Therefore

$$
W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}} - \frac{2\ln(2)\,x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)
$$

Which simplifies to

$$
W = 2x^3
$$

Which simplifies to

$$
W = 2x^3
$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) x(x\ln(x)+1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(-2\ln(x)+\ln(2))(x\ln(x)+1)}{2x^2} dx$$

Hence

$$u_1 = -\frac{4\ln(x)^3 x - 3\ln(2)\ln(x)^2 x - 12\ln(x) + 6\ln(2) - 12}{12x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\, x(x\ln(x)+1)}{2x^5} dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{x\ln(x)+1}{2x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\left(4\ln(x)^3 x - 3\ln(2)\ln(x)^2 x - 12\ln(x) + 6\ln(2) - 12\right)\sqrt{x^4}}{12x} + \left(\frac{\ln(x)^2}{4} - \frac{1}{2x}\right)\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)$$

Which simplifies to

$$y_p(x) = \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2}\right) + \left(\frac{x(\ln(x)^3 x + 6)}{6}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{x(\ln(x)^3 x + 6)}{6} \tag{1}
$$

Verification of solutions

$$
y = \frac{(c_2 \ln(x^4) - 2c_2 \ln(2) + c_1)\sqrt{x^4}}{2} + \frac{x(\ln(x)^3 x + 6)}{6}
$$

Verified OK. `{0 < x}`

12.1.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -3x, C = 4, f(x) = x(x\ln(x) + 1)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - 3xy' + 4y = 0
$$

In normal form the ode

$$
x^2y'' - 3xy' + 4y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{c\tau} c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3xy' + 4y = x(x \ln(x) + 1)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^4} \\
y_2 &= \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & \frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4} \\ \frac{2x^3}{\sqrt{x^4}} & \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} - \frac{2\ln(2)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(\frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}} - \frac{2\ln(2)\,x^3}{\sqrt{x^4}}\right) - \left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{\ln(x^4)\sqrt{x^4}}{2} - \ln(2)\sqrt{x^4}\right)x(x\ln(x)+1)}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{(-2\ln(x)+\ln(2))\,(x\ln(x)+1)}{2x^2}\,dx$$

Hence

$$u_1 = -\frac{4\ln(x)^3 x - 3\ln(2)\ln(x)^2 x - 12\ln(x) + 6\ln(2) - 12}{12x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\, x(x \ln(x) + 1)}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{x \ln(x) + 1}{2x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{4} - \frac{1}{2x}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\frac{\left(4 \ln(x)^3 x - 3 \ln(2) \ln(x)^2 x - 12 \ln(x) + 6 \ln(2) - 12\right) \sqrt{x^4}}{12x} \\ &+ \left(\frac{\ln(x)^2}{4} - \frac{1}{2x}\right) \left(\frac{\ln(x^4) \sqrt{x^4}}{2} - \ln(2) \sqrt{x^4}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + \left(\frac{x\left(\ln(x)^3 x + 6\right)}{6}\right) \\ &= \frac{x\left(\ln(x)^3 x + 6\right)}{6} + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{x^2 \ln(x)^3}{6} + c_1 x^2 + x$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 \ln(x)^3}{6} + c_1 x^2 + x \tag{1}$$

Verification of solutions

$$y = \frac{x^2 \ln(x)^3}{6} + c_1 x^2 + x$$

Verified OK. `{0 < x}`

12.1.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x(x\ln(x) + 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx\\
\ln(u) &= -\ln(x) + c_1\\
u &= \mathrm{e}^{-\ln(x)+c_1}\\
&= \frac{c_1}{x}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx + c_2\\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n\\
&= (c_1 \ln(x) + c_2)\,x^2\\
&= (c_1 \ln(x) + c_2)\,x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3xy' + 4y = x(x\ln(x) + 1)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2\\
y_2 &= x^2 \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^2 \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x \ln(x) + x \end{vmatrix}$$

Therefore

$$W = (x^2)(2x \ln(x) + x) - (x^2 \ln(x))(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \ln(x)(x \ln(x) + 1)}{x^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)(x \ln(x) + 1)}{x^2} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3(x\ln(x)+1)}{x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x\ln(x)+1}{x^2}dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}\right)x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x}\right)x^2\ln(x)$$

Which simplifies to

$$y_p(x) = \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1\ln(x) + c_2)\,x^2\right) + \left(\frac{x\left(\ln(x)^3 x + 6\right)}{6}\right) \\ &= \frac{x\left(\ln(x)^3 x + 6\right)}{6} + (c_1\ln(x) + c_2)\,x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{x^2\ln(x)^3}{6} + c_1x^2\ln(x) + c_2x^2 + x$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 \ln(x)^3}{6} + c_1 x^2 \ln(x) + c_2 x^2 + x \tag{1}$$

Verification of solutions

$$y = \frac{x^2 \ln(x)^3}{6} + c_1 x^2 \ln(x) + c_2 x^2 + x$$

Verified OK. `{0 < x}`

12.1.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 221: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-}\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0
$$

$$
0 = 0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} \, dx} \\
&= z_1 e^{\frac{3 \ln(x)}{2}} \\
&= z_1 \left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{3 \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 (\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 3xy' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1x^2 + c_2x^2 \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^2 \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^2 \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^2 \ln(x) \\ 2x & 2x \ln(x) + x \end{vmatrix}$$

Therefore

$$W = (x^2)(2x \ln(x) + x) - (x^2 \ln(x))(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \ln(x)(x \ln(x) + 1)}{x^5} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)(x \ln(x) + 1)}{x^2} dx$$

Hence

$$u_1 = -\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3(x \ln(x) + 1)}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{x \ln(x) + 1}{x^2} dx$$

Hence

$$u_2 = \frac{\ln(x)^2}{2} - \frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{\ln(x)^3}{3} + \frac{\ln(x)}{x} + \frac{1}{x}\right) x^2 + \left(\frac{\ln(x)^2}{2} - \frac{1}{x}\right) x^2 \ln(x)$$

Which simplifies to

$$y_p(x) = \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + c_2 x^2 \ln(x)\right) + \left(\frac{x\left(\ln(x)^3 x + 6\right)}{6}\right) \end{aligned}$$

Which simplifies to

$$y = x^2 (c_1 + c_2 \ln(x)) + \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Summary

The solution(s) found are the following

$$y = x^2 (c_1 + c_2 \ln(x)) + \frac{x\left(\ln(x)^3 x + 6\right)}{6} \tag{1}$$

Verification of solutions

$$y = x^2 (c_1 + c_2 \ln(x)) + \frac{x\left(\ln(x)^3 x + 6\right)}{6}$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x+x^2*ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)^3 x^2}{6} + \ln(x) x^2 c_1 + c_2 x^2 + x$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 30

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x+x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6} x \left(x \log^3(x) + 6 c_1 x + 12 c_2 x \log(x) + 6\right)$$

12.2 problem 7

12.2.1 Solving as second order euler ode ode 1811
12.2.2 Solving as linear second order ode solved by an integrating factor ode 1814
12.2.3 Solving as second order change of variable on x method 2 ode . 1815
12.2.4 Solving as second order change of variable on x method 1 ode . 1821
12.2.5 Solving as second order change of variable on y method 1 ode . 1827
12.2.6 Solving as second order change of variable on y method 2 ode . 1832
12.2.7 Solving as second order ode non constant coeff transformation on B ode 1837
12.2.8 Solving using Kovacic algorithm 1841

Internal problem ID [5407]
Internal file name [`OUTPUT/4898_Sunday_February_04_2024_12_46_57_AM_38265301/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)}$$

12.2.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^2 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln\left(x^2\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2\ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2\ln(x) - \ln(x^2)}{x}$$

$$u_2 = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x \tag{1}
$$

Verification of solutions

$$
y = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + c_2 x^2 + c_1 x
$$

Verified OK.

12.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$
y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)
$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{2}{x}\,dx} \\
&= \frac{1}{x}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \frac{\ln(x)^2 - \ln(x^2)}{x^3} \\
\left(\frac{y}{x}\right)'' &= \frac{\ln(x)^2 - \ln(x^2)}{x^3}
\end{aligned}
$$

Integrating once gives

$$
\left(\frac{y}{x}\right)' = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2} + c_1
$$

Integrating again gives

$$\left(\frac{y}{x}\right) = \frac{4c_1x^2 + 2\ln(x)^2 + 6\ln(x) - 2\ln(x^2) + 1}{4x} + c_2$$

Hence the solution is

$$y = \frac{\frac{4c_1x^2+2\ln(x)^2+6\ln(x)-2\ln(x^2)+1}{4x} + c_2}{\frac{1}{x}}$$

Or

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary

The solution(s) found are the following

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \tag{1}$$

Verification of solutions

$$y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\, \tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x) dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{x} dx\right)} dx \\
&= \int e^{2 \ln(x)}\, dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3}
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{x^4} \\
&= \frac{2}{x^6}
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{x^6} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{2}{x^6} = \frac{2}{9\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0
$$

Simplifying gives

$$
9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0
$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$
9r(r-1) + 0 + 2 = 0
$$

Or

$$
9r^2 - 9r + 2 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{3} \\
r_2 &= \frac{2}{3}
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
y(\tau) = c_1 y_1 + c_2 y_2
$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{3}} + c_2 \tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$

$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}}\right)\left(\frac{2x^2}{(x^3)^{\frac{1}{3}}}\right) - \left((x^3)^{\frac{2}{3}}\right)\left(\frac{x^2}{(x^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}}$$

Which simplifies to

$$y_p(x) = -(x^3)^{\frac{1}{3}} \left(-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3}\right) + \left(-(x^3)^{\frac{1}{3}} \left(-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} - (x^3)^{\frac{1}{3}} \left(-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 3^{\frac{2}{3}} (x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}} (x^3)^{\frac{2}{3}}}{3} - (x^3)^{\frac{1}{3}} \left(-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha \right) (x^3)^{\frac{1}{3}} + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha \right)$$

Verified OK.

12.2.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -2x, C = 2, f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the ode

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\
\tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{3c\sqrt{2}}{2}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^3\right)^{\frac{1}{3}} \\
y_2 &= \left(x^3\right)^{\frac{2}{3}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{d}{dx}\left((x^3)^{\frac{1}{3}}\right) & \frac{d}{dx}\left((x^3)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^3)^{\frac{1}{3}} & (x^3)^{\frac{2}{3}} \\ \frac{x^2}{(x^3)^{\frac{2}{3}}} & \frac{2x^2}{(x^3)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left((x^3)^{\frac{1}{3}}\right)\left(\frac{2x^2}{(x^3)^{\frac{1}{3}}}\right) - \left((x^3)^{\frac{2}{3}}\right)\left(\frac{x^2}{(x^3)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\ &+ \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}} \end{aligned}$$

Which simplifies to

$$\begin{aligned} y_p(x) = -(x^3)^{\frac{1}{3}} \Bigg(&-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\ &+ \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\Bigg) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{3}{2}} \left(c_1 \cosh\left(\frac{\ln (x)}{2}\right) + i c_2 \sinh\left(\frac{\ln (x)}{2}\right)\right)\right) \\ &\quad + \Bigg(-(x^3)^{\frac{1}{3}} \Bigg(-\left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\ &\quad + \int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\Bigg)\Bigg) \end{aligned}$$

$$
= -\left(x^3\right)^{\frac{1}{3}} \left(-\left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{1}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{1}{3}} + \int_0^x \frac{\left(\alpha^3\right)^{\frac{2}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) + x^{\frac{3}{2}} \left(c_1 \cosh \left(\frac{\ln \left(x\right)}{2}\right) + i c_2 \sinh \left(\frac{\ln \left(x\right)}{2}\right)\right)
$$

Which simplifies to

$$
y = i x^{\frac{3}{2}} \sinh \left(\frac{\ln \left(x\right)}{2}\right) c_2 + x^{\frac{3}{2}} \cosh \left(\frac{\ln \left(x\right)}{2}\right) c_1 + \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{1}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{2}{3}} - \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{2}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{1}{3}}
$$

Summary

The solution(s) found are the following

$$
y = i x^{\frac{3}{2}} \sinh \left(\frac{\ln \left(x\right)}{2}\right) c_2 + x^{\frac{3}{2}} \cosh \left(\frac{\ln \left(x\right)}{2}\right) c_1 + \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{1}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{2}{3}} - \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{2}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{1}{3}} \tag{1}
$$

Verification of solutions

$$
y = i x^{\frac{3}{2}} \sinh \left(\frac{\ln \left(x\right)}{2}\right) c_2 + x^{\frac{3}{2}} \cosh \left(\frac{\ln \left(x\right)}{2}\right) c_1 + \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{1}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{2}{3}} - \left(\int_0^x \frac{\left(\alpha^3\right)^{\frac{2}{3}} \left(\ln \left(\alpha\right)^2 - \ln \left(\alpha^2\right)\right)}{\alpha^4} d\alpha\right) \left(x^3\right)^{\frac{1}{3}}
$$

Verified OK.

12.2.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 2xy' + 2y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x^3 v''(x) = \ln(x)^2 - \ln(x^2)$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{\ln(x)^2 - \ln(x^2)}{x^3}$$

Integrating once gives

$$v'(x) = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2} + c_1$$

Integrating again gives

$$v(x) = \frac{1}{4x} + \frac{3\ln(x)}{2x} + \frac{\ln(x)^2}{2x} - \frac{\ln(x^2)}{2x} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(\frac{1}{4x} + \frac{3\ln(x)}{2x} + \frac{\ln(x)^2}{2x} - \frac{\ln(x^2)}{2x} + c_1 x + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = \left(\frac{1}{4x} + \frac{3\ln(x)}{2x} + \frac{\ln(x)^2}{2x} - \frac{\ln(x^2)}{2x} + c_1 x + c_2\right) x$$

Therefore the homogeneous solution y_h is

$$y_h = \left(\frac{1}{4x} + \frac{3\ln(x)}{2x} + \frac{\ln(x)^2}{2x} - \frac{\ln(x^2)}{2x} + c_1 x + c_2\right) x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^3\right)^{\frac{1}{3}}$$
$$y_2 = \left(x^3\right)^{\frac{2}{3}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{d}{dx}\left(\left(x^3\right)^{\frac{1}{3}}\right) & \frac{d}{dx}\left(\left(x^3\right)^{\frac{2}{3}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^3\right)^{\frac{1}{3}} & \left(x^3\right)^{\frac{2}{3}} \\ \frac{x^2}{\left(x^3\right)^{\frac{2}{3}}} & \frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^3\right)^{\frac{1}{3}}\right)\left(\frac{2x^2}{\left(x^3\right)^{\frac{1}{3}}}\right) - \left(\left(x^3\right)^{\frac{2}{3}}\right)\left(\frac{x^2}{\left(x^3\right)^{\frac{2}{3}}}\right)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^3)^{\frac{2}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Hence

$$u_1 = -\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^3)^{\frac{1}{3}} \left(\ln (x)^2 - \ln (x^2)\right)}{x^4} dx$$

Hence

$$u_2 = \int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\ &+ \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln (\alpha)^2 - \ln (\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}} \end{aligned}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{1}{4x} + \frac{3\ln(x)}{2x} + \frac{\ln(x)^2}{2x} - \frac{\ln(x^2)}{2x} + c_1 x + c_2\right) x\right) \\
&\quad + \left(-\left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \right. \\
&\qquad \left. + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y &= c_1 x^2 + c_2 x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\
&\quad - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\
&\quad + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^2 + c_2 x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\
&\quad - \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\
&\quad + \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = c_1x^2 + c_2x + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \\
- \left(\int_0^x \frac{(\alpha^3)^{\frac{2}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{1}{3}} \\
+ \left(\int_0^x \frac{(\alpha^3)^{\frac{1}{3}} \left(\ln(\alpha)^2 - \ln(\alpha^2)\right)}{\alpha^4} d\alpha\right) (x^3)^{\frac{2}{3}}
\end{aligned}
$$

Verified OK.

12.2.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = -2x, C = 2, f(x) = \ln(x)^2 - \ln(x^2)$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' - 2xy' + 2y = 0
$$

In normal form the ode

$$
x^2y'' - 2xy' + 2y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{2}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right)x^2 \\
&= (c_2 x - c_1)\,x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 2xy' + 2y = \ln(x)^2 - \ln(x^2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= x^2
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2\ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(\ln(x)^2 - \ln(x^2)\right)}{x^4}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2\ln(x) - \ln(x^2)}{x}$$

$$u_2 = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{x} + c_2\right)x^2\right) + \left(\frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}\right) \\
&= \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} + \left(-\frac{c_1}{x} + c_2\right)x^2
\end{aligned}$$

Which simplifies to

$$y = c_2 x^2 + \frac{\ln(x)^2}{2} - c_1 x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary
The solution(s) found are the following

$$y = c_2 x^2 + \frac{\ln(x)^2}{2} - c_1 x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \tag{1}$$

Verification of solutions

$$y = c_2 x^2 + \frac{\ln(x)^2}{2} - c_1 x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.7 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$AB u' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2 \\
F &= \ln(x)^2 - \ln\left(x^2\right)
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-2x)(-2) + (2)(-2x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3 v'' + (0)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3 u'(x) = 0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned}
u(x) &= \int 0 \, \mathrm{d}x \\
&= c_1
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int c_1 \, \mathrm{d}x \\
&= c_1 x + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)\,(c_1x + c_2) \\
&= -2x(c_1x + c_2)
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= x^2
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2\ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2\ln(x) - \ln(x^2)}{x}$$

$$u_2 = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= (-2x(c_1x + c_2)) + \left(\frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}\right) \\
&= -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 + \frac{\ln(x)^2}{2} - 2c_2x + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

12.2.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 \\
B &= -2x \\
C &= 2
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 222: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx} \\
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 2xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x^2 + c_1 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2)(1)$$

Which simplifies to

$$W = x^2$$

Which simplifies to

$$W = x^2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2 \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)^2 - \ln(x^2)}{x^2} dx$$

Hence

$$u_1 = \frac{\ln(x)^2}{x} + \frac{2\ln(x)}{x} - \frac{\ln(x^2)}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x \left(\ln(x)^2 - \ln(x^2)\right)}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x)^2 - \ln(x^2)}{x^3} dx$$

Hence

$$u_2 = -\frac{\ln(x)^2}{2x^2} - \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} + \frac{\ln(x^2)}{2x^2}$$

Which simplifies to

$$u_1 = \frac{\ln(x)^2 + 2\ln(x) - \ln(x^2)}{x}$$

$$u_2 = \frac{-2\ln(x)^2 - 2\ln(x) + 2\ln(x^2) + 1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x^2 + c_1 x) + \left(\frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_2 x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Summary
The solution(s) found are the following

$$y = x(c_2 x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4} \tag{1}$$

Verification of solutions

$$y = x(c_2 x + c_1) + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=(ln(x))^2-ln(x^2),y(x), singsol=all)
```

$$y(x) = c_2x + c_1x^2 + \frac{\ln(x)^2}{2} + \frac{3\ln(x)}{2} - \frac{\ln(x^2)}{2} + \frac{1}{4}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 38

```
DSolve[x^2*y''[x]-2*x*y'[x]+2*y[x]==(Log[x])^2-Log[x^2],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{4}\left(-2\log\left(x^2\right) + 2\log^2(x) + 6\log(x) + 1\right) + c_2x^2 + c_1x$$

12.3 problem 8

12.3.1 Maple step by step solution 1853

Internal problem ID [5408]
Internal file name [`OUTPUT/4899_Sunday_February_04_2024_12_46_59_AM_3205007/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 8.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{x^3y''' + 2x^2y'' = x + \sin(\ln(x))}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + 2x^2y'' = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^3y''' + 2x^2y'' = x + \sin(\ln(x))$$

gives

$$2x^2\lambda(\lambda-1)\,x^{\lambda-2}+x^3\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda-3}=0$$

Which simplifies to

$$2\lambda(\lambda-1)\,x^{\lambda}+\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda}=0$$

And since $x^{\lambda}\neq 0$ then dividing through by x^{λ}, the above becomes

$$2\lambda(\lambda-1)+\lambda(\lambda-1)\,(\lambda-2)=0$$

Simplifying gives the characteristic equation as

$$\lambda^2(\lambda-1)=0$$

Solving the above gives the following roots

$$\lambda_1=1$$
$$\lambda_2=0$$
$$\lambda_3=0$$

This table summarises the result

root	multiplicity	type of root
0	2	real root
1	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^{λ} basis solution. Each real root of multiplicty two, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^{λ} and $c_2x^{\lambda}\ln(x)$ and $c_3x^{\lambda}\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^{\alpha}(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=c_1+c_2\ln(x)+c_3x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1=1$$
$$y_2=\ln(x)$$
$$y_3=x$$

Now the particular solution to the given ODE is found

$$x^3y''' + 2x^2y'' = x + \sin(\ln(x))$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\,dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} 1 & \ln(x) & x \\ 0 & \frac{1}{x} & 1 \\ 0 & -\frac{1}{x^2} & 0 \end{bmatrix}$$
$$|W| = \frac{1}{x^2}$$

The determinant simplifies to

$$|W| = \frac{1}{x^2}$$

Now we determine W_i for each U_i.

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} \ln(x) & x \\ \frac{1}{x} & 1 \end{bmatrix} \\ &= \ln(x) - 1 \end{aligned}$$

$$
\begin{aligned}
W_2(x) &= \det \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det \begin{bmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{bmatrix} \\
&= \frac{1}{x}
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x) W_1(x)}{a W(x)} \, dx \\
&= (-1)^2 \int \frac{(x + \sin(\ln(x)))(\ln(x) - 1)}{(x^3)\left(\frac{1}{x^2}\right)} \, dx \\
&= \int \frac{(x + \sin(\ln(x)))(\ln(x) - 1)}{x} \, dx \\
&= \int \left(\frac{(x + \sin(\ln(x)))(\ln(x) - 1)}{x} \right) \, dx \\
&= -2x + \cos(\ln(x)) + x\ln(x) + \sin(\ln(x)) - \cos(\ln(x))\ln(x)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2} \int \frac{F(x) W_2(x)}{a W(x)} \, dx \\
&= (-1)^1 \int \frac{(x + \sin(\ln(x)))(1)}{(x^3)\left(\frac{1}{x^2}\right)} \, dx \\
&= -\int \frac{x + \sin(\ln(x))}{x} \, dx \\
&= -\int \left(\frac{x + \sin(\ln(x))}{x} \right) \, dx \\
&= -x + \cos(\ln(x))
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x) W_3(x)}{a W(x)}\, dx \\
&= (-1)^0 \int \frac{(x+\sin(\ln(x)))\left(\frac{1}{x}\right)}{(x^3)\left(\frac{1}{x^2}\right)}\, dx \\
&= \int \frac{\frac{x+\sin(\ln(x))}{x}}{x}\, dx \\
&= \int \left(\frac{x+\sin(\ln(x))}{x^2}\right) dx \\
&= \ln(x) + \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right) x}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = {} & (-2x + \cos(\ln(x)) + x\ln(x) + \sin(\ln(x)) - \cos(\ln(x))\ln(x)) \\
& + (-x + \cos(\ln(x)))(\ln(x)) \\
& + \left(\ln(x) + \frac{-\frac{1}{2} + \frac{\tan\left(\frac{\ln(x)}{2}\right)^2}{2} - \tan\left(\frac{\ln(x)}{2}\right)}{\left(1+\tan\left(\frac{\ln(x)}{2}\right)^2\right) x}\right)(x)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{\cos(\ln(x))}{2} - 2x + x\ln(x) + \frac{\sin(\ln(x))}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 + c_2\ln(x) + c_3 x) + \left(\frac{\cos(\ln(x))}{2} - 2x + x\ln(x) + \frac{\sin(\ln(x))}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + c_2\ln(x) + c_3 x + \frac{\cos(\ln(x))}{2} - 2x + x\ln(x) + \frac{\sin(\ln(x))}{2} \tag{1}
$$

Verification of solutions

$$y = c_1 + c_2 \ln(x) + c_3 x + \frac{\cos(\ln(x))}{2} - 2x + x\ln(x) + \frac{\sin(\ln(x))}{2}$$

Verified OK.

12.3.1 Maple step by step solution

Let's solve

$x^3 y''' + 2x^2 y'' = x + \sin(\ln(x))$

- Highest derivative means the order of the ODE is 3

y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(2*_a^2*_b(_a)-sin(ln(_a))-_a)/_a^3, _
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(x^3*diff(y(x),x$3)+2*x^2*diff(y(x),x$2)=x+sin(ln(x)),y(x), singsol=all)
```

$$y(x) = -c_1 \ln(x) + \ln(x)\, x + c_2 x + c_3 - x + \frac{\tan\left(\frac{\ln(x)}{2}\right) + 1}{1 + \tan\left(\frac{\ln(x)}{2}\right)^2}$$

✓ Solution by Mathematica

Time used: 0.173 (sec). Leaf size: 36

```
DSolve[x^3*y'''[x]+2*x^2*y''[x]==x+Sin[Log[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\sin(\log(x)) + \cos(\log(x)) + 2((-1 + c_3)x + (x - c_1)\log(x) + c_2))$$

12.4 problem 9

12.4.1 Maple step by step solution 1860

Internal problem ID [5409]
Internal file name [`OUTPUT/4900_Sunday_February_04_2024_12_47_00_AM_45729767/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 9.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_ODE_non_constant_coefficients_of_type_Euler"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

$$\boxed{x^3y''' + xy' - y = 3x^4}$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$x^3y''' + xy' - y = 0$$

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^3y''' + xy' - y = 3x^4$$

gives

$$x\lambda\,x^{\lambda-1}+x^3\lambda(\lambda-1)\,(\lambda-2)\,x^{\lambda-3}-x^\lambda=0$$

Which simplifies to

$$\lambda\,x^\lambda+\lambda(\lambda-1)\,(\lambda-2)\,x^\lambda-x^\lambda=0$$

And since $x^\lambda\neq 0$ then dividing through by x^λ, the above becomes

$$\lambda+\lambda(\lambda-1)\,(\lambda-2)-1=0$$

Simplifying gives the characteristic equation as

$$(\lambda-1)^3=0$$

Solving the above gives the following roots

$$\lambda_1=1$$
$$\lambda_2=1$$
$$\lambda_3=1$$

This table summarises the result

root	multiplicity	type of root
1	3	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda\ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda\ln(x)$ and $c_3x^\lambda\ln(x)^2$ basis solutions, and so on. Each complex root $\alpha\pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity two generates $\ln(x)\,x^\alpha(c_1\cos(\beta\ln(x))+c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha\pm i\beta$ of multiplicity three generates $\ln(x)^2\,x^\alpha(c_1\cos(\beta\ln(x))+c_2$ basis solutions. And so on. Using the above show that the solution is

$$y=c_1x+c_2x\ln(x)+c_3\ln(x)^2\,x$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1=x$$
$$y_2=x\ln(x)$$
$$y_3=x\ln(x)^2$$

Now the particular solution to the given ODE is found

$$x^3y''' + xy' - y = 3x^4$$

Let the particular solution be

$$y_p = U_1y_1 + U_2y_2 + U_3y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \begin{bmatrix} x & x\ln(x) & x\ln(x)^2 \\ 1 & \ln(x)+1 & \ln(x)(\ln(x)+2) \\ 0 & \frac{1}{x} & \frac{2\ln(x)+2}{x} \end{bmatrix}$$
$$|W| = 2$$

The determinant simplifies to

$$|W| = 2$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} x\ln(x) & x\ln(x)^2 \\ \ln(x)+1 & \ln(x)(\ln(x)+2) \end{bmatrix}$$
$$= x\ln(x)^2$$

$$
\begin{aligned}
W_2(x) &= \det\begin{bmatrix} x & x\ln(x)^2 \\ 1 & \ln(x)(\ln(x)+2) \end{bmatrix} \\
&= 2x\ln(x)
\end{aligned}
$$

$$
\begin{aligned}
W_3(x) &= \det\begin{bmatrix} x & x\ln(x) \\ 1 & \ln(x)+1 \end{bmatrix} \\
&= x
\end{aligned}
$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1}\int \frac{F(x)W_1(x)}{aW(x)}\,dx \\
&= (-1)^2\int \frac{(3x^4)\left(x\ln(x)^2\right)}{(x^3)(2)}\,dx \\
&= \int \frac{3x^5\ln(x)^2}{2x^3}\,dx \\
&= \int \left(\frac{3x^2\ln(x)^2}{2}\right)\,dx \\
&= \frac{x^3\ln(x)^2}{2} - \frac{x^3\ln(x)}{3} + \frac{x^3}{9}
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= (-1)^{3-2}\int \frac{F(x)W_2(x)}{aW(x)}\,dx \\
&= (-1)^1\int \frac{(3x^4)(2x\ln(x))}{(x^3)(2)}\,dx \\
&= -\int \frac{6x^5\ln(x)}{2x^3}\,dx \\
&= -\int \left(3x^2\ln(x)\right)\,dx \\
&= -x^3\ln(x) + \frac{x^3}{3}
\end{aligned}
$$

$$
\begin{aligned}
U_3 &= (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\, dx \\
&= (-1)^{0} \int \frac{(3x^4)\,(x)}{(x^3)\,(2)}\, dx \\
&= \int \frac{3x^5}{2x^3}\, dx \\
&= \int \left(\frac{3x^2}{2}\right)\, dx \\
&= \frac{x^3}{2}
\end{aligned}
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1 y_1 + U_2 y_2 + U_3 y_3
$$

Hence

$$
\begin{aligned}
y_p = &\left(\frac{x^3 \ln(x)^2}{2} - \frac{x^3 \ln(x)}{3} + \frac{x^3}{9}\right)(x) \\
&+ \left(-x^3 \ln(x) + \frac{x^3}{3}\right)(x \ln(x)) \\
&+ \left(\frac{x^3}{2}\right)\left(x \ln(x)^2\right)
\end{aligned}
$$

Therefore the particular solution is

$$
y_p = \frac{x^4}{9}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x + c_2 x \ln(x) + c_3 \ln(x)^2 x\right) + \left(\frac{x^4}{9}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9} \tag{1}
$$

Verification of solutions

$$y = x\left(c_1 + c_2 \ln(x) + c_3 \ln(x)^2\right) + \frac{x^4}{9}$$

Verified OK.

12.4.1 Maple step by step solution

Let's solve

$x^3 y''' + x y' - y = 3x^4$

- Highest derivative means the order of the ODE is 3

y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x^3*diff(y(x),x$3)+x*diff(y(x),x)-y(x)=3*x^4,y(x), singsol=all)
```

$$y(x) = \frac{x\left(9c_3 \ln(x)^2 + x^3 + 9c_2 \ln(x) + 9c_1\right)}{9}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 31

```
DSolve[x^3*y'''[x]+x*y'[x]-y[x]==3*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{9} + c_1 x + c_3 x \log^2(x) + c_2 x \log(x)$$

12.5 problem 10

12.5.1 Solving as second order change of variable on x method 2 ode . 1863
12.5.2 Solving as second order change of variable on x method 1 ode . 1868
12.5.3 Solving as second order integrable as is ode 1874
12.5.4 Solving as second order ode non constant coeff transformation on B ode 1875
12.5.5 Solving as type second_order_integrable_as_is (not using ABC version) 1880
12.5.6 Solving using Kovacic algorithm 1882
12.5.7 Solving as exact linear second order ode ode 1890

Internal problem ID [5410]
Internal file name [`OUTPUT/4901_Sunday_February_04_2024_12_47_00_AM_5832621/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)^2 y'' + (x+1) y' - y = \ln (x+1)^2 + x - 1}$$

12.5.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$(x+1)^2 y'' + (x+1) y' - y = 0$$

In normal form the ode

$$(x+1)^2 y'' + (x+1) y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x+1} \\ q(x) &= -\frac{1}{(x+1)^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x+1}dx\right)}dx \\
&= \int e^{-\ln(x+1)}\,dx \\
&= \int \frac{1}{x+1}dx \\
&= \ln(x+1) \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{(x+1)^2}}{\frac{1}{(x+1)^2}} \\
&= -1 \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \qquad (1)
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - 1 = 0 \qquad (2)
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}$$
$$= \pm 1$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(x+1)^2 c_1 + c_2}{x+1}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(x+1)^2 c_1 + c_2}{x+1}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x+1} \\
y_2 &= \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \end{vmatrix}
$$

Therefore

$$
\begin{aligned}
W = {} & \left(\frac{1}{x+1}\right)\left(-\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2}\right) \\
& - \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)
\end{aligned}
$$

Which simplifies to

$$
W = \frac{2}{x+1}
$$

Which simplifies to

$$W=\frac{2}{x+1}$$

Therefore Eq. (2) becomes

$$u_1=-\int \frac{\left(\frac{x^2}{x+1}+\frac{2x}{x+1}+\frac{1}{x+1}\right)\left(\ln\left(x+1\right)^2+x-1\right)}{2+2x}\,dx$$

Which simplifies to

$$u_1=-\int \left(\frac{\ln\left(x+1\right)^2}{2}+\frac{x}{2}-\frac{1}{2}\right)dx$$

Hence

$$u_1=-\frac{x}{2}-\frac{x^2}{4}-\frac{\ln\left(x+1\right)^2\left(x+1\right)}{2}+\left(x+1\right)\ln\left(x+1\right)-1$$

And Eq. (3) becomes

$$u_2=\int \frac{\frac{\ln(x+1)^2+x-1}{x+1}}{2+2x}\,dx$$

Which simplifies to

$$u_2=\int \frac{\ln\left(x+1\right)^2+x-1}{2\left(x+1\right)^2}dx$$

Hence

$$u_2=-\frac{\ln\left(x+1\right)^2}{2\left(x+1\right)}-\frac{\ln\left(x+1\right)}{x+1}+\frac{\ln\left(x+1\right)}{2}$$

Which simplifies to

$$u_1=-1+\frac{\left(-x-1\right)\ln\left(x+1\right)^2}{2}+\left(x+1\right)\ln\left(x+1\right)-\frac{x^2}{4}-\frac{x}{2}$$
$$u_2=\frac{\ln\left(x+1\right)\left(-\ln\left(x+1\right)-1+x\right)}{2+2x}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = & \frac{-1 + \frac{(-x-1)\ln(x+1)^2}{2} + (x+1)\ln(x+1) - \frac{x^2}{4} - \frac{x}{2}}{x+1} \\
& + \frac{\ln(x+1)\left(-\ln(x+1) - 1 + x\right)\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right)}{2+2x}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{(x+1)^2 c_1 + c_2}{x+1}\right) + \left(\frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x+1)^2 c_1 + c_2}{x+1} + \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4} \tag{1}
$$

Verification of solutions

$$
y = \frac{(x+1)^2 c_1 + c_2}{x+1} + \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) - x^2 - 2x - 4}{4x+4}
$$

Verified OK.

12.5.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = (x+1)^2, B = x+1, C = -1, f(x) = \ln(x+1)^2 + x - 1$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(x+1)^2 y''+(x+1)\,y'-y=0$$

In normal form the ode

$$(x+1)^2 y''+(x+1)\,y'-y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{1}{x+1}\\ q(x)&=-\frac{1}{(x+1)^2}\end{aligned}$$

Applying change of variables $\tau=g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau)=\frac{\tau''(x)+p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau)=\frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1=c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau'&=\frac{1}{c}\sqrt{q}\\ &=\frac{\sqrt{-\frac{1}{(x+1)^2}}}{c} \qquad (6)\\ \tau''&=\frac{1}{c\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3}\end{aligned}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{c\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)^3} + \frac{1}{x+1}\,\frac{\sqrt{-\frac{1}{(x+1)^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{(x+1)^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{(x+1)^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{(x+1)^2}}\,(x+1)\ln(x+1)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(c_2 i + c_1)\,x^2 + (2c_2 i + 2c_1)\,x + 2c_1}{2 + 2x}
$$

Now the particular solution to this ODE is found

$$
(x+1)^2\, y'' + (x+1)\, y' - y = \ln(x+1)^2 + x - 1
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{1}{x+1} \\
y_2 &= \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ \frac{d}{dx}\left(\frac{1}{x+1}\right) & \frac{d}{dx}\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1} \\ -\frac{1}{(x+1)^2} & -\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2} \end{vmatrix}$$

Therefore

$$\begin{aligned}
W = {} & \left(\frac{1}{x+1}\right)\left(-\frac{x^2}{(x+1)^2} + \frac{2x}{x+1} - \frac{2x}{(x+1)^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2}\right) \\
& - \left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right)\left(-\frac{1}{(x+1)^2}\right)
\end{aligned}$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Which simplifies to

$$W = \frac{2}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{x^2}{x+1} + \frac{2x}{x+1} + \frac{1}{x+1}\right)\left(\ln(x+1)^2 + x - 1\right)}{2+2x}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{\ln(x+1)^2}{2} + \frac{x}{2} - \frac{1}{2}\right) dx$$

Hence

$$u_1 = -\frac{x}{2} - \frac{x^2}{4} - \frac{\ln(x+1)^2 (x+1)}{2} + (x+1)\ln(x+1) - 1$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2 + x - 1}{x+1}}{2+2x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2 + x - 1}{2(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} + \frac{\ln(x+1)}{2}$$

Which simplifies to

$$\begin{aligned} u_1 &= -1 + \frac{(-x-1)\ln(x+1)^2}{2} + (x+1)\ln(x+1) - \frac{x^2}{4} - \frac{x}{2} \\ u_2 &= \frac{\ln(x+1)\left(-\ln(x+1) - 1 + x\right)}{2+2x} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-1+\frac{(-x-1)\ln(x+1)^2}{2}+(x+1)\ln(x+1)-\frac{x^2}{4}-\frac{x}{2}}{x+1} + \frac{\ln(x+1)\left(-\ln(x+1)-1+x\right)\left(\frac{x^2}{x+1}+\frac{2x}{x+1}+\frac{1}{x+1}\right)}{2+2x}$$

Which simplifies to

$$y_p(x) = \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)-x^2-2x-4}{4x+4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_2 i + c_1)x^2 + (2c_2 i + 2c_1)x + 2c_1}{2+2x}\right) \\ &\quad + \left(\frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)-x^2-2x-4}{4x+4}\right) \\ &= \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)-x^2-2x-4}{4x+4} \\ &\quad + \frac{(c_2 i + c_1)x^2 + (2c_2 i + 2c_1)x + 2c_1}{2+2x} \end{aligned}$$

Which simplifies to

$$y = \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)+(2c_2 i+2c_1-1)x^2+(4c_2 i+4c_1-2)x+4c_1-4}{4x+4}$$

Summary

The solution(s) found are the following

$$y = \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)+(2c_2 i+2c_1-1)x^2+(4c_2 i+4c_1-2)x+4c_1-4}{4x+4} \tag{1}$$

Verification of solutions

$$y = \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)+(2c_2 i+2c_1-1)x^2+(4c_2 i+4c_1-2)x+4c_1-4}{4x+4}$$

Verified OK.

12.5.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(\left(x+1\right)^{2} y''+\left(x+1\right) y'-y\right) dx = \int \left(\ln \left(x+\right.\right.$$

$$\left(-x-1\right) y+\left(x^{2}+2x+1\right) y' = x+\frac{x^{2}}{2}+\ln \left(x+1\right)^{2} \left(x+1\right)-2(x+1) \ln \left(x+1\right)+2+c_{1}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x+1} \\ q(x) &= \frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{2}} \end{aligned}$$

Hence the ode is

$$y'-\frac{y}{x+1} = \frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{2}}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x+1}dx} \\ &= \frac{1}{x+1} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu) \left(\frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{2}}\right) \\ \frac{d}{dx}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right) \left(\frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{2}}\right) \\ d\left(\frac{y}{x+1}\right) &= \left(\frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{3}}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x+1} &= \int \frac{(2+2x) \ln \left(x+1\right)^{2}+(-4x-4) \ln \left(x+1\right)+x^{2}+2x+2c_{1}+4}{2 \left(x+1\right)^{3}} dx \\ \frac{y}{x+1} &= -\frac{\ln \left(x+1\right)^{2}}{x+1}+\frac{\ln \left(x+1\right)}{2}-\frac{c_{1}}{2 \left(x+1\right)^{2}}-\frac{3}{4 \left(x+1\right)^{2}}+c_{2} \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = (x+1)\left(-\frac{\ln\left(x+1\right)^{2}}{x+1}+\frac{\ln\left(x+1\right)}{2}-\frac{c_1}{2\left(x+1\right)^{2}}-\frac{3}{4\left(x+1\right)^{2}}\right)+c_2(x+1)$$

Summary
The solution(s) found are the following

$$y = (x+1)\left(-\frac{\ln\left(x+1\right)^{2}}{x+1}+\frac{\ln\left(x+1\right)}{2}-\frac{c_1}{2\left(x+1\right)^{2}}-\frac{3}{4\left(x+1\right)^{2}}\right)+c_2(x+1) \quad (1)$$

Verification of solutions

$$y = (x+1)\left(-\frac{\ln\left(x+1\right)^{2}}{x+1}+\frac{\ln\left(x+1\right)}{2}-\frac{c_1}{2\left(x+1\right)^{2}}-\frac{3}{4\left(x+1\right)^{2}}\right)+c_2(x+1)$$

Verified OK.

12.5.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \quad (1)$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= (x+1)^2 \\
B &= x+1 \\
C &= -1 \\
F &= \ln (x+1)^2 + x - 1
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left((x+1)^2\right)(0) + (x+1)(1) + (-1)(x+1) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$(x+1)^3 v'' + \left(3(x+1)^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$(x+1)^2 \left((x+1) u'(x) + 3u(x)\right) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x+1}
\end{aligned}$$

Where $f(x) = -\frac{3}{x+1}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x+1}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x+1}\, dx \\
\ln (u) &= -3\ln (x+1) + c_1 \\
u &= \mathrm{e}^{-3\ln(x+1)+c_1} \\
&= \frac{c_1}{(x+1)^3}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{(x+1)^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{(x+1)^3}\, dx \\ &= -\frac{c_1}{2(x+1)^2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (x+1)\left(-\frac{c_1}{2(x+1)^2} + c_2\right) \\ &= \frac{2c_2(x+1)^2 - c_1}{2+2x} \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{2+2x} \\ y_2 &= \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{2+2x} & \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \\ \frac{d}{dx}\left(\frac{1}{2+2x}\right) & \frac{d}{dx}\left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{2+2x} & \frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x} \\ -\frac{2}{(2+2x)^2} & -\frac{4x^2}{(2+2x)^2} + \frac{4x}{2+2x} - \frac{8x}{(2+2x)^2} + \frac{4}{2+2x} - \frac{4}{(2+2x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{2+2x}\right)\left(-\frac{4x^2}{(2+2x)^2} + \frac{4x}{2+2x} - \frac{8x}{(2+2x)^2} + \frac{4}{2+2x} - \frac{4}{(2+2x)^2}\right) - \left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x}\right)\left(-\frac{2}{(2+2x)^2}\right)$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(\frac{2x^2}{2+2x} + \frac{4x}{2+2x} + \frac{2}{2+2x}\right)\left(\ln(x+1)^2 + x - 1\right)}{x+1}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\ln(x+1)^2 + x - 1\right) dx$$

Hence

$$u_1 = -x - \frac{x^2}{2} - \ln(x+1)^2(x+1) + 2(x+1)\ln(x+1) - 2$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2+x-1}{2+2x}}{x+1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\ln\left(x+1\right)^2+x-1}{2\left(x+1\right)^2}dx$$

Hence

$$u_2 = -\frac{\ln\left(x+1\right)^2}{2\left(x+1\right)} - \frac{\ln\left(x+1\right)}{x+1} + \frac{\ln\left(x+1\right)}{2}$$

Which simplifies to

$$u_1 = -2+(-x-1)\ln\left(x+1\right)^2+2(x+1)\ln\left(x+1\right)-\frac{x^2}{2}-x$$

$$u_2 = \frac{\ln\left(x+1\right)\left(-\ln\left(x+1\right)-1+x\right)}{2+2x}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) &= \frac{-2+(-x-1)\ln\left(x+1\right)^2+2(x+1)\ln\left(x+1\right)-\frac{x^2}{2}-x}{2+2x} \\ &\quad+\frac{\ln\left(x+1\right)\left(-\ln\left(x+1\right)-1+x\right)\left(\frac{2x^2}{2+2x}+\frac{4x}{2+2x}+\frac{2}{2+2x}\right)}{2+2x} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{(-4x-4)\ln\left(x+1\right)^2+2(x+1)^2\ln\left(x+1\right)-x^2-2x-4}{4x+4}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\frac{2c_2(x+1)^2-c_1}{2+2x}\right)+\left(\frac{(-4x-4)\ln\left(x+1\right)^2+2(x+1)^2\ln\left(x+1\right)-x^2-2x-4}{4x+4}\right) \\ &= \frac{(-4x-4)\ln\left(x+1\right)^2+2(x+1)^2\ln\left(x+1\right)+\left(4c_2-1\right)x^2+\left(8c_2-2\right)x-2c_1+4c_2-4}{4x+4} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (4c_2-1)x^2 + (8c_2-2)x - 2c_1 + 4c_2 - 4}{4x+4} \tag{1}$$

Verification of solutions

$$y = \frac{(-4x-4)\ln(x+1)^2 + 2(x+1)^2\ln(x+1) + (4c_2-1)x^2 + (8c_2-2)x - 2c_1 + 4c_2 - 4}{4x+4}$$

Verified OK.

12.5.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x+1)^2 y'' + (x+1) y' - y = \ln(x+1)^2 + x - 1$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x+1)^2 y'' + (x+1) y' - y\right) dx = \int (\ln (x +$$

$$(-x-1) y + \left(x^2+2x+1\right) y' = x + \frac{x^2}{2} + \ln(x+1)^2 (x+1) - 2(x+1)\ln(x+1) + 2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x+1}$$
$$q(x) = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x+1}dx} \\
&= \frac{1}{x+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(2+2x)\ln(x+1)^2+(-4x-4)\ln(x+1)+x^2+2x+2c_1+4}{2(x+1)^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right)\left(\frac{(2+2x)\ln(x+1)^2+(-4x-4)\ln(x+1)+x^2+2x+2c_1+4}{2(x+1)^2}\right) \\
\mathrm{d}\left(\frac{y}{x+1}\right) &= \left(\frac{(2+2x)\ln(x+1)^2+(-4x-4)\ln(x+1)+x^2+2x+2c_1+4}{2(x+1)^3}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x+1} &= \int \frac{(2+2x)\ln(x+1)^2+(-4x-4)\ln(x+1)+x^2+2x+2c_1+4}{2(x+1)^3}\mathrm{d}x \\
\frac{y}{x+1} &= -\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$
y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}\right)+c_2(x+1)
$$

Summary

The solution(s) found are the following

$$
y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}\right)+c_2(x+1) \quad (1)
$$

Verification of solutions

$$
y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}\right)+c_2(x+1)
$$

Verified OK.

12.5.6 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)^2 y'' + (x+1) y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= (x+1)^2 \\ B &= x+1 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4(x+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4(x+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4(x+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 225: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)^2}$$

For the pole at $x=-1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r=\frac{s}{t}=\frac{3}{4\left(x+1\right)^{2}}
$$

Since the $\gcd(s,t)=1$. This gives $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r=\frac{3}{4\left(x+1\right)^{2}}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d=\alpha_\infty^{s(\infty)}-\sum_{c\in\Gamma}\alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x+1)} + (-)(0) \\
&= -\frac{1}{2(x+1)} \\
&= -\frac{1}{2(x+1)}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x+1)^2}\right) + \left(-\frac{1}{2(x+1)}\right)^2 - \left(\frac{3}{4(x+1)^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2(x+1)} dx} \\
&= \frac{1}{\sqrt{x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x+1}{(x+1)^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x+1)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x+1}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x+1}{(x+1)^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{1}{2}x^2 + x\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x+1}\right) + c_2 \left(\frac{1}{x+1}\left(\frac{1}{2}x^2 + x\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$(x+1)^2\,y''+(x+1)\,y'-y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x+1}+\frac{c_2x(2+x)}{2+2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1&=\frac{1}{x+1}\\y_2&=\frac{x(2+x)}{2+2x}\end{aligned}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{x+1}&\frac{x(2+x)}{2+2x}\\\frac{d}{dx}\left(\frac{1}{x+1}\right)&\frac{d}{dx}\left(\frac{x(2+x)}{2+2x}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x+1} & \frac{x(2+x)}{2+2x} \\ -\frac{1}{(x+1)^2} & \frac{2+x}{2+2x} + \frac{x}{2+2x} - \frac{2x(2+x)}{(2+2x)^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x+1}\right)\left(\frac{2+x}{2+2x} + \frac{x}{2+2x} - \frac{2x(2+x)}{(2+2x)^2}\right) - \left(\frac{x(2+x)}{2+2x}\right)\left(-\frac{1}{(x+1)^2}\right)$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Which simplifies to

$$W = \frac{1}{x+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x(2+x)\left(\ln(x+1)^2+x-1\right)}{2+2x}}{x+1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x(2+x)\left(\ln(x+1)^2+x-1\right)}{2(x+1)^2} dx$$

Hence

$$\begin{aligned} u_1 = &-\frac{\ln(x+1)^2(x+1)}{2} + (x+1)\ln(x+1) \\ &- \frac{\ln(x+1)^2}{2(x+1)} - \frac{\ln(x+1)}{x+1} - \frac{(x+1)^2}{4} + \frac{\ln(x+1)}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\ln(x+1)^2+x-1}{x+1}}{x+1}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\ln(x+1)^2 + x - 1}{(x+1)^2} dx$$

Hence

$$u_2 = -\frac{\ln(x+1)^2}{x+1} - \frac{2\ln(x+1)}{x+1} + \ln(x+1)$$

Which simplifies to

$$u_1 = \frac{(-2x^2 - 4x - 4)\ln(x+1)^2 + (4x^2 + 10x + 2)\ln(x+1) - (x+1)^3}{4x+4}$$

$$u_2 = \frac{\ln(x+1)(-\ln(x+1) - 1 + x)}{x+1}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \frac{(-2x^2 - 4x - 4)\ln(x+1)^2 + (4x^2 + 10x + 2)\ln(x+1) - (x+1)^3}{(4x+4)(x+1)} \\ & + \frac{\ln(x+1)(-\ln(x+1) - 1 + x)\, x(2+x)}{(x+1)(2+2x)} \end{aligned}$$

Which simplifies to

$$y_p(x) = -\ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{1}{4} - \frac{x}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x+1} + \frac{c_2 x(2+x)}{2+2x}\right) + \left(-\ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{1}{4} - \frac{x}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x+1} + \frac{c_2 x(2+x)}{2+2x} - \ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{1}{4} - \frac{x}{4} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x+1} + \frac{c_2 x(2+x)}{2+2x} - \ln(x+1)^2 + \frac{(2+2x)\ln(x+1)}{4} - \frac{1}{4} - \frac{x}{4}$$

Verified OK.

