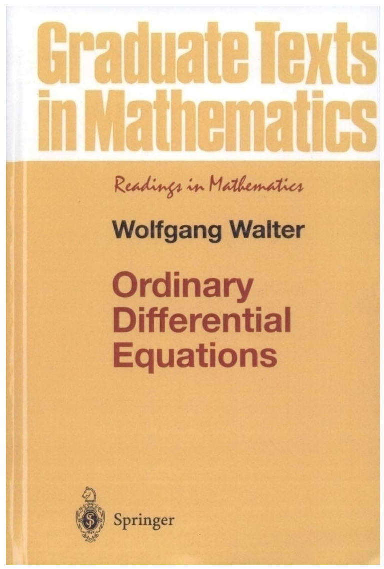
A Solution Manual For

Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998



Nasser M. Abbasi

December 3, 2025 Compiled on December 3, 2025 at 4:26am

Contents

1 Lookup tables for all problems in current book **5**

2 Book Solved Problems **15**

Chapter 1

Lookup tables for all problems in current book

Local contents

1.1 Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23 7
1.2 Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24 7
1.3 Chapter 1. First order equations: Some integrable cases. Excercises VI at page 33 8
1.4 Chapter 1. First order equations: Some integrable cases. Excercises VII at page 33 9
1.5 Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45 9
1.6 Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51 9
1.7 Chapter 2. Theory of First order differential equations. Excercises IV at page 89 10
1.8 Chapter 2. Theory of First order differential equations. Excercises XII at page 98 11
1.9 Chapter IV. Linear Differential Equations. Excercises IV at page 172 11
1.10 Chapter IV. Linear Differential Equations. Excercises V at page 173 11
1.11 Chapter IV. Linear Differential Equations. Excercise XII at page 189 12
1.12 Chapter IV. Linear Differential Equations. Excercise XIII at page 189 12
1.13 Chapter IV. Linear Differential Equations. Excercise VI at page 209 13
1.14 Chapter IV. Linear Differential Equations. Excercise VIII at page 210 13
1.15 Chapter V. Complex Linear Systems. Excercise VIII at page 221 13

1.16 Chapter V. Complex Linear Systems. Excercise XII at page 244 14

1.1 Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20962	(a)	$y' = \frac{1+y}{x+2} - e^{\frac{1+y}{x+2}}$	✓	✓	✓	✗
20963	(b)	$y' = \frac{1+y}{x+2} + e^{\frac{1+y}{x+2}}$	✓	✓	✓	✗
20964	(c)	$y' = \frac{x+y+1}{x+2} - e^{\frac{x+y+1}{x+2}}$	✓	✓	✓	✗
20965	(d)	$y' = \frac{x+2y+1}{2x+2+y}$	✓	✓	✓	✓
20966	(e)	$y' = \frac{2x+y+1}{x+2y+2}$	✓	✓	✓	✓

1.2 Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20967	(a)	$y' = 3y^{2/3}$	✓	✓	✓	✓
20968	(c)	$y' = \sqrt{y\,(1-y)}$	✓	✓	✓	✓
20969	(d)	$y' = \frac{e^{-y^2}}{y\,(x^2+2x)}$ $y(2) = 0$	✓	✓	✓	✓

Continued on next page

Table 1.2 Lookup table
Continued from previous page

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20970	(e)	$y' = \frac{y \ln(y)}{\sin(x)}$ $y\left(\frac{\pi}{2}\right) = e^{e}$	✓	✓	✓	✓
20971	(f)	$y' = \frac{\cos(x)}{\cos(y)^2}$ $y(\pi) = \frac{\pi}{4}$	✓	✓	✓	✓
20972	(g)	$y' = (x - y + 3)^2$	✓	✓	✓	✓
20973	(h)	$y' = \frac{2y(y-1)}{x(2-y)}$	✓	✓	✓	✓
20974	(i)	$y = y'x - \sqrt{x^2 + y^2}$	✓	✓	✓	✓

1.3 Chapter 1. First order equations: Some integrable cases. Excercises VI at page 33

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20975	(a)	$y' = f(x)\, y \ln\left(\frac{1}{y}\right)$	✓	✓	✓	✓

1.4 Chapter 1. First order equations: Some integrable cases. Excercises VII at page 33

Table 1.4: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20976	(a)	$y' - y + y^2 e^x + 5 e^{-x} = 0$ $y(0) = \eta$	✓	✓	✓	✓

1.5 Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45

Table 1.5: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20977	(a)	$\cos(x + y^2) + 3y + (2y\cos(x + y^2) + 3x)\, y' = 0$	✓	✓	✓	✗
20978	(b)	$xy^2 - y^3 + (1 - xy^2)\, y' = 0$	✓	✓	✓	✓
20979	(c)	$(yx + 1)\, y = y'x$	✓	✓	✓	✓
20980	(e)	$y' + p(x)\, y = q(x)$	✓	✓	✓	✓

1.6 Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51

Table 1.6: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20981	(a)	$y = y'x - \sqrt{y' - 1}$	✓	✓	✓	✗

Continued on next page

Table 1.6 Lookup table

Continued from previous page

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20982	(b)	$y = y'x + {y'}^2$	✓	✓	✓	✓
20983	(c)	$y = y'x + ay' + b$	✓	✓	✓	✓
20984	(d)	$y = {y'}^2 x + \ln\left({y'}^2\right)$	✓	✓	✓	✗
20985	(e)	$x = y\left(y' + \frac{1}{y'}\right) + {y'}^5$	✓	✓	✓	✗

1.7 Chapter 2. Theory of First order differential equations. Excercises IV at page 89

Table 1.7: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20986	(a)	$y' = e^x + x\cos(y)$ $y(0) = 0$ Series expansion around $x = 0$.	✓	✓	✓	✓
20987	(b.1)	$y' = x^3 + y^3$ $y(0) = 1$ Series expansion around $x = 0$.	✓	✓	✓	✓
20988	(b.2)	$u' = u^3$ $u(0) = 1$ Series expansion around $x = 0$.	✓	✓	✓	✓

1.8 Chapter 2. Theory of First order differential equations. Excercises XII at page 98

Table 1.8: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20989	(a)	$y' = x^3 + y^3$ $y(0) = 1$	✗	✗	✗	✗
20990	(b)	$y' = x + \sqrt{1 + y^2}$ $y(0) = 1$	✗	✗	✓	✗

1.9 Chapter IV. Linear Differential Equations. Excercises IV at page 172

Table 1.9: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20991	(a)	$x'(t) = x(t)\cos(t) - \sin(t)\,y(t)$ $y'(t) = \sin(t)\,x(t) + \cos(t)\,y(t)$	✗	✓	✓	✓

1.10 Chapter IV. Linear Differential Equations. Excercises V at page 173

Table 1.10: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20992	(a)	$x'(t) = (3t-1)\,x(t) - (-t+1)\,y(t) + t\,e^{t^2}$ $y'(t) = -(t+2)\,x(t) + (t-2)\,y(t) - e^{t^2}$	✗	✓	✓	✗

1.11 Chapter IV. Linear Differential Equations. Excercise XII at page 189

Table 1.11: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20993	(c)	$x'(t) = 2x(t) - 4y(t)$ $y'(t) = -x(t) + 2y(t)$	✓	✓	✓	✓

1.12 Chapter IV. Linear Differential Equations. Excercise XIII at page 189

Table 1.12: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20994	(a)	$x'(t) = 3x(t) + 6y(t)$ $y'(t) = -2x(t) - 3y(t)$	✓	✓	✓	✓
20995	(b)	$x'(t) = 8x(t) + y(t)$ $y'(t) = -4x(t) + 4y(t)$	✓	✓	✓	✓
20996	(c)	$x'(t) = x(t) - y(t) + 2z(t)$ $y'(t) = -x(t) + y(t) + 2z(t)$ $z'(t) = x(t) + y(t)$	✓	✓	✓	✓
20997	(d)	$x'(t) = -x(t) + y(t) - z(t)$ $y'(t) = 2x(t) - y(t) + 2z(t)$ $z'(t) = 2x(t) + 2y(t) - z(t)$	✓	✓	✓	✓

1.13 Chapter IV. Linear Differential Equations. Excercise VI at page 209

Table 1.13: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
20998	(a)	$y'' + 4y' + 4y = e^x$ $y(0) = 1$ $y'(0) = 0$	✓	✓	✓	✓
20999	(b)	$y'' - 2y' + 5y = e^x$ $y(0) = 1$ $y'(0) = 0$	✓	✓	✓	✓

1.14 Chapter IV. Linear Differential Equations. Excercise VIII at page 210

Table 1.14: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
21000	(a)	$u'' + 2au' + \omega^2 u = c\cos(\omega t)$	✓	✓	✓	✓

1.15 Chapter V. Complex Linear Systems. Excercise VIII at page 221

Table 1.15: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
21001	(a)	$w_1'(z) = w_2(z)$ $w_2'(z) = \frac{a w_1(z)}{z^2}$	✗	✓	✓	✗

1.16 Chapter V. Complex Linear Systems. Excercise XII at page 244

Table 1.16: Lookup table for all problems in current section

ID	problem	ODE	Solved?	Maple	Mma	Sympy
21002	(a)	$z^2 u'' + (3z + 1) u' + u = 0$	✓	✓	✓	✗

Chapter 2

Book Solved Problems

Local contents

2.1 Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23 16
2.2 Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24 106
2.3 Chapter 1. First order equations: Some integrable cases. Excercises VI at page 33 224
2.4 Chapter 1. First order equations: Some integrable cases. Excercises VII at page 33 233
2.5 Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45 248
2.6 Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51 285
2.7 Chapter 2. Theory of First order differential equations. Excercises IV at page 89 316
2.8 Chapter 2. Theory of First order differential equations. Excercises XII at page 98 333
2.9 Chapter IV. Linear Differential Equations. Excercises IV at page 172 340
2.10 Chapter IV. Linear Differential Equations. Excercises V at page 173 343
2.11 Chapter IV. Linear Differential Equations. Excercise XII at page 189 346
2.12 Chapter IV. Linear Differential Equations. Excercise XIII at page 189 355
2.13 Chapter IV. Linear Differential Equations. Excercise VI at page 209 394
2.14 Chapter IV. Linear Differential Equations. Excercise VIII at page 210 418
2.15 Chapter V. Complex Linear Systems. Excercise VIII at page 221 428
2.16 Chapter V. Complex Linear Systems. Excercise XII at page 244 431

2.1 Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23

Local contents

2.1.1 Problem (a) 17
2.1.2 Problem (b) 29
2.1.3 Problem (c) 41
2.1.4 Problem (d) 58
2.1.5 Problem (e) 84

2.1.1 Problem (a)

Local contents

Solved using first_order_ode_homog_type_maple_C 17
Solved using first_order_ode_LIE 20
✓Maple 27
✓Mathematica 28
✗Sympy 28

Internal problem ID [20962]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:09:30 AM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solved using first_order_ode_homog_type_maple_C

Time used: 0.928 (sec)

Solve

$$y' = \frac{y+1}{x+2} - e^{\frac{y+1}{x+2}}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{e^{\frac{Y(X)+y_0+1}{X+x_0+2}}(X+x_0) + 2\,e^{\frac{Y(X)+y_0+1}{X+x_0+2}} - Y(X) - y_0 - 1}{X+x_0+2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -2$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-e^{\frac{Y(X)}{X}} e^{-\frac{1}{X}} e^{\frac{1}{X}} X + Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{-e^{\frac{Y}{X}} e^{-\frac{1}{X}} e^{\frac{1}{X}} X + Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -e^{\frac{Y}{X}} e^{-\frac{1}{X}} e^{\frac{1}{X}} X + Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dX}X + u &= -e^{u} + u \\ \frac{du}{dX} &= -\frac{e^{u(X)}}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) + \frac{e^{u(X)}}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) X + e^{u(X)} = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{e^{u(X)}}{X} \tag{2.1}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{e^{u(X)}}{X} \\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X} \\ g(u) &= e^{u}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\ \int e^{-u}\, du &= \int -\frac{1}{X}\, dX\end{aligned}$$

$$-e^{-u(X)} = \ln\left(\frac{1}{X}\right) + c_1$$

Converting $-e^{-u(X)} = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$-e^{-\frac{Y(X)}{X}} = \ln\left(\frac{1}{X}\right) + c_1$$

Using the solution for $Y(X)$

$$-e^{-\frac{Y(X)}{X}} = \ln\left(\frac{1}{X}\right) + c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 1$$
$$X = x - 2$$

Then the solution in y becomes using EQ (A)

$$-e^{-\frac{y+1}{x+2}} = \ln\left(\frac{1}{x+2}\right) + c_1$$

Simplifying the above gives

$$-e^{\frac{-y-1}{x+2}} = \ln\left(\frac{1}{x+2}\right) + c_1$$

Solving for y gives

$$y = -\ln\left(\ln\left(x+2\right) - c_1\right) x - 2\ln\left(\ln\left(x+2\right) - c_1\right) - 1$$

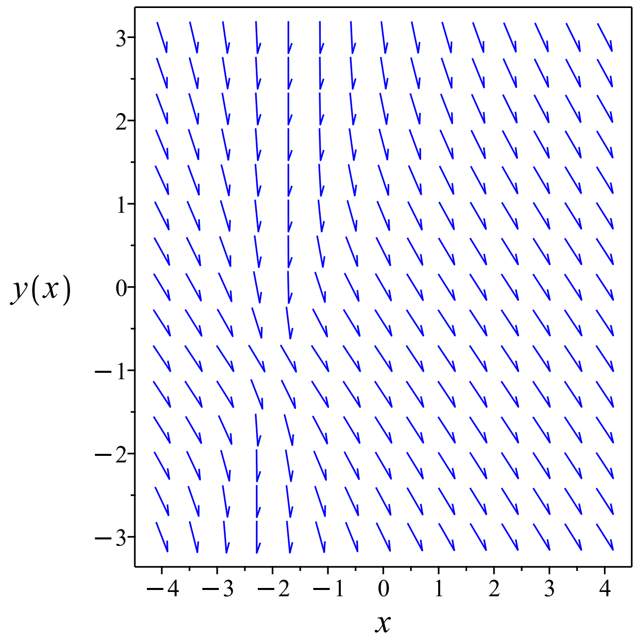


Figure 2.1: Slope field $y' = \frac{y+1}{x+2} - \mathrm{e}^{\frac{y+1}{x+2}}$

Summary of solutions found

$$y = -\ln\left(\ln\left(x+2\right) - c_1\right) x - 2\ln\left(\ln\left(x+2\right) - c_1\right) - 1$$

Solved using first_order_ode_LIE

Time used: 1.509 (sec)

Solve

$$y' = \frac{y+1}{x+2} - \mathrm{e}^{\frac{y+1}{x+2}}$$

Writing the ode as

$$y' = -\frac{\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} - y - 1}{x+2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{\left(e^{\frac{y+1}{x+2}} x + 2\,e^{\frac{y+1}{x+2}} - y - 1\right)(b_3 - a_2)}{x+2} - \frac{\left(e^{\frac{y+1}{x+2}} x + 2\,e^{\frac{y+1}{x+2}} - y - 1\right)^2 a_3}{(x+2)^2} \\
&- \left(-\frac{-\frac{(y+1)e^{\frac{y+1}{x+2}} x}{(x+2)^2} + e^{\frac{y+1}{x+2}} - \frac{2(y+1)e^{\frac{y+1}{x+2}}}{(x+2)^2}}{x+2} + \frac{e^{\frac{y+1}{x+2}} x + 2\,e^{\frac{y+1}{x+2}} - y - 1}{(x+2)^2}\right)(x a_2 \\
&+ y a_3 + a_1) + \frac{\left(\frac{e^{\frac{y+1}{x+2}} x}{x+2} + \frac{2\,e^{\frac{y+1}{x+2}}}{x+2} - 1\right)(x b_2 + y b_3 + b_1)}{x+2} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{e^{\frac{2y+2}{x+2}} x^2 a_3 + 4\,e^{\frac{2y+2}{x+2}} x a_3 + e^{\frac{y+1}{x+2}} x y a_2 - 2\,e^{\frac{y+1}{x+2}} x y a_3 - e^{\frac{y+1}{x+2}} x y b_3 - e^{\frac{y+1}{x+2}} x^2 a_2 + x b_1 - x b_3 - y a_1 + 2 y a_2 + e^{\frac{y+1}{x+2}} a}{}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-e^{\frac{2y+2}{x+2}} x^2 a_3 - 4\,e^{\frac{2y+2}{x+2}} x a_3 - e^{\frac{y+1}{x+2}} x y a_2 + 2\,e^{\frac{y+1}{x+2}} x y a_3 + e^{\frac{y+1}{x+2}} x y b_3 + e^{\frac{y+1}{x+2}} x^2 a_2 \\
&- x b_1 + x b_3 + y a_1 - 2 y a_2 - e^{\frac{y+1}{x+2}} a_1 + 4\,e^{\frac{y+1}{x+2}} a_2 + 4\,e^{\frac{y+1}{x+2}} a_3 + 2\,e^{\frac{y+1}{x+2}} b_1 \\
&- 4\,e^{\frac{y+1}{x+2}} b_3 + e^{\frac{y+1}{x+2}} x^2 b_2 - e^{\frac{y+1}{x+2}} x^2 b_3 - e^{\frac{y+1}{x+2}} y^2 a_3 + 3\,e^{\frac{y+1}{x+2}} x a_2 + 2\,e^{\frac{y+1}{x+2}} x a_3 \\
&+ e^{\frac{y+1}{x+2}} x b_1 + 2\,e^{\frac{y+1}{x+2}} x b_2 - 4\,e^{\frac{y+1}{x+2}} x b_3 - e^{\frac{y+1}{x+2}} y a_1 + 3\,e^{\frac{y+1}{x+2}} y a_3 + 2\,e^{\frac{y+1}{x+2}} y b_3 \\
&- 4\,e^{\frac{2y+2}{x+2}} a_3 - y a_3 + 2 x b_2 + a_1 - 2 a_2 - a_3 - 2 b_1 + 4 b_2 + 2 b_3 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-6\,e^{\frac{2y+2}{x+2}} x^2 a_3 - 12\,e^{\frac{2y+2}{x+2}} x a_3 - 2\,e^{\frac{y+1}{x+2}} x y a_2 + 7\,e^{\frac{y+1}{x+2}} x y a_3 + 4\,e^{\frac{y+1}{x+2}} x y b_3 \\
&- e^{\frac{y+1}{x+2}} x^2 y a_2 + 2\,e^{\frac{y+1}{x+2}} x^2 y a_3 + e^{\frac{y+1}{x+2}} x^2 y b_3 - e^{\frac{y+1}{x+2}} x\, y^2 a_3 - e^{\frac{y+1}{x+2}} x y a_1 + x y a_1 \\
&- 2 x y a_2 - x y a_3 + 5\,e^{\frac{y+1}{x+2}} x^2 a_2 - x^2 b_1 + 2 x^2 b_2 + x^2 b_3 + x a_1 - x a_3 - 4 x b_1 \\
&+ 4 x b_3 + 2 y a_1 - 4 y a_2 - 2\,e^{\frac{y+1}{x+2}} a_1 + 8\,e^{\frac{y+1}{x+2}} a_2 + 8\,e^{\frac{y+1}{x+2}} a_3 + 4\,e^{\frac{y+1}{x+2}} b_1 - 8\,e^{\frac{y+1}{x+2}} b_3 \\
&+ 4\,e^{\frac{y+1}{x+2}} x^2 b_2 - 6\,e^{\frac{y+1}{x+2}} x^2 b_3 - 2\,e^{\frac{y+1}{x+2}} y^2 a_3 + 10\,e^{\frac{y+1}{x+2}} x a_2 + 8\,e^{\frac{y+1}{x+2}} x a_3 + 4\,e^{\frac{y+1}{x+2}} x b_1 \\
&+ 4\,e^{\frac{y+1}{x+2}} x b_2 - 12\,e^{\frac{y+1}{x+2}} x b_3 - 2\,e^{\frac{y+1}{x+2}} y a_1 + 6\,e^{\frac{y+1}{x+2}} y a_3 + 4\,e^{\frac{y+1}{x+2}} y b_3 + e^{\frac{y+1}{x+2}} x^3 a_2 \\
&+ e^{\frac{y+1}{x+2}} x^3 b_2 - e^{\frac{y+1}{x+2}} x^3 b_3 + 2\,e^{\frac{y+1}{x+2}} x^2 a_3 + e^{\frac{y+1}{x+2}} x^2 b_1 - e^{\frac{y+1}{x+2}} x a_1 - e^{\frac{2y+2}{x+2}} x^3 a_3 \\
&- 8\,e^{\frac{2y+2}{x+2}} a_3 - 2 x a_2 - 2 y a_3 + 8 x b_2 + 2 a_1 - 4 a_2 - 2 a_3 - 4 b_1 + 8 b_2 + 4 b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, e^{\frac{y+1}{x+2}}, e^{\frac{2y+2}{x+2}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, e^{\frac{y+1}{x+2}} = v_3, e^{\frac{2y+2}{x+2}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&v_3v_1^3a_2 - v_3v_1^2v_2a_2 - v_4v_1^3a_3 + 2v_3v_1^2v_2a_3 - v_3v_1v_2^2a_3 + v_3v_1^3b_2 - v_3v_1^3b_3\\&+ v_3v_1^2v_2b_3 - v_3v_1v_2a_1 + 5v_3v_1^2a_2 - 2v_3v_1v_2a_2 + 2v_3v_1^2a_3 - 6v_4v_1^2a_3 + 7v_3v_1v_2a_3\\&- 2v_3v_2^2a_3 + v_3v_1^2b_1 + 4v_3v_1^2b_2 - 6v_3v_1^2b_3 + 4v_3v_1v_2b_3 + v_1v_2a_1 - v_3v_1a_1\\&- 2v_3v_2a_1 - 2v_1v_2a_2 + 10v_3v_1a_2 - v_1v_2a_3 + 8v_3v_1a_3 - 12v_4v_1a_3 + 6v_3v_2a_3\\&- v_1^2b_1 + 4v_3v_1b_1 + 2v_1^2b_2 + 4v_3v_1b_2 + v_1^2b_3 - 12v_3v_1b_3 + 4v_3v_2b_3 + v_1a_1 + 2v_2a_1\\&- 2v_3a_1 - 2v_1a_2 - 4v_2a_2 + 8v_3a_2 - v_1a_3 - 2v_2a_3 + 8v_3a_3 - 8v_4a_3 - 4v_1b_1\\&+ 4v_3b_1 + 8v_1b_2 + 4v_1b_3 - 8v_3b_3 + 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}&(a_2 + b_2 - b_3)\, v_1^3v_3 - v_4v_1^3a_3 + (-a_2 + 2a_3 + b_3)\, v_1^2v_2v_3\\&+ (5a_2 + 2a_3 + b_1 + 4b_2 - 6b_3)\, v_1^2v_3 - 6v_4v_1^2a_3 + (-b_1 + 2b_2 + b_3)\, v_1^2\\&- v_3v_1v_2^2a_3 + (-a_1 - 2a_2 + 7a_3 + 4b_3)\, v_1v_2v_3 + (a_1 - 2a_2 - a_3)\, v_1v_2\\&+ (-a_1 + 10a_2 + 8a_3 + 4b_1 + 4b_2 - 12b_3)\, v_1v_3 - 12v_4v_1a_3\\&+ (a_1 - 2a_2 - a_3 - 4b_1 + 8b_2 + 4b_3)\, v_1 - 2v_3v_2^2a_3 + (-2a_1 + 6a_3 + 4b_3)\, v_2v_3\\&+ (2a_1 - 4a_2 - 2a_3)\, v_2 + (-2a_1 + 8a_2 + 8a_3 + 4b_1 - 8b_3)\, v_3\\&- 8v_4a_3 + 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_3 &= 0 \\
-8a_3 &= 0 \\
-6a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-2a_1 + 6a_3 + 4b_3 &= 0 \\
a_1 - 2a_2 - a_3 &= 0 \\
2a_1 - 4a_2 - 2a_3 &= 0 \\
-a_2 + 2a_3 + b_3 &= 0 \\
a_2 + b_2 - b_3 &= 0 \\
-b_1 + 2b_2 + b_3 &= 0 \\
-a_1 - 2a_2 + 7a_3 + 4b_3 &= 0 \\
-2a_1 + 8a_2 + 8a_3 + 4b_1 - 8b_3 &= 0 \\
5a_2 + 2a_3 + b_1 + 4b_2 - 6b_3 &= 0 \\
-a_1 + 10a_2 + 8a_3 + 4b_1 + 4b_2 - 12b_3 &= 0 \\
a_1 - 2a_2 - a_3 - 4b_1 + 8b_2 + 4b_3 &= 0 \\
2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2b_3 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 2 \\
\eta &= y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 1 - \left(-\frac{e^{\frac{y+1}{x+2}} x + 2\, e^{\frac{y+1}{x+2}} - y - 1}{x+2}\right)(x+2) \\
&= (x+2)\, e^{\frac{y+1}{x+2}} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{(x+2)\, e^{\frac{y+1}{x+2}}} dy
\end{aligned}
$$

Which results in

$$
S = -e^{-\frac{y+1}{x+2}}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{e^{\frac{y+1}{x+2}} x + 2\, e^{\frac{y+1}{x+2}} - y - 1}{x+2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{(-y-1)\,\mathrm{e}^{\frac{-y-1}{x+2}}}{(x+2)^2} \\
S_y &= \frac{\mathrm{e}^{\frac{-y-1}{x+2}}}{x+2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x+2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R+2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

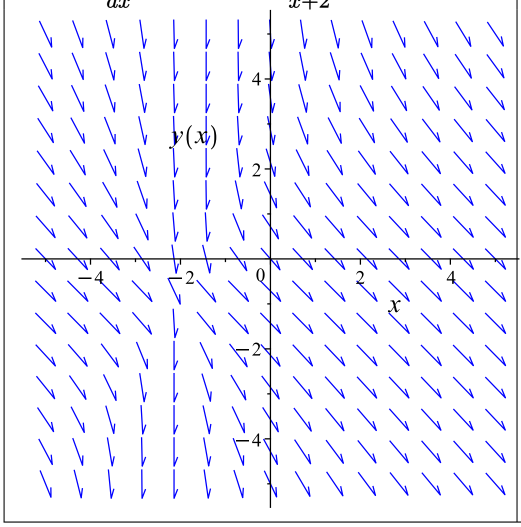
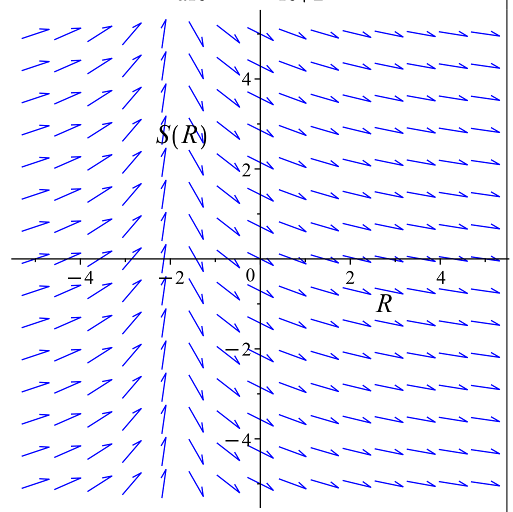
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{1}{R+2}\,dR \\
S(R) &= -\ln(R+2) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
-\mathrm{e}^{\frac{-y-1}{x+2}} = -\ln(x+2) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{\frac{y+1}{x+2}} x + 2 e^{\frac{y+1}{x+2}} - y - 1}{x+2}$ 	$R = x$ $S = -e^{\frac{-y-1}{x+2}}$	$\frac{dS}{dR} = -\frac{1}{R+2}$ 

Solving for y gives

$$y = -\ln(\ln(x+2) - c_2)\, x - 2\ln(\ln(x+2) - c_2) - 1$$

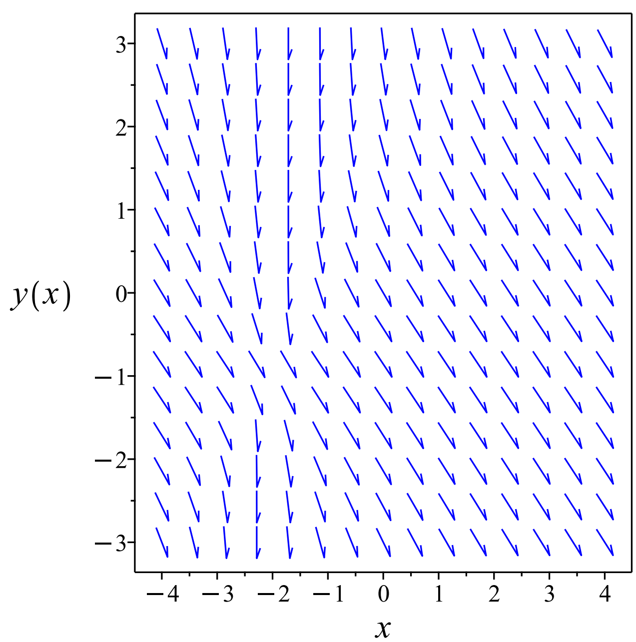


Figure 2.2: Slope field $y' = \frac{y+1}{x+2} - e^{\frac{y+1}{x+2}}$

Summary of solutions found

$$y = -\ln\left(\ln\left(x+2\right)-c_2\right)x - 2\ln\left(\ln\left(x+2\right)-c_2\right) - 1$$

✓ **Maple.** Time used: 0.007 (sec). Leaf size: 19

```
ode:=diff(y(x),x) = (1+y(x))/(x+2)-exp((1+y(x))/(x+2));
dsolve(ode,y(x), singsol=all);
```

$$y = -1 + (-x-2)\ln\left(\ln\left(x+2\right)+c_1\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \frac{y(x)+1}{x+2} - \mathrm{e}^{\frac{y(x)+1}{x+2}}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{y(x)+1}{x+2} - \mathrm{e}^{\frac{y(x)+1}{x+2}}$

✓ **Mathematica.** Time used: 1.335 (sec). Leaf size: 22

```
ode=D[y[x],x]==(y[x]+1)/(x+2)-Exp[(y[x]+1)/(x+2)];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -1 - ((x+2)\log(\log(x+2) - c_1))$$

✗ **Sympy**

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(exp((y(x) + 1)/(x + 2)) + Derivative(y(x), x) - (y(x) + 1)/(x + 2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : argument of type Mul is not iterable
```

2.1.2 Problem (b)

Local contents

Solved using first_order_ode_homog_type_maple_C 29
Solved using first_order_ode_LIE 32
✓Maple 39
✓Mathematica 40
✗Sympy 40

Internal problem ID [20963]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:10:19 AM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solved using first_order_ode_homog_type_maple_C

Time used: 0.871 (sec)

Solve

$$y' = \frac{y+1}{x+2} + \mathrm{e}^{\frac{y+1}{x+2}}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{\mathrm{e}^{\frac{Y(X)+y_0+1}{X+x_0+2}}(X+x_0) + 2\,\mathrm{e}^{\frac{Y(X)+y_0+1}{X+x_0+2}} + Y(X) + y_0 + 1}{X + x_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -2$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{\mathrm{e}^{\frac{Y(X)}{X}}\mathrm{e}^{-\frac{1}{X}}\mathrm{e}^{\frac{1}{X}}X + Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{\mathrm{e}^{\frac{Y}{X}}\mathrm{e}^{-\frac{1}{X}}\mathrm{e}^{\frac{1}{X}}X + Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = \mathrm{e}^{\frac{Y}{X}} \mathrm{e}^{-\frac{1}{X}} \mathrm{e}^{\frac{1}{X}} X + Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X} X + u = \mathrm{e}^u + u$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{\mathrm{e}^{u(X)}}{X}$$

Or

$$\frac{d}{dX} u(X) - \frac{\mathrm{e}^{u(X)}}{X} = 0$$

Or

$$\left(\frac{d}{dX} u(X)\right) X - \mathrm{e}^{u(X)} = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX} u(X) = \frac{\mathrm{e}^{u(X)}}{X} \tag{2.2}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX} u(X) &= \frac{\mathrm{e}^{u(X)}}{X} \\ &= f(X) g(u) \end{aligned}$$

Where

$$\begin{aligned} f(X) &= \frac{1}{X} \\ g(u) &= \mathrm{e}^u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(X)\, dX$$
$$\int \mathrm{e}^{-u}\, du = \int \frac{1}{X}\, dX$$

$$-\mathrm{e}^{-u(X)} = \ln(X) + c_1$$

Converting $-\mathrm{e}^{-u(X)} = \ln(X) + c_1$ back to $Y(X)$ gives

$$-\mathrm{e}^{-\frac{Y(X)}{X}} = \ln(X) + c_1$$

Using the solution for $Y(X)$

$$-\mathrm{e}^{-\frac{Y(X)}{X}} = \ln(X) + c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y - 1$$
$$X = x - 2$$

Then the solution in y becomes using EQ (A)

$$-\mathrm{e}^{-\frac{y+1}{x+2}} = \ln(x+2) + c_1$$

Simplifying the above gives

$$-\mathrm{e}^{\frac{-y-1}{x+2}} = \ln(x+2) + c_1$$

Solving for y gives

$$y = -\ln(-\ln(x+2) - c_1)\,x - 2\ln(-\ln(x+2) - c_1) - 1$$

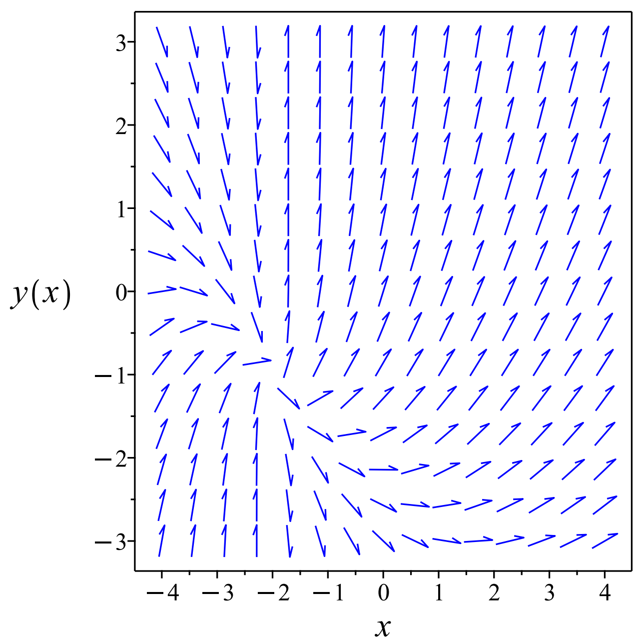


Figure 2.3: Slope field $y' = \frac{y+1}{x+2} + \mathrm{e}^{\frac{y+1}{x+2}}$

Summary of solutions found

$$y = -\ln\left(-\ln\left(x+2\right)-c_1\right)x - 2\ln\left(-\ln\left(x+2\right)-c_1\right) - 1$$

Solved using first_order_ode_LIE

Time used: 1.483 (sec)

Solve

$$y' = \frac{y+1}{x+2} + \mathrm{e}^{\frac{y+1}{x+2}}$$

Writing the ode as

$$\begin{aligned} y' &= \frac{\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1}{x+2} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{\left(\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1\right)(b_3 - a_2)}{x+2} - \frac{\left(\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1\right)^2 a_3}{(x+2)^2} \\ &- \left(\frac{-\frac{(y+1)\mathrm{e}^{\frac{y+1}{x+2}} x}{(x+2)^2} + \mathrm{e}^{\frac{y+1}{x+2}} - \frac{2(y+1)\mathrm{e}^{\frac{y+1}{x+2}}}{(x+2)^2}}{x+2} - \frac{\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1}{(x+2)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \frac{\left(\frac{\mathrm{e}^{\frac{y+1}{x+2}} x}{x+2} + \frac{2\,\mathrm{e}^{\frac{y+1}{x+2}}}{x+2} + 1\right)(xb_2 + yb_3 + b_1)}{x+2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-e^{\frac{y+1}{x+2}} x y a_2 + 2 e^{\frac{y+1}{x+2}} x y a_3 + e^{\frac{y+1}{x+2}} x y b_3 - a_1 + 2a_2 + a_3 + 2b_1 - 4b_2 - 2b_3 + e^{\frac{2y+2}{x+2}} x^2 a_3 + 4 e^{\frac{2y+2}{x+2}} x a_3 + e^{\frac{y+1}{x+2}} x^2 a}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & e^{\frac{y+1}{x+2}} x y a_2 - 2 e^{\frac{y+1}{x+2}} x y a_3 - e^{\frac{y+1}{x+2}} x y b_3 + a_1 - 2a_2 - a_3 - 2b_1 + 4b_2 + 2b_3 \\ & - e^{\frac{2y+2}{x+2}} x^2 a_3 - 4 e^{\frac{2y+2}{x+2}} x a_3 - e^{\frac{y+1}{x+2}} x^2 a_2 - e^{\frac{y+1}{x+2}} x^2 b_2 + e^{\frac{y+1}{x+2}} x^2 b_3 + e^{\frac{y+1}{x+2}} y^2 a_3 \\ & - 3 e^{\frac{y+1}{x+2}} x a_2 - 2 e^{\frac{y+1}{x+2}} x a_3 - e^{\frac{y+1}{x+2}} x b_1 - 2 e^{\frac{y+1}{x+2}} x b_2 + 4 e^{\frac{y+1}{x+2}} x b_3 + e^{\frac{y+1}{x+2}} y a_1 \\ & - 3 e^{\frac{y+1}{x+2}} y a_3 - 2 e^{\frac{y+1}{x+2}} y b_3 - y a_3 + 2 x b_2 - 4 e^{\frac{2y+2}{x+2}} a_3 - x b_1 + x b_3 + y a_1 \\ & - 2 y a_2 + e^{\frac{y+1}{x+2}} a_1 - 4 e^{\frac{y+1}{x+2}} a_2 - 4 e^{\frac{y+1}{x+2}} a_3 - 2 e^{\frac{y+1}{x+2}} b_1 + 4 e^{\frac{y+1}{x+2}} b_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & 2 e^{\frac{y+1}{x+2}} x y a_2 - 7 e^{\frac{y+1}{x+2}} x y a_3 - 4 e^{\frac{y+1}{x+2}} x y b_3 + e^{\frac{y+1}{x+2}} x^2 y a_2 - 2 e^{\frac{y+1}{x+2}} x^2 y a_3 \\ & - e^{\frac{y+1}{x+2}} x^2 y b_3 + e^{\frac{y+1}{x+2}} x\, y^2 a_3 + e^{\frac{y+1}{x+2}} x y a_1 - 2 e^{\frac{y+1}{x+2}} x^2 a_3 - e^{\frac{y+1}{x+2}} x^2 b_1 + e^{\frac{y+1}{x+2}} x a_1 \\ & - e^{\frac{y+1}{x+2}} x^3 a_2 - e^{\frac{y+1}{x+2}} x^3 b_2 + e^{\frac{y+1}{x+2}} x^3 b_3 + 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 \\ & - 6 e^{\frac{2y+2}{x+2}} x^2 a_3 - 12 e^{\frac{2y+2}{x+2}} x a_3 + x y a_1 - 2 x y a_2 - x y a_3 - 5 e^{\frac{y+1}{x+2}} x^2 a_2 - 4 e^{\frac{y+1}{x+2}} x^2 b_2 \\ & + 6 e^{\frac{y+1}{x+2}} x^2 b_3 + 2 e^{\frac{y+1}{x+2}} y^2 a_3 - 10 e^{\frac{y+1}{x+2}} x a_2 - 8 e^{\frac{y+1}{x+2}} x a_3 - 4 e^{\frac{y+1}{x+2}} x b_1 - 4 e^{\frac{y+1}{x+2}} x b_2 \\ & + 12 e^{\frac{y+1}{x+2}} x b_3 + 2 e^{\frac{y+1}{x+2}} y a_1 - 6 e^{\frac{y+1}{x+2}} y a_3 - 4 e^{\frac{y+1}{x+2}} y b_3 - e^{\frac{2y+2}{x+2}} x^3 a_3 - 2 x a_2 \\ & - 2 y a_3 + 8 x b_2 - 8 e^{\frac{2y+2}{x+2}} a_3 - x^2 b_1 + 2 x^2 b_2 + x^2 b_3 + x a_1 - x a_3 - 4 x b_1 + 4 x b_3 \\ & + 2 y a_1 - 4 y a_2 + 2 e^{\frac{y+1}{x+2}} a_1 - 8 e^{\frac{y+1}{x+2}} a_2 - 8 e^{\frac{y+1}{x+2}} a_3 - 4 e^{\frac{y+1}{x+2}} b_1 + 8 e^{\frac{y+1}{x+2}} b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, e^{\frac{y+1}{x+2}}, e^{\frac{2y+2}{x+2}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, e^{\frac{y+1}{x+2}} = v_3, e^{\frac{2y+2}{x+2}} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-v_3v_1^3a_2 + v_3v_1^2v_2a_2 - v_4v_1^3a_3 - 2v_3v_1^2v_2a_3 + v_3v_1v_2^2a_3 - v_3v_1^3b_2 + v_3v_1^3b_3 \\
&- v_3v_1^2v_2b_3 + v_3v_1v_2a_1 - 5v_3v_1^2a_2 + 2v_3v_1v_2a_2 - 2v_3v_1^2a_3 - 6v_4v_1^2a_3 - 7v_3v_1v_2a_3 \\
&+ 2v_3v_2^2a_3 - v_3v_1^2b_1 - 4v_3v_1^2b_2 + 6v_3v_1^2b_3 - 4v_3v_1v_2b_3 + v_1v_2a_1 + v_3v_1a_1 \\
&+ 2v_3v_2a_1 - 2v_1v_2a_2 - 10v_3v_1a_2 - v_1v_2a_3 - 8v_3v_1a_3 - 12v_4v_1a_3 - 6v_3v_2a_3 \\
&- v_1^2b_1 - 4v_3v_1b_1 + 2v_1^2b_2 - 4v_3v_1b_2 + v_1^2b_3 + 12v_3v_1b_3 - 4v_3v_2b_3 + v_1a_1 + 2v_2a_1 \\
&+ 2v_3a_1 - 2v_1a_2 - 4v_2a_2 - 8v_3a_2 - v_1a_3 - 2v_2a_3 - 8v_3a_3 - 8v_4a_3 - 4v_1b_1 \\
&- 4v_3b_1 + 8v_1b_2 + 4v_1b_3 + 8v_3b_3 + 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - b_2 + b_3)\, v_1^3v_3 - v_4v_1^3a_3 + (a_2 - 2a_3 - b_3)\, v_1^2v_2v_3 \\
&+ (-5a_2 - 2a_3 - b_1 - 4b_2 + 6b_3)\, v_1^2v_3 - 6v_4v_1^2a_3 + (-b_1 + 2b_2 + b_3)\, v_1^2 \\
&+ v_3v_1v_2^2a_3 + (a_1 + 2a_2 - 7a_3 - 4b_3)\, v_1v_2v_3 + (a_1 - 2a_2 - a_3)\, v_1v_2 \\
&+ (a_1 - 10a_2 - 8a_3 - 4b_1 - 4b_2 + 12b_3)\, v_1v_3 - 12v_4v_1a_3 \\
&+ (a_1 - 2a_2 - a_3 - 4b_1 + 8b_2 + 4b_3)\, v_1 + 2v_3v_2^2a_3 + (2a_1 - 6a_3 - 4b_3)\, v_2v_3 \\
&+ (2a_1 - 4a_2 - 2a_3)\, v_2 + (2a_1 - 8a_2 - 8a_3 - 4b_1 + 8b_3)\, v_3 \\
&- 8v_4a_3 + 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_3 &= 0\\
-12a_3 &= 0\\
-8a_3 &= 0\\
-6a_3 &= 0\\
-a_3 &= 0\\
2a_3 &= 0\\
a_1 - 2a_2 - a_3 &= 0\\
2a_1 - 4a_2 - 2a_3 &= 0\\
2a_1 - 6a_3 - 4b_3 &= 0\\
-a_2 - b_2 + b_3 &= 0\\
a_2 - 2a_3 - b_3 &= 0\\
-b_1 + 2b_2 + b_3 &= 0\\
a_1 + 2a_2 - 7a_3 - 4b_3 &= 0\\
2a_1 - 8a_2 - 8a_3 - 4b_1 + 8b_3 &= 0\\
-5a_2 - 2a_3 - b_1 - 4b_2 + 6b_3 &= 0\\
a_1 - 10a_2 - 8a_3 - 4b_1 - 4b_2 + 12b_3 &= 0\\
a_1 - 2a_2 - a_3 - 4b_1 + 8b_2 + 4b_3 &= 0\\
2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2b_3\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= b_3\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 2\\
\eta &= y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y + 1 - \left(\frac{\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1}{x+2}\right)(x+2) \\
&= (-x-2)\,\mathrm{e}^{\frac{y+1}{x+2}} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{(-x-2)\,\mathrm{e}^{\frac{y+1}{x+2}}} dy
\end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\frac{y+1}{x+2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\mathrm{e}^{\frac{y+1}{x+2}} x + 2\,\mathrm{e}^{\frac{y+1}{x+2}} + y + 1}{x+2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{e^{\frac{-y-1}{x+2}}(y+1)}{(x+2)^2} \\
S_y &= -\frac{e^{\frac{-y-1}{x+2}}}{x+2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x+2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R+2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