12.5.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=(x+1)^2\\ q(x)&=x+1\\ r(x)&=-1\\ s(x)&=\ln\left(x+1\right)^2+x-1\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=2\\ q'(x)&=1\end{aligned}$$

Therefore (1) becomes

$$2-(1)+(-1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$\left(x+1\right)^2y'+\left(-x-1\right)y=\int\ln\left(x+1\right)^2+x-1\,dx$$

We now have a first order ode to solve which is

$$\left(x+1\right)^2y'+\left(-x-1\right)y=x+\frac{x^2}{2}+\ln\left(x+1\right)^2\left(x+1\right)-2(x+1)\ln\left(x+1\right)+2+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x+1}\\ q(x) &= \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x+1} = \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x+1}dx}\\ &= \frac{1}{x+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x+1}\right) &= \left(\frac{1}{x+1}\right)\left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^2}\right)\\ \mathrm{d}\left(\frac{y}{x+1}\right) &= \left(\frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x+1} &= \int \frac{(2+2x)\ln(x+1)^2 + (-4x-4)\ln(x+1) + x^2 + 2x + 2c_1 + 4}{2(x+1)^3}\,\mathrm{d}x\\ \frac{y}{x+1} &= -\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x+1}$ results in

$$y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1} + \frac{\ln(x+1)}{2} - \frac{c_1}{2(x+1)^2} - \frac{3}{4(x+1)^2}\right) + c_2(x+1)$$

Summary

The solution(s) found are the following

$$y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}\right)+c_2(x+1) \quad (1)$$

Verification of solutions

$$y = (x+1)\left(-\frac{\ln(x+1)^2}{x+1}+\frac{\ln(x+1)}{2}-\frac{c_1}{2(x+1)^2}-\frac{3}{4(x+1)^2}\right)+c_2(x+1)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 53

```
dsolve((x+1)^2*diff(y(x),x$2)+(x+1)*diff(y(x),x)-y(x)=(ln(x+1))^2+x-1,y(x), singsol=all)
```

$$y(x) = \frac{(-4x-4)\ln(x+1)^2+2(x+1)^2\ln(x+1)+4c_2x^2+8c_2x+4c_1+4c_2-3}{4x+4}$$

✓ Solution by Mathematica

Time used: 0.262 (sec). Leaf size: 72

```
DSolve[(x+1)^2*y''[x]+(x+1)*y'[x]-y[x]==(Log[x+1])^2+x-1,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{(-1+2c_1+2ic_2)x^2-4(x+1)\log^2(x+1)+2(x+1)^2\log(x+1)+(-2+4c_1+4ic_2)x-1+4c_1}{4(x+1)}$$

12.6 problem 11

12.6.1 Solving as second order integrable as is ode 1893
12.6.2 Solving as type second_order_integrable_as_is (not using ABC version) 1895
12.6.3 Solving using Kovacic algorithm 1897
12.6.4 Solving as exact linear second order ode ode 1905

Internal problem ID [5411]
Internal file name [`OUTPUT/4902_Sunday_February_04_2024_12_47_02_AM_76936695/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 17. Linear equations with variable coefficients (Cauchy and Legndre). Supplemetary problems. Page 110
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{(1+2x)^2 y'' - 2(1+2x) y' - 12y = 6x}$$

12.6.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(4\left(\frac{1}{2}+x\right)^2 y'' + (-4x-2) y' - 12y\right) dx = \int 6x dx$$
$$(-12x-6) y + \left(4x^2+4x+1\right) y' = 3x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{6}{1+2x} \\
q(x) &= \frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{6y}{1+2x} = \frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{6}{1+2x}dx} \\
&= \frac{1}{(1+2x)^3}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}\right) \\
\frac{d}{dx}\left(\frac{y}{(1+2x)^3}\right) &= \left(\frac{1}{(1+2x)^3}\right)\left(\frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}\right) \\
d\left(\frac{y}{(1+2x)^3}\right) &= \left(\frac{3x^2+c_1}{(1+2x)^5}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{(1+2x)^3} &= \int \frac{3x^2+c_1}{(1+2x)^5}\, dx \\
\frac{y}{(1+2x)^3} &= -\frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} - \frac{\frac{3}{8}+\frac{c_1}{2}}{4(1+2x)^4} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$
y = (1+2x)^3\left(-\frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} - \frac{\frac{3}{8}+\frac{c_1}{2}}{4(1+2x)^4}\right) + c_2(1+2x)^3
$$

which simplifies to

$$
y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)x^2 + (256c_2-8)x - 4c_1 + 32c_2 - 1}{32+64x}
$$

Summary
The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)\,x^2 + (256c_2 - 8)\,x - 4c_1 + 32c_2 - 1}{32 + 64x} \tag{1}$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2 - 24)\,x^2 + (256c_2 - 8)\,x - 4c_1 + 32c_2 - 1}{32 + 64x}$$

Verified OK.

12.6.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$4\left(\frac{1}{2} + x\right)^2 y'' + (-4x - 2)\,y' - 12y = 6x$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(4\left(\frac{1}{2} + x\right)^2 y'' + (-4x - 2)\,y' - 12y\right) dx = \int 6x dx$$

$$(-12x - 6)\,y + \left(4x^2 + 4x + 1\right) y' = 3x^2 + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{6}{1 + 2x}$$
$$q(x) = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

Hence the ode is

$$y' - \frac{6y}{1 + 2x} = \frac{3x^2 + c_1}{4\left(\frac{1}{2} + x\right)^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{6}{1+2x}dx}$$
$$= \frac{1}{(1+2x)^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{(1+2x)^3}\right) = \left(\frac{1}{(1+2x)^3}\right)\left(\frac{3x^2+c_1}{4\left(\frac{1}{2}+x\right)^2}\right)$$
$$d\left(\frac{y}{(1+2x)^3}\right) = \left(\frac{3x^2+c_1}{(1+2x)^5}\right) dx$$

Integrating gives

$$\frac{y}{(1+2x)^3} = \int \frac{3x^2+c_1}{(1+2x)^5} dx$$
$$\frac{y}{(1+2x)^3} = -\frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} - \frac{\frac{3}{8}+\frac{c_1}{2}}{4(1+2x)^4} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$y = (1+2x)^3\left(-\frac{3}{16(1+2x)^2} + \frac{1}{4(1+2x)^3} - \frac{\frac{3}{8}+\frac{c_1}{2}}{4(1+2x)^4}\right) + c_2(1+2x)^3$$

which simplifies to

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)x^2 + (256c_2-8)x - 4c_1 + 32c_2 - 1}{32+64x}$$

Summary

The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)x^2 + (256c_2-8)x - 4c_1 + 32c_2 - 1}{32+64x} \tag{1}$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)x^2 + (256c_2-8)x - 4c_1 + 32c_2 - 1}{32+64x}$$

Verified OK.

12.6.3 Solving using Kovacic algorithm

Writing the ode as

$$4\left(\frac{1}{2}+x\right)^{2} y''+(-4x-2)\,y'-12y=0 \tag{1}$$

$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=4\left(\frac{1}{2}+x\right)^{2}\\ B&=-4x-2\\ C&=-12\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r&=\frac{s}{t}\\ &=\frac{2AB'-2BA'+B^2-4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{15}{(1+2x)^{2}} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s&=15\\ t&=(1+2x)^{2}\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{15}{(1+2x)^{2}}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 226: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (1 + 2x)^2$. There is a pole at $x = -\frac{1}{2}$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4\left(\frac{1}{2}+x\right)^2}$$

For the pole at $x = -\frac{1}{2}$ let b be the coefficient of $\frac{1}{\left(\frac{1}{2}+x\right)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{2}\\ \alpha_c^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{(1+2x)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2}+\sqrt{1+4b} = \frac{5}{2}\\ \alpha_\infty^- &= \frac{1}{2}-\sqrt{1+4b} = -\frac{3}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{(1+2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$-\frac{1}{2}$	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{3}{2\left(\frac{1}{2}+x\right)} + (-)\,(0) \\
&= -\frac{3}{2\left(\frac{1}{2}+x\right)} \\
&= -\frac{3}{1+2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2\left(\frac{1}{2}+x\right)}\right)(0) + \left(\left(\frac{3}{2\left(\frac{1}{2}+x\right)^2}\right) + \left(-\frac{3}{2\left(\frac{1}{2}+x\right)}\right)^2 - \left(\frac{15}{(1+2x)^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{3}{2\left(\frac{1}{2}+x\right)} dx} \\
&= \frac{1}{(1+2x)^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x-2}{4\left(\frac{1}{2}+x\right)^2} \, dx} \\
&= z_1 e^{\frac{\ln(1+2x)}{2}} \\
&= z_1 \left(\sqrt{1+2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{1+2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x-2}{4\left(\frac{1}{2}+x\right)^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(1+2x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{(1+2x)^4}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{1+2x}\right) + c_2 \left(\frac{1}{1+2x}\left(\frac{(1+2x)^4}{8}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
4\left(\frac{1}{2}+x\right)^2 y'' + (-4x-2)\, y' - 12y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{1+2x} + \frac{c_2(1+2x)^3}{8}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{1+2x} \\
y_2 &= \frac{(1+2x)^3}{8}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{1+2x} & \frac{(1+2x)^3}{8} \\ \frac{d}{dx}\left(\frac{1}{1+2x}\right) & \frac{d}{dx}\left(\frac{(1+2x)^3}{8}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{1+2x} & \frac{(1+2x)^3}{8} \\ -\frac{2}{(1+2x)^2} & \frac{3(1+2x)^2}{4} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{1+2x}\right)\left(\frac{3(1+2x)^2}{4}\right) - \left(\frac{(1+2x)^3}{8}\right)\left(-\frac{2}{(1+2x)^2}\right)$$

Which simplifies to

$$W = 1 + 2x$$

Which simplifies to

$$W = 1 + 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{3x(1+2x)^3}{4}}{4\left(\frac{1}{2}+x\right)^2(1+2x)}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{3x}{4}dx$$

Hence

$$u_1 = -\frac{3x^2}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{6x}{1+2x}}{4\left(\frac{1}{2}+x\right)^2(1+2x)}\,dx$$

Which simplifies to

$$u_2 = \int \frac{6x}{(1+2x)^4} dx$$

Hence

$$u_2 = -\frac{3}{4(1+2x)^2} + \frac{1}{2(1+2x)^3}$$

Which simplifies to

$$u_1 = -\frac{3x^2}{8}$$
$$u_2 = \frac{-6x-1}{4(1+2x)^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{3x^2}{8(1+2x)} - \frac{3x}{16} - \frac{1}{32}$$

Which simplifies to

$$y_p(x) = \frac{-24x^2-8x-1}{32+64x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{1+2x} + \frac{c_2(1+2x)^3}{8}\right) + \left(\frac{-24x^2-8x-1}{32+64x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{2\left(\frac{1}{2}+x\right)^4 c_2 + c_1}{1+2x} + \frac{-24x^2-8x-1}{32+64x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\left(\frac{1}{2}+x\right)^4 c_2 + c_1}{1+2x} + \frac{-24x^2-8x-1}{32+64x} \tag{1}$$

Verification of solutions

$$y = \frac{2\left(\frac{1}{2}+x\right)^{4} c_2 + c_1}{1+2x} + \frac{-24x^2-8x-1}{32+64x}$$

Verified OK.

12.6.4 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 4\left(\frac{1}{2}+x\right)^{2} \\ q(x) &= -4x-2 \\ r(x) &= -12 \\ s(x) &= 6x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 8 \\ q'(x) &= -4 \end{aligned}$$

Therefore (1) becomes

$$8 - (-4) + (-12) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + \left(q(x) - p'(x)\right)y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + \left(q(x) - p'(x)\right)y = \int s(x)\; dx$$

Substituting the above values for p, q, r, s gives

$$4\left(\frac{1}{2}+x\right)^{2} y' + (-12x-6)\,y = \int 6x\,dx$$

We now have a first order ode to solve which is

$$4\left(\frac{1}{2}+x\right)^{2} y'+\left(-12x-6\right) y = 3x^{2}+c_{1}$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{6}{1+2x}\\
q(x) &= \frac{3x^{2}+c_{1}}{\left(1+2x\right)^{2}}
\end{aligned}$$

Hence the ode is

$$y' - \frac{6y}{1+2x} = \frac{3x^{2}+c_{1}}{\left(1+2x\right)^{2}}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{6}{1+2x}dx}\\
&= \frac{1}{\left(1+2x\right)^{3}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3x^{2}+c_{1}}{\left(1+2x\right)^{2}}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\left(1+2x\right)^{3}}\right) &= \left(\frac{1}{\left(1+2x\right)^{3}}\right)\left(\frac{3x^{2}+c_{1}}{\left(1+2x\right)^{2}}\right)\\
\mathrm{d}\left(\frac{y}{\left(1+2x\right)^{3}}\right) &= \left(\frac{3x^{2}+c_{1}}{\left(1+2x\right)^{5}}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{\left(1+2x\right)^{3}} &= \int \frac{3x^{2}+c_{1}}{\left(1+2x\right)^{5}}\mathrm{d}x\\
\frac{y}{\left(1+2x\right)^{3}} &= -\frac{3}{16\left(1+2x\right)^{2}}+\frac{1}{4\left(1+2x\right)^{3}}-\frac{\frac{3}{8}+\frac{c_{1}}{2}}{4\left(1+2x\right)^{4}}+c_{2}
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(1+2x)^3}$ results in

$$y = (1+2x)^3 \left(-\frac{3}{16\,(1+2x)^2} + \frac{1}{4\,(1+2x)^3} - \frac{\frac{3}{8}+\frac{c_1}{2}}{4\,(1+2x)^4}\right) + c_2(1+2x)^3$$

which simplifies to

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)\,x^2 + (256c_2-8)\,x - 4c_1 + 32c_2 - 1}{32+64x}$$

Summary

The solution(s) found are the following

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)\,x^2 + (256c_2-8)\,x - 4c_1 + 32c_2 - 1}{32+64x} \tag{1}$$

Verification of solutions

$$y = \frac{512c_2x^4 + 1024c_2x^3 + (768c_2-24)\,x^2 + (256c_2-8)\,x - 4c_1 + 32c_2 - 1}{32+64x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
dsolve((2*x+1)^2*diff(y(x),x$2)-2*(2*x+1)*diff(y(x),x)-12*y(x)=6*x,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{2x+1} + (2x+1)^3\,c_2 + \frac{-24x^2-8x-1}{64x+32}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 41

```
DSolve[(2*x+1)^2*y''[x]-2*(2*x+1)*y'[x]-12*y[x]==6*x,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{-24x^2 - 8x + 32c_1(2x+1)^4 - 1 + 32c_2}{32(2x+1)}$$

13 Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120

13.1 problem 21 1910
13.2 problem 22 1924
13.3 problem 23 1943
13.4 problem 24 1962
13.5 problem 25 1972
13.6 problem 26 1979
13.7 problem 27 1989
13.8 problem 28 1998
13.9 problem 29 2008
13.10 problem 30 2024
13.11 problem 31 2048
13.12 problem 32 2072
13.13 problem 33 2079
13.14 problem 35 2112
13.15 problem 36 2128
13.16 problem 37 2139
13.17 problem 38 2156

13.1 problem 21

13.1.1 Solving using Kovacic algorithm 1910
13.1.2 Solving as second order ode lagrange adjoint equation method ode 1916
13.1.3 Maple step by step solution 1920

Internal problem ID [5412]
Internal file name [`OUTPUT/4903_Sunday_February_04_2024_12_47_04_AM_2677231/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[_Laguerre]`

$$\boxed{xy'' - (2 + x)\, y' + 2y = 0}$$

13.1.1 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + (-x - 2)\, y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -x - 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 - 4x + 8}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 - 4x + 8 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{x^2 - 4x + 8}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 227: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{1}{4} - \frac{1}{x} + \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{1}{2} - \frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{2}{x^5} - \frac{6}{x^6} - \frac{28}{x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\ &= \frac{1}{2} \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 - 4x + 8}{4x^2} \\ &= Q + \frac{R}{4x^2} \\ &= \left(\frac{1}{4}\right) + \left(\frac{-4x + 8}{4x^2}\right) \\ &= \frac{1}{4} + \frac{-4x + 8}{4x^2} \end{aligned}$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -4. Dividing this by leading coefficient in t which is 4 gives -1. Now b can be found.

$$\begin{aligned} b &= (-1) - (0) \\ &= -1 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{1}{2} \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-1}{\frac{1}{2}} - 0\right) &= -1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-1}{\frac{1}{2}} - 0\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{x^2 - 4x + 8}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
0	$\frac{1}{2}$	-1	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = -1$ then

$$\begin{aligned} d &= \alpha_\infty^+ - (\alpha_{c_1}^-) \\ &= -1 - (-1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (+)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + \left(\frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{1}{x} \\
&= \frac{-2+x}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2} - \frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(\frac{1}{2} - \frac{1}{x}\right)^2 - \left(\frac{x^2-4x+8}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(\frac{1}{2} - \frac{1}{x}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{x}{2}}}{x}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x-2}{x} \, dx} \\
&= z_1 e^{\frac{x}{2} + \ln(x)} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}} x\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x-2}{x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{x+2\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(-\left(x^2+2x+2\right)\mathrm{e}^{-x}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\mathrm{e}^{x} + c_2\left(-x^2-2x-2\right) \tag{1}$$

Verification of solutions

$$y = c_1\mathrm{e}^{x} + c_2\left(-x^2-2x-2\right)$$

Verified OK.

13.1.2 Solving as second order ode lagrange adjoint equation method ode

In normal form the ode

$$xy'' + (-x-2)\,y' + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-x-2}{x} \\
q(x) &= \frac{2}{x} \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(-x-2)\,\xi(x)}{x}\right)' + \left(\frac{2\xi(x)}{x}\right) &= 0 \\
\xi''(x) - \frac{(-x-2)\,\xi'(x)}{x} + \left(\frac{3}{x} + \frac{-x-2}{x^2}\right)\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the ode

$$
\xi''(x)\, x^2 + \left(x^2+2x\right)\xi'(x) + (2x-2)\,\xi(x) = 0 \tag{1}
$$

Becomes

$$
\xi''(x) + p(x)\,\xi'(x) + q(x)\,\xi(x) = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{2+x}{x} \\
q(x) &= \frac{2x-2}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $\xi(x) = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $\xi(x)$.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n(2+x)}{x^2} + \frac{2x-2}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$n = -2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(-\frac{4}{x} + \frac{2+x}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(-2+x)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2+x)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(-2+x)\, u}{x} \end{aligned}$$

Where $f(x) = -\frac{-2+x}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{-2+x}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{-2+x}{x}\, dx \\ \ln(u) &= -x + 2\ln(x) + c_1 \\ u &= \mathrm{e}^{-x+2\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{-x+2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1 x^2 \mathrm{e}^{-x}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\left(x^2+2x+2\right) c_1 \mathrm{e}^{-x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\xi(x) &= v(x)\,x^n \\
&= \frac{-(x^2+2x+2)\,c_1\mathrm{e}^{-x} + c_2}{x^2} \\
&= \frac{-(x^2+2x+2)\,c_1\mathrm{e}^{-x} + c_2}{x^2}
\end{aligned}
$$

The original ode (2) now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

$$
y' + y\left(\frac{-x-2}{x} - \frac{\left(\frac{-(2+2x)c_1\mathrm{e}^{-x}+(x^2+2x+2)c_1\mathrm{e}^{-x}}{x^2} - \frac{2\left(-(x^2+2x+2)c_1\mathrm{e}^{-x}+c_2\right)}{x^3}\right)x^2}{-\left(x^2+2x+2\right)c_1\mathrm{e}^{-x}+c_2}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y(c_2\mathrm{e}^{x} - 2c_1x - 2c_1)}{-c_1x^2 + c_2\mathrm{e}^{x} - 2c_1x - 2c_1}
\end{aligned}
$$

Where $f(x) = \frac{c_2\mathrm{e}^{x}-2c_1x-2c_1}{-c_1x^2+c_2\mathrm{e}^{x}-2c_1x-2c_1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{c_2\mathrm{e}^{x} - 2c_1x - 2c_1}{-c_1x^2 + c_2\mathrm{e}^{x} - 2c_1x - 2c_1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{c_2\mathrm{e}^{x} - 2c_1x - 2c_1}{-c_1x^2 + c_2\mathrm{e}^{x} - 2c_1x - 2c_1}\,dx \\
\ln(y) &= \ln\left(-c_1x^2 + c_2\mathrm{e}^{x} - 2c_1x - 2c_1\right) + c_3 \\
y &= \mathrm{e}^{\ln(-c_1x^2+c_2\mathrm{e}^{x}-2c_1x-2c_1)+c_3} \\
&= c_3\left(-c_1x^2 + c_2\mathrm{e}^{x} - 2c_1x - 2c_1\right)
\end{aligned}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3\left(-c_1x^2 + c_2 e^x - 2c_1x - 2c_1\right)$$

Summary
The solution(s) found are the following

$$y = c_3\left(-c_1x^2 + c_2 e^x - 2c_1x - 2c_1\right) \tag{1}$$

Verification of solutions

$$y = c_3\left(-c_1x^2 + c_2 e^x - 2c_1x - 2c_1\right)$$

Verified OK.

13.1.3 Maple step by step solution

Let's solve

$y''x + (-x-2)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x} + \frac{(2+x)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(2+x)y'}{x} + \frac{2y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2+x}{x}, P_3(x) = \frac{2}{x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = -2$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

  $y''x + (-x-2)\,y' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$

  - Shift index using $k- >k+1-m$

    $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-2) - a_k(k+r-2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-2)\,(a_{k+1}(k+1+r) - a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+1+r}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+1}$

- Solution for $r=0$

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k}{k+1}\right]$$

- Recursion relation for $r=3$

$$a_{k+1}=\frac{a_k}{k+4}$$

- Solution for $r=3$

$$\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1}=\frac{a_k}{k+4}\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+4}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x$2)-(x+2)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x)=\mathrm{e}^{x} c_1+c_2\left(x^2+2x+2\right)$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[x*y''[x]-(x+2)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x - c_2 (x^2 + 2x + 2)$$

13.2 problem 22

13.2.1 Solving as second order change of variable on y method 2 ode . 1924
13.2.2 Solving as second order ode non constant coeff transformation on B ode 1929
13.2.3 Solving using Kovacic algorithm 1934

Internal problem ID [5413]
Internal file name [`OUTPUT/4904_Sunday_February_04_2024_12_47_05_AM_64049652/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=2}$$

13.2.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2+1, B=-2x, C=2, f(x)=2$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(x^2+1)\,y''-2xy'+2y=0$$

In normal form the ode

$$\left(x^{2}+1\right) y''-2 x y'+2 y=0 \tag{1}$$

Becomes

$$y''+p(x) y'+q(x) y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2 x}{x^{2}+1} \\ q(x) &= \frac{2}{x^{2}+1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x) x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2 n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{n p}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^{2}}+\frac{n p}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^{2}}-\frac{2 n}{x^{2}+1}+\frac{2}{x^{2}+1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}-\frac{2 x}{x^{2}+1}\right) v'(x) &= 0 \\ v''(x)+\frac{2 v'(x)}{x\left(x^{2}+1\right)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{2u(x)}{x\left(x^{2}+1\right)}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u}{x\left(x^{2}+1\right)}
\end{aligned}$$

Where $f(x)=-\frac{2}{x(x^2+1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\ln\left(u\right) &= \ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}\\
u &= \mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}}\\
&= c_{1}\mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_{1}\left(1+\frac{1}{x^{2}}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_{2}\\
&= c_{1}\left(x-\frac{1}{x}\right)+c_{2}
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^{n}\\
&= \left(c_{1}\left(x-\frac{1}{x}\right)+c_{2}\right)x\\
&= c_{1}x^{2}+c_{2}x-c_{1}
\end{aligned}$$

Now the particular solution to this ODE is found

$$
(x^2+1)\,y''-2xy'+2y=2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x)=u_1y_1+u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^2-1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W=\begin{vmatrix} x & x^2-1 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2-1) \end{vmatrix}
$$

Which gives

$$
W=\begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix}
$$

Therefore

$$
W=(x)\,(2x)-(x^2-1)\,(1)
$$

Which simplifies to

$$W = x^2 + 1$$

Which simplifies to

$$W = x^2 + 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^2 - 2}{\left(x^2 + 1\right)^2} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{2x^2 - 2}{\left(x^2 + 1\right)^2} dx$$

Hence

$$u_1 = \frac{2x}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x}{\left(x^2 + 1\right)^2} \, dx$$

Which simplifies to

$$u_2 = \int \frac{2x}{\left(x^2 + 1\right)^2} dx$$

Hence

$$u_2 = -\frac{1}{x^2 + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x^2}{x^2 + 1} - \frac{x^2 - 1}{x^2 + 1}$$

Which simplifies to

$$y_p(x) = 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x\right) + (1) \\
&= 1 + \left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x
\end{aligned}
$$

Which simplifies to

$$
y = c_1x^2 + c_2x - c_1 + 1
$$

Summary
The solution(s) found are the following

$$
y = c_1x^2 + c_2x - c_1 + 1 \tag{1}
$$

Verification of solutions

$$
y = c_1x^2 + c_2x - c_1 + 1
$$

Verified OK.

13.2.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)\,v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= -2x \\ C &= 2 \\ F &= 2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2 + 1\right)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x\left(x^2 + 1\right) v'' + (-4) v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(-2x^3 - 2x\right) u'(x) - 4u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= -\frac{2u}{x\left(x^2 + 1\right)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^2+1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^2+1\right)}\,dx \\
\ln\left(u\right) &= \ln\left(x^2+1\right) - 2\ln\left(x\right) + c_1 \\
u &= \mathrm{e}^{\ln\left(x^2+1\right) - 2\ln\left(x\right) + c_1} \\
&= c_1 \mathrm{e}^{\ln\left(x^2+1\right) - 2\ln\left(x\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(1 + \frac{1}{x^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1\left(1 + \frac{1}{x^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{\left(x^2+1\right)c_1}{x^2}\,\mathrm{d}x \\
&= c_1\left(x - \frac{1}{x}\right) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)\left(c_1\left(x - \frac{1}{x}\right) + c_2\right) \\
&= -2c_1x^2 - 2c_2x + 2c_1
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= -2x^2 + 2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & -2x^2 + 2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 2) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x & -2x^2 + 2 \\ 1 & -4x \end{vmatrix}
$$

Therefore

$$
W = (x)(-4x) - (-2x^2 + 2)(1)
$$

Which simplifies to

$$
W = -2x^2 - 2
$$

Which simplifies to

$$
W = -2x^2 - 2
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{-4x^2 + 4}{(x^2 + 1)(-2x^2 - 2)}\, dx
$$

Which simplifies to

$$u_1 = -\int \frac{2x^2 - 2}{\left(x^2+1\right)^2} dx$$

Hence

$$u_1 = \frac{2x}{x^2+1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x}{\left(x^2+1\right)\left(-2x^2-2\right)}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{x}{\left(x^2+1\right)^2} dx$$

Hence

$$u_2 = \frac{1}{2x^2+2}$$

Which simplifies to

$$u_1 = \frac{2x}{x^2+1}$$
$$u_2 = \frac{1}{2x^2+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2x^2}{x^2+1} + \frac{-2x^2+2}{2x^2+2}$$

Which simplifies to

$$y_p(x) = 1$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-2c_1x^2 - 2c_2x + 2c_1\right) + (1) \\ &= -2c_1x^2 - 2c_2x + 2c_1 + 1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -2c_1x^2 - 2c_2x + 2c_1 + 1 \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 - 2c_2x + 2c_1 + 1$$

Verified OK.

13.2.3 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*}(x^2+1)\,y'' - 2xy' + 2y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2}\end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*}A &= x^2+1\\ B &= -2x \tag{3}\\ C &= 2\end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*}r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*}s &= -3\\ t &= (x^2+1)^2\end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 229: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\ &= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\ &= \frac{-2i+x}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(\frac{1}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)}+\frac{3}{2(x+i)}\right)dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\, dx \\
&= y_1\left(-\frac{x}{(x+i)^2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2}\right) + c_2 \left(\frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2} \left(-\frac{x}{\left(x+i\right)^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
\left(x^2+1\right) y'' - 2xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 \left(x^2+1\right)^2}{\left(ix+1\right)^2} + \frac{c_2 \left(x^2+1\right)^2 x}{\left(x-i\right)^2 \left(x+i\right)^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{\left(x^2+1\right)^2}{\left(ix+1\right)^2} \\
y_2 &= \frac{\left(x^2+1\right)^2 x}{\left(x-i\right)^2 \left(x+i\right)^2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2} \\ \frac{d}{dx}\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) & \frac{d}{dx}\left(\frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2} \\ \frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3} & \frac{4(x^2+1)x^2}{(x-i)^2(x+i)^2} + \frac{(x^2+1)^2}{(x-i)^2(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^3(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^2(x+i)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2+1)^2}{(ix+1)^2}\right)\left(\frac{4(x^2+1)\,x^2}{(x-i)^2\,(x+i)^2} + \frac{(x^2+1)^2}{(x-i)^2\,(x+i)^2} - \frac{2(x^2+1)^2\,x}{(x-i)^3\,(x+i)^2} - \frac{2(x^2+1)^2\,x}{(x-i)^2\,(x+i)^3}\right) - \left(\frac{(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2}\right)\left(\frac{4(x^2+1)\,x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3}\right)$$

Which simplifies to

$$W = \frac{(x^2+1)^3\,(ix^5+3x^4-2ix^3+2x^2-3ix-1)}{(ix+1)^3\,(-x+i)^3\,(x+i)^3}$$

Which simplifies to

$$W = \frac{(x^2+1)^4\,(-x^3+3ix^2+3x-i)}{(-ix-1)^6\,(x+i)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2(x^2+1)^2 x}{(x-i)^2(x+i)^2}}{\frac{(x^2+1)^5(-x^3+3ix^2+3x-i)}{(-ix-1)^6(x+i)^3}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2x(x+i)\,(x-i)^4}{(x^3-3ix^2-3x+i)\,(x^2+1)^3}dx$$

Hence

$$u_1 = -\frac{x^2}{x^2+1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2(x^2+1)^2}{(ix+1)^2}}{\frac{(x^2+1)^5(-x^3+3ix^2+3x-i)}{(-ix-1)^6(x+i)^3}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2(x+i)^3(x-i)^4}{(x^2+1)^3(-x^3+3ix^2+3x-i)}dx$$

Hence

$$u_2 = \frac{2ix}{-x+i}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2(x^2+1)}{(ix+1)^2} + \frac{2ix^2(x^2+1)^2}{(-x+i)(x-i)^2(x+i)^2}$$

Which simplifies to

$$y_p(x) = \frac{x^2(x^2+1)^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(x+i)^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2(x+i)^2}\right) + \left(\frac{x^2(x^2+1)^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(x+i)^2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2(x+i)^2} + \frac{x^2(x^2+1)^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(x+i)^2} \tag{1}$$

Verification of solutions

$$y=\frac{c_1(x^2+1)^2}{(ix+1)^2}+\frac{c_2(x^2+1)^2 x}{(x-i)^2(x+i)^2}+\frac{x^2(x^2+1)^2(-x^3+3ix^2+3x-i)}{(-x+i)^5(x+i)^2}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=2,y(x), singsol=all)
```

$$y(x)=c_1x^2+c_2x-c_1+1$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 22

```
DSolve[(1+x^2)*y''[x]-2*x*y'[x]+2*y[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -c_1(x-i)^2+c_2x+1$$

13.3 problem 23

13.3.1 Solving as second order change of variable on y method 2 ode . 1943
13.3.2 Solving as second order ode non constant coeff transformation on B ode 1948
13.3.3 Solving using Kovacic algorithm 1953

Internal problem ID [5414]
Internal file name [`OUTPUT/4905_Sunday_February_04_2024_12_47_07_AM_82207300/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+4)\,y''-2xy'+2y=8}$$

13.3.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2+4, B=-2x, C=2, f(x)=8$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(x^2+4)\,y''-2xy'+2y=0$$

In normal form the ode

$$\left(x^2+4\right) y''-2xy'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2x}{x^2+4} \\ q(x) &= \frac{2}{x^2+4} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{2n}{x^2+4}+\frac{2}{x^2+4}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}-\frac{2x}{x^2+4}\right) v'(x) &= 0 \\ v''(x)+\frac{8v'(x)}{\left(x^2+4\right) x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x)+\frac{8u(x)}{(x^2+4)\,x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{8u}{(x^2+4)\,x}
\end{aligned}$$

Where $f(x)=-\frac{8}{(x^2+4)x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{8}{(x^2+4)\,x}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{8}{(x^2+4)\,x}\,dx\\
\ln(u) &= \ln\left(x^2+4\right)-2\ln(x)+c_1\\
u &= \mathrm{e}^{\ln(x^2+4)-2\ln(x)+c_1}\\
&= c_1\mathrm{e}^{\ln(x^2+4)-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(1+\frac{4}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_2\\
&= c_1\left(x-\frac{4}{x}\right)+c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n\\
&= \left(c_1\left(x-\frac{4}{x}\right)+c_2\right)x\\
&= c_1x^2+c_2x-4c_1
\end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2+4)\,y''-2xy'+2y=8$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x^2-4 \end{aligned}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2=\int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x & x^2-4 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2-4) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x & x^2-4 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W=(x)\,(2x)-(x^2-4)\,(1)$$

Which simplifies to

$$W = x^2 + 4$$

Which simplifies to

$$W = x^2 + 4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^2 - 32}{\left(x^2 + 4\right)^2} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{8x^2 - 32}{\left(x^2 + 4\right)^2} dx$$

Hence

$$u_1 = \frac{8x}{x^2 + 4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x}{\left(x^2 + 4\right)^2} \, dx$$

Which simplifies to

$$u_2 = \int \frac{8x}{\left(x^2 + 4\right)^2} dx$$

Hence

$$u_2 = -\frac{4}{x^2 + 4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8x^2}{x^2 + 4} - \frac{4(x^2 - 4)}{x^2 + 4}$$

Which simplifies to

$$y_p(x) = 4$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1 \left(x - \frac{4}{x}\right) + c_2\right) x\right) + (4) \\
&= 4 + \left(c_1 \left(x - \frac{4}{x}\right) + c_2\right) x
\end{aligned}
$$

Which simplifies to

$$
y = c_1 x^2 + c_2 x - 4c_1 + 4
$$

Summary
The solution(s) found are the following

$$
y = c_1 x^2 + c_2 x - 4c_1 + 4 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 + c_2 x - 4c_1 + 4
$$

Verified OK.

13.3.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 + 4 \\
B &= -2x \\
C &= 2 \\
F &= 8
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2 + 4\right)(0) + (-2x)(-2) + (2)(-2x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2\left(x^2 + 4\right) x v'' + (-16) v' = 0$$

Now by applying $v' = u$ the above becomes

$$\left(-2x^3 - 8x\right) u'(x) - 16u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{8u}{\left(x^2 + 4\right) x}
\end{aligned}$$

Where $f(x) = -\frac{8}{(x^2+4)x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{8}{(x^2+4)\,x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{8}{(x^2+4)\,x}\,dx \\
\ln(u) &= \ln(x^2+4) - 2\ln(x) + c_1 \\
u &= \mathrm{e}^{\ln(x^2+4)-2\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{\ln(x^2+4)-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(1 + \frac{4}{x^2}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u \\
&= c_1\left(1 + \frac{4}{x^2}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{(x^2+4)\,c_1}{x^2}\,\mathrm{d}x \\
&= c_1\left(x - \frac{4}{x}\right) + c_2
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)\left(c_1\left(x - \frac{4}{x}\right) + c_2\right) \\
&= -2c_1x^2 - 2c_2x + 8c_1
\end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= -2x^2 + 8
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & -2x^2 + 8 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 8) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x & -2x^2 + 8 \\ 1 & -4x \end{vmatrix}
$$

Therefore

$$
W = (x)(-4x) - (-2x^2 + 8)(1)
$$

Which simplifies to

$$
W = -2x^2 - 8
$$

Which simplifies to

$$
W = -2x^2 - 8
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{-16x^2 + 64}{(x^2 + 4)(-2x^2 - 8)}\, dx
$$

Which simplifies to

$$u_1 = -\int \frac{8x^2 - 32}{(x^2+4)^2} dx$$

Hence

$$u_1 = \frac{8x}{x^2+4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x}{(x^2+4)(-2x^2-8)} dx$$

Which simplifies to

$$u_2 = \int -\frac{4x}{(x^2+4)^2} dx$$

Hence

$$u_2 = \frac{2}{x^2+4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{8x^2}{x^2+4} + \frac{-4x^2+16}{x^2+4}$$

Which simplifies to

$$y_p(x) = 4$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-2c_1x^2 - 2c_2x + 8c_1\right) + (4) \\ &= -2c_1x^2 - 2c_2x + 8c_1 + 4 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -2c_1x^2 - 2c_2x + 8c_1 + 4 \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 - 2c_2x + 8c_1 + 4$$

Verified OK.

13.3.3 Solving using Kovacic algorithm

Writing the ode as

$$(x^2+4)\,y''-2xy'+2y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+4 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{-12}{(x^2+4)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -12 \\ t &= (x^2+4)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(-\frac{12}{(x^2+4)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 230: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+4)^2$. There is a pole at $x = 2i$ of order 2. There is a pole at $x = -2i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(-2i+x)^2} + \frac{3}{4(x+2i)^2} + \frac{3i}{8(-2i+x)} - \frac{3i}{8(x+2i)}$$

For the pole at $x = 2i$ let b be the coefficient of $\frac{1}{(-2i+x)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

For the pole at $x = -2i$ let b be the coefficient of $\frac{1}{(x+2i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{12}{(x^2+4)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
$2i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-2i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(-2i+x)} + \frac{3}{2(x+2i)} + (-)(0) \\
&= -\frac{1}{2(-2i+x)} + \frac{3}{2(x+2i)} \\
&= \frac{x-4i}{x^2+4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(-\frac{1}{2(-2i+x)} + \frac{3}{2(x+2i)}\right)(0) + \left(\left(\frac{1}{2(-2i+x)^2} - \frac{3}{2(x+2i)^2}\right) + \left(-\frac{1}{2(-2i+x)} + \frac{3}{2(x+2
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{1}{2(-2i+x)} + \frac{3}{2(x+2i)}\right) dx} \\
&= \frac{(x^2+4)^{\frac{3}{2}}}{(ix+2)^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+4}\,dx} \\
&= z_1 e^{\frac{\ln\left(x^2+4\right)}{2}} \\
&= z_1\left(\sqrt{x^2+4}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{\left(x^2+4\right)^2}{\left(ix+2\right)^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+4}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{\ln\left(x^2+4\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1\left(-\frac{x}{\left(x+2i\right)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{\left(x^2+4\right)^2}{\left(ix+2\right)^2}\right) + c_2\left(\frac{\left(x^2+4\right)^2}{\left(ix+2\right)^2}\left(-\frac{x}{\left(x+2i\right)^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x^2+4)\, y'' - 2xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 (x^2+4)^2}{(ix+2)^2} + c_2 x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{(x^2+4)^2}{(ix+2)^2} \\
y_2 &= x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{(x^2+4)^2}{(ix+2)^2} & x \\ \frac{d}{dx}\left(\frac{(x^2+4)^2}{(ix+2)^2}\right) & \frac{d}{dx}(x) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \frac{(x^2+4)^2}{(ix+2)^2} & x \\ \frac{4(x^2+4)x}{(ix+2)^2} - \frac{2i(x^2+4)^2}{(ix+2)^3} & 1 \end{vmatrix}
$$

Therefore

$$
W = \left(\frac{(x^2+4)^2}{(ix+2)^2}\right)(1) - (x)\left(\frac{4(x^2+4)\,x}{(ix+2)^2} - \frac{2i(x^2+4)^2}{(ix+2)^3}\right)
$$

Which simplifies to

$$
W = -\frac{(x^2+4)\,(ix^3+6x^2-12ix-8)}{(ix+2)^3}
$$

Which simplifies to

$$
W = -\frac{(x^2+4)\,(ix^3+6x^2-12ix-8)}{(ix+2)^3}
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{8x}{-\frac{(x^2+4)^2(ix^3+6x^2-12ix-8)}{(ix+2)^3}}\,dx
$$

Which simplifies to

$$
u_1 = -\int -\frac{8x(ix+2)^3}{(x^2+4)^2\,(ix^3+6x^2-12ix-8)}dx
$$

Hence

$$
u_1 = -\frac{x^2}{x^2+4}
$$

And Eq. (3) becomes

$$
u_2 = \int \frac{\frac{8(x^2+4)^2}{(ix+2)^2}}{-\frac{(x^2+4)^2(ix^3+6x^2-12ix-8)}{(ix+2)^3}}\,dx
$$

Which simplifies to

$$
u_2 = \int \frac{-16i+8x}{-x^3+6ix^2+12x-8i}dx
$$

Hence

$$u_2 = \frac{4ix}{2i - x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(x^2+4)\,x^2}{(ix+2)^2} + \frac{4ix^2}{2i - x}$$

Which simplifies to

$$y_p(x) = \frac{(x^2+4)\,x^2}{(-2i+x)^2} + \frac{4ix^2}{2i - x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1(x^2+4)^2}{(ix+2)^2} + c_2x\right) + \left(\frac{(x^2+4)\,x^2}{(-2i+x)^2} + \frac{4ix^2}{2i - x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1(x^2+4)^2}{(ix+2)^2} + c_2x + \frac{(x^2+4)\,x^2}{(-2i+x)^2} + \frac{4ix^2}{2i - x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1(x^2+4)^2}{(ix+2)^2} + c_2x + \frac{(x^2+4)\,x^2}{(-2i+x)^2} + \frac{4ix^2}{2i - x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 17

```
dsolve((x^2+4)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=8,y(x), singsol=all)
```

$$y(x) = c_1x^2 + c_2x - 4c_1 + 4$$

✓ Solution by Mathematica
Time used: 0.048 (sec). Leaf size: 22

```
DSolve[(x^2+4)*y''[x]-2*x*y'[x]+2*y[x]==8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x - 2i)^2 - c_2x + 4$$

13.4 problem 24

13.4.1 Solving using Kovacic algorithm 1962

Internal problem ID [5415]
Internal file name [OUTPUT/4906_Sunday_February_04_2024_12_47_08_AM_16356909/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{(x+1)\,y''-(2x+3)\,y'+(2+x)\,y=\left(x^2+2x+1\right)\mathrm{e}^{2x}}$$

13.4.1 Solving using Kovacic algorithm

Writing the ode as

$$(x+1)\,y''+(-2x-3)\,y'+(2+x)\,y=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=x+1\\B&=-2x-3\\C&=2+x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4\left(x+1\right)^{2}} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4(x+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4\left(x+1\right)^{2}}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x+1)^2}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4(x+1)^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4(x+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2(x+1)} + (-)(0) \\
&= -\frac{1}{2(x+1)} \\
&= -\frac{1}{2(x+1)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\left(x+1\right)}\right)(0) + \left(\left(\frac{1}{2\left(x+1\right)^2}\right) + \left(-\frac{1}{2\left(x+1\right)}\right)^2 - \left(\frac{3}{4\left(x+1\right)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2(x+1)} dx} \\ &= \frac{1}{\sqrt{x+1}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-3}{x+1}\, dx} \\ &= z_1 e^{x + \frac{\ln(x+1)}{2}} \\ &= z_1\left(\sqrt{x+1}\,\mathrm{e}^{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-3}{x+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x+\ln(x+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x(2+x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2 \left(\mathrm{e}^x \left(\frac{x(2+x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
(x+1)\,y'' + (-2x-3)\,y' + (2+x)\,y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + \frac{c_2 \mathrm{e}^x x(2+x)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^x$$
$$y_2 = \frac{e^x x(2+x)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{e^x x(2+x)}{2} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{e^x x(2+x)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{e^x x(2+x)}{2} \\ e^x & \frac{e^x x(2+x)}{2} + \frac{e^x (2+x)}{2} + \frac{x\, e^x}{2} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(\frac{e^x x(2+x)}{2} + \frac{e^x (2+x)}{2} + \frac{x\, e^x}{2}\right) - \left(\frac{e^x x(2+x)}{2}\right)(e^x)$$

Which simplifies to

$$W = e^{2x} x + e^{2x}$$

Which simplifies to

$$W = (x+1)\, e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^x x(2+x)(x+1)^2 e^{2x}}{2}}{(x+1)^2 e^{2x}} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^x x(2+x)}{2} dx$$

Hence

$$u_1 = -\frac{x^2 e^x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^x (x+1)^2 e^{2x}}{(x+1)^2 e^{2x}} dx$$

Which simplifies to

$$u_2 = \int e^x dx$$

Hence

$$u_2 = e^x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^2 e^{2x}}{2} + \frac{e^{2x} x(2+x)}{2}$$

Which simplifies to

$$y_p(x) = e^{2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 e^x x(2+x)}{2}\right) + \left(e^{2x} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + \frac{c_2 e^x x(2+x)}{2} + e^{2x} x \tag{1}$$

Verification of solutions

$$y = c_1 e^x + \frac{c_2 e^x x(2+x)}{2} + e^{2x} x$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve((x+1)*diff(y(x),x$2)-(2*x+3)*diff(y(x),x)+(x+2)*y(x)=(x^2+2*x+1)*exp(2*x),y(x), sings
```

$$y(x) = e^{2x} x + e^x \left(c_1 x^2 + 2 c_1 x + c_2\right)$$

✓ Solution by Mathematica

Time used: 0.117 (sec). Leaf size: 32

```
DSolve[(x+1)*y''[x]-(2*x+3)*y'[x]+(x+2)*y[x]==(x^2+2*x+1)*Exp[2*x],y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{2}e^x(2e^x x + ec_2(x+2)x + 2ec_1)$$

13.5 problem 25

13.5.1 Solving as second order change of variable on y method 1 ode . 1972
13.5.2 Solving using Kovacic algorithm 1975

Internal problem ID [5416]
Internal file name [`OUTPUT/4907_Sunday_February_04_2024_12_47_09_AM_15884514/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2\tan(x)\, y' - 10y = 0}$$

13.5.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -2\tan(x)$$
$$q(x) = -10$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -10 - \frac{(-2\tan(x))'}{2} - \frac{(-2\tan(x))^2}{4} \\
&= -10 - \frac{(-2 - 2\tan(x)^2)}{2} - \frac{(4\tan(x)^2)}{4} \\
&= -10 - \left(-\tan(x)^2 - 1\right) - \tan(x)^2 \\
&= -9
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-2\tan(x)}{2}} \\
&= \sec(x)
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x) \sec(x) \tag{4}
$$

Applying this change of variable to the original ode results in

$$
\sec(x)\,(v''(x) - 9v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - 9 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)}\\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3\\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}v(x) &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ v(x) &= c_1 e^{(3)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$v(x) = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x)\\ &= \left(\mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}\right)(z(x))\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sec(x)$$

Hence (7) becomes

$$y = \left(\mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}\right)\sec(x)$$

Summary
The solution(s) found are the following

$$y = \left(e^{3x} c_1 + c_2 e^{-3x}\right) \sec(x) \tag{1}$$

Verification of solutions

$$y = \left(e^{3x} c_1 + c_2 e^{-3x}\right) \sec(x)$$

Verified OK.

13.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2\tan(x) y' - 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2\tan(x) \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 232: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2\tan(x)}{1}\,dx} \\
&= z_1 e^{-\ln(\cos(x))} \\
&= z_1(\sec(x))
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}\sec(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2\tan(x)}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(\cos(x))}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{6x}}{6}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\sec(x)\right) + c_2\left(\mathrm{e}^{-3x}\sec(x)\left(\frac{\mathrm{e}^{6x}}{6}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-3x}\sec(x) + \frac{c_2\mathrm{e}^{3x}\sec(x)}{6} \tag{1}
$$

Verification of solutions

$$y = c_1 e^{-3x} \sec(x) + \frac{c_2 e^{3x} \sec(x)}{6}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-2*tan(x)*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sec(x) (c_1 \sinh(3x) + c_2 \cosh(3x))$$

✓ Solution by Mathematica

Time used: 0.07 (sec). Leaf size: 29

```
DSolve[y''[x]-2*Tan[x]*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6} e^{-3x} \left(c_2 e^{6x} + 6c_1\right) \sec(x)$$

13.6 problem 26

13.6.1 Solving using Kovacic algorithm 1979

Internal problem ID [5417]
Internal file name [`OUTPUT/4908_Sunday_February_04_2024_12_47_09_AM_12356748/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' - x(2x+3)\,y' + \left(x^2+3x+3\right)y = \left(-x^2+6\right)\mathrm{e}^x}$$

13.6.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + \left(-2x^2-3x\right)y' + \left(x^2+3x+3\right)y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x^2-3x \\ C &= x^2+3x+3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 233: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x^2 - 3x}{x^2}\, dx} \\ &= z_1 e^{x + \frac{3\ln(x)}{2}} \\ &= z_1 \left(x^{\frac{3}{2}} e^x\right) \end{aligned}$$

Which simplifies to

$$y_1 = x\, e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x^2-3x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+3\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (x\,\mathrm{e}^x) + c_2 \left(x\,\mathrm{e}^x \left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + \left(-2x^2 - 3x\right) y' + \left(x^2 + 3x + 3\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x\,\mathrm{e}^x + \frac{c_2 x^3 \mathrm{e}^x}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x\,\mathrm{e}^{x}$$

$$y_2 = \frac{x^3 \mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & \frac{x^3 \mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(x\,\mathrm{e}^{x}\right) & \frac{d}{dx}\left(\frac{x^3 \mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x\,\mathrm{e}^{x} & \frac{x^3 \mathrm{e}^{x}}{2} \\ \mathrm{e}^{x} + x\,\mathrm{e}^{x} & \frac{3x^2 \mathrm{e}^{x}}{2} + \frac{x^3 \mathrm{e}^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(x\,\mathrm{e}^{x}\right)\left(\frac{3x^2 \mathrm{e}^{x}}{2} + \frac{x^3 \mathrm{e}^{x}}{2}\right) - \left(\frac{x^3 \mathrm{e}^{x}}{2}\right)\left(\mathrm{e}^{x} + x\,\mathrm{e}^{x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{2x} x^3$$

Which simplifies to

$$W = \mathrm{e}^{2x} x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3 \mathrm{e}^{2x}(-x^2+6)}{2}}{x^5 \mathrm{e}^{2x}} dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+6}{2x^2} dx$$

Hence

$$u_1 = \frac{x}{2} + \frac{3}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\,\mathrm{e}^{2x}(-x^2+6)}{x^5 \mathrm{e}^{2x}} dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2+6}{x^4} dx$$

Hence

$$u_2 = \frac{1}{x} - \frac{2}{x^3}$$

Which simplifies to

$$u_1 = \frac{x}{2} + \frac{3}{x}$$
$$u_2 = \frac{x^2-2}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{2} + \frac{3}{x}\right) x\,\mathrm{e}^x + \frac{(x^2-2)\,\mathrm{e}^x}{2}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^x (x^2+2)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x\,\mathrm{e}^{x} + \frac{c_2 x^3 \mathrm{e}^{x}}{2}\right) + \left(\mathrm{e}^{x}\left(x^2+2\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{x} + \frac{c_2 x^3 \mathrm{e}^{x}}{2} + \mathrm{e}^{x}\left(x^2+2\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\,\mathrm{e}^{x} + \frac{c_2 x^3 \mathrm{e}^{x}}{2} + \mathrm{e}^{x}\left(x^2+2\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)-x*(2*x+3)*diff(y(x),x)+(x^2+3*x+3)*y(x)=(6-x^2)*exp(x),y(x), sings
```

$$y(x) = \mathrm{e}^{x}\left(c_1 x^3 + c_2 x + x^2 + 2\right)$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 30

```
DSolve[x^2*y''[x]-x*(2*x+3)*y'[x]+(x^2+3*x+3)*y[x]==(6-x^2)*Exp[x],y[x],x,IncludeSingularSol
```

$$y(x) \to \frac{1}{2} e^{x}\left(c_2 x^3 + 2x^2 + 2c_1 x + 4\right)$$

13.7 problem 27

13.7.1 Solving using Kovacic algorithm 1989
13.7.2 Maple step by step solution 1994

Internal problem ID [5418]
Internal file name [`OUTPUT/4909_Sunday_February_04_2024_12_47_10_AM_63012554/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' + 4y'x^3 + \left(x^2+1\right)^2 y = 0}$$

13.7.1 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + 4y'x^3 + \left(x^2+1\right)^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 4x^3 \\ C &= \left(x^2+1\right)^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4x^3}{4x^2}\, dx} \\ &= z_1 e^{-\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}\, \mathrm{e}^{-\frac{x^2}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^3}{4x^2}dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\right) + c_2\left(\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}(\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\ln(x) \tag{1}
$$

Verification of solutions

$$
y = c_1\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}} + c_2\sqrt{x}\,\mathrm{e}^{-\frac{x^2}{4}}\ln(x)
$$

Verified OK.

13.7.2 Maple step by step solution

Let's solve

$4x^2y'' + 4y'x^3 + (x^2+1)^2 y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -xy' - \frac{(x^2+1)^2 y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + xy' + \frac{(x^2+1)^2 y}{4x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- Define functions

  $\left[P_2(x) = x, P_3(x) = \frac{\left(x^2+1\right)^2}{4x^2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = \frac{1}{4}$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + 4y'x^3 + \left(x^2+1\right)^2 y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..4$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k- >k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x^3 \cdot y'$ to series expansion

  $x^3 \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r+2}$

- Shift index using $k- >k - 2$

  $x^3 \cdot y' = \sum_{k=2}^{\infty} a_{k-2}(k-2+r)\, x^{k+r}$

- Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

Rewrite ODE with series expansions

$a_0(-1+2r)^2 x^r + a_1(1+2r)^2 x^{1+r} + \left(a_2(3+2r)^2 + 2a_0(1+2r)\right) x^{2+r} + \left(a_3(5+2r)^2 + 2a_1(3+\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)^2 = 0$
- Values of r that satisfy the indicial equation

  $r = \frac{1}{2}$
- The coefficients of each power of x must be 0

  $\left[a_1(1+2r)^2 = 0, a_2(3+2r)^2 + 2a_0(1+2r) = 0, a_3(5+2r)^2 + 2a_1(3+2r) = 0\right]$
- Solve for the dependent coefficient(s)

  $\left\{a_1 = 0, a_2 = -\frac{2a_0(1+2r)}{4r^2+12r+9}, a_3 = 0\right\}$
- Each term in the series must be 0, giving the recursion relation

  $a_k(2k+2r-1)^2 + (4k+4r-6)\, a_{k-2} + a_{k-4} = 0$
- Shift index using $k- >k+4$

  $a_{k+4}(2k+7+2r)^2 + (4k+10+4r)\, a_{k+2} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4ka_{k+2}+4ra_{k+2}+a_k+10a_{k+2}}{(2k+7+2r)^2}$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+4} = -\frac{4ka_{k+2}+a_k+12a_{k+2}}{(2k+8)^2}$
- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+4} = -\frac{4ka_{k+2}+a_k+12a_{k+2}}{(2k+8)^2}, a_1 = 0, a_2 = -\frac{a_0}{4}, a_3 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 20

```
dsolve(4*x^2*diff(y(x),x$2)+4*x^3*diff(y(x),x)+(x^2+1)^2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\, e^{-\frac{x^2}{4}}(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica
Time used: 0.037 (sec). Leaf size: 28

```
DSolve[4*x^2*y''[x]+4*x^3*y'[x]+(x^2+1)^2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{x^2}{4}}\sqrt{x}(c_2 \log(x) + c_1)$$

13.8 problem 28

13.8.1 Solving using Kovacic algorithm 1998

Internal problem ID [5419]
Internal file name [`OUTPUT/4910_Sunday_February_04_2024_12_47_11_AM_42613447/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x^2y'' + (-4x^2+x)\,y' + (4x^2-2x+1)\,y = (x^2-x+1)\,e^x}$$

13.8.1 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + (-4x^2+x)\,y' + (4x^2-2x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x^2+x \\ C &= 4x^2-2x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-5}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -5 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 236: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-i}{x} + (-)(0) \\
&= \frac{\frac{1}{2}-i}{x} \\
&= \frac{\frac{1}{2}-i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - i}{x} dx} \\
&= x^{\frac{1}{2} - i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4x^2 + x}{x^2} \, dx} \\
&= z_1 e^{2x - \frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{2x}}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-i} \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x^2+x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-i} \mathrm{e}^{2x}\right) + c_2 \left(x^{-i} \mathrm{e}^{2x} \left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + \left(-4x^2 + x\right) y' + \left(4x^2 - 2x + 1\right) y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 x^{-i} \mathrm{e}^{2x} - \frac{i c_2 x^{i} \mathrm{e}^{2x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-i} \mathrm{e}^{2x}$$
$$y_2 = -\frac{i x^{i} \mathrm{e}^{2x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-i} \mathrm{e}^{2x} & -\frac{i x^{i} \mathrm{e}^{2x}}{2} \\ \frac{d}{dx}\left(x^{-i} \mathrm{e}^{2x}\right) & \frac{d}{dx}\left(-\frac{i x^{i} \mathrm{e}^{2x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-i} \mathrm{e}^{2x} & -\frac{i x^{i} \mathrm{e}^{2x}}{2} \\ -\frac{i x^{-i} \mathrm{e}^{2x}}{x} + 2 x^{-i} \mathrm{e}^{2x} & \frac{x^{i} \mathrm{e}^{2x}}{2x} - i x^{i} \mathrm{e}^{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-i} \mathrm{e}^{2x}\right) \left(\frac{x^{i} \mathrm{e}^{2x}}{2x} - i x^{i} \mathrm{e}^{2x}\right) - \left(-\frac{i x^{i} \mathrm{e}^{2x}}{2}\right) \left(-\frac{i x^{-i} \mathrm{e}^{2x}}{x} + 2 x^{-i} \mathrm{e}^{2x}\right)$$

Which simplifies to

$$W = \frac{x^{-i} \mathrm{e}^{4x} x^{i}}{x}$$

Which simplifies to

$$W = \frac{\mathrm{e}^{4x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i x^i e^{2x} (x^2-x+1) e^x}{2}}{x\, e^{4x}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i x^{-1+i} e^{-x} (x^2 - x + 1)}{2} dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-i} e^{2x} (x^2 - x + 1)\, e^x}{x\, e^{4x}}\, dx$$

Which simplifies to

$$u_2 = \int x^{-1-i} e^{-x} \left(x^2 - x + 1\right) dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined}\ x^{-i} e^{2x} - i \, \text{undefined}\ x^i e^{2x}$$

Which simplifies to

$$y_p(x) = e^{2x} \left(i x^i + x^{-i}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{-i} e^{2x} - \frac{i c_2 x^i e^{2x}}{2}\right) + \left(e^{2x} \left(i x^i + x^{-i}\right) \text{undefined}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}\left(x^{-i} c_1 - \frac{i x^i c_2}{2}\right) + e^{2x}\left(i x^i + x^{-i}\right) \text{undefined}$$

Summary

The solution(s) found are the following

$$y = e^{2x}\left(x^{-i} c_1 - \frac{i x^i c_2}{2}\right) + e^{2x}\left(i x^i + x^{-i}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = e^{2x}\left(x^{-i} c_1 - \frac{i x^i c_2}{2}\right) + e^{2x}\left(i x^i + x^{-i}\right) \text{undefined}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 27