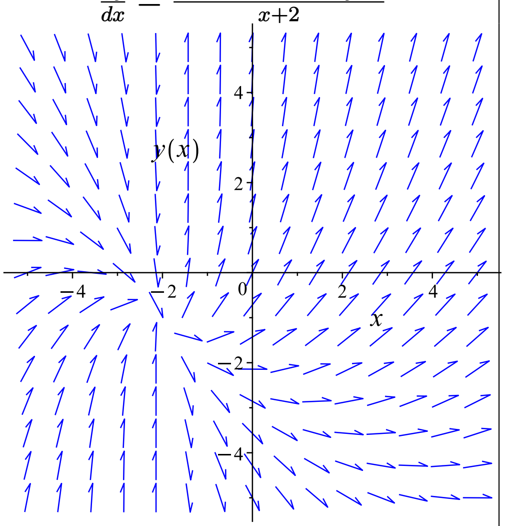
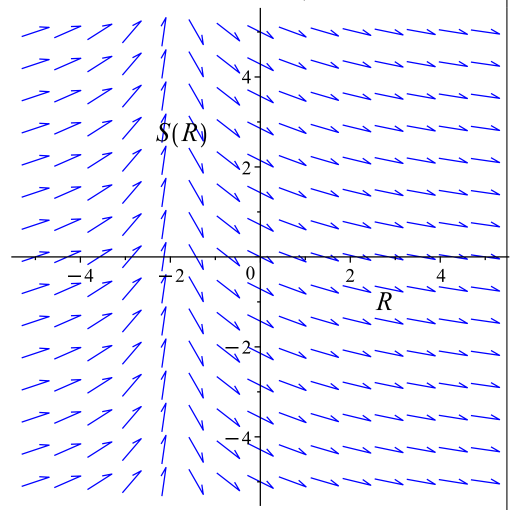
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{1}{R+2}\, dR \\
S(R) &= -\ln(R+2) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
e^{\frac{-y-1}{x+2}} = -\ln(x+2) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{e^{\frac{y+1}{x+2}} x+2\, e^{\frac{y+1}{x+2}}+y+1}{x+2}$ 	$R = x$ $S = e^{\frac{-y-1}{x+2}}$	$\frac{dS}{dR} = -\frac{1}{R+2}$ 

Solving for y gives

$$y = -\ln\left(-\ln\left(x+2\right)+c_2\right) x - 2\ln\left(-\ln\left(x+2\right)+c_2\right) - 1$$

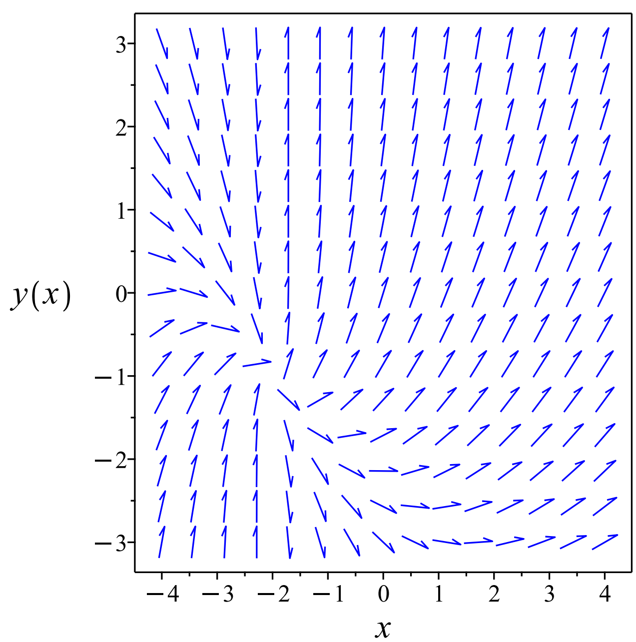


Figure 2.4: Slope field $y' = \frac{y+1}{x+2} + e^{\frac{y+1}{x+2}}$

Summary of solutions found

$$y = -\ln\left(-\ln\left(x+2\right)+c_{2}\right)x-2\ln\left(-\ln\left(x+2\right)+c_{2}\right)-1$$

✓ **Maple.** `Time used: 0.006 (sec). Leaf size: 23`

```
ode:=diff(y(x),x) = (1+y(x))/(x+2)+exp((1+y(x))/(x+2));
dsolve(ode,y(x), singsol=all);
```

$$y = -1+\left(-x-2\right)\ln\left(-\ln\left(x+2\right)-c_{1}\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \frac{y(x)+1}{x+2}+\mathrm{e}^{\frac{y(x)+1}{x+2}}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{y(x)+1}{x+2}+\mathrm{e}^{\frac{y(x)+1}{x+2}}$

✓ **Mathematica.** Time used: 0.761 (sec). Leaf size: 22

```
ode=D[y[x],x]==(y[x]+1)/(x+2)+Exp[(y[x]+1)/(x+2)];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -1 - ((x+2)\log(-\log(x+2) + c_1))$$

✗ **Sympy**

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-exp((y(x) + 1)/(x + 2)) + Derivative(y(x), x) - (y(x) + 1)/(x + 2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : argument of type Mul is not iterable
```

2.1.3 Problem (c)

Local contents

Solved using first_order_ode_dAlembert 41
Solved using first_order_ode_homog_type_maple_C 45
Solved using first_order_ode_LIE 49
✓Maple 56
✓Mathematica 57
✗Sympy 57

Internal problem ID [20964]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:10:48 AM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solved using first_order_ode_dAlembert

Time used: 1.543 (sec)

Solve

$$y' = \frac{x+y+1}{x+2} - e^{\frac{x+y+1}{x+2}}$$

Let $p = y'$ the ode becomes

$$p = \frac{x+y+1}{x+2} - e^{\frac{x+y+1}{x+2}}$$

Solving for y from the above results in

$$y = (-\operatorname{LambertW}(-e^{p}) + p - 1)x - 2\operatorname{LambertW}(-e^{p}) + 2p - 1 \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\operatorname{LambertW}(-e^p) + p - 1 \\
g &= -2\operatorname{LambertW}(-e^p) + 2p - 1
\end{aligned}
$$

Hence (2) becomes

$$
1+\operatorname{LambertW}(-e^p) = \left(-\frac{x \operatorname{LambertW}(-e^p)}{1+\operatorname{LambertW}(-e^p)} + x - \frac{2\operatorname{LambertW}(-e^p)}{1+\operatorname{LambertW}(-e^p)} + 2\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
1+\operatorname{LambertW}(-e^p) = 0
$$

Solving the above for p results in

$$
p_1 = -1
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y = -1 - x
$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{1+\operatorname{LambertW}\left(-e^{p(x)}\right)}{-\frac{x \operatorname{LambertW}\left(-e^{p(x)}\right)}{1+\operatorname{LambertW}\left(-e^{p(x)}\right)} + x - \frac{2\operatorname{LambertW}\left(-e^{p(x)}\right)}{1+\operatorname{LambertW}\left(-e^{p(x)}\right)} + 2} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$
p'(x) = \frac{\left(1+\operatorname{LambertW}\left(-e^{p(x)}\right)\right)^2}{x+2} \tag{2.3}
$$

is separable as it can be written as

$$
\begin{aligned}
p'(x) &= \frac{\left(1+\operatorname{LambertW}\left(-e^{p(x)}\right)\right)^2}{x+2} \\
&= f(x)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x+2} \\
g(p) &= \left(1+\operatorname{LambertW}(-e^p)\right)^2
\end{aligned}
$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{1}{(1+\text{LambertW}(-e^{p}))^2}\,dp = \int \frac{1}{x+2}\,dx$$

$$-\ln\left(1+\text{LambertW}\left(-e^{p(x)}\right)\right) + \ln\left(-e^{p(x)}\right) - \text{LambertW}\left(-e^{p(x)}\right) = \ln(x+2) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$(1+\text{LambertW}(-e^{p}))^2 = 0$$

for $p(x)$ gives

$$p(x) = -1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(1+\text{LambertW}\left(-e^{p(x)}\right)\right) + \ln\left(-e^{p(x)}\right) - \text{LambertW}\left(-e^{p(x)}\right) = \ln(x+2) + c_1$$

$$p(x) = -1$$

Substituing the above solution for p in (2A) gives

$$y = \left(-\text{LambertW}\left(-e^{\frac{i\pi e^{c_1}x+2i\pi e^{c_1}+e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)x+e^{c_1}c_1x-i\pi+2e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)+2e^{c_1}c_1-e^{c_1}x-2e^{c_1}-\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right.}{e^{c_1}x+2e^{c_1}-1}}\right.\right.$$

$$+\frac{i\pi e^{c_1}x+2i\pi e^{c_1}+e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)x+e^{c_1}c_1x-i\pi+2e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)+2e^{c_1}c_1-e^{c_1}x-2e^{c_1}}{e^{c_1}x+2e^{c_1}-1}$$

$$\left.-1\right)x$$

$$-2\,\text{LambertW}\left(-e^{\frac{i\pi e^{c_1}x+2i\pi e^{c_1}+e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)x+e^{c_1}c_1x-i\pi+2e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)+2e^{c_1}c_1-e^{c_1}x-2e^{c_1}-\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right.}{e^{c_1}x+2e^{c_1}-1}}\right.$$

$$+\frac{2i\pi e^{c_1}x+4i\pi e^{c_1}+2e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)x+2e^{c_1}c_1x-2i\pi+4e^{c_1}\ln\left(-\frac{x+2}{e^{c_1}x+2e^{c_1}-1}\right)+4e^{c_1}c_1-2e^{c_1}x}{e^{c_1}x+2e^{c_1}-1}$$

$$-1$$

$$y = -1-x$$

Simplifying the above gives

$$y = -1 - x$$

$$y = \frac{-(x+2)\left(e^{c_1}(x+2)-1\right) \operatorname{LambertW}\left(-e^{\frac{\left(e^{c_1}(x+2)-1\right)\ln\left(\frac{-x-2}{e^{c_1}(x+2)-1}\right)+(x+2)(i\pi+c_1-1)e^{c_1}-i\pi-c_1}{e^{c_1}(x+2)-1}}\right) + (x+2)\left(e^{c_1}(x+2\right.}{e^{c_1}(x+}$$

$$y = -1 - x$$

The solution

$$y = \frac{-(x+2)\left(e^{c_1}(x+2)-1\right) \operatorname{LambertW}\left(-e^{\frac{\left(e^{c_1}(x+2)-1\right)\ln\left(\frac{-x-2}{e^{c_1}(x+2)-1}\right)+(x+2)(i\pi+c_1-1)e^{c_1}-i\pi-c_1}{e^{c_1}(x+2)-1}}\right) + (x+2)\left(e^{c_1}(x+2\right.}{e^{c_1}(x+}$$

was found not to satisfy the ode or the IC. Hence it is removed.

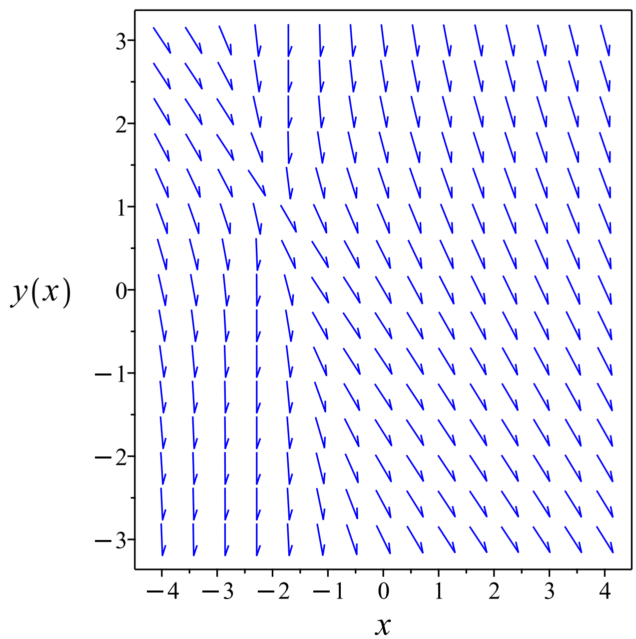


Figure 2.5: Slope field $y' = \frac{x+y+1}{x+2} - e^{\frac{x+y+1}{x+2}}$

Summary of solutions found

$$y = -1 - x$$

Solved using first_order_ode_homog_type_maple_C

Time used: 1.050 (sec)

Solve

$$y' = \frac{x+y+1}{x+2} - \mathrm{e}^{\frac{x+y+1}{x+2}}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-\mathrm{e}^{\frac{X+x_0+Y(X)+y_0+1}{X+x_0+2}}(X+x_0) + Y(X) + y_0 - 2\,\mathrm{e}^{\frac{X+x_0+Y(X)+y_0+1}{X+x_0+2}} + X + x_0 + 1}{X+x_0+2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -2$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-\mathrm{e}\,\mathrm{e}^{-\frac{2}{X}}\mathrm{e}^{\frac{Y(X)}{X}}\mathrm{e}^{\frac{2}{X}}X + Y(X) + X}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{\mathrm{e}\,\mathrm{e}^{-\frac{2}{X}}\mathrm{e}^{\frac{Y}{X}}\mathrm{e}^{\frac{2}{X}}X - Y - X}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -\mathrm{e}\,\mathrm{e}^{-\frac{2}{X}}\mathrm{e}^{\frac{Y}{X}}\mathrm{e}^{\frac{2}{X}}X + Y + X$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = -\mathrm{e}\,\mathrm{e}^{u} + 1 + u$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{-\mathrm{e}\,\mathrm{e}^{u(X)} + 1}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{-\mathrm{e}\,\mathrm{e}^{u(X)} + 1}{X} = 0$$

Or

$$\mathrm{e}\,\mathrm{e}^{u(X)} + \left(\frac{d}{dX}u(X)\right)X - 1 = 0$$

Or

$$\mathrm{e}^{1+u(X)} + \left(\frac{d}{dX}u(X)\right)X - 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{\mathrm{e}^{1+u(X)} - 1}{X} \tag{2.4}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{\mathrm{e}^{1+u(X)} - 1}{X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X}\\ g(u) &= -\mathrm{e}^{1+u} + 1\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$

$$\int \frac{1}{-\mathrm{e}^{1+u} + 1}\,du = \int \frac{1}{X}\,dX$$

$$-\ln\left(\mathrm{e}^{1+u(X)} - 1\right) + 1 + u(X) = \ln(X) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-\mathrm{e}^{1+u} + 1 = 0$$

for $u(X)$ gives

$$u(X) = -1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(e^{1+u(X)}-1\right)+1+u(X)=\ln(X)+c_1$$
$$u(X)=-1$$

Converting $-\ln\left(e^{1+u(X)}-1\right)+1+u(X)=\ln(X)+c_1$ back to $Y(X)$ gives

$$\frac{-\ln\left(e^{\frac{X+Y(X)}{X}}-1\right)X+Y(X)+X}{X}=\ln(X)+c_1$$

Converting $u(X)=-1$ back to $Y(X)$ gives

$$Y(X)=-X$$

Using the solution for $Y(X)$

$$\frac{-\ln\left(e^{\frac{X+Y(X)}{X}}-1\right)X+Y(X)+X}{X}=\ln(X)+c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y=y+1$$
$$X=x-2$$

Then the solution in y becomes using EQ (A)

$$\frac{-\ln\left(e^{\frac{x+y+1}{x+2}}-1\right)(x+2)+y+1+x}{x+2}=\ln(x+2)+c_1$$

Using the solution for $Y(X)$

$$Y(X)=-X \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x_0+x$$

Or

$$Y = y + 1$$
$$X = x - 2$$

Then the solution in y becomes using EQ (A)

$$y - 1 = -x - 2$$

Simplifying the above gives

$$\frac{(-x-2)\ln\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right) + x + y + 1}{x+2} = \ln(x+2) + c_1$$
$$y - 1 = -x - 2$$

Solving for y gives

$$y = -1 - x$$
$$y = c_1 x + \ln(x+2)\,x + \ln\left(\frac{1}{\mathrm{e}^{c_1} x + 2\,\mathrm{e}^{c_1} - 1}\right) x + 2c_1 - x + 2\ln(x+2) + 2\ln\left(\frac{1}{\mathrm{e}^{c_1} x + 2\,\mathrm{e}^{c_1} - 1}\right) - 1$$

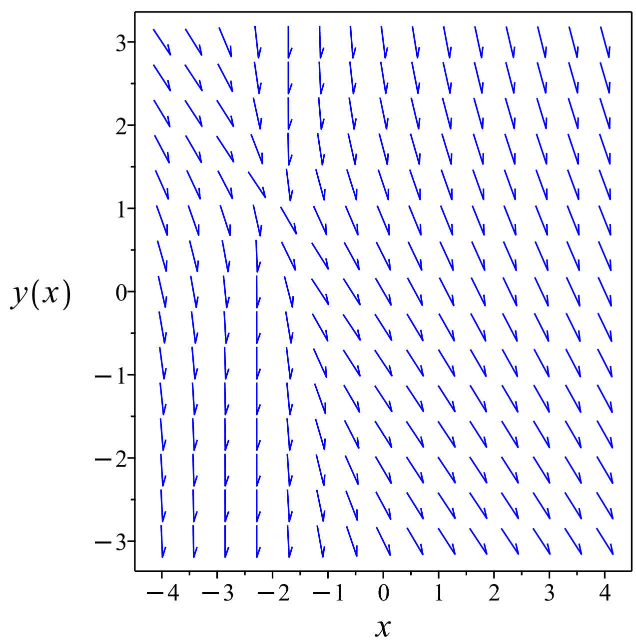


Figure 2.6: Slope field $y' = \frac{x+y+1}{x+2} - \mathrm{e}^{\frac{x+y+1}{x+2}}$

Summary of solutions found

$$y = -1 - x$$

$$y = c_1 x + \ln(x+2)\,x + \ln\left(\frac{1}{\mathrm{e}^{c_1} x + 2\,\mathrm{e}^{c_1} - 1}\right) x + 2c_1 - x + 2\ln(x+2) + 2\ln\left(\frac{1}{\mathrm{e}^{c_1} x + 2\,\mathrm{e}^{c_1} - 1}\right) - 1$$

Solved using first_order_ode_LIE

Time used: 5.635 (sec)

Solve

$$y' = \frac{x+y+1}{x+2} - \mathrm{e}^{\frac{x+y+1}{x+2}}$$

Writing the ode as

$$y' = \frac{-\mathrm{e}^{\frac{x+y+1}{x+2}} x - 2\,\mathrm{e}^{\frac{x+y+1}{x+2}} + x + y + 1}{x+2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{\left(-e^{\frac{x+y+1}{x+2}} x - 2\,e^{\frac{x+y+1}{x+2}} + x + y + 1\right)(b_3 - a_2)}{x+2} \\
&- \frac{\left(-e^{\frac{x+y+1}{x+2}} x - 2\,e^{\frac{x+y+1}{x+2}} + x + y + 1\right)^2 a_3}{(x+2)^2} \\
&- \left(\frac{-\left(\frac{1}{x+2} - \frac{x+y+1}{(x+2)^2}\right) e^{\frac{x+y+1}{x+2}} x - e^{\frac{x+y+1}{x+2}} - 2\left(\frac{1}{x+2} - \frac{x+y+1}{(x+2)^2}\right) e^{\frac{x+y+1}{x+2}} + 1}{x+2}\right. \\
&\left. - \frac{-e^{\frac{x+y+1}{x+2}} x - 2\,e^{\frac{x+y+1}{x+2}} + x + y + 1}{(x+2)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \frac{\left(-\frac{e^{\frac{x+y+1}{x+2}} x}{x+2} - \frac{2\,e^{\frac{x+y+1}{x+2}}}{x+2} + 1\right)(x b_2 + y b_3 + b_1)}{x+2} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{e^{\frac{x+y+1}{x+2}} x y a_2 - 2\,e^{\frac{x+y+1}{x+2}} x y a_3 - e^{\frac{x+y+1}{x+2}} x y b_3 + 4 x a_2 + 3 y a_3 - 2 x b_2 + 4\,e^{\frac{2x+2y+2}{x+2}} a_3 + 2 x y a_3 - e^{\frac{x+y+1}{x+2}} x^2 a_2 - 2\,e^{\cdots}}{\cdots}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-e^{\frac{x+y+1}{x+2}} x y a_2 + 2\,e^{\frac{x+y+1}{x+2}} x y a_3 + e^{\frac{x+y+1}{x+2}} x y b_3 - 4 x a_2 - 3 y a_3 + 2 x b_2 - 4\,e^{\frac{2x+2y+2}{x+2}} a_3 \\
&- 2 x y a_3 + e^{\frac{x+y+1}{x+2}} x^2 a_2 + 2\,e^{\frac{x+y+1}{x+2}} x^2 a_3 + e^{\frac{x+y+1}{x+2}} x^2 b_2 - e^{\frac{x+y+1}{x+2}} x^2 b_3 - e^{\frac{x+y+1}{x+2}} y^2 a_3 \\
&+ 5\,e^{\frac{x+y+1}{x+2}} x a_2 + 6\,e^{\frac{x+y+1}{x+2}} x a_3 + e^{\frac{x+y+1}{x+2}} x b_1 + 2\,e^{\frac{x+y+1}{x+2}} x b_2 - 4\,e^{\frac{x+y+1}{x+2}} x b_3 \\
&- e^{\frac{x+y+1}{x+2}} y a_1 + 5\,e^{\frac{x+y+1}{x+2}} y a_3 + 2\,e^{\frac{x+y+1}{x+2}} y b_3 - e^{\frac{2x+2y+2}{x+2}} x^2 a_3 - 4\,e^{\frac{2x+2y+2}{x+2}} x a_3 - a_1 \\
&- 2 a_2 - a_3 - 2 b_1 + 4 b_2 + 2 b_3 - x^2 a_2 - x^2 a_3 + x^2 b_3 - 2 x a_3 - x b_1 + 3 x b_3 + y a_1 \\
&- 2 y a_2 + e^{\frac{x+y+1}{x+2}} a_1 + 4\,e^{\frac{x+y+1}{x+2}} a_2 + 4\,e^{\frac{x+y+1}{x+2}} a_3 + 2\,e^{\frac{x+y+1}{x+2}} b_1 - 4\,e^{\frac{x+y+1}{x+2}} b_3 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& e^{\frac{x+y+1}{x+2}} x^3 a_2 + 2\,e^{\frac{x+y+1}{x+2}} x^3 a_3 + e^{\frac{x+y+1}{x+2}} x^3 b_2 - e^{\frac{x+y+1}{x+2}} x^3 b_3 + e^{\frac{x+y+1}{x+2}} x^2 b_1 \\
& + e^{\frac{x+y+1}{x+2}} x a_1 - 2\,e^{\frac{x+y+1}{x+2}} x y a_2 + 9\,e^{\frac{x+y+1}{x+2}} x y a_3 + 4\,e^{\frac{x+y+1}{x+2}} x y b_3 - e^{\frac{x+y+1}{x+2}} x^2 y a_2 \\
& + 2\,e^{\frac{x+y+1}{x+2}} x^2 y a_3 + e^{\frac{x+y+1}{x+2}} x^2 y b_3 - e^{\frac{x+y+1}{x+2}} x\, y^2 a_3 - e^{\frac{x+y+1}{x+2}} x y a_1 - e^{\frac{2x+2y+2}{x+2}} x^3 a_3 \\
& - 10 x a_2 - 6 y a_3 + 8 x b_2 - 8\,e^{\frac{2x+2y+2}{x+2}} a_3 - 2 x^2 y a_3 + x y a_1 - 2 x y a_2 \\
& - 7 x y a_3 + 7\,e^{\frac{x+y+1}{x+2}} x^2 a_2 + 10\,e^{\frac{x+y+1}{x+2}} x^2 a_3 + 4\,e^{\frac{x+y+1}{x+2}} x^2 b_2 - 6\,e^{\frac{x+y+1}{x+2}} x^2 b_3 \\
& - 2\,e^{\frac{x+y+1}{x+2}} y^2 a_3 + 14\,e^{\frac{x+y+1}{x+2}} x a_2 + 16\,e^{\frac{x+y+1}{x+2}} x a_3 + 4\,e^{\frac{x+y+1}{x+2}} x b_1 + 4\,e^{\frac{x+y+1}{x+2}} x b_2 \\
& - 12\,e^{\frac{x+y+1}{x+2}} x b_3 - 2\,e^{\frac{x+y+1}{x+2}} y a_1 + 10\,e^{\frac{x+y+1}{x+2}} y a_3 + 4\,e^{\frac{x+y+1}{x+2}} y b_3 - 6\,e^{\frac{2x+2y+2}{x+2}} x^2 a_3 \\
& - 12\,e^{\frac{2x+2y+2}{x+2}} x a_3 - 2 a_1 - 4 a_2 - 2 a_3 - 4 b_1 + 8 b_2 + 4 b_3 - x^3 a_2 - x^3 a_3 + x^3 b_3 \\
& - 6 x^2 a_2 - 4 x^2 a_3 - x^2 b_1 + 2 x^2 b_2 + 5 x^2 b_3 - x a_1 - 5 x a_3 - 4 x b_1 + 8 x b_3 + 2 y a_1 \\
& - 4 y a_2 + 2\,e^{\frac{x+y+1}{x+2}} a_1 + 8\,e^{\frac{x+y+1}{x+2}} a_2 + 8\,e^{\frac{x+y+1}{x+2}} a_3 + 4\,e^{\frac{x+y+1}{x+2}} b_1 - 8\,e^{\frac{x+y+1}{x+2}} b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, e^{\frac{x+y+1}{x+2}}, e^{\frac{2x+2y+2}{x+2}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, e^{\frac{x+y+1}{x+2}} = v_3, e^{\frac{2x+2y+2}{x+2}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& v_3 v_1^3 a_2 - v_3 v_1^2 v_2 a_2 + 2 v_3 v_1^3 a_3 - v_4 v_1^3 a_3 + 2 v_3 v_1^2 v_2 a_3 - v_3 v_1 v_2^2 a_3 + v_3 v_1^3 b_2 \\
& - v_3 v_1^3 b_3 + v_3 v_1^2 v_2 b_3 - v_3 v_1 v_2 a_1 - v_1^3 a_2 + 7 v_3 v_1^2 a_2 - 2 v_3 v_1 v_2 a_2 - v_1^3 a_3 \\
& - 2 v_1^2 v_2 a_3 + 10 v_3 v_1^2 a_3 - 6 v_4 v_1^2 a_3 + 9 v_3 v_1 v_2 a_3 - 2 v_3 v_2^2 a_3 + v_3 v_1^2 b_1 + 4 v_3 v_1^2 b_2 \\
& + v_1^3 b_3 - 6 v_3 v_1^2 b_3 + 4 v_3 v_1 v_2 b_3 + v_1 v_2 a_1 + v_3 v_1 a_1 - 2 v_3 v_2 a_1 - 6 v_1^2 a_2 - 2 v_1 v_2 a_2 \\
& + 14 v_3 v_1 a_2 - 4 v_1^2 a_3 - 7 v_1 v_2 a_3 + 16 v_3 v_1 a_3 - 12 v_4 v_1 a_3 + 10 v_3 v_2 a_3 - v_1^2 b_1 \\
& + 4 v_3 v_1 b_1 + 2 v_1^2 b_2 + 4 v_3 v_1 b_2 + 5 v_1^2 b_3 - 12 v_3 v_1 b_3 + 4 v_3 v_2 b_3 - v_1 a_1 + 2 v_2 a_1 \\
& + 2 v_3 a_1 - 10 v_1 a_2 - 4 v_2 a_2 + 8 v_3 a_2 - 5 v_1 a_3 - 6 v_2 a_3 + 8 v_3 a_3 - 8 v_4 a_3 - 4 v_1 b_1 \\
& + 4 v_3 b_1 + 8 v_1 b_2 + 8 v_1 b_3 - 8 v_3 b_3 - 2 a_1 - 4 a_2 - 2 a_3 - 4 b_1 + 8 b_2 + 4 b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-a_2 - a_3 + b_3)\, v_1^3 + (-6a_2 - 4a_3 - b_1 + 2b_2 + 5b_3)\, v_1^2 \\
&+ (-a_1 - 10a_2 - 5a_3 - 4b_1 + 8b_2 + 8b_3)\, v_1 + (2a_1 - 4a_2 - 6a_3)\, v_2 \\
&+ (2a_1 + 8a_2 + 8a_3 + 4b_1 - 8b_3)\, v_3 - v_3 v_1 v_2^2 a_3 - 6v_4 v_1^2 a_3 - 12v_4 v_1 a_3 \\
&+ (a_2 + 2a_3 + b_2 - b_3)\, v_1^3 v_3 + (7a_2 + 10a_3 + b_1 + 4b_2 - 6b_3)\, v_1^2 v_3 \\
&+ (a_1 - 2a_2 - 7a_3)\, v_1 v_2 + (a_1 + 14a_2 + 16a_3 + 4b_1 + 4b_2 - 12b_3)\, v_1 v_3 \\
&+ (-2a_1 + 10a_3 + 4b_3)\, v_2 v_3 - 8v_4 a_3 - v_4 v_1^3 a_3 - 2v_1^2 v_2 a_3 \\
&- 2v_3 v_2^2 a_3 - 2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 \\
&+ (-a_2 + 2a_3 + b_3)\, v_1^2 v_2 v_3 + (-a_1 - 2a_2 + 9a_3 + 4b_3)\, v_1 v_2 v_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-12a_3 &= 0 \\
-8a_3 &= 0 \\
-6a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-2a_1 + 10a_3 + 4b_3 &= 0 \\
a_1 - 2a_2 - 7a_3 &= 0 \\
2a_1 - 4a_2 - 6a_3 &= 0 \\
-a_2 - a_3 + b_3 &= 0 \\
-a_2 + 2a_3 + b_3 &= 0 \\
-a_1 - 2a_2 + 9a_3 + 4b_3 &= 0 \\
a_2 + 2a_3 + b_2 - b_3 &= 0 \\
2a_1 + 8a_2 + 8a_3 + 4b_1 - 8b_3 &= 0 \\
-6a_2 - 4a_3 - b_1 + 2b_2 + 5b_3 &= 0 \\
7a_2 + 10a_3 + b_1 + 4b_2 - 6b_3 &= 0 \\
-2a_1 - 4a_2 - 2a_3 - 4b_1 + 8b_2 + 4b_3 &= 0 \\
-a_1 - 10a_2 - 5a_3 - 4b_1 + 8b_2 + 8b_3 &= 0 \\
a_1 + 14a_2 + 16a_3 + 4b_1 + 4b_2 - 12b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 2b_3 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= -b_3 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + 2 \\
\eta &= y - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - 1 - \left(\frac{-\mathrm{e}^{\frac{x+y+1}{x+2}} x - 2\,\mathrm{e}^{\frac{x+y+1}{x+2}} + x + y + 1}{x + 2}\right)(x + 2) \\
&= (x + 2)\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{(x + 2)\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right)} dy
\end{aligned}
$$

Which results in

$$S = \ln\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right) - \ln\left(\mathrm{e}^{\frac{x+y+1}{x+2}}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-\mathrm{e}^{\frac{x+y+1}{x+2}} x - 2\,\mathrm{e}^{\frac{x+y+1}{x+2}} + x + y + 1}{x+2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{-y+1}{(x+2)^2\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right)} \\
S_y &= \frac{1}{(x+2)\left(\mathrm{e}^{\frac{x+y+1}{x+2}} - 1\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x+2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R+2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

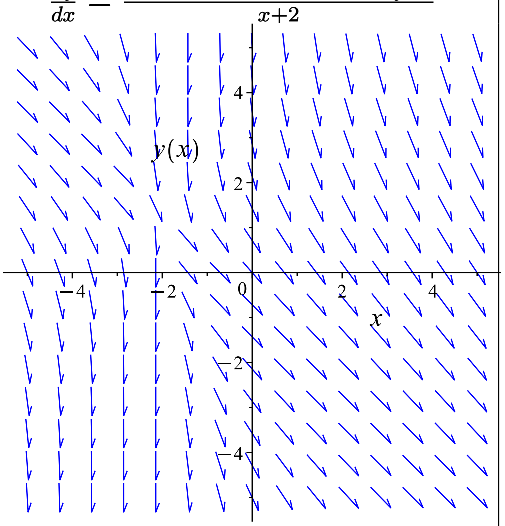
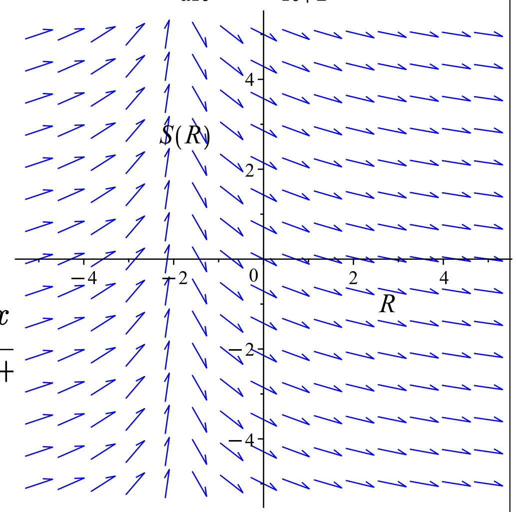
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -\frac{1}{R+2}\, dR \\
S(R) &= -\ln(R+2) + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(\mathrm{e}^{\frac{x+y+1}{x+2}}-1\right)(x+2)-y-1-x}{x+2}=-\ln(x+2)+c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{-\mathrm{e}^{\frac{x+y+1}{x+2}}x-2\,\mathrm{e}^{\frac{x+y+1}{x+2}}+x+y+1}{x+2}$ 	$R = x$ $S = \frac{\ln\left(\mathrm{e}^{\frac{x+y+1}{x+2}}-1\right)(x}{x+}$	$\frac{dS}{dR}=-\frac{1}{R+2}$ 

Solving for y gives

$$y=\ln(x+2)\,x+x\ln\left(-\frac{1}{\mathrm{e}^{c_2}-x-2}\right)+2\ln(x+2)+2\ln\left(-\frac{1}{\mathrm{e}^{c_2}-x-2}\right)-x-1$$

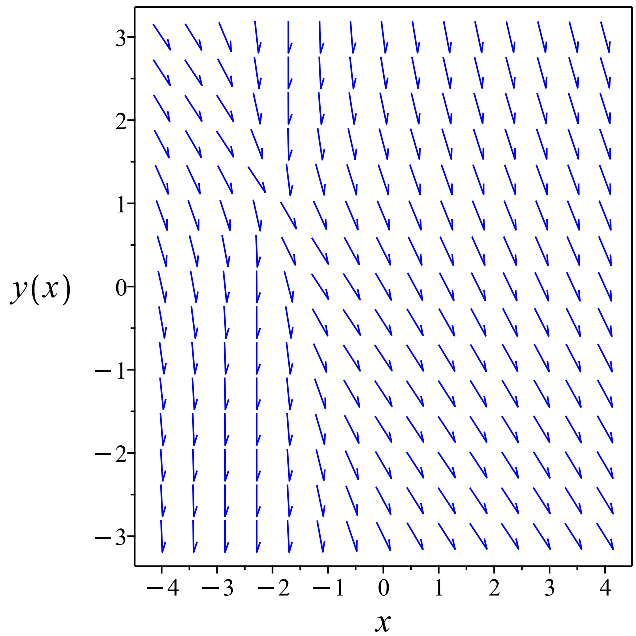


Figure 2.7: Slope field $y' = \frac{x+y+1}{x+2} - \mathrm{e}^{\frac{x+y+1}{x+2}}$

Summary of solutions found

$$y = \ln(x+2)\,x + x\ln\left(-\frac{1}{\mathrm{e}^{c_2} - x - 2}\right) + 2\ln(x+2) + 2\ln\left(-\frac{1}{\mathrm{e}^{c_2} - x - 2}\right) - x - 1$$

✓ **Maple.** `Time used: 0.029 (sec). Leaf size: 31`

```
ode:=diff(y(x),x) = (x+y(x)+1)/(x+2)-exp((x+y(x)+1)/(x+2));
dsolve(ode,y(x), singsol=all);
```

$$y = 1 + \left(-\ln\left(\frac{(x+2)\,\mathrm{e}^{c_1+1} - 1}{x+2}\right) + c_1\right)(x+2)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
```

```
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve

$\frac{d}{dx}y(x) = \frac{x+y(x)+1}{x+2} - \mathrm{e}^{\frac{x+y(x)+1}{x+2}}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{x+y(x)+1}{x+2} - \mathrm{e}^{\frac{x+y(x)+1}{x+2}}$

✓ **Mathematica.** `Time used: 0.632 (sec). Leaf size: 41`

```
ode=D[y[x],x]==(x+y[x]+1)/(x+2)-Exp[(x+y[x]+1)/(x+2)];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1-y(x)}{x+2} + \log\left(1 - e^{\frac{y(x)+x+1}{x+2}}\right) + \log(x+2) = c_1, y(x)\right]$$

✗ **Sympy**

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(exp((x + y(x) + 1)/(x + 2)) + Derivative(y(x), x) - (x + y(x) + 1)/(x +
    2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
TypeError : argument of type Mul is not iterable
```

2.1.4 Problem (d)

Local contents

Solved using first_order_ode_dAlembert 58
Solved using first_order_ode_homog_type_maple_C 62
Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind 67
Solved using first_order_ode_LIE 76
✓Maple 81
✓Mathematica 82
✓Sympy 83

Internal problem ID [20965]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23
Problem number : (d)
Date solved : `Saturday, November 29, 2025 at 01:11:33 AM`
CAS classification :
`[[_homogeneous, 'class C'], _rational, [_Abel, '2nd type', 'class A']]`

Solved using first_order_ode_dAlembert

Time used: 2.704 (sec)

Solve

$$y' = \frac{2y+x+1}{2x+y+2}$$

Let $p = y'$ the ode becomes

$$p = \frac{2y+x+1}{2x+y+2}$$

Solving for y from the above results in

$$y = -\frac{(2p-1)\,x}{p-2} - \frac{2p-1}{p-2} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p+1}{p-2} \\ g &= \frac{-2p+1}{p-2} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p+1}{p-2} = \left(-\frac{2x}{p-2} + \frac{2xp}{(p-2)^2} - \frac{x}{(p-2)^2} - \frac{2}{p-2} + \frac{2p}{(p-2)^2} - \frac{1}{(p-2)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p+1}{p-2} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= 1 \\ p_2 &= -1 \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= 1 + x \\ y &= -1 - x \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)+1}{p(x)-2}}{-\frac{2x}{p(x)-2} + \frac{2xp(x)}{(p(x)-2)^2} - \frac{x}{(p(x)-2)^2} - \frac{2}{p(x)-2} + \frac{2p(x)}{(p(x)-2)^2} - \frac{1}{(p(x)-2)^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = \frac{(p(x)-1)(p(x)-2)(p(x)+1)}{3x+3} \tag{2.5}$$

is separable as it can be written as

$$\begin{aligned}
p'(x) &= \frac{(p(x)-1)(p(x)-2)(p(x)+1)}{3x+3} \\
&= f(x)g(p)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \frac{1}{3x+3} \\
g(p) &= (p-1)(p-2)(p+1)
\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\,dp = \int f(x)\,dx$$

$$\int \frac{1}{(p-1)(p-2)(p+1)}\,dp = \int \frac{1}{3x+3}\,dx$$

$$\frac{\ln(p(x)+1)}{6} + \frac{\ln(p(x)-2)}{3} - \frac{\ln(p(x)-1)}{2} = \ln\left((1+x)^{1/3}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or

$$(p-1)(p-2)(p+1) = 0$$

for $p(x)$ gives

$$\begin{aligned}
p(x) &= -1 \\
p(x) &= 1
\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\frac{\ln(p(x)+1)}{6} + \frac{\ln(p(x)-2)}{3} - \frac{\ln(p(x)-1)}{2} &= \ln\left((1+x)^{1/3}\right) + c_1 \\
p(x) &= -1 \\
p(x) &= 1
\end{aligned}$$

Substituing the above solution for p in (2A) gives

Expression too large to display

$$y = -1 - x$$

$$y = 1 + x$$

Simplifying the above gives

$$y = 1 + x$$

$$y = -1 - x$$

$$y = -\frac{2(1+x)\left(1 + \frac{\left(-1+e^{6c_1}(1+x)^2\right)\left(\left(-1+e^{6c_1}(1+x)^2\right)^2\left(e^{3c_1}\sqrt{\frac{(1+x)^2}{-1+e^{6c_1}(1+x)^2}}+1\right)\right)^{1/3}}{2} - e^{6c_1}(1+x)^2 + \left(\left(-1+e^{6c_1}(1\right.\right.\right.}{\left(1-e^{6c_1}(1+x)^2\right)\left(\left(-1+e^{6c_1}(1+x)^2\right)^2\left(e^{3c_1}\sqrt{\frac{(1+x)^2}{-1+e^{6c_1}(1+x)^2}}+1\right)\right)^{1/3} - e^{6c_1}(1+x)^2 + \left(\left(-1+e^{6c_1}(1\right.\right.}$$

$$y = -1 - x$$

$$y = 1 + x$$

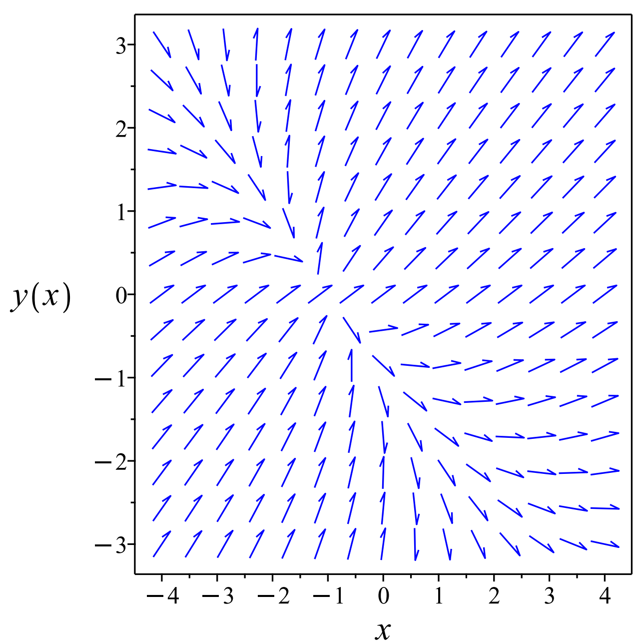


Figure 2.8: Slope field $y' = \frac{2y+x+1}{2x+y+2}$

Summary of solutions found

$$y = -\frac{2(1+x)\left(1+\frac{\left(-1+e^{6c_1}(1+x)^2\right)\left(\left(-1+e^{6c_1}(1+x)^2\right)^2\left(e^{3c_1}\sqrt{\frac{(1+x)^2}{-1+e^{6c_1}(1+x)^2}}+1\right)\right)^{1/3}}{2} - e^{6c_1}(1+x)^2 + \left(\left(-1+e^{6c_1}(1\right.\right.\right.}{\left(1-e^{6c_1}(1+x)^2\right)\left(\left(-1+e^{6c_1}(1+x)^2\right)^2\left(e^{3c_1}\sqrt{\frac{(1+x)^2}{-1+e^{6c_1}(1+x)^2}}+1\right)\right)^{1/3} - e^{6c_1}(1+x)^2 + \left(\left(-1+e^{6c_1}(1\right.\right.}$$

$$y = -1 - x$$

$$y = 1 + x$$

Solved using first_order_ode_homog_type_maple_C

Time used: 1.125 (sec)

Solve

$$y' = \frac{2y + x + 1}{2x + y + 2}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 + X + x_0 + 1}{2X + 2x_0 + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + X}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2Y + X}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2Y+X$ and $N = 2X+Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u+1}{u+2}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)+1}{u(X)+2} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)+1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + u(X)^2 + 2\left(\frac{d}{dX}u(X)\right)X - 1 = 0$$

Or

$$(u(X)+2)\,X\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{(u(X)-1)\,(u(X)+1)}{(u\,(X)+2)\,X} \tag{2.6}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{(u(X)-1)\,(u(X)+1)}{(u\,(X)+2)\,X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{(u-1)\,(u+1)}{u+2}\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$

$$\int \frac{u+2}{(u-1)\,(u+1)}\,du = \int -\frac{1}{X}\,dX$$

$$-\frac{\ln \left(u(X)+1\right)}{2}+\frac{3 \ln \left(u(X)-1\right)}{2} = \ln \left(\frac{1}{X}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\frac{\left(u-1\right)\left(u+1\right)}{u+2} = 0$$

for $u(X)$ gives

$$u(X) = -1$$
$$u(X) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln \left(u(X)+1\right)}{2}+\frac{3 \ln \left(u(X)-1\right)}{2} = \ln \left(\frac{1}{X}\right)+c_1$$
$$u(X) = -1$$
$$u(X) = 1$$

Converting $-\frac{\ln(u(X)+1)}{2}+\frac{3\ln(u(X)-1)}{2} = \ln\left(\frac{1}{X}\right)+c_1$ back to $Y(X)$ gives

$$-\frac{\ln \left(\frac{Y(X)+X}{X}\right)}{2}+\frac{3 \ln \left(\frac{Y(X)-X}{X}\right)}{2} = \ln \left(\frac{1}{X}\right)+c_1$$

Converting $u(X) = -1$ back to $Y(X)$ gives

$$Y(X) = -X$$

Converting $u(X) = 1$ back to $Y(X)$ gives

$$Y(X) = X$$

Using the solution for $Y(X)$

$$-\frac{\ln \left(\frac{Y(X)+X}{X}\right)}{2}+\frac{3 \ln \left(\frac{Y(X)-X}{X}\right)}{2} = \ln \left(\frac{1}{X}\right)+c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y$$
$$X = x - 1$$

Then the solution in y becomes using EQ (A)

$$-\frac{\ln\left(\frac{x+y+1}{1+x}\right)}{2} + \frac{3\ln\left(\frac{y-x-1}{1+x}\right)}{2} = \ln\left(\frac{1}{1+x}\right) + c_1$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y$$
$$X = x - 1$$

Then the solution in y becomes using EQ (A)

$$y = -1 - x$$

Using the solution for $Y(X)$

$$Y(X) = X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y$$
$$X = x - 1$$

Then the solution in y becomes using EQ (A)

$$y = 1 + x$$

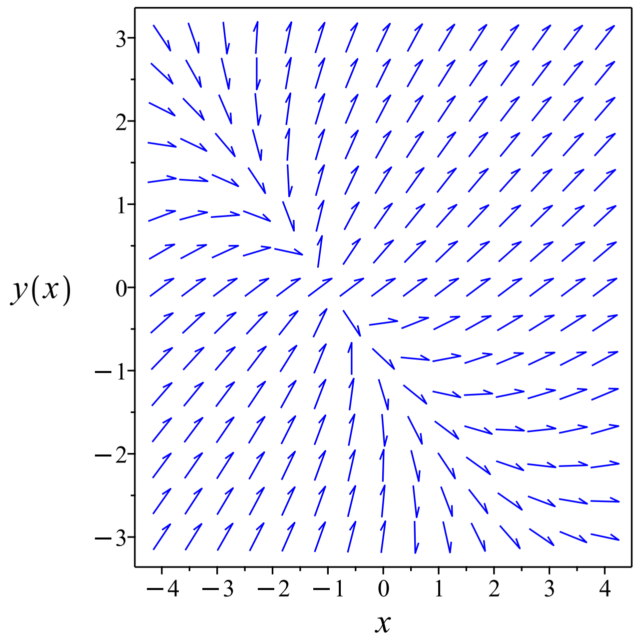


Figure 2.9: Slope field $y' = \frac{2y+x+1}{2x+y+2}$

Summary of solutions found

$$-\frac{\ln\left(\frac{x+y+1}{1+x}\right)}{2} + \frac{3\ln\left(\frac{y-x-1}{1+x}\right)}{2} = \ln\left(\frac{1}{1+x}\right) + c_1$$

$$y = -1 - x$$

$$y = 1 + x$$

Solve

$$y' = \frac{2y + x + 1}{2x + y + 2}$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u(x) + xu'(x) = \frac{2xu(x) + x + 1}{2x + xu(x) + 2}$$

Which is now solved Unknown ode type.

Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

Time used: 25.183 (sec)

This is Abel second kind ODE, it has the form

$$(y(x)+g)\,y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y(x)^2 + f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$y'(x) = \frac{2y(x)+x+1}{2x+y(x)+2} \tag{1}$$

Shows that

$$\begin{aligned} g &= 2x+2 \\ f_0 &= 1+x \\ f_1 &= 2 \\ f_2 &= 0 \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u'(x) = (3x+3)\,u(x)^3 - 4u(x)^2$$

Which is now solved.

Solve This is Abel first kind ODE, it has the form

$$u'(x) = f_0(x) + f_1(x)u(x) + f_2(x)u(x)^2 + f_3(x)u(x)^3$$

Comparing the above to given ODE which is

$$u'(x) = (3x+3)\,u(x)^3 - 4u(x)^2 \tag{1}$$

Therefore

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 0 \\ f_2 &= -4 \\ f_3 &= 3x+3 \end{aligned}$$

Hence

$$\begin{aligned}f_0' &= 0\\ f_3' &= 3\end{aligned}$$

Since $f_2(x) = -4$ is not zero, then the followingtransformation is used to remove f_2. Let $u(x) = u(x) - \frac{f_2}{3f_3}$ or

$$\begin{aligned}u(x) &= u(x) - \left(\frac{-4}{9x+9}\right)\\ &= u(x) + \frac{4}{9x+9}\end{aligned}$$

The above transformation applied to (1) gives a new ODE as

$$u'(x) = \frac{(729x^3 + 2187x^2 + 2187x + 729)\,u(x)^3}{243\,(1+x)^2} + \frac{(-432x - 432)\,u(x)}{243\,(1+x)^2} - \frac{20}{243\,(1+x)^2} \tag{2}$$

The above ODE (2) can now be solved.

Solve Writing the ode as

$$\begin{aligned}u'(x) &= \frac{729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20}{243\,(1+x)^2}\\ u'(x) &= \omega(x, u(x))\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_{u(x)} - \xi_x) - \omega^2\xi_{u(x)} - \omega_x\xi - \omega_{u(x)}\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ua_3 + xa_2 + a_1 \tag{1E}$$

$$\eta = ub_3 + xb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 \\
&\quad + \frac{(729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20)\,(b_3 - a_2)}{243\,(1+x)^2} \\
&\quad - \frac{(729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20)^2\,a_3}{59049\,(1+x)^4} \\
&\quad - \left(\frac{2187u^3x^2 + 4374u^3x + 2187u^3 - 432u}{243\,(1+x)^2}\right. \\
&\quad \left. - \frac{2(729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20)}{243\,(1+x)^3}\right)(ua_3 \\
&\quad + xa_2 + a_1) \\
&\quad - \frac{(2187u^2x^3 + 6561u^2x^2 + 6561u^2x + 2187u^2 - 432x - 432)\,(ub_3 + xb_2 + b_1)}{243\,(1+x)^2} \\
&\quad = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{531441u^6x^6a_3 + 3188646u^6x^5a_3 + 7971615u^6x^4a_3 + 10628820u^6x^3a_3 + 7971615u^6x^2a_3 - 452709u^4x^4a_3 + \cdots}{\cdots}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-531441u^6x^6a_3 - 3188646u^6x^5a_3 - 7971615u^6x^4a_3 \\
&- 10628820u^6x^3a_3 - 7971615u^6x^2a_3 + 452709u^4x^4a_3 \\
&- 354294u^3x^5a_2 - 354294u^3x^5b_3 - 531441u^2x^6b_2 - 3188646u^6xa_3 \\
&+ 1810836u^4x^3a_3 - 177147u^3x^4a_1 - 1594323u^3x^4a_2 \\
&- 1771470u^3x^4b_3 - 531441u^2x^5b_1 - 2657205u^2x^5b_2 - 531441u^6a_3 \\
&+ 2716254u^4x^2a_3 - 708588u^3x^3a_1 - 2834352u^3x^3a_2 \\
&+ 29160u^3x^3a_3 - 3542940u^3x^3b_3 - 2657205u^2x^4b_1 \\
&- 5314410u^2x^4b_2 + 1810836u^4xa_3 - 1062882u^3x^2a_1 \\
&- 2480058u^3x^2a_2 + 87480u^3x^2a_3 - 3542940u^3x^2b_3 \\
&- 5314410u^2x^3b_1 - 5314410u^2x^3b_2 + 452709u^4a_3 - 708588u^3xa_1 \\
&- 1062882u^3xa_2 + 87480u^3xa_3 - 1771470u^3xb_3 - 291600u^2x^2a_3 \\
&- 5314410u^2x^2b_1 - 2657205u^2x^2b_2 + 164025x^4b_2 - 177147u^3a_1 \\
&- 177147u^3a_2 + 29160u^3a_3 - 354294u^3b_3 - 583200u^2xa_3 \\
&- 2657205u^2xb_1 - 531441u^2xb_2 - 104976u\,x^2a_1 + 104976u\,x^2a_2 \\
&+ 104976x^3b_1 + 551124x^3b_2 - 291600u^2a_3 - 531441u^2b_1 \\
&- 209952uxa_1 + 209952uxa_2 - 27000uxa_3 - 4860x^2a_2 \\
&+ 314928x^2b_1 + 669222x^2b_2 - 4860x^2b_3 - 104976ua_1 + 104976ua_2 \\
&- 27000ua_3 - 9720xa_1 + 314928xb_1 + 341172xb_2 - 9720xb_3 \\
&- 9720a_1 + 4860a_2 - 400a_3 + 104976b_1 + 59049b_2 - 4860b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{u, x\}$ in them.

$$\{u, x\}$$

The following substitution is now made to be able to collect on all terms with $\{u, x\}$ in them

$$\{u = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-531441a_3v_1^6v_2^6 - 3188646a_3v_1^6v_2^5 - 7971615a_3v_1^6v_2^4 \\
&- 10628820a_3v_1^6v_2^3 - 354294a_2v_1^3v_2^5 - 7971615a_3v_1^6v_2^2 \\
&+ 452709a_3v_1^4v_2^4 - 531441b_2v_1^2v_2^6 - 354294b_3v_1^3v_2^5 - 177147a_1v_1^3v_2^4 \\
&- 1594323a_2v_1^3v_2^4 - 3188646a_3v_1^6v_2 + 1810836a_3v_1^4v_2^3 \\
&- 531441b_1v_1^2v_2^5 - 2657205b_2v_1^2v_2^5 - 1771470b_3v_1^3v_2^4 \\
&- 708588a_1v_1^3v_2^3 - 2834352a_2v_1^3v_2^3 - 531441a_3v_1^6 + 2716254a_3v_1^4v_2^2 \\
&+ 29160a_3v_1^3v_2^3 - 2657205b_1v_1^2v_2^4 - 5314410b_2v_1^2v_2^4 - 3542940b_3v_1^3v_2^3 \\
&- 1062882a_1v_1^3v_2^2 - 2480058a_2v_1^3v_2^2 + 1810836a_3v_1^4v_2 \\
&+ 87480a_3v_1^3v_2^2 - 5314410b_1v_1^2v_2^3 - 5314410b_2v_1^2v_2^3 \\
&- 3542940b_3v_1^3v_2^2 - 708588a_1v_1^3v_2 - 1062882a_2v_1^3v_2 + 452709a_3v_1^4 \\
&+ 87480a_3v_1^3v_2 - 291600a_3v_1^2v_2^2 - 5314410b_1v_1^2v_2^2 - 2657205b_2v_1^2v_2^2 \\
&+ 164025b_2v_2^4 - 1771470b_3v_1^3v_2 - 177147a_1v_1^3 - 104976a_1v_1v_2^2 \\
&- 177147a_2v_1^3 + 104976a_2v_1v_2^2 + 29160a_3v_1^3 - 583200a_3v_1^2v_2 \\
&- 2657205b_1v_1^2v_2 + 104976b_1v_2^3 - 531441b_2v_1^2v_2 + 551124b_2v_2^3 \\
&- 354294b_3v_1^3 - 209952a_1v_1v_2 + 209952a_2v_1v_2 - 4860a_2v_2^2 \\
&- 291600a_3v_1^2 - 27000a_3v_1v_2 - 531441b_1v_1^2 + 314928b_1v_2^2 \\
&+ 669222b_2v_2^2 - 4860b_3v_2^2 - 104976a_1v_1 - 9720a_1v_2 + 104976a_2v_1 \\
&- 27000a_3v_1 + 314928b_1v_2 + 341172b_2v_2 - 9720b_3v_2 - 9720a_1 \\
&+ 4860a_2 - 400a_3 + 104976b_1 + 59049b_2 - 4860b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-354294a_2 - 354294b_3)\, v_1^3 v_2^5 \\
&+ (-177147a_1 - 1594323a_2 - 1771470b_3)\, v_1^3 v_2^4 \\
&+ (-708588a_1 - 2834352a_2 + 29160a_3 - 3542940b_3)\, v_1^3 v_2^3 \\
&+ (-1062882a_1 - 2480058a_2 + 87480a_3 - 3542940b_3)\, v_1^3 v_2^2 \\
&+ (-708588a_1 - 1062882a_2 + 87480a_3 - 1771470b_3)\, v_1^3 v_2 \\
&+ (-531441b_1 - 2657205b_2)\, v_1^2 v_2^5 + (-2657205b_1 - 5314410b_2)\, v_1^2 v_2^4 \\
&+ (-5314410b_1 - 5314410b_2)\, v_1^2 v_2^3 \\
&+ (-291600a_3 - 5314410b_1 - 2657205b_2)\, v_1^2 v_2^2 \\
&+ (-583200a_3 - 2657205b_1 - 531441b_2)\, v_1^2 v_2 \\
&+ (-104976a_1 + 104976a_2)\, v_1 v_2^2 \\
&+ (-209952a_1 + 209952a_2 - 27000a_3)\, v_1 v_2 \\
&+ (-177147a_1 - 177147a_2 + 29160a_3 - 354294b_3)\, v_1^3 \\
&+ (-291600a_3 - 531441b_1)\, v_1^2 \\
&+ (-104976a_1 + 104976a_2 - 27000a_3)\, v_1 + (104976b_1 + 551124b_2)\, v_2^3 \\
&+ (-4860a_2 + 314928b_1 + 669222b_2 - 4860b_3)\, v_2^2 \\
&+ (-9720a_1 + 314928b_1 + 341172b_2 - 9720b_3)\, v_2 - 531441a_3 v_1^6 \\
&+ 452709a_3 v_1^4 + 164025b_2 v_2^4 + 1810836a_3 v_1^4 v_2 - 531441a_3 v_1^6 v_2^6 \\
&- 3188646a_3 v_1^6 v_2^5 - 7971615a_3 v_1^6 v_2^4 - 10628820a_3 v_1^6 v_2^3 \\
&- 7971615a_3 v_1^6 v_2^2 + 452709a_3 v_1^4 v_2^4 - 531441b_2 v_1^2 v_2^6 \\
&- 3188646a_3 v_1^6 v_2 + 1810836a_3 v_1^4 v_2^3 + 2716254a_3 v_1^4 v_2^2 - 9720a_1 \\
&+ 4860a_2 - 400a_3 + 104976b_1 + 59049b_2 - 4860b_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-10628820a_3 &= 0 \\
-7971615a_3 &= 0 \\
-3188646a_3 &= 0 \\
-531441a_3 &= 0 \\
452709a_3 &= 0 \\
1810836a_3 &= 0 \\
2716254a_3 &= 0 \\
-531441b_2 &= 0 \\
164025b_2 &= 0 \\
-104976a_1 + 104976a_2 &= 0 \\
-354294a_2 - 354294b_3 &= 0 \\
-291600a_3 - 531441b_1 &= 0 \\
-5314410b_1 - 5314410b_2 &= 0 \\
-2657205b_1 - 5314410b_2 &= 0 \\
-531441b_1 - 2657205b_2 &= 0 \\
104976b_1 + 551124b_2 &= 0 \\
-209952a_1 + 209952a_2 - 27000a_3 &= 0 \\
-177147a_1 - 1594323a_2 - 1771470b_3 &= 0 \\
-104976a_1 + 104976a_2 - 27000a_3 &= 0 \\
-583200a_3 - 2657205b_1 - 531441b_2 &= 0 \\
-291600a_3 - 5314410b_1 - 2657205b_2 &= 0 \\
-1062882a_1 - 2480058a_2 + 87480a_3 - 3542940b_3 &= 0 \\
-708588a_1 - 2834352a_2 + 29160a_3 - 3542940b_3 &= 0 \\
-708588a_1 - 1062882a_2 + 87480a_3 - 1771470b_3 &= 0 \\
-177147a_1 - 177147a_2 + 29160a_3 - 354294b_3 &= 0 \\
-9720a_1 + 314928b_1 + 341172b_2 - 9720b_3 &= 0 \\
-4860a_2 + 314928b_1 + 669222b_2 - 4860b_3 &= 0 \\
-9720a_1 + 4860a_2 - 400a_3 + 104976b_1 + 59049b_2 - 4860b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_3 \\
a_2 &= -b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 - x \\
\eta &= u
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,u)\,\xi \\
&= u - \left(\frac{729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20}{243\,(1+x)^2}\right)(-1-x) \\
&= \frac{-20 + 729(1+x)^3\,u^3 + (-189x - 189)\,u}{243 + 243x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{du}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u}\right) S(x,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-20+729(1+x)^3u^3+(-189x-189)u}{243+243x}} dy
\end{aligned}
$$

Which results in

$$S = (243 + 243x)\left(-\frac{\ln(9ux + 9u + 1)}{18(9x + 9)} + \frac{\ln(9ux + 9u + 4)}{243 + 243x} + \frac{\ln(9ux + 9u - 5)}{486x + 486}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, u)S_u}{R_x + \omega(x, u)R_u} \tag{2}$$

Where in the above R_x, R_u, S_x, S_u are all partial derivatives and $\omega(x, u)$ is the right hand side of the original ode given by

$$\omega(x, u) = \frac{729u^3x^3 + 2187u^3x^2 + 2187u^3x + 729u^3 - 432ux - 432u - 20}{243(1 + x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_u &= 0 \\ S_x &= \frac{243u}{-20 + (9x + 9)^3 u^3 + (-189x - 189)u} \\ S_u &= \frac{243 + 243x}{-20 + (9x + 9)^3 u^3 + (-189x - 189)u} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{1 + x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{1}{R + 1}\, dR \\ S(R) &= \ln(R + 1) + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, u coordinates. This results in

$$-\frac{3\ln\left(1+u(x)\left(9x+9\right)\right)}{2}+\ln\left(u(x)\left(9x+9\right)+4\right)+\frac{\ln\left(-5+u(x)\left(9x+9\right)\right)}{2}=\ln\left(1+x\right)+c_2$$

Substituting $u = u(x) - \frac{4}{3(3x+3)}$ in the above solution gives

$$-\frac{3\ln\left(1+\left(u(x)-\frac{4}{3(3x+3)}\right)\left(9x+9\right)\right)}{2}+\ln\left(\left(u(x)-\frac{4}{3\left(3x+3\right)}\right)\left(9x+9\right)+4\right)+\frac{\ln\left(-5+\left(u(x)-\frac{4}{3(3x+\right.\right.}}{2}$$

Simplifying the above gives

$$\frac{3\ln\left(3\right)}{2}-\frac{3\ln\left(-1+\left(3x+3\right)u(x)\right)}{2}+\ln\left(u(x)\left(x+1\right)\right)+\frac{\ln\left(u(x)\,x+u(x)-1\right)}{2}=\ln\left(x+1\right)+c_2$$

Substituting $u(x) = \frac{1}{2x+y(x)+2}$ in the above solution gives

$$\frac{3\ln\left(3\right)}{2}-\frac{3\ln\left(-1+\frac{3x+3}{2x+y(x)+2}\right)}{2}+\ln\left(\frac{x+1}{2x+y\left(x\right)+2}\right)+\frac{\ln\left(\frac{x}{2x+y(x)+2}+\frac{1}{2x+y(x)+2}-1\right)}{2}$$
$$=\ln\left(x+1\right)+c_2$$

Simplifying the above gives

$$\frac{3\ln\left(3\right)}{2}-\frac{3\ln\left(\frac{-y(x)+x+1}{2x+y(x)+2}\right)}{2}+\ln\left(\frac{x+1}{2x+y\left(x\right)+2}\right)+\frac{\ln\left(\frac{-x-y(x)-1}{2x+y(x)+2}\right)}{2}=\ln\left(x+1\right)+c_2$$

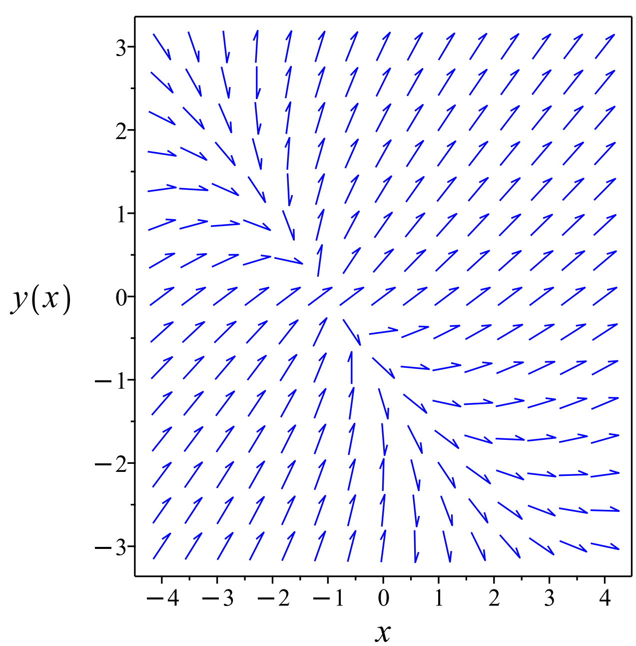


Figure 2.10: Slope field $y'(x) = \frac{2y(x)+x+1}{2x+y(x)+2}$

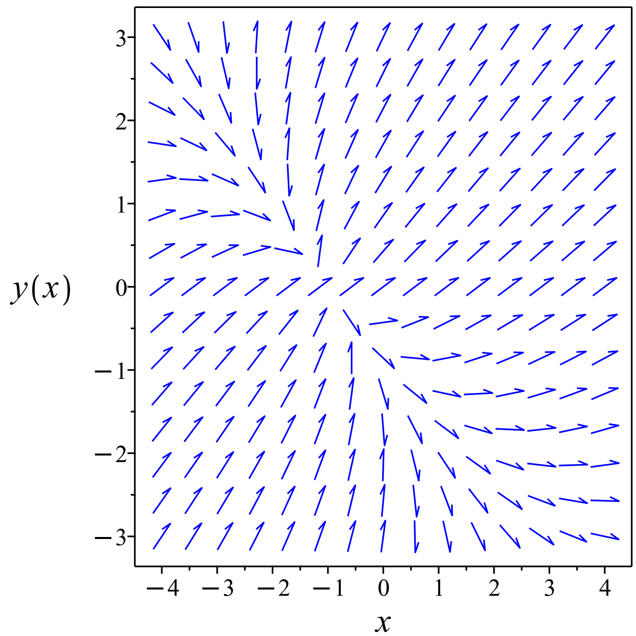


Figure 2.11: Slope field $y'(x) = \frac{2y(x)+x+1}{2x+y(x)+2}$

Summary of solutions found

$$\frac{3\ln(3)}{2} - \frac{3\ln\left(\frac{-y(x)+x+1}{2x+y(x)+2}\right)}{2} + \ln\left(\frac{x+1}{2x+y(x)+2}\right) + \frac{\ln\left(\frac{-x-y(x)-1}{2x+y(x)+2}\right)}{2} = \ln(x+1) + c_2$$

Solved using first_order_ode_LIE

Time used: 16.804 (sec)

Solve

$$y'(x) = \frac{2y(x)+x+1}{2x+y(x)+2}$$

Writing the ode as

$$y'(x) = \frac{2y+x+1}{2x+y+2}$$
$$y'(x) = \omega(x, y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_{y(x)} - \xi_x) - \omega^2\xi_{y(x)} - \omega_x\xi - \omega_{y(x)}\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2 + \frac{(2y+x+1)(b_3-a_2)}{2x+y+2} - \frac{(2y+x+1)^2 a_3}{(2x+y+2)^2} \\
&- \left(\frac{1}{2x+y+2} - \frac{2(2y+x+1)}{(2x+y+2)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{2}{2x+y+2} - \frac{2y+x+1}{(2x+y+2)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2 + x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 - 4xyb_2 - 2xyb_3 + 2y^2a_2 + y^2a_3 - y^2b_2 - 2y^2b_3 + 4xa_2 + 2}{(2x+y+2)^2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-2x^2a_2 - x^2a_3 + x^2b_2 + 2x^2b_3 - 2xya_2 - 4xya_3 + 4xyb_2 + 2xyb_3 \\
&- 2y^2a_2 - y^2a_3 + y^2b_2 + 2y^2b_3 - 4xa_2 - 2xa_3 - 3xb_1 + 5xb_2 + 4xb_3 \\
&+ 3ya_1 - 5ya_2 - 4ya_3 + 4yb_2 + 2yb_3 - 2a_2 - a_3 - 3b_1 + 4b_2 + 2b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&-2a_2v_1^2 - 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 - 4a_3v_1v_2 - a_3v_2^2 + b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 \\
&+ 2b_3v_1^2 + 2b_3v_1v_2 + 2b_3v_2^2 + 3a_1v_2 - 4a_2v_1 - 5a_2v_2 - 2a_3v_1 - 4a_3v_2 \\
&- 3b_1v_1 + 5b_2v_1 + 4b_2v_2 + 4b_3v_1 + 2b_3v_2 - 2a_2 - a_3 - 3b_1 + 4b_2 + 2b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-2a_2 - a_3 + b_2 + 2b_3)\, v_1^2 + (-2a_2 - 4a_3 + 4b_2 + 2b_3)\, v_1 v_2 \\
&+ (-4a_2 - 2a_3 - 3b_1 + 5b_2 + 4b_3)\, v_1 + (-2a_2 - a_3 + b_2 + 2b_3)\, v_2^2 \\
&+ (3a_1 - 5a_2 - 4a_3 + 4b_2 + 2b_3)\, v_2 - 2a_2 - a_3 - 3b_1 + 4b_2 + 2b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0 \\
-2a_2 - a_3 + b_2 + 2b_3 &= 0 \\
3a_1 - 5a_2 - 4a_3 + 4b_2 + 2b_3 &= 0 \\
-4a_2 - 2a_3 - 3b_1 + 5b_2 + 4b_3 &= 0 \\
-2a_2 - a_3 - 3b_1 + 4b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_3 \\
a_2 &= b_3 \\
a_3 &= b_2 \\
b_1 &= b_2 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= y \\
\eta &= x + 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x + 1 - \left(\frac{2y + x + 1}{2x + y + 2}\right)(y) \\
&= \frac{2x^2 - 2y^2 + 4x + 2}{2x + y + 2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x^2-2y^2+4x+2}{2x+y+2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x+y+1)}{4} - \frac{3\ln(-1-x+y)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y + x + 1}{2x + y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-2y - x - 1}{2(x+y+1)(x-y+1)} \\ S_y &= \frac{2x + y + 2}{2(x+y+1)(x-y+1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

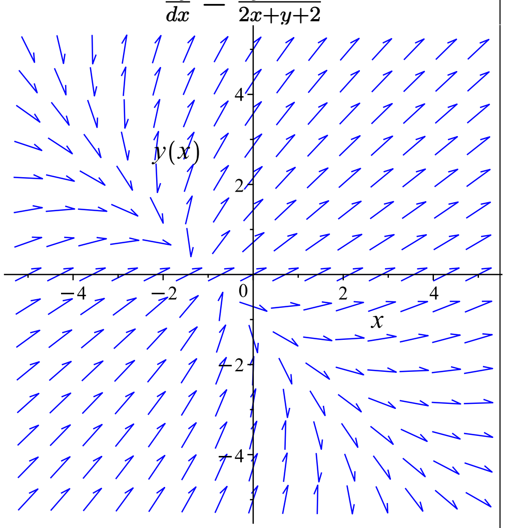
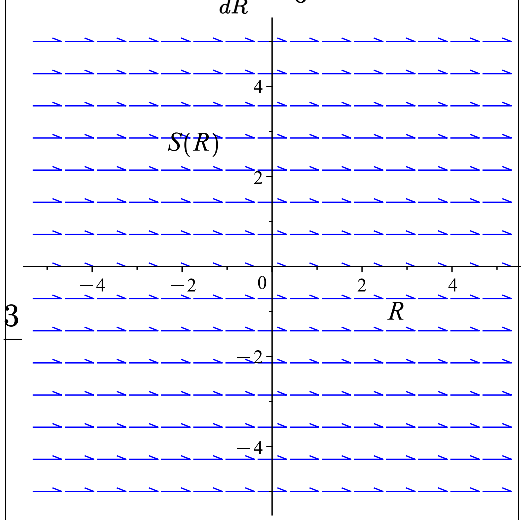
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(x + y(x) + 1)}{4} - \frac{3\ln(y(x) - x - 1)}{4} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y+x+1}{2x+y+2}$ 	$R = x$ $S = \frac{\ln(x+y+1)}{4} - 3$	$\frac{dS}{dR} = 0$ 

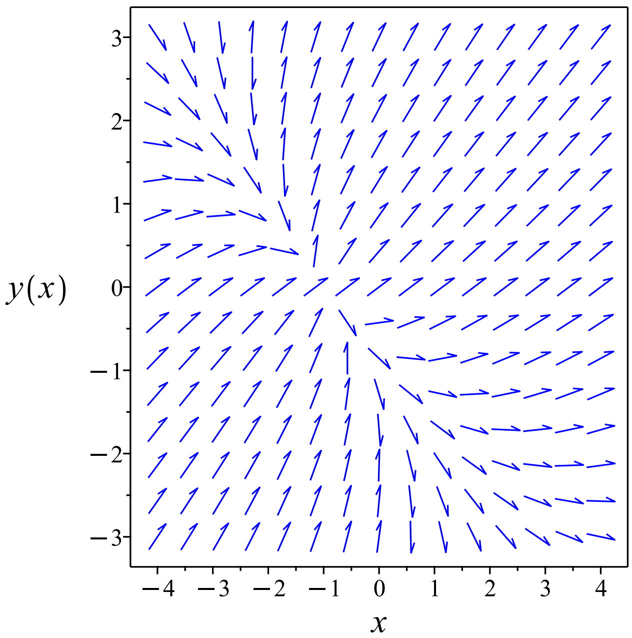


Figure 2.12: Slope field $y'(x) = \frac{2y(x)+x+1}{2x+y(x)+2}$

Summary of solutions found

$$\frac{\ln\left(x+y(x)+1\right)}{4}-\frac{3\ln\left(y(x)-x-1\right)}{4}=c_2$$

✓ **Maple.** `Time used: 0.096 (sec). Leaf size: 117`

```
ode:=diff(y(x),x) = (x+2*y(x)+1)/(2*x+2+y(x));
dsolve(ode,y(x), singsol=all);
```

$y =$

$$-\frac{\left(i\sqrt{3}-1\right)\left(3\sqrt{3}\sqrt{27c_1^2\left(x+1\right)^2-1}-27(x+1)\,c_1\right)^{2/3}-3i\sqrt{3}-3-6\left(3\sqrt{3}\sqrt{27c_1^2\left(x+1\right)^2-1}-27\right.}{6\left(3\sqrt{3}\sqrt{27c_1^2\left(x+1\right)^2-1}-27\left(x+1\right)c_1\right)^{1/3}c_1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
```

```
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \frac{x+2y(x)+1}{2x+y(x)+2}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{x+2y(x)+1}{2x+y(x)+2}$