```
dsolve(x^2*diff(y(x),x$2)+(x-4*x^2)*diff(y(x),x)+(1-2*x+4*x^2)*y(x)=(x^2-x+1)*exp(x),y(x), s
```

$$y(x) = e^{2x} x^i c_2 + e^{2x} x^{-i} c_1 + e^x$$

✓ Solution by Mathematica

Time used: 0.184 (sec). Leaf size: 104

```
DSolve[x^2*y''[x]+(x-4*x^2)*y'[x]+(1-2*x+4*x^2)*y[x]==(x^2-x+1)*Exp[x],y[x],x,IncludeSingula
```

$$y(x) \to \frac{1}{2}e^{2x}x^{-i}\left(ix^{2i}\Gamma(-i,x) - ix^{2i}\Gamma(1-i,x) + ix^{2i}\Gamma(2-i,x) - ic_2x^{2i} - i\Gamma(i,x) + i\Gamma(1+i,x) - i\Gamma(2+i,x) + 2c_1\right)$$

13.9 problem 29

13.9.1 Solving as second order change of variable on x method 2 ode . 2008
13.9.2 Solving as second order change of variable on x method 1 ode . 2011
13.9.3 Solving as second order bessel ode ode 2013
13.9.4 Solving using Kovacic algorithm 2014
13.9.5 Maple step by step solution 2020

Internal problem ID [5420]
Internal file name [`OUTPUT/4911_Sunday_February_04_2024_12_47_12_AM_81941235/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{xy'' - y' + 4yx^3 = 0}$$

13.9.1 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{4x^2}{x^2} \\
&= 4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + 4\, e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x^2) + c_2 \sin(x^2) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(x^2) + c_2 \sin(x^2)$$

Verified OK.

13.9.2 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$xy'' - y' + 4yx^3 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 4x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{x^2}}{c} \\
\tau'' &= \frac{2x}{c\sqrt{x^2}}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2x}{c\sqrt{x^2}} - \frac{1}{x}\frac{2\sqrt{x^2}}{c}}{\left(\frac{2\sqrt{x^2}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{x^2}dx}{c} \\
&= \frac{x\sqrt{x^2}}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(x^2) + c_2 \sin(x^2)
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x^2) + c_2 \sin(x^2) \tag{1}
$$

Verification of solutions

$$y = c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

Verified OK.

13.9.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - x y' + 4 y x^4 = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= 1\\ n &= \frac{1}{2}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x \sqrt{2}\, \sin\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}} - \frac{c_2 x \sqrt{2}\, \cos\left(x^2\right)}{\sqrt{\pi}\, \sqrt{x^2}}$$

Verified OK.

13.9.4 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
xy'' - y' + 4yx^3 &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x\\
B &= -1 && (3)\\
C &= 4x^3
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} && (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16x^4 + 3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16x^4 + 3\\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{-16x^4 + 3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 4 \\
&= -2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4x^2 + \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx 2ix - \frac{3i}{16x^3} - \frac{9i}{1024x^7} - \frac{27i}{32768x^{11}} - \frac{405i}{4194304x^{15}} - \frac{1701i}{134217728x^{19}} - \frac{15309i}{8589934592x^{23}} - \frac{72171i}{274877906944x^{27}} + \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 2i
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= 2ix
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = -4x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (-4x^2) + \left(\frac{3}{4x^2}\right) \\
&= -4x^2 + \frac{3}{4x^2}
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 2ix \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{-16x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2ix$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(2ix) \\
&= -\frac{1}{2x} - 2ix \\
&= -\frac{1}{2x} - 2ix
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - 2ix\right)(0) + \left(\left(\frac{1}{2x^2} - 2i\right) + \left(-\frac{1}{2x} - 2ix\right)^2 - \left(\frac{-16x^4+3}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - 2ix\right) dx} \\
&= \frac{\mathrm{e}^{-ix^2}}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-i x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-i x^2}\right) + c_2\left(\mathrm{e}^{-i x^2}\left(-\frac{i\,\mathrm{e}^{2 i x^2}}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-i x^2} - \frac{i c_2 \mathrm{e}^{i x^2}}{4} \tag{1}
$$

Verification of solutions

$$
y = c_1 e^{-i x^2} - \frac{i c_2 e^{i x^2}}{4}
$$

Verified OK.

13.9.5 Maple step by step solution

Let's solve

$y''x - y' + 4y x^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - 4x^2 y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + 4x^2 y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{1}{x}, P_3(x) = 4x^2\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -1$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x - y' + 4y x^3 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x^3 \cdot y$ to series expansion

    $x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+3}$
  - Shift index using $k- >k-3$

    $x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, x^{-1+r} + a_1(1+r)\,(-1+r)\, x^r + a_2(2+r)\, r\, x^{1+r} + a_3(3+r)\,(1+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (a_k\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- The coefficients of each power of x must be 0

  $[a_1(1+r)\,(-1+r) = 0, a_2(2+r)\, r = 0, a_3(3+r)\,(1+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r-1) + 4a_{k-3} = 0$

- Shift index using $k- >k+3$

  $a_{k+4}(k+4+r)(k+2+r)+4a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+4}=-\frac{4a_k}{(k+4+r)(k+2+r)}$
- Recursion relation for $r=0$

  $a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}$
- Solution for $r=0$

  $\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0\right]$
- Recursion relation for $r=2$

  $a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}$
- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+4}=-\frac{4a_k}{(k+6)(k+4)}, a_1=0, a_2=0, a_3=0\right]$
- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+2}\right), a_{k+4}=-\frac{4a_k}{(k+4)(k+2)}, a_1=0, a_2=0, a_3=0, b_{k+4}=-\frac{4b_k}{(k+6)(k+4)}\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)+4*x^3*y(x)=0,y(x), singsol=all)
```

$$y(x)=c_1\sin\left(x^2\right)+c_2\cos\left(x^2\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 20

```
DSolve[x*y''[x]-y'[x]+4*x^3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(x^2\right) + c_2 \sin\left(x^2\right)$$

13.10 problem 30

13.10.1 Solving as second order change of variable on x method 2 ode . 2024
13.10.2 Solving as second order change of variable on x method 1 ode . 2030
13.10.3 Solving as second order bessel ode ode 2035
13.10.4 Solving using Kovacic algorithm 2038

Internal problem ID [5421]
Internal file name [`OUTPUT/4912_Sunday_February_04_2024_12_47_13_AM_96616221/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^4y'' + 2y'x^3 + y = \frac{x+1}{x}}$$

13.10.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^4y'' + 2y'x^3 + y = 0$$

In normal form the ode

$$x^4y'' + 2y'x^3 + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{1}{x^4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\ &= \int e^{-2\ln(x)}\, dx \\ &= \int \frac{1}{x^2}dx \\ &= -\frac{1}{x} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= 1 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} + e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{x}\right)$$
$$y_2 = \sin\left(\frac{1}{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right) - \left(\sin\left(\frac{1}{x}\right)\right)\left(\frac{\sin\left(\frac{1}{x}\right)}{x^2}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{x}\right)^2 + \sin\left(\frac{1}{x}\right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{1}{x}\right)\left(1 + \frac{1}{x}\right)}{-x^2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{x}\right)(x+1)}{x^3}dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}{-x^2}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right)(x+1)}{x^3} dx$$

Hence

$$u_2 = \cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)\right)\cos\left(\frac{1}{x}\right) \\ &+ \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)\right)\sin\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{x+1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)\right) + \left(\frac{x+1}{x}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x} \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + \frac{x+1}{x}$$

Verified OK.

13.10.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4, B = 2x^3, C = 1, f(x) = 1 + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^4y'' + 2y'x^3 + y = 0$$

In normal form the ode

$$x^4y'' + 2y'x^3 + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{1}{x^4} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^4}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^4}}\, x^5}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{2}{c\sqrt{\frac{1}{x^4}}\, x^5} + \frac{2}{x}\frac{\sqrt{\frac{1}{x^4}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^4}}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{1}{x^4}} dx}{c} \\ &= -\frac{x\sqrt{\frac{1}{x^4}}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right)$$

Now the particular solution to this ODE is found

$$x^4y'' + 2y'x^3 + y = 1 + \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{x}\right)$$
$$y_2 = \sin\left(\frac{1}{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right) - \left(\sin\left(\frac{1}{x}\right)\right)\left(\frac{\sin\left(\frac{1}{x}\right)}{x^2}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{x}\right)^2 + \sin\left(\frac{1}{x}\right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{1}{x}\right)\left(1 + \frac{1}{x}\right)}{-x^2}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{x}\right)(x+1)}{x^3}dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{1}{x}\right)\left(1 + \frac{1}{x}\right)}{-x^2}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right)(x+1)}{x^3}dx$$

Hence

$$u_2 = \cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)\right)\cos\left(\frac{1}{x}\right) \\
& + \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)\right)\sin\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = \frac{x+1}{x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos\left(\frac{1}{x}\right) - c_2\sin\left(\frac{1}{x}\right)\right) + \left(\frac{x+1}{x}\right) \\
&= c_1\cos\left(\frac{1}{x}\right) - c_2\sin\left(\frac{1}{x}\right) + \frac{x+1}{x}
\end{aligned}
$$

Which simplifies to

$$
y = c_1\cos\left(\frac{1}{x}\right) - c_2\sin\left(\frac{1}{x}\right) + \frac{x+1}{x}
$$

Summary

The solution(s) found are the following

$$
y = c_1\cos\left(\frac{1}{x}\right) - c_2\sin\left(\frac{1}{x}\right) + \frac{x+1}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\cos\left(\frac{1}{x}\right) - c_2\sin\left(\frac{1}{x}\right) + \frac{x+1}{x}
$$

Verified OK.

13.10.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2xy' + \frac{y}{x^2} = \frac{1+\frac{1}{x}}{x^2} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha) xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\, x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\, x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= 1 \\
n &= \frac{1}{2} \\
\gamma &= -1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sin\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{2}\,\sin\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{x}\right)$$

$$y_2 = \sin\left(\frac{1}{x}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{x}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{x}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{x}\right) & \sin\left(\frac{1}{x}\right) \\ \frac{\sin\left(\frac{1}{x}\right)}{x^2} & -\frac{\cos\left(\frac{1}{x}\right)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{\cos\left(\frac{1}{x}\right)}{x^2}\right) - \left(\sin\left(\frac{1}{x}\right)\right)\left(\frac{\sin\left(\frac{1}{x}\right)}{x^2}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{x}\right)^2 + \sin\left(\frac{1}{x}\right)^2}{x^2}$$

Which simplifies to

$$W = -\frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}{x^2}}{-1}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{x}\right)(x+1)}{x^3}dx$$

Hence

$$u_1 = -\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}{x^2}}{-1}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{x}\right)(x+1)}{x^3}dx$$

Hence

$$u_2 = \cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left(-\sin\left(\frac{1}{x}\right) + \frac{\cos\left(\frac{1}{x}\right)}{x} + \cos\left(\frac{1}{x}\right)\right)\cos\left(\frac{1}{x}\right) \\ & + \left(\cos\left(\frac{1}{x}\right) + \frac{\sin\left(\frac{1}{x}\right)}{x} + \sin\left(\frac{1}{x}\right)\right)\sin\left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$
y_p(x) = \frac{x+1}{x}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1\sqrt{2}\,\sin\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}}\right) + \left(\frac{x+1}{x}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1\sqrt{2}\,\sin\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} + \frac{x+1}{x} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{c_1\sqrt{2}\,\sin\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} - \frac{c_2\sqrt{2}\,\cos\left(\frac{1}{x}\right)}{\sqrt{x}\,\sqrt{\pi}\,\sqrt{\frac{1}{x}}} + \frac{x+1}{x}
$$

Verified OK.

13.10.4 Solving using Kovacic algorithm

Writing the ode as

$$
x^4 y'' + 2y'x^3 + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^4 \\
B &= 2x^3 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^4}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 239: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^4$. There is a pole at $x = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_2^v \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = -\frac{1}{x^4}$$

There is pole in r at $x = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i}{x^2} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{i}{x^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^2}$ is

$$a = i$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned} b &= (0) - (0) \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{i}{x^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &= \frac{1}{2}\left(\frac{0}{i} + 2\right) &= 1 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &= \frac{1}{2}\left(-\frac{0}{i} + 2\right) &= 1 \end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{x^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{i}{x^2}$	1	1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{i}{x^2} + \frac{1}{x} + (-)\,(0) \\
&= -\frac{i}{x^2} + \frac{1}{x} \\
&= \frac{x-i}{x^2}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{i}{x^2} + \frac{1}{x}\right)(0) + \left(\left(\frac{2i}{x^3} - \frac{1}{x^2}\right) + \left(-\frac{i}{x^2} + \frac{1}{x}\right)^2 - \left(-\frac{1}{x^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{i}{x^2} + \frac{1}{x}\right) dx} \\
&= x\, \mathrm{e}^{\frac{i}{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x^3}{x^4}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{i}{x}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x^3}{x^4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i\mathrm{e}^{-\frac{2i}{x}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{i}{x}}\right) + c_2\left(\mathrm{e}^{\frac{i}{x}}\left(-\frac{i\mathrm{e}^{-\frac{2i}{x}}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^4y''+2y'x^3+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{\frac{i}{x}} - \frac{i c_2 \mathrm{e}^{-\frac{i}{x}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\frac{i}{x}}$$
$$y_2 = -\frac{i\mathrm{e}^{-\frac{i}{x}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\frac{i}{x}} & -\frac{i\mathrm{e}^{-\frac{i}{x}}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{\frac{i}{x}}\right) & \frac{d}{dx}\left(-\frac{i\mathrm{e}^{-\frac{i}{x}}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{i}{x}} & -\frac{i e^{-\frac{i}{x}}}{2} \\ -\frac{i e^{\frac{i}{x}}}{x^2} & \frac{e^{-\frac{i}{x}}}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{i}{x}}\right)\left(\frac{e^{-\frac{i}{x}}}{2x^2}\right) - \left(-\frac{i e^{-\frac{i}{x}}}{2}\right)\left(-\frac{i e^{\frac{i}{x}}}{x^2}\right)$$

Which simplifies to

$$W = \frac{e^{\frac{i}{x}} e^{-\frac{i}{x}}}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i e^{-\frac{i}{x}}\left(1+\frac{1}{x}\right)}{2}}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i e^{-\frac{i}{x}}(x+1)}{2x^3} dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{\frac{i}{x}}\left(1+\frac{1}{x}\right)}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{\frac{i}{x}}(x+1)}{x^3} dx$$

Hence

$$u_2 = \mathit{undefined}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } e^{\frac{i}{x}} - i \,\text{undefined } e^{-\frac{i}{x}}$$

Which simplifies to

$$y_p(x) = \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2}\right) + \left(\left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}}\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2} + \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{i}{x}} - \frac{i c_2 e^{-\frac{i}{x}}}{2} + \left(i e^{-\frac{i}{x}} + e^{\frac{i}{x}}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(x^4*diff(y(x),x$2)+2*x^3*diff(y(x),x)+y(x)=(1+x)/x,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{x}\right) c_2 + \cos\left(\frac{1}{x}\right) c_1 + \frac{x+1}{x}$$

✓ Solution by Mathematica

Time used: 0.072 (sec). Leaf size: 25

```
DSolve[x^4*y''[x]+2*x^3*y'[x]+y[x]==(1+x)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x} + c_1 \cos\left(\frac{1}{x}\right) - c_2 \sin\left(\frac{1}{x}\right) + 1$$

13.11 problem 31

13.11.1 Solving as second order change of variable on x method 2 ode . 2048
13.11.2 Solving as second order change of variable on x method 1 ode . 2054
13.11.3 Solving as second order bessel ode ode 2058
13.11.4 Solving using Kovacic algorithm 2062

Internal problem ID [5422]
Internal file name [`OUTPUT/4913_Sunday_February_04_2024_12_47_14_AM_75733031/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^8 y'' + 4x^7 y' + y = \frac{1}{x^3}}$$

13.11.1 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^8 y'' + 4x^7 y' + y = 0$$

In normal form the ode

$$x^8y'' + 4x^7y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{4}{x}\\ q(x) &= \frac{1}{x^8}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx\\ &= \int \mathrm{e}^{-\left(\int \frac{4}{x}dx\right)}dx\\ &= \int e^{-4\ln(x)}\,dx\\ &= \int \frac{1}{x^4}dx\\ &= -\frac{1}{3x^3}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^8}}{\frac{1}{x^8}} \\
&= 1
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^{0}(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{3x^3}\right)$$
$$y_2 = \sin\left(\frac{1}{3x^3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right)\left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right)\left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}}{-x^4}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}}{-x^4}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_2 = 3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}\right)\cos\left(\frac{1}{3x^3}\right) + \left(3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}\right)\sin\left(\frac{1}{3x^3}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(\frac{1}{3x^3}\right) - c_2\sin\left(\frac{1}{3x^3}\right)\right) + \left(\frac{1}{x^3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos\left(\frac{1}{3x^3}\right) - c_2\sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3} \tag{1}$$

Verification of solutions

$$y = c_1\cos\left(\frac{1}{3x^3}\right) - c_2\sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}$$

Verified OK.

13.11.2 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^8, B = 4x^7, C = 1, f(x) = \frac{1}{x^3}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^8y'' + 4x^7y' + y = 0$$

In normal form the ode

$$x^8y'' + 4x^7y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{1}{x^8} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^8}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{4}{c\sqrt{\frac{1}{x^8}}\, x^9}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{c\sqrt{\frac{1}{x^8}}\,x^9} + \frac{4}{x}\frac{\sqrt{\frac{1}{x^8}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^8}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^8}}dx}{c} \\
&= -\frac{x\sqrt{\frac{1}{x^8}}}{3c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)
$$

Now the particular solution to this ODE is found

$$
x^8 y'' + 4x^7 y' + y = \frac{1}{x^3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{3x^3}\right)$$

$$y_2 = \sin\left(\frac{1}{3x^3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right)\left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right)\left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}}{-x^4}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}}{-x^4}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_2 = 3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}\right)\cos\left(\frac{1}{3x^3}\right) + \left(3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}\right)\sin\left(\frac{1}{3x^3}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)\right) + \left(\frac{1}{x^3}\right) \\
&= c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}
\end{aligned}$$

Which simplifies to

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3} \tag{1}$$

Verification of solutions

$$y = c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right) + \frac{1}{x^3}$$

Verified OK.

13.11.3 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' + 4xy' + \frac{y}{x^6} = \frac{1}{x^9} \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1-2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1 \operatorname{BesselJ}(n, \beta\,x^\gamma) + c_2 \operatorname{BesselY}(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= -\frac{3}{2}\\ \beta &= \frac{1}{3}\\ n &= \frac{1}{2}\\ \gamma &= -3\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1\sqrt{2}\,\sqrt{3}\,\sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} - \frac{c_2\sqrt{2}\,\sqrt{3}\,\cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1\sqrt{2}\,\sqrt{3}\,\sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} - \frac{c_2\sqrt{2}\,\sqrt{3}\,\cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos\left(\frac{1}{3x^3}\right)$$

$$y_2 = \sin\left(\frac{1}{3x^3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{1}{3x^3}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{1}{3x^3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{1}{3x^3}\right) & \sin\left(\frac{1}{3x^3}\right) \\ \frac{\sin\left(\frac{1}{3x^3}\right)}{x^4} & -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4} \end{vmatrix}$$

Therefore

$$W = \left(\cos\left(\frac{1}{3x^3}\right)\right)\left(-\frac{\cos\left(\frac{1}{3x^3}\right)}{x^4}\right) - \left(\sin\left(\frac{1}{3x^3}\right)\right)\left(\frac{\sin\left(\frac{1}{3x^3}\right)}{x^4}\right)$$

Which simplifies to

$$W = -\frac{\cos\left(\frac{1}{3x^3}\right)^2 + \sin\left(\frac{1}{3x^3}\right)^2}{x^4}$$

Which simplifies to

$$W = -\frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\sin\left(\frac{1}{3x^3}\right)}{x^9}}{-\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_1 = -3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\cos\left(\frac{1}{3x^3}\right)}{x^9}}{-\frac{1}{x^2}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(\frac{1}{3x^3}\right)}{x^7} dx$$

Hence

$$u_2 = 3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-3\sin\left(\frac{1}{3x^3}\right) + \frac{\cos\left(\frac{1}{3x^3}\right)}{x^3}\right)\cos\left(\frac{1}{3x^3}\right) + \left(3\cos\left(\frac{1}{3x^3}\right) + \frac{\sin\left(\frac{1}{3x^3}\right)}{x^3}\right)\sin\left(\frac{1}{3x^3}\right)$$

Which simplifies to

$$y_p(x) = \frac{1}{x^3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\sqrt{2}\,\sqrt{3}\,\sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} - \frac{c_2\sqrt{2}\,\sqrt{3}\,\cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}}\right) + \left(\frac{1}{x^3}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{c_1\sqrt{2}\,\sqrt{3}\,\sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} - \frac{c_2\sqrt{2}\,\sqrt{3}\,\cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} + \frac{1}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\sqrt{2}\,\sqrt{3}\,\sin\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} - \frac{c_2\sqrt{2}\,\sqrt{3}\,\cos\left(\frac{1}{3x^3}\right)}{x^{\frac{3}{2}}\sqrt{\pi}\,\sqrt{\frac{1}{x^3}}} + \frac{1}{x^3}$$

Verified OK.

13.11.4 Solving using Kovacic algorithm

Writing the ode as

$$x^8y'' + 4x^7y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^8 \\ B &= 4x^7 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2x^6 - 1}{x^8} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2x^6 - 1 \\ t &= x^8 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2x^6 - 1}{x^8}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 240: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 8 - 6 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^8$. There is a pole at $x = 0$ of order 8. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(x-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(x-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(x-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(x-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned}\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a} + v\right)\\ \alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a} + v\right)\end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{2}{x^2} - \frac{1}{x^8}$$

There is pole in r at $x = 0$ of order 8, hence $v = 4$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{i}{x^4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 4$ the above becomes

$$[\sqrt{r}]_c = \frac{i}{x^4} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(x-0)^4}$ is

$$a = i$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(x-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{x^5}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 0. Therefore

$$\begin{aligned}b &= (0) - (0)\\ &= 0\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= \frac{i}{x^4} \\
\alpha_c^{+} &= \frac{1}{2}\left(\frac{b}{a}+v\right) &&= \frac{1}{2}\left(\frac{0}{i}+4\right) &&= 2 \\
\alpha_c^{-} &= \frac{1}{2}\left(-\frac{b}{a}+v\right) &&= \frac{1}{2}\left(-\frac{0}{i}+4\right) &&= 2
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{2x^6-1}{x^8}
$$

Since the $\gcd(s,t)=1$. This gives $b=2$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2}+\sqrt{1+4b} = 2 \\
\alpha_\infty^{-} &= \frac{1}{2}-\sqrt{1+4b} = -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{2x^6-1}{x^8}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	8	$\frac{i}{x^4}$	2	2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{+} = 2$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= 2 - (2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{i}{x^4} + \frac{2}{x} + (0) \\
&= -\frac{i}{x^4} + \frac{2}{x} \\
&= \frac{2x^3 - i}{x^4}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{gathered}
(0) + 2\left(-\frac{i}{x^4} + \frac{2}{x}\right)(0) + \left(\left(\frac{4i}{x^5} - \frac{2}{x^2}\right) + \left(-\frac{i}{x^4} + \frac{2}{x}\right)^2 - \left(\frac{2x^6 - 1}{x^8}\right) \right) = 0 \\
0 = 0
\end{gathered}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \left(-\frac{i}{x^4} + \frac{2}{x}\right) dx} \\
&= x^2 e^{\frac{i}{3x^3}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4x^7}{x^8}\,dx} \\
&= z_1 e^{-2\ln(x)} \\
&= z_1 \left(\frac{1}{x^2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{i}{3x^3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4x^7}{x^8}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-4\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i\mathrm{e}^{-\frac{2i}{3x^3}}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{i}{3x^3}}\right) + c_2 \left(\mathrm{e}^{\frac{i}{3x^3}} \left(-\frac{i\mathrm{e}^{-\frac{2i}{3x^3}}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^8y'' + 4x^7y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{\frac{i}{3x^3}} - \frac{ic_2 e^{-\frac{i}{3x^3}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{i}{3x^3}}$$

$$y_2 = -\frac{i e^{-\frac{i}{3x^3}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{i}{3x^3}} & -\frac{i e^{-\frac{i}{3x^3}}}{2} \\ \frac{d}{dx}\left(e^{\frac{i}{3x^3}}\right) & \frac{d}{dx}\left(-\frac{i e^{-\frac{i}{3x^3}}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{i}{3x^3}} & -\frac{i e^{-\frac{i}{3x^3}}}{2} \\ -\frac{i e^{\frac{i}{3x^3}}}{x^4} & \frac{e^{-\frac{i}{3x^3}}}{2x^4} \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{i}{3x^3}}\right)\left(\frac{e^{-\frac{i}{3x^3}}}{2x^4}\right) - \left(-\frac{i e^{-\frac{i}{3x^3}}}{2}\right)\left(-\frac{i e^{\frac{i}{3x^3}}}{x^4}\right)$$

Which simplifies to

$$W = \frac{e^{\frac{i}{3x^3}} e^{-\frac{i}{3x^3}}}{x^4}$$

Which simplifies to

$$W = \frac{1}{x^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i e^{-\frac{i}{3x^3}}}{2x^3}}{x^4} \, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i e^{-\frac{i}{3x^3}}}{2x^7} dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{e^{\frac{i}{3x^3}}}{x^3}}{x^4} \, dx$$

Which simplifies to

$$u_2 = \int \frac{e^{\frac{i}{3x^3}}}{x^7} dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined}\, e^{\frac{i}{3x^3}} - i\, \text{undefined}\, e^{-\frac{i}{3x^3}}$$

Which simplifies to

$$y_p(x) = \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2}\right) + \left(\left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}}\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2} + \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = c_1 e^{\frac{i}{3x^3}} - \frac{i c_2 e^{-\frac{i}{3x^3}}}{2} + \left(i e^{-\frac{i}{3x^3}} + e^{\frac{i}{3x^3}}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve(x^8*diff(y(x),x$2)+4*x^7*diff(y(x),x)+y(x)=1/x^3,y(x), singsol=all)
```

$$y(x) = \sin\left(\frac{1}{3x^3}\right) c_2 + \cos\left(\frac{1}{3x^3}\right) c_1 + \frac{1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 32

```
DSolve[x^8*y''[x]+4*x^7*y'[x]+y[x]==1/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^3} + c_1 \cos\left(\frac{1}{3x^3}\right) - c_2 \sin\left(\frac{1}{3x^3}\right)$$

13.12 problem 32

13.12.1 Solving as second order change of variable on y method 2 ode . 2072

Internal problem ID [5423]
Internal file name [OUTPUT/4914_Sunday_February_04_2024_12_47_15_AM_24853985/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(\sin(x)x+\cos(x))y''-y'\cos(x)x+y\cos(x)=x}$$

13.12.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=\sin(x)x+\cos(x), B=-\cos(x)x, C=\cos(x), f(x)=x$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$(\sin(x)x+\cos(x))y''-y'\cos(x)x+y\cos(x)=0$$

In normal form the ode

$$(\sin(x)\, x+\cos(x))\, y''-y' \cos(x)\, x+y \cos(x)=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{\cos(x)\, x}{\sin(x)\, x+\cos(x)}\\
q(x) &= \frac{\cos(x)}{\sin(x)\, x+\cos(x)}
\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\, x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^{2}}-\frac{n\cos(x)}{\sin(x)\, x+\cos(x)}+\frac{\cos(x)}{\sin(x)\, x+\cos(x)}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}
v''(x)+\left(\frac{2}{x}-\frac{\cos(x)\, x}{\sin(x)\, x+\cos(x)}\right) v'(x)&=0\\
v''(x)+\left(\frac{2}{x}-\frac{\cos(x)\, x}{\sin(x)\, x+\cos(x)}\right) v'(x)&=0 \qquad (7)
\end{aligned}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) + \left(\frac{2}{x} - \frac{\cos(x)\, x}{\sin(x)\, x + \cos(x)}\right) u(x) = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u(2\sin(x)\, x + 2\cos(x) - \cos(x)\, x^2)}{x(\sin(x)\, x + \cos(x))}
\end{aligned}$$

Where $f(x) = -\frac{2\sin(x)x + 2\cos(x) - \cos(x)x^2}{x(\sin(x)x + \cos(x))}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2\sin(x)\, x + 2\cos(x) - \cos(x)\, x^2}{x(\sin(x)\, x + \cos(x))}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2\sin(x)\, x + 2\cos(x) - \cos(x)\, x^2}{x(\sin(x)\, x + \cos(x))}\, dx \\
\ln(u) &= -2\ln(x) - \ln\left(\tan\left(\frac{x}{2}\right)^2 + 1\right) + \ln\left(2\tan\left(\frac{x}{2}\right) x - \tan\left(\frac{x}{2}\right)^2 + 1\right) + c_1 \\
u &= \mathrm{e}^{-2\ln(x) - \ln\left(\tan\left(\frac{x}{2}\right)^2 + 1\right) + \ln\left(2\tan\left(\frac{x}{2}\right) x - \tan\left(\frac{x}{2}\right)^2 + 1\right) + c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x) - \ln\left(\tan\left(\frac{x}{2}\right)^2 + 1\right) + \ln\left(2\tan\left(\frac{x}{2}\right) x - \tan\left(\frac{x}{2}\right)^2 + 1\right)}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1 \cos(x)}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1 \cos(x)}{x} + c_2\right) x \\
&= -\cos(x)\, c_1 + c_2 x
\end{aligned}$$

Now the particular solution to this ODE is found

$$(\sin(x)\,x+\cos(x))\,y''-y'\cos(x)\,x+y\cos(x)=x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1&=x\\ y_2&=\cos(x)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}x & \cos(x)\\ \frac{d}{dx}(x) & \frac{d}{dx}(\cos(x))\end{vmatrix}$$

Which gives

$$W=\begin{vmatrix}x & \cos(x)\\ 1 & -\sin(x)\end{vmatrix}$$

Therefore

$$W=(x)(-\sin(x))-(\cos(x))(1)$$

Which simplifies to

$$W = -\sin(x)\,x - \cos(x)$$

Which simplifies to

$$W = -\sin(x)\,x - \cos(x)$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\cos(x)\,x}{(\sin(x)\,x + \cos(x))\,(-\sin(x)\,x - \cos(x))}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\cos(x)}{(\sin(x)\,x + \cos(x))^2}dx$$

Hence

$$u_1 = -\frac{1}{\sin(x)\,x + \cos(x)}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{(\sin(x)\,x + \cos(x))\,(-\sin(x)\,x - \cos(x))}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{x^2}{(\sin(x)\,x + \cos(x))^2}dx$$

Hence

$$u_2 = \frac{x - x\tan\left(\frac{x}{2}\right)^2 - 2\tan\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right)x - \tan\left(\frac{x}{2}\right)^2 + 1}$$

Which simplifies to

$$u_1 = -\frac{1}{\sin(x)\,x + \cos(x)}$$

$$u_2 = \frac{\cos(x)\,x - \sin(x)}{\sin(x)\,x + \cos(x)}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{\sin(x)\,x+\cos(x)}+\frac{(\cos(x)\,x-\sin(x))\cos(x)}{\sin(x)\,x+\cos(x)}$$

Which simplifies to

$$y_p(x) = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1\cos(x)}{x}+c_2\right)x\right)+(-\sin(x)) \\
&= -\sin(x)+\left(-\frac{c_1\cos(x)}{x}+c_2\right)x
\end{aligned}$$

Which simplifies to

$$y = -\cos(x)\,c_1+c_2x-\sin(x)$$

Summary

The solution(s) found are the following

$$y = -\cos(x)\,c_1+c_2x-\sin(x) \tag{1}$$

Verification of solutions

$$y = -\cos(x)\,c_1+c_2x-\sin(x)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
trying symmetries linear in x and y(x)
Try integration with the canonical coordinates of the symmetry [0, x]
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (cos(_a)*_a^2*_b(_a)-2*sin(_a)*_a*_b(_a
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- differential order: 2; canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 17

```
dsolve((x*sin(x)+cos(x))*diff(y(x),x$2)-x*cos(x)*diff(y(x),x)+y(x)*cos(x)=x,y(x), singsol=al
```

$$y(x) = -\cos(x) c_1 + c_2 x - \sin(x)$$

✓ Solution by Mathematica

Time used: 0.764 (sec). Leaf size: 20

```
DSolve[(x*Sin[x]+Cos[x])*y''[x]-x*Cos[x]*y'[x]+y[x]*Cos[x]==x,y[x],x,IncludeSingularSolution
```

$$y(x) \to -\sin(x) + c_1 x - c_2 \cos(x)$$

13.13 problem 33

13.13.1 Solving as second order euler ode ode 2080
13.13.2 Solving as second order change of variable on x method 2 ode . 2083
13.13.3 Solving as second order change of variable on x method 1 ode . 2089
13.13.4 Solving as second order change of variable on y method 2 ode . 2093
13.13.5 Solving as second order ode non constant coeff transformation on B ode 2098
13.13.6 Solving using Kovacic algorithm 2103

Internal problem ID [5424]
Internal file name [`OUTPUT/4915_Sunday_February_04_2024_12_47_20_AM_85124973/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{xy'' - 3y' + \frac{3y}{x} = 2 + x}$$

The ode can be written as

$$x^2y'' - 3xy' + 3y = x(2 + x)$$

Which shows it is a Euler ODE.

13.13.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = x(2 + x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2x^3 + c_1x$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 3y = x(2 + x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4(2+x)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2+x}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(2+x)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2+x}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x} - \frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right) x + \left(-\frac{1}{2x} - \frac{1}{2x^2}\right) x^3$$

Which simplifies to

$$y_p(x) = -x^2 - x\ln(x) - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) - 2c_1 + 2x + 1)}{2}$$

Verified OK.

13.13.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\,dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{x^2}}{x^6} \\ &= \frac{3}{x^8}\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{4}} + c_2\tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}}\left(c_2\sqrt{x^4} + 2c_1\right)}{4}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\sqrt{2}\left(x^4\right)^{\frac{1}{4}}\left(c_2\sqrt{x^4}+2c_1\right)}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^4\right)^{\frac{1}{4}}$$

$$y_2 = \left(x^4\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^4\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^4\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{x^3}{\left(x^4\right)^{\frac{3}{4}}} & \frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W=\left(\left(x^{4}\right)^{\frac{1}{4}}\right)\left(\frac{3x^{3}}{\left(x^{4}\right)^{\frac{1}{4}}}\right)-\left(\left(x^{4}\right)^{\frac{3}{4}}\right)\left(\frac{x^{3}}{\left(x^{4}\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W=2x^{3}$$

Which simplifies to

$$W=2x^{3}$$

Therefore Eq. (2) becomes

$$u_1=-\int \frac{\left(x^{4}\right)^{\frac{3}{4}} x(2+x)}{2x^{5}}\,dx$$

Which simplifies to

$$u_1=-\int \frac{\left(x^{4}\right)^{\frac{3}{4}} (2+x)}{2x^{4}}dx$$

Hence

$$u_1=-\frac{\left(x^{4}\right)^{\frac{3}{4}}}{2x^{2}}-\frac{\left(x^{4}\right)^{\frac{3}{4}}\ln\left(x\right)}{x^{3}}$$

And Eq. (3) becomes

$$u_2=\int \frac{\left(x^{4}\right)^{\frac{1}{4}} x(2+x)}{2x^{5}}\,dx$$

Which simplifies to

$$u_2=\int \frac{\left(x^{4}\right)^{\frac{1}{4}} (2+x)}{2x^{4}}dx$$

Hence

$$u_2=-\frac{\left(x+1\right)\left(x^{4}\right)^{\frac{1}{4}}}{2x^{3}}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\frac{(x^4)^{\frac{3}{4}}\left(x+2\ln\left(x\right)\right)}{2x^3} \\
u_2 &= -\frac{(x+1)\left(x^4\right)^{\frac{1}{4}}}{2x^3}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = -\frac{x(x+2\ln\left(x\right))}{2} - \frac{x(x+1)}{2}
$$

Which simplifies to

$$
y_p(x) = -x^2 - x\ln\left(x\right) - \frac{x}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{2}\left(x^4\right)^{\frac{1}{4}}\left(c_2\sqrt{x^4}+2c_1\right)}{4}\right) + \left(-x^2 - x\ln\left(x\right) - \frac{x}{2}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{\sqrt{2}\left(x^4\right)^{\frac{1}{4}}\left(c_2\sqrt{x^4}+2c_1\right)}{4} - x^2 - x\ln\left(x\right) - \frac{x}{2} \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \frac{\sqrt{2}\left(x^4\right)^{\frac{1}{4}}\left(c_2\sqrt{x^4}+2c_1\right)}{4} - x^2 - x\ln\left(x\right) - \frac{x}{2}
$$

Verified OK.

13.13.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = x(2 + x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \\
&= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{4c\sqrt{3}}{3}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1\cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2\sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{3}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{3}\sqrt{\frac{1}{x^2}}\,x\ln\left(x\right)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{\left(\left(ic_2 + c_1\right)x^2 - ic_2 + c_1\right)x}{2}
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 3xy' + 3y = x(2+x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^4\right)^{\frac{1}{4}}$$
$$y_2 = \left(x^4\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^4\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^4\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^4\right)^{\frac{1}{4}} & \left(x^4\right)^{\frac{3}{4}} \\ \frac{x^3}{\left(x^4\right)^{\frac{3}{4}}} & \frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^4\right)^{\frac{1}{4}}\right)\left(\frac{3x^3}{\left(x^4\right)^{\frac{1}{4}}}\right) - \left(\left(x^4\right)^{\frac{3}{4}}\right)\left(\frac{x^3}{\left(x^4\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}} x(2+x)}{2x^5} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^4)^{\frac{3}{4}} (2+x)}{2x^4} dx$$

Hence

$$u_1 = -\frac{(x^4)^{\frac{3}{4}}}{2x^2} - \frac{(x^4)^{\frac{3}{4}} \ln(x)}{x^3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} x(2+x)}{2x^5} \, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^4)^{\frac{1}{4}} (2+x)}{2x^4} dx$$

Hence

$$u_2 = -\frac{(x+1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Which simplifies to

$$u_1 = -\frac{(x^4)^{\frac{3}{4}} (x + 2\ln(x))}{2x^3}$$

$$u_2 = -\frac{(x+1)(x^4)^{\frac{1}{4}}}{2x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x(x + 2\ln(x))}{2} - \frac{x(x+1)}{2}$$

Which simplifies to

$$y_p(x) = -x^2 - x \ln(x) - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\left((ic_2 + c_1)\, x^2 - ic_2 + c_1\right) x}{2}\right) + \left(-x^2 - x \ln(x) - \frac{x}{2}\right) \\
&= -x^2 - x \ln(x) - \frac{x}{2} + \frac{\left((ic_2 + c_1)\, x^2 - ic_2 + c_1\right) x}{2}
\end{aligned}$$

Which simplifies to

$$y = \frac{x\left(-2 \ln(x) + (ic_2 + c_1)\, x^2 - 2x - ic_2 + c_1 - 1\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x\left(-2 \ln(x) + (ic_2 + c_1)\, x^2 - 2x - ic_2 + c_1 - 1\right)}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x\left(-2 \ln(x) + (ic_2 + c_1)\, x^2 - 2x - ic_2 + c_1 - 1\right)}{2}$$

Verified OK.

13.13.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 3, f(x) = x(2 + x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 3y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{3u}{x}
\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \\
&= c_2 x^3 - \frac{1}{2} c_1 x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 3y = x(2 + x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^4(2+x)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2+x}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2(2+x)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2+x}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x} - \frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right) x + \left(-\frac{1}{2x} - \frac{1}{2x^2}\right) x^3$$

Which simplifies to

$$y_p(x) = -x^2 - x\ln(x) - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x^3\right) + \left(-x^2 - x\ln(x) - \frac{x}{2}\right) \\ &= -x^2 - x\ln(x) - \frac{x}{2} + \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \end{aligned}$$

Which simplifies to

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x(-2c_2x^2 + 2\ln(x) + c_1 + 2x + 1)}{2}$$

Verified OK.

13.13.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -3x \\
C &= 3 \\
F &= x^2 + 2x
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-3x)(-3) + (3)(-3x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3x^3 v'' + \left(3x^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
-3x^2 (u'(x)\, x - u(x)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{1}{x}\, dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= e^{\ln(x) + c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 x \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1 x \, \mathrm{d}x \\ &= \frac{c_1 x^2}{2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-3x)\left(\frac{c_1 x^2}{2} + c_2\right) \\ &= -\frac{3x(c_1 x^2 + 2c_2)}{2} \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x^3 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x^3 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x)\left(3x^2\right) - \left(x^3\right)(1)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3(x^2 + 2x)}{2x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2+x}{2x} dx$$

Hence

$$u_1 = -\frac{x}{2} - \ln(x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2 + 2x)}{2x^5}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2+x}{2x^3} dx$$

Hence

$$u_2 = -\frac{1}{2x} - \frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{2} - \ln(x)\right) x + \left(-\frac{1}{2x} - \frac{1}{2x^2}\right) x^3$$

Which simplifies to

$$y_p(x) = -x^2 - x\ln(x) - \frac{x}{2}$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(-\frac{3x(c_1x^2 + 2c_2)}{2}\right) + \left(-x^2 - x\ln(x) - \frac{x}{2}\right) \\ &= -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2} \tag{1}$$

Verification of solutions

$$y = -\frac{x(3c_1x^2 + 2\ln(x) + 6c_2 + 2x + 1)}{2}$$

Verified OK.

13.13.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 3xy' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x + \frac{1}{2} c_2 x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = \frac{x^3}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{x^3}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{x^3}{2} \\ 1 & \frac{3x^2}{2} \end{vmatrix}$$

Therefore

$$W=(x)\left(\frac{3x^{2}}{2}\right)-\left(\frac{x^{3}}{2}\right)(1)$$

Which simplifies to

$$W=x^{3}$$

Which simplifies to

$$W=x^{3}$$

Therefore Eq. (2) becomes

$$u_{1}=-\int\frac{\frac{x^{3}(x^{2}+2x)}{2}}{x^{5}}\,dx$$

Which simplifies to

$$u_{1}=-\int\frac{2+x}{2x}dx$$

Hence

$$u_{1}=-\frac{x}{2}-\ln(x)$$

And Eq. (3) becomes

$$u_{2}=\int\frac{x(x^{2}+2x)}{x^{5}}\,dx$$

Which simplifies to

$$u_{2}=\int\frac{2+x}{x^{3}}dx$$

Hence

$$u_{2}=-\frac{1}{x}-\frac{1}{x^{2}}$$

Therefore the particular solution, from equation (1) is

$$y_{p}(x)=\left(-\frac{x}{2}-\ln(x)\right)x+\frac{\left(-\frac{1}{x}-\frac{1}{x^{2}}\right)x^{3}}{2}$$

Which simplifies to

$$y_p(x) = -x^2 - x\ln(x) - \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x + \frac{1}{2}c_2 x^3\right) + \left(-x^2 - x\ln(x) - \frac{x}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x + \frac{c_2 x^3}{2} - x^2 - x\ln(x) - \frac{x}{2} \tag{1}$$

Verification of solutions

$$y = c_1 x + \frac{c_2 x^3}{2} - x^2 - x\ln(x) - \frac{x}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x*diff(y(x),x$2)-3*diff(y(x),x)+3*y(x)/x=x+2,y(x), singsol=all)
```

$$y(x) = \frac{(-2x - 2\ln(x) + c_1 x^2 + 2c_2)\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 30

```
DSolve[x*y''[x]-3*y'[x]+3*y[x]/x==x+2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x\left(2c_2x^2 - 2x - 2\log(x) - 1 + 2c_1\right)$$

13.14 problem 35

13.14.1 Solving as second order ode non constant coeff transformation on B ode 2112
13.14.2 Solving using Kovacic algorithm 2117

Internal problem ID [5425]
Internal file name [`OUTPUT/4916_Sunday_February_04_2024_12_47_22_AM_74213593/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x+1)\,y''-(3x+4)\,y'+3y=(2+3x)\,\mathrm{e}^{3x}}$$

13.14.1 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay''+By'+Cy=F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y=Bv$$

This results in

$$\begin{aligned}
y'&=B'v+v'B\\
y''&=B''v+B'v'+v''B+v'B'\\
&=v''B+2v'+B'+B''v
\end{aligned}$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x + 1 \\
B &= -3x - 4 \\
C &= 3 \\
F &= (2 + 3x)\, e^{3x}
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (x + 1)\,(0) + (-3x - 4)\,(-3) + (3)\,(-3x - 4) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3x^2 - 7x - 4v'' + \left(9x^2 + 18x + 10\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(-3x^2 - 7x - 4\right) u'(x) + 9\left(x^2 + 2x + \frac{10}{9}\right) u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u(9x^2 + 18x + 10)}{3x^2 + 7x + 4}
\end{aligned}
$$

Where $f(x) = \frac{9x^2+18x+10}{3x^2+7x+4}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{9x^2+18x+10}{3x^2+7x+4}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{9x^2+18x+10}{3x^2+7x+4}\,dx \\
\ln(u) &= 3x - 2\ln(3x+4) + \ln(x+1) + c_1 \\
u &= \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)+c_1} \\
&= c_1 \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)}
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)}
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)}\,\mathrm{d}x \\
&= \frac{(3x+4)\,c_1 \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)}}{9+9x} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-3x-4)\left(\frac{(3x+4)\,c_1 \mathrm{e}^{3x-2\ln(3x+4)+\ln(x+1)}}{9+9x} + c_2\right) \\
&= -\frac{\mathrm{e}^{3x} c_1}{9} + (-3x-4)\,c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{3x} \\
y_2 &= -3x - 4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & -3x-4 \\ \frac{d}{dx}\left(e^{3x}\right) & \frac{d}{dx}(-3x-4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & -3x-4 \\ 3\,e^{3x} & -3 \end{vmatrix}$$

Therefore

$$W = \left(e^{3x}\right)(-3) - (-3x-4)\left(3\,e^{3x}\right)$$

Which simplifies to

$$W = 9x\,e^{3x} + 9\,e^{3x}$$

Which simplifies to

$$W = 9(x+1)\,e^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-3x-4)\,(2+3x)\,e^{3x}}{9\,(x+1)^2\,e^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-9x^2-18x-8}{9\,(x+1)^2}dx$$

Hence

$$u_1 = x + \frac{1}{9+9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(\mathrm{e}^{3x}\right)^2 (2+3x)}{9\left(x+1\right)^2 \mathrm{e}^{3x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{3x}(2+3x)}{9\left(x+1\right)^2} dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{3x}}{9+9x}$$

Which simplifies to

$$u_1 = x + \frac{1}{9+9x}$$

$$u_2 = \frac{\mathrm{e}^{3x}}{9+9x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x + \frac{1}{9+9x}\right) \mathrm{e}^{3x} + \frac{\mathrm{e}^{3x}(-3x-4)}{9+9x}$$

Which simplifies to

$$y_p(x) = \frac{(3x-1)\,\mathrm{e}^{3x}}{3}$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-\frac{\mathrm{e}^{3x} c_1}{9} + (-3x-4)\, c_2\right) + \left(\frac{(3x-1)\,\mathrm{e}^{3x}}{3}\right) \\
&= \frac{(9x-c_1-3)\,\mathrm{e}^{3x}}{9} + (-3x-4)\, c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(9x - c_1 - 3)\,e^{3x}}{9} + (-3x - 4)\,c_2 \tag{1}$$

Verification of solutions

$$y = \frac{(9x - c_1 - 3)\,e^{3x}}{9} + (-3x - 4)\,c_2$$

Verified OK.

13.14.2 Solving using Kovacic algorithm

Writing the ode as

$$(x + 1)\,y'' + (-3x - 4)\,y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x + 1 \\ B &= -3x - 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9x^2 + 12x + 6}{4\,(x + 1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9x^2 + 12x + 6 \\
t &= 4(x+1)^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{9x^2 + 12x + 6}{4\left(x+1\right)^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 242: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 2 \\
&= 0
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4(x+1)^2$. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{9}{4} + \frac{3}{4(x+1)^2} - \frac{3}{2(x+1)}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty) = 0$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{0}{2} = 0$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{0} a_i x^i
\end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^0$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{3}{2} - \frac{1}{2x} + \frac{2}{3x^2} - \frac{7}{9x^3} + \frac{91}{108x^4} - \frac{283}{324x^5} + \frac{143}{162x^6} - \frac{214}{243x^7} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{3}{2}$$

From Eq. (9) the sum up to $v = 0$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{0} a_i x^i \\
&= \frac{3}{2}
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^{-1} = \frac{1}{x}$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = \frac{9}{4}
$$

This shows that the coefficient of $\frac{1}{x}$ in the above is 0. Now we need to find the coefficient of $\frac{1}{x}$ in r. How this is done depends on if $v = 0$ or not. Since $v = 0$ then starting from $r = \frac{s}{t}$ and doing long division in the form

$$
r = Q + \frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of $\frac{1}{x}$ in r will be the coefficient in R of the term in x of degree of t minus one, divided by the leading coefficient in t. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{9x^2 + 12x + 6}{4x^2 + 8x + 4} \\
&= Q + \frac{R}{4x^2 + 8x + 4} \\
&= \left(\frac{9}{4}\right) + \left(\frac{-3 - 6x}{4x^2 + 8x + 4}\right) \\
&= \frac{9}{4} + \frac{-3 - 6x}{4x^2 + 8x + 4}
\end{aligned}
$$

Since the degree of t is 2, then we see that the coefficient of the term x in the remainder R is -6. Dividing this by leading coefficient in t which is 4 gives $-\frac{3}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{3}{2}\right) - (0) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{3}{2} \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{3}{2}} - 0\right) &&= \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{9x^2 + 12x + 6}{4\left(x+1\right)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^{+} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (+)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2\,(x+1)} + \left(\frac{3}{2}\right) \\
&= -\frac{1}{2\,(x+1)} + \frac{3}{2} \\
&= \frac{2+3x}{2+2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2\,(x+1)} + \frac{3}{2}\right)(0) + \left(\left(\frac{1}{2\,(x+1)^2}\right) + \left(-\frac{1}{2\,(x+1)} + \frac{3}{2}\right)^2 - \left(\frac{9x^2+12x+6}{4\,(x+1)^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x+1)} + \frac{3}{2}\right) dx} \\
&= \frac{\mathrm{e}^{\frac{3x}{2}}}{\sqrt{x+1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x-4}{x+1}\, dx} \\
&= z_1 e^{\frac{3x}{2} + \frac{\ln(x+1)}{2}} \\
&= z_1\left(\sqrt{x+1}\,\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3x-4}{x+1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{3x+\ln(x+1)}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{(3x+4)\, e^{-3x}}{9}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{3x}\right) + c_2 \left(e^{3x} \left(-\frac{(3x+4)\, e^{-3x}}{9}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$(x+1)\, y'' + (-3x-4)\, y' + 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{3x} c_1 + c_2 \left(-\frac{x}{3} - \frac{4}{9}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{3x}$$
$$y_2 = -\frac{x}{3} - \frac{4}{9}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{3x} & -\frac{x}{3} - \frac{4}{9} \\ \frac{d}{dx}(e^{3x}) & \frac{d}{dx}\left(-\frac{x}{3} - \frac{4}{9}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{3x} & -\frac{x}{3} - \frac{4}{9} \\ 3\,e^{3x} & -\frac{1}{3} \end{vmatrix}$$

Therefore

$$W = \left(e^{3x}\right)\left(-\frac{1}{3}\right) - \left(-\frac{x}{3} - \frac{4}{9}\right)\left(3\,e^{3x}\right)$$

Which simplifies to

$$W = x\,e^{3x} + e^{3x}$$

Which simplifies to

$$W = (x+1)\,\mathrm{e}^{3x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\frac{x}{3}-\frac{4}{9}\right)(2+3x)\,\mathrm{e}^{3x}}{(x+1)^2\,\mathrm{e}^{3x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-9x^2-18x-8}{9\,(x+1)^2}dx$$

Hence

$$u_1 = x + \frac{1}{9+9x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{6x}(2+3x)}{(x+1)^2\,\mathrm{e}^{3x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{e}^{3x}(2+3x)}{(x+1)^2}dx$$

Hence

$$u_2 = \frac{\mathrm{e}^{3x}}{x+1}$$

Which simplifies to

$$u_1 = x + \frac{1}{9+9x}$$

$$u_2 = \frac{\mathrm{e}^{3x}}{x+1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(x + \frac{1}{9+9x}\right)\mathrm{e}^{3x} + \frac{\mathrm{e}^{3x}\left(-\frac{x}{3}-\frac{4}{9}\right)}{x+1}$$

Which simplifies to

$$y_p(x) = \frac{(3x-1)\,\mathrm{e}^{3x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{3x} c_1 + c_2\left(-\frac{x}{3} - \frac{4}{9}\right)\right) + \left(\frac{(3x-1)\,\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x} c_1 + c_2\left(-\frac{x}{3} - \frac{4}{9}\right) + \frac{(3x-1)\,\mathrm{e}^{3x}}{3} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{3x} c_1 + c_2\left(-\frac{x}{3} - \frac{4}{9}\right) + \frac{(3x-1)\,\mathrm{e}^{3x}}{3}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Reducible group (found another exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x+1)*diff(y(x),x$2)-(3*x+4)*diff(y(x),x)+3*y(x)=(3*x+2)*exp(3*x),y(x), singsol=all)
```

$$y(x) = (x + c_1) e^{3x} + \frac{(3x + 4) c_2}{3}$$

✓ Solution by Mathematica

Time used: 0.364 (sec). Leaf size: 48

```
DSolve[(x+1)*y''[x]-(3*x+4)*y'[x]+3*y[x]==(3*x+2)*Exp[3*x],y[x],x,IncludeSingularSolutions -
```

$$y(x) \to e^{3x}\left(x + \frac{2}{3}\right) + \frac{c_1 e^{3x+3}}{\sqrt{2}} - \frac{1}{9}\sqrt{2}c_2(3x + 4)$$

13.15 problem 36

13.15.1 Solving as second order change of variable on y method 1 ode . 2128
13.15.2 Solving as second order bessel ode ode 2131
13.15.3 Solving using Kovacic algorithm 2132
13.15.4 Maple step by step solution 2135

Internal problem ID [5426]
Internal file name [`OUTPUT/4917_Sunday_February_04_2024_12_47_23_AM_18303722/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second__order__bessel__ode", "second__order__change__of__variable__on__y__method__1"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4xy' + (9x^2 + 6)\, y = 0}$$

13.15.1 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{9x^2 + 6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{9x^2+6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\
&= \frac{9x^2+6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\
&= \frac{9x^2+6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\
&= 9
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-\frac{4}{x}}{2}} \\
&= x^2
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = v(x)\, x^2 \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^4(v''(x) + 9v(x)) = 0
$$

Which is now solved for $v(x)$ This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda x} + 9\, \mathrm{e}^{\lambda x} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 + 9 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$v(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

Now that $v(x)$ is known, then

$$\begin{aligned}y &= v(x)\, z(x) \\ &= (c_1 \cos(3x) + c_2 \sin(3x))\,(z(x)) \qquad (7)\end{aligned}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = (c_1 \cos(3x) + c_2 \sin(3x))\, x^2$$

Summary

The solution(s) found are the following

$$y = (c_1 \cos(3x) + c_2 \sin(3x))\, x^2 \tag{1}$$

Verification of solutions

$$y = (c_1 \cos(3x) + c_2 \sin(3x))\, x^2$$

Verified OK.

13.15.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2 y'' - 4xy' + (9x^2 + 6)\, y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + xy' + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, xy' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{5}{2}$$
$$\beta = 3$$
$$n = -\frac{1}{2}$$
$$\gamma = 1$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 x^2 \sqrt{2}\,\sqrt{3}\,\cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\,\sqrt{3}\,\sin(3x)}{3\sqrt{\pi}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2 \sqrt{2}\,\sqrt{3}\,\cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\,\sqrt{3}\,\sin(3x)}{3\sqrt{\pi}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 \sqrt{2}\,\sqrt{3}\,\cos(3x)}{3\sqrt{\pi}} + \frac{c_2 x^2 \sqrt{2}\,\sqrt{3}\,\sin(3x)}{3\sqrt{\pi}}$$

Verified OK.

13.15.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 4xy' + (9x^2 + 6)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 9x^2 + 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 243: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx} \\ &= z_1 e^{2\ln(x)} \\ &= z_1 \left(x^2\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2 \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{4\ln(x)}}{\left(y_1\right)^2}\,dx \\ &= y_1 \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2 \cos(3x)\right) + c_2 \left(x^2 \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \cos(3x) + \frac{c_2 \sin(3x) x^2}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2 \cos(3x) + \frac{c_2 \sin(3x) x^2}{3}
$$

Verified OK.

13.15.4 Maple step by step solution

Let's solve

$x^2 y'' - 4xy' + (9x^2 + 6) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3(3x^2+2)y}{x^2} + \frac{4y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{x} + \frac{3(3x^2+2)y}{x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x) = -\frac{4}{x}, P_3(x) = \frac{3(3x^2+2)}{x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = -4$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 6$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2y''-4xy'+(9x^2+6)\,y=0$

- Assume series solution for y

  $y=\sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m\cdot y$ to series expansion for $m=0..2$

  $x^m\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+m}$

- Shift index using $k->k-m$

  $x^m\cdot y=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x\cdot y'$ to series expansion

  $x\cdot y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r}$

- Convert $x^2\cdot y''$ to series expansion

  $x^2\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-2+r)\,(-3+r)\,x^r+a_1(-1+r)\,(-2+r)\,x^{1+r}+\left(\sum_{k=2}^{\infty} (a_k(k+r-2)\,(k+r-3)+9a_{k-2})\,x$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-2+r)\,(-3+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{2,3\}$

- Each term must be 0

  $a_1(-1+r)\,(-2+r)=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r-2)(k+r-3)+9a_{k-2}=0$

- Shift index using $k->k+2$

  $a_{k+2}(k+r)(k+r-1)+9a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{9a_k}{(k+r)(k+r-1)}$

- Recursion relation for $r=2$

  $a_{k+2}=-\frac{9a_k}{(k+2)(k+1)}$

- Solution for $r=2$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{9a_k}{(k+2)(k+1)}, a_1=0\right]$

- Recursion relation for $r=3$

  $a_{k+2}=-\frac{9a_k}{(k+3)(k+2)}$

- Solution for $r=3$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+2}=-\frac{9a_k}{(k+3)(k+2)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k+2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+3}\right), a_{k+2}=-\frac{9a_k}{(k+2)(k+1)}, a_1=0, b_{k+2}=-\frac{9b_k}{(k+3)(k+2)}, b_1=0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+(6+9*x^2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]-4*x*y'[x]+(6+9*x^2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-3ix}x^2\left(6c_1 - ic_2e^{6ix}\right)$$

13.16 problem 37

13.16.1 Solving as second order change of variable on y method 1 ode . 2139
13.16.2 Solving as second order bessel ode ode 2146
13.16.3 Solving using Kovacic algorithm 2149

Internal problem ID [5427]
Internal file name [`OUTPUT/4918_Sunday_February_04_2024_12_47_23_AM_15867267/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_bessel_ode", "second_order_change_of_variable_on_y_method_1"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{xy'' + 2y' + 4yx = 4}$$

13.16.1 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' + 4yx = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= 4
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4 - \frac{\left(\frac{2}{x}\right)'}{2} - \frac{\left(\frac{2}{x}\right)^2}{4} \\
&= 4 - \frac{\left(-\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\
&= 4 - \left(-\frac{1}{x^2}\right) - \frac{1}{x^2} \\
&= 4
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{\frac{2}{x}}{2}} \\
&= \frac{1}{x}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{x} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
v''(x) + 4v(x) = 4
$$

Which is now solved for $v(x)$ This is second order non-homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = f(x)
$$

Where $A = 1, B = 0, C = 4, f(x) = 4$. Let the solution be

$$v(x) = v_h + v_p$$

Where v_h is the solution to the homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = 0$, and v_p is a particular solution to the non-homogeneous ODE $Av''(x)+Bv'(x)+Cv(x) = f(x)$. v_h is the solution to

$$v''(x) + 4v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$v(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution v_h is

$$v_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$v_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution v_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution v_p, gives the particular solution

$$v_p = 1$$

Therefore the general solution is

$$\begin{aligned} v &= v_h + v_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + (1) \end{aligned}$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 \cos(2x) + c_2 \sin(2x) + 1)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{x}$$

Hence (7) becomes

$$y = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \cos(2x) + c_2 \sin(2x) + 1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{\cos(2x)}{x} \\ y_2 &= \frac{\sin(2x)}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ \frac{d}{dx}\left(\frac{\cos(2x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(2x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ -\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x}\right)\left(\frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2}\right) - \left(\frac{\sin(2x)}{x}\right)\left(-\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{2\cos(2x)^2 + 2\sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{4\sin(2x)}{x}}{\frac{2}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int 2\sin(2x)\,dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\cos(2x)}{x}}{\frac{2}{x}}\, dx$$

Which simplifies to

$$u_2 = \int 2\cos(2x)\, dx$$

Hence

$$u_2 = \sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\cos(2x) + c_2\sin(2x) + 1}{x}\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\cos(2x) + c_2\sin(2x) + 1}{x} + \frac{1}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\cos(2x) + c_2\sin(2x) + 1}{x} + \frac{1}{x}$$

Warning, solution could not be verified

13.16.2 Solving as second order bessel ode ode

Writing the ode as

$$x^2y'' + 2xy' + 4x^2y = 4x \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= -\frac{1}{2} \\
\beta &= 2 \\
n &= \frac{1}{2} \\
\gamma &= 1
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 \sin\left(2x\right)}{x\sqrt{\pi}} - \frac{c_2 \cos\left(2x\right)}{x\sqrt{\pi}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 \sin\left(2x\right)}{x\sqrt{\pi}} - \frac{c_2 \cos\left(2x\right)}{x\sqrt{\pi}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(2x)}{x}$$
$$y_2 = \frac{\sin(2x)}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ \frac{d}{dx}\left(\frac{\cos(2x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(2x)}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{x} \\ -\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x}\right)\left(\frac{2\cos(2x)}{x} - \frac{\sin(2x)}{x^2}\right) - \left(\frac{\sin(2x)}{x}\right)\left(-\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{2\cos(2x)^2 + 2\sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{2}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\sin(2x)}{2} \, dx$$

Which simplifies to

$$u_1 = -\int 2\sin(2x) \, dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\cos(2x)}{2} \, dx$$

Which simplifies to

$$u_2 = \int 2\cos(2x) \, dx$$

Hence

$$u_2 = \sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}}\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}} + \frac{1}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sin(2x)}{x\sqrt{\pi}} - \frac{c_2 \cos(2x)}{x\sqrt{\pi}} + \frac{1}{x}$$

Verified OK.

13.16.3 Solving using Kovacic algorithm

Writing the ode as

$$xy'' + 2y' + 4yx = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= 2 \\ C &= 4x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r=-4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z''=rz$ as one solution is

$$z_1(x)=\cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{x}\,dx}\\ &= z_1 e^{-\ln(x)}\\ &= z_1\left(\frac{1}{x}\right)\end{aligned}$$

Which simplifies to

$$y_1=\frac{\cos(2x)}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2=y_1\int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1\int \frac{e^{\int -\frac{2}{x}\,dx}}{(y_1)^2}\,dx\\ &= y_1\int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1y_1+c_2y_2\\ &= c_1\left(\frac{\cos(2x)}{x}\right)+c_2\left(\frac{\cos(2x)}{x}\left(\frac{\tan(2x)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$xy'' + 2y' + 4yx = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1 \cos(2x)}{x} + \frac{c_2 \sin(2x)}{2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{\cos(2x)}{x}$$
$$y_2 = \frac{\sin(2x)}{2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{2x} \\ \frac{d}{dx}\left(\frac{\cos(2x)}{x}\right) & \frac{d}{dx}\left(\frac{\sin(2x)}{2x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{\cos(2x)}{x} & \frac{\sin(2x)}{2x} \\ -\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2} & \frac{\cos(2x)}{x} - \frac{\sin(2x)}{2x^2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{\cos(2x)}{x}\right)\left(\frac{\cos(2x)}{x} - \frac{\sin(2x)}{2x^2}\right) - \left(\frac{\sin(2x)}{2x}\right)\left(-\frac{2\sin(2x)}{x} - \frac{\cos(2x)}{x^2}\right)$$

Which simplifies to

$$W = \frac{\cos(2x)^2 + \sin(2x)^2}{x^2}$$

Which simplifies to

$$W = \frac{1}{x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\sin(2x)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_1 = -\int 2\sin(2x)\,dx$$

Hence

$$u_1 = \cos(2x)$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{4\cos(2x)}{x}}{\frac{1}{x}}\,dx$$

Which simplifies to

$$u_2 = \int 4\cos(2x)\, dx$$

Hence

$$u_2 = 2\sin(2x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\cos(2x)^2}{x} + \frac{\sin(2x)^2}{x}$$

Which simplifies to

$$y_p(x) = \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1\cos(2x)}{x} + \frac{c_2\sin(2x)}{2x}\right) + \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y = \frac{c_1\cos(2x) + \frac{c_2\sin(2x)}{2}}{x} + \frac{1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1\cos(2x) + \frac{c_2\sin(2x)}{2}}{x} + \frac{1}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1\cos(2x) + \frac{c_2\sin(2x)}{2}}{x} + \frac{1}{x}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+4*x*y(x)=4,y(x), singsol=all)
```

$$y(x) = \frac{1 + \sin(2x)\, c_2 + \cos(2x)\, c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 38

```
DSolve[x*y''[x]+2*y'[x]+4*x*y[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4c_1 e^{-2ix} - ic_2 e^{2ix} + 4}{4x}$$

13.17 problem 38

13.17.1 Solving as second order change of variable on y method 2 ode . 2156
13.17.2 Solving as second order ode non constant coeff transformation on B ode 2161
13.17.3 Solving using Kovacic algorithm 2166

Internal problem ID [5428]
Internal file name [`OUTPUT/4919_Sunday_February_04_2024_12_47_25_AM_26764961/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 18. Linear equations with variable coefficients (Equations of second order). Supplemetary problems. Page 120
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^{2}+1\right) y^{\prime \prime}-2 x y^{\prime}+2 y=\frac{-x^{2}+1}{x}}$$

13.17.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$A y^{\prime \prime}(x)+B y^{\prime}(x)+C y(x)=f(x)$$

Where $A=x^{2}+1, B=-2 x, C=2, f(x)=-x+\frac{1}{x}$. Let the solution be

$$y=y_{h}+y_{p}$$

Where y_h is the solution to the homogeneous ODE $A y''(x)+B y'(x)+C y(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $A y''(x)+B y'(x)+C y(x)=f(x)$.

Solving for y_h from

$$\left(x^2+1\right) y''-2xy'+2y=0$$

In normal form the ode

$$\left(x^2+1\right) y''-2xy'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2x}{x^2+1} \\ q(x) &= \frac{2}{x^2+1} \end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{2n}{x^2+1}+\frac{2}{x^2+1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}-\frac{2x}{x^2+1}\right)v'(x) &= 0 \\ v''(x)+\frac{2v'(x)}{x\left(x^2+1\right)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x\left(x^2+1\right)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x\left(x^2+1\right)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{2}{x\left(x^2+1\right)}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^2+1\right)}\,dx \\ \ln(u) &= \ln\left(x^2+1\right) - 2\ln(x) + c_1 \\ u &= \mathrm{e}^{\ln(x^2+1)-2\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{\ln(x^2+1)-2\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(1 + \frac{1}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1\left(x - \frac{1}{x}\right) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(c_1 \left(x - \frac{1}{x}\right) + c_2\right) x \\
&= c_1 x^2 + c_2 x - c_1
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
(x^2 + 1)\, y'' - 2xy' + 2y = -x + \frac{1}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= x^2 - 1
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x & x^2 - 1 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2 - 1) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x & x^2 - 1 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W = (x)(2x) - (x^2 - 1)(1)$$

Which simplifies to

$$W = x^2 + 1$$

Which simplifies to

$$W = x^2 + 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^2 - 1)\left(-x + \frac{1}{x}\right)}{(x^2 + 1)^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{(x^2 - 1)^2}{x (x^2 + 1)^2} dx$$

Hence

$$u_1 = \ln(x) + \frac{2}{x^2 + 1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(-x + \frac{1}{x}\right)}{(x^2 + 1)^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-x^2 + 1}{(x^2 + 1)^2} dx$$

Hence

$$u_2 = \frac{x}{x^2 + 1}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\ln(x) + \frac{2}{x^2+1}\right)x + \frac{x(x^2-1)}{x^2+1}$$

Which simplifies to

$$y_p(x) = (\ln(x) + 1)\,x$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x\right) + ((\ln(x) + 1)\,x) \\
&= (\ln(x) + 1)\,x + \left(c_1\left(x - \frac{1}{x}\right) + c_2\right)x
\end{aligned}$$

Which simplifies to

$$y = c_1x^2 + x\ln(x) + c_2x - c_1 + x$$

Summary
The solution(s) found are the following

$$y = c_1x^2 + x\ln(x) + c_2x - c_1 + x \tag{1}$$

Verification of solutions

$$y = c_1x^2 + x\ln(x) + c_2x - c_1 + x$$

Verified OK.

13.17.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 + 1 \\
B &= -2x \\
C &= 2 \\
F &= -x + \frac{1}{x}
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2 + 1\right)(0) + (-2x)(-2) + (2)(-2x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2x\left(x^2 + 1\right) v'' + (-4)\, v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
\left(-2x^3 - 2x\right) u'(x) - 4u(x) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u}{x\left(x^{2}+1\right)}
\end{aligned}
$$

Where $f(x)=-\frac{2}{x(x^2+1)}$ and $g(u)=u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\ln\left(u\right) &= \ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}\\
u &= \mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}}\\
&= c_{1}\mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_{1}\left(1+\frac{1}{x^{2}}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u\\
&= c_{1}\left(1+\frac{1}{x^{2}}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{\left(x^{2}+1\right)c_{1}}{x^{2}}\,\mathrm{d}x\\
&= c_{1}\left(x-\frac{1}{x}\right)+c_{2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv\\
&= \left(-2x\right)\left(c_{1}\left(x-\frac{1}{x}\right)+c_{2}\right)\\
&= -2c_{1}x^{2}-2c_{2}x+2c_{1}
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= -2x^2 + 2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 + 2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -2x^2 + 2 \\ 1 & -4x \end{vmatrix}$$

Therefore

$$W = (x)(-4x) - \left(-2x^2 + 2\right)(1)$$

Which simplifies to

$$W = -2x^2 - 2$$

Which simplifies to

$$W = -2x^2 - 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-2x^2+2)\left(-x+\frac{1}{x}\right)}{(x^2+1)(-2x^2-2)}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{(x^2-1)^2}{x(x^2+1)^2}dx$$

Hence

$$u_1 = \ln(x) + \frac{2}{x^2+1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(-x+\frac{1}{x}\right)}{(x^2+1)(-2x^2-2)}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2-1}{2(x^2+1)^2}dx$$

Hence

$$u_2 = -\frac{x}{2(x^2+1)}$$

Which simplifies to

$$u_1 = \ln(x) + \frac{2}{x^2+1}$$
$$u_2 = -\frac{x}{2x^2+2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\ln(x) + \frac{2}{x^2+1}\right)x - \frac{x(-2x^2+2)}{2x^2+2}$$

Which simplifies to

$$y_p(x) = (\ln(x) + 1)\,x$$

Hence the complete solution is

$$\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-2c_1x^2 - 2c_2x + 2c_1\right) + \left((\ln(x) + 1)\,x\right) \\
&= -2c_1x^2 + x\ln(x) - 2c_2x + 2c_1 + x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -2c_1x^2 + x\ln(x) - 2c_2x + 2c_1 + x \tag{1}$$

Verification of solutions

$$y = -2c_1x^2 + x\ln(x) - 2c_2x + 2c_1 + x$$

Verified OK.

13.17.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2 + 1\right)y'' - 2xy' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= x^2 + 1 \\
B &= -2x \\
C &= 2
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{(x^2+1)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(x^2+1\right)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4\left(x-i\right)^2} + \frac{3}{4\left(x+i\right)^2} + \frac{3i}{4\left(x-i\right)} - \frac{3i}{4\left(x+i\right)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2}\\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0\\
\alpha_\infty^{+} &= 0\\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{\left(x^2+1\right)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} + (-)\,(0) \\
&= -\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)} \\
&= \frac{-2i + x}{x^2+1}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)(0) + \left(\left(\frac{1}{2(x-i)^2} - \frac{3}{2(x+i)^2}\right) + \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right)^2 - \left(-\frac{}{(}\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\ &= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x}{x^2+1} \, dx} \\ &= z_1 e^{\frac{\ln(x^2+1)}{2}} \\ &= z_1 \left(\sqrt{x^2+1}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
(x^2+1)\, y'' - 2xy' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2\, x}{(x-i)^2\,(x+i)^2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

$$y_2 = \frac{(x^2+1)^2 x}{(x-i)^2 (x+i)^2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2} \\ \frac{d}{dx}\left(\frac{(x^2+1)^2}{(ix+1)^2}\right) & \frac{d}{dx}\left(\frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{(x^2+1)^2}{(ix+1)^2} & \frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2} \\ \frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3} & \frac{4(x^2+1)x^2}{(x-i)^2(x+i)^2} + \frac{(x^2+1)^2}{(x-i)^2(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^3(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^2(x+i)^3} \end{vmatrix}$$

Therefore

$$W = \left(\frac{(x^2+1)^2}{(ix+1)^2}\right)\left(\frac{4(x^2+1)x^2}{(x-i)^2(x+i)^2} + \frac{(x^2+1)^2}{(x-i)^2(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^3(x+i)^2} - \frac{2(x^2+1)^2 x}{(x-i)^2(x+i)^3}\right) - \left(\frac{(x^2+1)^2 x}{(x-i)^2(x+i)^2}\right)\left(\frac{4(x^2+1)x}{(ix+1)^2} - \frac{2i(x^2+1)^2}{(ix+1)^3}\right)$$

Which simplifies to

$$W = \frac{\left(x^2+1\right)^3 \left(i x^5+3 x^4-2 i x^3+2 x^2-3 i x-1\right)}{\left(i x+1\right)^3 \left(-x+i\right)^3 \left(x+i\right)^3}$$

Which simplifies to

$$W = \frac{\left(x^2+1\right)^4 \left(-x^3+3 i x^2+3 x-i\right)}{\left(-i x-1\right)^6 \left(x+i\right)^3}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\left(x^2+1\right)^2 x \left(-x+\frac{1}{x}\right)}{\left(x-i\right)^2 \left(x+i\right)^2}}{\frac{\left(x^2+1\right)^5 \left(-x^3+3 i x^2+3 x-i\right)}{\left(-i x-1\right)^6 \left(x+i\right)^3}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^2-1\right) \left(x+i\right) \left(x-i\right)^4}{\left(x^2+1\right)^3 \left(-x^3+3 i x^2+3 x-i\right)} dx$$

Hence

$$u_1 = -\frac{x}{x^2+1}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{\left(x^2+1\right)^2 \left(-x+\frac{1}{x}\right)}{\left(i x+1\right)^2}}{\frac{\left(x^2+1\right)^5 \left(-x^3+3 i x^2+3 x-i\right)}{\left(-i x-1\right)^6 \left(x+i\right)^3}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{\left(x+i\right)^3 \left(i x+1\right)^4 \left(x^2-1\right)}{\left(-x^3+3 i x^2+3 x-i\right) x \left(x^2+1\right)^3} dx$$

Hence

$$u_2 = \int_0^x -\frac{\left(\alpha+i\right)^3 \left(i\alpha+1\right)^4 \left(\alpha^2-1\right)}{\left(-\alpha^3+3 i\alpha^2+3\alpha-i\right) \alpha \left(\alpha^2+1\right)^3} d\alpha$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\frac{x}{x^2+1} \\
u_2 &= -\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\,\alpha\,(\alpha^2+1)^3}d\alpha\right)
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = -\frac{(x^2+1)\,x}{(ix+1)^2} - \frac{\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\alpha(\alpha^2+1)^3}d\alpha\right)(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2}
$$

Which simplifies to

$$
y_p(x) = \frac{2x(x^2+1)^2\left(\left(ix-\frac{1}{2}x^2+\frac{1}{2}\right)\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\alpha(\alpha^2+1)^3}d\alpha\right)+\frac{x^2}{2}+\frac{1}{2}\right)}{(-x+i)^4\,(x+i)^2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2}\right) \\
&\quad + \left(\frac{2x(x^2+1)^2\left(\left(ix-\frac{1}{2}x^2+\frac{1}{2}\right)\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\alpha(\alpha^2+1)^3}d\alpha\right)+\frac{x^2}{2}+\frac{1}{2}\right)}{(-x+i)^4\,(x+i)^2}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2\,x}{(x-i)^2\,(x+i)^2} \\
&\quad + \frac{2x(x^2+1)^2\left(\left(ix-\frac{1}{2}x^2+\frac{1}{2}\right)\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\alpha(\alpha^2+1)^3}d\alpha\right)+\frac{x^2}{2}+\frac{1}{2}\right)}{(-x+i)^4\,(x+i)^2}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1(x^2+1)^2}{(ix+1)^2} + \frac{c_2(x^2+1)^2 x}{(x-i)^2 (x+i)^2}
+ \frac{2x(x^2+1)^2 \left(\left(ix - \frac{1}{2}x^2 + \frac{1}{2}\right)\left(\int_0^x \frac{(\alpha^2-1)(\alpha+i)^3(i-\alpha)^4}{(-\alpha^3+3i\alpha^2+3\alpha-i)\alpha(\alpha^2+1)^3} d\alpha\right) + \frac{x^2}{2} + \frac{1}{2}\right)}{(-x+i)^4 (x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=(1-x^2)/x,y(x), singsol=all)
```

$$
y(x) = c_1x^2 + \ln(x)\,x + c_2x - c_1 + x
$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 27

```
DSolve[(1+x^2)*y''[x]-2*x*y'[x]+2*y[x]==(1-x^2)/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + 1) - c_1(x - i)^2 + c_2 x$$

14 Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132

14.1 problem 22 2178
14.2 problem 23 2183
14.3 problem 24 2198
14.4 problem 25 2221
14.5 problem 26 2225
14.6 problem 27 2230
14.7 problem 28 2238
14.8 problem 29 2242
14.9 problem 30 2244
14.10 problem 31 2251
14.11 problem 32 2256
14.12 problem 33 2267
14.13 problem 34 2271
14.14 problem 35 2274
14.15 problem 36 2276

14.1 problem 22

14.1.1 Solving as second order ode missing y ode 2178
14.1.2 Solving as second order ode missing x ode 2179
14.1.3 Maple step by step solution 2181

Internal problem ID [5429]
Internal file name [`OUTPUT/4920_Tuesday_February_06_2024_10_14_16_PM_16279652/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x", "second__order__ode__missing__y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{y'' + y'^2 = -1}$$

14.1.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x)^2 + 1 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{-p^2 - 1} dp = x + c_1$$
$$-\arctan(p) = x + c_1$$

Solving for p gives these solutions

$$p_1 = -\tan(x + c_1)$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\tan(x + c_1)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\tan(x + c_1)\, \mathrm{d}x \\ &= -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = -\frac{\ln\left(1 + \tan(x + c_1)^2\right)}{2} + c_2$$

Verified OK.

14.1.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = -1$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\int -\frac{p}{p^2+1}dp = \int dy$$
$$-\frac{\ln(p^2+1)}{2} = y + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{p^2+1}} = e^{y+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{p^2+1}} = c_2 e^{y}$$

Solving for $p(y)$ gives

$$p(y) = \text{RootOf}\left(\_Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1\right)$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \text{RootOf}\left(\_Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1\right)$$

Integrating both sides gives

$$\int \frac{1}{\text{RootOf}\left(\_Z^2 c_2^2 e^{2y} + c_2^2 e^{2y} - 1\right)}dy = \int dx$$
$$\int^{y} \frac{1}{\text{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)}d\_a = c_3 + x$$

Summary

The solution(s) found are the following

$$\int^{y} \frac{1}{\text{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)}d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^{y} \frac{1}{\text{RootOf}\left(\_Z^2 c_2^2 e^{2\_a} + c_2^2 e^{2\_a} - 1\right)}d\_a = c_3 + x$$

Verified OK.

14.1.3 Maple step by step solution

Let's solve

$y'' + y'^2 = -1$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x) + u(x)^2 = -1$

- Separate variables

  $\frac{u'(x)}{-u(x)^2-1} = 1$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{-u(x)^2-1}dx = \int 1dx + c_1$

- Evaluate integral

  $-\arctan(u(x)) = x + c_1$

- Solve for $u(x)$

  $u(x) = -\tan(x + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = -\tan(x + c_1)$

- Make substitution $u = y'$

  $y' = -\tan(x + c_1)$

- Integrate both sides to solve for y

  $\int y'dx = \int -\tan(x + c_1)\,dx + c_2$

- Compute integrals

  $y = -\frac{\ln\left(1+\tan(x+c_1)^2\right)}{2} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+diff(y(x),x)^2+1=0,y(x), singsol=all)
```

$$y(x) = \ln(\cos(x) c_2 - c_1 \sin(x))$$

✓ Solution by Mathematica

Time used: 1.79 (sec). Leaf size: 16

```
DSolve[y''[x]+y'[x]^2+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(\cos(x - c_1)) + c_2$$

14.2 problem 23

14.2.1 Solving as second order integrable as is ode 2183
14.2.2 Solving as second order ode missing y ode 2184
14.2.3 Solving as type second_order_integrable_as_is (not using ABC version) 2186
14.2.4 Solving using Kovacic algorithm 2187
14.2.5 Solving as exact linear second order ode ode 2194
14.2.6 Maple step by step solution 2195

Internal problem ID [5430]
Internal file name [`OUTPUT/4921_Tuesday_February_06_2024_10_14_17_PM_52449057/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{(x^2+1)\, y''+2xy'=\frac{2}{x^3}}$$

14.2.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2+1)\, y''+2xy'\right) dx = \int \frac{2}{x^3} dx$$
$$(x^2+1)\, y' = -\frac{1}{x^2}+c_1$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{c_1 x^2 - 1}{(x^2+1)\,x^2}\, \mathrm{d}x \\
&= (c_1+1)\arctan(x) + \frac{1}{x} + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = (c_1+1)\arctan(x) + \frac{1}{x} + c_2 \tag{1}
$$

Verification of solutions

$$
y = (c_1+1)\arctan(x) + \frac{1}{x} + c_2
$$

Verified OK.

14.2.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$
p(x) = y'
$$

Then

$$
p'(x) = y''
$$

Hence the ode becomes

$$
(x^2+1)\,p'(x) + 2p(x)\,x - \frac{2}{x^3} = 0
$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2x}{x^2+1} \\
q(x) &= \frac{2}{(x^2+1)\,x^3}
\end{aligned}
$$

Hence the ode is

$$p'(x)+\frac{2xp(x)}{x^2+1}=\frac{2}{(x^2+1)\,x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2x}{x^2+1}dx}\\ &= x^2+1\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{2}{(x^2+1)\,x^3}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left((x^2+1)\,p\right) &= (x^2+1)\left(\frac{2}{(x^2+1)\,x^3}\right)\\ \mathrm{d}\left((x^2+1)\,p\right) &= \left(\frac{2}{x^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}(x^2+1)\,p &= \int \frac{2}{x^3}\,\mathrm{d}x\\ (x^2+1)\,p &= -\frac{1}{x^2}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=x^2+1$ results in

$$p(x)=-\frac{1}{(x^2+1)\,x^2}+\frac{c_1}{x^2+1}$$

which simplifies to

$$p(x)=\frac{c_1x^2-1}{(x^2+1)\,x^2}$$

Since $p=y'$ then the new first order ode to solve is

$$y'=\frac{c_1x^2-1}{(x^2+1)\,x^2}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_1x^2-1}{(x^2+1)\,x^2}\,\mathrm{d}x\\ &= (c_1+1)\arctan(x)+\frac{1}{x}+c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = (c_1 + 1)\arctan(x) + \frac{1}{x} + c_2 \tag{1}$$

Verification of solutions

$$y = (c_1 + 1)\arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$(x^2 + 1)\, y'' + 2xy' = \frac{2}{x^3}$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left((x^2 + 1)\, y'' + 2xy'\right) dx = \int \frac{2}{x^3} dx$$
$$(x^2 + 1)\, y' = -\frac{1}{x^2} + c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int \frac{c_1 x^2 - 1}{(x^2 + 1)\, x^2}\, \mathrm{d}x$$
$$= (c_1 + 1)\arctan(x) + \frac{1}{x} + c_2$$

Summary
The solution(s) found are the following

$$y = (c_1 + 1)\arctan(x) + \frac{1}{x} + c_2 \tag{1}$$

Verification of solutions

$$y = (c_1 + 1)\arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.4 Solving using Kovacic algorithm

Writing the ode as

$$(x^2+1)\,y''+2xy'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2+1 \\ B &= 2x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x)=rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB'-2BA'+B^2-4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A,B,C from (3) in the above and simplifying gives

$$r=\frac{1}{(x^2+1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= (x^2+1)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x)=\left(\frac{1}{(x^2+1)^2}\right)z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y=z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 248: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(x-i)^2} - \frac{1}{4(x+i)^2} - \frac{i}{4(x-i)} + \frac{i}{4x+4i}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $4 > 2$ then

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{(x^2+1)^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right)\\
&= 1 - (1)\\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{x-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2}\right) + (-)[\sqrt{r}]_\infty\\
&= \frac{1}{2x+2i} + \frac{1}{2x-2i} + (-)\,(0)\\
&= \frac{1}{2x+2i} + \frac{1}{2x-2i}\\
&= \frac{x}{x^2+1}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2x+2i} + \frac{1}{2x-2i}\right)(0) + \left(\left(-\frac{1}{2\,(x+i)^2} - \frac{1}{2\,(x-i)^2}\right) + \left(\frac{1}{2x+2i} + \frac{1}{2x-2i}\right)^2 - \left(\frac{1}{(x^2+1)^2}\right.\right.
$$
$$
0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx}\\
&= \mathrm{e}^{\int \left(\frac{1}{2x+2i} + \frac{1}{2x-2i}\right) dx}\\
&= \sqrt{x^2+1}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2+1}\,dx} \\
&= z_1 e^{-\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x^2+1}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2+1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln\left(x^2+1\right)}}{\left(y_1\right)^2}\,dx \\
&= y_1(\arctan\left(x\right))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\arctan\left(x\right)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
\left(x^2+1\right)y'' + 2xy' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2 \arctan(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \arctan(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \arctan(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\arctan(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \arctan(x) \\ 0 & \frac{1}{x^2+1} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x^2+1}\right) - (\arctan(x))(0)$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Which simplifies to

$$W = \frac{1}{x^2+1}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\arctan(x)}{x^3}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\arctan(x)}{x^3}dx$$

Hence

$$u_1 = \frac{\arctan(x)}{x^2} + \arctan(x) + \frac{1}{x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2}{x^3}}{1}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^3}dx$$

Hence

$$u_2 = -\frac{1}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \arctan(x) + \frac{1}{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \arctan(x)) + \left(\arctan(x) + \frac{1}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 \arctan(x) + \arctan(x) + \frac{1}{x} \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \arctan(x) + \arctan(x) + \frac{1}{x}$$

Verified OK.

14.2.5 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 + 1 \\ q(x) &= 2x \\ r(x) &= 0 \\ s(x) &= \frac{2}{x^3} \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 2 \end{aligned}$$

Therefore (1) becomes

$$2 - (2) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\, y' + (q(x) - p'(x))\, y)' = s(x)$$

Integrating gives

$$p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx$$

Substituting the above values for p, q, r, s gives

$$(x^2+1)\,y' = \int \frac{2}{x^3}\, dx$$

We now have a first order ode to solve which is

$$(x^2+1)\,y' = -\frac{1}{x^2} + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1 x^2 - 1}{(x^2+1)\,x^2}\, \mathrm{d}x \\ &= (c_1+1)\arctan(x) + \frac{1}{x} + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (c_1+1)\arctan(x) + \frac{1}{x} + c_2 \tag{1}$$

Verification of solutions

$$y = (c_1+1)\arctan(x) + \frac{1}{x} + c_2$$

Verified OK.

14.2.6 Maple step by step solution

Let's solve

$(x^2+1)\,y'' + 2xy' = \frac{2}{x^3}$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $(x^2+1)\,u'(x) + 2xu(x) = \frac{2}{x^3}$

- Isolate the derivative

  $u'(x) = -\frac{2xu(x)}{x^2+1} + \frac{2}{(x^2+1)x^3}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{2xu(x)}{x^2+1} = \frac{2}{(x^2+1)x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x)+\frac{2xu(x)}{x^2+1}\right)=\frac{2\mu(x)}{(x^2+1)x^3}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x)+\frac{2xu(x)}{x^2+1}\right)=\mu'(x)\,u(x)+\mu(x)\,u'(x)$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)x}{x^2+1}$
- Solve to find the integrating factor

  $\mu(x)=x^2+1$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,u(x))\right)dx=\int\frac{2\mu(x)}{(x^2+1)x^3}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,u(x)=\int\frac{2\mu(x)}{(x^2+1)x^3}dx+c_1$
- Solve for $u(x)$

  $u(x)=\frac{\int\frac{2\mu(x)}{(x^2+1)x^3}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=x^2+1$

  $u(x)=\frac{\int\frac{2}{x^3}dx+c_1}{x^2+1}$
- Evaluate the integrals on the rhs

  $u(x)=\frac{-\frac{1}{x^2}+c_1}{x^2+1}$
- Solve 1st ODE for $u(x)$

  $u(x)=\frac{-\frac{1}{x^2}+c_1}{x^2+1}$
- Make substitution $u=y'$

  $y'=\frac{-\frac{1}{x^2}+c_1}{x^2+1}$
- Integrate both sides to solve for y

  $\int y'dx=\int\frac{-\frac{1}{x^2}+c_1}{x^2+1}dx+c_2$
- Compute integrals

  $y=(c_1+1)\arctan(x)+\frac{1}{x}+c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -2*(_b(_a)*_a^4-1)/(_a^3*(_a^2+1)), _b(
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x)=2*x^(-3),y(x), singsol=all)
```

$$y(x) = \frac{1}{x} + (c_1 + 1) \arctan(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 18

```
DSolve[(1+x^2)*y''[x]+2*x*y'[x]==2*x^(-3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (1 + c_1) \arctan(x) + \frac{1}{x} + c_2$$

14.3 problem 24

14.3.1 Solving as second order integrable as is ode 2199
14.3.2 Solving as second order ode missing y ode 2200
14.3.3 Solving as second order ode non constant coeff transformation on B ode 2202
14.3.4 Solving as type second_order_integrable_as_is (not using ABC version) 2206
14.3.5 Solving using Kovacic algorithm 2208
14.3.6 Solving as exact linear second order ode ode 2216
14.3.7 Maple step by step solution 2218

Internal problem ID [5431]
Internal file name [`OUTPUT/4922_Tuesday_February_06_2024_10_14_18_PM_47983974/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' - y' = -\frac{2}{x} - \ln(x)}$$

14.3.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xy'' - y')\,dx = \int \left(-\frac{2}{x} - \ln(x)\right) dx$$

$$xy' - 2y = -2\ln(x) - x\ln(x) + x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{(-x-2)\ln(x) + x + c_1}{x} \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x-2)\ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x} dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{(-x-2)\ln(x) + x + c_1}{x^3}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^2} &= \int \frac{(-x-2)\ln(x) + x + c_1}{x^3}\,\mathrm{d}x \\ \frac{y}{x^2} &= \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} \right) + c_2 x^2$$

which simplifies to

$$y = c_2 x^2 + x \ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = c_2 x^2 + x \ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_2 x^2 + x \ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x - p(x) + \frac{2}{x} + \ln(x) = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{-x \ln(x) - 2}{x^2} \end{aligned}$$

Hence the ode is

$$p'(x) - \frac{p(x)}{x} = \frac{-x\ln(x) - 2}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)\left(\frac{-x\ln(x) - 2}{x^2}\right) \\ \frac{d}{dx}\left(\frac{p}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{-x\ln(x) - 2}{x^2}\right) \\ d\left(\frac{p}{x}\right) &= \left(\frac{-x\ln(x) - 2}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{p}{x} &= \int \frac{-x\ln(x) - 2}{x^3} dx \\ \frac{p}{x} &= \frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$p(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2}\right) + c_1 x$$

which simplifies to

$$p(x) = \frac{c_1 x^2 + x\ln(x) + x + 1}{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1 x^2 + x\ln(x) + x + 1}{x}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_1 x^2 + x\ln(x) + x + 1}{x} dx \\ &= \frac{c_1 x^2}{2} + x\ln(x) + \ln(x) + c_2\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2}{2} + x \ln(x) + \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2}{2} + x \ln(x) + \ln(x) + c_2$$

Verified OK.

14.3.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x \\
B &= -1 \\
C &= 0 \\
F &= -\frac{2}{x}-\ln(x)
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB &= (x)(0)+(-1)(0)+(0)(-1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-xv''+(1)\,v'=0
$$

Now by applying $v'=u$ the above becomes

$$
-xu'(x)+u(x)=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u}{x}
\end{aligned}
$$

Where $f(x)=\frac{1}{x}$ and $g(u)=u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x)+c_1 \\
u &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \, \mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-1)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{c_1 x^2}{2} - c_2
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= -1 \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} -1 & x^2 \\ \frac{d}{dx}(-1) & \frac{d}{dx}(x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} -1 & x^2 \\ 0 & 2x \end{vmatrix}$$

Therefore

$$W = (-1)(2x) - (x^2)(0)$$

Which simplifies to

$$W = -2x$$

Which simplifies to

$$W = -2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2\left(-\frac{2}{x} - \ln(x)\right)}{-2x^2}\, dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{1}{x} + \frac{\ln(x)}{2}\right) dx$$

Hence

$$u_1 = -\ln(x) - \frac{x \ln(x)}{2} + \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2}{x} + \ln(x)}{-2x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{-x\ln(x) - 2}{2x^3} dx$$

Hence

$$u_2 = \frac{\ln(x)}{2x} + \frac{1}{2x} + \frac{1}{2x^2}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\ln(x) - \frac{x\ln(x)}{2} + \frac{x}{2} \\
u_2 &= \frac{x\ln(x) + x + 1}{2x^2}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \ln(x) + x\ln(x) + \frac{1}{2}
$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(-\frac{c_1 x^2}{2} - c_2\right) + \left(\ln(x) + x\ln(x) + \frac{1}{2}\right) \\
&= -\frac{c_1 x^2}{2} - c_2 + \ln(x) + x\ln(x) + \frac{1}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{c_1 x^2}{2} - c_2 + \ln(x) + x\ln(x) + \frac{1}{2} \tag{1}
$$

Verification of solutions

$$
y = -\frac{c_1 x^2}{2} - c_2 + \ln(x) + x\ln(x) + \frac{1}{2}
$$

Verified OK.

14.3.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
xy'' - y' = -\frac{2}{x} - \ln(x)
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (xy'' - y')\, dx &= \int \left(-\frac{2}{x} - \ln(x)\right) dx \\
xy' - 2y &= -2\ln(x) - x\ln(x) + x + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{2}{x}\\ q(x) &= \frac{(-x-2)\ln(x) + x + c_1}{x}\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x-2)\ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx}\\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right)\\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{(-x-2)\ln(x) + x + c_1}{x^3}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \frac{(-x-2)\ln(x) + x + c_1}{x^3}\,\mathrm{d}x\\ \frac{y}{x^2} &= \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = c_2x^2 + x\ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary
The solution(s) found are the following

$$y = c_2 x^2 + x \ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_2 x^2 + x \ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.5 Solving using Kovacic algorithm

Writing the ode as

$$xy'' - y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 3 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 250: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right)=0
$$
$$
0=0
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{x}\, dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{x}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' - y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 x^2}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= \frac{x^2}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{x^2}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{x^2}{2} \\ 0 & x \end{vmatrix}$$

Therefore

$$W = (1)\,(x) - \left(\frac{x^2}{2}\right)(0)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2\left(-\frac{2}{x}-\ln(x)\right)}{2}}{x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-\frac{1}{x} - \frac{\ln(x)}{2}\right) dx$$

Hence

$$u_1 = \ln(x) + \frac{x\ln(x)}{2} - \frac{x}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{-\frac{2}{x} - \ln(x)}{x^2}\,dx$$

Which simplifies to

$$u_2 = \int \frac{-x \ln(x) - 2}{x^3} dx$$

Hence

$$u_2 = \frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x) + \frac{x \ln(x)}{2} - \frac{x}{2} + \frac{\left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2}\right) x^2}{2}$$

Which simplifies to

$$y_p(x) = \ln(x) + x \ln(x) + \frac{1}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 x^2}{2}\right) + \left(\ln(x) + x \ln(x) + \frac{1}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 x^2}{2} + \ln(x) + x \ln(x) + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_1 + \frac{c_2 x^2}{2} + \ln(x) + x \ln(x) + \frac{1}{2}$$

Verified OK.

14.3.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y''+q(x)\,y'+r(x)\,y=s(x)$$

is exact if

$$p''(x)-q'(x)+r(x)=0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x)&=x\\ q(x)&=-1\\ r(x)&=0\\ s(x)&=-\frac{2}{x}-\ln\left(x\right)\end{aligned}$$

Hence

$$\begin{aligned}p''(x)&=0\\ q'(x)&=0\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)$$

Integrating gives

$$p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$xy'-2y=\int -\frac{2}{x}-\ln\left(x\right)\;dx$$

We now have a first order ode to solve which is

$$xy'-2y=-2\ln\left(x\right)-x\ln\left(x\right)+x+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{2}{x} \\
q(x) &= \frac{(-x-2)\ln(x) + x + c_1}{x}
\end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{(-x-2)\ln(x) + x + c_1}{x}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{(-x-2)\ln(x) + x + c_1}{x}\right) \\
d\left(\frac{y}{x^2}\right) &= \left(\frac{(-x-2)\ln(x) + x + c_1}{x^3}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x^2} &= \int \frac{(-x-2)\ln(x) + x + c_1}{x^3} dx \\
\frac{y}{x^2} &= \frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2} + c_2
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2\left(\frac{\ln(x)}{x} + \frac{\ln(x)}{x^2} + \frac{1}{2x^2} - \frac{c_1}{2x^2}\right) + c_2x^2$$

which simplifies to

$$y = c_2x^2 + x\ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Summary
The solution(s) found are the following

$$y = c_2x^2 + x\ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2} \tag{1}$$

Verification of solutions

$$y = c_2x^2 + x\ln(x) + \ln(x) - \frac{c_1}{2} + \frac{1}{2}$$

Verified OK.

14.3.7 Maple step by step solution

Let's solve

$y''x - y' = -\frac{2}{x} - \ln(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x - u(x) = -\frac{2}{x} - \ln(x)$

- Isolate the derivative

  $u'(x) = \frac{u(x)}{x} - \frac{x\ln(x)+2}{x^2}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) - \frac{u(x)}{x} = -\frac{x\ln(x)+2}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = -\frac{\mu(x)(x\ln(x)+2)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,u(x))$

  $\mu(x)\left(u'(x) - \frac{u(x)}{x}\right) = \mu'(x)\,u(x) + \mu(x)\,u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,u(x))\right) dx = \int -\frac{\mu(x)(x\ln(x)+2)}{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) u(x) = \int -\frac{\mu(x)(x \ln(x)+2)}{x^2} dx + c_1$
- Solve for $u(x)$

  $u(x) = \frac{\int -\frac{\mu(x)(x \ln(x)+2)}{x^2} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

  $u(x) = x\left(\int -\frac{x \ln(x)+2}{x^3} dx + c_1\right)$
- Evaluate the integrals on the rhs

  $u(x) = x\left(\frac{\ln(x)}{x} + \frac{1}{x} + \frac{1}{x^2} + c_1\right)$
- Simplify

  $u(x) = \frac{c_1 x^2 + x \ln(x) + x + 1}{x}$
- Solve 1st ODE for $u(x)$

  $u(x) = \frac{c_1 x^2 + x \ln(x) + x + 1}{x}$
- Make substitution $u = y'$

  $y' = \frac{c_1 x^2 + x \ln(x) + x + 1}{x}$
- Integrate both sides to solve for y

  $\int y' dx = \int \frac{c_1 x^2 + x \ln(x) + x + 1}{x} dx + c_2$
- Compute integrals

  $y = \frac{c_1 x^2}{2} + x \ln(x) + \ln(x) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (_b(_a)*_a-ln(_a)*_a-2)/_a^2, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x$2)-diff(y(x),x)=-2/x-ln(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2}{2} + \ln(x) x + \ln(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 23

```
DSolve[x*y''[x]-y'[x]==-2/x-Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 x^2}{2} + (x+1)\log(x) + c_2$$

14.4 problem 25

Internal problem ID [5432]
Internal file name [`OUTPUT/4923_Tuesday_February_06_2024_10_14_19_PM_17103159/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{y''' + y'' = x^2}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y'' = 0$$

The characteristic equation is

$$\lambda^3 + \lambda^2 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = 0$$
$$\lambda_3 = 0$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 e^{-x} + c_2 + c_3 x$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= 1 \\
y_3 &= x
\end{aligned}
$$

Now the particular solution to the given ODE is found

$$
y''' + y'' = x^2
$$

The particular solution is found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{1, x, \mathrm{e}^{-x}\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x, x^2, x^3\}]
$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x^2, x^3, x^4\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_3 x^4 + A_2 x^3 + A_1 x^2
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
12x^2 A_3 + 6x A_2 + 24x A_3 + 2A_1 + 6A_2 = x^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -\frac{1}{3}, A_3 = \frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{12}x^4 - \frac{1}{3}x^3 + x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{-x} + c_2 + c_3 x) + \left(\frac{1}{12}x^4 - \frac{1}{3}x^3 + x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + c_2 + c_3 x + \frac{x^4}{12} - \frac{x^3}{3} + x^2 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + c_2 + c_3 x + \frac{x^4}{12} - \frac{x^3}{3} + x^2$$

Verified OK.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _a^2-_b(_a), _b(_a)`   *** Sublevel 2 *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)=x^2,y(x), singsol=all)
```

$$y(x) = \frac{x^4}{12} + x^2 - \frac{x^3}{3} + c_1 \mathrm{e}^{-x} + c_2 x + c_3$$

✓ Solution by Mathematica

Time used: 0.09 (sec). Leaf size: 37

```
DSolve[y'''[x]+y''[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^4}{12} - \frac{x^3}{3} + x^2 + c_3 x + c_1 e^{-x} + c_2$$

14.5 problem 26

14.5.1 Solving as second order ode missing x ode 2225
14.5.2 Maple step by step solution 2227

Internal problem ID [5433]
Internal file name [`OUTPUT/4924_Tuesday_February_06_2024_10_14_19_PM_55034848/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1],
  [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' + {y'}^3 = 0}$$

14.5.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y p(y) \left(\frac{d}{dy} p(y)\right) + p(y)^3 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned} p' &= F(y, p) \\ &= f(y) g(p) \\ &= -\frac{p^2}{y} \end{aligned}$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p^2}\, dp &= -\frac{1}{y}\, dy \\ \int \frac{1}{p^2}\, dp &= \int -\frac{1}{y}\, dy \\ -\frac{1}{p} &= -\ln(y) + c_1 \end{aligned}$$

The solution is

$$-\frac{1}{p(y)} + \ln(y) - c_1 = 0$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{1}{y'} + \ln(y) - c_1 = 0$$

Integrating both sides gives

$$\begin{aligned} \int (\ln(y) - c_1)\, dy &= x + c_2 \\ -c_1 y + \ln(y)\, y - y &= x + c_2 \end{aligned}$$

Solving for y gives these solutions

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x+c_2)\mathrm{e}^{-1-c_1}\right)+1+c_1} \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{\mathrm{LambertW}\left((x+c_2)\mathrm{e}^{-1-c_1}\right)+1+c_1}$$

Verified OK.

14.5.2 Maple step by step solution

Let's solve

$y y'' + y'^3 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $y u(y)\left(\frac{d}{dy}u(y)\right) + u(y)^3 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)^2} = -\frac{1}{y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)^2}dy = \int -\frac{1}{y}dy + c_1$

- Evaluate integral

  $-\frac{1}{u(y)} = -\ln(y) + c_1$

- Solve for $u(y)$

  $u(y) = \frac{1}{\ln(y) - c_1}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{1}{\ln(y)-c_1}$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{1}{\ln(y)-c_1}$
- Separate variables

  $y'(\ln(y) - c_1) = 1$
- Integrate both sides with respect to x

  $\int y'(\ln(y) - c_1)\, dx = \int 1dx + c_2$
- Evaluate integral

  $-yc_1 + \ln(y)\, y - y = x + c_2$
- Solve for y

  $y = \mathrm{e}^{\mathit{LambertW}\left((x+c_2)\mathrm{e}^{-1-c_1}\right)+1+c_1}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^3=0,y(x), singsol=all)
```

$$
\begin{aligned}
y(x) &= 0 \\
y(x) &= c_1 \\
y(x) &= \frac{x + c_2}{\operatorname{LambertW}\left((x + c_2)\,\mathrm{e}^{c_1 - 1}\right)}
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 60.092 (sec). Leaf size: 26

```
DSolve[y[x]*y''[x]+y'[x]^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$
y(x) \to \frac{x + c_2}{W\left(e^{-1-c_1}(x + c_2)\right)}
$$

14.6 problem 27

14.6.1 Solving as second order integrable as is ode 2230
14.6.2 Solving as second order ode missing x ode 2231
14.6.3 Solving as type second_order_integrable_as_is (not using ABC version) 2233
14.6.4 Solving as exact nonlinear second order ode ode 2234
14.6.5 Maple step by step solution 2235

Internal problem ID [5434]
Internal file name [`OUTPUT/4925_Tuesday_February_06_2024_10_14_20_PM_13102016/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{yy'' + y'^2 = 0}$$

14.6.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$
\begin{aligned}
p' &= F(y,p) \\
&= f(y)g(p) \\
&= -\frac{p}{y}
\end{aligned}
$$

Where $f(y) = -\frac{1}{y}$ and $g(p) = p$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{p}\,dp &= -\frac{1}{y}\,dy \\
\int \frac{1}{p}\,dp &= \int -\frac{1}{y}\,dy \\
\ln(p) &= -\ln(y) + c_1 \\
p &= \mathrm{e}^{-\ln(y)+c_1} \\
&= \frac{c_1}{y}
\end{aligned}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = \frac{c_1}{y}
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{y}{c_1} dy &= x + c_2 \\
\frac{y^2}{2c_1} &= x + c_2
\end{aligned}
$$

Solving for y gives these solutions

$$
\begin{aligned}
y_1 &= \sqrt{2c_1c_2 + 2c_1x} \\
y_2 &= -\sqrt{2c_1c_2 + 2c_1x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}
$$

$$
y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}
$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = 0$$
$$yy' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{y}{c_1} dy = x + c_2$$
$$\frac{y^2}{2c_1} = x + c_2$$

Solving for y gives these solutions

$$y_1 = \sqrt{2c_1c_2 + 2c_1x}$$
$$y_2 = -\sqrt{2c_1c_2 + 2c_1x}$$

Summary

The solution(s) found are the following

$$y = \sqrt{2c_1c_2 + 2c_1x} \tag{1}$$
$$y = -\sqrt{2c_1c_2 + 2c_1x} \tag{2}$$

Verification of solutions

$$y = \sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

$$y = -\sqrt{2c_1c_2 + 2c_1x}$$

Verified OK.