✓ **Mathematica.** Time used: 60.123 (sec). Leaf size: 1598

```
ode=D[y[x],x]==(x+2*y[x]+1)/(2*x+y[x]+2);
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Too large to display

✓ **Sympy.** Time used: 98.379 (sec). Leaf size: 371

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq((-x - 2*y(x) - 1)/(2*x + y(x) + 2) + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$\left[y(x) = \frac{\frac{2 \cdot 3^{\frac{2}{3}} i C_1}{3 \sqrt[3]{C_1 \left(9x + \sqrt{3}\sqrt{C_1 + 27x^2 + 54x + 27} + 9\right)}} + \sqrt{3}x - ix + \frac{3^{\frac{5}{6}} \sqrt[3]{C_1 \left(9x + \sqrt{3}\sqrt{C_1 + 27x^2 + 54x\right.}}{3}}{\sqrt{3} - i} \right.$$

$$\left. + x - \frac{\sqrt[3]{3} \sqrt[3]{C_1 \left(9x + \sqrt{3}\sqrt{C_1 + 27x^2 + 54x + 27} + 9\right)}}{3} + 1 \right]$$

2.1.5 Problem (e)

Local contents

Solved using first_order_ode_dAlembert 84
Solved using first_order_ode_homog_type_maple_C 88
Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind 93
Solved using first_order_ode_LIE 99
✓Maple 104
✓Mathematica 105
✓Sympy 105

Internal problem ID [20966]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XII at page 23
Problem number : (e)
Date solved : `Saturday, November 29, 2025 at 01:13:03 AM`
CAS classification :
`[[_homogeneous, 'class C'], _rational, [_Abel, '2nd type', 'class A']]`

Solved using first_order_ode_dAlembert

Time used: 2.633 (sec)

Solve

$$y' = \frac{2x + y + 1}{x + 2y + 2}$$

Let $p = y'$ the ode becomes

$$p = \frac{2x + y + 1}{x + 2y + 2}$$

Solving for y from the above results in

$$y = -1 - \frac{(p - 2)\, x}{2p - 1} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-p+2}{2p-1} \\ g &= -1 \end{aligned}$$

Hence (2) becomes

$$p - \frac{-p+2}{2p-1} = \left(-\frac{x}{2p-1} + \frac{2xp}{(2p-1)^2} - \frac{4x}{(2p-1)^2}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-p+2}{2p-1} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= 1 \\ p_2 &= -1 \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= x - 1 \\ y &= -1 - x \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-p(x)+2}{2p(x)-1}}{-\frac{x}{2p(x)-1} + \frac{2xp(x)}{(2p(x)-1)^2} - \frac{4x}{(2p(x)-1)^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$p'(x) = -\frac{2(p(x)-1)(2p(x)-1)(p(x)+1)}{3x} \tag{2.7}$$

is separable as it can be written as

$$\begin{aligned}
p'(x) &= -\frac{2(p(x)-1)\,(2p(x)-1)\,(p(x)+1)}{3x} \\
&= f(x)g(p)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= -\frac{2}{3x} \\
g(p) &= (p-1)\,(2p-1)\,(p+1)
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\
\int \frac{1}{(p-1)\,(2p-1)\,(p+1)}\,dp &= \int -\frac{2}{3x}\,dx
\end{aligned}$$

$$\frac{\ln(p(x)+1)}{6} - \frac{2\ln(2p(x)-1)}{3} + \frac{\ln(p(x)-1)}{2} = \ln\left(\frac{1}{x^{2/3}}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$(p-1)\,(2p-1)\,(p+1) = 0$$

for $p(x)$ gives

$$\begin{aligned}
p(x) &= -1 \\
p(x) &= 1
\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\frac{\ln(p(x)+1)}{6} - \frac{2\ln(2p(x)-1)}{3} + \frac{\ln(p(x)-1)}{2} &= \ln\left(\frac{1}{x^{2/3}}\right) + c_1 \\
p(x) &= -1 \\
p(x) &= 1
\end{aligned}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(-\operatorname{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Ze^{6c_1}-e^{6c_1}\right)+1\right)}{2\operatorname{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Ze^{6c_1}-e^{6c_1}\right)+1} - 1$$

$$y = -1 - x$$

$$y = x - 1$$

Simplifying the above gives

$$y = x - 1$$

$$y = -1 - x$$

$$y = \frac{(-x-2)\operatorname{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Ze^{6c_1}-e^{6c_1}\right)+x-1}{2\operatorname{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Ze^{6c_1}-e^{6c_1}\right)+1}$$

$$y = -1 - x$$

$$y = x - 1$$

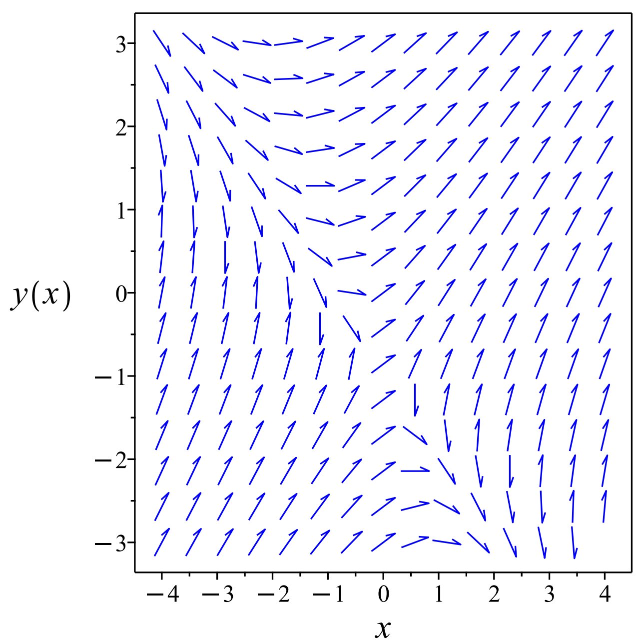


Figure 2.13: Slope field $y' = \frac{2x+y+1}{x+2y+2}$

Summary of solutions found

$$y = \frac{(-x-2)\,\mathrm{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Z e^{6c_1}-e^{6c_1}\right)+x-1}{2\,\mathrm{RootOf}\left(\left(-16\,e^{6c_1}+x^4\right)\_Z^4+\left(-32\,e^{6c_1}+2x^4\right)\_Z^3-24\_Z^2e^{6c_1}-8\_Z e^{6c_1}-e^{6c_1}\right)+1}$$

$$y = -1-x$$

$$y = x-1$$

Solved using first_order_ode_homog_type_maple_C

Time used: 1.127 (sec)

Solve

$$y' = \frac{2x+y+1}{x+2y+2}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2X+2x_0+Y(X)+y_0+1}{X+x_0+2Y(X)+2y_0+2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2X+Y(X)}{X+2Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{2X+Y}{X+2Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 2X+Y$ and $N = X+2Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u+2}{2u+1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+2}{2u(X)+1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+2}{2u(X)+1} - u(X)}{X} = 0$$

Or

$$2\left(\frac{d}{dX}u(X)\right)Xu(X) + \left(\frac{d}{dX}u(X)\right)X + 2u(X)^2 - 2 = 0$$

Or

$$-2 + (2u(X)+1)X\left(\frac{d}{dX}u(X)\right) + 2u(X)^2 = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = -\frac{2(u(X)-1)(u(X)+1)}{(2u(X)+1)X} \tag{2.8}$$

is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{2(u(X)-1)(u(X)+1)}{(2u(X)+1)X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{2}{X}\\ g(u) &= \frac{(u-1)(u+1)}{2u+1}\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$

$$\int \frac{2u+1}{(u-1)(u+1)}\,du = \int -\frac{2}{X}\,dX$$

$$\frac{\ln\left(u(X)+1\right)}{2}+\frac{3\ln\left(u(X)-1\right)}{2}=\ln\left(\frac{1}{X^{2}}\right)+c_{1}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{\left(u-1\right)\left(u+1\right)}{2u+1}=0$$

for $u(X)$ gives

$$u(X)=-1$$
$$u(X)=1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)+1\right)}{2}+\frac{3\ln\left(u(X)-1\right)}{2}=\ln\left(\frac{1}{X^{2}}\right)+c_{1}$$
$$u(X)=-1$$
$$u(X)=1$$

Converting $\frac{\ln(u(X)+1)}{2}+\frac{3\ln(u(X)-1)}{2}=\ln\left(\frac{1}{X^{2}}\right)+c_{1}$ back to $Y(X)$ gives

$$\frac{\ln\left(\frac{Y(X)+X}{X}\right)}{2}+\frac{3\ln\left(\frac{Y(X)-X}{X}\right)}{2}=\ln\left(\frac{1}{X^{2}}\right)+c_{1}$$

Converting $u(X)=-1$ back to $Y(X)$ gives

$$Y(X)=-X$$

Converting $u(X)=1$ back to $Y(X)$ gives

$$Y(X)=X$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)+X}{X}\right)}{2}+\frac{3\ln\left(\frac{Y(X)-X}{X}\right)}{2}=\ln\left(\frac{1}{X^{2}}\right)+c_{1} \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 1 \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
\frac{\ln\left(\frac{x+y+1}{x}\right)}{2} + \frac{3\ln\left(\frac{y-x+1}{x}\right)}{2} = \ln\left(\frac{1}{x^2}\right) + c_1
$$

Using the solution for $Y(X)$

$$
Y(X) = -X \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 1 \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y + 1 = -x
$$

Using the solution for $Y(X)$

$$
Y(X) = X \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x_0 + x
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y - 1 \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$y+1=x$$

Solving for y gives

$$\frac{\ln\left(\frac{x+y+1}{x}\right)}{2}+\frac{3\ln\left(\frac{y-x+1}{x}\right)}{2}=\ln\left(\frac{1}{x^2}\right)+c_1$$
$$y=-1-x$$
$$y=x-1$$

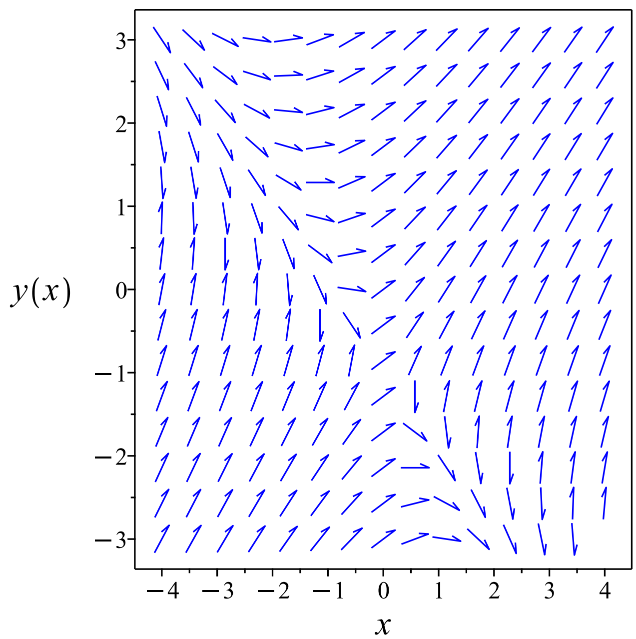


Figure 2.14: Slope field $y'=\frac{2x+y+1}{x+2y+2}$

Summary of solutions found

$$\frac{\ln\left(\frac{x+y+1}{x}\right)}{2}+\frac{3\ln\left(\frac{y-x+1}{x}\right)}{2}=\ln\left(\frac{1}{x^2}\right)+c_1$$
$$y=-1-x$$
$$y=x-1$$

Solve

$$y'=\frac{2x+y+1}{x+2y+2}$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u(x) + xu'(x) = \frac{2x + xu(x) + 1}{x + 2xu(x) + 2}$$

Which is now solved Unknown ode type.

Solved using first__order__ode__abel__second__kind__solved__by__converting__to__first__kind

Time used: 1.480 (sec)

This is Abel second kind ODE, it has the form

$$(y(x) + g)\, y'(x) = f_0(x) + f_1(x)y(x) + f_2(x)y(x)^2 + f_3(x)y(x)^3$$

Comparing the above to given ODE which is

$$y'(x) = \frac{2x + y(x) + 1}{x + 2y(x) + 2} \tag{1}$$

Shows that

$$\begin{aligned}
g &= 1 + \frac{x}{2} \\
f_0 &= x + \frac{1}{2} \\
f_1 &= \frac{1}{2} \\
f_2 &= 0 \\
f_3 &= 0
\end{aligned}$$

Applying transformation

$$y(x) = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u'(x) = -\frac{3u(x)^3 x}{4} - u(x)^2$$

Which is now solved.

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,u)\,\mathrm{d}x + N(x,u)\,\mathrm{d}u = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}u &= \left(-\frac{3}{4}u^3x - u^2\right)\mathrm{d}x\\ \left(\frac{3}{4}u^3x + u^2\right)\mathrm{d}x + \mathrm{d}u &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,u) &= \frac{3}{4}u^3x + u^2\\ N(x,u) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial u} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial}{\partial u}\left(\frac{3}{4}u^3x + u^2\right)\\ &= \frac{9}{4}x\,u^2 + 2u\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial u} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial u} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{9}{4}x\,u^2 + 2u\right) - (0)\right)\\ &= \frac{9}{4}x\,u^2 + 2u\end{aligned}$$

Since A depends on u, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial u}\right)\\ &= \frac{4}{u^2\,(3ux+4)}\left((0) - \left(\frac{9}{4}x\,u^2 + 2u\right)\right)\\ &= \frac{-9ux-8}{3x\,u^2+4u}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial u}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xu$. Therefore

$$\begin{aligned}R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial u}}{xM - yN}\\ &= \frac{(0) - \left(\frac{9}{4}x\,u^2 + 2u\right)}{x\left(\frac{3}{4}u^3x + u^2\right) - u\,(1)}\\ &= \frac{-9ux-8}{3u^2x^2+4ux-4}\end{aligned}$$

Replacing all powers of terms xu by t gives

$$R = \frac{-9t - 8}{3t^2 + 4t - 4}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}\mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(\frac{-9t-8}{3t^2+4t-4}\right) \mathrm{d}t}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{5\ln(t+2)}{4} - \frac{7\ln(3t-2)}{4}} \\ &= \frac{1}{(t+2)^{5/4}(3t-2)^{7/4}}\end{aligned}$$

Now t is replaced back with xu giving

$$\mu = \frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}\left(\frac{3}{4}u^3x + u^2\right) \\ &= \frac{u^2(3ux+4)}{4(ux+2)^{5/4}(3ux-2)^{7/4}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}(1) \\ &= \frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}u}{\mathrm{d}x} &= 0 \\ \left(\frac{u^2(3ux+4)}{4(ux+2)^{5/4}(3ux-2)^{7/4}}\right) + \left(\frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}\right)\frac{\mathrm{d}u}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,u)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial u} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{u^2(3ux+4)}{4\,(ux+2)^{5/4}\,(3ux-2)^{7/4}} \,\mathrm{d}x$$

$$\phi = -\frac{u}{4\,(ux+2)^{1/4}\,(3ux-2)^{3/4}} + f(u) \tag{3}$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both x and u. Taking derivative of equation (3) w.r.t u gives

$$\begin{aligned}\frac{\partial \phi}{\partial u} &= \frac{ux}{16\,(ux+2)^{5/4}\,(3ux-2)^{3/4}} + \frac{9ux}{16\,(ux+2)^{1/4}\,(3ux-2)^{7/4}} \\ &\quad - \frac{1}{4\,(ux+2)^{1/4}\,(3ux-2)^{3/4}} + f'(u) \\ &= \frac{1}{(ux+2)^{5/4}\,(3ux-2)^{7/4}} + f'(u)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial u} = \frac{1}{(ux+2)^{5/4}(3ux-2)^{7/4}}$. Therefore equation (4) becomes

$$\frac{1}{(ux+2)^{5/4}\,(3ux-2)^{7/4}} = \frac{1}{(ux+2)^{5/4}\,(3ux-2)^{7/4}} + f'(u) \tag{5}$$

Solving equation (5) for $f'(u)$ gives

$$f'(u) = 0$$

Therefore

$$f(u) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(u)$ into equation (3) gives ϕ

$$\phi = -\frac{u}{4\,(ux+2)^{1/4}\,(3ux-2)^{3/4}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{u}{4\left(ux+2\right)^{1/4}\left(3ux-2\right)^{3/4}}$$

Substituting $u(x) = \frac{1}{y(x)+1+\frac{x}{2}}$ in the above solution gives

$$-\frac{1}{4\left(\frac{x}{y(x)+1+\frac{x}{2}}+2\right)^{1/4}\left(\frac{3x}{y(x)+1+\frac{x}{2}}-2\right)^{3/4}\left(y\left(x\right)+1+\frac{x}{2}\right)} = c_1$$

Simplifying the above gives

$$-\frac{1}{\left(\frac{-y(x)+x-1}{x+2y(x)+2}\right)^{3/4}\left(\frac{x+y(x)+1}{x+2y(x)+2}\right)^{1/4}\left(8x+16y\left(x\right)+16\right)} = c_1$$

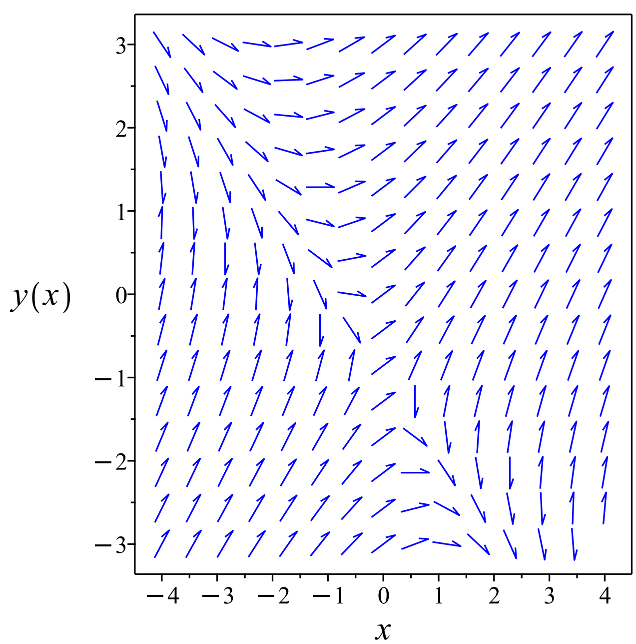


Figure 2.15: Slope field $y'(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$

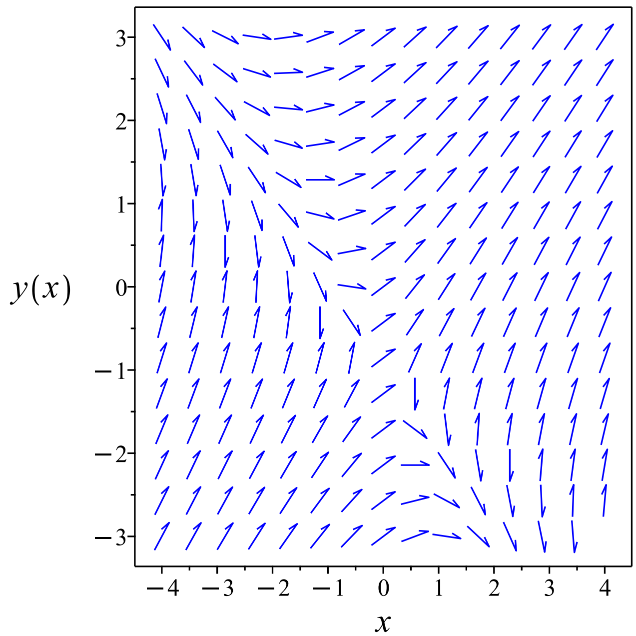


Figure 2.16: Slope field $y'(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$

Summary of solutions found

$$-\frac{1}{\left(\frac{-y(x)+x-1}{x+2y(x)+2}\right)^{3/4}\left(\frac{x+y(x)+1}{x+2y(x)+2}\right)^{1/4}(8x+16y(x)+16)} = c_1$$

Solved using first_order_ode_LIE

Time used: 18.617 (sec)

Solve

$$y'(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$$

Writing the ode as

$$y'(x) = \frac{2x+y+1}{x+2y+2}$$
$$y'(x) = \omega(x, y(x))$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega\left(\eta_{y(x)} - \xi_x\right) - \omega^2\xi_{y(x)} - \omega_x\xi - \omega_{y(x)}\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{(2x+y+1)\,(b_3-a_2)}{x+2y+2}-\frac{(2x+y+1)^2\,a_3}{(x+2y+2)^2} \\ & -\left(\frac{2}{x+2y+2}-\frac{2x+y+1}{(x+2y+2)^2}\right)(xa_2+ya_3+a_1) \\ & -\left(\frac{1}{x+2y+2}-\frac{2(2x+y+1)}{(x+2y+2)^2}\right)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2x^2a_2+4x^2a_3-4x^2b_2-2x^2b_3+8xya_2+4xya_3-4xyb_2-8xyb_3+2y^2a_2+4y^2a_3-4y^2b_2-2y^2b_3+8xa_2}{(x+2y+2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -2x^2a_2-4x^2a_3+4x^2b_2+2x^2b_3-8xya_2-4xya_3+4xyb_2+8xyb_3 \\ & -2y^2a_2-4y^2a_3+4y^2b_2+2y^2b_3-8xa_2-4xa_3+3xb_1+4xb_2+5xb_3 \\ & -3ya_1-4ya_2-5ya_3+8yb_2+4yb_3-3a_1-2a_2-a_3+4b_2+2b_3=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2-8a_2v_1v_2-2a_2v_2^2-4a_3v_1^2-4a_3v_1v_2-4a_3v_2^2+4b_2v_1^2+4b_2v_1v_2 \\ & +4b_2v_2^2+2b_3v_1^2+8b_3v_1v_2+2b_3v_2^2-3a_1v_2-8a_2v_1-4a_2v_2-4a_3v_1-5a_3v_2 \\ & +3b_1v_1+4b_2v_1+8b_2v_2+5b_3v_1+4b_3v_2-3a_1-2a_2-a_3+4b_2+2b_3=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-2a_2 - 4a_3 + 4b_2 + 2b_3)\, v_1^2 + (-8a_2 - 4a_3 + 4b_2 + 8b_3)\, v_1 v_2 \\&+ (-8a_2 - 4a_3 + 3b_1 + 4b_2 + 5b_3)\, v_1 + (-2a_2 - 4a_3 + 4b_2 + 2b_3)\, v_2^2 \\&+ (-3a_1 - 4a_2 - 5a_3 + 8b_2 + 4b_3)\, v_2 - 3a_1 - 2a_2 - a_3 + 4b_2 + 2b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-8a_2 - 4a_3 + 4b_2 + 8b_3 &= 0\\ -2a_2 - 4a_3 + 4b_2 + 2b_3 &= 0\\ -3a_1 - 4a_2 - 5a_3 + 8b_2 + 4b_3 &= 0\\ -3a_1 - 2a_2 - a_3 + 4b_2 + 2b_3 &= 0\\ -8a_2 - 4a_3 + 3b_1 + 4b_2 + 5b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= b_2\\ a_2 &= b_3\\ a_3 &= b_2\\ b_1 &= b_3\\ b_2 &= b_2\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y + 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi\\ &= y + 1 - \left(\frac{2x + y + 1}{x + 2y + 2}\right)(x)\\ &= \frac{-2x^2 + 2y^2 + 4y + 2}{x + 2y + 2}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2+2y^2+4y+2}{x+2y+2}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(x + y + 1)}{4} + \frac{3 \ln(-x + y + 1)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x + y + 1}{x + 2y + 2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + y + 1}{2(x + y + 1)(x - y - 1)} \\ S_y &= \frac{-x - 2y - 2}{2(x + y + 1)(x - y - 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

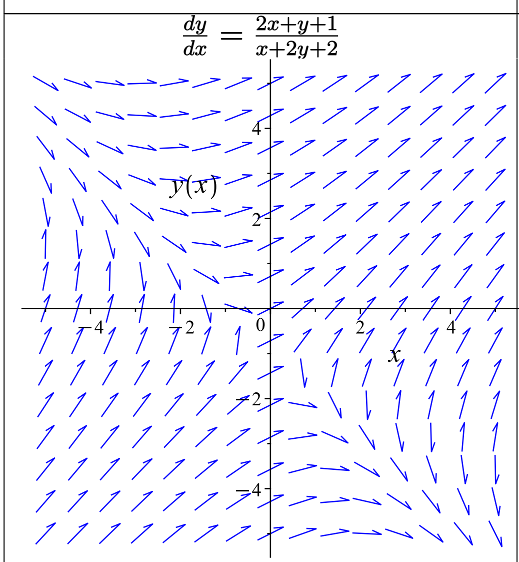
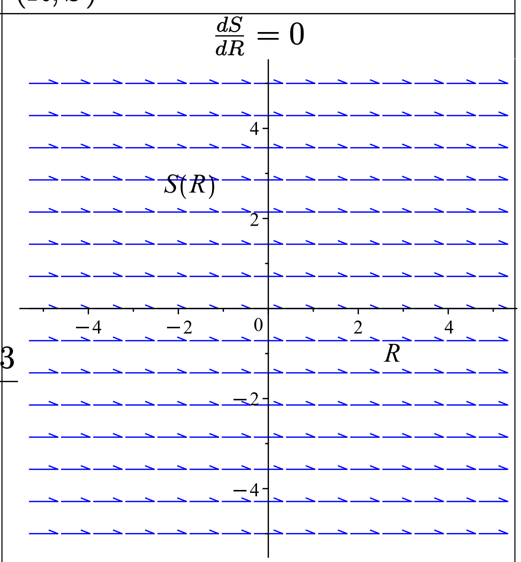
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0 \, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(x + y(x) + 1)}{4} + \frac{3\ln(y(x) - x + 1)}{4} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = x$ $S = \frac{\ln(x + y + 1)}{4} + 3$	

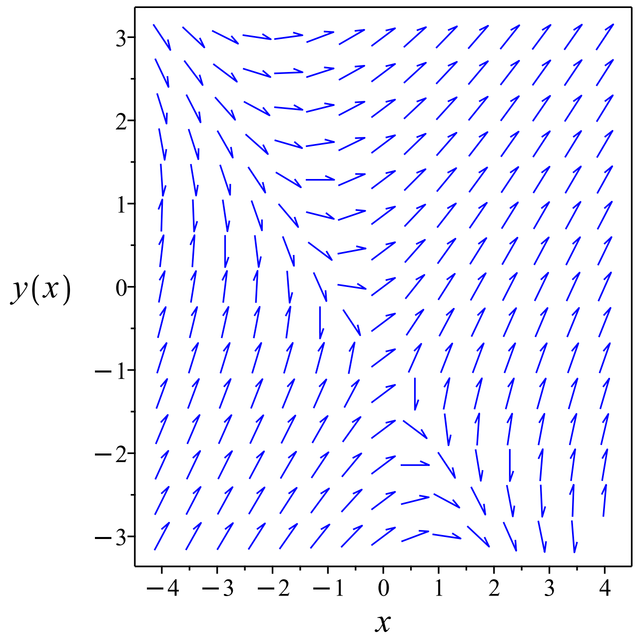


Figure 2.17: Slope field $y'(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$

Summary of solutions found

$$\frac{\ln (x+y(x)+1)}{4}+\frac{3 \ln (y(x)-x+1)}{4}=c_2$$

✓ **Maple.** Time used: 0.293 (sec). Leaf size: 29

```
ode:=diff(y(x),x) = (2*x+y(x)+1)/(x+2*y(x)+2);
dsolve(ode,y(x), singsol=all);
```

$$y=\frac{-\operatorname{RootOf}\left(-2 c_1 \_Z^3 x+\_Z^4-1\right)+(x-1) c_1}{c_1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
```

```
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{2x+y(x)+1}{x+2y(x)+2}$

✓ **Mathematica.** Time used: 60.121 (sec). Leaf size: 2637

```
ode=D[y[x],x]==(2*x+y[x]+1)/(x+2*y[x]+2);
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Too large to display

✓ **Sympy.** Time used: 48.169 (sec). Leaf size: 1122

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(Derivative(y(x), x) - (2*x + y(x) + 1)/(x + 2*y(x) + 2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

Solution too large to show

2.2 Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24

Local contents

2.2.1 Problem (a) . . . 107
2.2.2 Problem (c) . . . 120
2.2.3 Problem (d) . . . 129
2.2.4 Problem (e) . . . 146
2.2.5 Problem (f) . . . 154
2.2.6 Problem (g) . . . 163
2.2.7 Problem (h) . . . 180
2.2.8 Problem (i) . . . 203

2.2.1 Problem (a)

Local contents

Solved using first_order_ode_autonomous 107
Solved using first_order_ode_bernoulli 109
Solved using first_order_ode_exact 111
Solved using first_order_ode_dAlembert 115
✓Maple 118
✓Mathematica 119
✓Sympy 119

Internal problem ID [20967]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:14:10 AM`
CAS classification : `[_quadrature]`

Solved using first_order_ode_autonomous

Time used: 0.061 (sec)

Solve

$$y' = 3y^{2/3}$$

Integrating gives

$$\int \frac{1}{3y^{2/3}}dy = dx$$
$$y^{1/3} = x + c_1$$

Singular solutions are found by solving

$$3y^{2/3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

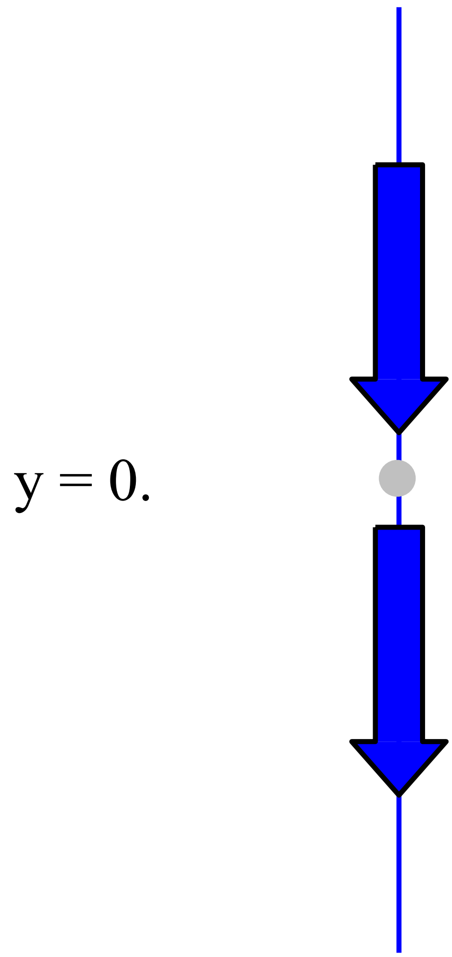


Figure 2.18: Phase line diagram

Solving for y gives

$$
\begin{aligned}
y &= 0 \\
y &= c_1^3 + 3c_1^2 x + 3c_1 x^2 + x^3
\end{aligned}
$$

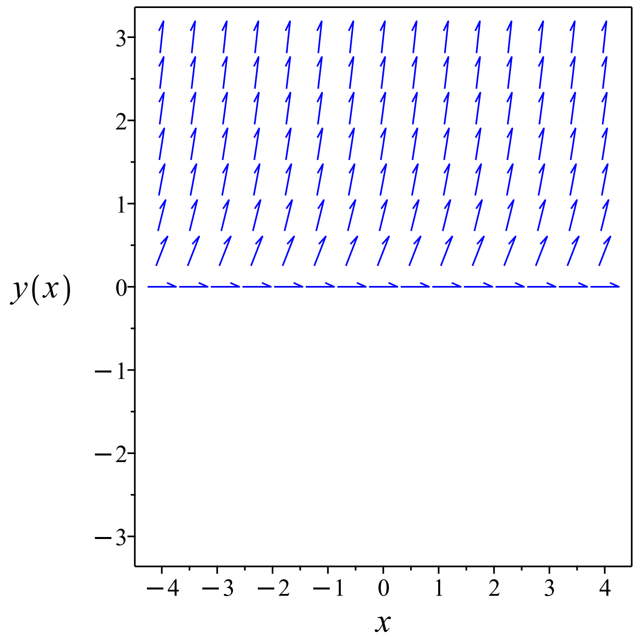


Figure 2.19: Slope field $y' = 3y^{2/3}$

Summary of solutions found

$$y = 0$$
$$y = c_1^3 + 3c_1^2 x + 3c_1 x^2 + x^3$$

Solved using first_order_ode_bernoulli

Time used: 0.170 (sec)

Solve

$$y' = 3y^{2/3}$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 3y^{2/3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = (3)\, y^{2/3} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$f_0 = 0$$
$$f_1 = 3$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 0 \\ f_1(x) &= 3 \\ n &= \frac{2}{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{2/3}$ gives

$$y' \frac{1}{y^{2/3}} = 0 + 3 \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= y^{1/3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = \frac{1}{3y^{2/3}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 3v'(x) &= 3 \\ v' &= 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dv &= \int 1 \, dx \\ v(x) &= x + c_1 \end{aligned}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$y^{1/3} = x + c_1$$

Solving for y gives

$$y = c_1^3 + 3c_1^2 x + 3c_1 x^2 + x^3$$

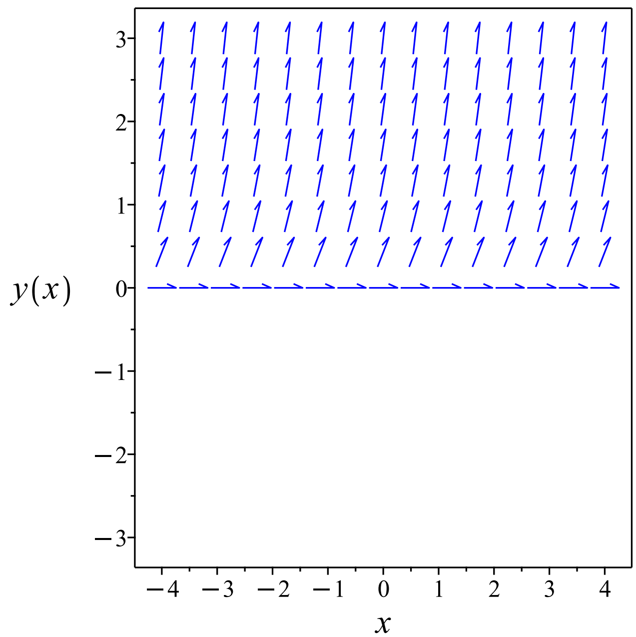


Figure 2.20: Slope field $y' = 3y^{2/3}$

Summary of solutions found

$$y = c_1^3 + 3c_1^2 x + 3c_1 x^2 + x^3$$

Solved using first_order_ode_exact

Time used: 0.127 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(3y^{2/3}\right)\mathrm{d}x\\ \left(-3y^{2/3}\right)\mathrm{d}x+\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3y^{2/3}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-3y^{2/3}\right)\\ &=-\frac{2}{y^{1/3}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=1\left(\left(-\frac{2}{y^{1/3}}\right)-(0)\right)\\ &=-\frac{2}{y^{1/3}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{1}{3y^{2/3}}\left((0) - \left(-\frac{2}{y^{1/3}}\right)\right) \\ &= -\frac{2}{3y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{3y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{2\ln(y)}{3}} \\ &= \frac{1}{y^{2/3}} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^{2/3}}\left(-3y^{2/3}\right) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^{2/3}}(1) \\ &= \frac{1}{y^{2/3}} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-3) + \left(\frac{1}{y^{2/3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -3\,\mathrm{d}x$$
$$\phi = -3x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y^{2/3}}$. Therefore equation (4) becomes

$$\frac{1}{y^{2/3}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^{2/3}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^{2/3}}\right)\mathrm{d}y$$
$$f(y) = 3y^{1/3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3x + 3y^{1/3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -3x + 3y^{1/3}$$

Solving for y gives

$$y = \frac{1}{27}c_1^3 + \frac{1}{3}c_1^2 x + c_1 x^2 + x^3$$

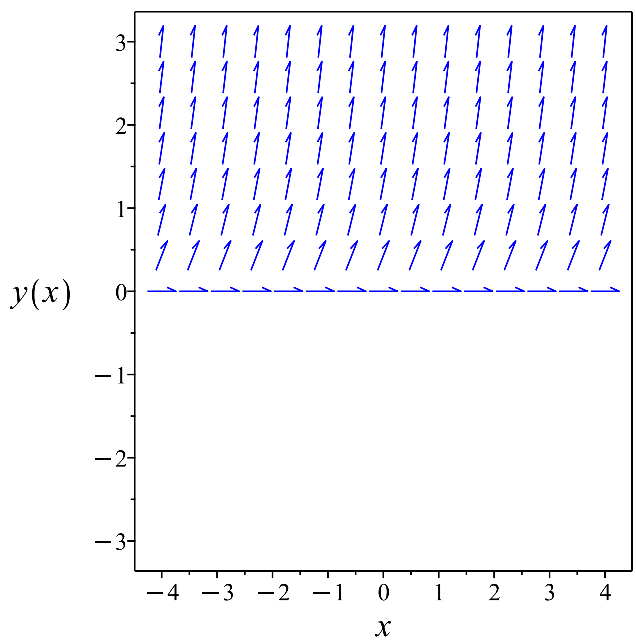


Figure 2.21: Slope field $y' = 3y^{2/3}$

Summary of solutions found

$$y = \frac{1}{27}c_1^3 + \frac{1}{3}c_1^2 x + c_1 x^2 + x^3$$

Solved using first_order_ode_dAlembert

Time used: 0.703 (sec)

Solve

$$y' = 3y^{2/3}$$

Let $p = y'$ the ode becomes

$$p = 3y^{2/3}$$

Solving for y from the above results in

$$y = \frac{\sqrt{3}\, p^{3/2}}{9} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved. Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= 0 \\ g &= \frac{\sqrt{3}\, p^{3/2}}{9} \end{aligned}$$

Hence (2) becomes

$$p = \frac{\sqrt{3}\, \sqrt{p}\, p'(x)}{6} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = 2\sqrt{3}\, \sqrt{p\,(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$\int \frac{\sqrt{3}}{6\sqrt{p}}dp = dx$$
$$\frac{\sqrt{3}\,\sqrt{p}}{3} = x + c_1$$

Singular solutions are found by solving

$$2\sqrt{3}\,\sqrt{p} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$

Substituing the above solution for p in (2A) gives

$$y = \left((x + c_1)^2\right)^{3/2}$$
$$y = 0$$

The solution

$$y = \left((x + c_1)^2\right)^{3/2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

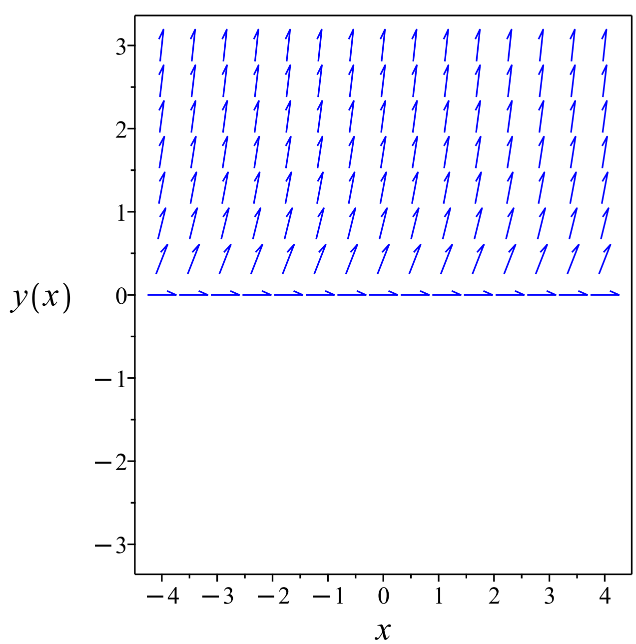


Figure 2.22: Slope field $y' = 3y^{2/3}$

Summary of solutions found

$$y = 0$$

✓ **Maple.** `Time used: 0.000 (sec). Leaf size: 14`

```
ode:=diff(y(x),x) = 3*y(x)^(2/3);
dsolve(ode,y(x), singsol=all);
```

$$y^{1/3} - x - c_1 = 0$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = 3y(x)^{2/3}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 3y(x)^{2/3}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)^{2/3}} = 3$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{y(x)^{2/3}}dx = \int 3dx + C1$
- Evaluate integral
  $3y(x)^{1/3} = 3x + C1$
- Solve for $y(x)$
  $y(x) = x^3 + C1\,x^2 + \frac{1}{3}x\,C1^2 + \frac{1}{27}C1^3$
- Simplify
  $y(x) = \left(x + \frac{C1}{3}\right)^3$
- Redefine the integration constant(s)
  $y(x) = (x + C1)^3$

✓ **Mathematica.** `Time used: 0.064 (sec). Leaf size: 22`

```
ode=D[y[x],x]==3*y[x]^(2/3);
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{27}(3x + c_1)^3$$
$$y(x) \to 0$$

✓ **Sympy.** `Time used: 0.104 (sec). Leaf size: 22`

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-3*y(x)**(2/3) + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{C_1^3}{27} + \frac{C_1^2 x}{3} + C_1 x^2 + x^3$$

2.2.2 Problem (c)

Local contents

Solved using first_order_ode_autonomous 120
Solved using first_order_ode_exact 122
✓Maple 127
✓Mathematica 128
✓Sympy 128

Internal problem ID [20968]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:14:28 AM`
CAS classification : `[_quadrature]`

Solved using first_order_ode_autonomous

Time used: 0.038 (sec)

Solve

$$y' = \sqrt{y\left(-y+1\right)}$$

Integrating gives

$$\int \frac{1}{\sqrt{y\left(-y+1\right)}}dy = dx$$
$$\arcsin\left(2y-1\right) = x + c_1$$

Singular solutions are found by solving

$$\sqrt{y\left(-y+1\right)} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$
$$y = 1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

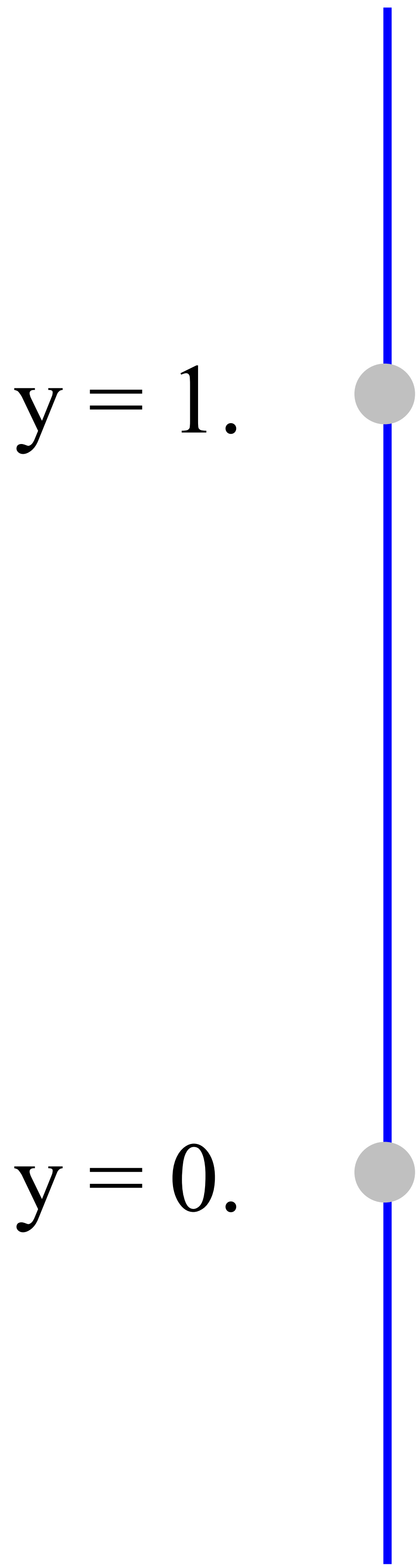


Figure 2.23: Phase line diagram

Solving for y gives

$$y = 0$$

$$y = 1$$

$$y = \frac{1}{2} + \frac{\sin(x + c_1)}{2}$$

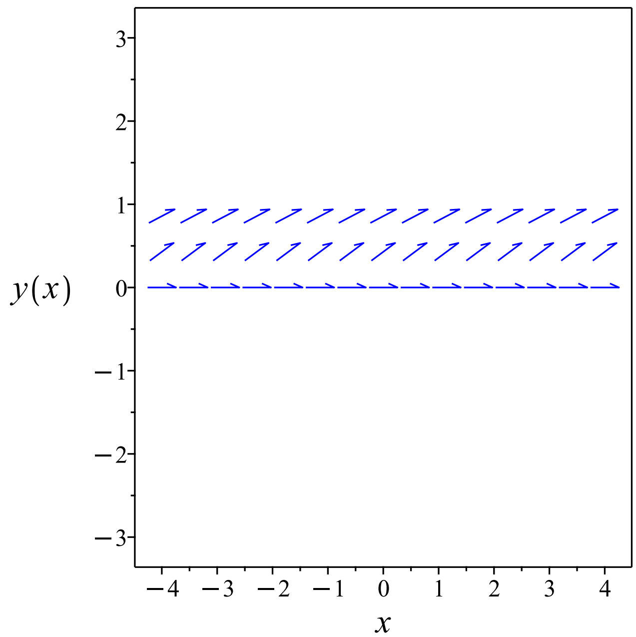


Figure 2.24: Slope field $y' = \sqrt{y(-y+1)}$

Summary of solutions found

$$y = 0$$

$$y = 1$$

$$y = \frac{1}{2} + \frac{\sin(x + c_1)}{2}$$

Solved using first_order_ode_exact

Time used: 0.160 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\sqrt{y\,(-y+1)}\right)\mathrm{d}x \\ \left(-\sqrt{y\,(-y+1)}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sqrt{y\,(-y+1)} \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sqrt{y\,(-y+1)}\right) \\ &= \frac{2y-1}{2\sqrt{-y\,(y-1)}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{1-2y}{2\sqrt{y(-y+1)}}\right) - (0)\right)\\ &= \frac{2y-1}{2\sqrt{-y(y-1)}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{\sqrt{-y(y-1)}}\left((0) - \left(-\frac{1-2y}{2\sqrt{y(-y+1)}}\right)\right)\\ &= \frac{1-2y}{2y(y-1)}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int \frac{1-2y}{2y(y-1)}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(y(y-1))}{2}}\\ &= \frac{1}{\sqrt{y(y-1)}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{y(y-1)}}\left(-\sqrt{y(-y+1)}\right)\\ &= -\frac{\sqrt{-y(y-1)}}{\sqrt{y(y-1)}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{\sqrt{y\left(y-1\right)}}(1)\\ &= \frac{1}{\sqrt{y\left(y-1\right)}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-\frac{\sqrt{-y\left(y-1\right)}}{\sqrt{y\left(y-1\right)}}\right) + \left(\frac{1}{\sqrt{y\left(y-1\right)}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int\overline{M}\,\mathrm{d}x$$

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\frac{\sqrt{-y\left(y-1\right)}}{\sqrt{y\left(y-1\right)}}\,\mathrm{d}x$$

$$\phi = -\frac{\sqrt{-y\left(y-1\right)}\,x}{\sqrt{y\left(y-1\right)}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{x(1-2y)}{2\sqrt{-y\left(y-1\right)}\,\sqrt{y\left(y-1\right)}} + \frac{\sqrt{-y\left(y-1\right)}\,x(2y-1)}{2\left(y\left(y-1\right)\right)^{3/2}} + f'(y)\\ &= 0 + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{\sqrt{y(y-1)}}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{y\left(y-1\right)}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y\,(y-1)}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{y\,(y-1)}}\right) \mathrm{d}y$$

$$f(y) = \ln\left(-\frac{1}{2} + y + \sqrt{y^2 - y}\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{-y\,(y-1)}\,x}{\sqrt{y\,(y-1)}} + \ln\left(-\frac{1}{2} + y + \sqrt{y^2 - y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{-y\,(y-1)}\,x}{\sqrt{y\,(y-1)}} + \ln\left(-\frac{1}{2} + y + \sqrt{y^2 - y}\right)$$

Simplifying the above gives

$$-\frac{\ln(2)\sqrt{y\,(y-1)} - \ln\left(-1 + 2y + 2\sqrt{y\,(y-1)}\right)\sqrt{y\,(y-1)} + \sqrt{-y\,(y-1)}\,x}{\sqrt{y\,(y-1)}} = c_1$$

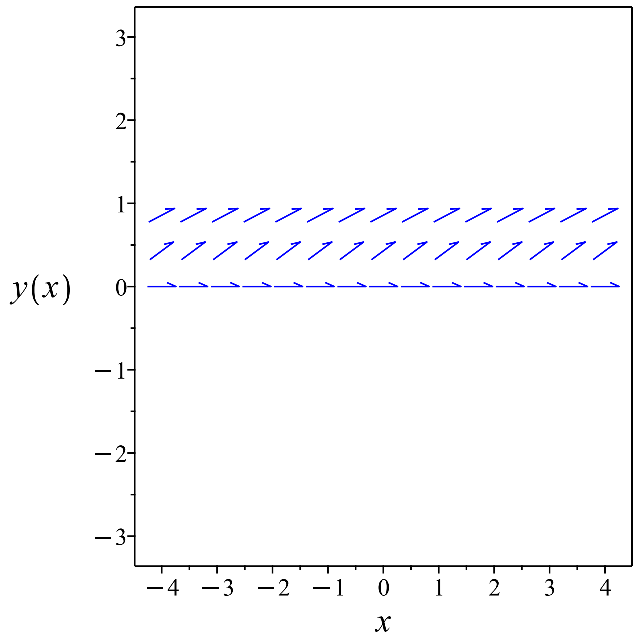


Figure 2.25: Slope field $y' = \sqrt{y\left(-y+1\right)}$

Summary of solutions found

$$-\frac{\ln\left(2\right)\sqrt{y\left(y-1\right)}-\ln\left(-1+2y+2\sqrt{y\left(y-1\right)}\right)\sqrt{y\left(y-1\right)}+\sqrt{-y\left(y-1\right)}\,x}{\sqrt{y\left(y-1\right)}} = c_1$$

✓ **Maple.** `Time used: 0.003 (sec). Leaf size: 12`

```
ode:=diff(y(x),x) = (y(x)*(1-y(x)))^(1/2);
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{1}{2}+\frac{\sin\left(x+c_1\right)}{2}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \sqrt{y(x)(1 - y(x))}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \sqrt{y(x)(1 - y(x))}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{\sqrt{y(x)(1-y(x))}} = 1$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{\sqrt{y(x)(1-y(x))}}dx = \int 1dx + C1$
- Evaluate integral
  $\arcsin(-1 + 2y(x)) = x + C1$
- Solve for $y(x)$
  $y(x) = \frac{1}{2} + \frac{\sin(x+C1)}{2}$