14.6.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= y \\ a_1 &= y' \\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\ \int y\, dy' + \int y'\, dy + \int 0\, dx &= c_1\end{aligned}$$

Which results in

$$2yy' = c_1$$

Which is now solved Integrating both sides gives

$$\begin{aligned}\int \frac{2y}{c_1} dy &= x + c_2 \\ \frac{y^2}{c_1} &= x + c_2\end{aligned}$$

Solving for y gives these solutions

$$\begin{aligned}y_1 &= \sqrt{c_1 c_2 + c_1 x} \\ y_2 &= -\sqrt{c_1 c_2 + c_1 x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{c_1 c_2 + c_1 x} \tag{1}$$
$$y = -\sqrt{c_1 c_2 + c_1 x} \tag{2}$$

Verification of solutions

$$y = \sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

$$y = -\sqrt{c_1 c_2 + c_1 x}$$

Verified OK.

14.6.5 Maple step by step solution

Let's solve

$yy'' + y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right) + u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = -\frac{1}{y}$

- Integrate both sides with respect to y

$$\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int -\frac{1}{y}dy + c_1$$

- Evaluate integral

  $$\ln(u(y)) = -\ln(y) + c_1$$

- Solve for $u(y)$

  $$u(y) = \frac{e^{c_1}}{y}$$

- Solve 1st ODE for $u(y)$

  $$u(y) = \frac{e^{c_1}}{y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

  $$y' = \frac{e^{c_1}}{y}$$

- Separate variables

  $$yy' = e^{c_1}$$

- Integrate both sides with respect to x

  $$\int yy'dx = \int e^{c_1}dx + c_2$$

- Evaluate integral

  $$\frac{y^2}{2} = x\,e^{c_1} + c_2$$

- Solve for y

  $$\left\{y = \sqrt{2x\,e^{c_1} + 2c_2}, y = -\sqrt{2x\,e^{c_1} + 2c_2}\right\}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
dsolve(y(x)*diff(y(x),x$2)+diff(y(x),x)^2=0,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \sqrt{2c_1x + 2c_2}$$
$$y(x) = -\sqrt{2c_1x + 2c_2}$$

✓ Solution by Mathematica

Time used: 0.172 (sec). Leaf size: 20

```
DSolve[y[x]*y''[x]+y'[x]^2==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{2x - c_1}$$

14.7 problem 28

14.7.1 Solving as second order ode missing x ode 2238

Internal problem ID [5435]
Internal file name [`OUTPUT/4926_Tuesday_February_06_2024_10_14_20_PM_11197644/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{yy'' - {y'}^2(1 - y' \cos(y) + yy' \sin(y)) = 0}$$

14.7.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) + \left(-y\sin(y)\,p(y)^2 + \cos(y)\,p(y)^2 - p(y)\right)p(y) = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\,y$ on the above ode results in new ode in $u(y)$

$$y^2u(y)\left(\left(\frac{d}{dy}u(y)\right)y+u(y)\right)+\left(-y^3\sin(y)\,u(y)^2+\cos(y)\,u(y)^2\,y^2-u(y)\,y\right)u(y)\,y=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(y,u)\\
&= f(y)g(u)\\
&= u^2(y\sin(y)-\cos(y))
\end{aligned}$$

Where $f(y)=y\sin(y)-\cos(y)$ and $g(u)=u^2$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u^2}\,du &= y\sin(y)-\cos(y)\;dy\\
\int\frac{1}{u^2}\,du &= \int y\sin(y)-\cos(y)\;dy\\
-\frac{1}{u} &= -y\cos(y)+c_2
\end{aligned}$$

The solution is

$$-\frac{1}{u(y)}+y\cos(y)-c_2=0$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\begin{aligned}
-\frac{y}{p(y)}+y\cos(y)-c_2&=0\\
-\frac{y}{p(y)}+y\cos(y)-c_2&=0
\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$-\frac{y}{y'}+y\cos(y)-c_2=0$$

Integrating both sides gives

$$\begin{aligned}
\int\frac{y\cos(y)-c_2}{y}dy &= \int dx\\
\int^y\frac{_a\cos(_a)-c_2}{_a}d_a &= c_3+x
\end{aligned}$$

Summary
The solution(s) found are the following

$$\int^{y} \frac{\_a \cos(\_a) - c_2}{\_a} d\_a = c_3 + x \tag{1}$$

Verification of solutions

$$\int^{y} \frac{\_a \cos(\_a) - c_2}{\_a} d\_a = c_3 + x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_b(_a)^2*(-sin(_a)*_a*_b(_a)+cos
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.109 (sec). Leaf size: 22

```
dsolve(y(x)*diff(y(x),x$2)=diff(y(x),x)^2*(1-diff(y(x),x)*cos(y(x))+y(x)*diff(y(x),x)*sin(y(
```

$$y(x) = c_1$$
$$\sin(y(x)) + c_1 \ln(y(x)) - x - c_2 = 0$$

✓ Solution by Mathematica

Time used: 0.425 (sec). Leaf size: 63

```
DSolve[y[x]*y''[x]==y'[x]^2*(1-y'[x]*Cos[y[x]]+y[x]*y'[x]*Sin[y[x]] ),y[x],x,IncludeSingular
```

$$y(x) \to \text{InverseFunction}[\sin(\#1) + c_1 \log(\#1)\&][x + c_2]$$
$$y(x) \to \text{InverseFunction}[\sin(\#1) - c_1 \log(\#1)\&][x + c_2]$$
$$y(x) \to \text{InverseFunction}[\sin(\#1) + c_1 \log(\#1)\&][x + c_2]$$

14.8 problem 29

14.8.1 Maple step by step solution 2242

Internal problem ID [5436]
Internal file name [`OUTPUT/4927_Tuesday_February_06_2024_10_14_22_PM_63877284/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 29.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{(2x-3)\,y''' - (6x-7)\,y'' + 4y'x - 4y = 8}$$

Unable to solve this ODE.

14.8.1 Maple step by step solution

Let's solve

$(2x-3)\,y''' - (6x-7)\,y'' + 4y'x - 4y = 8$

- Highest derivative means the order of the ODE is 3

  y'''

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
Equation is the LCLM of -1/x*y(x)+diff(y(x),x), -y(x)+diff(y(x),x), -2*y(x)+diff(y(x),x)
trying differential order: 1; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
trying differential order: 1; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful
<- solving the LCLM ode successful `
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve((2*x-3)*diff(y(x),x$3)-(6*x-7)*diff(y(x),x$2)+4*x*diff(y(x),x)-4*y(x)=8,y(x), singsol
```

$$y(x) = -2 + c_1 x + e^x c_2 + c_3 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 26

```
DSolve[(2*x-3)*y'''[x]-(6*x-7)*y''[x]+4*x*y'[x]-4*y[x]==8,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to c_1 x + c_2 e^x - c_3 e^{2x} - 2$$

14.9 problem 30

14.9.1 Maple step by step solution 2247

Internal problem ID [5437]
Internal file name [`OUTPUT/4928_Tuesday_February_06_2024_10_14_22_PM_25156565/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 30.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_missing_y"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_y]]
```

$$\boxed{(2x^3-1)\,y'''-6x^2y''+6xy'=0}$$

Since y is missing from the ode then we can use the substitution $y'=v(x)$ to reduce the order by one. The ODE becomes

$$(2x^3-1)\,v''(x)-6x^2v'(x)+6xv(x)=0$$

In normal form the ode

$$(2x^3-1)\,v''(x)-6x^2v'(x)+6xv(x)=0 \tag{1}$$

Becomes

$$v''(x)+p(x)\,v'(x)+q(x)\,v(x)=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{6x^2}{2x^3-1}\\ q(x)&=\frac{6x}{2x^3-1}\end{aligned}$$

Applying change of variables on the depndent variable $v(x) = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not $v(x)$.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{6nx}{2x^3-1} + \frac{6x}{2x^3-1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2}{x} - \frac{6x^2}{2x^3-1}\right) v'(x) &= 0 \\ v''(x) + \frac{(-2x^3-2)\, v'(x)}{2x^4 - x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(-2x^3-2)\, u(x)}{2x^4 - x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{2u(x^3+1)}{x\,(2x^3-1)} \end{aligned}$$

Where $f(x) = \frac{2x^3+2}{x(2x^3-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{2x^3+2}{x\,(2x^3-1)}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{2x^3+2}{x\,(2x^3-1)}\, dx \\
\ln(u) &= \ln\left(2x^3-1\right) - 2\ln(x) + c_1 \\
u &= \mathrm{e}^{\ln(2x^3-1) - 2\ln(x) + c_1} \\
&= c_1 \mathrm{e}^{\ln(2x^3-1) - 2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(2x - \frac{1}{x^2}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1\left(x^2 + \frac{1}{x}\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
v(x) &= v(x)\, x^n \\
&= \left(c_1\left(x^2 + \frac{1}{x}\right) + c_2\right) x \\
&= c_1 x^3 + c_2 x + c_1
\end{aligned}
$$

But since $y' = v(x)$ then we now need to solve the ode $y' = \left(c_1\left(x^2 + \frac{1}{x}\right) + c_2\right) x$. Integrating both sides gives

$$
\begin{aligned}
y &= \int c_1 x^3 + c_2 x + c_1 \,\mathrm{d}x \\
&= \frac{1}{4} c_1 x^4 + \frac{1}{2} c_2 x^2 + c_1 x + c_3
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{4} c_1 x^4 + \frac{1}{2} c_2 x^2 + c_1 x + c_3 \tag{1}
$$

Verification of solutions

$$y = \frac{1}{4}c_1x^4 + \frac{1}{2}c_2x^2 + c_1x + c_3$$

Verified OK.

14.9.1 Maple step by step solution

Let's solve

$(2x^3-1)\,y''' - 6x^2y'' + 6xy' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{6x^2}{2x^3-1}, P_3(x) = \frac{6x}{2x^3-1}, P_4(x) = 0\right]$
  - $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

    $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)\cdot P_2(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}} = 0$
  - $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2\cdot P_3(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

    $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^2\cdot P_3(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}} = 0$
  - $\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^3\cdot P_4(x)$ is analytic at $x = -\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

    $\left(\left(x+\frac{2^{\frac{2}{3}}}{4}+\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^3\cdot P_4(x)\right)\Bigg|_{x=-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}} = 0$
  - $x = -\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

- Change variables using $x = u-\frac{2^{\frac{2}{3}}}{4}-\frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$ so that the regular singular point is at $u = 0$

  $\left(2u^3-\frac{3u^2 2^{\frac{2}{3}}}{2}-\frac{3\,\mathrm{I}u^2\sqrt{3}\,2^{\frac{2}{3}}}{2}-\frac{3u2^{\frac{1}{3}}}{2}+\frac{3\,\mathrm{I}u2^{\frac{1}{3}}\sqrt{3}}{2}\right)\left(\frac{d^3}{du^3}y(u)\right)+\left(-6u^2+3u2^{\frac{2}{3}}+3\,\mathrm{I}u\sqrt{3}\,2^{\frac{2}{3}}+\frac{3\,2^{\frac{1}{3}}}{2}-3\right.$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 0..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^3}{du^3}y(u)\right)$ to series expansion for $m = 1..3$

  $$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,(k+r-2)\,u^{k+r-3+m}$$

- Shift index using $k- >k+3-m$

  $$u^m \cdot \left(\frac{d^3}{du^3}y(u)\right) = \sum_{k=-3+m}^{\infty} a_{k+3-m}(k+3-m+r)\,(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)r(-3+r)(r-1)a_0u^{-2+r}}{2} + \left(\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(1+r)(-2+r)ra_1}{2} - \frac{3\,2^{\frac{2}{3}}\left(1+\mathrm{I}\sqrt{3}\right)(r^2-5r+5)ra_0}{2}\right)u^{r-1} + \left(\sum_{k=0}^{\infty}\left(\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)r(-3+r)(r-1)}{2} = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 1, 3\}$$

- Each term in the series must be 0, giving the recursion relation

  $$\frac{3\,2^{\frac{1}{3}}\left(\mathrm{I}\sqrt{3}-1\right)(k+2+r)(k+r-1)(k+1+r)a_{k+2}}{2} - \frac{3(k^2+(2r-3)k+r^2-3r+1)(k+1+r)\left(1+\mathrm{I}\sqrt{3}\right)a_{k+1}2^{\frac{2}{3}}}{2} + 2a_k(k+r)\,(k+r$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = \frac{\left(9\,2^{\frac{2}{3}}\,k^2 r a_{k+1} + 9\,2^{\frac{2}{3}}\,k\,r^2 a_{k+1} - 12\,2^{\frac{2}{3}}\,k r a_{k+1} - 4k^3 a_k - 4r^3 a_k + 24k^2 a_k + 24r^2 a_k - 32k a_k - 32r a_k + 3\,2^{\frac{2}{3}}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1}}{}$$

- Recursion relation for $r = 0$

$$a_{k+2} = \frac{\left(-4k^3 a_k + 24k^2 a_k - 32k a_k + 3\,2^{\frac{2}{3}}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} - 6\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^3 a_{k+1} - 6\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^2 a_{k+1} - 6}{6\left(2 + 2\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 - 2\,\mathrm{I}\sqrt{3} - k^3 - \mathrm{I}\sqrt{3}\,k - 2k^2 + k\right)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

$$a_{k+2} = \frac{\left(-4k^3 a_k + 24k^2 a_k - 32k a_k + 3\,2^{\frac{2}{3}}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} - 6\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^3 a_{k+1} - 6\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^2 a_{k+1} - 6}{6\left(2 + 2\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 - 2\,\mathrm{I}\sqrt{3} - k^3 - \mathrm{I}\sqrt{3}\,k - 2k^2 + k\right)}$$

- Recursion relation for $r = 1$

$$a_{k+2} = \frac{\left(-12a_k - 4k^3 a_k + 12k^2 a_k + 4k a_k - 6\,2^{\frac{2}{3}}\,a_{k+1} - 6\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^2 a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} + 3\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^3 a_{k}}{6\left(5\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 + 6\,\mathrm{I}\sqrt{3}\,k - k^3 - 5k^2 - 6k\right)}$$

- Series not valid for $r = 1$, division by 0 in the recursion relation at $k = 0$

$$a_{k+2} = \frac{\left(-12a_k - 4k^3 a_k + 12k^2 a_k + 4k a_k - 6\,2^{\frac{2}{3}}\,a_{k+1} - 6\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^2 a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} + 3\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^3 a_{k}}{6\left(5\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 + 6\,\mathrm{I}\sqrt{3}\,k - k^3 - 5k^2 - 6k\right)}$$

- Recursion relation for $r = 3$

$$a_{k+2} = \frac{\left(12a_k - 4k^3 a_k - 12k^2 a_k + 4k a_k + 12\,2^{\frac{2}{3}}\,a_{k+1} + 12\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} + 21\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,k^3 a_{k+1} + 39\,2^{\frac{2}{3}}\,k a_{k+1}}{6\left(11\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 + 40\,\mathrm{I}\sqrt{3} + 38\,\mathrm{I}\sqrt{3}\,k - 40 - k^3 - 11k^2 - 38k\right)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+3}, a_{k+2} = \frac{\left(12a_k - 4k^3 a_k - 12k^2 a_k + 4k a_k + 12\,2^{\frac{2}{3}}\,a_{k+1} + 12\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1} + 21\,2^{\frac{2}{3}}\,k^2 a_{k+1} + 3\,\mathrm{I}\,2}{6\left(11\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 + 40\,\mathrm{I}\sqrt{3} + 38\,\mathrm{I}\sqrt{3}\,k - 40\right.}\right.$$

- Revert the change of variables $u = x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}$

$$\left[y = \sum_{k=0}^{\infty} a_k \left(x + \frac{2^{\frac{2}{3}}}{4} + \frac{\mathrm{I}\sqrt{3}\,2^{\frac{2}{3}}}{4}\right)^{k+3}, a_{k+2} = \frac{\left(12a_k - 4k^3 a_k - 12k^2 a_k + 4k a_k + 12\,2^{\frac{2}{3}}\,a_{k+1} + 12\,\mathrm{I}\,2^{\frac{2}{3}}\,\sqrt{3}\,a_{k+1} + 3\,2^{\frac{2}{3}}\,k^3 a_{k+1}\right.}{6\left(11\,\mathrm{I}\sqrt{3}\,k^2 + \mathrm{I}\sqrt{3}\,k^3 + 4\right.}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying high order exact linear fully integrable
trying to convert to a linear ODE with constant coefficients
trying differential order: 3; missing the dependent variable
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- missing the dependent variable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve((2*x^3-1)*diff(y(x),x$3)-6*x^2*diff(y(x),x$2)+6*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_3x^4 + c_2x^2 + 4c_3x + c_1$$

✓ Solution by Mathematica

Time used: 1.357 (sec). Leaf size: 31

```
DSolve[(2*x^3-1)*y'''[x]-6*x^2*y''[x]+6*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_2x^4}{4} + \frac{c_1x^2}{2} - c_2x + c_3$$

14.10 problem 31

14.10.1 Solving as second order ode missing x ode 2251

Internal problem ID [5438]
Internal file name [`OUTPUT/4929_Tuesday_February_06_2024_10_14_22_PM_49348237/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{yy'' - {y'}^2 - y^2 \ln(y) = 0}$$

14.10.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) - p(y)^2 - \ln(y)\, y^2 = 0$$

Which is now solved as first order ode for $p(y)$. Using the change of variables $p(y) = u(y)\,y$ on the above ode results in new ode in $u(y)$

$$y^2u(y)\left(\left(\frac{d}{dy}u(y)\right)y+u(y)\right)-u(y)^2\,y^2=\ln\left(y\right)y^2$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(y,u)\\ &= f(y)g(u)\\ &= \frac{\ln\left(y\right)}{uy}\end{aligned}$$

Where $f(y) = \frac{\ln(y)}{y}$ and $g(u) = \frac{1}{u}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{u}}\,du &= \frac{\ln\left(y\right)}{y}\,dy\\ \int\frac{1}{\frac{1}{u}}\,du &= \int\frac{\ln\left(y\right)}{y}\,dy\\ \frac{u^2}{2} &= \frac{\ln\left(y\right)^2}{2}+c_2\end{aligned}$$

The solution is

$$\frac{u(y)^2}{2}-\frac{\ln\left(y\right)^2}{2}-c_2=0$$

Replacing $u(y)$ in the above solution by $\frac{p(y)}{y}$ results in the solution for $p(y)$ in implicit form

$$\begin{aligned}\frac{p(y)^2}{2y^2}-\frac{\ln\left(y\right)^2}{2}-c_2=0\\ \frac{p(y)^2}{2y^2}-\frac{\ln\left(y\right)^2}{2}-c_2=0\end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{y'^2}{2y^2}-\frac{\ln\left(y\right)^2}{2}-c_2=0$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{\ln(y)^2 + 2c_2}\, y \tag{1}$$
$$y' = -\sqrt{\ln(y)^2 + 2c_2}\, y \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\int \frac{1}{\sqrt{\ln(y)^2 + 2c_2}\, y} dy = \int dx$$
$$\ln\left(\ln(y) + \sqrt{\ln(y)^2 + 2c_2}\right) = c_3 + x$$

Raising both side to exponential gives

$$\ln(y) + \sqrt{\ln(y)^2 + 2c_2} = \mathrm{e}^{c_3 + x}$$

Which simplifies to

$$\ln(y) + \sqrt{\ln(y)^2 + 2c_2} = c_4 \mathrm{e}^{x}$$

Solving for y gives

$$y = \mathrm{e}^{\frac{\left(\mathrm{e}^{2x} c_4^2 - 2c_2\right)\mathrm{e}^{-x}}{2c_4}}$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\int -\frac{1}{\sqrt{\ln(y)^2 + 2c_2}\, y} dy = \int dx$$
$$-\ln\left(\ln(y) + \sqrt{\ln(y)^2 + 2c_2}\right) = x + c_5$$

Raising both side to exponential gives

$$\frac{1}{\ln(y) + \sqrt{\ln(y)^2 + 2c_2}} = \mathrm{e}^{x + c_5}$$

Which simplifies to

$$\frac{1}{\ln(y)+\sqrt{\ln(y)^{2}+2c_{2}}}=c_{6}\mathrm{e}^{x}$$

Solving for y gives

$$y=\mathrm{e}^{-\frac{\left(2c_{2}c_{6}^{2}\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_{6}}}$$

Summary

The solution(s) found are the following

$$y=\mathrm{e}^{\frac{\left(\mathrm{e}^{2x}c_{4}^{2}-2c_{2}\right)\mathrm{e}^{-x}}{2c_{4}}} \tag{1}$$

$$y=\mathrm{e}^{-\frac{\left(2c_{2}c_{6}^{2}\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_{6}}} \tag{2}$$

Verification of solutions

$$y=\mathrm{e}^{\frac{\left(\mathrm{e}^{2x}c_{4}^{2}-2c_{2}\right)\mathrm{e}^{-x}}{2c_{4}}}$$

Verified OK.

$$y=\mathrm{e}^{-\frac{\left(2c_{2}c_{6}^{2}\mathrm{e}^{2x}-1\right)\mathrm{e}^{-x}}{2c_{6}}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve(y(x)*diff(y(x),x$2)-diff(y(x),x)^2=y(x)^2*ln(y(x)),y(x), singsol=all)
```

$$y(x) = e^{\frac{c_1 e^{-x}}{2} - \frac{e^x c_2}{2}}$$

✓ Solution by Mathematica

Time used: 2.66 (sec). Leaf size: 73

```
DSolve[y[x]*y''[x]-y'[x]^2==y[x]^2*Log[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \exp\left(-\frac{1}{2}\sqrt{c_1} e^{-x-c_2}\left(-1+e^{2(x+c_2)}\right)\right)$$
$$y(x) \to \exp\left(\frac{1}{2}\sqrt{c_1} e^{-x-c_2}\left(-1+e^{2(x+c_2)}\right)\right)$$

14.11 problem 32

14.11.1 Solving as second order integrable as is ode 2256
14.11.2 Solving as type second_order_integrable_as_is (not using ABC version) 2261

Internal problem ID [5439]
Internal file name [`OUTPUT/4930_Tuesday_February_06_2024_10_14_24_PM_44523640/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], [_2nd_order,
  _with_linear_symmetries], [_2nd_order, _reducible, _mu_x_y1],
   [_2nd_order, _reducible, _mu_y_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{(2y+x)\,y''+2{y'}^2+2y'=2}$$

14.11.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int ((2y+x)\,y''+(2y'+2)\,y')\,dx=\int 2dx$$
$$y+(2y+x)\,y'=2x+c_1$$

Which is now solved for y. Writing the ode as

$$y'=-\frac{y-c_1-2x}{x+2y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{(y - c_1 - 2x)(b_3 - a_2)}{x + 2y} - \frac{(y - c_1 - 2x)^2 a_3}{(x + 2y)^2} \\ &- \left(\frac{2}{x + 2y} + \frac{y - c_1 - 2x}{(x + 2y)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{1}{x + 2y} + \frac{2y - 2c_1 - 4x}{(x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{c_1^2 a_3 + 4c_1 x a_3 - 2c_1 x b_2 - c_1 x b_3 + 2c_1 y a_2 - 3c_1 y a_3 - 4c_1 y b_3 + 2x^2 a_2 + 4x^2 a_3 - 6x^2 b_2 - 2x^2 b_3 + 8xy a_2 -}{(x + 2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-c_1^2 a_3 - 4c_1 x a_3 + 2c_1 x b_2 + c_1 x b_3 - 2c_1 y a_2 + 3c_1 y a_3 + 4c_1 y b_3 \\ &- 2x^2 a_2 - 4x^2 a_3 + 6x^2 b_2 + 2x^2 b_3 - 8xy a_2 + 4xy a_3 + 4xy b_2 + 8xy b_3 \\ &+ 2y^2 a_2 - 6y^2 a_3 + 4y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 + 5x b_1 - 5y a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-c_1^2 a_3 - 2c_1 a_2 v_2 - 4c_1 a_3 v_1 + 3c_1 a_3 v_2 + 2c_1 b_2 v_1 + c_1 b_3 v_1 + 4c_1 b_3 v_2 \\ &- 2a_2 v_1^2 - 8a_2 v_1 v_2 + 2a_2 v_2^2 - 4a_3 v_1^2 + 4a_3 v_1 v_2 - 6a_3 v_2^2 + 6b_2 v_1^2 + 4b_2 v_1 v_2 \\ &+ 4b_2 v_2^2 + 2b_3 v_1^2 + 8b_3 v_1 v_2 - 2b_3 v_2^2 + c_1 a_1 + 2c_1 b_1 - 5a_1 v_2 + 5b_1 v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 - 4a_3 + 6b_2 + 2b_3) v_1^2 + (-8a_2 + 4a_3 + 4b_2 + 8b_3) v_1 v_2 \\ &+ (-4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1) v_1 + (2a_2 - 6a_3 + 4b_2 - 2b_3) v_2^2 \\ &+ (-2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1) v_2 - c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_2 + 4a_3 + 4b_2 + 8b_3 &= 0 \\ -2a_2 - 4a_3 + 6b_2 + 2b_3 &= 0 \\ 2a_2 - 6a_3 + 4b_2 - 2b_3 &= 0 \\ -c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 &= 0 \\ -2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1 &= 0 \\ -4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= \frac{1}{5} c_1 b_2 + \frac{2}{5} c_1 b_3 \\ a_2 &= b_2 + b_3 \\ a_3 &= b_2 \\ b_1 &= \frac{2}{5} c_1 b_2 - \frac{1}{5} c_1 b_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x + \frac{2c_1}{5}$$
$$\eta = y - \frac{c_1}{5}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \frac{c_1}{5} - \left(-\frac{y - c_1 - 2x}{x + 2y}\right)\left(x + \frac{2c_1}{5}\right) \\
&= \frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{5x + 10y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2c_1^2 - 10c_1x - 10x^2 + 10xy + 10y^2}{5x + 10y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-c_1^2 - 5c_1x - 5x^2 + 5xy + 5y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y - c_1 - 2x}{x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-5y + 5c_1 + 10x}{10x^2 + (-10y + 10c_1)\,x - 10y^2 + 2c_1^2} \\
S_y &= \frac{-5x - 10y}{10x^2 + (-10y + 10c_1)\,x - 10y^2 + 2c_1^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln\left(-5x^2 + (5y - 5c_1)\,x + 5y^2 - c_1^2\right)}{2} = c_1$$

Which simplifies to

$$
\frac{\ln \left(-5x^{2}+\left(5y-5c_{1}\right)x+5y^{2}-c_{1}^{2}\right)}{2}=c_{1}
$$

Summary

The solution(s) found are the following

$$
\frac{\ln \left(-5x^{2}+\left(5y-5c_{1}\right)x+5y^{2}-c_{1}^{2}\right)}{2}=c_{1} \tag{1}
$$

Verification of solutions

$$
\frac{\ln \left(-5x^{2}+\left(5y-5c_{1}\right)x+5y^{2}-c_{1}^{2}\right)}{2}=c_{1}
$$

Verified OK.

14.11.2 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
\left(2y+x\right)y''+\left(2y'+2\right)y'=2
$$

Integrating both sides of the ODE w.r.t x gives

$$
\int \left(\left(2y+x\right)y''+\left(2y'+2\right)y'\right)dx=\int 2dx
$$

$$
y+\left(2y+x\right)y'=2x+c_{1}
$$

Which is now solved for y. Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y-c_{1}-2x}{x+2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_{x}+\omega(\eta_{y}-\xi_{x})-\omega^{2}\xi_{y}-\omega_{x}\xi-\omega_{y}\eta=0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_{2}+ya_{3}+a_{1} \tag{1E}
$$

$$
\eta = xb_{2}+yb_{3}+b_{1} \tag{2E}
$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 - \frac{(y - c_1 - 2x)(b_3 - a_2)}{x + 2y} - \frac{(y - c_1 - 2x)^2 a_3}{(x + 2y)^2} \\
&- \left(\frac{2}{x + 2y} + \frac{y - c_1 - 2x}{(x + 2y)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{1}{x + 2y} + \frac{2y - 2c_1 - 4x}{(x + 2y)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{c_1^2 a_3 + 4c_1 x a_3 - 2c_1 x b_2 - c_1 x b_3 + 2c_1 y a_2 - 3c_1 y a_3 - 4c_1 y b_3 + 2x^2 a_2 + 4x^2 a_3 - 6x^2 b_2 - 2x^2 b_3 + 8xya_2 -}{(x + 2y)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-c_1^2 a_3 - 4c_1 x a_3 + 2c_1 x b_2 + c_1 x b_3 - 2c_1 y a_2 + 3c_1 y a_3 + 4c_1 y b_3 \\
&- 2x^2 a_2 - 4x^2 a_3 + 6x^2 b_2 + 2x^2 b_3 - 8xya_2 + 4xya_3 + 4xyb_2 + 8xyb_3 \\
&+ 2y^2 a_2 - 6y^2 a_3 + 4y^2 b_2 - 2y^2 b_3 + c_1 a_1 + 2c_1 b_1 + 5xb_1 - 5ya_1 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-c_1^2 a_3 - 2c_1 a_2 v_2 - 4c_1 a_3 v_1 + 3c_1 a_3 v_2 + 2c_1 b_2 v_1 + c_1 b_3 v_1 + 4c_1 b_3 v_2 \\
&- 2a_2 v_1^2 - 8a_2 v_1 v_2 + 2a_2 v_2^2 - 4a_3 v_1^2 + 4a_3 v_1 v_2 - 6a_3 v_2^2 + 6b_2 v_1^2 + 4b_2 v_1 v_2 \\
&+ 4b_2 v_2^2 + 2b_3 v_1^2 + 8b_3 v_1 v_2 - 2b_3 v_2^2 + c_1 a_1 + 2c_1 b_1 - 5a_1 v_2 + 5b_1 v_1 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-2a_2 - 4a_3 + 6b_2 + 2b_3)\, v_1^2 + (-8a_2 + 4a_3 + 4b_2 + 8b_3)\, v_1 v_2 \\
&+ (-4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1)\, v_1 + (2a_2 - 6a_3 + 4b_2 - 2b_3)\, v_2^2 \\
&+ (-2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1)\, v_2 - c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-8a_2 + 4a_3 + 4b_2 + 8b_3 &= 0 \\
-2a_2 - 4a_3 + 6b_2 + 2b_3 &= 0 \\
2a_2 - 6a_3 + 4b_2 - 2b_3 &= 0 \\
-c_1^2 a_3 + c_1 a_1 + 2c_1 b_1 &= 0 \\
-2c_1 a_2 + 3c_1 a_3 + 4c_1 b_3 - 5a_1 &= 0 \\
-4c_1 a_3 + 2c_1 b_2 + c_1 b_3 + 5b_1 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= \frac{1}{5}c_1 b_2 + \frac{2}{5}c_1 b_3 \\
a_2 &= b_2 + b_3 \\
a_3 &= b_2 \\
b_1 &= \frac{2}{5}c_1 b_2 - \frac{1}{5}c_1 b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x + \frac{2c_1}{5} \\
\eta &= y - \frac{c_1}{5}
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \frac{c_1}{5} - \left(-\frac{y - c_1 - 2x}{x + 2y}\right)\left(x + \frac{2c_1}{5}\right) \\
&= \frac{-2c_1^2 - 10c_1 x - 10x^2 + 10xy + 10y^2}{5x + 10y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-2c_1^2 - 10c_1 x - 10x^2 + 10xy + 10y^2}{5x + 10y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\ln\left(-c_1^2 - 5c_1 x - 5x^2 + 5xy + 5y^2\right)}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y - c_1 - 2x}{x + 2y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-5y + 5c_1 + 10x}{10x^2 + (-10y + 10c_1)\, x - 10y^2 + 2c_1^2} \\
S_y &= \frac{-5x - 10y}{10x^2 + (-10y + 10c_1)\, x - 10y^2 + 2c_1^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln\left(-5x^2 + (5y - 5c_1)\, x + 5y^2 - c_1^2\right)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln\left(-5x^2 + (5y - 5c_1)\, x + 5y^2 - c_1^2\right)}{2} = c_1
$$

Summary

The solution(s) found are the following

$$
\frac{\ln\left(-5x^2 + (5y - 5c_1)\, x + 5y^2 - c_1^2\right)}{2} = c_1 \tag{1}
$$

Verification of solutions

$$
\frac{\ln\left(-5x^2 + (5y - 5c_1)\, x + 5y^2 - c_1^2\right)}{2} = c_1
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 49

```
dsolve((x+2*y(x))*diff(y(x),x$2)+2*diff(y(x),x)^2+2*diff(y(x),x)=2,y(x), singsol=all)
```

$$y(x) = -\frac{x}{2} - \frac{\sqrt{-4c_1x + 5x^2 + 4c_2}}{2}$$
$$y(x) = -\frac{x}{2} + \frac{\sqrt{-4c_1x + 5x^2 + 4c_2}}{2}$$

✓ Solution by Mathematica

Time used: 0.645 (sec). Leaf size: 77

```
DSolve[(x+2*y[x])*y''[x]+2*y'[x]^2+2*y'[x]==2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}x\left(1 + \sqrt{\frac{1}{x^2}}\sqrt{5x^2 + 4c_2x + 4c_1}\right)$$
$$y(x) \to \frac{1}{2}x\left(-1 + \sqrt{\frac{1}{x^2}}\sqrt{5x^2 + 4c_2x + 4c_1}\right)$$

14.12 problem 33

Internal problem ID [5440]
Internal file name [`OUTPUT/4931_Tuesday_February_06_2024_10_14_25_PM_21281142/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _nonlinear]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*(1+2*_b(_a)+3*_b(_a)^2)+6*_b(
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying a quadrature
   <- quadrature successful
   <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 1110

```
dsolve((1+2*y(x)+3*y(x)^2)*diff(y(x),x$3)+6*diff(y(x),x)*( diff(y(x),x$2)+diff(y(x),x)^2+3*y
```

$y(x)$

$$= \frac{\left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 2592c_1^3}\right.}{}$$

$$- \frac{}{3\left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 + 1080c_1^2x^4 - 432c_2x^5 + 25}\right.}$$

$$- \frac{1}{3}$$

$y(x) =$

$$- \frac{\left(1 + i\sqrt{3}\right)\left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 - 432c_2x^5 + (1080}\right.}{}$$

$y(x)$

$$= \frac{\left(i\sqrt{3} - 1\right)\left(224 + 36x^4 - 432c_1x^2 - 432c_1^2 - 864c_2x + 864c_3 + 12\sqrt{9x^8 - 216c_1x^6 - 432c_2x^5 + (1080c_1^2}\right.}{}$$

✓ Solution by Mathematica

Time used: 0.369 (sec). Leaf size: 523

```
DSolve[(1+2*y[x]+3*y[x]^2)*y'''[x]+6*y'[x]*( y''[x]+y'[x]^2+3*y[x]*y''[x] )==x,y[x],x,Includ
```

$$y(x) \to \frac{2^{2/3}\left(9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2\right)^{2/3} - 4}{12\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2}}}$$

$$y(x) \to \frac{1}{24}\left(i2^{2/3}\left(\sqrt{3}+i\right)\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56}\right.$$
$$\left. + \frac{16\sqrt[3]{2}(1 + i\sqrt{3})}{\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2}} - 8\right)$$

$$y(x) \to \frac{1}{24}\left(-2^{2/3}\left(1+i\sqrt{3}\right)\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 5}\right.$$
$$\left. + \frac{16\sqrt[3]{2}(1 - i\sqrt{3})}{\sqrt[3]{9x^4 + 108c_1x^2 + \sqrt{2048 + (9x^4 + 108c_1x^2 + 27c_3x + 56 + 216c_2)^2} + 27c_3x + 56 + 216c_2}} - 8\right)$$

14.13 problem 34

Internal problem ID [5441]
Internal file name [`OUTPUT/4932_Tuesday_February_06_2024_10_14_25_PM_15270978/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _nonlinear], [_3rd_order,
   _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (diff(diff(_b(_a), _a), _a))*_b(_a)^2*_a+2*_a*_b(_a)*(diff
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
   <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 104

```
dsolve(3*x*( y(x)^2* diff(y(x),x$3)+6*y(x)*diff(y(x),x)*diff(y(x),x$2)+2*diff(y(x),x)^3   )-
```

$$y(x) = \frac{\left(8\ln\left(x\right)x-8c_3x^3+\left(12c_1-4\right)x+8c_2\right)^{\frac{1}{3}}}{2}$$

$$y(x) = -\frac{\left(8\ln\left(x\right)x-8c_3x^3+\left(12c_1-4\right)x+8c_2\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$

$$y(x) = \frac{\left(8\ln\left(x\right)x-8c_3x^3+\left(12c_1-4\right)x+8c_2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.297 (sec). Leaf size: 121

```
DSolve[3*x*( y[x]^2* y'''[x]+6*y[x]*y'[x]*y''[x]+2*y'[x]^3   )-3*y[x]* (y[x]*y''[x]+2* y'[x]
```

$$y(x) \to -\sqrt[3]{-\frac{1}{6}}\sqrt[3]{6c_3x^3 + 6x\log(x) + (3 + 9c_2)x + 2c_1}$$

$$y(x) \to \sqrt[3]{c_3x^3 + x\log(x) + \frac{1}{2}(1 + 3c_2)x + \frac{c_1}{3}}$$

$$y(x) \to (-1)^{2/3}\sqrt[3]{c_3x^3 + x\log(x) + \frac{1}{2}(1 + 3c_2)x + \frac{c_1}{3}}$$

14.14 problem 35

Internal problem ID [5442]
Internal file name [`OUTPUT/4933_Tuesday_February_06_2024_10_14_25_PM_14447332/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[[_3rd_order, _exact, _nonlinear], [_3rd_order,
   _with_linear_symmetries]]
```

Unable to solve or complete the solution.

Unable to parse ODE.

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying 3rd order ODE linearizable_by_differentiation
differential order: 3; trying a linearization to 4th order
trying differential order: 3; missing variables
trying differential order: 3; exact nonlinear
-> Calling odsolve with the ODE`, (1/2)*_b(_a)^2+(diff(_b(_a), _a))^2-2*(diff(_b(_a), _a))*_
   Methods for second order ODEs:
   --- Trying classification methods ---
   trying 2nd order Liouville
   trying 2nd order WeierstrassP
   trying 2nd order JacobiSN
   differential order: 2; trying a linearization to 3rd order
   trying 2nd order ODE linearizable_by_differentiation
   trying 2nd order, 2 integrating factors of the form mu(x,y)
   trying a quadrature
   checking if the LODE has constant coefficients
   <- constant coefficients successful
   <- 2nd order, 2 integrating factors of the form mu(x,y) successful
<- differential order: 3; exact nonlinear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 51

```
dsolve(y(x)*diff(y(x),x$3)+3*diff(y(x),x)*diff(y(x),x$2)-2*y(x)*diff(y(x),x$2)-2*diff(y(x),x
```

$$y(x) = \sqrt{-2c_3 x\, \mathrm{e}^{x} + \mathrm{e}^{2x} + 2\, \mathrm{e}^{x} c_2 - 2c_1}$$
$$y(x) = -\sqrt{\mathrm{e}^{2x} + (-2c_3 x + 2c_2)\, \mathrm{e}^{x} - 2c_1}$$

✓ Solution by Mathematica

Time used: 0.387 (sec). Leaf size: 65

```
DSolve[y[x]*y'''[x]+3*y'[x]*y''[x]-2*y[x]*y''[x]-2*y'[x]^2+y[x]*y'[x]==Exp[2*x],y[x],x,Inclu
```

$$y(x) \to -\sqrt{e^{2x} + e^{x}(c_3 x + 2c_2) + 2c_1}$$
$$y(x) \to \sqrt{e^{2x} + e^{x}(c_3 x + 2c_2) + 2c_1}$$

14.15 problem 36

14.15.1 Solving as second order ode missing x ode 2276

Internal problem ID [5443]
Internal file name [`OUTPUT/4934_Tuesday_February_06_2024_10_14_26_PM_76002521/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 19. Linear equations with variable coefficients (Misc. types). Supplemetary problems. Page 132
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__ode__missing__x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{2(y+1)\,y''+2y'^2+y^2+2y=0}$$

14.15.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y'=p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$(2y+2)\,p(y)\left(\frac{d}{dy}p(y)\right)+2p(y)^2+y(y+2)=0$$

Which is now solved as first order ode for $p(y)$. Writing the ode as

$$
\begin{aligned}
\frac{d}{dy}p(y) &= -\frac{2p^2+y^2+2y}{2\,(y+1)\,p} \\
\frac{d}{dy}p(y) &= \omega(y,p)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_y + \omega(\eta_p - \xi_y) - \omega^2\xi_p - \omega_y\xi - \omega_p\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 256: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(y,p) &= 0 \\
\eta(y,p) &= \frac{1}{p\left(y+1\right)^{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(y,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dy}{\xi} = \frac{dp}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial p}\right) S(y,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{p(y+1)^{2}}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{\left(y+1\right)^{2} p^{2}}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_y + \omega(y,p) S_p}{R_y + \omega(y,p) R_p} \tag{2}
$$

Where in the above R_y, R_p, S_y, S_p are all partial derivatives and $\omega(y,p)$ is the right hand side of the original ode given by

$$
\omega(y,p) = -\frac{2p^{2}+y^{2}+2y}{2\left(y+1\right)p}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_y &= 1 \\
R_p &= 0 \\
S_y &= (y+1)\,p^2 \\
S_p &= (y+1)^2\,p
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2}y^3 - \frac{3}{2}y^2 - y \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for y, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2}R^3 - \frac{3}{2}R^2 - R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{8}R^4 - \frac{1}{2}R^3 - \frac{1}{2}R^2 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to y, p coordinates. This results in

$$
\frac{p(y)^2\,(y+1)^2}{2} = -\frac{1}{8}y^4 - \frac{1}{2}y^3 - \frac{1}{2}y^2 + c_1
$$

Which simplifies to

$$
\frac{p(y)^2\,(y+1)^2}{2} = -\frac{1}{8}y^4 - \frac{1}{2}y^3 - \frac{1}{2}y^2 + c_1
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
\frac{y'^2(y+1)^2}{2} = -\frac{y^4}{8} - \frac{y^3}{2} - \frac{y^2}{2} + c_1
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$
y' = \frac{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}}{2y+2} \tag{1}
$$

$$
y' = -\frac{\sqrt{-y^4 - 4y^3 - 4y^2 + 8c_1}}{2\,(y+1)} \tag{2}
$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{2y+2}{\sqrt{-y^4-4y^3-4y^2+8c_1}}dy = \int dx$$
$$-i\ln\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right) = x+c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{-i\ln\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right)} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right)^{-i} = c_3\mathrm{e}^{x}$$

Solving for y gives

$$y = \frac{i\left(2i(c_3\mathrm{e}^{x})^{i}-\sqrt{2i\left(c_3\mathrm{e}^{x}\right)^{i}\left(c_3\mathrm{e}^{x}\right)^{2i}-16i\left(c_3\mathrm{e}^{x}\right)^{i}c_1-4\left(c_3\mathrm{e}^{x}\right)^{2i}}\right)\left(c_3\mathrm{e}^{x}\right)^{-i}}{2}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{2(y+1)}{\sqrt{-y^4-4y^3-4y^2+8c_1}}dy = \int dx$$
$$i\ln\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right) = x+c_4$$

Raising both side to exponential gives

$$\mathrm{e}^{i\ln\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right)} = \mathrm{e}^{x+c_4}$$

Which simplifies to

$$\left(iy^2+2iy+\sqrt{-y^4-4y^3-4y^2+8c_1}\right)^{i} = c_5\mathrm{e}^{x}$$

Solving for y gives

$$y = \frac{i\left(2i(c_5\mathrm{e}^{x})^{-i}-\sqrt{2i\left(c_5\mathrm{e}^{x}\right)^{-i}\left(c_5\mathrm{e}^{x}\right)^{-2i}-16i\left(c_5\mathrm{e}^{x}\right)^{-i}c_1-4\left(c_5\mathrm{e}^{x}\right)^{-2i}}\right)\left(c_5\mathrm{e}^{x}\right)^{i}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{i\left(2i(c_3 \mathrm{e}^x)^i - \sqrt{2i\left(c_3 \mathrm{e}^x\right)^i \left(c_3 \mathrm{e}^x\right)^{2i} - 16i\left(c_3 \mathrm{e}^x\right)^i c_1 - 4\left(c_3 \mathrm{e}^x\right)^{2i}}\right)\left(c_3 \mathrm{e}^x\right)^{-i}}{2} \tag{1}$$

$$y = \frac{i\left(2i(c_5 \mathrm{e}^x)^{-i} - \sqrt{2i\left(c_5 \mathrm{e}^x\right)^{-i} \left(c_5 \mathrm{e}^x\right)^{-2i} - 16i\left(c_5 \mathrm{e}^x\right)^{-i} c_1 - 4\left(c_5 \mathrm{e}^x\right)^{-2i}}\right)\left(c_5 \mathrm{e}^x\right)^{i}}{2} \tag{2}$$

Verification of solutions

$$y = \frac{i\left(2i(c_3 \mathrm{e}^x)^i - \sqrt{2i\left(c_3 \mathrm{e}^x\right)^i \left(c_3 \mathrm{e}^x\right)^{2i} - 16i\left(c_3 \mathrm{e}^x\right)^i c_1 - 4\left(c_3 \mathrm{e}^x\right)^{2i}}\right)\left(c_3 \mathrm{e}^x\right)^{-i}}{2}$$

Verified OK.

$$y = \frac{i\left(2i(c_5 \mathrm{e}^x)^{-i} - \sqrt{2i\left(c_5 \mathrm{e}^x\right)^{-i} \left(c_5 \mathrm{e}^x\right)^{-2i} - 16i\left(c_5 \mathrm{e}^x\right)^{-i} c_1 - 4\left(c_5 \mathrm{e}^x\right)^{-2i}}\right)\left(c_5 \mathrm{e}^x\right)^{i}}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 41

```
dsolve(2*(y(x)+1)*diff(y(x),x$2)+2*diff(y(x),x)^2+y(x)^2+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = -1 - \sqrt{1 + 2\cos(x) c_2 - 2c_1 \sin(x)}$$
$$y(x) = -1 + \sqrt{1 + 2\cos(x) c_2 - 2c_1 \sin(x)}$$

✓ Solution by Mathematica

Time used: 24.469 (sec). Leaf size: 5629

```
DSolve[2*(y[x]+1)*y''[x]+2*y'[x]^2+y[x]^2+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

15 Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163

15.1 problem 10 2284
15.2 problem 11 2296
15.3 problem 12 2307
15.4 problem 13 2319
15.5 problem 17 2322

15.1 problem 10

15.1.1 Solution using Matrix exponential method 2284
15.1.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2286
15.1.3 Maple step by step solution 2291

Internal problem ID [5444]
Internal file name [`OUTPUT/4935_Tuesday_February_06_2024_10_14_27_PM_74109112/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = -x(t) + e^{2t} - e^{t}$$
$$y'(t) = -x(t) + y(t) + e^{2t}$$

15.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e^{2t} - e^{t} \\ e^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} c_1 \\ \left(-\frac{e^t}{2} + \frac{e^{-t}}{2}\right) c_1 + e^t c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} c_1 \\ \frac{e^{-t} c_1}{2} - \frac{e^t (c_1 - 2c_2)}{2} \end{bmatrix}
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At} \vec{G}(t)\, dt$$

But

$$\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} e^t & 0 \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & e^{-t} \end{bmatrix}
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \int \begin{bmatrix} e^t & 0 \\ \frac{e^t}{2} - \frac{e^{-t}}{2} & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} - e^t \\ e^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix} \begin{bmatrix} \frac{e^{3t}}{3} - \frac{e^{2t}}{2} \\ \frac{t}{2} + \frac{e^{3t}}{6} - \frac{e^{2t}}{4} + \frac{e^t}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}}{3} - \frac{e^t}{2} \\ \frac{2\, e^{2t}}{3} + \frac{(2t-1) e^t}{4} \end{bmatrix}
\end{aligned}$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$
$$= \begin{bmatrix} \mathrm{e}^{-t}c_1 + \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{-t}c_1}{2} + \frac{2\,\mathrm{e}^{2t}}{3} + \frac{(-1+2t-2c_1+4c_2)\mathrm{e}^{t}}{4} \end{bmatrix}$$

15.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \mathrm{e}^{2t} - \mathrm{e}^{t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 0 \\ -1 & 1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(-1-\lambda)(1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \implies \left[\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-t} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} \end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^t \\
&= \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} 2\,\mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t)\, dt
$$

But

$$
\Phi^{-1} = \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} & 0 \\ -\frac{\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \end{bmatrix}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 2\,\mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{t}}{2} & 0 \\ -\frac{\mathrm{e}^{-t}}{2} & \mathrm{e}^{-t} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{2t} - \mathrm{e}^{t} \\ \mathrm{e}^{2t} \end{bmatrix} dt \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{2t}(\mathrm{e}^{t}-1)}{2} \\ \frac{1}{2} + \frac{\mathrm{e}^{t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} 2\,\mathrm{e}^{-t} & 0 \\ \mathrm{e}^{-t} & \mathrm{e}^{t} \end{bmatrix} \begin{bmatrix} \frac{\mathrm{e}^{3t}}{6} - \frac{\mathrm{e}^{2t}}{4} \\ \frac{t}{2} + \frac{\mathrm{e}^{t}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{t}}{2} \\ \frac{2\,\mathrm{e}^{2t}}{3} + \frac{(2t-1)\mathrm{e}^{t}}{4} \end{bmatrix}
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 2c_1\mathrm{e}^{-t} \\ c_1\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2\mathrm{e}^{t} \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{t}}{2} \\ \frac{2\,\mathrm{e}^{2t}}{3} + \frac{(2t-1)\mathrm{e}^{t}}{4} \end{bmatrix}
\end{aligned}
$$

Which becomes

$$
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2c_1\mathrm{e}^{-t} + \frac{\mathrm{e}^{2t}}{3} - \frac{\mathrm{e}^{t}}{2} \\ c_1\mathrm{e}^{-t} + \frac{2\,\mathrm{e}^{2t}}{3} + \frac{(-1+2t+4c_2)\mathrm{e}^{t}}{4} \end{bmatrix}
$$

15.1.3 Maple step by step solution

Let's solve

$\left[x'(t) = -x(t) + (\mathrm{e}^{t})^2 - \mathrm{e}^{t}, y'(t) = -x(t) + y(t) + (\mathrm{e}^{t})^2\right]$

- Define vector

$\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} (e^t)^2 - e^t \\ (e^t)^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = e^{-t} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = e^t \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p(t)$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \begin{bmatrix} 2\,e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \begin{bmatrix} 2\,e^{-t} & 0 \\ e^{-t} & e^t \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^t}{2} + \frac{e^{-t}}{2} & e^t \end{bmatrix}$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

- Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

- Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

$\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t)=\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t\frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $\vec{x}_p(t)=\left[\begin{array}{c}\frac{e^{2t}}{3}-\frac{e^t}{2}+\frac{e^{-t}}{6}\\ \frac{e^{-t}}{12}+\frac{2\,e^{2t}}{3}+\frac{(-3+2t)e^t}{4}\end{array}\right]$

- Plug particular solution back into general solution

  $\vec{x}(t)=c_1\vec{x}_1+c_2\vec{x}_2+\left[\begin{array}{c}\frac{e^{2t}}{3}-\frac{e^t}{2}+\frac{e^{-t}}{6}\\ \frac{e^{-t}}{12}+\frac{2\,e^{2t}}{3}+\frac{(-3+2t)e^t}{4}\end{array}\right]$

- Substitute in vector of dependent variables

  $\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right]=\left[\begin{array}{c}2\,e^{-t}c_1+\frac{e^{2t}}{3}-\frac{e^t}{2}+\frac{e^{-t}}{6}\\ \frac{(12c_1+1)e^{-t}}{12}+\frac{2\,e^{2t}}{3}+\frac{(-3+2t+4c_2)e^t}{4}\end{array}\right]$

- Solution to the system of ODEs

  $\left\{x(t)=2\,e^{-t}c_1+\frac{e^{2t}}{3}-\frac{e^t}{2}+\frac{e^{-t}}{6},y(t)=\frac{(12c_1+1)e^{-t}}{12}+\frac{2\,e^{2t}}{3}+\frac{(-3+2t+4c_2)e^t}{4}\right\}$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 48

```
dsolve([diff(x(t),t)-diff(y(t),t)+y(t)=-exp(t),x(t)+diff(y(t),t)-y(t)=exp(2*t)],singsol=all)
```

$$x(t)=-\frac{e^t}{2}+\frac{e^{2t}}{3}+c_2e^{-t}$$
$$y(t)=\frac{c_2e^{-t}}{2}+\frac{2\,e^{2t}}{3}+c_1e^t+\frac{e^tt}{2}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 72

```
DSolve[{x'[t]-y'[t]+y[t]==-Exp[t],x[t]+y'[t]-y[t]==Exp[2*t]},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to \frac{1}{6}e^t\left(2e^t - 3\right) + c_1e^{-t}$$

$$y(t) \to \frac{2e^{2t}}{3} + \frac{c_1e^{-t}}{2} + \frac{1}{4}e^t(2t - 1 - 2c_1 + 4c_2)$$

15.2 problem 11

15.2.1 Solution using Matrix exponential method 2296
15.2.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2298
15.2.3 Maple step by step solution 2303

Internal problem ID [5445]
Internal file name [`OUTPUT/4936_Tuesday_February_06_2024_10_14_28_PM_79830252/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 3x(t) - t^2 + 2y(t) + t$$
$$y'(t) = t^2 - 5x(t) - 3y(t)$$

15.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} (\cos(t) + 3\sin(t))\, c_1 + 2\sin(t)\, c_2 \\ -5\sin(t)\, c_1 + (\cos(t) - 3\sin(t))\, c_2 \end{bmatrix} \\ &= \begin{bmatrix} (3c_1 + 2c_2)\sin(t) + c_1\cos(t) \\ (-5c_1 - 3c_2)\sin(t) + c_2\cos(t) \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} \cos(t) - 3\sin(t) & -2\sin(t) \\ 5\sin(t) & \cos(t) + 3\sin(t) \end{bmatrix} \end{aligned}$$

Hence

$$\begin{aligned} \vec{x}_p(t) &= \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix} \int \begin{bmatrix} \cos(t) - 3\sin(t) & -2\sin(t) \\ 5\sin(t) & \cos(t) + 3\sin(t) \end{bmatrix} \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix} \begin{bmatrix} (-t^2 + t + 3)\cos(t) - \sin(t)(t^2 - 3t + 1) \\ (2t^2 - 3t - 4)\cos(t) + \sin(t)(t - 1)(t - 3) \end{bmatrix} \\ &= \begin{bmatrix} -t^2 + t + 3 \\ 2t^2 - 3t - 4 \end{bmatrix} \end{aligned}$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} (3c_1 + 2c_2)\sin(t) + c_1\cos(t) - t^2 + t + 3 \\ (-5c_1 - 3c_2)\sin(t) + c_2\cos(t) + 2t^2 - 3t - 4 \end{bmatrix}
\end{aligned}
$$

15.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & 2 \\ -5 & -3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 + 1 = 0
$$

The roots of the above are the eigenvalues.