✓ **Mathematica.** `Time used: 0.22 (sec). Leaf size: 34`

```
ode=D[y[x],x]==Sqrt[ y[x]*(1-y[x])];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sin^2\left(\frac{x + c_1}{2}\right)$$
$$y(x) \to 0$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{0, 1\}]$$

✓ **Sympy.** `Time used: 0.231 (sec). Leaf size: 12`

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-sqrt((1 - y(x))*y(x)) + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{\sin(C_1 + x)}{2} + \frac{1}{2}$$

2.2.3 Problem (d)

Local contents

Existence and uniqueness analysis . . . 129
Solved using first_order_ode_separable . . . 130
Solved using first_order_ode_exact . . . 131
Solved using first_order_ode_LIE . . . 136
✓Maple . . . 143
✓Mathematica . . . 145
✓Sympy . . . 145

Internal problem ID [20969]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (d)
Date solved : `Saturday, November 29, 2025 at 01:14:50 AM`
CAS classification : `[_separable]`

Existence and uniqueness analysis

$$y' = \frac{e^{-y^2}}{y\left(x^2+2x\right)}$$
$$y(2) = 0$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{e^{-y^2}}{yx\left(x+2\right)} \end{aligned}$$

$f(x,y)$ is not defined at $y=0$ therefore existence and uniqueness theorem do not apply.

Solved using first_order_ode_separable

Time used: 0.191 (sec)

Solve

$$
\begin{aligned}
y' &= \frac{\mathrm{e}^{-y^2}}{y\left(x^2+2x\right)} \\
y(2) &= 0
\end{aligned}
$$

The ode

$$
y' = \frac{\mathrm{e}^{-y^2}}{yx\left(x+2\right)} \tag{2.9}
$$

is separable as it can be written as

$$
\begin{aligned}
y' &= \frac{\mathrm{e}^{-y^2}}{yx\left(x+2\right)} \\
&= f(x)g(y)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x\left(x+2\right)} \\
g(y) &= \frac{\mathrm{e}^{-y^2}}{y}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\
\int \mathrm{e}^{y^2} y\,dy &= \int \frac{1}{x\left(x+2\right)}\,dx
\end{aligned}
$$

$$
\frac{\mathrm{e}^{y^2}}{2} = \ln\left(\frac{\sqrt{x}}{\sqrt{x+2}}\right) + c_1
$$

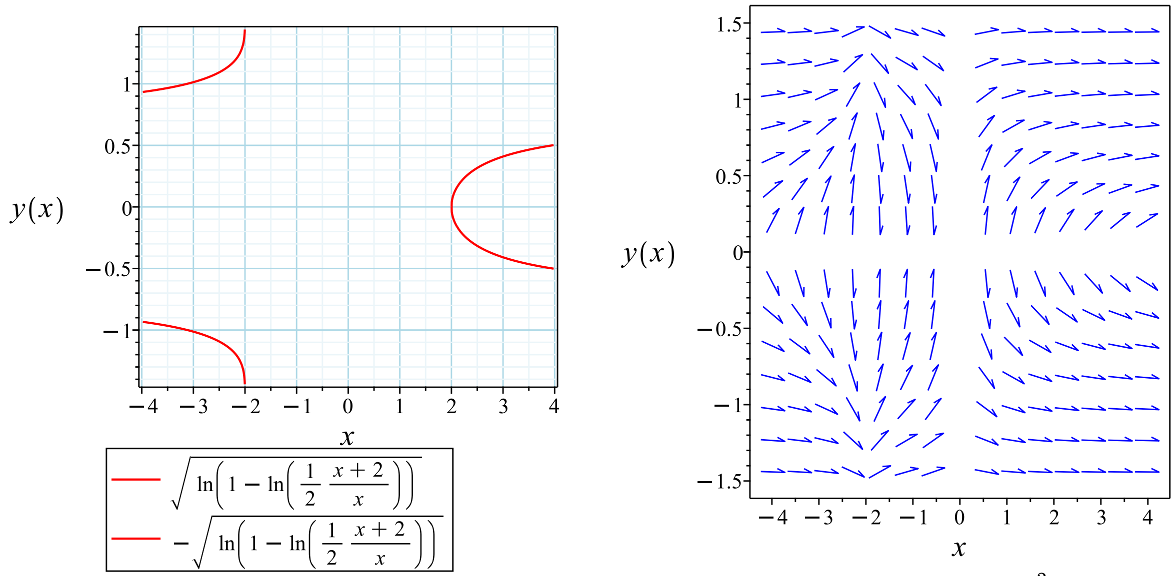
Solving for initial conditions the solution is

$$
\frac{\mathrm{e}^{y^2}}{2} = \ln\left(\frac{\sqrt{x}}{\sqrt{x+2}}\right) + \frac{1}{2} + \frac{\ln\left(2\right)}{2}
$$

Solving for y gives

$$y=\sqrt{\ln\left(1-\ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y=-\sqrt{\ln\left(1-\ln\left(\frac{x+2}{2x}\right)\right)}$$



(a) Solutions plot

(b) Slope field $y'=\frac{e^{-y^2}}{y(x^2+2x)}$

Summary of solutions found

$$y=\sqrt{\ln\left(1-\ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y=-\sqrt{\ln\left(1-\ln\left(\frac{x+2}{2x}\right)\right)}$$

Solved using first_order_ode_exact

Time used: 0.163 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{\mathrm{e}^{-y^2}}{y\,(x^2 + 2x)}\right)\mathrm{d}x$$

$$\left(-\frac{\mathrm{e}^{-y^2}}{y\,(x^2 + 2x)}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{\mathrm{e}^{-y^2}}{y\,(x^2 + 2x)}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\mathrm{e}^{-y^2}}{y\,(x^2 + 2x)}\right)\\ &= \frac{\mathrm{e}^{-y^2}\left(2 + \frac{1}{y^2}\right)}{x^2 + 2x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{2\,\mathrm{e}^{-y^2}}{x^2+2x} + \frac{\mathrm{e}^{-y^2}}{y^2\,(x^2+2x)}\right) - (0)\right)\\ &= \frac{\mathrm{e}^{-y^2}\left(2+\frac{1}{y^2}\right)}{x^2+2x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\mathrm{e}^{y^2} yx(x+2)\left((0) - \left(\frac{2\,\mathrm{e}^{-y^2}}{x^2+2x} + \frac{\mathrm{e}^{-y^2}}{y^2\,(x^2+2x)}\right)\right)\\ &= \frac{2y^2+1}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int \frac{2y^2+1}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{y^2+\ln(y)}\\ &= \mathrm{e}^{y^2} y\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{y^2} y\left(-\frac{\mathrm{e}^{-y^2}}{y\,(x^2+2x)}\right)\\ &= -\frac{1}{x\,(x+2)}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= e^{y^2} y(1) \\
&= e^{y^2} y
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N} \frac{dy}{dx} &= 0 \\
\left(-\frac{1}{x(x+2)}\right) + \left(e^{y^2} y\right) \frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\
\int \frac{\partial \phi}{\partial x} dx &= \int -\frac{1}{x(x+2)} dx
\end{aligned}
$$

$$
\phi = \frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{y^2} y$. Therefore equation (4) becomes

$$
e^{y^2} y = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = e^{y^2} y
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\mathrm{e}^{y^2} y\right) \mathrm{d}y$$

$$f(y) = \frac{\mathrm{e}^{y^2}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} + \frac{\mathrm{e}^{y^2}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} + \frac{\mathrm{e}^{y^2}}{2}$$

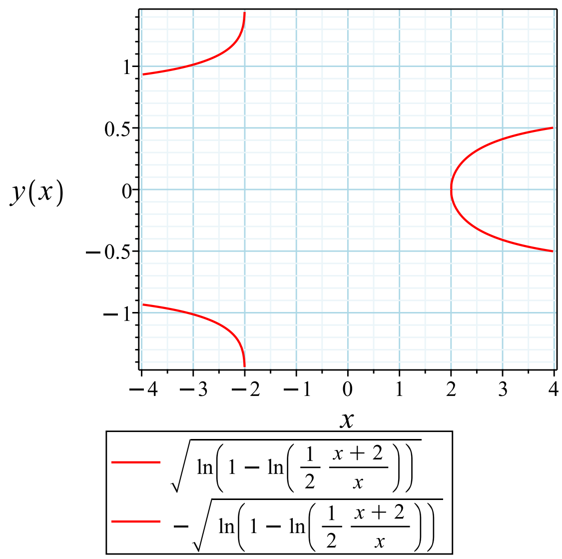
Solving for initial conditions the solution is

$$\frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} + \frac{\mathrm{e}^{y^2}}{2} = \frac{1}{2} + \frac{\ln(2)}{2}$$

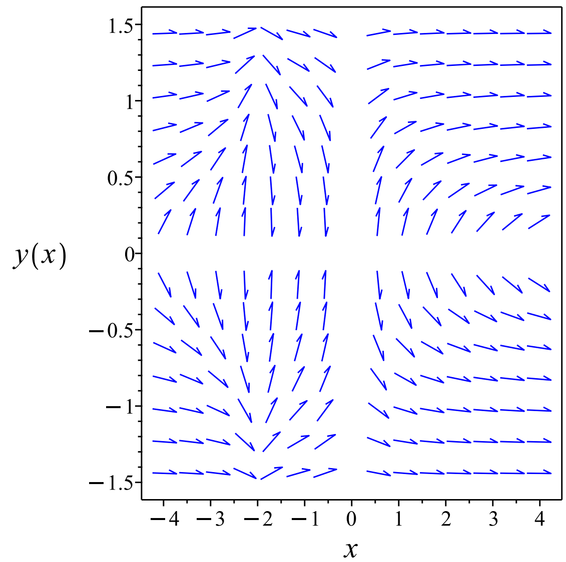
Solving for y gives

$$y = \sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y = -\sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$



(a) Solutions plot



(b) Slope field $y' = \frac{e^{-y^2}}{y(x^2+2x)}$

Summary of solutions found

$$y = \sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y = -\sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

Solved using first_order_ode_LIE

Time used: 1.281 (sec)

Solve

$$y' = \frac{e^{-y^2}}{y\,(x^2+2x)}$$
$$y(2) = 0$$

Writing the ode as

$$y' = \frac{e^{-y^2}}{yx\,(x+2)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 2 to use as anstaz gives

$$\xi = x^2 a_4 + yxa_5 + y^2 a_6 + xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = x^2 b_4 + yxb_5 + y^2 b_6 + xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&2xb_4 + yb_5 + b_2 + \frac{\mathrm{e}^{-y^2}(-2xa_4 + xb_5 - ya_5 + 2yb_6 - a_2 + b_3)}{yx\,(x+2)} \\
&- \frac{\mathrm{e}^{-2y^2}(xa_5 + 2ya_6 + a_3)}{y^2x^2\,(x+2)^2} \\
&- \left(-\frac{\mathrm{e}^{-y^2}}{y\,x^2\,(x+2)} - \frac{\mathrm{e}^{-y^2}}{yx\,(x+2)^2}\right)\left(x^2 a_4 + yxa_5 + y^2 a_6 + xa_2 + ya_3 + a_1\right) \\
&- \left(-\frac{2\,\mathrm{e}^{-y^2}}{x\,(x+2)} - \frac{\mathrm{e}^{-y^2}}{y^2x\,(x+2)}\right)\left(x^2 b_4 + yxb_5 + y^2 b_6 + xb_2 + yb_3 + b_1\right) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-4b_2y^2x^2 - 2\,\mathrm{e}^{-y^2}x^3y^2b_2 - 2\,\mathrm{e}^{-y^2}x^2y^3b_3 - 2\,\mathrm{e}^{-y^2}x^2y^2b_1 - 4\,\mathrm{e}^{-y^2}x^2y^2b_2 - 4\,\mathrm{e}^{-y^2}x\,y^3b_3 - \mathrm{e}^{-y^2}x^2ya_2 - 2\,\mathrm{e}^{-y^2}}{}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&4b_2y^2x^2 + 2\,\mathrm{e}^{-y^2}x^3y^2b_2 + 2\,\mathrm{e}^{-y^2}x^2y^3b_3 + 2\,\mathrm{e}^{-y^2}x^2y^2b_1 + 4\,\mathrm{e}^{-y^2}x^2y^2b_2 \\
&+ 4\,\mathrm{e}^{-y^2}x\,y^3b_3 + \mathrm{e}^{-y^2}x^2ya_2 + 2\,\mathrm{e}^{-y^2}x^2yb_3 + 2\,\mathrm{e}^{-y^2}x\,y^2a_3 + 4\,\mathrm{e}^{-y^2}x\,y^2b_1 \\
&+ 2\,\mathrm{e}^{-y^2}xya_1 + 4\,\mathrm{e}^{-y^2}xyb_3 + 4\,\mathrm{e}^{-y^2}x^3y^2b_4 + 4\,\mathrm{e}^{-y^2}x^2y^3b_5 + 4\,\mathrm{e}^{-y^2}x\,y^4b_6 \\
&+ 2\,\mathrm{e}^{-y^2}x^3yb_5 + \mathrm{e}^{-y^2}x^2y^2a_5 + 3\,\mathrm{e}^{-y^2}x^2y^2b_6 + 2\,\mathrm{e}^{-y^2}x\,y^3a_6 \\
&- 2\,\mathrm{e}^{-y^2}x^2ya_4 + 4\,\mathrm{e}^{-y^2}x^2yb_5 + 6\,\mathrm{e}^{-y^2}x\,y^2b_6 + 8x^3b_4y^2 + 4y^3b_5x^2 \\
&+ 2\,\mathrm{e}^{-y^2}x^4y^2b_4 + 2\,\mathrm{e}^{-y^2}x^3y^3b_5 + 2\,\mathrm{e}^{-y^2}x^2y^4b_6 - \mathrm{e}^{-2y^2}xa_5 - 2\,\mathrm{e}^{-2y^2}ya_6 \\
&+ 8x^4y^2b_4 + 4x^3y^3b_5 + \mathrm{e}^{-y^2}x^4b_4 + 2\,\mathrm{e}^{-y^2}x^3b_4 + 2\,\mathrm{e}^{-y^2}y^3a_6 + x^4y^2b_2 \\
&+ 4x^3y^2b_2 + \mathrm{e}^{-y^2}x^3b_2 + \mathrm{e}^{-y^2}x^2b_1 + 2\,\mathrm{e}^{-y^2}x^2b_2 + 2\,\mathrm{e}^{-y^2}y^2a_3 \\
&+ 2\,\mathrm{e}^{-y^2}xb_1 + 2\,\mathrm{e}^{-y^2}ya_1 + 2x^5y^2b_4 + x^4y^3b_5 - \mathrm{e}^{-2y^2}a_3 = 0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&4b_2y^2x^2 + 2\,\mathrm{e}^{-y^2}x^3y^2b_2 + 2\,\mathrm{e}^{-y^2}x^2y^3b_3 + 2\,\mathrm{e}^{-y^2}x^2y^2b_1 + 4\,\mathrm{e}^{-y^2}x^2y^2b_2 \\
&+ 4\,\mathrm{e}^{-y^2}x\,y^3b_3 + \mathrm{e}^{-y^2}x^2ya_2 + 2\,\mathrm{e}^{-y^2}x^2yb_3 + 2\,\mathrm{e}^{-y^2}x\,y^2a_3 + 4\,\mathrm{e}^{-y^2}x\,y^2b_1 \\
&+ 2\,\mathrm{e}^{-y^2}xya_1 + 4\,\mathrm{e}^{-y^2}xyb_3 + 4\,\mathrm{e}^{-y^2}x^3y^2b_4 + 4\,\mathrm{e}^{-y^2}x^2y^3b_5 + 4\,\mathrm{e}^{-y^2}x\,y^4b_6 \\
&+ 2\,\mathrm{e}^{-y^2}x^3yb_5 + \mathrm{e}^{-y^2}x^2y^2a_5 + 3\,\mathrm{e}^{-y^2}x^2y^2b_6 + 2\,\mathrm{e}^{-y^2}x\,y^3a_6 \\
&- 2\,\mathrm{e}^{-y^2}x^2ya_4 + 4\,\mathrm{e}^{-y^2}x^2yb_5 + 6\,\mathrm{e}^{-y^2}x\,y^2b_6 + 8x^3b_4y^2 + 4y^3b_5x^2 \\
&+ 2\,\mathrm{e}^{-y^2}x^4y^2b_4 + 2\,\mathrm{e}^{-y^2}x^3y^3b_5 + 2\,\mathrm{e}^{-y^2}x^2y^4b_6 - \mathrm{e}^{-2y^2}xa_5 - 2\,\mathrm{e}^{-2y^2}ya_6 \\
&+ 8x^4y^2b_4 + 4x^3y^3b_5 + \mathrm{e}^{-y^2}x^4b_4 + 2\,\mathrm{e}^{-y^2}x^3b_4 + 2\,\mathrm{e}^{-y^2}y^3a_6 + x^4y^2b_2 \\
&+ 4x^3y^2b_2 + \mathrm{e}^{-y^2}x^3b_2 + \mathrm{e}^{-y^2}x^2b_1 + 2\,\mathrm{e}^{-y^2}x^2b_2 + 2\,\mathrm{e}^{-y^2}y^2a_3 \\
&+ 2\,\mathrm{e}^{-y^2}xb_1 + 2\,\mathrm{e}^{-y^2}ya_1 + 2x^5y^2b_4 + x^4y^3b_5 - \mathrm{e}^{-2y^2}a_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, \mathrm{e}^{-2y^2}, \mathrm{e}^{-y^2}\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2, \mathrm{e}^{-2y^2} = v_3, \mathrm{e}^{-y^2} = v_4\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&2v_1^5v_2^2b_4 + 2v_4v_1^4v_2^2b_4 + v_1^4v_2^3b_5 + 2v_4v_1^3v_2^3b_5 + 2v_4v_1^2v_2^4b_6 + v_1^4v_2^2b_2 + 2v_4v_1^3v_2^2b_2 \\
&+ 2v_4v_1^2v_2^3b_3 + 8v_1^4v_2^2b_4 + 4v_4v_1^3v_2^2b_4 + 4v_1^3v_2^3b_5 + 4v_4v_1^2v_2^3b_5 + 4v_4v_1v_2^4b_6 \\
&+ v_4v_1^2v_2^2a_5 + 2v_4v_1v_2^3a_6 + 2v_4v_1^2v_2^2b_1 + 4v_1^3v_2^2b_2 + 4v_4v_1^2v_2^2b_2 + 4v_4v_1v_2^3b_3 \\
&+ v_4v_1^4b_4 + 8v_1^3b_4v_2^2 + 2v_4v_1^3v_2b_5 + 4v_2^3b_5v_1^2 + 3v_4v_1^2v_2^2b_6 + v_4v_1^2v_2a_2 + 2v_4v_1v_2^2a_3 \\
&- 2v_4v_1^2v_2a_4 + 2v_4v_2^3a_6 + 4v_4v_1v_2^2b_1 + v_4v_1^3b_2 + 4b_2v_2^2v_1^2 + 2v_4v_1^2v_2b_3 \\
&+ 2v_4v_1^3b_4 + 4v_4v_1^2v_2b_5 + 6v_4v_1v_2^2b_6 + 2v_4v_1v_2a_1 + 2v_4v_2^2a_3 + v_4v_1^2b_1 \\
&+ 2v_4v_1^2b_2 + 4v_4v_1v_2b_3 + 2v_4v_2a_1 - v_3v_1a_5 - 2v_3v_2a_6 + 2v_4v_1b_1 - v_3a_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(b_2 + 8b_4)\,v_1^4v_2^2 + (4b_2 + 8b_4)\,v_1^3v_2^2 + (b_2 + 2b_4)\,v_1^3v_4 + (b_1 + 2b_2)\,v_1^2v_4 + 4v_4v_1v_2^4b_6 \\
&+ 2v_4v_1^3v_2b_5 + 2v_4v_1^4v_2^2b_4 + 2v_4v_1^3v_2^3b_5 + 2v_4v_1^2v_2^4b_6 + 4b_2v_2^2v_1^2 + 4v_2^3b_5v_1^2 \\
&- v_3v_1a_5 - 2v_3v_2a_6 + 4v_1^3v_2^3b_5 + v_4v_1^4b_4 + 2v_4v_2^3a_6 + 2v_4v_2^2a_3 + 2v_4v_1b_1 \\
&+ 2v_4v_2a_1 + 2v_1^5v_2^2b_4 + v_1^4v_2^3b_5 + (2b_2 + 4b_4)\,v_1^3v_2^2v_4 + (2b_3 + 4b_5)\,v_1^2v_2^3v_4 \\
&+ (a_5 + 2b_1 + 4b_2 + 3b_6)\,v_1^2v_2^2v_4 + (a_2 - 2a_4 + 2b_3 + 4b_5)\,v_1^2v_2v_4 \\
&+ (2a_6 + 4b_3)\,v_1v_2^3v_4 + (2a_3 + 4b_1 + 6b_6)\,v_1v_2^2v_4 + (2a_1 + 4b_3)\,v_1v_2v_4 - v_3a_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_4 &= 0\\
b_5 &= 0\\
2a_1 &= 0\\
-a_3 &= 0\\
2a_3 &= 0\\
-a_5 &= 0\\
-2a_6 &= 0\\
2a_6 &= 0\\
2b_1 &= 0\\
4b_2 &= 0\\
2b_4 &= 0\\
2b_5 &= 0\\
4b_5 &= 0\\
2b_6 &= 0\\
4b_6 &= 0\\
2a_1 + 4b_3 &= 0\\
2a_6 + 4b_3 &= 0\\
b_1 + 2b_2 &= 0\\
b_2 + 2b_4 &= 0\\
b_2 + 8b_4 &= 0\\
2b_2 + 4b_4 &= 0\\
4b_2 + 8b_4 &= 0\\
2b_3 + 4b_5 &= 0\\
2a_3 + 4b_1 + 6b_6 &= 0\\
a_2 - 2a_4 + 2b_3 + 4b_5 &= 0\\
a_5 + 2b_1 + 4b_2 + 3b_6 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 2a_4 \\ a_3 &= 0 \\ a_4 &= a_4 \\ a_5 &= 0 \\ a_6 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \\ b_4 &= 0 \\ b_5 &= 0 \\ b_6 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x^2 + 2x \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi \\ &= 0 - \left(\frac{e^{-y^2}}{yx\,(x+2)}\right)(x^2+2x) \\ &= -\frac{e^{-y^2}}{y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{e^{-y^2}}{y}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{y^2}}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{e^{-y^2}}{yx(x + 2)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= -e^{y^2} y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x(x + 2)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R(R + 2)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

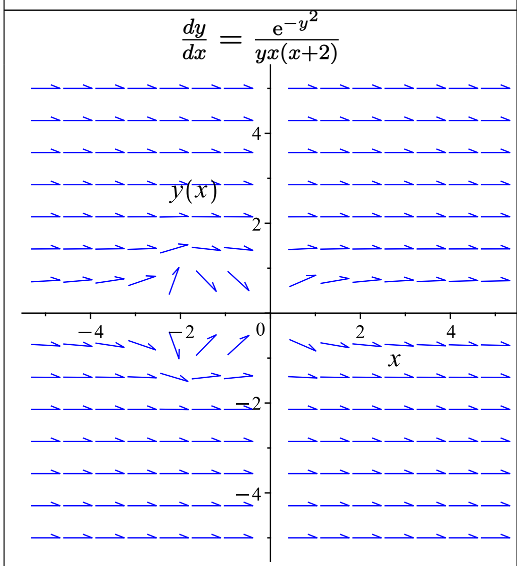
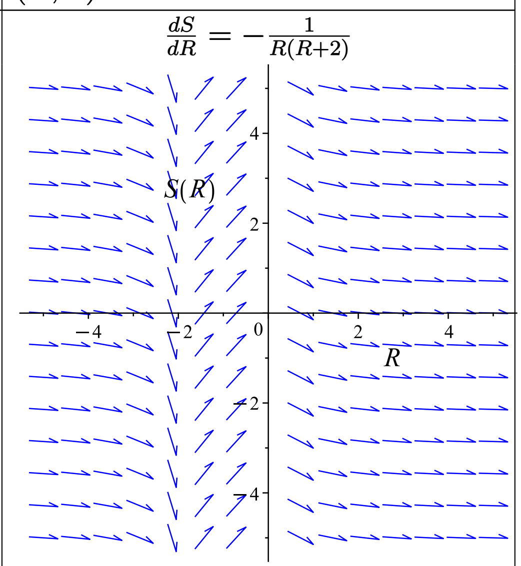
$$\int dS = \int -\frac{1}{R(R+2)}\, dR$$

$$S(R) = \frac{\ln(R+2)}{2} - \frac{\ln(R)}{2} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{e^{y^2}}{2} = \frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = x$ $S = -\frac{e^{y^2}}{2}$	

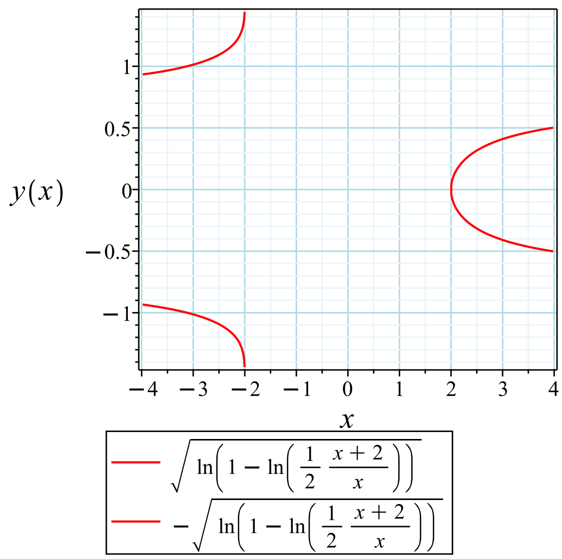
Solving for initial conditions the solution is

$$-\frac{e^{y^2}}{2} = \frac{\ln(x+2)}{2} - \frac{\ln(x)}{2} - \frac{1}{2} - \frac{\ln(2)}{2}$$

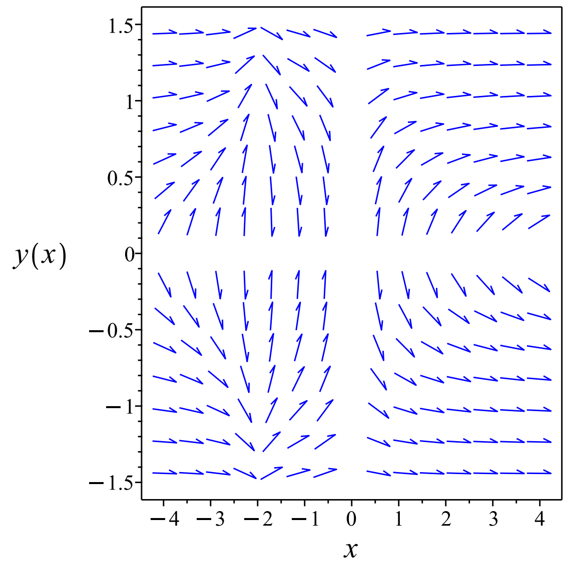
Solving for y gives

$$y = \sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y = -\sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$



(a) Solutions plot



(b) Slope field $y' = \frac{e^{-y^2}}{y(x^2+2x)}$

Summary of solutions found

$$y = \sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

$$y = -\sqrt{\ln\left(1 - \ln\left(\frac{x+2}{2x}\right)\right)}$$

✓ **Maple.** Time used: 0.159 (sec). Leaf size: 43

```
ode:=diff(y(x),x) = exp(-y(x)^2)/y(x)/(x^2+2*x);
ic:=[y(2) = 0];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \sqrt{\ln\left(1 + \ln(2) - \ln\left(\frac{x+2}{x}\right)\right)}$$
$$y = -\sqrt{\ln\left(1 + \ln(2) - \ln\left(\frac{x+2}{x}\right)\right)}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
```

```
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}y(x) = \frac{e^{-y(x)^2}}{y(x)(x^2+2x)}, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{e^{-y(x)^2}}{y(x)(x^2+2x)}$
- Separate variables
  $\frac{\left(\frac{d}{dx}y(x)\right)y(x)}{e^{-y(x)^2}} = \frac{1}{x^2+2x}$
- Integrate both sides with respect to x
  $\int \frac{\left(\frac{d}{dx}y(x)\right)y(x)}{e^{-y(x)^2}}dx = \int \frac{1}{x^2+2x}dx + C1$
- Evaluate integral
  $\frac{1}{2\,e^{-y(x)^2}} = -\frac{\ln(x+2)}{2} + \frac{\ln(x)}{2} + C1$
- Solve for $y(x)$
  $\left\{y(x) = \sqrt{-\ln\left(-\frac{1}{-2C1+\ln\left(\frac{x+2}{x}\right)}\right)}, y(x) = -\sqrt{-\ln\left(-\frac{1}{-2C1+\ln\left(\frac{x+2}{x}\right)}\right)}\right\}$
- Simplify
  $\left\{y(x) = \sqrt{-\ln\left(\frac{1}{2C1-\ln\left(\frac{x+2}{x}\right)}\right)}, y(x) = -\sqrt{-\ln\left(\frac{1}{2C1-\ln\left(\frac{x+2}{x}\right)}\right)}\right\}$
- Redefine the integration constant(s)
  $\left\{y(x) = \sqrt{-\ln\left(\frac{1}{C1-\ln\left(\frac{x+2}{x}\right)}\right)}, y(x) = -\sqrt{-\ln\left(\frac{1}{C1-\ln\left(\frac{x+2}{x}\right)}\right)}\right\}$
- Use initial condition $y(2) = 0$
  $0 = \sqrt{-\ln\left(\frac{1}{C1-\ln(2)}\right)}$
- Solve for $\_C1$
  $C1 = 1 + \ln(2)$
- Substitute $\_C1 = 1 + \ln(2)$ into general solution and simplify
  $y(x) = \sqrt{-\ln\left(\frac{1}{1+\ln(2)-\ln\left(\frac{x+2}{x}\right)}\right)}$
- Use initial condition $y(2) = 0$
  $0 = -\sqrt{-\ln\left(\frac{1}{C1-\ln(2)}\right)}$
- Solve for $\_C1$

$$C1 = 1 + \ln(2)$$

- Substitute $\_C1 = 1 + \ln(2)$ into general solution and simplify
$$y(x) = -\sqrt{-\ln\left(\frac{1}{1+\ln(2)-\ln\left(\frac{x+2}{x}\right)}\right)}$$
- Solutions to the IVP
$$\left[y(x) = \sqrt{-\ln\left(\frac{1}{1+\ln(2)-\ln\left(\frac{x+2}{x}\right)}\right)}, y(x) = -\sqrt{-\ln\left(\frac{1}{1+\ln(2)-\ln\left(\frac{x+2}{x}\right)}\right)}\right]$$

✓ **Mathematica.** Time used: 1.563 (sec). Leaf size: 45

```
ode=D[y[x],x]== Exp[-y[x]^2]/(y[x]*(2*x+x^2)  );
ic={y[2]==0};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{\log(\log(2x) - \log(x+2) + 1)}$$
$$y(x) \to \sqrt{\log(\log(2x) - \log(x+2) + 1)}$$

✓ **Sympy.** Time used: 0.759 (sec). Leaf size: 42

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(Derivative(y(x), x) - exp(-y(x)**2)/((x**2 + 2*x)*y(x)),0)
ics = {y(2): 0}
dsolve(ode,func=y(x),ics=ics)
```

$$\left[y(x) = -\sqrt{\log(\log(x) - \log(x+2) + \log(2) + 1)},\ y(x) = \sqrt{\log(\log(x) - \log(x+2) + \log(2) + 1)}\right]$$

2.2.4 Problem (e)

Local contents

Existence and uniqueness analysis 146
Solved using first_order_ode_exact 147
✓Maple 151
✓Mathematica 152
✓Sympy 153

Internal problem ID [20970]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (e)
Date solved : `Saturday, November 29, 2025 at 01:15:23 AM`
CAS classification : `[_separable]`

Existence and uniqueness analysis

$$y' = \frac{y \ln(y)}{\sin(x)}$$

$$y\left(\frac{\pi}{2}\right) = e^{e}$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y \ln(y)}{\sin(x)} \end{aligned}$$

The x domain of $f(x,y)$ when $y = e^{e}$ is

$$\{x < \pi\_Z354 \vee \pi\_Z354 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{\pi}{2}$ is

$$\{0 < y\}$$

And the point $y_0 = e^{e}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y \ln(y)}{\sin(x)}\right) \\ &= \frac{\ln(y)}{\sin(x)} + \frac{1}{\sin(x)} \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = e^e$ is

$$\{x < \pi\_Z354 \vee \pi\_Z354 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{2}$ is

$$\{0 < y\}$$

And the point $y_0 = e^e$ is inside this domain. Therefore solution exists and is unique.

Solved using first_order_ode_exact

Time used: 0.141 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y\ln(y)}{\sin(x)}\right)\mathrm{d}x$$

$$\left(-\frac{y\ln(y)}{\sin(x)}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{y\ln(y)}{\sin(x)} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y\ln(y)}{\sin(x)}\right) \\ &= -(1+\ln(y))\csc(x) \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{\ln(y)}{\sin(x)} - \frac{1}{\sin(x)}\right) - (0)\right) \\ &= -(1+\ln(y))\csc(x) \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{\sin(x)}{y\ln(y)}\left((0) - \left(-\frac{\ln(y)}{\sin(x)} - \frac{1}{\sin(x)}\right)\right) \\ &= \frac{-\ln(y) - 1}{y\ln(y)} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int \frac{-\ln(y)-1}{y\ln(y)} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\ln(y))-\ln(y)} \\ &= \frac{1}{y\ln(y)}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y\ln(y)}\left(-\frac{y\ln(y)}{\sin(x)}\right) \\ &= -\csc(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y\ln(y)}(1) \\ &= \frac{1}{y\ln(y)}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-\csc(x)) + \left(\frac{1}{y\ln(y)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -\csc(x)\,\mathrm{d}x$$

$$\phi = \ln(\csc(x) + \cot(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y \ln(y)}$. Therefore equation (4) becomes

$$\frac{1}{y \ln(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y \ln(y)}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y \ln(y)} \right) \mathrm{d}y$$
$$f(y) = \ln(\ln(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(\csc(x) + \cot(x)) + \ln(\ln(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \ln(\csc(x) + \cot(x)) + \ln(\ln(y))$$

Solving for initial conditions the solution is

$$\ln(\csc(x) + \cot(x)) + \ln(\ln(y)) = 1$$

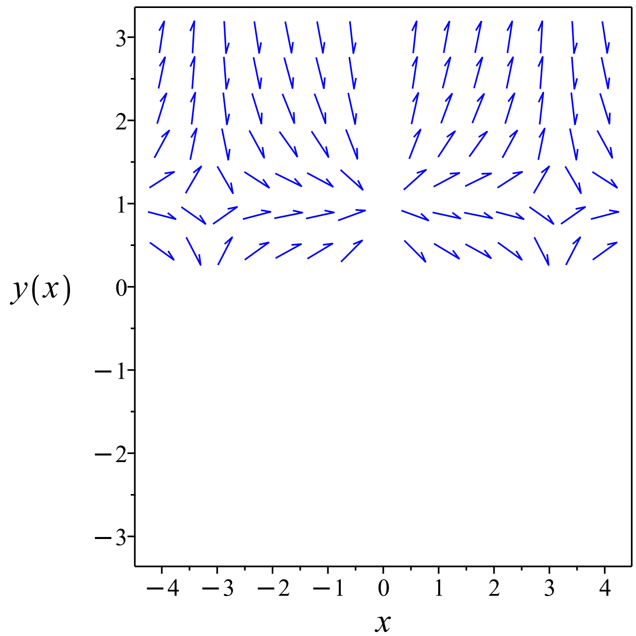


Figure 2.29: Slope field $y' = \frac{y\ln(y)}{\sin(x)}$

Summary of solutions found

$$\ln\left(\csc\left(x\right)+\cot\left(x\right)\right)+\ln\left(\ln\left(y\right)\right) = 1$$

✓ **Maple.** `Time used: 0.172 (sec). Leaf size: 15`

```
ode:=diff(y(x),x) = y(x)*ln(y(x))/sin(x);
ic:=[y(1/2*Pi) = exp(exp(1))];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = e^{e^{(-\cot(x)+\csc(x))}}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}y(x) = \frac{y(x)\ln(y(x))}{\sin(x)}, y\left(\frac{\pi}{2}\right) = \mathrm{e}^{\mathrm{e}}\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{y(x)\ln(y(x))}{\sin(x)}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)\ln(y(x))} = \frac{1}{\sin(x)}$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{y(x)\ln(y(x))}dx = \int \frac{1}{\sin(x)}dx + C1$
- Evaluate integral
  $\ln(\ln(y(x))) = \ln(\csc(x) - \cot(x)) + C1$
- Solve for $y(x)$
  $y(x) = \mathrm{e}^{-\frac{\mathrm{e}^{C1}(-1+\cos(x))}{\sin(x)}}$
- Simplify
  $y(x) = \mathrm{e}^{-\mathrm{e}^{C1}(-\csc(x)+\cot(x))}$
- Redefine the integration constant(s)
  $y(x) = \mathrm{e}^{C1(-\csc(x)+\cot(x))}$
- Use initial condition $y\left(\frac{\pi}{2}\right) = \mathrm{e}^{\mathrm{e}}$
  $\mathrm{e}^{\mathrm{e}} = \mathrm{e}^{-C1}$
- Solve for $\_C1$
  $C1 = -\mathrm{e}$
- Removesolutionsthatdon'tsatisfytheODE
  $\varnothing$
- Solution does not satisfy initial condition

✓ **Mathematica.** `Time used: 0.147 (sec). Leaf size: 16`

```
ode=D[y[x],x]== y[x]*Log[y[x]]/Sin[x];
ic={y[Pi/2]==Exp[Exp[1]]};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{e^{1-\operatorname{arctanh}(\cos(x))}}$$

✓ **Sympy.** Time used: 0.311 (sec). Leaf size: 27

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-y(x)*log(y(x))/sin(x) + Derivative(y(x), x),0)
ics = {y(pi/2): exp(E)}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = e^{-\frac{ei\sqrt{\cos(x)-1}}{\sqrt{\cos(x)+1}}}$$

2.2.5 Problem (f)

Local contents

Existence and uniqueness analysis 154
Solved using first_order_ode_separable 155
Solved using first_order_ode_exact 157
✓Maple 160
✓Mathematica 161
✓Sympy 162

Internal problem ID [20971]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (f)
Date solved : `Saturday, November 29, 2025 at 01:16:07 AM`
CAS classification : `[_separable]`

Existence and uniqueness analysis

$$y' = \frac{\cos(x)}{\cos(y)^2}$$
$$y(\pi) = \frac{\pi}{4}$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\cos(x)}{\cos(y)^2} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{\pi}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0=\pi$ is inside this domain. The y domain of $f(x,y)$ when $x=\pi$ is

$$\left\{y < \frac{1}{2}\pi + \pi\_Z357 \vee \frac{1}{2}\pi + \pi\_Z357 < y\right\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\cos(x)}{\cos(y)^2}\right) \\ = \frac{2\cos(x)\sin(y)}{\cos(y)^3}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \pi$ is

$$\left\{y < \frac{1}{2}\pi + \pi\_Z357 \vee \frac{1}{2}\pi + \pi\_Z357 < y\right\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

Solved using first_order_ode_separable

Time used: 0.171 (sec)

Solve

$$y' = \frac{\cos(x)}{\cos(y)^2}$$
$$y(\pi) = \frac{\pi}{4}$$

The ode

$$y' = \frac{\cos(x)}{\cos(y)^2} \tag{2.10}$$

is separable as it can be written as

$$y' = \frac{\cos(x)}{\cos(y)^2} \\ = f(x)g(y)$$

Where

$$f(x) = \cos(x)$$
$$g(y) = \frac{1}{\cos(y)^2}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$\int \cos(y)^2\,dy = \int \cos(x)\,dx$$

$$\frac{\cos(y)\sin(y)}{2} + \frac{y}{2} = \sin(x) + c_1$$

Simplifying the above gives

$$\frac{\sin(2y)}{4} + \frac{y}{2} = \sin(x) + c_1$$

Solving for initial conditions the solution is

$$\frac{\sin(2y)}{4} + \frac{y}{2} = \sin(x) + \frac{1}{4} + \frac{\pi}{8}$$

Solving for y gives

$$y = \frac{\operatorname{RootOf}(2\_Z + 2\sin(\_Z) - 2 - 8\sin(x) - \pi)}{2}$$

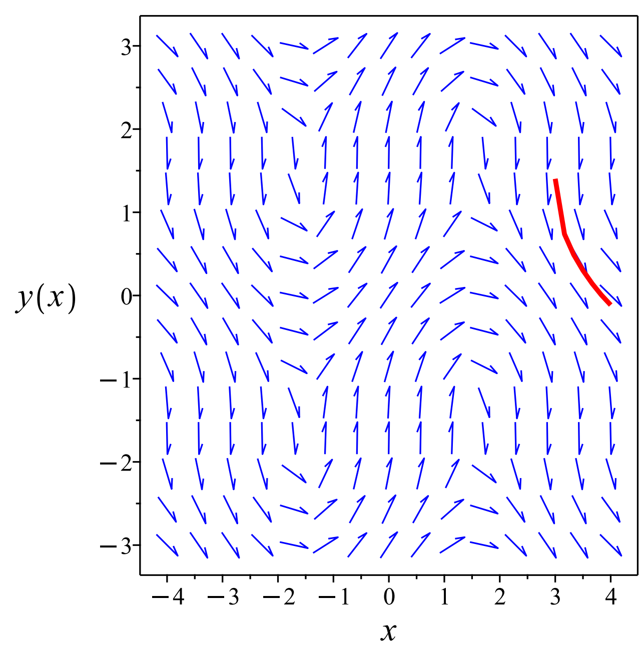


Figure 2.30: Slope field $y' = \frac{\cos(x)}{\cos(y)^2}$

Summary of solutions found

$$y = \frac{\mathrm{RootOf}\left(2\_Z + 2\sin\left(\_Z\right) - 2 - 8\sin\left(x\right) - \pi\right)}{2}$$

Solved using first_order_ode_exact

Time used: 0.088 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\cos\left(y\right)^2\right)\mathrm{d}y = \left(\cos\left(x\right)\right)\mathrm{d}x$$
$$\left(-\cos\left(x\right)\right)\mathrm{d}x + \left(\cos\left(y\right)^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\cos(x)$$
$$N(x,y) = \cos(y)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\cos(y)^2) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\cos(x)\,\mathrm{d}x$$
$$\phi = -\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(y)^2$. Therefore equation (4) becomes

$$\cos(y)^2 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \cos(y)^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\, dy = \int \left(\cos(y)^2\right) dy$$

$$f(y) = \frac{\cos(y)\sin(y)}{2} + \frac{y}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\sin(x) + \frac{\cos(y)\sin(y)}{2} + \frac{y}{2}$$

Simplifying the above gives

$$-\sin(x) + \frac{\sin(2y)}{4} + \frac{y}{2} = c_1$$

Solving for initial conditions the solution is

$$-\sin(x) + \frac{\sin(2y)}{4} + \frac{y}{2} = \frac{1}{4} + \frac{\pi}{8}$$

Solving for y gives

$$y = \frac{\mathrm{RootOf}\left(2\_Z + 2\sin(\_Z) - 2 - 8\sin(x) - \pi\right)}{2}$$

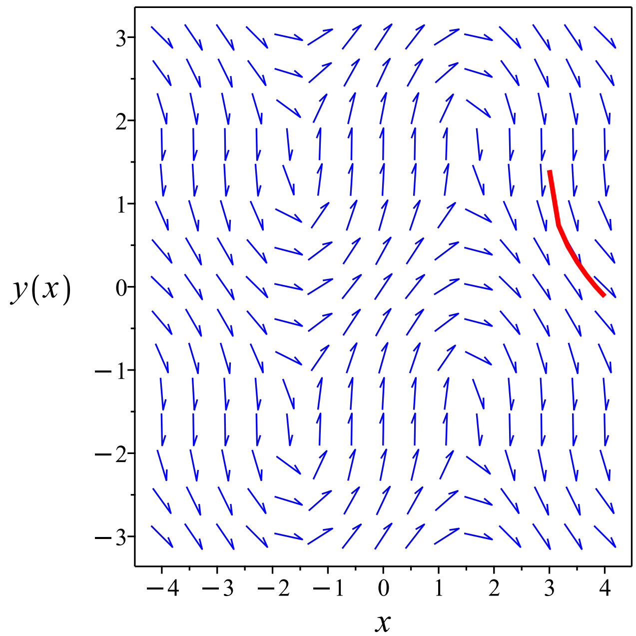


Figure 2.31: Slope field $y' = \frac{\cos(x)}{\cos(y)^2}$

Summary of solutions found

$$y = \frac{\text{RootOf}\left(2\_Z + 2\sin\left(\_Z\right) - 2 - 8\sin\left(x\right) - \pi\right)}{2}$$

✓ **Maple.** `Time used: 0.122 (sec). Leaf size: 23`

```
ode:=diff(y(x),x) = cos(x)/cos(y(x))^2;
ic:=[y(Pi) = 1/4*Pi];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \frac{\text{RootOf}\left(2\_Z - \pi - 2 - 8\sin\left(x\right) + 2\sin\left(\_Z\right)\right)}{2}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}y(x) = \frac{\cos(x)}{\cos(y(x))^2}, y(\pi) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{\cos(x)}{\cos(y(x))^2}$
- Separate variables
  $\left(\frac{d}{dx}y(x)\right)\cos\left(y(x)\right)^2 = \cos\left(x\right)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right)\cos\left(y(x)\right)^2 dx = \int \cos\left(x\right) dx + C1$
- Evaluate integral
  $\frac{\cos(y(x))\sin(y(x))}{2} + \frac{y(x)}{2} = \sin\left(x\right) + C1$
- Use initial condition $y(\pi) = \frac{\pi}{4}$
  $\frac{1}{4} + \frac{\pi}{8} = C1$
- Solve for $\_C1$
  $C1 = \frac{1}{4} + \frac{\pi}{8}$
- Substitute $\_C1 = \frac{1}{4} + \frac{\pi}{8}$ into general solution and simplify
  $\frac{\sin(2y(x))}{4} + \frac{y(x)}{2} = \sin\left(x\right) + \frac{1}{4} + \frac{\pi}{8}$
- Solution to the IVP
  $\frac{\sin(2y(x))}{4} + \frac{y(x)}{2} = \sin\left(x\right) + \frac{1}{4} + \frac{\pi}{8}$

✓ **Mathematica.** `Time used: 0.227 (sec). Leaf size: 36`