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-i$	1	complex eigenvalue
i	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (-i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3+i & 2 \\ -5 & -3+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3+i & 2 & 0 \\ -5 & -3+i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cc|c} 3+i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3+i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{5} + \frac{i}{5}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5}+\frac{i}{5}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{3}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$$

Let $t=1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}+\frac{i}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{3}{5}+\frac{I}{5}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -3+i \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} - (i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3-i & 2 \\ -5 & -3-i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3-i & 2 & 0 \\ -5 & -3-i & 0 \end{array}\right]$$

$$R_2 = R_2 + \left(\frac{3}{2}+\frac{i}{2}\right)R_1 \Longrightarrow \left[\begin{array}{cc|c} 3-i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3-i & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{3}{5} - \frac{i}{5}\right)t\right\}$

Hence the solution is

$$\left[\begin{array}{c} \left(-\frac{3}{5} - \frac{I}{5}\right)t \\ t \end{array}\right] = \left[\begin{array}{c} \left(-\frac{3}{5} - \frac{i}{5}\right)t \\ t \end{array}\right]$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\left[\begin{array}{c} \left(-\frac{3}{5} - \frac{I}{5}\right)t \\ t \end{array}\right] = t\left[\begin{array}{c} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{array}\right]$$

Let $t = 1$ the eigenvector becomes

$$\left[\begin{array}{c} \left(-\frac{3}{5} - \frac{I}{5}\right)t \\ t \end{array}\right] = \left[\begin{array}{c} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{array}\right]$$

Which is normalized to

$$\left[\begin{array}{c} \left(-\frac{3}{5} - \frac{I}{5}\right)t \\ t \end{array}\right] = \left[\begin{array}{c} -3 - i \\ 5 \end{array}\right]$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
i	1	1	No	$\left[\begin{array}{c} -\frac{3}{5} - \frac{i}{5} \\ 1 \end{array}\right]$
$-i$	1	1	No	$\left[\begin{array}{c} -\frac{3}{5} + \frac{i}{5} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} \\ e^{it} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{-it} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{5i e^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5i e^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} \frac{5i e^{-it}}{2} & \left(\frac{1}{2} + \frac{3i}{2}\right) e^{-it} \\ -\frac{5i e^{it}}{2} & \left(\frac{1}{2} - \frac{3i}{2}\right) e^{it} \end{bmatrix} \begin{bmatrix} -t^2 + t \\ t^2 \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \int \begin{bmatrix} -t\left(-\frac{5i}{2} + \left(-\frac{1}{2} + i\right) t\right) e^{-it} \\ \frac{e^{it} t(2it+t-5i)}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) e^{it} & \left(-\frac{3}{5} + \frac{i}{5}\right) e^{-it} \\ e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} -\frac{(-5+(2+i)t)(-t^2+(2+i)t+1-2i) e^{-it}}{2t-4+2i} \\ \frac{(5+(-2+i)t)(t^2+(-2+i)t-1-2i) e^{it}}{-2t+4+2i} \end{bmatrix} \\
&= \begin{bmatrix} \frac{t^4-5t^3+6t^2+7t-15}{(-t+2+i)(t-2+i)} \\ \frac{-2t^4+11t^3-18t^2-t+20}{(-t+2+i)(t-2+i)} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \left(-\frac{3}{5} - \frac{i}{5}\right) c_1 \mathrm{e}^{it} \\ c_1 \mathrm{e}^{it} \end{bmatrix} + \begin{bmatrix} \left(-\frac{3}{5} + \frac{i}{5}\right) c_2 \mathrm{e}^{-it} \\ c_2 \mathrm{e}^{-it} \end{bmatrix} + \begin{bmatrix} \frac{t^4-5t^3+6t^2+7t-15}{(-t+2+i)(t-2+i)} \\ \frac{-2t^4+11t^3-18t^2-t+20}{(-t+2+i)(t-2+i)} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{((3-i)c_2 \mathrm{e}^{-it}+(3+i)c_1 \mathrm{e}^{it}+5t^2-5t-15)(t^2-4t+5)}{5(-t+2+i)(t-2+i)} \\ -\frac{(c_2 \mathrm{e}^{-it}+c_1 \mathrm{e}^{it}+2t^2-3t-4)(t^2-4t+5)}{(-t+2+i)(t-2+i)} \end{bmatrix}$$

15.2.3 Maple step by step solution

Let's solve

$[x'(t) = 3x(t) - t^2 + 2y(t) + t, y'(t) = t^2 - 5x(t) - 3y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2+t \\ t^2 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} -t^2+t \\ t^2 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} -t^2+t \\ t^2 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $$\left[\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right], \left[\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}-\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]\right]$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $$\left[-\mathrm{I}, \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]\right]$$
- Solution from eigenpair

  $$\mathrm{e}^{-\mathrm{I}t} \cdot \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$
- Use Euler identity to write solution in terms of $\sin$ and $\cos$

  $$(\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\frac{3}{5}+\frac{\mathrm{I}}{5} \\ 1 \end{array}\right]$$
- Simplify expression

  $$\left[\begin{array}{c} \left(-\frac{3}{5}+\frac{\mathrm{I}}{5}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$
- Both real and imaginary parts are solutions to the homogeneous system

  $$\left[\overrightarrow{x}_1(t) = \left[\begin{array}{c} -\frac{3\cos(t)}{5}+\frac{\sin(t)}{5} \\ \cos(t) \end{array}\right], \overrightarrow{x}_2(t) = \left[\begin{array}{c} \frac{3\sin(t)}{5}+\frac{\cos(t)}{5} \\ -\sin(t) \end{array}\right]\right]$$
- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

  $\overrightarrow{x}(t) = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t) + \overrightarrow{x}_p(t)$

□ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

  $$\phi(t) = \left[\begin{array}{cc} -\frac{3\cos(t)}{5}+\frac{\sin(t)}{5} & \frac{3\sin(t)}{5}+\frac{\cos(t)}{5} \\ \cos(t) & -\sin(t) \end{array}\right]$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

  $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$
- Substitute the value of $\phi(t)$ and $\phi(0)$

  $\Phi(t) = \begin{bmatrix} -\frac{3\cos(t)}{5} + \frac{\sin(t)}{5} & \frac{3\sin(t)}{5} + \frac{\cos(t)}{5} \\ \cos(t) & -\sin(t) \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}}$
- Evaluate and simplify to get the fundamental matrix

  $\Phi(t) = \begin{bmatrix} \cos(t) + 3\sin(t) & 2\sin(t) \\ -5\sin(t) & \cos(t) - 3\sin(t) \end{bmatrix}$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

  $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$
- Take the derivative of the particular solution

  $\vec{x}'_p(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \begin{bmatrix} -t^2 - 3\cos(t) - \sin(t) + t + 3 \\ 2t^2 + 4\cos(t) + 3\sin(t) - 3t - 4 \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \begin{bmatrix} -t^2 - 3\cos(t) - \sin(t) + t + 3 \\ 2t^2 + 4\cos(t) + 3\sin(t) - 3t - 4 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{(-3c_1+c_2-15)\cos(t)}{5} + \frac{(c_1+3c_2-5)\sin(t)}{5} - t^2 + t + 3 \\ (4 + c_1)\cos(t) + (-c_2 + 3)\sin(t) + 2t^2 - 3t - 4 \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \frac{(-3c_1+c_2-15)\cos(t)}{5} + \frac{(c_1+3c_2-5)\sin(t)}{5} - t^2 + t + 3, y(t) = (4 + c_1)\cos(t) + (-c_2 + 3)\sin(t) -\right.$$

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
dsolve([diff(x(t),t)+2*x(t)+diff(y(t),t)+y(t)=t,5*x(t)+diff(y(t),t)+3*y(t)=t^2],singsol=all)
```

$$x(t) = c_2 \sin(t) + c_1 \cos(t) - t^2 + t + 3$$
$$y(t) = 2t^2 - \frac{3c_2 \sin(t)}{2} - \frac{3c_1 \cos(t)}{2} - 3t - 4 + \frac{c_2 \cos(t)}{2} - \frac{c_1 \sin(t)}{2}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 61

```
DSolve[{x'[t]+2*x[t]+y'[t]+y[t]==t,5*x[t]+y'[t]+3*y[t]==t^2},{x[t],y[t]},t,IncludeSingularSo
```

$$x(t) \to -t^2 + t + c_1 \cos(t) + (3c_1 + 2c_2)\sin(t) + 3$$
$$y(t) \to 2t^2 - 3t + c_2 \cos(t) - (5c_1 + 3c_2)\sin(t) - 4$$

15.3 problem 12

15.3.1 Solution using Matrix exponential method 2307
15.3.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2309
15.3.3 Maple step by step solution 2314

Internal problem ID [5446]
Internal file name [`OUTPUT/4937_Tuesday_February_06_2024_10_14_29_PM_15147259/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t) &= 4 - 5x(t) - y(t) - e^t \\
y'(t) &= 2x(t) - 3y(t) + e^t - 1
\end{aligned}
$$

15.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4 - e^t \\ e^t - 1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation

of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} \mathrm{e}^{-4t}\cos(t) - \mathrm{e}^{-4t}\sin(t) & -\mathrm{e}^{-4t}\sin(t) \\ 2\,\mathrm{e}^{-4t}\sin(t) & \mathrm{e}^{-4t}\cos(t) + \mathrm{e}^{-4t}\sin(t) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(\cos(t) - \sin(t)) & -\mathrm{e}^{-4t}\sin(t) \\ 2\,\mathrm{e}^{-4t}\sin(t) & \mathrm{e}^{-4t}(\sin(t) + \cos(t)) \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(\cos(t) - \sin(t)) & -\mathrm{e}^{-4t}\sin(t) \\ 2\,\mathrm{e}^{-4t}\sin(t) & \mathrm{e}^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(\cos(t) - \sin(t))\,c_1 - \mathrm{e}^{-4t}\sin(t)\,c_2 \\ 2\,\mathrm{e}^{-4t}\sin(t)\,c_1 + \mathrm{e}^{-4t}(\sin(t) + \cos(t))\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} ((-c_2 - c_1)\sin(t) + c_1\cos(t))\,\mathrm{e}^{-4t} \\ ((2c_1 + c_2)\sin(t) + c_2\cos(t))\,\mathrm{e}^{-4t} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1} \\
&= \begin{bmatrix} \mathrm{e}^{4t}(\sin(t) + \cos(t)) & \mathrm{e}^{4t}\sin(t) \\ -2\,\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}(\cos(t) - \sin(t)) \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \mathrm{e}^{-4t}(\cos(t) - \sin(t)) & -\mathrm{e}^{-4t}\sin(t) \\ 2\,\mathrm{e}^{-4t}\sin(t) & \mathrm{e}^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \int \begin{bmatrix} \mathrm{e}^{4t}(\sin(t) + \cos(t)) & \mathrm{e}^{4t}\sin(t) \\ -2\,\mathrm{e}^{4t}\sin(t) & \mathrm{e}^{4t}(\cos(t) - \sin(t)) \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{-4t}(\cos(t) - \sin(t)) & -\mathrm{e}^{-4t}\sin(t) \\ 2\,\mathrm{e}^{-4t}\sin(t) & \mathrm{e}^{-4t}(\sin(t) + \cos(t)) \end{bmatrix} \begin{bmatrix} \frac{(13\cos(t)+16\sin(t))\mathrm{e}^{4t}}{17} - \frac{5\left(\cos(t)+\frac{\sin(t)}{5}\right)\mathrm{e}^{5t}}{26} \\ \frac{(3\cos(t)-29\sin(t))\mathrm{e}^{4t}}{17} + \frac{2\left(\cos(t)+\frac{3\sin(t)}{2}\right)\mathrm{e}^{5t}}{13} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{5\,\mathrm{e}^{t}}{26} + \frac{13}{17} \\ \frac{2\,\mathrm{e}^{t}}{13} + \frac{3}{17} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
&= \begin{bmatrix} \left(\frac{13\,\mathrm{e}^{4t}}{17} - \frac{5\,\mathrm{e}^{5t}}{26} + (-c_2 - c_1)\sin(t) + c_1\cos(t)\right)\mathrm{e}^{-4t} \\ \frac{\left(39\,\mathrm{e}^{4t} + 34\,\mathrm{e}^{5t} + 221(2c_1 + c_2)\sin(t) + 221 c_2 \cos(t)\right)\mathrm{e}^{-4t}}{221} \end{bmatrix}
\end{aligned}
$$

15.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)
$$

Or

$$
\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 4 - \mathrm{e}^{t} \\ \mathrm{e}^{t} - 1 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det(A - \lambda I) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} -5 - \lambda & -1 \\ 2 & -3 - \lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 + 8\lambda + 17 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -4 + i$$
$$\lambda_2 = -4 - i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-4 - i$	1	complex eigenvalue
$-4 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -4 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - (-4 - i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1+i & -1 \\ 2 & 1+i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1+i & -1 & 0 \\ 2 & 1+i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1+i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1+i & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} - \frac{i}{2}\right)t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} - \frac{I}{2}\right)t \\ t \end{bmatrix} = \begin{bmatrix} -1 - i \\ 2 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -4 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -5 & -1 \\ 2 & -3 \end{bmatrix} - (-4 + i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 - i & -1 \\ 2 & 1 - i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -1 - i & -1 & 0 \\ 2 & 1 - i & 0 \end{array}\right]$$

$$R_2 = R_2 + (1 - i)\,R_1 \Longrightarrow \left[\begin{array}{cc|c} -1 - i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \left(-\frac{1}{2} + \frac{i}{2}\right) t\right\}$

Hence the solution is

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} \left(-\frac{1}{2} + \frac{i}{2}\right) t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \left(-\frac{1}{2} + \frac{I}{2}\right) t \\ t \end{bmatrix} = \begin{bmatrix} -1+i \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$-4+i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}+\frac{i}{2} \\ 1 \end{bmatrix}$
$-4-i$	1	1	No	$\begin{bmatrix} -\frac{1}{2}-\frac{i}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-4+i)t} \\ \mathrm{e}^{(-4+i)t} \end{bmatrix} + c_2 \begin{bmatrix} \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-4-i)t} \\ \mathrm{e}^{(-4-i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-4+i)t} & \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-4-i)t} \\ \mathrm{e}^{(-4+i)t} & \mathrm{e}^{(-4-i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -i\mathrm{e}^{(4-i)t} & \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(4-i)t} \\ i\mathrm{e}^{(4+i)t} & \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(4+i)t} \end{bmatrix}$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-4+i) t} & \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-4-i) t} \\ \mathrm{e}^{(-4+i) t} & \mathrm{e}^{(-4-i) t} \end{array}\right] \int \left[\begin{array}{cc} -i \mathrm{e}^{(4-i) t} & \left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(4-i) t} \\ i \mathrm{e}^{(4+i) t} & \left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(4+i) t} \end{array}\right] \left[\begin{array}{c} 4-\mathrm{e}^{t} \\ \mathrm{e}^{t}-1 \end{array}\right] d t \\
&= \left[\begin{array}{cc} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-4+i) t} & \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-4-i) t} \\ \mathrm{e}^{(-4+i) t} & \mathrm{e}^{(-4-i) t} \end{array}\right] \int \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{7 i}{2}\right) \mathrm{e}^{(4-i) t}+\left(\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(5-i) t} \\ \left(-\frac{1}{2}+\frac{7 i}{2}\right) \mathrm{e}^{(4+i) t}+\left(\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(5+i) t} \end{array}\right] d t \\
&= \left[\begin{array}{cc} \left(-\frac{1}{2}+\frac{i}{2}\right) \mathrm{e}^{(-4+i) t} & \left(-\frac{1}{2}-\frac{i}{2}\right) \mathrm{e}^{(-4-i) t} \\ \mathrm{e}^{(-4+i) t} & \mathrm{e}^{(-4-i) t} \end{array}\right] \left[\begin{array}{c} \left(\frac{3}{34}-\frac{29 i}{34}\right) \mathrm{e}^{(4-i) t}+\left(\frac{1}{13}+\frac{3 i}{26}\right) \mathrm{e}^{(5-i) t} \\ \left(\frac{3}{34}+\frac{29 i}{34}\right) \mathrm{e}^{(4+i) t}+\left(\frac{1}{13}-\frac{3 i}{26}\right) \mathrm{e}^{(5+i) t} \end{array}\right] \\
&= \left[\begin{array}{c} -\frac{5 \mathrm{e}^{t}}{26}+\frac{13}{17} \\ \frac{2 \mathrm{e}^{t}}{13}+\frac{3}{17} \end{array}\right]
\end{aligned}
$$

Now that we found particular solution, the final solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] &= \left[\begin{array}{c} \left(-\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(-4+i) t} \\ c_1 \mathrm{e}^{(-4+i) t} \end{array}\right]+\left[\begin{array}{c} \left(-\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(-4-i) t} \\ c_2 \mathrm{e}^{(-4-i) t} \end{array}\right]+\left[\begin{array}{c} -\frac{5 \mathrm{e}^{t}}{26}+\frac{13}{17} \\ \frac{2 \mathrm{e}^{t}}{13}+\frac{3}{17} \end{array}\right]
\end{aligned}
$$

Which becomes

$$
\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]=\left[\begin{array}{c} \left(-\frac{1}{2}+\frac{i}{2}\right) c_1 \mathrm{e}^{(-4+i) t}+\left(-\frac{1}{2}-\frac{i}{2}\right) c_2 \mathrm{e}^{(-4-i) t}-\frac{5 \mathrm{e}^{t}}{26}+\frac{13}{17} \\ c_1 \mathrm{e}^{(-4+i) t}+c_2 \mathrm{e}^{(-4-i) t}+\frac{2 \mathrm{e}^{t}}{13}+\frac{3}{17} \end{array}\right]
$$

15.3.3 Maple step by step solution

Let's solve

$[x'(t) = 4-5x(t)-y(t)-\mathrm{e}^{t}, y'(t) = 2x(t)-3y(t)+\mathrm{e}^{t}-1]$

- Define vector

$$
\overrightarrow{x}(t)=\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]
$$

- Convert system into a vector equation

$$
\overrightarrow{x}'(t)=\left[\begin{array}{cc} -5 & -1 \\ 2 & -3 \end{array}\right] \cdot \overrightarrow{x}(t)+\left[\begin{array}{c} 4-\mathrm{e}^{t} \\ \mathrm{e}^{t}-1 \end{array}\right]
$$

- System to solve

$$\overrightarrow{x}'(t) = \left[\begin{array}{cc} -5 & -1 \\ 2 & -3 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 4-e^{t} \\ e^{t}-1 \end{array}\right]$$

- Define the forcing function

$$\overrightarrow{f}(t) = \left[\begin{array}{c} 4-e^{t} \\ e^{t}-1 \end{array}\right]$$

- Define the coefficient matrix

$$A = \left[\begin{array}{cc} -5 & -1 \\ 2 & -3 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-4-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right], \left[-4+\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-4-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-4-\mathrm{I})t} \cdot \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-4t} \cdot (\cos(t) - \mathrm{I}\sin(t)) \cdot \left[\begin{array}{c} -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{-4t} \cdot \left[\begin{array}{c} \left(-\frac{1}{2}-\frac{\mathrm{I}}{2}\right)(\cos(t) - \mathrm{I}\sin(t)) \\ \cos(t) - \mathrm{I}\sin(t) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = e^{-4t} \cdot \left[\begin{array}{c} -\frac{\cos(t)}{2} - \frac{\sin(t)}{2} \\ \cos(t) \end{array}\right], \vec{x}_2(t) = e^{-4t} \cdot \left[\begin{array}{c} \frac{\sin(t)}{2} - \frac{\cos(t)}{2} \\ -\sin(t) \end{array}\right]\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$

☐ Fundamental matrix

  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} e^{-4t}\left(-\frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right) & e^{-4t}\left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right) \\ e^{-4t}\cos(t) & -e^{-4t}\sin(t) \end{array}\right]$$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} e^{-4t}\left(-\frac{\cos(t)}{2} - \frac{\sin(t)}{2}\right) & e^{-4t}\left(\frac{\sin(t)}{2} - \frac{\cos(t)}{2}\right) \\ e^{-4t}\cos(t) & -e^{-4t}\sin(t) \end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{array}\right]}$$

  - Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} e^{-4t}(\cos(t) - \sin(t)) & -e^{-4t}\sin(t) \\ 2\,e^{-4t}\sin(t) & e^{-4t}(\sin(t) + \cos(t)) \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

$\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

$\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

$A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

- Cancel like terms

  $\Phi(t) \cdot \vec{v}\,'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}\,'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$

- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$

- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds\right)$

- Plug in the fundamental matrix and the forcing function and compute

  $$\vec{x}_p(t) = \left[\begin{array}{c} \frac{e^{-4t}\left(-85\,e^{5t}+338\,e^{4t}+399\sin(t)-253\cos(t)\right)}{442} \\ -\frac{e^{-4t}\left(-34\,e^{5t}-39\,e^{4t}+326\sin(t)+73\cos(t)\right)}{221} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \left[\begin{array}{c} \frac{e^{-4t}\left(-85\,e^{5t}+338\,e^{4t}+399\sin(t)-253\cos(t)\right)}{442} \\ -\frac{e^{-4t}\left(-34\,e^{5t}-39\,e^{4t}+326\sin(t)+73\cos(t)\right)}{221} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{\left(-\frac{26\,e^{4t}}{17}+\frac{5\,e^{5t}}{13}+\left(c_1+c_2+\frac{253}{221}\right)\cos(t)+\left(c_1-c_2-\frac{399}{221}\right)\sin(t)\right)e^{-4t}}{2} \\ \frac{\left(39\,e^{4t}+34\,e^{5t}+(221c_1-73)\cos(t)+(-221c_2-326)\sin(t)\right)e^{-4t}}{221} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = -\frac{\left(-\frac{26\,e^{4t}}{17}+\frac{5\,e^{5t}}{13}+\left(c_1+c_2+\frac{253}{221}\right)\cos(t)+\left(c_1-c_2-\frac{399}{221}\right)\sin(t)\right)e^{-4t}}{2}, y(t) = \frac{(39\,e^{4t}+34\,e^{5t}+(221c_1-73)\cos(t)+(-22}{221}\right.$$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 71

```
dsolve([diff(x(t),t)+x(t)+2*diff(y(t),t)+7*y(t)=exp(t)+2,-2*x(t)+diff(y(t),t)+3*y(t)=exp(t)-
```

$$x(t) = e^{-4t}\sin(t)\,c_2 + e^{-4t}\cos(t)\,c_1 + \frac{13}{17} - \frac{5\,e^{t}}{26}$$
$$y(t) = -e^{-4t}\sin(t)\,c_2 - e^{-4t}\cos(t)\,c_2 - e^{-4t}\cos(t)\,c_1 + e^{-4t}\sin(t)\,c_1 + \frac{2\,e^{t}}{13} + \frac{3}{17}$$

✓ Solution by Mathematica

Time used: 0.239 (sec). Leaf size: 79

```
DSolve[{x'[t]+x[t]+2*y'[t]+7*y[t]==Exp[t]+2,-2*x[t]+y'[t]+3*y[t]==Exp[t]-1},{x[t],y[t]},t,In
```

$$x(t) \to -\frac{5e^t}{26} + c_1 e^{-4t}\cos(t) - (c_1 + c_2)e^{-4t}\sin(t) + \frac{13}{17}$$
$$y(t) \to \frac{2e^t}{13} + c_2 e^{-4t}\cos(t) + (2c_1 + c_2)e^{-4t}\sin(t) + \frac{3}{17}$$

15.4 problem 13

Internal problem ID [5447]
Internal file name [`OUTPUT/4938_Tuesday_February_06_2024_10_14_29_PM_76667389/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x'(t)+y'(t) &= x(t)-3y(t)-1+\mathrm{e}^{-t} \\
x'(t)+y'(t) &= -2x(t)-3y(t)+\mathrm{e}^{2t}+1
\end{aligned}
$$

The system is

$$
x'(t)+y'(t) = x(t)-3y(t)-1+\mathrm{e}^{-t} \tag{1}
$$

$$
x'(t)+y'(t) = -2x(t)-3y(t)+\mathrm{e}^{2t}+1 \tag{2}
$$

Since the left side is the same, this implies

$$
x(t)-3y(t)-1+\mathrm{e}^{-t} = -2x(t)-3y(t)+\mathrm{e}^{2t}+1
$$

$$
x(t) = \frac{(\mathrm{e}^{3t}+2\,\mathrm{e}^{t}-1)\,\mathrm{e}^{-t}}{3} \tag{3}
$$

Taking derivative of the above w.r.t. t gives

$$
x'(t) = \frac{(3\,\mathrm{e}^{3t}+2\,\mathrm{e}^{t})\,\mathrm{e}^{-t}}{3} - \frac{(\mathrm{e}^{3t}+2\,\mathrm{e}^{t}-1)\,\mathrm{e}^{-t}}{3} \tag{4}
$$

Substituting (3,4) in (1) to eliminate $x(t)\,,x'(t)$ gives

$$
\frac{(3\,\mathrm{e}^{3t}+2\,\mathrm{e}^{t})\,\mathrm{e}^{-t}}{3} - \frac{(\mathrm{e}^{3t}+2\,\mathrm{e}^{t}-1)\,\mathrm{e}^{-t}}{3} + y'(t) = \frac{(\mathrm{e}^{3t}+2\,\mathrm{e}^{t}-1)\,\mathrm{e}^{-t}}{3} - 3y(t) - 1 + \mathrm{e}^{-t}
$$

$$
x'(t) = -\frac{(\mathrm{e}^{3t}+9\,\mathrm{e}^{t}y(t)+\mathrm{e}^{t}-1)\,\mathrm{e}^{-t}}{3} \tag{5}
$$

Which is now solved for $y(t)$.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'(t) + p(t)y(t) = q(t)$$

Where here

$$\begin{aligned} p(t) &= 3 \\ q(t) &= -\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3} \end{aligned}$$

Hence the ode is

$$y'(t) + 3y(t) = -\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 3dt} \\ &= e^{3t} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dt}(\mu y) &= (\mu)\left(-\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3}\right) \\ \frac{d}{dt}(e^{3t} y) &= (e^{3t})\left(-\frac{e^{2t}}{3} - \frac{1}{3} + \frac{e^{-t}}{3}\right) \\ d(e^{3t} y) &= \left(-\frac{e^{5t}}{3} - \frac{e^{3t}}{3} + \frac{e^{2t}}{3}\right) dt \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{3t} y &= \int -\frac{e^{5t}}{3} - \frac{e^{3t}}{3} + \frac{e^{2t}}{3} dt \\ e^{3t} y &= -\frac{e^{3t}}{9} - \frac{e^{5t}}{15} + \frac{e^{2t}}{6} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{3t}$ results in

$$y(t) = e^{-3t}\left(-\frac{e^{3t}}{9} - \frac{e^{5t}}{15} + \frac{e^{2t}}{6}\right) + c_1 e^{-3t}$$

which simplifies to

$$y(t) = \frac{(-6\, e^{5t} - 10\, e^{3t} + 15\, e^{2t} + 90 c_1)\, e^{-3t}}{90}$$

Given now that we have the solution

$$y(t) = \frac{(-6\,e^{5t} - 10\,e^{3t} + 15\,e^{2t} + 90c_1)\,e^{-3t}}{90} \tag{6}$$

Then substituting (6) into (3) gives

$$x(t) = \frac{(e^{3t} + 2\,e^{t} - 1)\,e^{-t}}{3} \tag{7}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 42

```
dsolve([diff(x(t),t)-x(t)+diff(y(t),t)+3*y(t)=exp(-t)-1,diff(x(t),t)+2*x(t)+diff(y(t),t)+3*y
```

$$x(t) = \frac{e^{2t}}{3} + \frac{2}{3} - \frac{e^{-t}}{3}$$
$$y(t) = -\frac{1}{9} - \frac{e^{2t}}{15} + \frac{e^{-t}}{6} + c_1 e^{-3t}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 62

```
DSolve[{x'[t]-x[t]+y'[t]+3*y[t]==Exp[-t]-1,x'[t]+2*x[t]+y'[t]+3*y[t]==Exp[2*t]+1},{x[t],y[t]
```

$$x(t) \to \frac{1}{3}e^{-t}\left(2e^{t} + e^{3t} - 1\right)$$
$$y(t) \to \frac{e^{-t}}{6} - \frac{e^{2t}}{15} + \frac{1}{16}c_1 e^{-3t} - \frac{1}{9}$$

15.5 problem 17

15.5.1 Solution using Matrix exponential method 2322
15.5.2 Solution using explicit Eigenvalue and Eigenvector method . . . 2324
15.5.3 Maple step by step solution 2333

Internal problem ID [5448]
Internal file name [`OUTPUT/4939_Wednesday_February_14_2024_02_05_46_AM_5854370/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 21. System of simultaneous linear equations. Supplemetary problems. Page 163
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$x'(t) = 1 + x(t) + \frac{e^t}{2}$$
$$y'(t) = -2y(t) + \frac{e^t}{2}$$
$$z'(t) = 2 - z(t) + \frac{e^t}{2}$$

15.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1 + \frac{e^t}{2} \\ \frac{e^t}{2} \\ 2 + \frac{e^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned} \vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} e^{t}c_1 \\ e^{-2t}c_2 \\ e^{-t}c_3 \end{bmatrix} \end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\,dt$$

But

$$\begin{aligned} e^{-At} &= (e^{At})^{-1} \\ &= \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{t} \end{bmatrix} \end{aligned}$$

Hence

$$\vec{x}_p(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \int \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1+\frac{e^t}{2} \\ \frac{e^t}{2} \\ 2+\frac{e^t}{2} \end{bmatrix} dt$$

$$= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{t}{2} - e^{-t} \\ \frac{e^{3t}}{6} \\ 2\,e^t + \frac{e^{2t}}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{e^t t}{2} - 1 \\ \frac{e^t}{6} \\ 2 + \frac{e^t}{4} \end{bmatrix}$$

Hence the complete solution is

$$\begin{aligned} \vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\ &= \begin{bmatrix} -1 + \frac{(t+2c_1)e^t}{2} \\ \frac{(e^{3t}+6c_2)e^{-2t}}{6} \\ e^{-t}c_3 + 2 + \frac{e^t}{4} \end{bmatrix} \end{aligned}$$

15.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 1+\frac{e^t}{2} \\ \frac{e^t}{2} \\ 2+\frac{e^t}{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}\right) = 0$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(-2-\lambda)(-1-\lambda) = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = -2$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
-2	1	real eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the current pivot $A(2, 3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1, v_3\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$
\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

Let $t = 1$ the eigenvector becomes

$$
\begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$
\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$
\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

Therefore the system in Echelon form is

$$
\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(1, 2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,3)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1\}$ and the leading variables are $\{v_2, v_3\}$. Let $v_1 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0, v_3 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	 geometric k	defective?	eigenvectors
-1	1	1	No	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
-2	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{-t} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}
\end{aligned}
$$

Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-2t} \\
&= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-2t}
\end{aligned}
$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ \mathrm{e}^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{-2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt
$$

But

$$\Phi^{-1} = \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{-2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \int \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1+\frac{\mathrm{e}^{t}}{2} \\ \frac{\mathrm{e}^{t}}{2} \\ 2+\frac{\mathrm{e}^{t}}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{-2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \int \begin{bmatrix} \frac{\mathrm{e}^{t}(4+\mathrm{e}^{t})}{2} \\ \frac{\mathrm{e}^{3t}}{2} \\ \mathrm{e}^{-t}+\frac{1}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} 0 & 0 & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{-2t} & 0 \\ \mathrm{e}^{-t} & 0 & 0 \end{bmatrix} \begin{bmatrix} 2\,\mathrm{e}^{t}+\frac{\mathrm{e}^{2t}}{4} \\ \frac{\mathrm{e}^{3t}}{6} \\ \frac{t}{2}-\mathrm{e}^{-t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\mathrm{e}^{t}t}{2}-1 \\ \frac{\mathrm{e}^{t}}{6} \\ 2+\frac{\mathrm{e}^{t}}{4} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c_1\mathrm{e}^{-t} \end{bmatrix} + \begin{bmatrix} 0 \\ c_2\mathrm{e}^{-2t} \\ 0 \end{bmatrix} + \begin{bmatrix} c_3\mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\mathrm{e}^{t}t}{2}-1 \\ \frac{\mathrm{e}^{t}}{6} \\ 2+\frac{\mathrm{e}^{t}}{4} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -1+\frac{(t+2c_3)\mathrm{e}^{t}}{2} \\ \frac{(\mathrm{e}^{3t}+6c_2)\mathrm{e}^{-2t}}{6} \\ c_1\mathrm{e}^{-t}+2+\frac{\mathrm{e}^{t}}{4} \end{bmatrix}$$

15.5.3 Maple step by step solution

Let's solve

$$\left[x'(t)=1+x(t)+\frac{e^{t}}{2}, y'(t)=-2y(t)+\frac{e^{t}}{2}, z'(t)=2-z(t)+\frac{e^{t}}{2}\right]$$

- Define vector

$$\vec{x}(t)=\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\cdot\vec{x}(t)+\begin{bmatrix} 1+\frac{e^{t}}{2} \\ \frac{e^{t}}{2} \\ 2+\frac{e^{t}}{2} \end{bmatrix}$$

- System to solve

$$\vec{x}'(t)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}\cdot\vec{x}(t)+\begin{bmatrix} 1+\frac{e^{t}}{2} \\ \frac{e^{t}}{2} \\ 2+\frac{e^{t}}{2} \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t)=\begin{bmatrix} 1+\frac{e^{t}}{2} \\ \frac{e^{t}}{2} \\ 2+\frac{e^{t}}{2} \end{bmatrix}$$

- Define the coefficient matrix

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t)=A\cdot\vec{x}(t)+\vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right], \left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right], \left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{-2t} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[-1, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{-t} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[1, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{t} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \vec{x}_p(t)$

- ☐ Fundamental matrix
  - Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

    $\phi(t) = \begin{bmatrix} 0 & 0 & e^{t} \\ e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \end{bmatrix}$

  - The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

    $\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

  - Substitute the value of $\phi(t)$ and $\phi(0)$

    $\Phi(t) = \begin{bmatrix} 0 & 0 & e^{t} \\ e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \end{bmatrix} \cdot \frac{1}{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}$

  - Evaluate and simplify to get the fundamental matrix

    $\Phi(t) = \begin{bmatrix} e^{t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$

  - Take the derivative of the particular solution

    $\vec{x}_p'(t) = \Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t)$

  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A \cdot \Phi(t) \cdot \vec{v}(t) + \Phi(t) \cdot \vec{v}'(t) = A \cdot \Phi(t) \cdot \vec{v}(t) + \vec{f}(t)$

  - Cancel like terms

    $\Phi(t) \cdot \vec{v}'(t) = \vec{f}(t)$

- Multiply by the inverse of the fundamental matrix

  $\vec{v}'(t) = \frac{1}{\Phi(t)} \cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$

  $\vec{v}(t) = \int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution

  $\vec{x}_p(t) = \Phi(t) \cdot \left(\int_0^t \frac{1}{\Phi(s)} \cdot \vec{f}(s)\, ds \right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t) = \left[\begin{array}{c} \frac{\ln(\mathrm{e}^t)\mathrm{e}^t}{2} + \mathrm{e}^t - 1 \\ \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{6} \\ \frac{\mathrm{e}^t}{4} + 2 - \frac{9\,\mathrm{e}^{-t}}{4} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + \left[\begin{array}{c} \frac{\ln(\mathrm{e}^t)\mathrm{e}^t}{2} + \mathrm{e}^t - 1 \\ \frac{(\mathrm{e}^{3t}-1)\mathrm{e}^{-2t}}{6} \\ \frac{\mathrm{e}^t}{4} + 2 - \frac{9\,\mathrm{e}^{-t}}{4} \end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right] = \left[\begin{array}{c} c_3\mathrm{e}^t + \frac{\ln(\mathrm{e}^t)\mathrm{e}^t}{2} + \mathrm{e}^t - 1 \\ \frac{(\mathrm{e}^{3t}+6c_1-1)\mathrm{e}^{-2t}}{6} \\ c_2\mathrm{e}^{-t} + \frac{\mathrm{e}^t}{4} + 2 - \frac{9\,\mathrm{e}^{-t}}{4} \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t) = c_3\mathrm{e}^t + \frac{\ln(\mathrm{e}^t)\mathrm{e}^t}{2} + \mathrm{e}^t - 1, y(t) = \frac{(\mathrm{e}^{3t}+6c_1-1)\mathrm{e}^{-2t}}{6}, z(t) = c_2\mathrm{e}^{-t} + \frac{\mathrm{e}^t}{4} + 2 - \frac{9\,\mathrm{e}^{-t}}{4}\right\}$$

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 45

```
dsolve([diff(x(t),t)-x(t)+diff(y(t),t)+2*y(t)=1+exp(t),diff(y(t),t)+2*y(t)+diff(z(t),t)+z(t)
```

$$x(t) = -1 + \frac{e^{t}(2c_3 + t)}{2}$$
$$y(t) = \frac{e^{t}}{6} + c_2 e^{-2t}$$
$$z(t) = 2 + \frac{e^{t}}{4} + e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 60

```
DSolve[{x'[t]-x[t]+y'[t]+2*y[t]==1+Exp[t],y'[t]+2*y[t]+z'[t]+z[t]==2+Exp[t],x'[t]-x[t]+z'[t]
```

$$x(t) \to -1 + e^{t}\left(\frac{t}{2} + c_1\right)$$
$$y(t) \to \frac{e^{t}}{6} + c_2 e^{-2t}$$
$$z(t) \to \frac{e^{t}}{4} + (4 + c_3)e^{-t} + 2$$

16 Chapter 25. Integration in series. Supplemetary problems. Page 205

16.1 problem 9 2339
16.2 problem 10A 2349
16.3 problem 10B 2360
16.4 problem 11 2376
16.5 problem 12 2386
16.6 problem 13 2396
16.7 problem 14 2405
16.8 problem 15 2414
16.9 problem 16 2424
16.10 problem 17 2436

16.1 problem 9

16.1.1 Solving as series ode 2339
16.1.2 Maple step by step solution 2346

Internal problem ID [5449]
Internal file name [`OUTPUT/4940_Wednesday_February_14_2024_02_05_47_AM_53084472/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(1-x)\,y'+y=x^2}$$

With the expansion point for the power series method at $x=0$.

16.1.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\frac{df}{dx}\bigg|_{x_0,y_0}+\frac{x^3}{3!}\frac{d^2f}{dx^2}\bigg|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\begin{aligned}
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{-x^2+y}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= -\frac{2x}{x-1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \frac{2}{(x-1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= -\frac{4}{(x-1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \frac{12}{(x-1)^4}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= -y(0) \\
F_1 &= 0 \\
F_2 &= 2 \\
F_3 &= 4 \\
F_4 &= 12
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (1-x)\, y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{x-1} &= -\frac{x^2}{x-1} \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{x-1} \\ p(x) &= -\frac{x^2}{x-1} \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x-1)\, y' - y = -x^2$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x-1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=-x^2 \tag{1}$$

Expanding $-x^2$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}-x^2 &= -x^2+\dots\\ &= -x^2\end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(x-1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=-x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=-x^2 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty}\left(-n a_n x^{n-1}\right)=\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-(n+1) a_{n+1} x^n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=-x^2 \tag{3}$$

$n = 0$ gives

$$\begin{aligned}-a_1-a_0 &= 0\\ a_1 &= -a_0\end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$\left(n a_n-(n+1) a_{n+1}-a_n\right) x^n=-x^2 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(-2a_2)\,x &= 0\\ -2a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = 0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(a_2 - 3a_3)\,x^2 &= -x^2\\ a_2 - 3a_3 &= -1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{1}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(2a_3 - 4a_4)\,x^3 &= 0\\ 2a_3 - 4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{6}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(3a_4 - 5a_5)\,x^4 &= 0\\ 3a_4 - 5a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{10}$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(4a_5 - 6a_6)\, x^5 &= 0 \\
4a_5 - 6a_6 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_6 = \frac{1}{15}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 - a_0 x + \frac{1}{3}x^3 + \frac{1}{6}x^4 + \frac{1}{10}x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = (1 - x)\, a_0 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O\left(x^6\right) \tag{3}
$$

Summary

The solution(s) found are the following

$$
y = (1 - x)\, y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O\left(x^6\right) \tag{1}
$$

$$
y = (1 - x)\, c_1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O\left(x^6\right) \tag{2}
$$

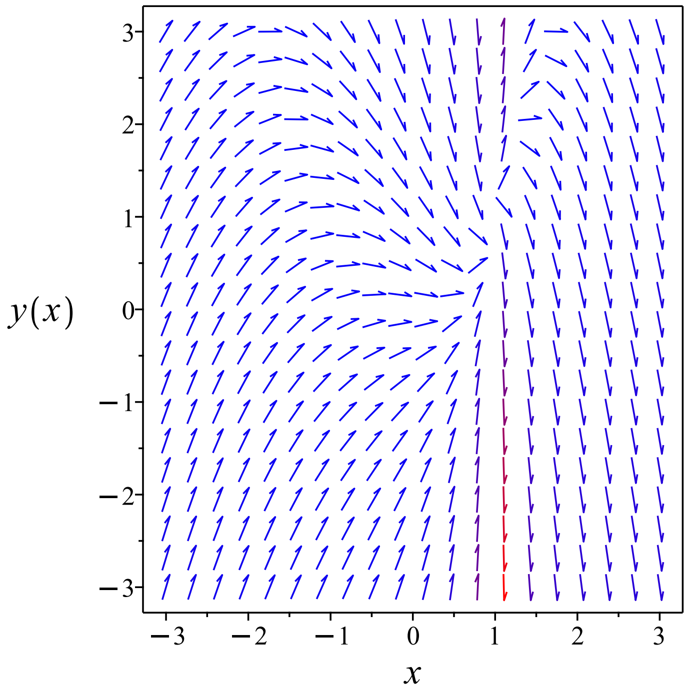


Figure 283: Slope field plot

Verification of solutions

$$y = (1 - x)\, y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Verified OK.

$$y = (1 - x)\, c_1 + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

Verified OK.

16.1.2 Maple step by step solution

Let's solve

$(x - 1)\, y' - y = -x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x-1} - \frac{x^2}{x-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x-1} = -\frac{x^2}{x-1}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{y}{x-1}\right) = -\frac{\mu(x) x^2}{x-1}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{y}{x-1}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x-1}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x-1}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x) x^2}{x-1} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{\mu(x) x^2}{x-1} dx + c_1$
- Solve for y

  $y = \frac{\int -\frac{\mu(x) x^2}{x-1} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x-1}$

  $y = (x-1) \left(\int -\frac{x^2}{(x-1)^2} dx + c_1\right)$
- Evaluate the integrals on the rhs

  $y = (x-1) \left(-x - 2\ln(x-1) + \frac{1}{x-1} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=6;
dsolve((1-x)*diff(y(x),x)=x^2-y(x),y(x),type='series',x=0);
```

$$y(x) = (1-x)\,y(0) + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{10} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 33

```
AsymptoticDSolveValue[(1-x)*y'[x]==x^2-y[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{10} + \frac{x^4}{6} + \frac{x^3}{3} + c_1(1-x)$$

16.2 problem 10A

16.2.1 Solving as series ode 2349
16.2.2 Maple step by step solution 2357

Internal problem ID [5450]
Internal file name [`OUTPUT/4941_Wednesday_February_14_2024_02_05_48_AM_84468907/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 10A.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - 2y = 1 - x}$$

With the expansion point for the power series method at $x = 1$.

16.2.1 Solving as series ode

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t + 1)\left(\frac{d}{dt}y(t)\right) - 2y(t) = -t$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f \qquad (3) \\
&\ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y) \qquad (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y} \right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y} \right) F_0 \\
&= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \right) f \\
&= \frac{\partial}{\partial x} \left(\frac{df}{dx} \right) + \frac{\partial}{\partial y} \left(\frac{df}{dx} \right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n \big|_{x_0, y_0} \tag{6}
$$

Hence

$$
\begin{aligned}
F_0 &= \frac{2y(t) - t}{t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{-1 + 2y(t) - 2t}{(t+1)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= -1 + 2y(0) \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y(t) = \left(t^2 + 2t + 1\right) y(0) - \frac{t^2}{2} + O\left(t^6\right)
$$

Since $t = 0$ is also an ordinary point, then standard power series can also be used.

Writing the ODE as

$$\frac{d}{dt}y(t) + q(t)y(t) = p(t)$$
$$\frac{d}{dt}y(t) - \frac{2y(t)}{t+1} = -\frac{t}{t+1}$$

Where

$$q(t) = -\frac{2}{t+1}$$
$$p(t) = -\frac{t}{t+1}$$

Next, the type of the expansion point $t = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $t = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $t = 0$ is called an ordinary point $q(t)$ has a Taylor series expansion around the point $t = 0$. $t = 0$ is called a regular singular point if $q(t)$ is not not analytic at $t = 0$ but $tq(t)$ has Taylor series expansion. And finally, $t = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $t = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(t+1)\left(\frac{d}{dt}y(t)\right) - 2y(t) = -t$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$

Substituting the above back into the ode gives

$$(t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = -t \tag{1}$$

Expanding $-t$ as Taylor series around $t = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -t &= -t + \dots \\ &= -t \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(t+1)\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) - 2\left(\sum_{n=0}^{\infty} a_n t^n\right) = -t \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right) + \sum_{n=0}^{\infty} (-2a_n t^n) = -t \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n t^n\right) + \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} t^n\right) + \sum_{n=0}^{\infty} (-2a_n t^n) = -t \tag{3}$$

$n = 0$ gives

$$\begin{aligned} a_1 - 2a_0 &= 0 \\ a_1 &= 2a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$(n a_n + (n+1) a_{n+1} - 2a_n) t^n = -t \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned} (-a_1 + 2a_2) t &= -t \\ -a_1 + 2a_2 &= -1 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{1}{2} + a_0$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(3a_3)\, t^2 &= 0 \\ 3a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(a_3 + 4a_4)\, t^3 &= 0 \\ a_3 + 4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(2a_4 + 5a_5)\, t^4 &= 0 \\ 2a_4 + 5a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(3a_5 + 6a_6)\, t^5 &= 0 \\ 3a_5 + 6a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} a_n t^n \\
&= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y(t) = a_0 + 2a_0 t + \left(-\frac{1}{2} + a_0\right) t^2 + \dots
$$

Collecting terms, the solution becomes

$$
y(t) = \left(t^2 + 2t + 1\right) a_0 - \frac{t^2}{2} + O\left(t^6\right) \tag{3}
$$

Replacing t in the above with the original independent variable xs using $t = x - 1$ results in

$$
y = \left((x-1)^2 + 2x - 1\right) y(1) - \frac{(x-1)^2}{2} + O\left((x-1)^6\right)
$$

Summary
The solution(s) found are the following

$$
y = \left((x-1)^2 + 2x - 1\right) y(1) - \frac{(x-1)^2}{2} + O\left((x-1)^6\right) \tag{1}
$$

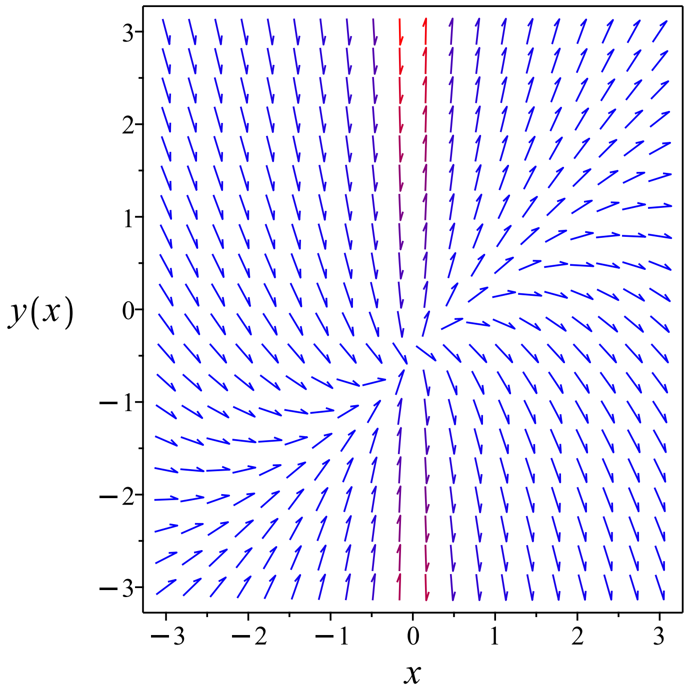


Figure 284: Slope field plot

Verification of solutions

$$y = \left((x-1)^2 + 2x - 1\right) y(1) - \frac{(x-1)^2}{2} + O\left((x-1)^6\right)$$

Verified OK.

16.2.2 Maple step by step solution

Let's solve

$xy' - 2y = 1 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} - \frac{x-1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = -\frac{x-1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-\frac{2y}{x}\right)=-\frac{\mu(x)(x-1)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{2y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{2\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x^2}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int -\frac{\mu(x)(x-1)}{x}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int -\frac{\mu(x)(x-1)}{x}dx+c_1$
- Solve for y

  $y=\frac{\int -\frac{\mu(x)(x-1)}{x}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x^2}$

  $y=x^2\left(\int -\frac{x-1}{x^3}dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=x^2\left(\frac{1}{x}-\frac{1}{2x^2}+c_1\right)$
- Simplify

  $y=x-\frac{1}{2}+c_1x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(x*diff(y(x),x)=1-x+2*y(x),y(x),type='series',x=1);
```

$$y(x) = y(1)\,x^2 - \frac{(x-1)^2}{2}$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 28

```
AsymptoticDSolveValue[x*y'[x]==1-x+2*y[x],y[x],{x,1,5}]
```

$$y(x) \to -\frac{1}{2}(x-1)^2 + c_1\left((x-1)^2 + 2(x-1) + 1\right)$$

16.3 problem 10B

16.3.1 Solving as linear ode 2360
16.3.2 Solving as homogeneousTypeMapleC ode 2362
16.3.3 Solving as first order ode lie symmetry lookup ode 2365
16.3.4 Solving as exact ode 2369
16.3.5 Maple step by step solution 2374

Internal problem ID [5451]
Internal file name [`OUTPUT/4942_Wednesday_February_14_2024_02_05_48_AM_69199443/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 10B.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' - 2y = 1 - x}$$

16.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{1-x}{x}$$

Hence the ode is

$$y' - \frac{2y}{x} = \frac{1-x}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1-x}{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\frac{1-x}{x}\right) \\
\mathrm{d}\left(\frac{y}{x^2}\right) &= \left(\frac{1-x}{x^3}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int \frac{1-x}{x^3}\,\mathrm{d}x \\
\frac{y}{x^2} &= \frac{1}{x} - \frac{1}{2x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = x^2\left(\frac{1}{x} - \frac{1}{2x^2}\right) + c_1 x^2
$$

which simplifies to

$$
y = x - \frac{1}{2} + c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = x - \frac{1}{2} + c_1 x^2 \tag{1}
$$

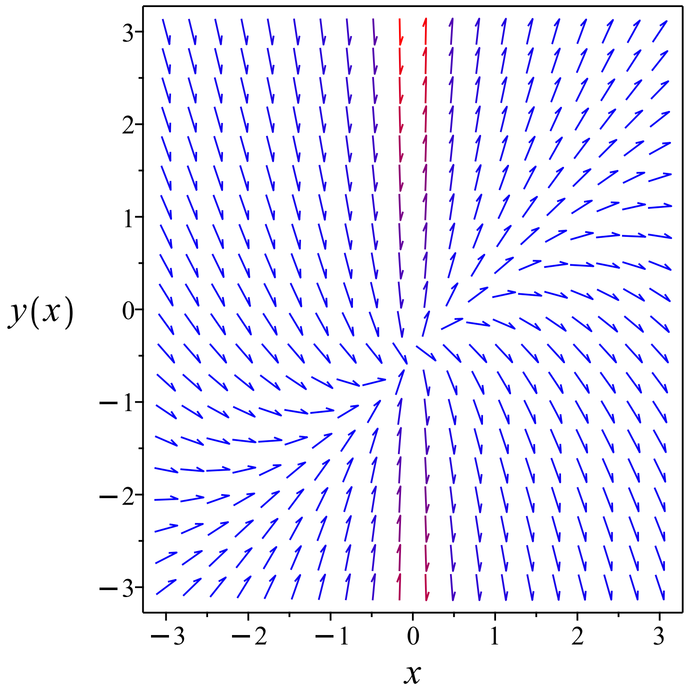


Figure 285: Slope field plot

Verification of solutions

$$y = x - \frac{1}{2} + c_1 x^2$$

Verified OK.

16.3.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{1 - X - x_0 + 2Y(X) + 2y_0}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -\frac{1}{2}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + 2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{-X+2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -X+2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -1 + 2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-1+u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-1+u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X) + 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= \frac{-1+u}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = -1+u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-1+u}\,du &= \frac{1}{X}\,dX \\ \int \frac{1}{-1+u}\,du &= \int \frac{1}{X}\,dX \\ \ln(-1+u) &= \ln(X) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$-1+u = e^{\ln(X)+c_2}$$

Which simplifies to

$$-1+u = c_3 X$$

Which simplifies to

$$u(X) = c_3 X\,e^{c_2}+1$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X(c_3 X\,e^{c_2}+1)$$

Using the solution for $Y(X)$

$$Y(X) = X(c_3 X\,e^{c_2}+1)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{1}{2}$$
$$X = x$$

Then the solution in y becomes

$$y + \frac{1}{2} = x(c_3 x\,e^{c_2}+1)$$

Summary
The solution(s) found are the following

$$y + \frac{1}{2} = x(c_3 x\,e^{c_2}+1) \tag{1}$$

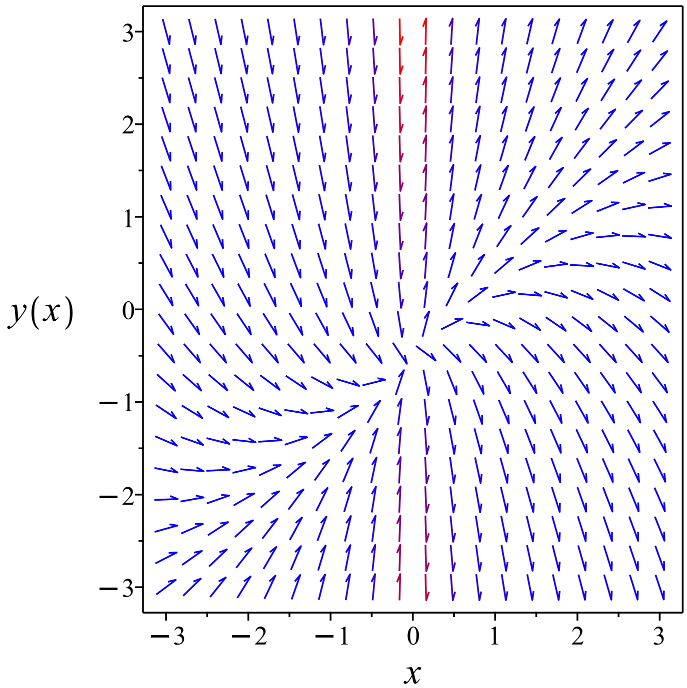


Figure 286: Slope field plot

Verification of solutions

$$y+\frac{1}{2}=x\left(c_3x\,\mathrm{e}^{c_2}+1\right)$$

Verified OK.