```
ode=D[y[x],x]== Cos[x]/Cos[y[x]]^2;
ic={y[Pi]==Pi/4};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[2\left(\frac{\#1}{2} + \frac{1}{4}\sin(2\#1)\right)\&\right]\left[\frac{1}{4}(8\sin(x) + \pi + 2)\right]$$

✓ **Sympy.** Time used: 3.051 (sec). Leaf size: 26

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-cos(x)/cos(y(x))**2 + Derivative(y(x), x),0)
ics = {y(pi): pi/4}
dsolve(ode,func=y(x),ics=ics)
```

$$\frac{y(x)}{2} - \sin(x) + \frac{\sin(y(x))\cos(y(x))}{2} = \frac{1}{4} + \frac{\pi}{8}$$

2.2.6 Problem (g)

Local contents

Solved using first_order_ode_dAlembert 163
Solved using first_order_ode_LIE 167
Solved using first_order_ode_riccati 172
Solved using first_order_ode_riccati_by_guessing_particular_solution 176
✓ Maple 178
✓ Mathematica 179
✓ Sympy 179

Internal problem ID [20972]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (g)
Date solved : `Saturday, November 29, 2025 at 01:16:24 AM`
CAS classification : `[[_homogeneous, ‘class C‘], _Riccati]`

Solved using first_order_ode_dAlembert

Time used: 0.560 (sec)

Solve

$$y' = (x - y + 3)^2$$

Let $p = y'$ the ode becomes

$$p = (x - y + 3)^2$$

Solving for y from the above results in

$$y = x + 3 + \sqrt{p} \tag{1}$$

$$y = x + 3 - \sqrt{p} \tag{2}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

<u>Solving ode 1A</u>

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 1 \\
g &= 3 + \sqrt{p}
\end{aligned}
$$

Hence (2) becomes

$$
p - 1 = \frac{p'(x)}{2\sqrt{p}} \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - 1 = 0
$$

Solving the above for p results in

$$
p_1 = 1
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y = x + 4
$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = 2(p(x) - 1)\sqrt{p(x)} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$
\begin{aligned}
\int \frac{1}{2(p-1)\sqrt{p}} dp &= dx \\
-\operatorname{arctanh}\left(\sqrt{p}\right) &= x + c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
2(p - 1)\sqrt{p} = 0
$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = 1$$

Substituing the above solution for p in (2A) gives

$$y = x + 3 + \sqrt{\tanh(x + c_1)^2}$$
$$y = x + 3$$
$$y = x + 4$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 1$$
$$g = 3 - \sqrt{p}$$

Hence (2) becomes

$$p - 1 = -\frac{p'(x)}{2\sqrt{p}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving the above for p results in

$$p_1 = 1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = x + 2$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -2(p(x) - 1)\sqrt{p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed.

Integrating gives

$$\int -\frac{1}{2\left(p-1\right)\sqrt{p}}dp = dx$$
$$\operatorname{arctanh}\left(\sqrt{p}\right) = x + c_2$$

Singular solutions are found by solving

$$-2(p-1)\sqrt{p} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = 1$$

Substituting the above solution for p in (2A) gives

$$y = x + 3 - \sqrt{\tanh\left(x + c_2\right)^2}$$
$$y = x + 3$$
$$y = x + 2$$

The solution

$$y = x + 3$$

was found not to satisfy the ode or the IC. Hence it is removed.

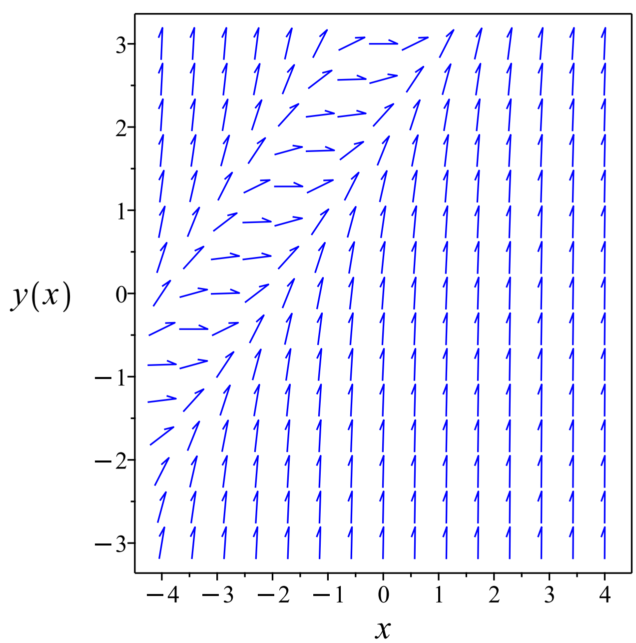


Figure 2.32: Slope field $y' = (x - y + 3)^2$

Summary of solutions found

$$y = x + 2$$
$$y = x + 4$$
$$y = x + 3 + \sqrt{\tanh\left(x + c_1\right)^2}$$
$$y = x + 3 - \sqrt{\tanh\left(x + c_2\right)^2}$$

Solved using first_order_ode_LIE

Time used: 0.386 (sec)

Solve

$$y' = (x - y + 3)^2$$

Writing the ode as

$$y' = (-x + y - 3)^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Using these anstaz

$$\xi = 1 \tag{1E}$$
$$\eta = \frac{Ax + By}{Cx} \tag{2E}$$

Where the unknown coefficients are

$$\{A, B, C\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\frac{A}{Cx} - \frac{Ax + By}{C\,x^2} + \frac{(-x + y - 3)^2 B}{Cx} - 6 - 2x + 2y - \frac{(-2x + 2y - 6)(Ax + By)}{Cx} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2A\,x^3 - 2A\,x^2 y + B\,x^3 - Bx\,y^2 - 2C\,x^3 + 2yC\,x^2 + 6A\,x^2 + 6B\,x^2 - 6C\,x^2 + 9Bx - By}{C\,x^2} = 0$$

Setting the numerator to zero gives

$$2A\,x^3-2A\,x^2y+B\,x^3-Bx\,y^2-2C\,x^3+2yC\,x^2+6A\,x^2+6B\,x^2-6C\,x^2+9Bx-By=0 \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$2Av_1^3-2Av_1^2v_2+Bv_1^3-Bv_1v_2^2-2Cv_1^3+2Cv_1^2v_2+6Av_1^2+6Bv_1^2-6Cv_1^2+9Bv_1-Bv_2=0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(2A+B-2C)\,v_1^3+(-2A+2C)\,v_1^2v_2+(6A+6B-6C)\,v_1^2-Bv_1v_2^2+9Bv_1-Bv_2=0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-B&=0\\9B&=0\\-2A+2C&=0\\2A+B-2C&=0\\6A+6B-6C&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}A&=C\\B&=0\\C&=C\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi=1$$

$$\eta=1$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

This is easily solved to give

$$y = x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = y - x$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (-x + y - 3)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -1\\R_y &= 1\\S_x &= 1\\S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(-x+y-3)^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(R-3)^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

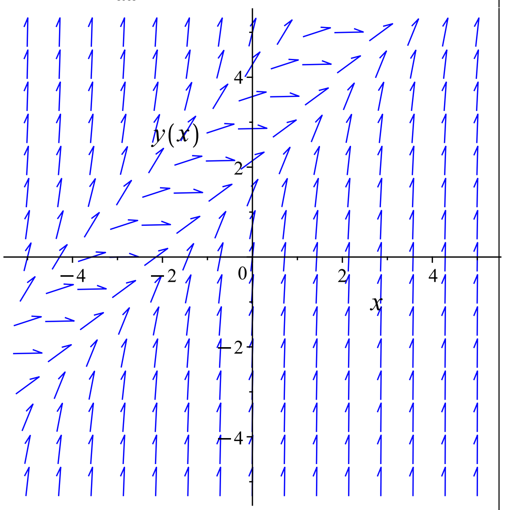
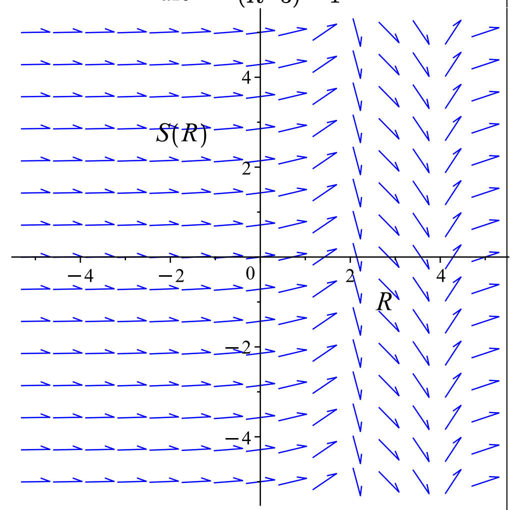
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int \frac{1}{R^2 - 6R + 8}\, dR\\S(R) &= -\frac{\ln(R-2)}{2} + \frac{\ln(R-4)}{2} + c_2\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$x = -\frac{\ln(-2-x+y)}{2} + \frac{\ln(y-x-4)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (-x+y-3)^2$ 	$R = y - x$ $S = x$	$\frac{dS}{dR} = \frac{1}{(R-3)^2-1}$ 

Solving for y gives

$$y = \frac{x\,\mathrm{e}^{-2x+2c_2} + 4\,\mathrm{e}^{-2x+2c_2} - x - 2}{-1 + \mathrm{e}^{-2x+2c_2}}$$

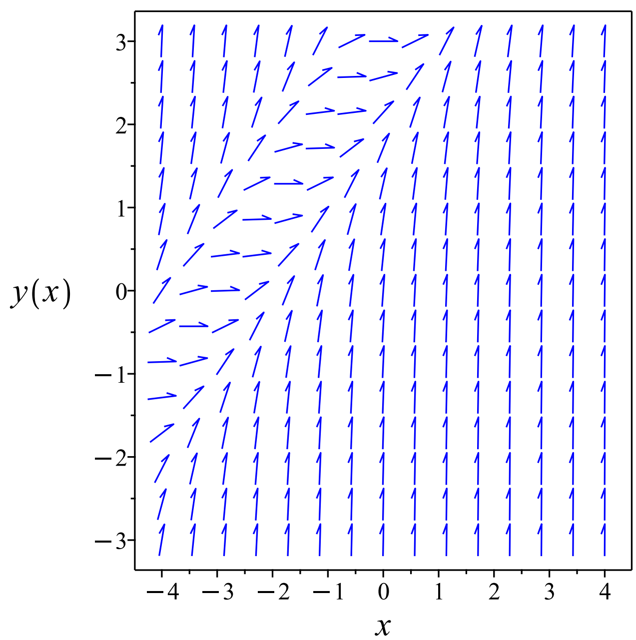


Figure 2.33: Slope field $y' = (x - y + 3)^2$

Summary of solutions found

$$y = \frac{x\,\mathrm{e}^{-2x+2c_2} + 4\,\mathrm{e}^{-2x+2c_2} - x - 2}{-1 + \mathrm{e}^{-2x+2c_2}}$$

Solved using first_order_ode_riccati

Time used: 0.530 (sec)

Solve

$$y' = (x - y + 3)^2$$

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (-x + y - 3)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - 2yx + y^2 + 6x - 6y + 9$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = (-x - 3)^2$, $f_1(x) = -2x - 6$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -2x - 6 \\ f_2^2 f_0 &= (-x - 3)^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - (-2x - 6)\, u'(x) + (-x - 3)^2\, u(x) = 0$$

In normal form the given ode is written as

$$\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\,u = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2x + 6 \\ q(x) &= (x+3)^2 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= (x+3)^2 - \frac{(2x+6)'}{2} - \frac{(2x+6)^2}{4} \\ &= (x+3)^2 - \frac{(2)}{2} - \frac{\left((2x+6)^2\right)}{4} \\ &= (x+3)^2 - (1) - \frac{(2x+6)^2}{4} \\ &= -1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$u = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2x+6}{2}} \\ &= \mathrm{e}^{-\frac{x(x+6)}{2}} \end{aligned} \tag{5}$$

Hence (3) becomes

$$u = v(x)\,\mathrm{e}^{-\frac{x(x+6)}{2}} \tag{4}$$

Applying this change of variable to the original ode results in

$$\mathrm{e}^{-\frac{x(x+6)}{2}}\left(\frac{d^2}{dx^2}v(x) - v(x)\right) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$\frac{d^2}{dx^2}v(x) - v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are distinct, then the solution is

$$v(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$v(x) = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$v(x) = c_1\,\mathrm{e}^{x} + c_2\,\mathrm{e}^{-x}$$

Now that $v(x)$ is known, then

$$\begin{aligned} u &= v(x)\, z(x) \\ &= \left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right) (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-\frac{x(x+6)}{2}}$$

Hence (7) becomes

$$u = \left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right) \mathrm{e}^{-\frac{x(x+6)}{2}}$$

Taking derivative gives

$$u'(x) = \left(c_1 \,\mathrm{e}^{x} - c_2 \,\mathrm{e}^{-x}\right) \mathrm{e}^{-\frac{x(x+6)}{2}} + \left(c_1 \,\mathrm{e}^{x} + c_2 \,\mathrm{e}^{-x}\right) (-x-3)\, \mathrm{e}^{-\frac{x(x+6)}{2}} \tag{4}$$

Substituting equations (3,4) into (1) results in

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ y &= \frac{-u'}{u} \\ y &= \frac{c_1 (x+2)\,\mathrm{e}^{2x} + c_2 (x+4)}{c_1 \,\mathrm{e}^{2x} + c_2} \end{aligned}$$

Doing change of constants, the above solution becomes

$$y = -\frac{\left(\left(\mathrm{e}^{x} - \mathrm{e}^{-x} c_3\right) \mathrm{e}^{-\frac{x(x+6)}{2}} + \left(\mathrm{e}^{x} + \mathrm{e}^{-x} c_3\right) (-x-3)\, \mathrm{e}^{-\frac{x(x+6)}{2}}\right) \mathrm{e}^{\frac{x(x+6)}{2}}}{\mathrm{e}^{x} + \mathrm{e}^{-x} c_3}$$

Simplifying the above gives

$$y = \frac{(x+2)\,\mathrm{e}^{2x} + c_3 (x+4)}{\mathrm{e}^{2x} + c_3}$$

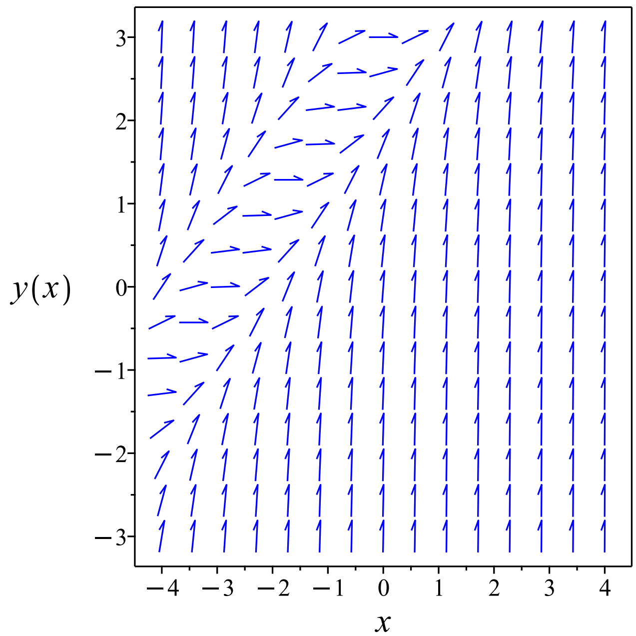


Figure 2.34: Slope field $y' = (x - y + 3)^2$

Summary of solutions found

$$y = \frac{(x+2)\,\mathrm{e}^{2x} + c_3(x+4)}{\mathrm{e}^{2x} + c_3}$$

Solved using first_order_ode_riccati_by_guessing_particular_solution

Time used: 0.115 (sec)

Solve

$$y' = (x - y + 3)^2$$

This is a Riccati ODE. Comparing the above ODE to solve with the Riccati standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that

$$f_0(x) = (-x-3)^2$$
$$f_1(x) = -2x - 6$$
$$f_2(x) = 1$$

Using trial and error, the following particular solution was found

$$y_p = x + 2$$

Since a particular solution is known, then the general solution is given by

$$y = y_p + \frac{\phi(x)}{c_1 - \int \phi(x) f_2 \, dx}$$

Where

$$\phi(x) = e^{\int 2 f_2 y_p + f_1 \, dx}$$

Evaluating the above gives the general solution as

$$y = x + 2 + \frac{1}{c_1 \, \mathrm{e}^{2x} + \frac{1}{2}}$$

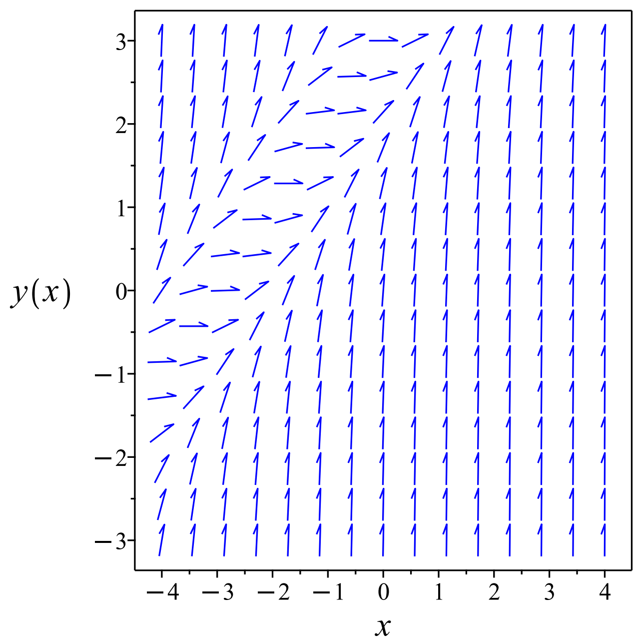


Figure 2.35: Slope field $y' = (x - y + 3)^2$

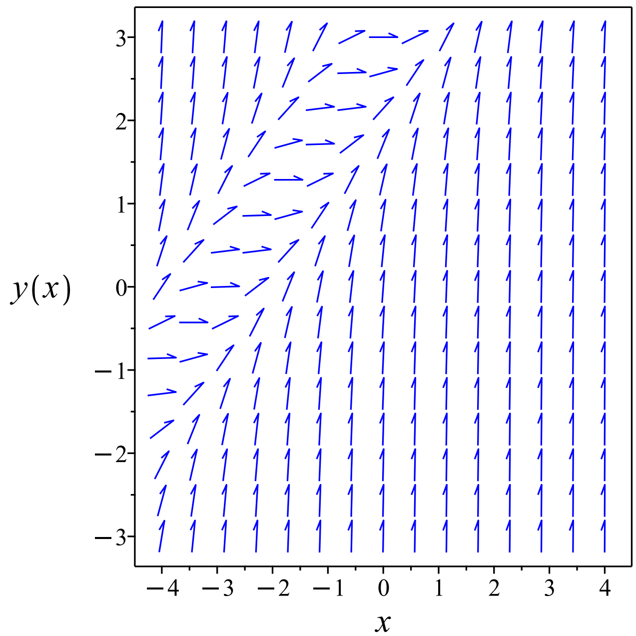


Figure 2.36: Slope field $y' = (x - y + 3)^2$

Summary of solutions found

$$y = x + 2 + \frac{1}{c_1 \,\mathrm{e}^{2x} + \frac{1}{2}}$$

✓ **Maple.** Time used: 0.009 (sec). Leaf size: 29

```
ode:=diff(y(x),x) = (x-y(x)+3)^2;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{c_1 (x+2)\,\mathrm{e}^{2x} - x - 4}{\mathrm{e}^{2x} c_1 - 1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
```

```
   -> Calling odsolve with the ODE, diff(y(x),x) = 1, y(x)
      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = (x - y(x) + 3)^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = (x - y(x) + 3)^2$

✓ **Mathematica.** Time used: 0.108 (sec). Leaf size: 29

```
ode=D[y[x],x]== (x-y[x]+3)^2;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + \frac{1}{\frac{1}{2} + c_1 e^{2x}} + 2$$
$$y(x) \to x + 2$$

✓ **Sympy.** Time used: 0.201 (sec). Leaf size: 29

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-(x - y(x) + 3)**2 + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{C_1 x + 4C_1 - xe^{2x} - 2e^{2x}}{C_1 - e^{2x}}$$

2.2.7 Problem (h)

Local contents

Solved using first_order_ode_separable 180
Solved using first_order_ode_exact 182
Solved using first_order_ode_isobaric 187
Solved using first_order_ode_homog_type_G 190
Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind 192
Solved using first_order_ode_LIE 195
✓Maple 201
✓Mathematica 202
✓Sympy 202

Internal problem ID [20973]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (h)
Date solved : `Saturday, November 29, 2025 at 01:16:48 AM`
CAS classification : `[_separable]`

Solved using first_order_ode_separable

Time used: 0.232 (sec)

Solve

$$y' = \frac{2y(y-1)}{x\,(2-y)}$$

The ode

$$y' = -\frac{2y(y-1)}{x\,(y-2)} \tag{2.11}$$

is separable as it can be written as

$$\begin{aligned} y' &= -\frac{2y(y-1)}{x\,(y-2)} \\ &= f(x)g(y) \end{aligned}$$

Where

$$f(x) = -\frac{2}{x}$$
$$g(y) = \frac{y(y-1)}{y-2}$$

Integrating gives

$$\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$
$$\int \frac{y-2}{y\,(y-1)}\, dy = \int -\frac{2}{x}\, dx$$

$$-\ln(y-1) + 2\ln(y) = \ln\left(\frac{1}{x^2}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$\frac{y(y-1)}{y-2} = 0$$

for y gives

$$y = 0$$
$$y = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln(y-1) + 2\ln(y) = \ln\left(\frac{1}{x^2}\right) + c_1$$
$$y = 0$$
$$y = 1$$

Solving for y gives

$$y = 0$$
$$y = 1$$
$$y = \frac{e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$
$$y = -\frac{-e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$

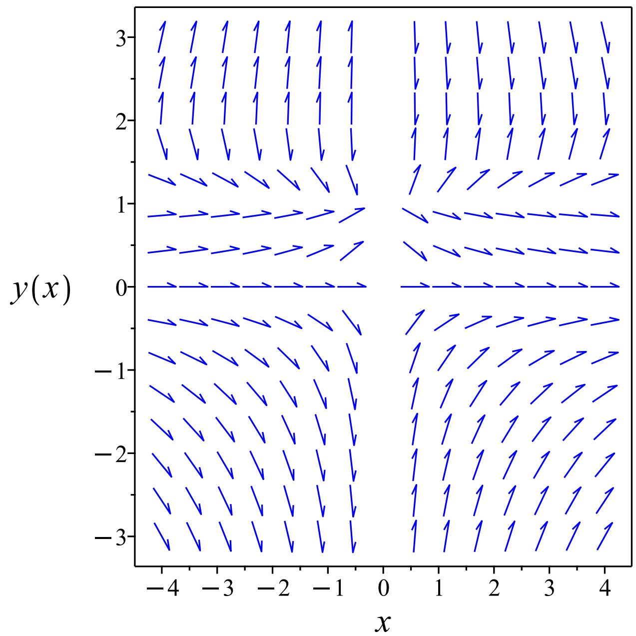


Figure 2.37: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

Summary of solutions found

$$y = 0$$

$$y = 1$$

$$y = \frac{e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1}x^2}}{2x^2}$$

$$y = -\frac{-e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1}x^2}}{2x^2}$$

Solved using first_order_ode_exact

Time used: 0.145 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{2y(y-1)}{x\,(2-y)}\right) \mathrm{d}x\\ \left(-\frac{2y(y-1)}{x\,(2-y)}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{2y(y-1)}{x\,(2-y)}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y(y-1)}{x\,(2-y)}\right)\\ &= \frac{2y^2 - 8y + 4}{x\,(y-2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{2(y-1)}{x\left(2-y\right)} - \frac{2y}{x\left(2-y\right)} - \frac{2y(y-1)}{x\left(2-y\right)^2}\right) - (0)\right) \\ &= \frac{2y^2 - 8y + 4}{x\left(y-2\right)^2} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{x(y-2)}{2y\left(y-1\right)}\left((0) - \left(-\frac{2(y-1)}{x\left(2-y\right)} - \frac{2y}{x\left(2-y\right)} - \frac{2y(y-1)}{x\left(2-y\right)^2}\right)\right) \\ &= \frac{-y^2 + 4y - 2}{\left(y-2\right)y\left(y-1\right)} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{-y^2+4y-2}{(y-2)y(y-1)} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(y-2) - \ln(y-1) - \ln(y)} \\ &= \frac{y-2}{y\left(y-1\right)} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{y-2}{y\left(y-1\right)}\left(-\frac{2y(y-1)}{x\left(2-y\right)}\right) \\ &= \frac{2}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{y-2}{y\left(y-1\right)}(1) \\ &= \frac{y-2}{y\left(y-1\right)} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{2}{x}\right) + \left(\frac{y-2}{y\,(y-1)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}x$$
$$\phi = 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{y-2}{y(y-1)}$. Therefore equation (4) becomes

$$\frac{y-2}{y\,(y-1)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y-2}{y\,(y-1)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y-2}{y\,(y-1)}\right)\mathrm{d}y$$
$$f(y) = -\ln(y-1) + 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\ln(x) - \ln(y-1) + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = 2\ln(x) - \ln(y-1) + 2\ln(y)$$

Solving for y gives

$$y = -\frac{e^{\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} + \sqrt{-4x^2 + e^{c_1}}\right)}{2x^2}$$

$$y = \frac{e^{\frac{c_1}{2}}\left(e^{\frac{c_1}{2}} + \sqrt{-4x^2 + e^{c_1}}\right)}{2x^2}$$

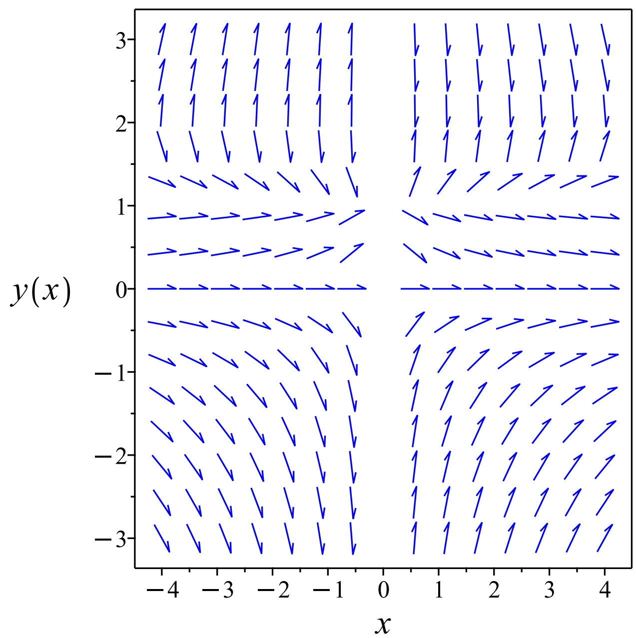


Figure 2.38: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

Summary of solutions found

$$y = -\frac{e^{\frac{c_1}{2}}\left(-e^{\frac{c_1}{2}} + \sqrt{-4x^2 + e^{c_1}}\right)}{2x^2}$$

$$y = \frac{e^{\frac{c_1}{2}}\left(e^{\frac{c_1}{2}} + \sqrt{-4x^2 + e^{c_1}}\right)}{2x^2}$$

Solved using first_order_ode_isobaric

Time used: 0.625 (sec)

Solve

$$y' = \frac{2y(y-1)}{x(2-y)}$$

Solving for y' gives

$$y' = -\frac{2y(y-1)}{x(y-2)} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{2y(y-1)}{x(y-2)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 0$$

Since the ode is isobaric of order $m = 0$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= u \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u'(x) = -\frac{2u(x)(u(x)-1)}{x(u(x)-2)}$$

The ode

$$u'(x) = -\frac{2u(x)(u(x)-1)}{x(u(x)-2)} \tag{2.12}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{2u(x)(u(x)-1)}{x(u(x)-2)} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{2}{x} \\ g(u) &= \frac{u(u-1)}{u-2} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$

$$\int \frac{u-2}{u\,(u-1)}\,du = \int -\frac{2}{x}\,dx$$

$$-\ln\left(u(x)-1\right)+2\ln\left(u(x)\right)=\ln\left(\frac{1}{x^2}\right)+c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or

$$\frac{u(u-1)}{u-2}=0$$

for $u(x)$ gives

$$u(x)=0$$
$$u(x)=1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(u(x)-1\right)+2\ln\left(u(x)\right)=\ln\left(\frac{1}{x^2}\right)+c_1$$

$$u(x)=0$$

$$u(x)=1$$

Converting $-\ln\left(u(x)-1\right)+2\ln\left(u(x)\right)=\ln\left(\frac{1}{x^2}\right)+c_1$ back to y gives

$$-\ln\left(y-1\right)+2\ln\left(y\right)=\ln\left(\frac{1}{x^2}\right)+c_1$$

Converting $u(x)=0$ back to y gives

$$y=0$$

Converting $u(x)=1$ back to y gives

$$y=1$$

Solving for y gives

$$y = 0$$

$$y = 1$$

$$y = \frac{e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$

$$y = -\frac{-e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$

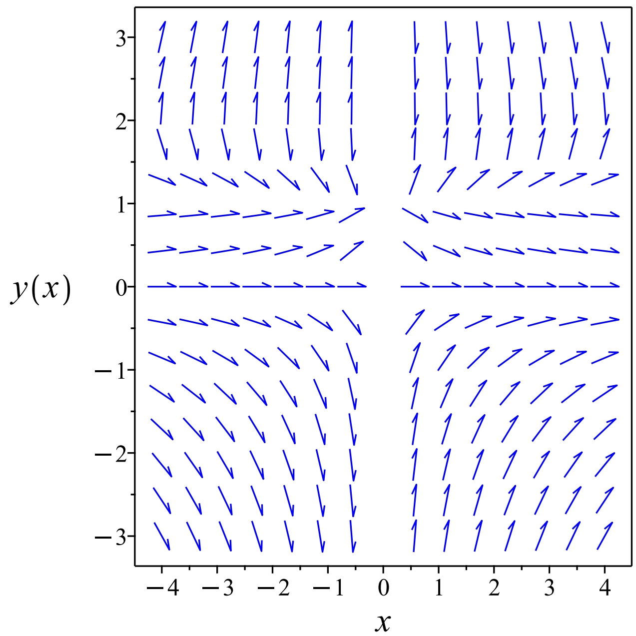


Figure 2.39: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

<u>Summary of solutions found</u>

$$y = 0$$

$$y = 1$$

$$y = \frac{e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$

$$y = -\frac{-e^{c_1} + \sqrt{e^{2c_1} - 4\,e^{c_1} x^2}}{2x^2}$$

Solved using first_order_ode_homog_type_G

Time used: 0.082 (sec)

Solve

$$y' = \frac{2y(y-1)}{x(2-y)}$$

Multiplying the right side of the ode, which is $-\frac{2y(y-1)}{x(y-2)}$ by $\frac{x}{y}$ gives

$$\begin{aligned}
y' &= \left(\frac{x}{y}\right) - \frac{2y(y-1)}{x(y-2)} \\
&= -\frac{2(y-1)}{y-2} \\
&= F(x,y)
\end{aligned}$$

Since $F(x,y)$ has y, then let

$$\begin{aligned}
f_x &= x\left(\frac{\partial}{\partial x}F(x,y)\right) \\
&= 0 \\
f_y &= y\left(\frac{\partial}{\partial y}F(x,y)\right) \\
&= \frac{2y}{(y-2)^2} \\
\alpha &= \frac{f_x}{f_y} \\
&= 0
\end{aligned}$$

Since α is independent of x,y then this is Homogeneous type G.

Let

$$\begin{aligned}
y &= \frac{z}{x^\alpha} \\
&= \frac{z}{1}
\end{aligned}$$

Substituting the above back into $F(x,y)$ gives

$$F(z) = -\frac{2(z-1)}{z-2}$$

We see that $F(z)$ does not depend on x nor on y. If this was not the case, then this method will not work.

Therefore, the implicit solution is given by

$$\ln(x) - c_1 - \int^{yx^\alpha} \frac{1}{z(\alpha + F(z))}\, dz = 0$$

Which gives

$$\ln(x) - c_1 + \int^{y} \frac{z-2}{2z(z-1)} dz = 0$$

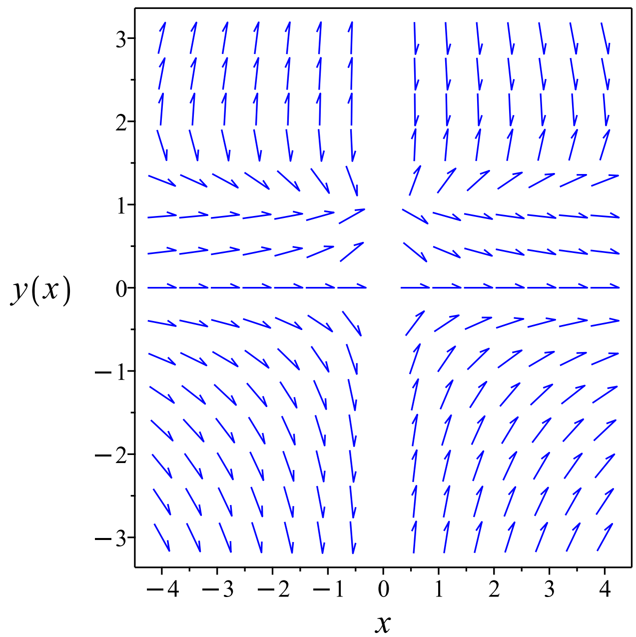


Figure 2.40: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

Summary of solutions found

$$\ln(x) - c_1 + \int^{y} \frac{z-2}{2z(z-1)} dz = 0$$

Solve

$$y' = \frac{2y(y-1)}{x(2-y)}$$

Solved using first_order_ode_abel_second_kind_solved_by_converting_to_first_kind

Time used: 18.353 (sec)

This is Abel second kind ODE, it has the form

$$(y+g)\,y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3$$

Comparing the above to given ODE which is

$$y' = \frac{2y(y-1)}{x\,(2-y)} \tag{1}$$

Shows that

$$\begin{aligned} g &= -2 \\ f_0 &= 0 \\ f_1 &= \frac{2}{x} \\ f_2 &= -\frac{2}{x} \\ f_3 &= 0 \end{aligned}$$

Applying transformation

$$y = \frac{1}{u(x)} - g$$

Results in the new ode which is Abel first kind

$$u'(x) = \frac{4u(x)^3}{x} + \frac{6u(x)^2}{x} + \frac{2u(x)}{x}$$

Which is now solved.

Solve The ode

$$u'(x) = \frac{2u(x)\,(u(x)+1)\,(2u(x)+1)}{x} \tag{2.13}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{2u(x)\,(u(x)+1)\,(2u(x)+1)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{2}{x} \\ g(u) &= u(u+1)\,(2u+1) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$

$$\int \frac{1}{u\,(u+1)\,(2u+1)}\, du = \int \frac{2}{x}\, dx$$

$$\ln\left(u(x)+1\right) - 2\ln\left(2u(x)+1\right) + \ln\left(u(x)\right) = \ln\left(x^2\right) + c_2$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u(u+1)\,(2u+1) = 0$$

for $u(x)$ gives

$$u(x) = -1$$
$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(u(x)+1\right) - 2\ln\left(2u(x)+1\right) + \ln\left(u(x)\right) = \ln\left(x^2\right) + c_2$$
$$u(x) = -1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = -1$$

$$u(x) = 0$$

$$u(x) = -\frac{\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1} - 1}{2\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1}}$$

$$u(x) = -\frac{\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1} + 1}{2\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1}}$$

Now we transform the solution $u(x) = -1$ to y using $u(x) = \frac{1}{y-2}$ which gives

$$y = 1$$

Now we transform the solution $u(x) = -\frac{\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}-1}{2\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}}$ to y using $u(x) = \frac{1}{y-2}$ which gives

$$y = -\frac{2}{\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}-1}$$

Now we transform the solution $u(x) = -\frac{\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}+1}{2\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}}$ to y using $u(x) = \frac{1}{y-2}$ which gives

$$y = \frac{2}{\sqrt{-4\,\mathrm{e}^{c_2} x^2+1}+1}$$

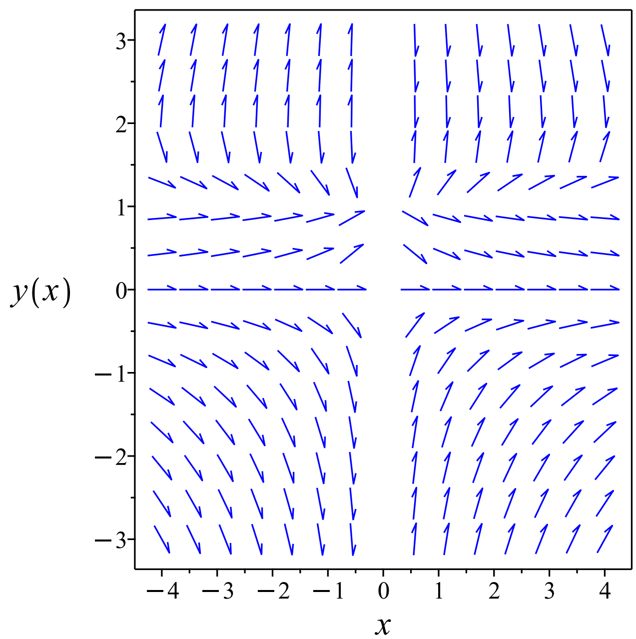


Figure 2.41: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

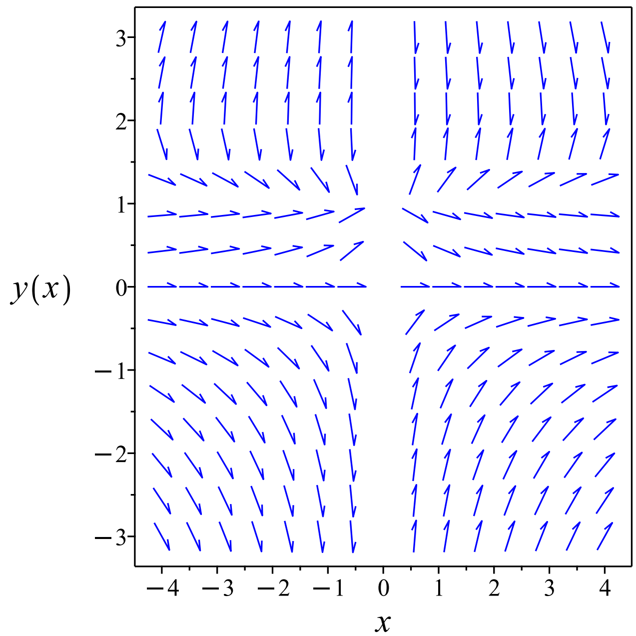


Figure 2.42: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

Summary of solutions found

$$y = 1$$

$$y = -\frac{2}{\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1} - 1}$$

$$y = \frac{2}{\sqrt{-4\,\mathrm{e}^{c_2} x^2 + 1} + 1}$$

Solved using first_order_ode_LIE

Time used: 4.656 (sec)

Solve

$$y' = \frac{2y(y-1)}{x\,(2-y)}$$

Writing the ode as

$$y' = -\frac{2y(y-1)}{x\,(y-2)}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{2y(y-1)(b_3-a_2)}{x(y-2)} - \frac{4y^2(y-1)^2 a_3}{x^2(y-2)^2} - \frac{2y(y-1)(xa_2+ya_3+a_1)}{x^2(y-2)} \\ - \left(-\frac{2(y-1)}{x(y-2)} - \frac{2y}{x(y-2)} + \frac{2y(y-1)}{x(y-2)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2y^2b_2 - 6y^4a_3 - 12x^2yb_2 + 2x\,y^2b_1 - 2x\,y^2b_3 - 2y^3a_1 + 14y^3a_3 + 8b_2x^2 - 8xyb_1 + 6y^2a_1 - 8y^2a_3 + 4xb_1 -}{(y-2)^2\,x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} 3x^2y^2b_2 - 6y^4a_3 - 12x^2yb_2 + 2x\,y^2b_1 - 2x\,y^2b_3 - 2y^3a_1 \\ + 14y^3a_3 + 8b_2x^2 - 8xyb_1 + 6y^2a_1 - 8y^2a_3 + 4xb_1 - 4ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -6a_3v_2^4 + 3b_2v_1^2v_2^2 - 2a_1v_2^3 + 14a_3v_2^3 + 2b_1v_1v_2^2 - 12b_2v_1^2v_2 \\ - 2b_3v_1v_2^2 + 6a_1v_2^2 - 8a_3v_2^2 - 8b_1v_1v_2 + 8b_2v_1^2 - 4a_1v_2 + 4b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1^2v_2^2 - 12b_2v_1^2v_2 + 8b_2v_1^2 + (2b_1 - 2b_3)\,v_1v_2^2 - 8b_1v_1v_2 + 4b_1v_1 \\ - 6a_3v_2^4 + (-2a_1 + 14a_3)\,v_2^3 + (6a_1 - 8a_3)\,v_2^2 - 4a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_1 &= 0 \\ -6a_3 &= 0 \\ -8b_1 &= 0 \\ 4b_1 &= 0 \\ -12b_2 &= 0 \\ 3b_2 &= 0 \\ 8b_2 &= 0 \\ -2a_1 + 14a_3 &= 0 \\ 6a_1 - 8a_3 &= 0 \\ 2b_1 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$

$$\eta = 0$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 0 - \left(-\frac{2y(y-1)}{x\,(y-2)}\right)(x) \\
&= \frac{2y(y-1)}{y-2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{2y(y-1)}{y-2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y-1)}{2} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(y-1)}{x\,(y-2)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{y-2}{2y\,(y-1)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

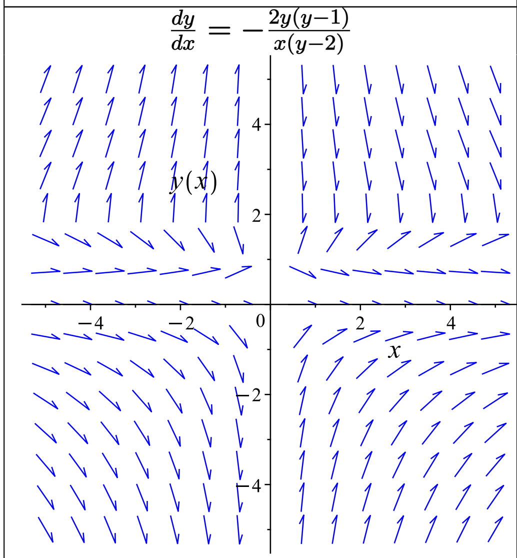
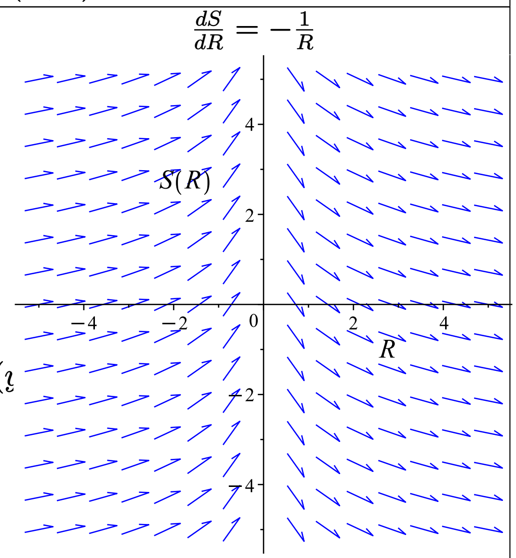
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{1}{R}\,dR \\
S(R) &= -\ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{\ln(y-1)}{2} + \ln(y) = -\ln(x) + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = x$ $S = -\frac{\ln(y-1)}{2} + \ln($	

Solving for y gives

$$y = -\frac{e^{c_2}\left(-e^{c_2} + \sqrt{e^{2c_2} - 4x^2}\right)}{2x^2}$$

$$y = \frac{e^{c_2}\left(e^{c_2} + \sqrt{e^{2c_2} - 4x^2}\right)}{2x^2}$$

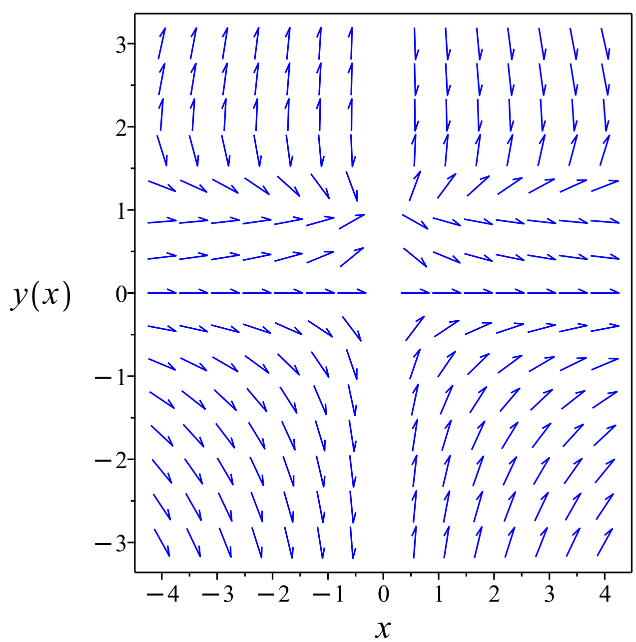


Figure 2.43: Slope field $y' = \frac{2y(y-1)}{x(2-y)}$

Summary of solutions found

$$y = -\frac{e^{c_2}\left(-e^{c_2}+\sqrt{e^{2c_2}-4x^2}\right)}{2x^2}$$
$$y = \frac{e^{c_2}\left(e^{c_2}+\sqrt{e^{2c_2}-4x^2}\right)}{2x^2}$$

✓ **Maple.** `Time used: 0.023 (sec). Leaf size: 85`