16.3.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=\frac{1-x+2y}{x}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 264: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1 - x + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1 - x}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1 - R}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{1}{R} - \frac{1}{2R^2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = \frac{1}{x} - \frac{1}{2x^2} + c_1$$

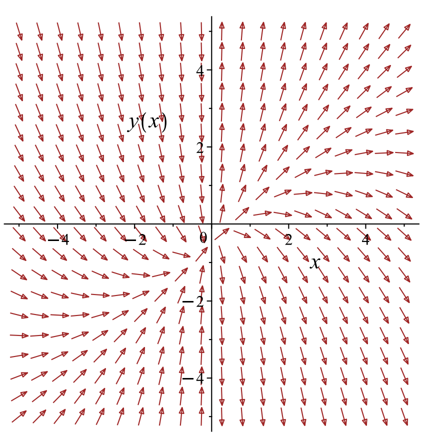
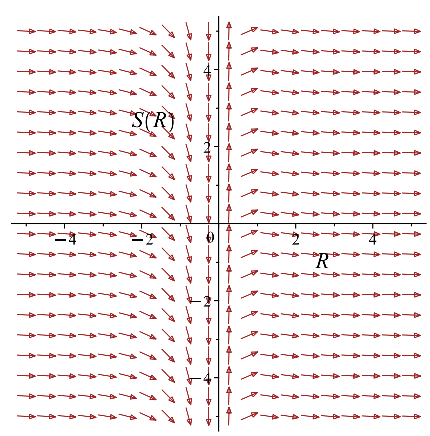
Which simplifies to

$$\frac{y}{x^2} = \frac{1}{x} - \frac{1}{2x^2} + c_1$$

Which gives

$$y = x - \frac{1}{2} + c_1 x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1-x+2y}{x}$ 	$R = x$ $S = \frac{y}{x^2}$	$\frac{dS}{dR} = \frac{1-R}{R^3}$ 

Summary

The solution(s) found are the following

$$y = x - \frac{1}{2} + c_1 x^2 \tag{1}$$

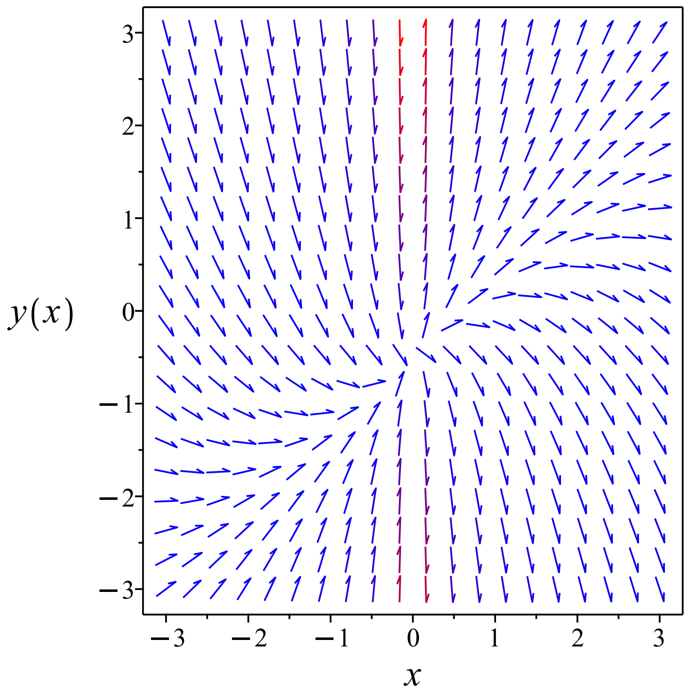


Figure 287: Slope field plot

Verification of solutions

$$y = x - \frac{1}{2} + c_1 x^2$$

Verified OK.

16.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (1 - x + 2y)\,\mathrm{d}x \\ (-1 + x - 2y)\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -1 + x - 2y \\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 + x - 2y) \\ &= -2\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-2) - (1)) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}(-1 + x - 2y) \\
&= \frac{-1 + x - 2y}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(x) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-1+x-2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-1+x-2y}{x^3}\,\mathrm{d}x \\
\phi &= \frac{-2x+1+2y}{2x^2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$
\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-2x + 1 + 2y}{2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-2x + 1 + 2y}{2x^2}$$

The solution becomes

$$y = x - \frac{1}{2} + c_1 x^2$$

Summary

The solution(s) found are the following

$$y = x - \frac{1}{2} + c_1 x^2 \tag{1}$$

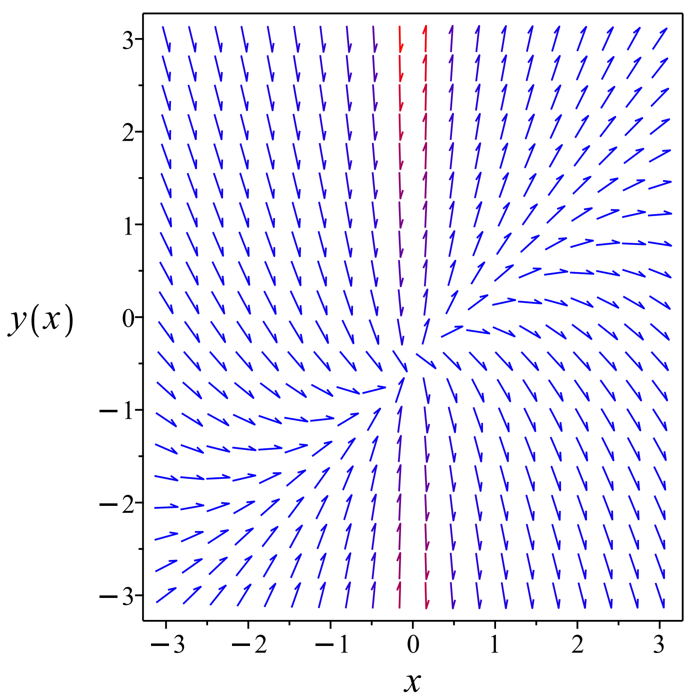


Figure 288: Slope field plot

Verification of solutions

$$
y = x - \frac{1}{2} + c_1 x^2
$$

Verified OK.

16.3.5 Maple step by step solution

Let's solve

$x y' - 2y = 1 - x$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = \frac{2y}{x} - \frac{x-1}{x}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = -\frac{x-1}{x}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = -\frac{\mu(x)(x-1)}{x}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int -\frac{\mu(x)(x-1)}{x} dx + c_1$
- Solve for y

  $y = \frac{\int -\frac{\mu(x)(x-1)}{x} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x^2}$

$y = x^2 \left(\int -\frac{x-1}{x^3} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2 \left(\frac{1}{x} - \frac{1}{2x^2} + c_1\right)$

- Simplify

  $y = x - \frac{1}{2} + c_1 x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)=1-x+2*y(x),y(x), singsol=all)
```

$$y(x) = x - \frac{1}{2} + c_1 x^2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 16

```
DSolve[x*y'[x]==1-x+2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2 + x - \frac{1}{2}$$

16.4 problem 11

16.4.1 Solving as series ode 2376
16.4.2 Maple step by step solution 2383

Internal problem ID [5452]
Internal file name [`OUTPUT/4943_Wednesday_February_14_2024_02_05_49_AM_48988675/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 3y = 2x^2}$$

With the expansion point for the power series method at $x = 0$.

16.4.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\ &= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\ &= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= 2x^2 + 3y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= 4x + 6x^2 + 9y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 4 + 12x + 18x^2 + 27y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= 12 + 36x + 54x^2 + 81y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= 36 + 108x + 162x^2 + 243y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 3y(0) \\
F_1 &= 9y(0) \\
F_2 &= 27y(0) + 4 \\
F_3 &= 81y(0) + 12 \\
F_4 &= 243y(0) + 36
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - 3y &= 2x^2
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -3 \\
p(x) &= 2x^2
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 2x^2 \tag{1}
$$

Expanding $2x^2$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
2x^2 &= 2x^2 + \dots \\
&= 2x^2
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) = 2x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} (-3 a_n x^n) = 2x^2 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=0}^{\infty} (-3 a_n x^n) = 2x^2 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$((n+1) a_{n+1} - 3a_n) x^n = 2x^2 \tag{4}$$

For $n = 0$ the recurrence equation gives

$$\begin{aligned} (a_1 - 3a_0) 1 &= 0 \\ a_1 - 3a_0 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_1 = 3a_0$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned} (2a_2 - 3a_1) x &= 0 \\ 2a_2 - 3a_1 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{9a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned} (3a_3 - 3a_2) x^2 &= 2x^2 \\ 3a_3 - 3a_2 &= 2 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2}{3} + \frac{9a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned} (4a_4 - 3a_3) x^3 &= 0 \\ 4a_4 - 3a_3 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{2} + \frac{27a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned} (5a_5 - 3a_4) x^4 &= 0 \\ 5a_5 - 3a_4 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3}{10} + \frac{81a_0}{40}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned} (6a_6 - 3a_5) x^5 &= 0 \\ 6a_6 - 3a_5 &= 0 \end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{3}{20} + \frac{81a_0}{80}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + 3a_0x + \frac{9a_0x^2}{2} + \left(\frac{2}{3} + \frac{9a_0}{2}\right)x^3 + \left(\frac{1}{2} + \frac{27a_0}{8}\right)x^4 + \left(\frac{3}{10} + \frac{81a_0}{40}\right)x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right)a_0 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right)y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \tag{1}$$

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right)c_1 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6) \tag{2}$$

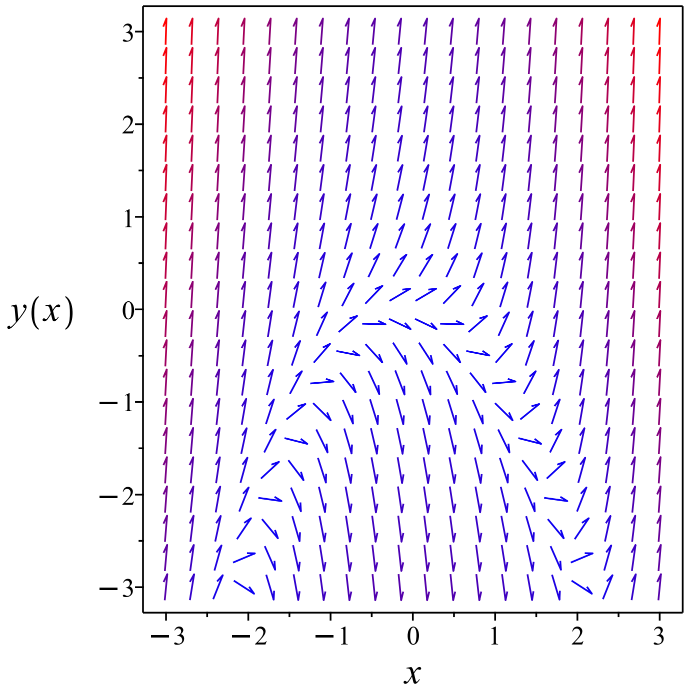


Figure 289: Slope field plot

Verification of solutions

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) y(0) + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Verified OK.

$$y = \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \frac{27}{8}x^4 + \frac{81}{40}x^5\right) c_1 + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{3x^5}{10} + O(x^6)$$

Verified OK.

16.4.2 Maple step by step solution

Let's solve

$y' - 3y = 2x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2x^2 + 3y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 3y = 2x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - 3y) = 2\mu(x)\,x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' - 3y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -3\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-3x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,x^2 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,x^2 dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x) x^2 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-3x}$

  $y = \frac{\int 2\,\mathrm{e}^{-3x} x^2 dx + c_1}{\mathrm{e}^{-3x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-\frac{2\left(9x^2+6x+2\right)\mathrm{e}^{-3x}}{27} + c_1}{\mathrm{e}^{-3x}}$
- Simplify

  $y = \mathrm{e}^{3x} c_1 - \frac{2x^2}{3} - \frac{4x}{9} - \frac{4}{27}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 42

```
Order:=6;
dsolve(diff(y(x),x)=2*x^2+3*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{27}{8}x^4+\frac{81}{40}x^5\right)y(0)+\frac{2x^3}{3}+\frac{x^4}{2}+\frac{3x^5}{10}+O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 61

```
AsymptoticDSolveValue[y'[x]==2*x^2+3*y[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{3x^5}{10}+\frac{x^4}{2}+\frac{2x^3}{3}+c_1\left(\frac{81x^5}{40}+\frac{27x^4}{8}+\frac{9x^3}{2}+\frac{9x^2}{2}+3x+1\right)$$

16.5 problem 12

16.5.1 Solving as series ode 2386
16.5.2 Maple step by step solution 2393

Internal problem ID [5453]
Internal file name [`OUTPUT/4944_Wednesday_February_14_2024_02_05_50_AM_32247685/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{(x+1)\, y'-y=x^{2}-2x}$$

With the expansion point for the power series method at $x=0$.

16.5.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y'=f(x,y)$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y''(x_0)+\cdots \\
&= y_0+xf+\frac{x^2}{2}\left.\frac{df}{dx}\right|_{x_0,y_0}+\frac{x^3}{3!}\left.\frac{d^2f}{dx^2}\right|_{x_0,y_0}+\cdots \\
&= y_0+\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)\,!}\left.\frac{d^n f}{dx^n}\right|_{x_0,y_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= \frac{x^2 - 2x + y}{x+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \frac{2x-2}{x+1} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= \frac{4}{(x+1)^2} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= -\frac{8}{(x+1)^3} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \frac{24}{(x+1)^4}
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= y(0) \\
F_1 &= -2 \\
F_2 &= 4 \\
F_3 &= -8 \\
F_4 &= 24
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = (x+1)\, y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$\begin{aligned} y' + q(x)y &= p(x) \\ y' - \frac{y}{x+1} &= \frac{x(-2+x)}{x+1} \end{aligned}$$

Where

$$\begin{aligned} q(x) &= -\frac{1}{x+1} \\ p(x) &= \frac{x(-2+x)}{x+1} \end{aligned}$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Now the ode is normalized by writing it as

$$(x+1)\, y' - y = x(-2+x)$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$(x+1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=x(-2+x) \tag{1}$$

Expanding $x(-2+x)$ as Taylor series around $x=0$ and keeping only the first 6 terms gives

$$\begin{aligned} x(-2+x) &= x^2-2x+\dots \\ &= x^2-2x \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$(x+1)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^n\right)=x^2-2x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=x^2-2x \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} n a_n x^n\right)+\left(\sum_{n=0}^{\infty}(n+1) a_{n+1} x^n\right)+\sum_{n=0}^{\infty}\left(-a_n x^n\right)=x^2-2x \tag{3}$$

$n=0$ gives

$$\begin{aligned} a_1-a_0 &= 0 \\ a_1 &= a_0 \end{aligned}$$

For $1 \leq n$, the recurrence equation is

$$(na_n + (n+1)\, a_{n+1} - a_n)\, x^n = x^2 - 2x \tag{4}$$

For $n = 1$ the recurrence equation gives

$$\begin{aligned}(2a_2)\, x &= -2x\\ 2a_2 &= -2\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_2 = -1$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(a_2 + 3a_3)\, x^2 &= x^2\\ a_2 + 3a_3 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{2}{3}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(2a_3 + 4a_4)\, x^3 &= 0\\ 2a_3 + 4a_4 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{1}{3}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(3a_4 + 5a_5)\, x^4 &= 0\\ 3a_4 + 5a_5 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{5}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(4a_5 + 6a_6) x^5 &= 0\\ 4a_5 + 6a_6 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{2}{15}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_0 x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \dots$$

Collecting terms, the solution becomes

$$y = (x+1)\, a_0 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = (x+1)\, y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \tag{1}$$

$$y = (x+1)\, c_1 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6) \tag{2}$$

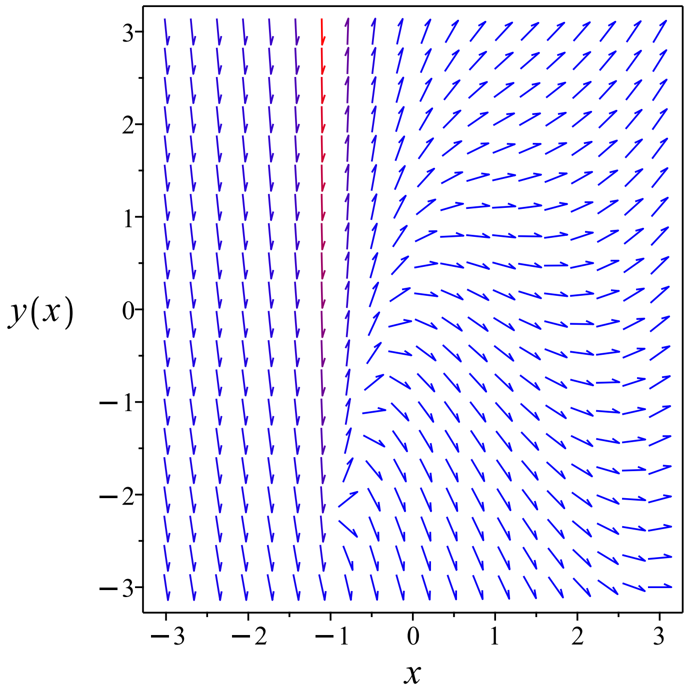


Figure 290: Slope field plot

Verification of solutions

$$y = (x+1)\, y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

$$y = (x+1)\, c_1 - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

Verified OK.

16.5.2 Maple step by step solution

Let's solve

$(x+1)\, y' - y = x(-2+x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x+1} + \frac{x(-2+x)}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $$y' - \frac{y}{x+1} = \frac{x(-2+x)}{x+1}$$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $$\mu(x)\left(y' - \frac{y}{x+1}\right) = \frac{\mu(x)x(-2+x)}{x+1}$$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $$\mu(x)\left(y' - \frac{y}{x+1}\right) = \mu'(x)\,y + \mu(x)\,y'$$
- Isolate $\mu'(x)$

  $$\mu'(x) = -\frac{\mu(x)}{x+1}$$
- Solve to find the integrating factor

  $$\mu(x) = \frac{1}{x+1}$$
- Integrate both sides with respect to x

  $$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1$$
- Evaluate the integral on the lhs

  $$\mu(x)\,y = \int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1$$
- Solve for y

  $$y = \frac{\int \frac{\mu(x)x(-2+x)}{x+1} dx + c_1}{\mu(x)}$$
- Substitute $\mu(x) = \frac{1}{x+1}$

  $$y = (x+1)\left(\int \frac{x(-2+x)}{(x+1)^2} dx + c_1\right)$$
- Evaluate the integrals on the rhs

  $$y = (x+1)\left(x - \frac{3}{x+1} - 4\ln(x+1) + c_1\right)$$
- Simplify

  $$y = (-4x-4)\ln(x+1) + x^2 + (c_1+1)\,x + c_1 - 3$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
Order:=6;
dsolve((x+1)*diff(y(x),x)=x^2-2*x+y(x),y(x),type='series',x=0);
```

$$y(x) = (x+1)\,y(0) - x^2 + \frac{2x^3}{3} - \frac{x^4}{3} + \frac{x^5}{5} + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 36

```
AsymptoticDSolveValue[(x+1)*y'[x]==x^2-2*x+y[x],y[x],{x,0,5}]
```

$$y(x) \to \frac{x^5}{5} - \frac{x^4}{3} + \frac{2x^3}{3} - x^2 + c_1(x+1)$$

16.6 problem 13

16.6.1 Maple step by step solution 2403

Internal problem ID [5454]
Internal file name [OUTPUT/4945_Wednesday_February_14_2024_02_05_50_AM_84918379/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (470)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (471)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -xy' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= x^2 y - 2y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^3 + 6xy' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -y(0) \\
F_2 &= -2y'(0) \\
F_3 &= 0 \\
F_4 &= 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6\right) y(0) + \left(x - \frac{1}{12}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n)+a_{n-1} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^3 - \frac{1}{12} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^3}{6}\right) a_0 + \left(x - \frac{1}{12} x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{6} x^3 + \frac{1}{180} x^6\right) y(0) + \left(x - \frac{1}{12} x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^3}{6}\right) c_1 + \left(x - \frac{1}{12} x^4\right) c_2 + O(x^6)$$

Verified OK.

16.6.1 Maple step by step solution

Let's solve

$y'' = -yx$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$

  - Shift index using $k- >k - 1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$

  - Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  - Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} + a_{k-1} = 0$

- Shift index using $k- >k + 1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty} a_k x^k, a_{k+3}=-\frac{a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^3}{6}\right)y(0)+\left(x-\frac{1}{12}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{12}\right)+c_1\left(1-\frac{x^3}{6}\right)$$

16.7 problem 14

16.7.1 Maple step by step solution 2411

Internal problem ID [5455]
Internal file name [`OUTPUT/4946_Wednesday_February_14_2024_02_05_51_AM_18391819/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + 2x^2 y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{473}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{474}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -2x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4yx^4 - 8xy' - 4y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 4x^4 y' + 32yx^3 - 12y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 48y'x^3 - 8x^2 y\left(x^4 - 15\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -4y(0) \\
F_3 &= -12y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -2x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+2} a_n = \sum_{n=2}^{\infty} 2a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} 2a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 2a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{6}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 2a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{6} a_0 x^4 - \frac{1}{10} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{6}\right) a_0 + \left(x - \frac{1}{10} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10} x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{6}\right) c_1 + \left(x - \frac{1}{10} x^5\right) c_2 + O(x^6)$$

Verified OK.

16.7.1 Maple step by step solution

Let's solve

$y'' = -2x^2 y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$$y'' + 2x^2 y = 0$$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

 ◦ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

 ◦ Shift index using $k- >k-2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

 ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

 ◦ Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{6}\right) y(0) + \left(x - \frac{1}{10}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+2*x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^5}{10}\right) + c_1\left(1 - \frac{x^4}{6}\right)$$

16.8 problem 15

16.8.1 Maple step by step solution 2421

Internal problem ID [5456]
Internal file name [OUTPUT/4947_Wednesday_February_14_2024_02_05_51_AM_85677221/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - xy' + x^2y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{476}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{477}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= xy' - x^2 y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' + \left(-x^3 - 2x\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -y'x^3 - 4x^2 y - xy' - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(-x^4 - 8x^2 - 3\right) y' + yx\left(x^4 + x^2 - 8\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(-11x^3 - 27x\right) y' + y\left(x^6 + 13x^4 + 6x^2 - 8\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y'(0) \\
F_2 &= -2y(0) \\
F_3 &= -3y'(0) \\
F_4 &= -8y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{12}x^4 - \frac{1}{90}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-n x^n a_n) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} x^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-n x^n a_n) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - n a_n + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{n a_n - a_{n-2}}{(n+2)\,(n+1)} \\ &= \frac{n a_n}{(n+2)\,(n+1)} - \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 3a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{40}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 4a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{90}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 5a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{144}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{6} a_1 x^3 - \frac{1}{12} a_0 x^4 - \frac{1}{40} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1 - \frac{1}{12} x^4 - \frac{1}{90} x^6\right) y(0) + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{12} x^4 - \frac{1}{90} x^6\right) y(0) + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) c_2 + O(x^6)$$

Verified OK.

16.8.1 Maple step by step solution

Let's solve

$y'' = xy' - x^2y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - xy' + x^2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$

  ◦ Shift index using $k- >k - 2$

    $x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

    $y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

  ◦ Shift index using $k- >k + 2$

    $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

  Rewrite ODE with series expansions

  $2a_2 + (6a_3 - a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - a_k k + a_{k-2})\, x^k\right) = 0$

- The coefficients of each power of x must be 0

  $[2a_2 = 0, 6a_3 - a_1 = 0]$

- Solve for the dependent coefficient(s)

$\left\{a_2 = 0, a_3 = \frac{a_1}{6}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\, a_{k+2} - a_k k + a_{k-2} = 0$

- Shift index using $k- > k + 2$

  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - a_{k+2}(k+2) + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{k a_{k+2} - a_k + 2a_{k+2}}{k^2 + 7k + 12}, a_2 = 0, a_3 = \frac{a_1}{6}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
Order:=6;
dsolve(diff(y(x),x$2)-x*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x + \frac{1}{6} x^3 - \frac{1}{40} x^5\right) D(y)(0) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[y''[x]-x*y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{x^4}{12}\right) + c_2 \left(-\frac{x^5}{40} + \frac{x^3}{6} + x\right)$$

16.9 problem 16

16.9.1 Maple step by step solution 2432

Internal problem ID [5457]
Internal file name [OUTPUT/4948_Wednesday_February_14_2024_02_05_52_AM_58150107/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

`[_Gegenbauer]`

$$\boxed{\left(-x^{2}+1\right) y''-2 x y'+p (p+1) y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (480)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (481)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{p^2 y + p y - 2 x y'}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(x^2 p^2 + x^2 p - p^2 + 6x^2 - p + 2\right) y' - 4pxy(p+1)}{\left(x^2 - 1\right)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{\left(-8x\left(\left(p^2 + p + 3\right) x^2 - p^2 - p + 3\right) y' + py(p+1)\left(\left(p^2 + p + 18\right) x^2 - p^2 - p + 6\right)\right)(x-1)(x+1)}{\left(x^2 - 1\right)^4} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(x-1)(x+1)\left(\left(\left(p^4 + 2p^3 + 59p^2 + 58p + 120\right) x^4 + \left(-2p^4 - 4p^3 - 46p^2 - 44p + 240\right) x^2 + p^4 + 2p\right.\right.}{\left(x^2 - 1\right)^5} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(x-1)^2 (x+1)^2 \left(-18\left(\left(p^4 + 2p^3 + \frac{77}{3} p^2 + \frac{74}{3} p + 40\right) x^4 - 2\left(p^2 + p - \frac{20}{3}\right)\left(p^2 + p + 10\right) x^2 + p^4 + 2p\right.\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)(p+1)p \\
F_1 &= -y'(0) p^2 - y'(0) p + 2y'(0) \\
F_2 &= y(0) p^4 + 2y(0) p^3 - 5y(0) p^2 - 6y(0) p \\
F_3 &= y'(0) p^4 + 2y'(0) p^3 - 13y'(0) p^2 - 14y'(0) p + 24y'(0) \\
F_4 &= -y(0) p^6 - 3y(0) p^5 + 23y(0) p^4 + 51y(0) p^3 - 94y(0) p^2 - 120y(0) p
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}p^{2}-\frac{1}{2}x^{2}p+\frac{1}{24}p^{4}x^{4}+\frac{1}{12}p^{3}x^{4}-\frac{5}{24}p^{2}x^{4}-\frac{1}{4}p\,x^{4}-\frac{1}{720}x^{6}p^{6}-\frac{1}{240}x^{6}p^{5}\right.\\
&\left.+\frac{23}{720}x^{6}p^{4}+\frac{17}{240}x^{6}p^{3}-\frac{47}{360}x^{6}p^{2}-\frac{1}{6}x^{6}p\right)y(0)\\
&+\left(x-\frac{1}{6}p^{2}x^{3}-\frac{1}{6}p\,x^{3}+\frac{1}{3}x^{3}+\frac{1}{120}x^{5}p^{4}+\frac{1}{60}x^{5}p^{3}-\frac{13}{120}x^{5}p^{2}-\frac{7}{60}x^{5}p+\frac{1}{5}x^{5}\right)y'(0)\\
&+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-x^{2}+1\right)y''-2xy'+\left(p^{2}+p\right)y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty}a_{n}x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty}na_{n}x^{n-1}\\
y'' &= \sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)-2x\left(\sum_{n=1}^{\infty}na_{n}x^{n-1}\right)+\left(p^{2}+p\right)\left(\sum_{n=0}^{\infty}a_{n}x^{n}\right)=0 \tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left(-x^{n}a_{n}n(n-1)\right)+\left(\sum_{n=2}^{\infty}n(n-1)\,a_{n}x^{n-2}\right)\\
&+\sum_{n=1}^{\infty}\left(-2na_{n}x^{n}\right)+\left(\sum_{n=0}^{\infty}\left(p^{2}+p\right)a_{n}x^{n}\right)=0
\end{aligned} \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} (-x^n a_n n(n-1)) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) \\ + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} (p^2+p) a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 p(p+1) = 0$$

$$a_2 = -\frac{1}{2} a_0 p^2 - \frac{1}{2} a_0 p$$

$n = 1$ gives

$$6a_3 - 2a_1 + a_1 p(p+1) = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 p^2 - \frac{1}{6} a_1 p + \frac{1}{3} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n (n-1) + (n+2) a_{n+2}(n+1) - 2n a_n + a_n p(p+1) = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - p^2 + n - p)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$-6a_2 + 12a_4 + a_2 p(p+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{5}{24}a_0 p^2 - \frac{1}{4}a_0 p + \frac{1}{24}a_0 p^4 + \frac{1}{12}a_0 p^3$$

For $n = 3$ the recurrence equation gives

$$-12a_3 + 20a_5 + a_3 p(p+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{13}{120}a_1 p^2 - \frac{7}{60}a_1 p + \frac{1}{5}a_1 + \frac{1}{120}a_1 p^4 + \frac{1}{60}a_1 p^3$$

For $n = 4$ the recurrence equation gives

$$-20a_4 + 30a_6 + a_4 p(p+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{47}{360}a_0 p^2 - \frac{1}{6}a_0 p + \frac{23}{720}a_0 p^4 + \frac{17}{240}a_0 p^3 - \frac{1}{720}a_0 p^6 - \frac{1}{240}a_0 p^5$$

For $n = 5$ the recurrence equation gives

$$-30a_5 + 42a_7 + a_5 p(p+1) = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{5}{63}a_1 p^2 - \frac{37}{420}a_1 p + \frac{1}{7}a_1 + \frac{41}{5040}a_1 p^4 + \frac{29}{1680}a_1 p^3 - \frac{1}{5040}a_1 p^6 - \frac{1}{1680}a_1 p^5$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y = a_0 + a_1 x &+ \left(-\frac{1}{2} a_0 p^2 - \frac{1}{2} a_0 p\right) x^2 + \left(-\frac{1}{6} a_1 p^2 - \frac{1}{6} a_1 p + \frac{1}{3} a_1\right) x^3 \\
&+ \left(-\frac{5}{24} a_0 p^2 - \frac{1}{4} a_0 p + \frac{1}{24} a_0 p^4 + \frac{1}{12} a_0 p^3\right) x^4 \\
&+ \left(-\frac{13}{120} a_1 p^2 - \frac{7}{60} a_1 p + \frac{1}{5} a_1 + \frac{1}{120} a_1 p^4 + \frac{1}{60} a_1 p^3\right) x^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = {}& \left(1 + \left(-\frac{1}{2} p^2 - \frac{1}{2} p\right) x^2 + \left(-\frac{5}{24} p^2 - \frac{1}{4} p + \frac{1}{24} p^4 + \frac{1}{12} p^3\right) x^4\right) a_0 \\
&+ \left(x + \left(-\frac{1}{6} p^2 - \frac{1}{6} p + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} p^2 - \frac{7}{60} p + \frac{1}{5} + \frac{1}{120} p^4 + \frac{1}{60} p^3\right) x^5\right) a_1 + O(x^6)
\end{aligned} \tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = {}& \left(1 + \left(-\frac{1}{2} p^2 - \frac{1}{2} p\right) x^2 + \left(-\frac{5}{24} p^2 - \frac{1}{4} p + \frac{1}{24} p^4 + \frac{1}{12} p^3\right) x^4\right) c_1 \\
&+ \left(x + \left(-\frac{1}{6} p^2 - \frac{1}{6} p + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} p^2 - \frac{7}{60} p + \frac{1}{5} + \frac{1}{120} p^4 + \frac{1}{60} p^3\right) x^5\right) c_2 + O(x^6)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = {}& \left(1 - \frac{1}{2} x^2 p^2 - \frac{1}{2} x^2 p + \frac{1}{24} p^4 x^4 + \frac{1}{12} p^3 x^4 - \frac{5}{24} p^2 x^4 - \frac{1}{4} p\, x^4 - \frac{1}{720} x^6 p^6 - \frac{1}{240} x^6 p^5 \right.\\
&\left. + \frac{23}{720} x^6 p^4 + \frac{17}{240} x^6 p^3 - \frac{47}{360} x^6 p^2 - \frac{1}{6} x^6 p\right) y(0) + \left(x - \frac{1}{6} p^2 x^3 - \frac{1}{6} p\, x^3 + \frac{1}{3} x^3 \right.\\
&\left. + \frac{1}{120} x^5 p^4 + \frac{1}{60} x^5 p^3 - \frac{13}{120} x^5 p^2 - \frac{7}{60} x^5 p + \frac{1}{5} x^5\right) y'(0) + O(x^6)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {}& \left(1 + \left(-\frac{1}{2} p^2 - \frac{1}{2} p\right) x^2 + \left(-\frac{5}{24} p^2 - \frac{1}{4} p + \frac{1}{24} p^4 + \frac{1}{12} p^3\right) x^4\right) c_1 \\
&+ \left(x + \left(-\frac{1}{6} p^2 - \frac{1}{6} p + \frac{1}{3}\right) x^3 + \left(-\frac{13}{120} p^2 - \frac{7}{60} p + \frac{1}{5} + \frac{1}{120} p^4 + \frac{1}{60} p^3\right) x^5\right) c_2 \\
&+ O(x^6)
\end{aligned} \tag{2}
$$

Verification of solutions

$$y = \left(1-\frac{1}{2}x^2p^2-\frac{1}{2}x^2p+\frac{1}{24}p^4x^4+\frac{1}{12}p^3x^4-\frac{5}{24}p^2x^4-\frac{1}{4}p\,x^4-\frac{1}{720}x^6p^6-\frac{1}{240}x^6p^5 +\frac{23}{720}x^6p^4+\frac{17}{240}x^6p^3-\frac{47}{360}x^6p^2-\frac{1}{6}x^6p\right)y(0) + \left(x-\frac{1}{6}p^2x^3-\frac{1}{6}p\,x^3+\frac{1}{3}x^3+\frac{1}{120}x^5p^4+\frac{1}{60}x^5p^3-\frac{13}{120}x^5p^2-\frac{7}{60}x^5p+\frac{1}{5}x^5\right)y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1+\left(-\frac{1}{2}p^2-\frac{1}{2}p\right)x^2+\left(-\frac{5}{24}p^2-\frac{1}{4}p+\frac{1}{24}p^4+\frac{1}{12}p^3\right)x^4\right)c_1 + \left(x+\left(-\frac{1}{6}p^2-\frac{1}{6}p+\frac{1}{3}\right)x^3+\left(-\frac{13}{120}p^2-\frac{7}{60}p+\frac{1}{5}+\frac{1}{120}p^4+\frac{1}{60}p^3\right)x^5\right)c_2+O(x^6)$$

Verified OK.

16.9.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-2xy'+(p^2+p)\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{p(p+1)y}{x^2-1}-\frac{2xy'}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2xy'}{x^2-1}-\frac{p(p+1)y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2x}{x^2-1}, P_3(x)=-\frac{p(p+1)}{x^2-1}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

    $\left.((x+1)\cdot P_2(x))\right|_{x=-1}=1$

  - $(x+1)^2\cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $y''(x^2-1)+2xy'-p(p+1)\,y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2u-2)\left(\frac{d}{du}y(u)\right)+(-p^2-p)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty} a_k u^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m=0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$

- ◦ Shift index using $k->k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right)=\sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m=1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$

- ◦ Shift index using $k->k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)=\sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0r^2u^{-1+r}+\left(\sum_{k=0}^{\infty}\left(-2a_{k+1}(k+1+r)^2+a_k(r+1+p+k)\,(r-p+k)\right)u^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r^2=0$

- Values of r that satisfy the indicial equation

$r = 0$

- Each term in the series must be 0, giving the recursion relation

  $-2a_{k+1}(k+1)^2 + a_k(1+p+k)(-p+k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}$

- Solution for $r = 0$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}\right]$

- Revert the change of variables $u = x + 1$

  $\left[y = \sum_{k=0}^{\infty} a_k(x+1)^k, a_{k+1} = \frac{a_k(1+p+k)(-p+k)}{2(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Legendre successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 101

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+p*(p+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{p(p+1)\, x^2}{2} + \frac{p(p^3+2p^2-5p-6)\, x^4}{24}\right) y(0) + \left(x - \frac{(p^2+p-2)\, x^3}{6} + \frac{(p^4+2p^3-13p^2-14p+24)\, x^5}{120}\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 120

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-2*x*y'[x]+p*(p+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{1}{120}(p^2+p)^2 x^5 + \frac{7}{60}(-p^2-p)\, x^5 + \frac{1}{6}(-p^2-p)\, x^3 + \frac{x^5}{5} + \frac{x^3}{3} + x\right) + c_1\left(\frac{1}{24}(p^2+p)^2 x^4 + \frac{1}{4}(-p^2-p)\, x^4 + \frac{1}{2}(-p^2-p)\, x^2 + 1\right)$$

16.10 problem 17

Internal problem ID [5458]
Internal file name [`OUTPUT/4949_Wednesday_February_14_2024_02_05_52_AM_71386219/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 25. Integration in series. Supplemetary problems. Page 205
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + x^2 y = x^2 + x + 1}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{483}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{484}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -x^2 y + x^2 + x + 1 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2yx - x^2 y' + 2x + 1 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - x^4 - x^3 - 4xy' - x^2 - 2y + 2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(x^4 - 6\right) y' + 8yx^3 - 8x^3 - 7x^2 - 6x \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -yx^6 + x^6 + x^5 + 12y'x^3 + x^4 + 30x^2 y - 30x^2 - 20x - 12
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= -2y(0) + 2 \\
F_3 &= -6y'(0) \\
F_4 &= -12
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O\left(x^6\right)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) + x^2 + x + 1 \tag{1}$$

Expanding $x^2 + x + 1$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} x^2 + x + 1 &= x^2 + x + 1 + \dots \\ &= x^2 + x + 1 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + x^2 \left(\sum_{n=0}^{\infty} a_n x^n\right) = x^2 + x + 1$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = x^2 + x + 1 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=0}^{\infty} x^{n+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^{n}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n}\right) = x^{2}+x+1 \tag{3}$$

$n = 0$ gives

$$\begin{aligned}(2a_{2})\, 1 &= 1\\ 2a_{2} &= 1\end{aligned}$$

Or

$$a_{2} = \frac{1}{2}$$

$n = 1$ gives

$$\begin{aligned}(6a_{3})\, x &= x\\ 6a_{3} &= 1\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$6a_{3} = 1$$

Or

$$a_{3} = \frac{1}{6}$$

For $2 \leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1) + a_{n-2})\, x^{n} = x^{2}+x+1 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\begin{aligned}(12a_{4} + a_{0})\, x^{2} &= x^{2}\\ 12a_{4} + a_{0} &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_{4} = \frac{1}{12} - \frac{a_{0}}{12}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 + a_1)\, x^3 &= 0\\ 20a_5 + a_1 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 + a_2)\, x^4 &= 0\\ 30a_6 + a_2 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{60}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 + a_3)\, x^5 &= 0\\ 42a_7 + a_3 &= 0\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{252}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n\\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{x^2}{2} + \frac{x^3}{6} + \left(\frac{1}{12} - \frac{a_0}{12}\right) x^4 - \frac{a_1 x^5}{20} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20}x^5\right) a_1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) y'(0) + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{60} + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20}x^5\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=1+x+x^2,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{x^4}{12}\right)y(0)+\left(x-\frac{1}{20}x^5\right)D(y)(0)+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{12}+O(x^6)$$

✓ Solution by Mathematica
Time used: 0.013 (sec). Leaf size: 49

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==1+x+x^2,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^5}{20}\right)+\frac{x^4}{12}+c_1\left(1-\frac{x^4}{12}\right)+\frac{x^3}{6}+\frac{x^2}{2}$$

17 Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218

17.1 problem 11 . . . 2446
17.2 problem 12 . . . 2459
17.3 problem 13 . . . 2473
17.4 problem 14 . . . 2485
17.5 problem 15 . . . 2496
17.6 problem 16 . . . 2507
17.7 problem 17 . . . 2523
17.8 problem 18 . . . 2536
17.9 problem 19 . . . 2550
17.10 problem 20 . . . 2565
17.11 problem 21 . . . 2577

17.1 problem 11

17.1.1 Maple step by step solution 2455

Internal problem ID [5459]
Internal file name [`OUTPUT/4950_Wednesday_February_14_2024_02_05_53_AM_19838076/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2(x^3+x^2)\,y''-\left(-3x^2+x\right)y'+y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+2x^2\right)y''+\left(3x^2-x\right)y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{3x-1}{2x\,(x+1)}\\ q(x)&=\frac{1}{2x^2\,(x+1)}\end{aligned}$$

Table 273: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x-1}{2x(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

$q(x) = \frac{1}{2x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2y''x^2(x+1) + (3x^2 - x)\, y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x+1) + (3x^2 - x)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) \\ + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-1}(n + r - 1)(n + r - 2) + 2a_n(n + r)(n + r - 1) \\ &+ 3a_{n-1}(n + r - 1) - a_n(n + r) + a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -a_{n-1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -a_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -1$$

Which for the root $r = 1$ becomes

$$a_1 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$a_2 = 1$$

Which for the root $r = 1$ becomes

$$a_2 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1

For $n = 3$, using the above recursive equation gives

$$a_3 = -1$$

Which for the root $r = 1$ becomes

$$a_3 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$a_4 = 1$$

Which for the root $r = 1$ becomes

$$a_4 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1

For $n = 5$, using the above recursive equation gives

$$a_5 = -1$$

Which for the root $r = 1$ becomes

$$a_5 = -1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	-1	-1
a_2	1	1
a_3	-1	-1
a_4	1	1
a_5	-1	-1

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - x + x^2 - x^3 + x^4 - x^5 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &2b_{n-1}(n + r - 1)(n + r - 2) + 2b_n(n + r)(n + r - 1) \\ &+ 3b_{n-1}(n + r - 1) - b_n(n + r) + b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -b_{n-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = -b_{n-1} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1

For $n = 3$, using the above recursive equation gives

$$b_3 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_3 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1

For $n = 4$, using the above recursive equation gives

$$b_4 = 1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1

For $n = 5$, using the above recursive equation gives

$$b_5 = -1$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_5 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	-1	-1
b_2	1	1
b_3	-1	-1
b_4	1	1
b_5	-1	-1

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - x + x^2 - x^3 + x^4 - x^5 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)+c_2\sqrt{x}\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)+c_2\sqrt{x}\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)+c_2\sqrt{x}\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)+c_2\sqrt{x}\left(1-x+x^2-x^3+x^4-x^5+O(x^6)\right)
$$

Verified OK.

17.1.1 Maple step by step solution

Let's solve

$2y''x^2(x+1)+(3x^2-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{y}{2x^2(x+1)}-\frac{(3x-1)y'}{2x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{(3x-1)y'}{2x(x+1)}+\frac{y}{2x^2(x+1)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{3x-1}{2x(x+1)}, P_3(x)=\frac{1}{2x^2(x+1)}\right]$

  - $(x+1)\cdot P_2(x)$ is analytic at $x=-1$

$$\left((x+1)\cdot P_2(x)\right)\Big|_{x=-1} = 2$$

- $(x+1)^2 \cdot P_3(x)$ is analytic at $x=-1$

$$\left((x+1)^2\cdot P_3(x)\right)\Big|_{x=-1} = 0$$

- $x=-1$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0 = -1$

- Multiply by denominators

$2y''x^2(x+1) + x(3x-1)\,y' + y = 0$

- Change variables using $x = u-1$ so that the regular singular point is at $u=0$

$$(2u^3-4u^2+2u)\left(\frac{d^2}{du^2}y(u)\right) + (3u^2-7u+4)\left(\frac{d}{du}y(u)\right) + y(u) = 0$$

- Assume series solution for $y(u)$

$$y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$2a_0r(1+r)\,u^{-1+r} + \left(2a_1(1+r)\,(2+r) - a_0(1+r)\,(-1+4r)\right)u^r + \left(\sum_{k=1}^{\infty} \left(2a_{k+1}(k+r+1)\,(k+\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$2r(1+r)=0$$

- Values of r that satisfy the indicial equation

$$r\in\{-1,0\}$$

- Each term must be 0

$$2a_1(1+r)(2+r)-a_0(1+r)(-1+4r)=0$$

- Each term in the series must be 0, giving the recursion relation

$$(-4a_k+2a_{k-1}+2a_{k+1})k^2+((-8a_k+4a_{k-1}+4a_{k+1})r-3a_k-3a_{k-1}+6a_{k+1})k+(-4a_k+2a$$

- Shift index using $k->k+1$

$$(-4a_{k+1}+2a_k+2a_{k+2})(k+1)^2+((-8a_{k+1}+4a_k+4a_{k+2})r-3a_{k+1}-3a_k+6a_{k+2})(k+1)+$$

- Recursion relation that defines series solution to ODE

$$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+4kra_k-8kra_{k+1}+2r^2a_k-4r^2a_{k+1}+ka_k-11ka_{k+1}+ra_k-11ra_{k+1}-6a_{k+1}}{2(k^2+2kr+r^2+5k+5r+6)}$$

- Recursion relation for $r=-1$

$$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)}$$

- Solution for $r=-1$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k-1},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+1)^{k-1},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0\right]$$

- Recursion relation for $r=0$

$$a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)}$$

- Solution for $r=0$

$$\left[y(u)=\sum_{k=0}^{\infty}a_ku^{k},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)},4a_1+a_0=0\right]$$

- Revert the change of variables $u=x+1$

$$\left[y=\sum_{k=0}^{\infty}a_k(x+1)^{k},a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}+ka_k-11ka_{k+1}-6a_{k+1}}{2(k^2+5k+6)},4a_1+a_0=0\right]$$

- Combine solutions and rename parameters

$$\left[y=\left(\sum_{k=0}^{\infty}a_k(x+1)^{k-1}\right)+\left(\sum_{k=0}^{\infty}b_k(x+1)^{k}\right),a_{k+2}=-\frac{2k^2a_k-4k^2a_{k+1}-3ka_k-3ka_{k+1}+a_k+a_{k+1}}{2(k^2+3k+2)},0=0,\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(2*(x^2+x^3)*diff(y(x),x$2)-(x-3*x^2)*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (-x^5 + x^4 - x^3 + x^2 - x + 1)\left(c_1\sqrt{x} + c_2 x\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 58

```
AsymptoticDSolveValue[2*(x^2+x^3)*y''[x]-(x-3*x^2)*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x(-x^5 + x^4 - x^3 + x^2 - x + 1) + c_2\sqrt{x}(-x^5 + x^4 - x^3 + x^2 - x + 1)$$

17.2 problem 12

17.2.1 Maple step by step solution 2469

Internal problem ID [5460]
Internal file name [OUTPUT/4951_Wednesday_February_14_2024_02_05_54_AM_44143317/index.tex]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4xy'' + 2(1-x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + (2-2x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-1}{2x}$$
$$q(x) = -\frac{1}{4x}$$

Table 275: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + (2 - 2x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 4\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) x \\ + (2 - 2x) \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)
+\left(\sum_{n=0}^{\infty} 2(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0 \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty}\left(-2x^{n+r}a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)
$$

$$
\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\left(\sum_{n=0}^{\infty} 4x^{n+r-1}a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)
+\left(\sum_{n=0}^{\infty} 2(n+r)\,a_n x^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right)=0 \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
4x^{n+r-1}a_n(n+r)(n+r-1)+2(n+r)\,a_n x^{n+r-1}=0
$$

When $n=0$ the above becomes

$$
4x^{-1+r}a_0 r(-1+r)+2ra_0x^{-1+r}=0
$$

Or

$$
\left(4x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0
$$

Since $a_0\neq 0$ then the above simplifies to

$$
\left(4r^2-2r\right)x^{-1+r}=0
$$

Since the above is true for all x then the indicial equation becomes

$$4r^2 - 2r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2 - 2r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, 0\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) - 2a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + 2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4 (1 + r) (2 + r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8 (1 + r) (2 + r) (3 + r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{105}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\,(1+r)\,(2+r)\,(3+r)\,(4+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{945}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32\,(1+r)\,(2+r)\,(3+r)\,(4+r)\,(5+r)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{10395}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+2r}$	$\frac{1}{3}$
a_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{15}$
a_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{105}$
a_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{945}$
a_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{10395}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
4b_n(n+r)(n+r-1) - 2b_{n-1}(n+r-1) + 2(n+r)b_n - b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = \frac{b_{n-1}}{2n+2r} \tag{4}
$$

Which for the root $r = 0$ becomes

$$
b_n = \frac{b_{n-1}}{2n} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2 + 2r}$$

Which for the root $r = 0$ becomes

$$b_1 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4 (1 + r) (2 + r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8 (1 + r) (2 + r) (3 + r)}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{48}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16 (1 + r) (2 + r) (3 + r) (4 + r)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{384}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32 (1 + r) (2 + r) (3 + r) (4 + r) (5 + r)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{3840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+2r}$	$\frac{1}{2}$
b_2	$\frac{1}{4(1+r)(2+r)}$	$\frac{1}{8}$
b_3	$\frac{1}{8(1+r)(2+r)(3+r)}$	$\frac{1}{48}$
b_4	$\frac{1}{16(1+r)(2+r)(3+r)(4+r)}$	$\frac{1}{384}$
b_5	$\frac{1}{32(1+r)(2+r)(3+r)(4+r)(5+r)}$	$\frac{1}{3840}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6)\right) \\
&\quad + c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{15} + \frac{x^3}{105} + \frac{x^4}{945} + \frac{x^5}{10395} + O(x^6)\right) \\
+ c_2 \left(1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\sqrt{x}\left(1+\frac{x}{3}+\frac{x^2}{15}+\frac{x^3}{105}+\frac{x^4}{945}+\frac{x^5}{10395}+O(x^6)\right) + c_2\left(1+\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}+O(x^6)\right)$$

Verified OK.

17.2.1 Maple step by step solution

Let's solve

$4y''x+(2-2x)\,y'-y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=\frac{y}{4x}+\frac{(x-1)y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{(x-1)y'}{2x}-\frac{y}{4x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=-\frac{x-1}{2x}, P_3(x)=-\frac{1}{4x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left(x\cdot P_2(x)\right)\Big|_{x=0}=\frac{1}{2}$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4y''x+(2-2x)\,y'-y=0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $x^m \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- Shift index using $k- > k+1-m$

  $x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- Convert $x \cdot y''$ to series expansion

  $x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

- Shift index using $k- > k+1$

  $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$2a_0 r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(2a_{k+1}(k+1+r)\,(2k+2r+1) - a_k(2k+2r+1)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(a_{k+1}(k+1+r) - \frac{a_k}{2}\right)\left(k+r+\frac{1}{2}\right) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{2(k+1+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{2(k+1)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{2(k+1)}\right]$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k}{2\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k}{2(k+1)}, b_{k+1} = \frac{b_k}{2\left(k+\frac{3}{2}\right)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form is not straightforward to achieve - returning special function solu
   <- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(4*x*diff(y(x),x$2)+2*(1-x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{1}{3}x+\frac{1}{15}x^2+\frac{1}{105}x^3+\frac{1}{945}x^4+\frac{1}{10395}x^5+\mathrm{O}\left(x^6\right)\right)$$
$$+c_2\left(1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{48}x^3+\frac{1}{384}x^4+\frac{1}{3840}x^5+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 85

```
AsymptoticDSolveValue[4*x*y''[x]+2*(1-x)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^5}{10395}+\frac{x^4}{945}+\frac{x^3}{105}+\frac{x^2}{15}+\frac{x}{3}+1\right)+c_2\left(\frac{x^5}{3840}+\frac{x^4}{384}+\frac{x^3}{48}+\frac{x^2}{8}+\frac{x}{2}+1\right)$$

17.3 problem 13

17.3.1 Maple step by step solution 2481

Internal problem ID [5461]
Internal file name [`OUTPUT/4952_Wednesday_February_14_2024_02_05_55_AM_88251085/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2 - 1}{2x^2}$$

Table 277: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) - a_n(n + r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2+4nr+2r^2-3n-3r+1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2+5r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
$$

Verified OK.

17.3.1 Maple step by step solution

Let's solve

$2x^2 y'' - x y' + (-x^2 + 1)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{(x^2-1)y}{2x^2} + \frac{y'}{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $$y'' - \frac{y'}{2x} - \frac{(x^2-1)y}{2x^2} = 0$$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $$\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{x^2-1}{2x^2}\right]$$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $$(x \cdot P_2(x))\bigg|_{x=0} = -\frac{1}{2}$$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$(x^2 \cdot P_3(x))\bigg|_{x=0} = \frac{1}{2}$$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $$x_0 = 0$$

- Multiply by denominators

  $$2x^2y'' - xy' + (-x^2+1)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

- □ Rewrite ODE with series expansions
  - Convert $x^m \cdot y$ to series expansion for $m = 0..2$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

  - Shift index using $k- >k - m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

  - Convert $x \cdot y'$ to series expansion

    $$x \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

  - Convert $x^2 \cdot y''$ to series expansion

    $$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-1+2r)(-1+r)x^r + a_1(1+2r)r\,x^{1+r} + \left(\sum_{k=2}^{\infty}(a_k(2k+2r-1)(k+r-1)-a_{k-2})\,x^{k+r}\right) =$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(1+2r)\,r=0$

- Solve for the dependent coefficient(s)

  $a_1=0$

- Each term in the series must be 0, giving the recursion relation

  $2(k+r-1)\left(k+r-\frac{1}{2}\right)a_k - a_{k-2}=0$

- Shift index using $k->k+2$

  $2(k+1+r)\left(k+\frac{3}{2}+r\right)a_{k+2}-a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(k+1+r)(2k+3+2r)}$

- Recursion relation for $r=1$

  $a_{k+2}=\frac{a_k}{(k+2)(2k+5)}$

- Solution for $r=1$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+1}, a_{k+2}=\frac{a_k}{(k+2)(2k+5)}, a_1=0\right]$

- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}$

- Solution for $r=\frac{1}{2}$

  $\left[y=\sum_{k=0}^{\infty}a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{a_k}{\left(k+\frac{3}{2}\right)(2k+4)}, a_1=0\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty}a_k x^{k+1}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=\frac{a_k}{(k+2)(2k+5)}, a_1=0, b_{k+2}=\frac{b_k}{\left(k+\frac{3}{2}\right)(2k+4)}, b_1=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 33

```
Order:=6;
dsolve(2*x^2*diff(y(x),x$2)-x*diff(y(x),x)+(1-x^2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right)+c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 48

```
AsymptoticDSolveValue[2*x^2*y''[x]-x*y'[x]+(1-x^2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1x\left(\frac{x^4}{360}+\frac{x^2}{10}+1\right)+c_2\sqrt{x}\left(\frac{x^4}{168}+\frac{x^2}{6}+1\right)$$

17.4 problem 14

17.4.1 Maple step by step solution 2492

Internal problem ID [5462]
Internal file name [`OUTPUT/4953_Wednesday_February_14_2024_02_05_55_AM_39957459/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[_Lienard]
```

$$\boxed{xy'' + y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = 1$$

Table 279: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+2)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(r+2)^2}$	$-\frac{1}{4}$	$\frac{2}{(r+2)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(r+2)^2(r+4)^2}$	$\frac{1}{64}$	$\frac{-4r-12}{(r+2)^3(r+4)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \ldots \\
&= \left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
& + c_2\left(\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + \frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)+c_2\left(\left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right)\ln(x)+\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)$$

Verified OK.

17.4.1 Maple step by step solution

Let's solve

$y''x + y' + yx = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + y = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = 1\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x + y' + yx = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $x \cdot y$ to series expansion

    $x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+1}$
  - Shift index using $k- >k-1$

    $x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$
  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + a_1 (1+r)^2 x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)^2 + a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term must be 0

  $a_1 (1+r)^2 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_{k-1} = 0$
- Shift index using $k- >k+1$

  $a_{k+2}(k+2)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 41

```
Order:=6;
dsolve(x*diff(y(x),x$2)+diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 60

```
AsymptoticDSolveValue[x*y''[x]+y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right) + c_2 \left(-\frac{3x^4}{128} + \frac{x^2}{4} + \left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right) \log(x)\right)$$

17.5 problem 15

17.5.1 Maple step by step solution 2503

Internal problem ID [5463]
Internal file name [`OUTPUT/4954_Wednesday_February_14_2024_02_05_56_AM_55353319/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - x y' + y(x^2+1) = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' - x y' + y(x^2+1) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^2+1}{x^2}$$

Table 281: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - x y' + y(x^2 + 1) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) - x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) (x^2 + 1) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as $[1, 1]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-2}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{(1+r)^2}$$

Which for the root $r = 1$ becomes

$$a_2 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 (r+3)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{64}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$
a_3	0	0
a_4	$\frac{1}{(1+r)^2(r+3)^2}$	$\frac{1}{64}$
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	$-\frac{1}{(1+r)^2}$	$-\frac{1}{4}$	$\frac{2}{(1+r)^3}$	$\frac{1}{4}$
b_3	0	0	0	0
b_4	$\frac{1}{(1+r)^2 (r+3)^2}$	$\frac{1}{64}$	$\frac{-4r-8}{(1+r)^3 (r+3)^3}$	$-\frac{3}{128}$
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right) \\
&\quad + c_2\left(x\left(1 - \frac{x^2}{4} + \frac{x^4}{64} + O(x^6)\right)\ln(x) + x\left(\frac{x^2}{4} - \frac{3x^4}{128} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\ &\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x\left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\ &\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x\left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y &= c_1 x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \\ &\quad + c_2 \left(x \left(1-\frac{x^2}{4}+\frac{x^4}{64}+O(x^6)\right) \ln(x) + x\left(\frac{x^2}{4}-\frac{3x^4}{128}+O(x^6)\right)\right) \end{aligned}$$

Verified OK.

17.5.1 Maple step by step solution

Let's solve

$x^2y''-xy'+y(x^2+1)=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{(x^2+1)y}{x^2}+\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{y'}{x}+\frac{(x^2+1)y}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = -\frac{1}{x}, P_3(x) = \frac{x^2+1}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\bigg|_{x=0} = -1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2y'' - xy' + y(x^2 + 1) = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k(k + r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

  $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k(k + r)\,(k + r - 1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1 + r)^2\, x^r + a_1 r^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k + r - 1)^2 + a_{k-2}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$(-1+r)^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 1$

- Each term must be 0

  $a_1 r^2 = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k (k+r-1)^2 + a_{k-2} = 0$

- Shift index using $k- > k+2$

  $a_{k+2} (k+1+r)^2 + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+1+r)^2}$

- Recursion relation for $r = 1$

  $a_{k+2} = -\frac{a_k}{(k+2)^2}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{a_k}{(k+2)^2}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 45

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+(x^2+1)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left((c_2 \ln(x) + c_1)\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 65

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+(x^2+1)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x\left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right) + c_2\left(x\left(\frac{x^2}{4} - \frac{3x^4}{128}\right) + x\left(\frac{x^4}{64} - \frac{x^2}{4} + 1\right)\log(x)\right)$$

17.6 problem 16

17.6.1 Maple step by step solution 2519

Internal problem ID [5464]
Internal file name [`OUTPUT/4955_Wednesday_February_14_2024_02_05_56_AM_11051350/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' - 2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' - 2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{1}{x}$$

Table 283: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' - 2y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x - 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) - 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) - 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-3+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-3+r) = 0$$

Solving for r gives the roots of the indicial equation as $[3, 0]$.

Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^{3}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - 2a_n(n+r) + a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 - 3n - 3r} \tag{4}
$$

Which for the root $r = 3$ becomes

$$
a_n = -\frac{a_{n-1}}{n(n+3)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 - r - 2}$$

Which for the root $r = 3$ becomes

$$a_1 = -\frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{r^4 - 5r^2 + 4}$$

Which for the root $r = 3$ becomes

$$a_2 = \frac{1}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)}$$

Which for the root $r = 3$ becomes

$$a_3 = -\frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^4 - 5r^2 + 4) r (r + 3) (r^2 + 5r + 4)}$$

Which for the root $r = 3$ becomes

$$a_4 = \frac{1}{20160}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r + 1)^2 (r - 2) (r + 2)^2 (-1 + r) r (r + 3) (r + 4) (r + 5)}$$

Which for the root $r = 3$ becomes

$$a_5 = -\frac{1}{806400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2-r-2}$	$-\frac{1}{4}$
a_2	$\frac{1}{r^4-5r^2+4}$	$\frac{1}{40}$
a_3	$-\frac{1}{(r^4-5r^2+4)r(r+3)}$	$-\frac{1}{720}$
a_4	$\frac{1}{(r^4-5r^2+4)r(r+3)(r^2+5r+4)}$	$\frac{1}{20160}$
a_5	$-\frac{1}{(r+1)^2(r-2)(r+2)^2(-1+r)r(r+3)(r+4)(r+5)}$	$-\frac{1}{806400}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^3\left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)} &= \lim_{r\to 0} -\frac{1}{(r^4 - 5r^2 + 4)\, r\, (r + 3)} \\
&= \mathit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $xy'' - 2y' + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x - 2Cy_1'(x)\ln(x) \\
&- \frac{2Cy_1(x)}{x} - 2\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1''(x)\,x + y_1(x) - 2y_1'(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x - \frac{2y_1(x)}{x}\right)C \\
&+ \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&- 2\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
\tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
y_1''(x)\,x + y_1(x) - 2y_1'(x) = 0
$$

Eq (7) simplifes to

$$
\begin{aligned}
&\left(\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)x-\frac{2y_1(x)}{x}\right)C \\
&+\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x \\
&-2\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned}
\tag{8}
$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)x-3\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} \\
&+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2-2\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x} \\
&=0
\end{aligned}
\tag{9}
$$

Since $r_1=3$ and $r_2=0$ then the above becomes

$$
\begin{aligned}
&\frac{\left(2\left(\sum_{n=0}^{\infty}x^{2+n}a_n(n+3)\right)x-3\left(\sum_{n=0}^{\infty}a_n x^{n+3}\right)\right)C}{x} \\
&+\frac{\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)x^2-2\left(\sum_{n=0}^{\infty}x^{n-1}b_n n\right)x+\left(\sum_{n=0}^{\infty}b_n x^n\right)x}{x}=0
\end{aligned}
\tag{10}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}2C\,x^{2+n}a_n(n+3)\right)+\sum_{n=0}^{\infty}\left(-3C\,x^{2+n}a_n\right) \\
&+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}\left(-2x^{n-1}b_n n\right)+\left(\sum_{n=0}^{\infty}b_n x^n\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{2+n} a_n(n+3) &= \sum_{n=3}^{\infty} 2C a_{n-3} n\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-3C\, x^{2+n} a_n\right) &= \sum_{n=3}^{\infty} \left(-3C a_{n-3} x^{n-1}\right) \\
\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\begin{aligned}
\left(\sum_{n=3}^{\infty} 2C a_{n-3} n\, x^{n-1}\right) + \sum_{n=3}^{\infty} \left(-3C a_{n-3} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n(n-1)\right) \\
+ \sum_{n=0}^{\infty} \left(-2x^{n-1} b_n n\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n=1$, Eq (2B) gives

$$-2b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = \frac{1}{2}$$

For $n=2$, Eq (2B) gives

$$-2b_2 + b_1 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2 + \frac{1}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{4}$$

For $n = N$, where $N = 3$ which is the difference between the two roots, we are free to choose $b_3 = 0$. Hence for $n = 3$, Eq (2B) gives

$$3C + \frac{1}{4} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{12}$$

For $n = 4$, Eq (2B) gives

$$5Ca_1 + b_3 + 4b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$4b_4 + \frac{5}{48} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{192}$$

For $n = 5$, Eq (2B) gives

$$7Ca_2 + b_4 + 10b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$10b_5 - \frac{13}{320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{13}{3200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{12}$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = &-\frac{1}{12}\left(x^3\left(1 - \frac{x}{4} + \frac{x^2}{40} - \frac{x^3}{720} + \frac{x^4}{20160} - \frac{x^5}{806400} + O(x^6)\right)\right)\ln(x) \\ &+ 1 + \frac{x}{2} + \frac{x^2}{4} - \frac{5x^4}{192} + \frac{13x^5}{3200} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^3 \left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{1}{12}\left(x^3\left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right)\right)\ln(x)+1 \right. \\
&\quad \left. + \frac{x}{2}+\frac{x^2}{4}-\frac{5x^4}{192}+\frac{13x^5}{3200}+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^3 \left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^3\left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right)\ln(x)}{12}+1+\frac{x}{2}+\frac{x^2}{4}-\frac{5x^4}{192} \right. \\
&\quad \left. + \frac{13x^5}{3200}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^3 \left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^3\left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right)\ln(x)}{12}+1+\frac{x}{2}+\frac{x^2}{4} \right. \\
&\quad \left. - \frac{5x^4}{192}+\frac{13x^5}{3200}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^3 \left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right) + c_2 \left(-\frac{x^3 \left(1-\frac{x}{4}+\frac{x^2}{40}-\frac{x^3}{720}+\frac{x^4}{20160}-\frac{x^5}{806400}+O(x^6)\right) \ln(x)}{12}+1+\frac{x}{2}+\frac{x^2}{4}-\frac{5x^4}{192} + \frac{13x^5}{3200}+O(x^6)\right)
$$

Verified OK.

17.6.1 Maple step by step solution

Let's solve

$y''x - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x} + \frac{2y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{y}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = -\frac{2}{x}, P_3(x) = \frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y''x - 2y' + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert y' to series expansion

    $y' = \sum_{k=0}^{\infty} a_k (k+r) x^{k+r-1}$

  - Shift index using $k- > k+1$

    $y' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) x^{k+r}$

  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r) (k+r-1) x^{k+r-1}$

  - Shift index using $k- > k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1} (k+1+r) (k+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-3+r) x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k-2+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-3+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 3\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k-2+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k-2+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)(k-2)}$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 2$

$$a_{k+1} = -\frac{a_k}{(k+1)(k-2)}$$

- Recursion relation for $r = 3$

$$a_{k+1} = -\frac{a_k}{(k+4)(k+1)}$$

- Solution for $r = 3$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+3}, a_{k+1} = -\frac{a_k}{(k+4)(k+1)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 56

```
Order:=6;
dsolve(x*diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x^3 \left(1 - \frac{1}{4}x + \frac{1}{40}x^2 - \frac{1}{720}x^3 + \frac{1}{20160}x^4 - \frac{1}{806400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x^3 + \frac{1}{4}x^4 - \frac{1}{40}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(12 + 6x + 3x^2 - \frac{5}{16}x^4 + \frac{39}{800}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 79

```
AsymptoticDSolveValue[x*y''[x]-2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{48}(x-4)x^3\log(x)+\frac{1}{576}\left(-19x^4+16x^3+144x^2+288x+576\right)\right)+c_2\left(\frac{x^7}{20160}-\frac{x^6}{720}+\frac{x^5}{40}-\frac{x^4}{4}+x^3\right)$$

17.7 problem 17

17.7.1 Maple step by step solution 2532

Internal problem ID [5465]
Internal file name [`OUTPUT/4956_Wednesday_February_14_2024_02_05_58_AM_87988338/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[_Lienard]`

$$\boxed{xy'' + 2y' + yx = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + yx = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 285: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + yx = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + 2(n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as $[0, -1]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= 0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} 0 &= \lim_{r\to -1} 0 \\
&= 0
\end{aligned}
$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$
b_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
b_n(n+r)(n+r-1) + 2(n+r)\, b_n + b_{n-2} = 0 \tag{4}
$$

Which for for the root $r = -1$ becomes

$$
b_n(n-1)(n-2) + 2(n-1)\, b_n + b_{n-2} = 0 \tag{4A}
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x} \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{x}
$$

Verified OK.

17.7.1 Maple step by step solution

Let's solve

$y''x+2y'+yx=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{x}-y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{x}+y=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{2}{x}, P_3(x)=1\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

$$\left(x^{2}\cdot P_{3}(x)\right)\bigg|_{x=0}=0$$

- $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $y''x+2y'+yx=0$

- Assume series solution for y

  $$y=\sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x\cdot y$ to series expansion

  $$x\cdot y=\sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k->k-1$

  $$x\cdot y=\sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert y' to series expansion

  $$y'=\sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$y'=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,x^{k+r}$$

- Convert $x\cdot y''$ to series expansion

  $$x\cdot y''=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x\cdot y''=\sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(1+r)\,x^{-1+r}+a_1(1+r)\,(2+r)\,x^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+r+1)\,(k+2+r)+a_{k-1}\right)x^{k+r}\right)=0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$r(1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$

- Each term must be 0

  $a_1(1+r)(2+r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+r+1)(k+2+r) + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)(k+3+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$

- Recursion relation for $r = -1$

  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$

- Solution for $r = -1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$

- Recursion relation for $r = 0$

  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
Order:=6;
dsolve(x*diff(y(x),x$2)+2*diff(y(x),x)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right)}{x}$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 42

```
AsymptoticDSolveValue[x*y''[x]+2*y'[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{24} - \frac{x}{2} + \frac{1}{x}\right) + c_2\left(\frac{x^4}{120} - \frac{x^2}{6} + 1\right)$$

17.8 problem 18

17.8.1 Maple step by step solution 2546

Internal problem ID [5466]
Internal file name [`OUTPUT/4957_Wednesday_February_14_2024_02_05_58_AM_3640080/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(x+1)\,y''+x(x+1)\,y'-y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+x^2\right)y''+\left(x^2+x\right)y'-y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=\frac{1}{x}\\q(x)&=-\frac{1}{x^2\,(x+1)}\end{aligned}$$

Table 287: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2(x+1)}$	
singularity	type
$x = -1$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -1, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x+1)\,y'' + \left(x^2 + x\right) y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x^2(x+1) + \left(x^2+x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{gathered}
\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
+\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{gathered} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r} \\
\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) &= \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{gathered}
\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) \\
+\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right)=0
\end{gathered} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
x^{n+r} a_n(n+r)(n+r-1)+x^{n+r} a_n(n+r)-a_n x^{n+r}=0
$$

When $n=0$ the above becomes

$$
x^r a_0 r(-1+r)+x^r a_0 r-a_0 x^r=0
$$

Or

$$
\left(x^r r(-1+r)+x^r r-x^r\right) a_0=0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(r^2-1\right) x^r=0
$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $[1, -1]$.

Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n + r - 1)(n + r - 2) + a_n(n + r)(n + r - 1) \\ &+ a_{n-1}(n + r - 1) + a_n(n + r) - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r-1)}{1+n+r} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{a_{n-1}n}{2+n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{r}{2+r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(r+1)\, r}{(2+r)\,(3+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{r(r+1)}{(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r(r+1)}{(4+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$
a_4	$\frac{r(r+1)}{(4+r)(5+r)}$	$\frac{1}{15}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{r(r+1)}{(5+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{r}{2+r}$	$-\frac{1}{3}$
a_2	$\frac{(r+1)r}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	$-\frac{1}{10}$
a_4	$\frac{r(r+1)}{(4+r)(5+r)}$	$\frac{1}{15}$
a_5	$-\frac{r(r+1)}{(5+r)(6+r)}$	$-\frac{1}{21}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{(r+1)\,r}{(2+r)\,(3+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} \frac{(r+1)\,r}{(2+r)\,(3+r)} &= \lim_{r \to -1} \frac{(r+1)\,r}{(2+r)\,(3+r)} \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
b_{n-1}(n+r-1)(n+r-2)+b_n(n+r)(n+r-1)+b_{n-1}(n+r-1)+b_n(n+r)-b_n=0 \quad (4)
$$

Which for for the root $r = -1$ becomes

$$
b_{n-1}(n-2)(n-3)+b_n(n-1)(n-2)+b_{n-1}(n-2)+b_n(n-1)-b_n=0 \quad (4\text{A})
$$

Solving for b_n from the recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}(n+r-1)}{1+n+r} \tag{5}
$$

Which for the root $r = -1$ becomes

$$
b_n = -\frac{b_{n-1}(n-2)}{n} \tag{6}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{r}{2+r}
$$

Which for the root $r = -1$ becomes

$$
b_1 = 1
$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{(r+1)\, r}{(2+r)\,(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{r(r+1)}{(3+r)\,(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r(r+1)}{(4+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0
b_4	$\frac{r(r+1)}{(4+r)(5+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{r(r+1)}{(5+r)(6+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{r}{2+r}$	1
b_2	$\frac{(r+1)r}{(2+r)(3+r)}$	0
b_3	$-\frac{r(r+1)}{(3+r)(4+r)}$	0
b_4	$\frac{r(r+1)}{(4+r)(5+r)}$	0
b_5	$-\frac{r(r+1)}{(5+r)(6+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) + \frac{c_2(1 + x + O(x^6))}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) + \frac{c_2(1 + x + O(x^6))}{x} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) + \frac{c_2(1 + x + O(x^6))}{x} \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 - \frac{x}{3} + \frac{x^2}{6} - \frac{x^3}{10} + \frac{x^4}{15} - \frac{x^5}{21} + O(x^6)\right) + \frac{c_2(1 + x + O(x^6))}{x}$$

Verified OK.

17.8.1 Maple step by step solution

Let's solve

$y'' x^2 (1 + x) + (x^2 + x)\, y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2(1+x)} - \frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2(1+x)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{1}{x}, P_3(x) = -\frac{1}{x^2(1+x)}\right]$
  - $(1+x) \cdot P_2(x)$ is analytic at $x = -1$

    $\left((1+x) \cdot P_2(x)\right)\bigg|_{x=-1} = 0$
  - $(1+x)^2 \cdot P_3(x)$ is analytic at $x = -1$

    $\left((1+x)^2 \cdot P_3(x)\right)\bigg|_{x=-1} = 0$
  - $x = -1$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -1$

- Multiply by denominators

  $y''x^2(1+x) + x(1+x)\, y' - y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^3 - 2u^2 + u)\left(\frac{d^2}{du^2}y(u)\right) + (u^2 - u)\left(\frac{d}{du}y(u)\right) - y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k->k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^{m}\cdot\left(\frac{d^{2}}{d u^{2}}y(u)\right)=\sum_{k=-2+m}^{\infty}a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$

  Rewrite ODE with series expansions

  $a_{0}r(-1+r)\,u^{-1+r}+\left(a_{1}(1+r)\,r-a_{0}(2r^{2}-r+1)\right)u^{r}+\left(\sum_{k=1}^{\infty}\left(a_{k+1}(k+1+r)\,(k+r)-a_{k}(2k^{2}\right.\right.$

- a_{0}cannot be 0 by assumption, giving the indicial equation

  $r(-1+r)=0$

- Values of r that satisfy the indicial equation

  $r\in\{0,1\}$

- Each term must be 0

  $a_{1}(1+r)\,r-a_{0}(2r^{2}-r+1)=0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)\,(k+r)-a_{k}(2k^{2}+4kr+2r^{2}-k-r+1)+a_{k-1}(k+r-1)^{2}=0$

- Shift index using $k- >k+1$

  $a_{k+2}(k+2+r)\,(k+1+r)-a_{k+1}\left(2(k+1)^{2}+4(k+1)\,r+2r^{2}-k-r\right)+a_{k}(k+r)^{2}=0$

- Recursion relation that defines series solution to ODE

  $a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}+2kra_{k}-4kra_{k+1}+r^{2}a_{k}-2r^{2}a_{k+1}-3ka_{k+1}-3ra_{k+1}-2a_{k+1}}{(k+2+r)(k+1+r)}$

- Recursion relation for $r=0$

  $a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-3ka_{k+1}-2a_{k+1}}{(k+2)(k+1)}$

- Solution for $r=0$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k},a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-3ka_{k+1}-2a_{k+1}}{(k+2)(k+1)},-a_{0}=0\right]$

- Revert the change of variables $u=1+x$

  $\left[y=\sum_{k=0}^{\infty}a_{k}(1+x)^{k},a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}-3ka_{k+1}-2a_{k+1}}{(k+2)(k+1)},-a_{0}=0\right]$

- Recursion relation for $r=1$

  $a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}+2ka_{k}-7ka_{k+1}+a_{k}-7a_{k+1}}{(k+3)(k+2)}$

- Solution for $r=1$

  $\left[y(u)=\sum_{k=0}^{\infty}a_{k}u^{k+1},a_{k+2}=-\frac{k^{2}a_{k}-2k^{2}a_{k+1}+2ka_{k}-7ka_{k+1}+a_{k}-7a_{k+1}}{(k+3)(k+2)},2a_{1}-2a_{0}=0\right]$

- Revert the change of variables $u = 1 + x$

$$\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+1}, a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} + 2k a_k - 7k a_{k+1} + a_k - 7a_{k+1}}{(k+3)(k+2)}, 2a_1 - 2a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^{k}\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - 2k^2 a_{k+1} - 3k a_{k+1} - 2a_{k+1}}{(k+2)(k+1)}, -a_0 = 0, b_{k+2}\right.$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*(x+1)*diff(y(x),x$2)+x*(x+1)*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{3}x + \frac{1}{6}x^2 - \frac{1}{10}x^3 + \frac{1}{15}x^4 - \frac{1}{21}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(-2 - 2x + \mathrm{O}\left(x^6\right))}{x}$$

✓ Solution by Mathematica
Time used: 0.021 (sec). Leaf size: 45

```
AsymptoticDSolveValue[x^2*(x+1)*y''[x]+x*(x+1)*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 \left(\frac{x^5}{15} - \frac{x^4}{10} + \frac{x^3}{6} - \frac{x^2}{3} + x\right) + c_1 \left(\frac{1}{x} + 1\right)$$

17.9 problem 19

Internal problem ID [5467]
Internal file name [`OUTPUT/4958_Wednesday_February_14_2024_02_05_59_AM_96706282/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2xy'' + y' - y = 1 + x}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2xy'' + y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{1}{2x}$$

Table 289: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2xy'' + y' - y = 1 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2xy'' + y' - y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\,x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\,x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= \frac{1}{2} \\
r_2 &= 0
\end{aligned}
$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$
\left(2x^{-1+m} m(-1+m) + m\,x^{-1+m}\right) c_0 = 1 + x
$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$
r\,x^{-1+r}(-1+2r) = 0
$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, 0\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}
$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 +}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+3r+1}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{22680}$
a_5	$\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + (n+r)b_n - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-1}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2r^2 + 3r + 1}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2184r^6 + 9072r^5 + 22449r^4 + 33642r^3 + 29531r^2 + 13698r + 2520}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^{10} + 880r^9 + 10560r^8 + 72600r^7 + 315546r^6 + 902055r^5 + 1708465r^4 + 2102375r^3 + 1594197r^2 -}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+3r+1}$	1
b_2	$\frac{1}{4r^4+20r^3+35r^2+25r+6}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520}$	$\frac{1}{2520}$
b_5	$\frac{1}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400}$	$\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \sqrt{x} \left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6)\right) \\
&\quad + c_2 \left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^{-1+m} m(-1+m) + m\, x^{-1+m}\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{array}{|l|}
\hline
a_1 = \frac{a_0}{2r^2+3r+1} \\
a_2 = \frac{a_0}{4r^4+20r^3+35r^2+25r+6} \\
a_3 = \frac{a_0}{8r^6+84r^5+350r^4+735r^3+812r^2+441r+90} \\
a_4 = \frac{a_0}{16r^8+288r^7+2184r^6+9072r^5+22449r^4+33642r^3+29531r^2+13698r+2520} \\
a_5 = \frac{a_0}{32r^{10}+880r^9+10560r^8+72600r^7+315546r^6+902055r^5+1708465r^4+2102375r^3+1594197r^2+664290r+113400} \\
\hline
\end{array}
$$

Since the $F = 1 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$
\left(2x^{-1+m} m(-1+m) + m\,x^{-1+m}\right) c_0 = 1
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= 1 \\
m &= 1
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+1}
\end{aligned}
$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$
\begin{array}{|l|}
\hline
c_0 = 1 \\
c_1 = \frac{1}{6} \\
c_2 = \frac{1}{90} \\
c_3 = \frac{1}{2520} \\
c_4 = \frac{1}{113400} \\
c_5 = \frac{1}{7484400} \\
\hline
\end{array}
$$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x\left(1+\frac{1}{6}x+\frac{1}{90}x^2+\frac{1}{2520}x^3+\frac{1}{113400}x^4+\frac{1}{7484400}x^5\right) \\
&= x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2520}x^4+\frac{1}{113400}x^5+\frac{1}{7484400}x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$
\left(2x^{-1+m} m(-1+m) + m\, x^{-1+m}\right) c_0 = x
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{6} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{6}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \frac{1}{6} \\ c_1 &= \frac{1}{90} \\ c_2 &= \frac{1}{2520} \\ c_3 &= \frac{1}{113400} \\ c_4 &= \frac{1}{7484400} \\ c_5 &= \frac{1}{681080400} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{6} + \frac{1}{90}x + \frac{1}{2520}x^2 + \frac{1}{113400}x^3 + \frac{1}{7484400}x^4 + \frac{1}{681080400}x^5\right) \\ &= \frac{1}{6}x^2 + \frac{1}{90}x^3 + \frac{1}{2520}x^4 + \frac{1}{113400}x^5 + \frac{1}{7484400}x^6 + \frac{1}{681080400}x^7 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + \frac{x^6}{3742200} + \frac{x^7}{681080400} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x + \frac{x^2}{3} + \frac{x^3}{45} + \frac{x^4}{1260} + \frac{x^5}{56700} + O(x^6) \\ &\quad + c_1\sqrt{x}\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6)\right) \\ &\quad + c_2\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & x+\frac{x^{2}}{3}+\frac{x^{3}}{45}+\frac{x^{4}}{1260}+\frac{x^{5}}{56700}+O\left(x^{6}\right)\\
& +c_{1}\sqrt{x}\left(1+\frac{x}{3}+\frac{x^{2}}{30}+\frac{x^{3}}{630}+\frac{x^{4}}{22680}+\frac{x^{5}}{1247400}+O\left(x^{6}\right)\right)\\
& +c_{2}\left(1+x+\frac{x^{2}}{6}+\frac{x^{3}}{90}+\frac{x^{4}}{2520}+\frac{x^{5}}{113400}+O\left(x^{6}\right)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & x+\frac{x^{2}}{3}+\frac{x^{3}}{45}+\frac{x^{4}}{1260}+\frac{x^{5}}{56700}+O\left(x^{6}\right)\\
& +c_{1}\sqrt{x}\left(1+\frac{x}{3}+\frac{x^{2}}{30}+\frac{x^{3}}{630}+\frac{x^{4}}{22680}+\frac{x^{5}}{1247400}+O\left(x^{6}\right)\right)\\
& +c_{2}\left(1+x+\frac{x^{2}}{6}+\frac{x^{3}}{90}+\frac{x^{4}}{2520}+\frac{x^{5}}{113400}+O\left(x^{6}\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 60

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)-y(x)=x+1,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1\sqrt{x}\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22680}x^4+\frac{1}{1247400}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2520}x^4+\frac{1}{113400}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + x\left(1+\frac{1}{3}x+\frac{1}{45}x^2+\frac{1}{1260}x^3+\frac{1}{56700}x^4+\mathrm{O}\left(x^5\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 246

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]-y[x]==x+1,y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(\frac{x^5}{113400}+\frac{x^4}{2520}+\frac{x^3}{90}+\frac{x^2}{6}+x+1\right) \\
& + c_2\sqrt{x}\left(\frac{x^5}{1247400}+\frac{x^4}{22680}+\frac{x^3}{630}+\frac{x^2}{30}+\frac{x}{3}+1\right)+\sqrt{x}\left(\frac{x^5}{1247400}+\frac{x^4}{22680}+\frac{x^3}{630}\right. \\
& \left.+\frac{x^2}{30}+\frac{x}{3}+1\right)\left(\frac{23x^{11/2}}{311850}+\frac{29x^{9/2}}{11340}+\frac{16x^{7/2}}{315}+\frac{7x^{5/2}}{15}+\frac{4x^{3/2}}{3}\right. \\
& \left.+2\sqrt{x}\right)+\left(\frac{x^5}{113400}+\frac{x^4}{2520}+\frac{x^3}{90}+\frac{x^2}{6}+x+1\right)\left(-\frac{x^6}{133650}-\frac{37x^5}{113400}-\frac{11x^4}{1260}-\frac{11x^3}{90}-\frac{2x^2}{3}-x\right)
\end{aligned}
$$

17.10 problem 20

Internal problem ID [5468]
Internal file name [`OUTPUT/4959_Wednesday_February_14_2024_02_06_00_AM_35920087/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2x^3y'' + x^2y' + y = 0}$$

With the expansion point for the power series method at $x = \infty$.

Since expansion is around ∞, then the independent variable x is replaced by $\frac{1}{t}$ and the expansion is made around $t = 0$ and after solving, the solution is changed back to x using $x = \frac{1}{t}$. Changing variables results in the new ode

$$-\frac{2\left(-\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 2t\left(\frac{d}{dt}y(t)\right)\right)}{t} - \frac{d}{dt}y(t) + y(t) = 0$$

The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2\left(\frac{d^2}{dt^2}y(t)\right)t + y(t) + 3\frac{d}{dt}y(t) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$
\begin{aligned}
p(t) &= \frac{3}{2t} \\
q(t) &= \frac{1}{2t}
\end{aligned}
$$

Table 290: Table $p(t), q(t)$ singularites.

$p(t) = \frac{3}{2t}$	
singularity	type
$t = 0$	"regular"

$q(t) = \frac{1}{2t}$	
singularity	type
$t = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
2\left(\frac{d^2}{dt^2}y(t)\right)t + y(t) + 3\frac{d}{dt}y(t) = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2}\right)t + \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) + 3\left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1}\right) = 0 \quad (1)
$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n t^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n t^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n t^{n+r} = \sum_{n=1}^{\infty} a_{n-1} t^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2t^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3(n+r) a_n t^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} t^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2t^{n+r-1} a_n (n+r)(n+r-1) + 3(n+r) a_n t^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2t^{-1+r} a_0 r(-1+r) + 3r a_0 t^{-1+r} = 0$$

Or

$$\left(2t^{-1+r} r(-1+r) + 3r\, t^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r\right) t^{-1+r} = 0$$

Since the above is true for all t then the indicial equation becomes

$$2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r\right) t^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as $\left[0, -\frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(t) = t^{r_1} \left(\sum_{n=0}^{\infty} a_n t^n\right)$$
$$y_2(t) = t^{r_2} \left(\sum_{n=0}^{\infty} b_n t^n\right)$$

Or

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^n$$
$$y_2(t) = \sum_{n=0}^{\infty} b_n t^{n-\frac{1}{2}}$$

We start by finding $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 3a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 87863}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+5r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$
a_5	$-\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$-\frac{1}{1247400}$

Using the above table, then the solution $y_1(t)$ is

$$
\begin{aligned}
y_1(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \dots \\
&= 1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6)
\end{aligned}
$$

Now the second solution $y_2(t)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + 3(n+r) b_n + b_{n-1} = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \tag{4}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$
b_n = -\frac{b_{n-1}}{n(2n-1)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1}{2r^2 + 5r + 3}
$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = -\frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 87863}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = -\frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+5r+3}$	-1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$-\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$-\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$
b_5	$-\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$-\frac{1}{113400}$

Using the above table, then the solution $y_2(t)$ is

$$
\begin{aligned}
y_2(t) &= 1\big(b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 \dots\big) \\
&= \frac{1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6)}{\sqrt{t}}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\
&= c_1 \left(1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6)\right) \\
&\quad + \frac{c_2 \left(1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6)\right)}{\sqrt{t}}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y(t) &= y_h \\
&= c_1 \left(1 - \frac{t}{3} + \frac{t^2}{30} - \frac{t^3}{630} + \frac{t^4}{22680} - \frac{t^5}{1247400} + O(t^6)\right) \\
&\quad + \frac{c_2 \left(1 - t + \frac{t^2}{6} - \frac{t^3}{90} + \frac{t^4}{2520} - \frac{t^5}{113400} + O(t^6)\right)}{\sqrt{t}}
\end{aligned}
$$

Replacing t by $\frac{1}{x}$ gives

$$
y = c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right)\right) + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \right.}{\sqrt{\frac{1}{x}}}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right)\right) \\
&\quad + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \frac{1}{113400x^5} + O\left(\frac{1}{x^6}\right)\right)}{\sqrt{\frac{1}{x}}}
\end{aligned} \tag{1}
$$

Verification of solutions

$$
y = c_1 \left(1 - \frac{1}{3x} + \frac{1}{30x^2} - \frac{1}{630x^3} + \frac{1}{22680x^4} - \frac{1}{1247400x^5} + O\left(\frac{1}{x^6}\right)\right) + \frac{c_2 \left(1 - \frac{1}{x} + \frac{1}{6x^2} - \frac{1}{90x^3} + \frac{1}{2520x^4} - \frac{1}{113400x^5} + O\left(\frac{1}{x^6}\right)\right)}{\sqrt{\frac{1}{x}}}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 117

```
Order:=6;
dsolve(2*x^3*diff(y(x),x$2)+x^2*diff(y(x),x)+y(x)=0,y(x),type='series',x=infinity);
```

$$y(x) = \left(1 - \frac{(x - \text{Infinity})^2}{4\,\text{Infinity}^3} + \frac{7(x - \text{Infinity})^3}{24\,\text{Infinity}^4} + \frac{(-59\,\text{Infinity} + 2)\,(x - \text{Infinity})^4}{192\,\text{Infinity}^6} + \frac{(605\,\text{Infinity} - 52)\,(x - \text{Infinity})^5}{1920\,\text{Infinity}^7}\right) y(\text{Infinity}) + \left(x - \text{Infinity} - \frac{(x - \text{Infinity})^2}{4\,\text{Infinity}} + \frac{(3\,\text{Infinity}^2 - 2\,\text{Infinity})\,(x - \text{Infinity})^3}{24\,\text{Infinity}^4} - \frac{5\left(\text{Infinity} - \frac{28}{15}\right)(x - \text{Infinity})^4}{64\,\text{Infinity}^4} + \frac{(105\,\text{Infinity}^3 - 370\,\text{Infinity}^2 + 4\,\text{Infinity})\,(x - \text{Infinity})^5}{1920\,\text{Infinity}^7}\right) D(y)\,(\text{Infinity}) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 96

```
AsymptoticDSolveValue[2*x^3*y''[x]+x^2*y'[x]+y[x]==0,y[x],{x,infinity,5}]
```

$$y(x) \to c_2\left(\frac{1}{6x^{3/2}} - \frac{1}{90x^{5/2}} + \frac{1}{2520x^{7/2}} - \frac{1}{113400x^{9/2}} + \sqrt{x} - \frac{1}{\sqrt{x}}\right) + c_1\left(-\frac{1}{1247400x^5} + \frac{1}{22680x^4} - \frac{1}{630x^3} + \frac{1}{30x^2} - \frac{1}{3x} + 1\right)$$

17.11 problem 21

Internal problem ID [5469]
Internal file name [`OUTPUT/4960_Wednesday_February_14_2024_02_06_01_AM_85208353/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 26. Integration in series (singular points). Supplemetary problems. Page 218
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^3y'' + (x^2 + x)\,y' - y = 0}$$

With the expansion point for the power series method at $x = \infty$.

Since expansion is around ∞, then the independent variable x is replaced by $\frac{1}{t}$ and the expansion is made around $t = 0$ and after solving, the solution is changed back to x using $x = \frac{1}{t}$. Changing variables results in the new ode

$$-\frac{-\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 2t\left(\frac{d}{dt}y(t)\right)}{t} - \left(\frac{1}{t^2} + \frac{1}{t}\right)\left(\frac{d}{dt}y(t)\right)t^2 - y(t) = 0$$

The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(\frac{d^2}{dt^2}y(t)\right)t + (-t + 1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{dt^2}y(t) + p(t)\frac{d}{dt}y(t) + q(t)y(t) = 0$$

Where

$$p(t) = -\frac{t-1}{t}$$
$$q(t) = -\frac{1}{t}$$

Table 291: Table $p(t), q(t)$ singularites.

$p(t) = -\frac{t-1}{t}$	
singularity	type
$t = 0$	"regular"

$q(t) = -\frac{1}{t}$	
singularity	type
$t = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $t = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$\left(\frac{d^2}{dt^2}y(t)\right)t + (-t+1)\left(\frac{d}{dt}y(t)\right) - y(t) = 0$$

Let the solution be represented as Frobenius power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+r}$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=0}^{\infty} (n+r)\, a_n t^{n+r-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n t^{n+r-2}$$

Substituting the above back into the ode gives

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n t^{n+r-2}\right) t \\
&+(-t+1)\left(\sum_{n=0}^{\infty}(n+r)\,a_n t^{n+r-1}\right)-\left(\sum_{n=0}^{\infty}a_n t^{n+r}\right)=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}t^{n+r-1}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-t^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=0}^{\infty}(n+r)\,a_n t^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-a_n t^{n+r}\right)=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of t be $n+r-1$ in each summation term. Going over each summation term above with power of t in it which is not already t^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(-t^{n+r}a_n(n+r)\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,t^{n+r-1}\right) \\
\sum_{n=0}^{\infty}\left(-a_n t^{n+r}\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1}t^{n+r-1}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of t are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty}t^{n+r-1}a_n(n+r)\,(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,t^{n+r-1}\right) \\
&+\left(\sum_{n=0}^{\infty}(n+r)\,a_n t^{n+r-1}\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}t^{n+r-1}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
t^{n+r-1}a_n(n+r)\,(n+r-1)+(n+r)\,a_n t^{n+r-1}=0
$$

When $n = 0$ the above becomes

$$t^{-1+r}a_0r(-1+r) + ra_0t^{-1+r} = 0$$

Or

$$\left(t^{-1+r}r(-1+r) + r\,t^{-1+r}\right)a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$t^{-1+r}r^2 = 0$$

Since the above is true for all t then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$t^{-1+r}r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(t) = \sum_{n=0}^{\infty} a_n t^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(t) = y_1(t)\ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(t)$. Eq (2B) derived above is now used to find all a_n coefficients. The case

$n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{1+r}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(3+r)(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(3+r)(1+r)(2+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(3+r)(1+r)(2+r)(4+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{1+r}$	1
a_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$
a_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$
a_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{24}$
a_5	$\frac{1}{(3+r)(1+r)(2+r)(4+r)(5+r)}$	$\frac{1}{120}$

Using the above table, then the first solution $y_1(t)$ becomes

$$\begin{aligned} y_1(t) &= a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 \dots \\ &= t + 1 + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + O(t^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(t) = y_1(t) \ln(t) + \left(\sum_{n=1}^{\infty} b_n t^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{1+r}$	1	$-\frac{1}{(1+r)^2}$	-1
b_2	$\frac{1}{(1+r)(2+r)}$	$\frac{1}{2}$	$\frac{-3-2r}{(1+r)^2(2+r)^2}$	$-\frac{3}{4}$
b_3	$\frac{1}{(3+r)(1+r)(2+r)}$	$\frac{1}{6}$	$\frac{-3r^2-12r-11}{(3+r)^2(1+r)^2(2+r)^2}$	$-\frac{11}{36}$
b_4	$\frac{1}{(3+r)(1+r)(2+r)(4+r)}$	$\frac{1}{24}$	$\frac{-4r^3-30r^2-70r-50}{(3+r)^2(1+r)^2(2+r)^2(4+r)^2}$	$-\frac{25}{288}$
b_5	$\frac{1}{(3+r)(1+r)(2+r)(4+r)(5+r)}$	$\frac{1}{120}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{(3+r)^2(1+r)^2(2+r)^2(4+r)^2(5+r)^2}$	$-\frac{137}{7200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}y_2(t) &= y_1(t)\ln(t) + b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 \dots \\ &= \left(t+1+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+O(t^6)\right)\ln(t) - t - \frac{3t^2}{4} - \frac{11t^3}{36} - \frac{25t^4}{288} - \frac{137t^5}{7200} + O(t^6)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(t) &= c_1 y_1(t) + c_2 y_2(t) \\ &= c_1\left(t+1+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+O(t^6)\right) \\ &\quad + c_2\left(\left(t+1+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+O(t^6)\right)\ln(t) - t - \frac{3t^2}{4} - \frac{11t^3}{36} - \frac{25t^4}{288} \right. \\ &\quad \left. - \frac{137t^5}{7200} + O(t^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y(t) &= y_h \\ &= c_1\left(t+1+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+O(t^6)\right) \\ &\quad + c_2\left(\left(t+1+\frac{t^2}{2}+\frac{t^3}{6}+\frac{t^4}{24}+\frac{t^5}{120}+O(t^6)\right)\ln(t) - t - \frac{3t^2}{4} \right. \\ &\quad \left. - \frac{11t^3}{36} - \frac{25t^4}{288} - \frac{137t^5}{7200} + O(t^6)\right)\end{aligned}$$

Replacing t by $\frac{1}{x}$ gives

$$y = c_1\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right)\right) + c_2\left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} +\right.\right.$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & c_1\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right)\right) \\ & + c_2\left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right)\right)\ln\left(\frac{1}{x}\right) - \frac{1}{x} - \frac{3}{4x^2}\right. \\ & \left. - \frac{11}{36x^3} - \frac{25}{288x^4} - \frac{137}{7200x^5} + O\left(\frac{1}{x^6}\right)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & c_1\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right)\right) \\ & + c_2\left(\left(\frac{1}{x} + 1 + \frac{1}{2x^2} + \frac{1}{6x^3} + \frac{1}{24x^4} + \frac{1}{120x^5} + O\left(\frac{1}{x^6}\right)\right)\ln\left(\frac{1}{x}\right) - \frac{1}{x} - \frac{3}{4x^2}\right. \\ & \left. - \frac{11}{36x^3} - \frac{25}{288x^4} - \frac{137}{7200x^5} + O\left(\frac{1}{x^6}\right)\right) \end{aligned}$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 179

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+(x^2+x)*diff(y(x),x)-y(x)=0,y(x),type='series',x=infinity);
```

$$y(x) = \left(1 + \frac{(x-\text{Infinity})^2}{2\,\text{Infinity}^3} + \frac{(-4\,\text{Infinity} - 1)\,(x-\text{Infinity})^3}{6\,\text{Infinity}^5} + \frac{(18\,\text{Infinity}^2 + 10\,\text{Infinity} + 1)\,(x-\text{Infinity})^4}{24\,\text{Infinity}^7} + \frac{(-96\,\text{Infinity}^3 - 86\,\text{Infinity}^2 - 18\,\text{Infinity} - 1)\,(x-\text{Infinity})^5}{120\,\text{Infinity}^9}\right) y(\text{Infinity}) + \left(x - \text{Infinity} + \frac{(-\text{Infinity}^2 - \text{Infinity})\,(x-\text{Infinity})^2}{2\,\text{Infinity}^3} + \frac{(2\,\text{Infinity}^3 + 5\,\text{Infinity}^2 + \text{Infinity})\,(x-\text{Infinity})^3}{6\,\text{Infinity}^5} + \frac{(-6\,\text{Infinity}^4 - 26\,\text{Infinity}^3 - 11\,\text{Infinity}^2 - \text{Infinity})\,(x-\text{Infinity})^4}{24\,\text{Infinity}^7} + \frac{(24\,\text{Infinity}^5 + 154\,\text{Infinity}^4 + 102\,\text{Infinity}^3 + 19\,\text{Infinity}^2 + \text{Infinity})\,(x-\text{Infinity})^5}{120\,\text{Infinity}^9}\right) D(y)(\text{Infinity}) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 124

```
AsymptoticDSolveValue[x^3*y''[x]+(x^2+x)*y'[x]-y[x]==0,y[x],{x,infinity,5}]
```

$$y(x) \to c_1\left(\frac{1}{120x^5} + \frac{1}{24x^4} + \frac{1}{6x^3} + \frac{1}{2x^2} + \frac{1}{x} + 1\right) + c_2\left(-\frac{137}{7200x^5} - \frac{\log(x)}{120x^5} - \frac{25}{288x^4} - \frac{\log(x)}{24x^4} - \frac{11}{36x^3} - \frac{\log(x)}{6x^3} - \frac{3}{4x^2} - \frac{\log(x)}{2x^2} - \frac{1}{x} - \frac{\log(x)}{x} - \log(x)\right)$$

18 Chapter 27. The Legendre, Bessel and Gauss Equations. Supplemetary problems. Page 230

18.1 problem 20 2588

18.1 problem 20

18.1.1 Maple step by step solution 2595

Internal problem ID [5470]
Internal file name [`OUTPUT/4961_Wednesday_February_14_2024_02_06_02_AM_23127676/index.tex`]

Book: Schaums Outline. Theory and problems of Differential Equations, 1st edition. Frank Ayres. McGraw Hill 1952
Section: Chapter 27. The Legendre, Bessel and Gauss Equations. Supplemetary problems. Page 230
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{z'' + tz' + \left(t^2 - \frac{1}{9}\right) z = 0}$$

With the expansion point for the power series method at $t = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (497) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (498) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -zt^2 - tz' + \frac{z}{9} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial z} z' + \frac{\partial F_0}{\partial z'} F_0 \\
&= -\frac{8z'}{9} + \frac{(9t^3 - 19t) z}{9} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial z} z' + \frac{\partial F_1}{\partial z'} F_1 \\
&= z' t^3 + \frac{35zt^2}{9} - \frac{11tz'}{9} - \frac{179z}{81} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial z} z' + \frac{\partial F_2}{\partial z'} F_2 \\
&= \frac{(-81t^4 + 657t^2 - 278) z'}{81} - t\left(t^4 - \frac{4}{3}t^2 - \frac{619}{81}\right) z \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial z} z' + \frac{\partial F_3}{\partial z'} F_3 \\
&= \frac{(-291t^3 + 737t) z'}{27} + \left(t^6 - \frac{119}{9}t^4 + \frac{25}{3}t^2 + \frac{5293}{729}\right) z
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $z(0) = z(0)$ and $z'(0) = z'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{z(0)}{9} \\
F_1 &= -\frac{8z'(0)}{9} \\
F_2 &= -\frac{179z(0)}{81} \\
F_3 &= -\frac{278z'(0)}{81} \\
F_4 &= \frac{5293z(0)}{729}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$z = \left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4+\frac{5293}{524880}t^6\right)z(0)+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)z'(0)+O(t^6)$$

Since the expansion point $t=0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$z=\sum_{n=0}^{\infty} a_n t^n$$

Then

$$z' = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$z'' = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -\left(\sum_{n=0}^{\infty} a_n t^n\right)t^2 - t\left(\sum_{n=1}^{\infty} n a_n t^{n-1}\right)+\frac{\left(\sum_{n=0}^{\infty} a_n t^n\right)}{9} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right)+\left(\sum_{n=1}^{\infty} n\, t^n a_n\right)+\left(\sum_{n=0}^{\infty} t^{n+2} a_n\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n t^n}{9}\right)=0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

$$\sum_{n=0}^{\infty} t^{n+2} a_n = \sum_{n=2}^{\infty} a_{n-2} t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right)+\left(\sum_{n=1}^{\infty} n\, t^n a_n\right)+\left(\sum_{n=2}^{\infty} a_{n-2} t^n\right)+\sum_{n=0}^{\infty}\left(-\frac{a_n t^n}{9}\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 - \frac{a_0}{9} = 0$$

$$a_2 = \frac{a_0}{18}$$

$n = 1$ gives

$$6a_3 + \frac{8a_1}{9} = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{4a_1}{27}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + n a_n + a_{n-2} - \frac{a_n}{9} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{9n a_n - a_n + 9a_{n-2}}{9\,(n+2)\,(n+1)} \\ &= -\frac{(9n-1)\, a_n}{9\,(n+2)\,(n+1)} - \frac{a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + \frac{17a_2}{9} + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{179a_0}{1944}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + \frac{26a_3}{9} + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{139a_1}{4860}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + \frac{35a_4}{9} + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{5293a_0}{524880}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + \frac{44a_5}{9} + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{3149a_1}{459270}$$

And so on. Therefore the solution is

$$\begin{aligned} z &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$z = a_0 + a_1 t + \frac{1}{18} a_0 t^2 - \frac{4}{27} a_1 t^3 - \frac{179}{1944} a_0 t^4 - \frac{139}{4860} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4\right) a_0 + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$z = \left(1 + \frac{1}{18}t^2 - \frac{179}{1944}t^4\right) c_1 + \left(t - \frac{4}{27}t^3 - \frac{139}{4860}t^5\right) c_2 + O(t^6)$$

Summary

The solution(s) found are the following

$$z = \left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4+\frac{5293}{524880}t^6\right)z(0)+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)z'(0)+O\left(t^6\right) \tag{1}$$

$$z = \left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4\right)c_1+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)c_2+O\left(t^6\right) \tag{2}$$

Verification of solutions

$$z = \left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4+\frac{5293}{524880}t^6\right)z(0)+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)z'(0)+O\left(t^6\right)$$

Verified OK.

$$z = \left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4\right)c_1+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)c_2+O\left(t^6\right)$$

Verified OK.

18.1.1 Maple step by step solution

Let's solve

$z'' = -zt^2 - tz' + \frac{z}{9}$

- Highest derivative means the order of the ODE is 2

  z''

- Isolate 2nd derivative

  $z'' = \left(-t^2+\frac{1}{9}\right)z - tz'$

- Group terms with z on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $z'' + tz' + \left(t^2-\frac{1}{9}\right)z = 0$

- Multiply by denominators

  $9z'' + 9tz' + (9t^2-1)z = 0$

- Assume series solution for z

  $z = \sum_{k=0}^{\infty} a_k t^k$

□ Rewrite ODE with series expansions

  ◦ Convert $t^m \cdot z$ to series expansion for $m = 0..2$

$$t^{m} \cdot z = \sum_{k=\max(0,-m)}^{\infty} a_{k} t^{k+m}$$

- Shift index using $k- >k - m$

$$t^{m} \cdot z = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} t^{k}$$

- Convert $t \cdot z'$ to series expansion

$$t \cdot z' = \sum_{k=0}^{\infty} a_{k} k\, t^{k}$$

- Convert z'' to series expansion

$$z'' = \sum_{k=2}^{\infty} a_{k} k(k-1)\, t^{k-2}$$

- Shift index using $k- >k + 2$

$$z'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, t^{k}$$

Rewrite ODE with series expansions

$$18a_{2} - a_{0} + (54a_{3} + 8a_{1})\, t + \left(\sum_{k=2}^{\infty} \left(9a_{k+2}(k+2)\,(k+1) + a_{k}(9k-1) + 9a_{k-2}\right) t^{k}\right) = 0$$

- The coefficients of each power of t must be 0

$$[18a_{2} - a_{0} = 0, 54a_{3} + 8a_{1} = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_{2} = \frac{a_{0}}{18}, a_{3} = -\frac{4a_{1}}{27}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$9(k^{2} + 3k + 2)\, a_{k+2} + 9a_{k}k - a_{k} + 9a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$9\left((k+2)^{2} + 3k + 8\right) a_{k+4} + 9a_{k+2}(k+2) - a_{k+2} + 9a_{k} = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[z = \sum_{k=0}^{\infty} a_{k} t^{k}, a_{k+4} = -\frac{9ka_{k+2} + 9a_{k} + 17a_{k+2}}{9(k^{2} + 7k + 12)}, a_{2} = \frac{a_{0}}{18}, a_{3} = -\frac{4a_{1}}{27}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
   <- Kummer successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(z(t),t$2)+t*diff(z(t),t)+(t^2-1/9)*z(t)=0,z(t),type='series',t=0);
```

$$z(t)=\left(1+\frac{1}{18}t^2-\frac{179}{1944}t^4\right)z(0)+\left(t-\frac{4}{27}t^3-\frac{139}{4860}t^5\right)D(z)(0)+O(t^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[z''[t]+t*z'[t]+(t^2-1/9)*z[t]==0,z[t],{t,0,5}]
```

$$z(t)\to c_2\left(-\frac{139t^5}{4860}-\frac{4t^3}{27}+t\right)+c_1\left(-\frac{179t^4}{1944}+\frac{t^2}{18}+1\right)$$