```
ode:=diff(y(x),x) = 2*y(x)*(y(x)-1)/x/(2-y(x));
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{1+\sqrt{-4x^2c_1^2+1}}{-2x^2c_1^2+\sqrt{-4x^2c_1^2+1}+1}$$
$$y = \frac{-1+\sqrt{-4x^2c_1^2+1}}{2x^2c_1^2+\sqrt{-4x^2c_1^2+1}-1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = \frac{2y(x)(y(x)-1)}{x(2-y(x))}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{2y(x)(y(x)-1)}{x(2-y(x))}$
- Separate variables
  $\frac{\left(\frac{d}{dx}y(x)\right)(2-y(x))}{y(x)(y(x)-1)} = \frac{2}{x}$
- Integrate both sides with respect to x
  $\int \frac{\left(\frac{d}{dx}y(x)\right)(2-y(x))}{y(x)(y(x)-1)}dx = \int \frac{2}{x}dx + C1$

- Evaluate integral
  $\ln(y(x)-1) - 2\ln(y(x)) = 2\ln(x) + C1$
- Solve for $y(x)$
  $\left\{y(x) = -\frac{-1+\sqrt{1-4x^2\left(e^{\frac{C1}{2}}\right)^2}}{2x^2\left(e^{\frac{C1}{2}}\right)^2}, y(x) = \frac{1+\sqrt{1-4x^2\left(e^{\frac{C1}{2}}\right)^2}}{2x^2\left(e^{\frac{C1}{2}}\right)^2}\right\}$
- Simplify
  $\left\{y(x) = -\frac{\left(-1+\sqrt{1-4x^2 e^{C1}}\right)e^{-C1}}{2x^2}, y(x) = \frac{\left(1+\sqrt{1-4x^2 e^{C1}}\right)e^{-C1}}{2x^2}\right\}$

✓ **Mathematica.** Time used: 0.766 (sec). Leaf size: 90

```
ode=D[y[x],x]== (2*y[x]*(y[x]-1))/( x*(2-y[x]) ) ;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{e^{\frac{c_1}{2}}\sqrt{4x^2+e^{c_1}}+e^{c_1}}{2x^2}$$
$$y(x) \to -\frac{e^{c_1}-e^{\frac{c_1}{2}}\sqrt{4x^2+e^{c_1}}}{2x^2}$$
$$y(x) \to 0$$
$$y(x) \to 1$$

✓ **Sympy.** Time used: 0.620 (sec). Leaf size: 41

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(Derivative(y(x), x) - (2*y(x) - 2)*y(x)/(x*(2 - y(x))),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$\left[y(x) = \frac{C_1 - \sqrt{C_1\left(C_1 - 4x^2\right)}}{2x^2}, \ y(x) = \frac{C_1 + \sqrt{C_1\left(C_1 - 4x^2\right)}}{2x^2}\right]$$

2.2.8 Problem (i)

Local contents

Solved using first_order_ode_isobaric 203
Solved using first_order_ode_homog_A 206
Solved using first_order_ode_dAlembert 209
Solved using first_order_ode_homog_type_maple_C 212
Solved using first_order_ode_LIE 217
✓Maple 222
✓Mathematica 222
✓Sympy 223

Internal problem ID [20974]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises XIII at page 24
Problem number : (i)
Date solved : `Saturday, November 29, 2025 at 01:18:17 AM`
CAS classification : `[[_homogeneous, ‘class A‘], _rational, _dAlembert]`

Solved using first_order_ode_isobaric

Time used: 44.257 (sec)

Solve

$$y = y'x - \sqrt{x^2 + y^2}$$

Solving for y' gives

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{xu(x) + \sqrt{x^2 + x^2 u(x)^2}}{x}$$

The ode

$$u'(x) = \frac{\sqrt{u(x)^2 + 1}}{x} \tag{2.14}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{\sqrt{u(x)^2 + 1}}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \sqrt{u^2 + 1} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{\sqrt{u^2 + 1}}\, du &= \int \frac{1}{x}\, dx \end{aligned}$$

$$\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\sqrt{u^2 + 1} = 0$$

for $u(x)$ gives

$$\begin{aligned} u(x) &= -i \\ u(x) &= i \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$$
$$u(x) = -i$$
$$u(x) = i$$

Converting $\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$ back to y gives

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

Converting $u(x) = -i$ back to y gives

$$\frac{y}{x} = -i$$

Converting $u(x) = i$ back to y gives

$$\frac{y}{x} = i$$

Solving for y gives

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$
$$y = -ix$$
$$y = ix$$

The solution

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

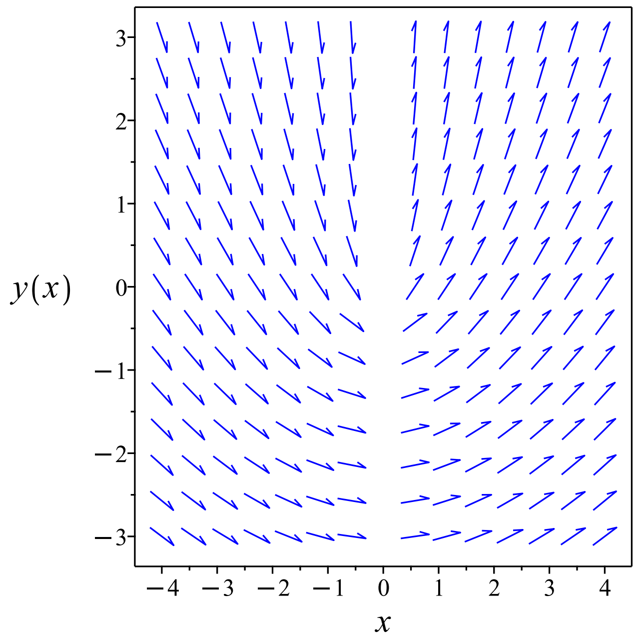


Figure 2.44: Slope field $y = y'x - \sqrt{x^2 + y^2}$

Summary of solutions found

$$y = -ix$$

$$y = ix$$

Solved using first_order_ode_homog_A

Time used: 49.945 (sec)

Solve

$$y = y'x - \sqrt{x^2 + y^2}$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y + \sqrt{x^2 + y^2}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = y + \sqrt{x^2 + y^2}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = u + \sqrt{u^2+1}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{u(x)^2+1}}{x}$$

Or

$$u'(x) - \frac{\sqrt{u(x)^2+1}}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{u(x)^2+1} = 0$$

Which is now solved as separable in $u(x)$.

The ode

$$u'(x) = \frac{\sqrt{u(x)^2+1}}{x} \tag{2.15}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{\sqrt{u(x)^2+1}}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$f(x) = \frac{1}{x}$$
$$g(u) = \sqrt{u^2+1}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{\sqrt{u^2+1}}\,du = \int \frac{1}{x}\,dx$$

$$\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\sqrt{u^2+1} = 0$$

for $u(x)$ gives

$$u(x) = -i$$
$$u(x) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$$
$$u(x) = -i$$
$$u(x) = i$$

Converting $\operatorname{arcsinh}(u(x)) = \ln(x) + c_1$ back to y gives

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

Converting $u(x) = -i$ back to y gives

$$y = -ix$$

Converting $u(x) = i$ back to y gives

$$y = ix$$

The solution

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

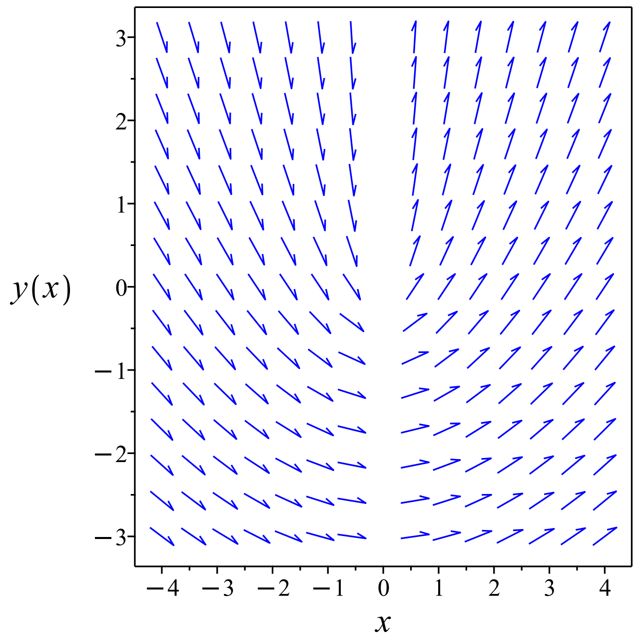


Figure 2.45: Slope field $y = y'x - \sqrt{x^2 + y^2}$

Summary of solutions found

$$y = -ix$$

$$y = ix$$

Solved using first_order_ode_dAlembert

Time used: 0.498 (sec)

Solve

$$y = y'x - \sqrt{x^2 + y^2}$$

Let $p = y'$ the ode becomes

$$y = px - \sqrt{x^2 + y^2}$$

Solving for y from the above results in

$$y = \frac{x(p^2 - 1)}{2p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p^2 - 1}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{p^2 - 1}{2p} = \left(\frac{x}{2} + \frac{x}{2p^2}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{p^2 - 1}{2p} = 0
$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= i \\
p_2 &= -i
\end{aligned}
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
\begin{aligned}
y &= ix \\
y &= -ix
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{p(x)^2 - 1}{2p(x)}}{\frac{x}{2} + \frac{x}{2p(x)^2}} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed.

The ode

$$
p'(x) = \frac{p(x)}{x} \tag{2.16}
$$

is separable as it can be written as

$$\begin{aligned}p'(x) &= \frac{p(x)}{x}\\ &= f(x)g(p)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \frac{1}{x}\\ g(p) &= p\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(x)\, dx$$

$$\int \frac{1}{p}\, dp = \int \frac{1}{x}\, dx$$

$$\ln(p(x)) = \ln(x) + c_1$$

Taking the exponential of both sides the solution becomes

$$p(x) = c_1 x$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or

$$p = 0$$

for $p(x)$ gives

$$p(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$p(x) = c_1 x$$

$$p(x) = 0$$

Substituing the above solution for p in (2A) gives

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

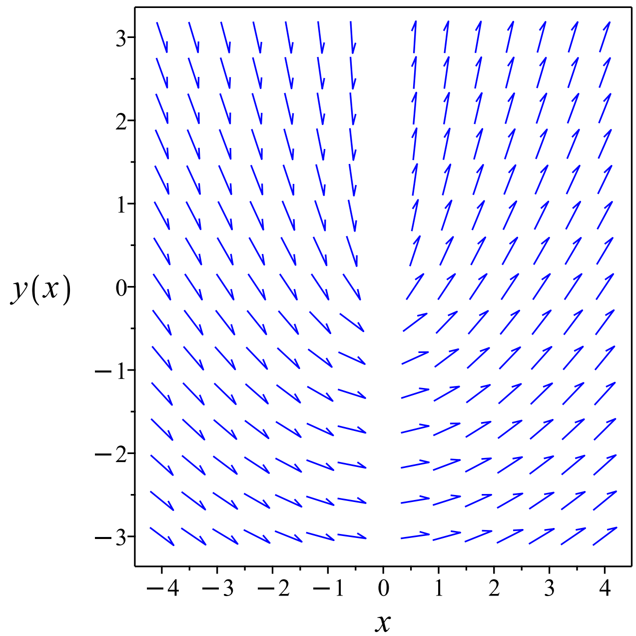


Figure 2.46: Slope field $y = y'x - \sqrt{x^2 + y^2}$

Summary of solutions found

$$y = -ix$$

$$y = ix$$

$$y = \frac{c_1^2 x^2 - 1}{2c_1}$$

Solved using first_order_ode_homog_type_maple_C

Time used: 1.770 (sec)

Solve

$$y = y'x - \sqrt{x^2 + y^2}$$

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + \sqrt{(X + x_0)^2 + (Y(X) + y_0)^2}}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X) + \sqrt{X^2 + Y(X)^2}}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{Y + \sqrt{X^2 + Y^2}}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = Y + \sqrt{X^2 + Y^2}$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u + \sqrt{u^2 + 1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\sqrt{u(X)^2 + 1}}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\sqrt{u(X)^2 + 1}}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - \sqrt{u(X)^2 + 1} = 0$$

Which is now solved as separable in $u(X)$.

The ode

$$\frac{d}{dX}u(X) = \frac{\sqrt{u(X)^2 + 1}}{X} \tag{2.17}$$

is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= \frac{\sqrt{u(X)^2 + 1}}{X} \\ &= f(X)g(u) \end{aligned}$$

Where

$$f(X) = \frac{1}{X}$$
$$g(u) = \sqrt{u^2+1}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(X)\,dX$$
$$\int \frac{1}{\sqrt{u^2+1}}\,du = \int \frac{1}{X}\,dX$$

$$\operatorname{arcsinh}(u(X)) = \ln(X) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$\sqrt{u^2+1} = 0$$

for $u(X)$ gives

$$u(X) = -i$$
$$u(X) = i$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\operatorname{arcsinh}(u(X)) = \ln(X) + c_1$$
$$u(X) = -i$$
$$u(X) = i$$

Converting $\operatorname{arcsinh}(u(X)) = \ln(X) + c_1$ back to $Y(X)$ gives

$$\operatorname{arcsinh}\left(\frac{Y(X)}{X}\right) = \ln(X) + c_1$$

Converting $u(X) = -i$ back to $Y(X)$ gives

$$Y(X) = -iX$$

Converting $u(X) = i$ back to $Y(X)$ gives

$$Y(X) = iX$$

Using the solution for $Y(X)$

$$\operatorname{arcsinh}\left(\frac{Y(X)}{X}\right) = \ln(X) + c_1 \tag{A}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x_0 + x \end{aligned}$$

Or

$$\begin{aligned} Y &= y \\ X &= x \end{aligned}$$

Then the solution in y becomes using EQ (A)

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

Using the solution for $Y(X)$

$$Y(X) = -iX \tag{A}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x_0 + x \end{aligned}$$

Or

$$\begin{aligned} Y &= y \\ X &= x \end{aligned}$$

Then the solution in y becomes using EQ (A)

$$y = -ix$$

Using the solution for $Y(X)$

$$Y(X) = iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x_0 + x$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = ix$$

The solution

$$\operatorname{arcsinh}\left(\frac{y}{x}\right) = \ln(x) + c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

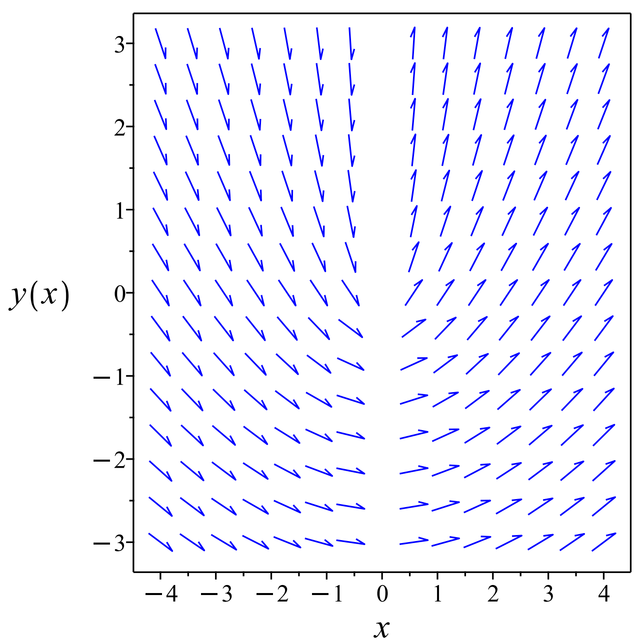


Figure 2.47: Slope field $y = y'x - \sqrt{x^2 + y^2}$

<u>Summary of solutions found</u>

$$y = -ix$$

$$y = ix$$

Solved using first_order_ode_LIE

Time used: 3.779 (sec)

Solve

$$y = y'x - \sqrt{x^2 + y^2}$$

Writing the ode as

$$\begin{aligned} y' &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Using these anstaz

$$\xi = \frac{Ax + By}{Cx} \tag{1E}$$

$$\eta = 1 \tag{2E}$$

Where the unknown coefficients are

$$\{A, B, C\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &\frac{\left(y + \sqrt{x^2 + y^2}\right)\left(-\frac{A}{Cx} + \frac{Ax + By}{C\,x^2}\right)}{x} - \frac{\left(y + \sqrt{x^2 + y^2}\right)^2 B}{x^3 C} \\ &- \frac{\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(Ax + By)}{Cx} - \frac{1 + \frac{y}{\sqrt{x^2+y^2}}}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{A\sqrt{x^2 + y^2}\,xy + Ax\,y^2 - B(x^2 + y^2)^{3/2} + B\sqrt{x^2 + y^2}\,y^2 - B\,x^2 y - x^2\sqrt{x^2 + y^2}\,C - C\,x^2 y}{\sqrt{x^2 + y^2}\,x^3 C} = 0$$

Setting the numerator to zero gives

$$A\sqrt{x^2 + y^2}\,xy + Ax\,y^2 - B\left(x^2 + y^2\right)^{3/2} + B\sqrt{x^2 + y^2}\,y^2 - B\,x^2 y - x^2\sqrt{x^2 + y^2}\,C - C\,x^2 y = 0 \tag{6E}$$

Simplifying the above gives

$$A(x^2+y^2)\,x + A\sqrt{x^2+y^2}\,xy - A\,x^3 - B(x^2+y^2)^{3/2} + B\sqrt{x^2+y^2}\,y^2 - B\,x^2y - x^2\sqrt{x^2+y^2}\,C - C\,x^2y = 0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$x\left(A\sqrt{x^2+y^2}\,y + A\,y^2 - B\sqrt{x^2+y^2}\,x - Bxy - C\sqrt{x^2+y^2}\,x - xCy\right) = 0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$v_1\left(Av_2^2 + Av_3v_2 - Bv_1v_2 - Bv_3v_1 - v_1Cv_2 - Cv_3v_1\right) = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$(-B-C)\,v_2v_1^2 + (-B-C)\,v_3v_1^2 + Av_2^2v_1 + Av_2v_3v_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} A &= 0 \\ -B-C &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} A &= 0 \\ B &= -C \\ C &= C \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -\frac{y}{x}$$
$$\eta = 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 1 - \left(\frac{y+\sqrt{x^2+y^2}}{x}\right)\left(-\frac{y}{x}\right) \\ &= \frac{\sqrt{x^2+y^2}\,y + x^2 + y^2}{x^2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{x^2+y^2}\,y+x^2+y^2}{x^2}} dy\end{aligned}$$

Which results in

$$S = x^2\left(\frac{y}{x^2} - \frac{\sqrt{x^2+y^2}}{x^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y+\sqrt{x^2+y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= -\frac{x}{\sqrt{x^2+y^2}}\\
S_y &= 1-\frac{y}{\sqrt{x^2+y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

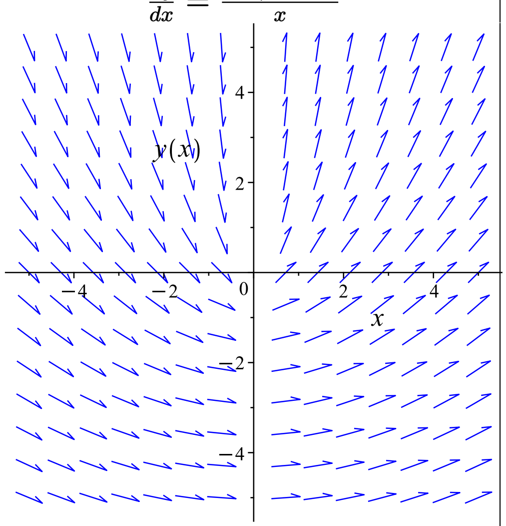
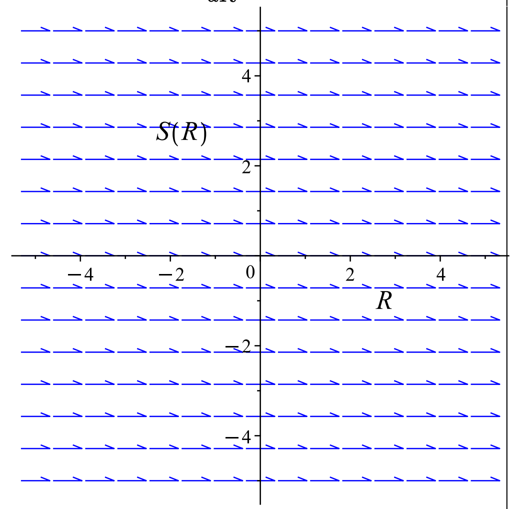
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\,dR + c_2\\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$y-\sqrt{x^2+y^2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\sqrt{x^2+y^2} + y$	$\frac{dS}{dR} = 0$ 

Solving for y gives

$$y = \frac{c_2^2 - x^2}{2c_2}$$

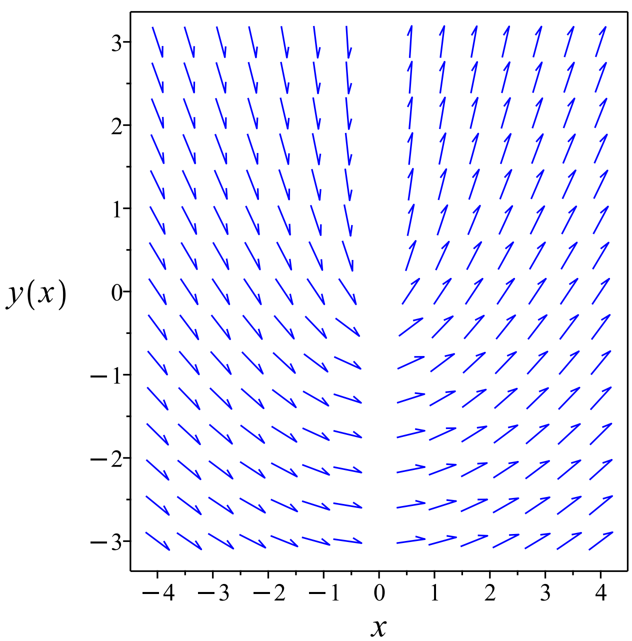


Figure 2.48: Slope field $y = y'x - \sqrt{x^2 + y^2}$

Summary of solutions found

$$y = \frac{c_2^2 - x^2}{2c_2}$$

✓ **Maple.** Time used: 0.004 (sec). Leaf size: 26

```
ode:=y(x) = diff(y(x),x)*x-(x^2+y(x)^2)^(1/2);
dsolve(ode,y(x), singsol=all);
```

$$\frac{-c_1 x^2 + y + \sqrt{x^2 + y^2}}{x^2} = 0$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful
```

Maple step by step

Let's solve

$y(x) = x\left(\frac{d}{dx}y(x)\right) - \sqrt{x^2 + y(x)^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{y(x)+\sqrt{x^2+y(x)^2}}{x}$

✓ **Mathematica.** Time used: 0.18 (sec). Leaf size: 13

```
ode=y[x]==x*D[y[x],x]- Sqrt[x^2 + y[x]^2];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x \sinh(\log(x) + c_1)$$

✓ **Sympy.** Time used: 0.733 (sec). Leaf size: 12

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*Derivative(y(x), x) + sqrt(x**2 + y(x)**2) + y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = -x \sinh \left(C_1 - \log \left(x\right)\right)$$

2.3 Chapter 1. First order equations: Some integrable cases. Excercises VI at page 33

Local contents

2.3.1 Problem (a) 225

2.3.1 Problem (a)

Local contents

Solved using first_order_ode_separable . . . 225
Solved using first_order_ode_exact . . . 227
✓Maple . . . 231
✓Mathematica . . . 232
✓Sympy . . . 232

Internal problem ID [20975]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VI at page 33
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:21:20 AM`
CAS classification : `[_separable]`

Solved using first_order_ode_separable

Time used: 0.225 (sec)

Solve

$$y' = f(x)\, y \ln\left(\frac{1}{y}\right)$$

The ode

$$y' = f(x)\, y \ln\left(\frac{1}{y}\right) \tag{2.18}$$

is separable as it can be written as

$$\begin{aligned} y' &= f(x)\, y \ln\left(\frac{1}{y}\right) \\ &= f(x) g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= f(x) \\ g(y) &= y \ln\left(\frac{1}{y}\right) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

$$\int \frac{1}{y\ln\left(\frac{1}{y}\right)}\,dy = \int f(x)\;dx$$

$$-\ln\left(\ln\left(\frac{1}{y}\right)\right) = \int f(x)\,dx + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$y\ln\left(\frac{1}{y}\right) = 0$$

for y gives

$$y = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln\left(\ln\left(\frac{1}{y}\right)\right) = \int f(x)\,dx + c_1$$
$$y = 1$$

Summary of solutions found

$$-\ln\left(\ln\left(\frac{1}{y}\right)\right) = \int f(x)\,dx + c_1$$
$$y = 1$$

Solved using first_order_ode_exact

Time used: 0.145 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(f(x)\,y\ln\left(\frac{1}{y}\right)\right)\mathrm{d}x$$

$$\left(-f(x)\,y\ln\left(\frac{1}{y}\right)\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -f(x)\,y\ln\left(\frac{1}{y}\right)$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-f(x)\, y \ln\left(\frac{1}{y}\right)\right)\\ &= -f(x)\left(\ln\left(\frac{1}{y}\right) - 1\right)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-f(x)\ln\left(\frac{1}{y}\right) + f(x)\right) - (0)\right)\\ &= -f(x)\left(\ln\left(\frac{1}{y}\right) - 1\right)\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{f(x)\, y \ln\left(\frac{1}{y}\right)}\left((0) - \left(-f(x)\ln\left(\frac{1}{y}\right) + f(x)\right)\right)\\ &= \frac{-\ln\left(\frac{1}{y}\right) + 1}{y \ln\left(\frac{1}{y}\right)}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int \frac{-\ln\left(\frac{1}{y}\right)+1}{y\ln\left(\frac{1}{y}\right)} \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln\left(\ln\left(\frac{1}{y}\right)\right)+\ln\left(\frac{1}{y}\right)}\\ &= \frac{1}{y\ln\left(\frac{1}{y}\right)}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y\ln\left(\frac{1}{y}\right)}\left(-f(x)\, y\ln\left(\frac{1}{y}\right)\right)\\ &= -f(x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{y\ln\left(\frac{1}{y}\right)}(1)\\ &= \frac{1}{y\ln\left(\frac{1}{y}\right)}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ (-f(x)) + \left(\frac{1}{y\ln\left(\frac{1}{y}\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -f(x)\,\mathrm{d}x\end{aligned}$$

$$\phi = \int -f(x)\,dx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y \ln\left(\frac{1}{y}\right)}$. Therefore equation (4) becomes

$$\frac{1}{y \ln\left(\frac{1}{y}\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y \ln\left(\frac{1}{y}\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{y \ln\left(\frac{1}{y}\right)}\right) \mathrm{d}y$$

$$f(y) = -\ln\left(\ln\left(\frac{1}{y}\right)\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int -f(x)\, dx - \ln\left(\ln\left(\frac{1}{y}\right)\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int -f(x)\, dx - \ln\left(\ln\left(\frac{1}{y}\right)\right)$$

<u>Summary of solutions found</u>

$$\int -f(x)\, dx - \ln\left(\ln\left(\frac{1}{y}\right)\right) = c_1$$

✓ **Maple.** Time used: 0.032 (sec). Leaf size: 17

```
ode:=diff(y(x),x) = f(x)*y(x)*ln(1/y(x));
dsolve(ode,y(x), singsol=all);
```

$$y = \mathrm{e}^{-\frac{\mathrm{e}^{-\int f(x)dx}}{c_1}}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) = f(x)\, y(x) \ln\left(\frac{1}{y(x)}\right)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = f(x)\, y(x) \ln\left(\frac{1}{y(x)}\right)$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)\ln\left(\frac{1}{y(x)}\right)} = f(x)$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{y(x)\ln\left(\frac{1}{y(x)}\right)}dx = \int f(x)\, dx + C1$
- Cannot compute integral
  $-\ln\left(\ln\left(\frac{1}{y(x)}\right)\right) = \int f(x)\, dx + C1$

✓ **Mathematica.** `Time used: 0.127 (sec). Leaf size: 33`

```
ode=D[y[x],x]==f[x]*y[x]*Log[1/y[x]];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(-\exp\left(-\int_1^x f(K[1])dK[1] - c_1\right)\right)$$
$$y(x) \to 1$$

✓ **Sympy.** `Time used: 0.248 (sec). Leaf size: 12`

```
from sympy import *
x = symbols("x")
y = Function("y")
f = Function("f")
ode = Eq(-f(x)*y(x)*log(1/y(x)) + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = e^{C_1 e^{-\int f(x)\,dx}}$$

2.4 Chapter 1. First order equations: Some integrable cases. Excercises VII at page 33

Local contents

2.4.1 Problem (a) 234

2.4.1 Problem (a)

Local contents

Existence and uniqueness analysis 234
Solved using first_order_ode_LIE 235
Solved using first_order_ode_riccati 239
Solved using first_order_ode_riccati_by_guessing_particular_solution 244
✓Maple 246
✓Mathematica 247
✓Sympy 247

Internal problem ID [20976]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VII at page 33
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:21:49 AM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _Riccati]`

Existence and uniqueness analysis

$$
\begin{aligned}
y' - y + e^{x} y^{2} + 5\,e^{-x} &= 0 \\
y(0) &= \eta
\end{aligned}
$$

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= y - e^{x} y^{2} - 5\,e^{-x}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=\eta$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$
\{-\infty < y < \infty\}
$$

But the point $y_0 = \eta$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

Solved using first_order_ode_LIE

Time used: 0.545 (sec)

Solve

$$y' - y + e^x y^2 + 5\,e^{-x} = 0$$
$$y(0) = \eta$$

Writing the ode as

$$y' = y - e^x y^2 - 5\,e^{-x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Using these anstaz

$$\xi = 1 \tag{1E}$$
$$\eta = Ax + By \tag{2E}$$

Where the unknown coefficients are

$$\{A, B\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$A + \left(y - e^x y^2 - 5\,e^{-x}\right) B + e^x y^2 - 5\,e^{-x} - (1 - 2\,e^x y)(Ax + By) = 0 \tag{5E}$$

Putting the above in normal form gives

$$2A\,e^x xy + B\,e^x y^2 + e^x y^2 - Ax - 5B\,e^{-x} + A - 5\,e^{-x} = 0$$

Setting the numerator to zero gives

$$2A\,e^x xy + B\,e^x y^2 + e^x y^2 - Ax - 5B\,e^{-x} + A - 5\,e^{-x} = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^x, e^{-x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^{x} = v_3, e^{-x} = v_4\}$$

The above PDE (6E) now becomes

$$2Av_3v_1v_2 + Bv_3v_2^2 + v_3v_2^2 - Av_1 - 5Bv_4 + A - 5v_4 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$2Av_3v_1v_2 - Av_1 + (B+1)\, v_3v_2^2 + (-5B-5)\, v_4 + A = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} A &= 0 \\ -A &= 0 \\ 2A &= 0 \\ -5B - 5 &= 0 \\ B + 1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} A &= 0 \\ B &= -1 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= -y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-y}{1}\\ &= -y\end{aligned}$$

This is easily solved to give

$$y = \mathrm{e}^{-x} c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \mathrm{e}^{x} y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y - \mathrm{e}^{x} y^2 - 5\,\mathrm{e}^{-x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= \mathrm{e}^{x} y\\ R_y &= \mathrm{e}^{x}\\ S_x &= 1\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2\,\mathrm{e}^{x} y - \mathrm{e}^{2x} y^{2} - 5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^{2} - 2R + 5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

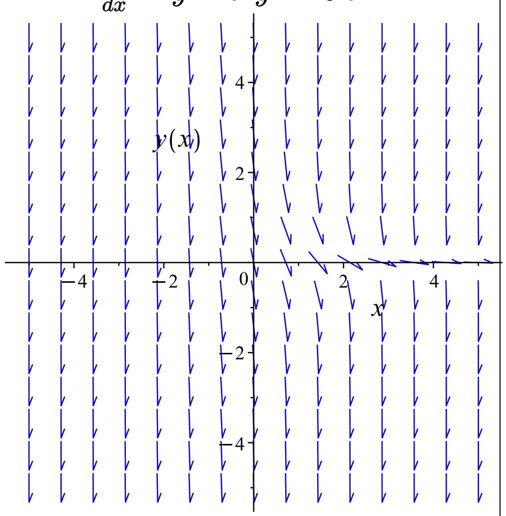
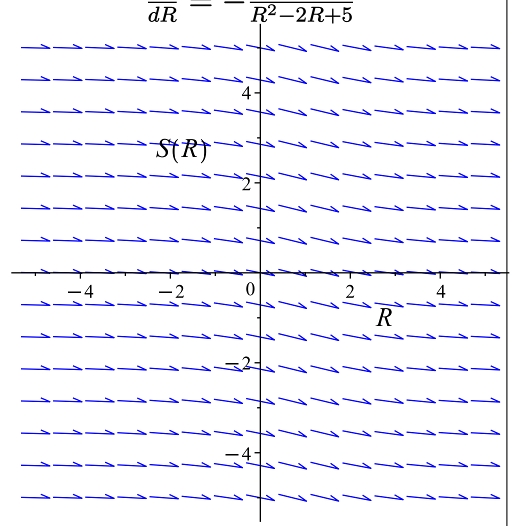
$$\int dS = \int -\frac{1}{R^{2} - 2R + 5}\, dR$$

$$S(R) = -\frac{\arctan\left(\frac{R}{2} - \frac{1}{2}\right)}{2} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$x = -\frac{\arctan\left(\frac{\mathrm{e}^{x} y}{2} - \frac{1}{2}\right)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = y - \mathrm{e}^{x} y^{2} - 5\,\mathrm{e}^{-x}$

</td><td>$R = \mathrm{e}^{x} y$
$S = x$</td><td>$\frac{dS}{dR} = -\frac{1}{R^{2}-2R+5}$

</td></tr>
</table>

Solving for initial conditions the solution is

$$x = -\frac{\arctan\left(\frac{e^{x} y}{2} - \frac{1}{2}\right)}{2} + \frac{\arctan\left(\frac{\eta}{2} - \frac{1}{2}\right)}{2}$$

Solving for y gives

$$y = -\left(2\tan\left(2x - \arctan\left(\frac{\eta}{2} - \frac{1}{2}\right)\right) - 1\right) e^{-x}$$

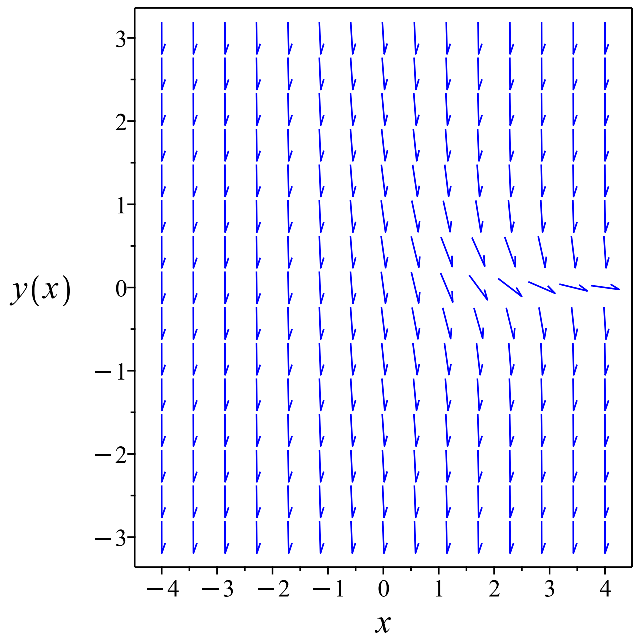


Figure 2.49: Slope field $y' - y + e^{x} y^{2} + 5\,e^{-x} = 0$

<u>Summary of solutions found</u>

$$y = -\left(2\tan\left(2x - \arctan\left(\frac{\eta}{2} - \frac{1}{2}\right)\right) - 1\right) e^{-x}$$

Solved using first_order_ode_riccati

Time used: 0.641 (sec)

Solve

$$y' - y + e^{x} y^{2} + 5\,e^{-x} = 0$$
$$y(0) = \eta$$

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y - e^x y^2 - 5\,e^{-x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y - e^x y^2 - 5\,e^{-x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -5\,e^{-x}$, $f_1(x) = 1$ and $f_2(x) = -e^x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-e^x u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification) in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -e^x \\ f_1 f_2 &= -e^x \\ f_2^2 f_0 &= -5\,e^{2x} e^{-x} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-e^x u''(x) + 2\,e^x u'(x) - 5\,e^{2x} e^{-x} u(x) = 0$$

In normal form the ode

$$-e^x \left(\frac{d^2u}{dx^2}\right) + 2\,e^x \left(\frac{du}{dx}\right) - 5\,e^{2x} e^{-x} u = 0 \tag{1}$$

Becomes

$$\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\,u = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= 5
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2}u(\tau) + p_1\left(\frac{d}{d\tau}u(\tau)\right) + q_1 u(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\
&= \int \mathrm{e}^{-\int (-2)dx} dx \\
&= \int e^{2x}\, dx \\
&= \int \mathrm{e}^{2x} dx \\
&= \frac{\mathrm{e}^{2x}}{2} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \\
&= \frac{5}{\mathrm{e}^{4x}} \\
&= 5\,\mathrm{e}^{-4x} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}u(\tau) + q_1 u(\tau) &= 0 \\
\frac{d^2}{d\tau^2}u(\tau) + 5\,\mathrm{e}^{-4x}u(\tau) &= 0
\end{aligned}
$$

But in terms of τ

$$5\,\mathrm{e}^{-4x} = \frac{5}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}u(\tau) + \frac{5u(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $u(\tau)$. This is Euler second order ODE. Let the solution be $u(\tau) = \tau^r$, then $u' = r\tau^{r-1}$ and $u'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 5\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + 5\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 5 = 0$$

Or

$$4r^2 - 4r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - i \\ r_2 &= \frac{1}{2} + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} u(\tau) &= c_1\tau^{r_1} + c_2\tau^{r_2} \\ &= c_1\tau^{\alpha+i\beta} + c_2\tau^{\alpha-i\beta} \\ &= \tau^{\alpha}\left(c_1\tau^{i\beta} + c_2\tau^{-i\beta}\right) \\ &= \tau^{\alpha}\left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^{\alpha}\left(c_1 e^{i(\beta\ln\tau)} + c_2 e^{-i(\beta\ln\tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -1$, the above becomes

$$u(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-i\ln(\tau)} + c_2 e^{i\ln(\tau)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$u(\tau) = \sqrt{\tau}(c_1 \cos(\ln(\tau)) + c_2 \sin(\ln(\tau)))$$

The above solution is now transformed back to u using (6) which results in

$$u = \frac{\sqrt{2}\sqrt{e^{2x}}\left(c_1 \cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + c_2 \sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}{2}$$

Taking derivative gives

$$u'(x) = \frac{\sqrt{2}\sqrt{e^{2x}}\left(c_1 \cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + c_2 \sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}{2} + \frac{\sqrt{2}\sqrt{e^{2x}}\left(-2c_1 \sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + 2c_2 \cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}{2} \tag{4}$$

Substituting equations (3,4) into (1) results in

$$y = \frac{-u'}{f_2 u}$$

$$y = \frac{-u'}{-e^x u}$$

$$y = \frac{\left((c_1 + 2c_2)\cos(-\ln(2) + \ln(e^{2x})) - 2\left(c_1 - \frac{c_2}{2}\right)\sin(-\ln(2) + \ln(e^{2x}))\right)e^{-x}}{c_1 \cos(-\ln(2) + \ln(e^{2x})) + c_2 \sin(-\ln(2) + \ln(e^{2x}))}$$

Doing change of constants, the above solution becomes

$$y = \frac{\left(\frac{\sqrt{2}\sqrt{e^{2x}}\left(\cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + c_3 \sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}{2} + \frac{\sqrt{2}\sqrt{e^{2x}}\left(-2\sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + 2c_3 \cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}{2}\right)e^{-x}\sqrt{2}}{\sqrt{e^{2x}}\left(\cos\left(\ln\left(\frac{e^{2x}}{2}\right)\right) + c_3 \sin\left(\ln\left(\frac{e^{2x}}{2}\right)\right)\right)}$$

Simplifying the above gives

$$y = \frac{e^{-x}\left((1 + 2i + (2 - i)c_3)\left(e^{2x}\right)^{2i} + 2^{1+2i}c_3 + (ic_3 - 2i + 1)2^{2i}\right)}{i2^{2i}c_3 - i\left(e^{2x}\right)^{2i} c_3 + 2^{2i} + \left(e^{2x}\right)^{2i}}$$

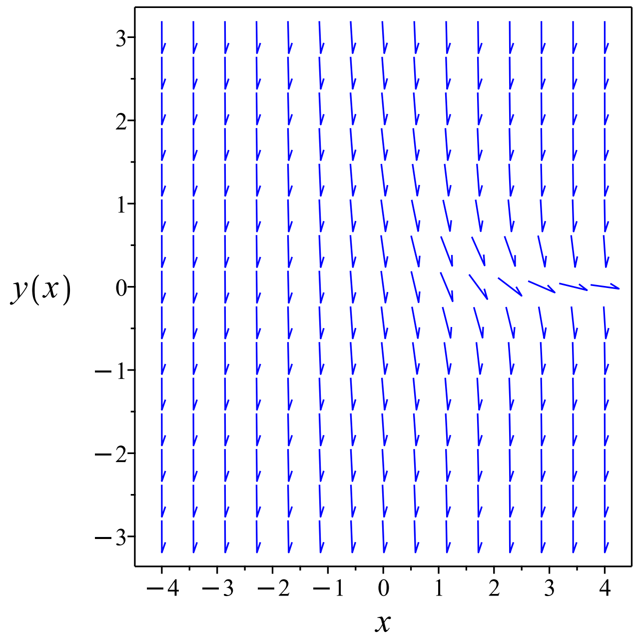


Figure 2.50: Slope field $y' - y + e^{x} y^{2} + 5\,e^{-x} = 0$

Summary of solutions found

$$y = \frac{2\,e^{-x} \left(\left(-\frac{5}{2} + \left(\frac{1}{2} + i\right) \eta\right) \left(e^{2x}\right)^{2i} + \frac{5}{2} + \left(-\frac{1}{2} + i\right) \eta\right)}{\left(\eta - 1 + 2i\right) \left(e^{2x}\right)^{2i} + 1 + 2i - \eta}$$

Solved using first_order_ode_riccati_by_guessing_particular_solution

Time used: 0.429 (sec)

Solve

$$y' - y + e^{x} y^{2} + 5\,e^{-x} = 0$$
$$y(0) = \eta$$

This is a Riccati ODE. Comparing the above ODE to solve with the Riccati standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that

$$f_0(x) = -5\,e^{-x}$$
$$f_1(x) = 1$$
$$f_2(x) = -e^{x}$$

Using trial and error, the following particular solution was found

$$y_p = (1 + 2i)\,e^{-x}$$

Since a particular solution is known, then the general solution is given by

$$y = y_p + \frac{\phi(x)}{c_1 - \int \phi(x) f_2 \, dx}$$

Where

$$\phi(x) = e^{\int 2 f_2 y_p + f_1 \, dx}$$

Evaluating the above gives the general solution as

$$y = \frac{2\,\mathrm{e}^{-x}\left(\left(\frac{5}{2} + \left(-\frac{1}{2} + i\right)\eta\right)\mathrm{e}^{-4ix} - \frac{5}{2} + \left(\frac{1}{2} + i\right)\eta\right)}{\left(1 + 2i - \eta\right)\mathrm{e}^{-4ix} - 1 + 2i + \eta}$$

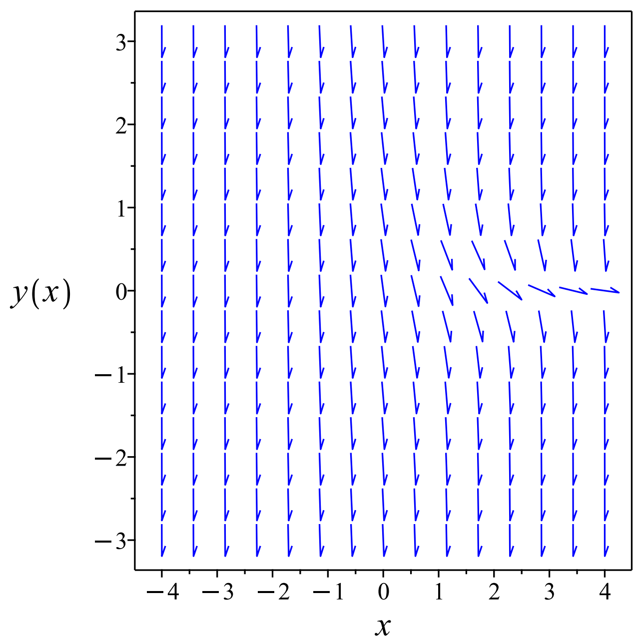


Figure 2.51: Slope field $y' - y + \mathrm{e}^{x} y^2 + 5\,\mathrm{e}^{-x} = 0$

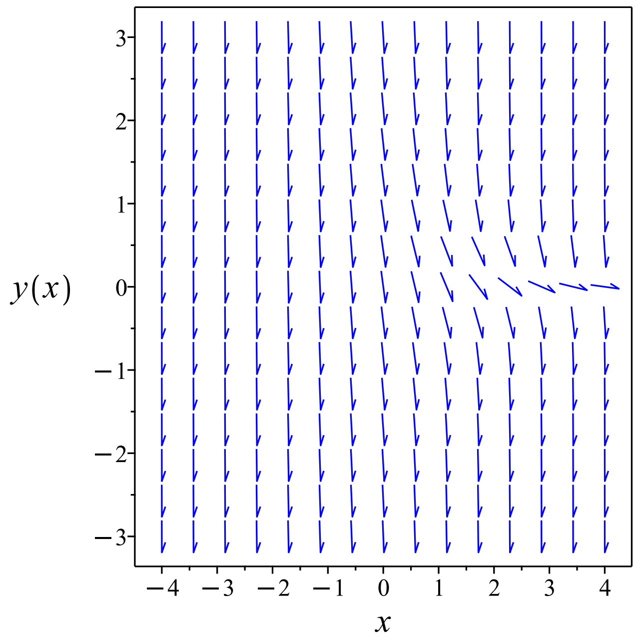


Figure 2.52: Slope field $y' - y + e^x y^2 + 5\,e^{-x} = 0$

Summary of solutions found

$$y = \frac{2\,e^{-x}\left(\left(\frac{5}{2} + \left(-\frac{1}{2} + i\right)\eta\right)e^{-4ix} - \frac{5}{2} + \left(\frac{1}{2} + i\right)\eta\right)}{\left(1 + 2i - \eta\right)e^{-4ix} - 1 + 2i + \eta}$$

✓ **Maple.** Time used: 0.085 (sec). Leaf size: 26

```
ode:=diff(y(x),x)-y(x)+y(x)^2*exp(x)+5*exp(-x) = 0;
ic:=[y(0) = eta];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \left(1 - 2\tan\left(2x - \arctan\left(\frac{\eta}{2} - \frac{1}{2}\right)\right)\right)e^{-x}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
```

```
trying Chini
<- Chini successful
```

✓ **Mathematica.** Time used: 0.184 (sec). Leaf size: 55

```
ode=D[y[x],x]-y[x]+Exp[x]*y[x]^2+5*Exp[-x]==0;
ic={y[0]==\[Eta]};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^{-x}((-1+2i)\eta + (-5+(1+2i)\eta)e^{4ix} + 5)}{-\eta + (\eta - (1-2i))e^{4ix} + (1+2i)}$$

✓ **Sympy.** Time used: 0.678 (sec). Leaf size: 22

```
from sympy import *
x = symbols("x")
a = symbols("a")
y = Function("y")
ode = Eq(y(x)**2*exp(x) - y(x) + Derivative(y(x), x) + 5*exp(-x),0)
ics = {y(0): a}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \left(1 - 2\tan\left(2x - \operatorname{atan}\left(\frac{a}{2} - \frac{1}{2}\right)\right)\right)e^{-x}$$

2.5 Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45

Local contents

2.5.1 Problem (a) 249
2.5.2 Problem (b) 254
2.5.3 Problem (c) 261
2.5.4 Problem (e) 282

2.5.1 Problem (a)

Local contents

Solved using first_order_ode_exact 249
✓Maple 252
✓Mathematica 253
✗Sympy 253

Internal problem ID [20977]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:22:19 AM`
CAS classification : `[_exact]`

Solved using first_order_ode_exact

Time used: 0.142 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} =$

$\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2y\cos\left(y^2+x\right)+3x\right)\mathrm{d}y &= \left(-\cos\left(y^2+x\right)-3y\right)\mathrm{d}x \\
\left(\cos\left(y^2+x\right)+3y\right)\mathrm{d}x + \left(2y\cos\left(y^2+x\right)+3x\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \cos\left(y^2+x\right)+3y \\
N(x,y) &= 2y\cos\left(y^2+x\right)+3x
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\cos\left(y^2+x\right)+3y\right) \\
&= -2y\sin\left(y^2+x\right)+3
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y\cos\left(y^2+x\right)+3x\right) \\
&= -2y\sin\left(y^2+x\right)+3
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \cos\left(y^2+x\right)+3y\,\mathrm{d}x
\end{aligned}$$

$$\phi = 3yx + \sin\left(y^2+x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\cos\left(y^2+x\right) + 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\cos\left(y^2+x\right) + 3x$. Therefore equation (4) becomes

$$2y\cos\left(y^2+x\right) + 3x = 2y\cos\left(y^2+x\right) + 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 3yx + \sin\left(y^2+x\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = 3yx + \sin\left(y^2+x\right)$$

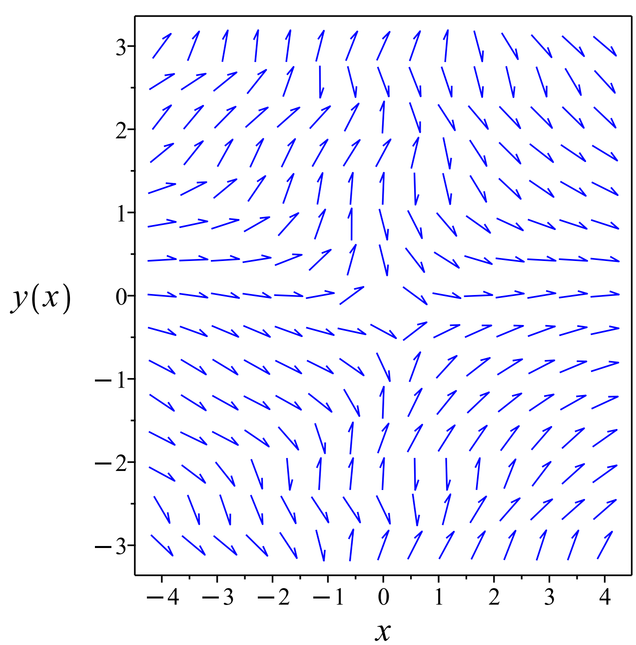


Figure 2.53: Slope field $\cos\left(x+y^2\right) + 3y + \left(2y\cos\left(x+y^2\right) + 3x\right)y' = 0$

Summary of solutions found

$$3yx + \sin\left(x + y^2\right) = c_1$$

✓ **Maple.** `Time used: 0.012 (sec). Leaf size: 41`

```
ode:=cos(x+y(x)^2)+3*y(x)+(2*y(x)*cos(x+y(x)^2)+3*x)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{-c_1 - \sin\left(\operatorname{RootOf}\left(-9x^2\_Z + 9x^3 + \sin\left(\_Z\right)^2 + 2c_1\sin\left(\_Z\right) + c_1^2\right)\right)}{3x}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful
```

Maple step by step

Let's solve
$\cos\left(x + y(x)^2\right) + 3y(x) + \left(2y(x)\cos\left(x + y(x)^2\right) + 3x\right)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function
    $\frac{d}{dx}G(x, y(x)) = 0$
  - Compute derivative of lhs
    $\frac{\partial}{\partial x}G(x, y) + \left(\frac{\partial}{\partial y}G(x, y)\right)\left(\frac{d}{dx}y(x)\right) = 0$
  - Evaluate derivatives
    $-2y\sin\left(y^2 + x\right) + 3 = -2y\sin\left(y^2 + x\right) + 3$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

$$\left[G(x,y)=C1, M(x,y)=\frac{\partial}{\partial x}G(x,y), N(x,y)=\frac{\partial}{\partial y}G(x,y)\right]$$

- Solve for $G(x,y)$ by integrating $M(x,y)$ with respect to x
  $G(x,y)=\int (\cos(y^2+x)+3y)\,dx+\_F1(y)$
- Evaluate integral
  $G(x,y)=3xy+\sin(y^2+x)+\_F1(y)$
- Take derivative of $G(x,y)$ with respect to y
  $N(x,y)=\frac{\partial}{\partial y}G(x,y)$
- Compute derivative
  $2y\cos(y^2+x)+3x=3x+2y\cos(y^2+x)+\frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$
  $\frac{d}{dy}\_F1(y)=0$
- Solve for $\_F1(y)$
  $\_F1(y)=0$
- Substitute $\_F1(y)$ into equation for $G(x,y)$
  $G(x,y)=3xy+\sin(y^2+x)$
- Substitute $G(x,y)$ into the solution of the ODE
  $3xy+\sin(y^2+x)=C1$
- Solve for $y(x)$
  $y(x)=\frac{C1-\sin\left(\mathit{RootOf}\left(-9x^2\_Z+9x^3+\sin(\_Z)^2-2C1\sin(\_Z)+C1^2\right)\right)}{3x}$

✓ Mathematica. Time used: 0.183 (sec). Leaf size: 28

```
ode=(Cos[x+y[x]^2]+3*y[x])+(2*y[x]*Cos[x+y[x]^2]+3*x)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[3xy(x)+\sin(x)\cos\left(y(x)^2\right)+\cos(x)\sin\left(y(x)^2\right)=c_1,y(x)\right]$$

✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq((3*x + 2*y(x)*cos(x + y(x)**2))*Derivative(y(x), x) + 3*y(x) + cos(x + y
    (x)**2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
Timed Out
```

2.5.2 Problem (b)

Local contents

Solved using first_order_ode_exact 254
✓Maple 258
✓Mathematica 259
✓Sympy 260

Internal problem ID [20978]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:22:53 AM`
CAS classification : `[_rational]`

Solved using first_order_ode_exact

Time used: 0.203 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=$

$\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x\,y^2+1)\,\mathrm{d}y &= (-x\,y^2+y^3)\,\mathrm{d}x\\ (x\,y^2-y^3)\,\mathrm{d}x + (-x\,y^2+1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= x\,y^2 - y^3\\ N(x,y) &= -x\,y^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x\,y^2-y^3)\\ &= y(2x-3y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x\,y^2+1)\\ &= -y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{-x\,y^2+1}\left((2yx-3y^2)-(-y^2)\right)\\ &= -\frac{2y(x-y)}{x\,y^2-1}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{y^2\,(x-y)}\left((-y^2)-(2yx-3y^2)\right)\\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^2}\left(x\,y^2 - y^3\right) \\ &= x - y\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^2}\left(-x\,y^2 + 1\right) \\ &= \frac{-x\,y^2 + 1}{y^2}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (x - y) + \left(\frac{-x\,y^2 + 1}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int x - y \, \mathrm{d}x$$

$$\phi = \frac{x(-2y + x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x\,y^2+1}{y^2}$. Therefore equation (4) becomes

$$\frac{-x\,y^2 + 1}{y^2} = -x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(-2y + x)}{2} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x(-2y + x)}{2} - \frac{1}{y}$$

Solving for y gives

$$y = \frac{x^2 - 2c_1 + \sqrt{x^4 - 4c_1\,x^2 + 4c_1^2 - 16x}}{4x}$$

$$y = -\frac{-x^2 + \sqrt{x^4 - 4c_1\,x^2 + 4c_1^2 - 16x} + 2c_1}{4x}$$

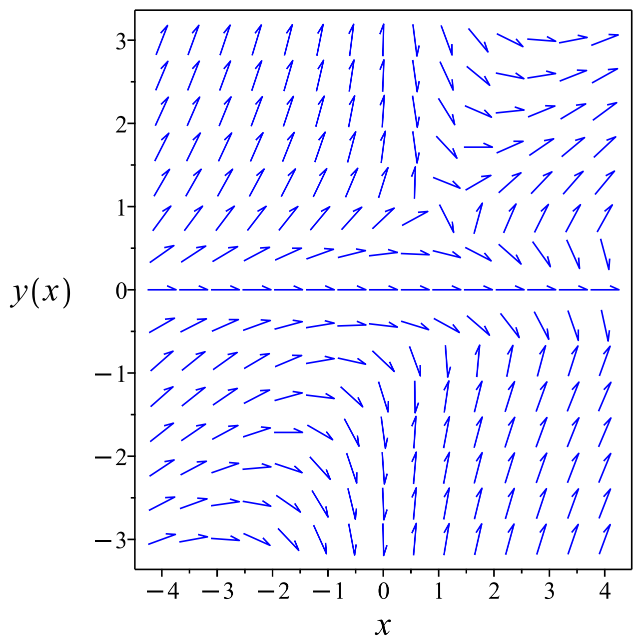


Figure 2.54: Slope field $xy^2 - y^3 + (1 - xy^2)\,y' = 0$

Summary of solutions found

$$y = \frac{x^2 - 2c_1 + \sqrt{x^4 - 4c_1\,x^2 + 4c_1^2 - 16x}}{4x}$$

$$y = -\frac{-x^2 + \sqrt{x^4 - 4c_1\,x^2 + 4c_1^2 - 16x} + 2c_1}{4x}$$

✓ **Maple.** `Time used: 0.005 (sec). Leaf size: 73`

```
ode:=x*y(x)^2-y(x)^3+(1-x*y(x)^2)*diff(y(x),x) = 0;
dsolve(ode,y(x), singsol=all);
```

$$y = \frac{x^2 + 2c_1 + \sqrt{x^4 + 4c_1\,x^2 + 4c_1^2 - 16x}}{4x}$$
$$y = \frac{x^2 - \sqrt{x^4 + 4c_1\,x^2 + 4c_1^2 - 16x} + 2c_1}{4x}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful
```

Maple step by step

Let's solve
$xy(x)^2 - y(x)^3 + (1 - xy(x)^2)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \frac{-xy(x)^2+y(x)^3}{1-xy(x)^2}$

✓ **Mathematica.** Time used: 0.342 (sec). Leaf size: 94

```
ode=(x*y[x]^2-y[x]^3)+(1-x*y[x]^2)*D[y[x],x]==0;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2 - \sqrt{x^4 + 4c_1x^2 - 16x + 4c_1{}^2} + 2c_1}{4x}$$
$$y(x) \to \frac{x^2 + \sqrt{x^4 + 4c_1x^2 - 16x + 4c_1{}^2} + 2c_1}{4x}$$
$$y(x) \to 0$$

✓ **Sympy.** Time used: 2.068 (sec). Leaf size: 34

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(x*y(x)**2 + (-x*y(x)**2 + 1)*Derivative(y(x), x) - y(x)**3,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{2C_1 + x^2 - \sqrt{4C_1^2 + 4C_1x^2 + x^4 - 16x}}{4x}$$

2.5.3 Problem (c)

Local contents

Solved using first_order_ode_bernoulli 261
Solved using first_order_ode_exact 264
Solved using first_order_ode_isobaric 268
Solved using first_order_ode_homog_type_D2 271
Solved using first_order_ode_homog_type_G 273
Solved using first_order_ode_LIE 275
✓Maple 280
✓Mathematica 281
✓Sympy 281

Internal problem ID [20979]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:23:18 AM`
CAS classification : `[[_homogeneous, ‘class D‘], _rational, _Bernoulli]`

Solved using first_order_ode_bernoulli

Time used: 0.121 (sec)

Solve

$$y(1 + yx) = y'x$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(yx + 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(\frac{1}{x}\right) y + (1)\, y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= \frac{1}{x} \\ f_1 &= 1 \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= 1 \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} + 1 \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -v'(x) &= \frac{v(x)}{x} + 1 \\ v' &= -\frac{v}{x} - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{x} \\ p(x) &= -1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)(-1) \\ \frac{\mathrm{d}}{\mathrm{d}x}(vx) &= (x)(-1) \\ \mathrm{d}(vx) &= (-x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}vx &= \int -x\,dx \\ &= -\frac{x^2}{2} + c_1\end{aligned}$$

Dividing throughout by the integrating factor x gives the final solution

$$v(x) = \frac{-\frac{x^2}{2} + c_1}{x}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$\frac{1}{y} = \frac{-\frac{x^2}{2} + c_1}{x}$$

Solving for y gives

$$y = \frac{2x}{-x^2 + 2c_1}$$

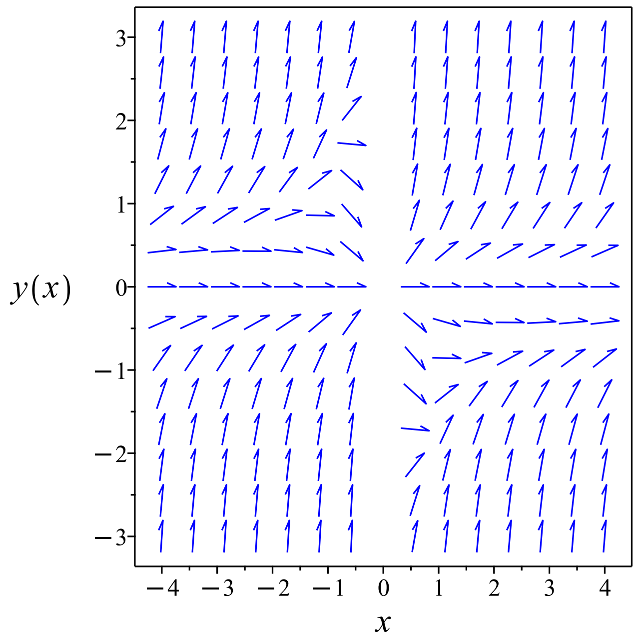


Figure 2.55: Slope field $y(1 + yx) = y'x$

Summary of solutions found

$$
y = \frac{2x}{-x^2 + 2c_1}
$$

Solved using first_order_ode_exact

Time used: 0.122 (sec)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= (-y(yx+1))\,\mathrm{d}x\\ (y(yx+1))\,\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y(yx+1)\\ N(x,y) &= -x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(yx+1))\\ &= 2yx+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{x}((2yx+1) - (-1))\\ &= \frac{-2yx-2}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y\,(yx+1)}((-1) - (2yx+1)) \\ &= -\frac{2}{y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{2}{y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-2\ln(y)} \\ &= \frac{1}{y^2} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^2}(y(yx+1)) \\ &= \frac{yx+1}{y} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^2}(-x) \\ &= -\frac{x}{y^2} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{yx+1}{y}\right) + \left(-\frac{x}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{yx+1}{y} \mathrm{d}x$$

$$\phi = \frac{x(yx+2)}{2y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2}{2y} - \frac{x(yx+2)}{2y^2} + f'(y) \tag{4}$$

$$= -\frac{x}{y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(yx+2)}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x(yx+2)}{2y}$$

Solving for y gives

$$y = \frac{2x}{-x^2 + 2c_1}$$

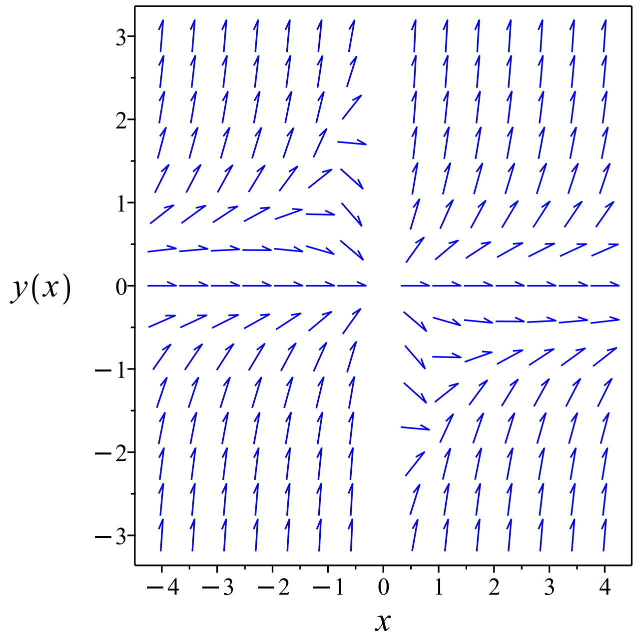


Figure 2.56: Slope field $y(1 + yx) = y'x$

Summary of solutions found

$$y = \frac{2x}{-x^2 + 2c_1}$$

Solved using first_order_ode_isobaric

Time used: 0.593 (sec)

Solve

$$y(1 + yx) = y'x$$

Solving for y' gives

$$y' = \frac{y(1 + yx)}{x} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y(1 + yx)}{x} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$-\frac{u(x)}{x^2} + \frac{u'(x)}{x} = \frac{u(x)(1 + u(x))}{x^2}$$

The ode

$$u'(x) = \frac{u(x)(u(x) + 2)}{x} \tag{2.19}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)(u(x) + 2)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u(u + 2) \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)} du = \int f(x) dx$$

$$\int \frac{1}{u(u + 2)} du = \int \frac{1}{x} dx$$

$$-\frac{\ln(u(x) + 2)}{2} + \frac{\ln(u(x))}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u(u + 2) = 0$$

for $u(x)$ gives

$$u(x) = -2$$
$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{\ln\left(u(x)+2\right)}{2}+\frac{\ln\left(u(x)\right)}{2} = \ln\left(x\right)+c_1$$
$$u(x) = -2$$
$$u(x) = 0$$

Converting $-\frac{\ln(u(x)+2)}{2}+\frac{\ln(u(x))}{2} = \ln\left(x\right)+c_1$ back to y gives

$$-\frac{\ln\left(yx+2\right)}{2}+\frac{\ln\left(yx\right)}{2} = \ln\left(x\right)+c_1$$

Converting $u(x) = -2$ back to y gives

$$yx = -2$$

Converting $u(x) = 0$ back to y gives

$$yx = 0$$

Solving for y gives

$$y = 0$$
$$y = -\frac{2}{x}$$
$$y = -\frac{2x\,\mathrm{e}^{2c_1}}{x^2\mathrm{e}^{2c_1}-1}$$

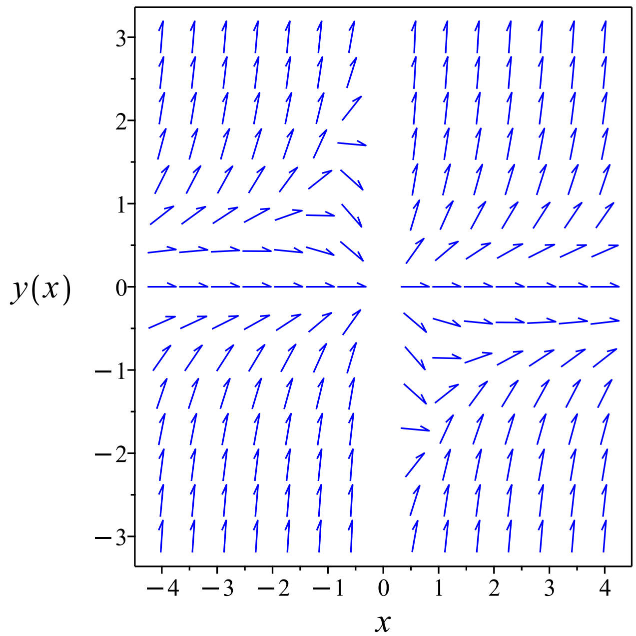


Figure 2.57: Slope field $y(1 + yx) = y'x$

Summary of solutions found

$$y = 0$$

$$y = -\frac{2}{x}$$

$$y = -\frac{2x\,\mathrm{e}^{2c_1}}{x^2\mathrm{e}^{2c_1} - 1}$$

Solved using first_order_ode_homog_type_D2

Time used: 0.181 (sec)

Solve

$$y(1 + yx) = y'x$$

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u(x)\,x\left(1 + u(x)\,x^2\right) = \left(u'(x)\,x + u(x)\right)x$$

Which is now solved The ode

$$u'(x) = u(x)^2\,x \tag{2.20}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= u(x)^2\,x \\ &= f(x)g(u) \end{aligned}$$

Where

$$f(x) = x$$
$$g(u) = u^2$$

Integrating gives

$$\int \frac{1}{g(u)} du = \int f(x) dx$$
$$\int \frac{1}{u^2} du = \int x dx$$

$$-\frac{1}{u(x)} = \frac{x^2}{2} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$u^2 = 0$$

for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\frac{1}{u(x)} = \frac{x^2}{2} + c_1$$
$$u(x) = 0$$

Converting $-\frac{1}{u(x)} = \frac{x^2}{2} + c_1$ back to y gives

$$-\frac{x}{y} = \frac{x^2}{2} + c_1$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Solving for y gives

$$y = 0$$

$$y = -\frac{2x}{x^2 + 2c_1}$$

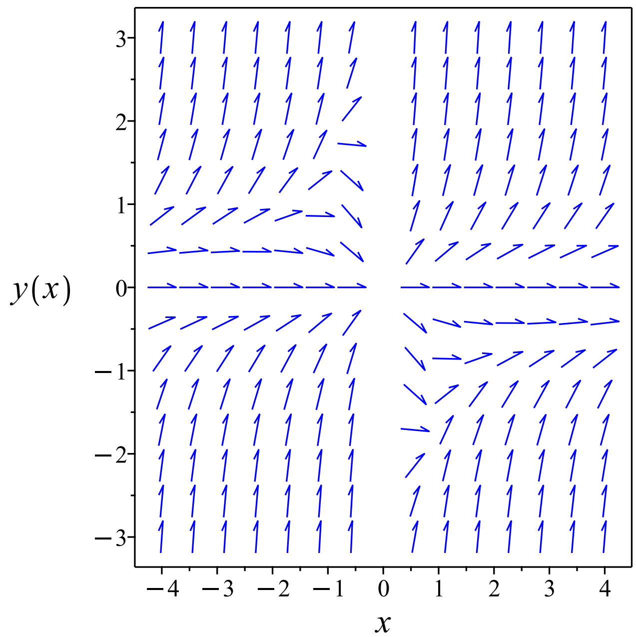


Figure 2.58: Slope field $y(1 + yx) = y'x$

Summary of solutions found

$$y = 0$$

$$y = -\frac{2x}{x^2 + 2c_1}$$

Solved using first_order_ode_homog_type_G

Time used: 0.089 (sec)

Solve

$$y(1 + yx) = y'x$$

Multiplying the right side of the ode, which is $\frac{y(yx+1)}{x}$ by $\frac{x}{y}$ gives

$$\begin{aligned} y' &= \left(\frac{x}{y}\right) \frac{y(yx+1)}{x} \\ &= yx + 1 \\ &= F(x, y) \end{aligned}$$

Since $F(x,y)$ has y, then let

$$\begin{aligned}
f_x &= x\left(\frac{\partial}{\partial x}F(x,y)\right)\\
&= yx\\
f_y &= y\left(\frac{\partial}{\partial y}F(x,y)\right)\\
&= yx\\
\alpha &= \frac{f_x}{f_y}\\
&= 1
\end{aligned}$$

Since α is independent of x, y then this is Homogeneous type G.

Let

$$\begin{aligned}
y &= \frac{z}{x^\alpha}\\
&= \frac{z}{x}
\end{aligned}$$

Substituting the above back into $F(x,y)$ gives

$$F(z) = z + 1$$

We see that $F(z)$ does not depend on x nor on y. If this was not the case, then this method will not work.

Therefore, the implicit solution is given by

$$\ln(x) - c_1 - \int^{yx^\alpha} \frac{1}{z(\alpha + F(z))}\, dz = 0$$

Which gives

$$\ln(x) - c_1 + \int^{yx} \frac{1}{z(-2-z)} dz = 0$$

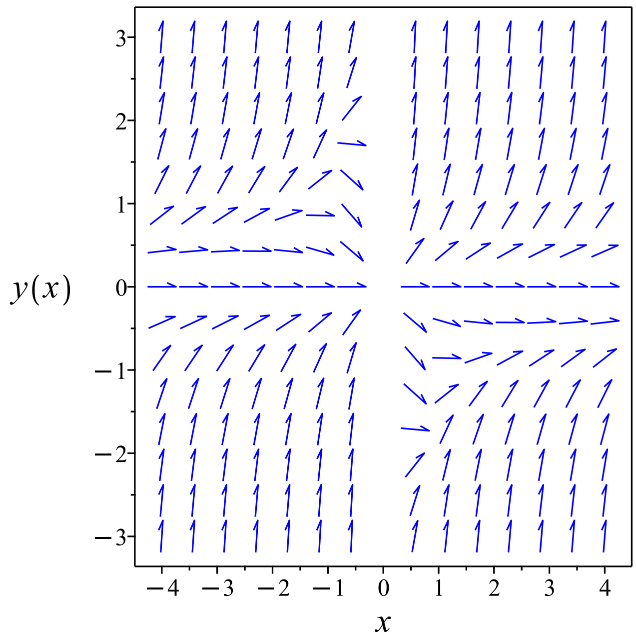


Figure 2.59: Slope field $y(1+yx) = y'x$

Summary of solutions found

$$\ln(x) - c_1 + \int^{yx} \frac{1}{z(-2-z)} dz = 0$$

Solved using first_order_ode_LIE

Time used: 2.971 (sec)

Solve

$$y(1+yx) = y'x$$

Writing the ode as

$$y' = \frac{y(yx+1)}{x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(yx+1)(b_3-a_2)}{x} - \frac{y^2(yx+1)^2 a_3}{x^2} - \left(\frac{y^2}{x} - \frac{y(yx+1)}{x^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{yx+1}{x} + y\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2y^4a_3 + 2x^3yb_2 + x^2y^2a_2 + x^2y^2b_3 + 2x\,y^3a_3 + 2x^2yb_1 + xb_1 - ya_1}{x^2} = 0$$

Setting the numerator to zero gives

$$-x^2y^4a_3 - 2x^3yb_2 - x^2y^2a_2 - x^2y^2b_3 - 2x\,y^3a_3 - 2x^2yb_1 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_1^2v_2^4 - a_2v_1^2v_2^2 - 2a_3v_1v_2^3 - 2b_2v_1^3v_2 - b_3v_1^2v_2^2 - 2b_1v_1^2v_2 + a_1v_2 - b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2v_1^3v_2 - a_3v_1^2v_2^4 + (-a_2 - b_3)\,v_1^2v_2^2 - 2b_1v_1^2v_2 - 2a_3v_1v_2^3 - b_1v_1 + a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0\\
-2a_3 &= 0\\
-a_3 &= 0\\
-2b_1 &= 0\\
-b_1 &= 0\\
-2b_2 &= 0\\
-a_2 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= y - \left(\frac{y(yx+1)}{x}\right)(-x)\\
&= y + y(yx+1)\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y + y\,(yx+1)} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(yx+2)}{2} + \frac{\ln(y)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(yx+1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{2yx+4} \\ S_y &= \frac{1}{y\,(yx+2)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

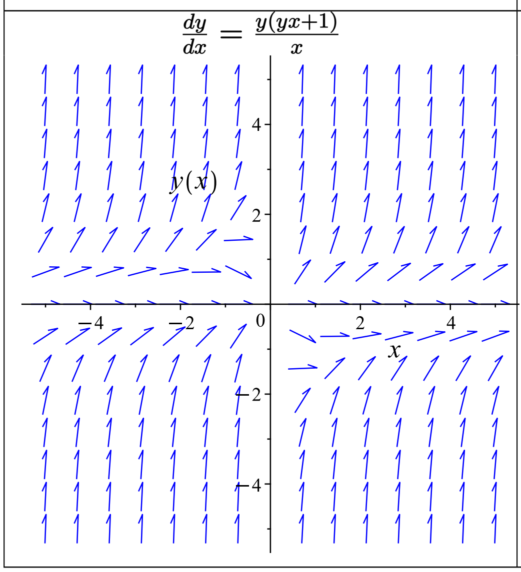
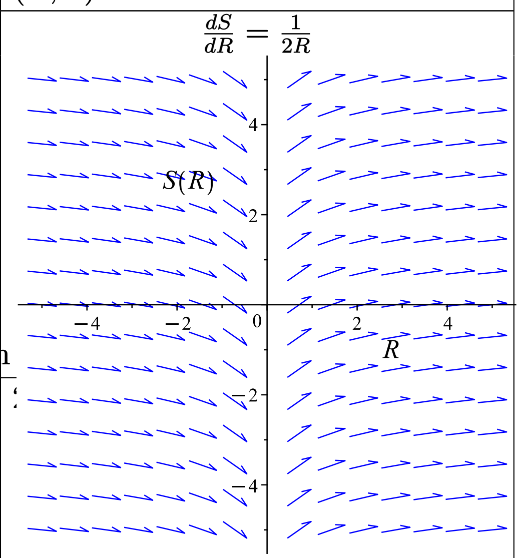
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{2R}\, dR$$
$$S(R) = \frac{\ln(R)}{2} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{\ln(yx+2)}{2} + \frac{\ln(y)}{2} = \frac{\ln(x)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
	$R = x$ $S = -\frac{\ln(yx+2)}{2} + \ln$	

Solving for y gives

$$y = \frac{2x}{-x^2 + e^{-2c_2}}$$

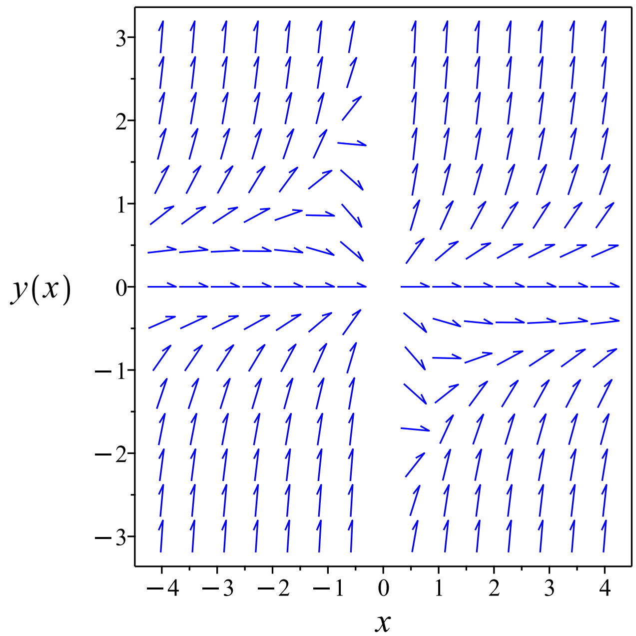


Figure 2.60: Slope field $y(1 + yx) = y'x$

Summary of solutions found

$$y = \frac{2x}{-x^2 + e^{-2c_2}}$$

✓ **Maple.** `Time used: 0.002 (sec). Leaf size: 16`

```
ode:=(y(x)*x+1)*y(x) = diff(y(x),x)*x;
dsolve(ode,y(x), singsol=all);
```

$$y = -\frac{2x}{x^2 - 2c_1}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
```

Maple step by step

Let's solve

$y(x)\,(1+xy(x)) = x\left(\frac{d}{dx}y(x)\right)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{y(x)(1+xy(x))}{x}$

✓ **Mathematica.** Time used: 0.085 (sec). Leaf size: 23

```
ode=y[x]*(1+x*y[x])==x*D[y[x],x];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2x}{x^2 - 2c_1}$$
$$y(x) \to 0$$

✓ **Sympy.** Time used: 0.114 (sec). Leaf size: 10

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*Derivative(y(x), x) + (x*y(x) + 1)*y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{2x}{C_1 - x^2}$$

2.5.4 Problem (e)

Local contents

Solved using first_order_ode_linear 282
✓Maple 283
✓Mathematica 284
✓Sympy 284

Internal problem ID [20980]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises IX at page 45
Problem number : (e)
Date solved : `Saturday, November 29, 2025 at 01:23:41 AM`
CAS classification : `[_linear]`

Solved using first_order_ode_linear

Time used: 0.022 (sec)

Solve

$$y' + p(x)\, y = q(x)$$

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = p(x)$$
$$p(x) = q(x)$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int p(x)dx}$$

Therefore the solution is

$$y = \left(\int q(x)\, \mathrm{e}^{\int p(x)dx} dx + c_1\right) \mathrm{e}^{-\int p(x)dx}$$

Summary of solutions found

$$y = \left(\int q(x)\, \mathrm{e}^{\int p(x)dx} dx + c_1\right) \mathrm{e}^{-\int p(x)dx}$$

✓ **Maple.** `Time used: 0.002 (sec). Leaf size: 24`

```
ode:=diff(y(x),x)+p(x)*y(x) = q(x);
dsolve(ode,y(x), singsol=all);
```

$$y = \left(\int q(x)\,\mathrm{e}^{\int p(x)dx}dx + c_1\right)\mathrm{e}^{-\int p(x)dx}$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
```

Maple step by step

Let's solve
$\frac{d}{dx}y(x) + p(x)\,y(x) = q(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = -p(x)\,y(x) + q(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) + p(x)\,y(x) = q(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) + p(x)\,y(x)\right) = \mu(x)\,q(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) + p(x)\,y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = \mu(x)\,p(x)$
- Solve to find the integrating factor
  $\mu(x) = \mathrm{e}^{\int p(x)dx}$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\,q(x)\,dx + C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x) = \int \mu(x)\,q(x)\,dx + C1$
- Solve for $y(x)$
  $y(x) = \frac{\int \mu(x)q(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{\int p(x)dx}$

$$y(x) = \frac{\int e^{\int p(x)dx} q(x)dx + C1}{e^{\int p(x)dx}}$$

- Simplify

$$y(x) = e^{-\int p(x)dx} \left(\int e^{\int p(x)dx} q(x)\, dx + C1 \right)$$

✓ **Mathematica.** `Time used: 0.055 (sec). Leaf size: 51`

```
ode=D[y[x],x]+p[x]*y[x]==q[x];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x -p(K[1])dK[1] \right) \left(\int_1^x \exp\left(- \int_1^{K[2]} -p(K[1])dK[1] \right) q(K[2])dK[2] + c_1 \right)$$

✓ **Sympy.** `Time used: 2.352 (sec). Leaf size: 41`

```
from sympy import *
x = symbols("x")
y = Function("y")
p = Function("p")
q = Function("q")
ode = Eq(p(x)*y(x) - q(x) + Derivative(y(x), x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$\left(e^{\int p(x)\, dx} - \int p(x) e^{\int p(x)\, dx}\, dx \right) y(x) + \int \left(p(x)y(x) - q(x) \right) e^{\int p(x)\, dx}\, dx = C_1$$

2.6 Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51

Local contents

2.6.1 Problem (a) 286
2.6.2 Problem (b) 290
2.6.3 Problem (c) 294
2.6.4 Problem (d) 308
2.6.5 Problem (e) 312

2.6.1 Problem (a)

Local contents

Solved using first_order_ode_clairaut 286
✓Maple 287
✓Mathematica 288
✗Sympy 289

Internal problem ID [20981]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:24:00 AM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _Clairaut]`

Solved using first_order_ode_clairaut

Time used: 0.241 (sec)

Solve

$$y = y'x - \sqrt{y' - 1}$$

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y = px - \sqrt{p - 1}$$

Solving for y from the above results in

$$y = px - \sqrt{p - 1} \tag{1A}$$

The above ode is a Clairaut ode which is now solved.

Writing the equation (1A) as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Where

$$g = -\sqrt{p-1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp+g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x+g')\frac{dp}{dx} \\
0 &= (x+g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \sqrt{c_1 - 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p-1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{1}{2\sqrt{p-1}} \\
&= 0
\end{aligned}$$

No valid singular solutions found.

Summary of solutions found

$$y = c_1 x - \sqrt{c_1 - 1}$$

✓ **Maple.** `Time used: 0.068 (sec). Leaf size: 30`

```
ode:=y(x) = diff(y(x),x)*x-(diff(y(x),x)-1)^(1/2);
dsolve(ode,y(x), singsol=all);
```

$$\begin{aligned}
y &= \frac{4x^2-1}{4x} \\
y &= c_1 x - \sqrt{c_1 - 1}
\end{aligned}$$

Maple trace

```
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful
```

Maple step by step

Let's solve
$y(x) = x\left(\frac{d}{dx}y(x)\right) - \sqrt{\frac{d}{dx}y(x) - 1}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{\frac{1+\sqrt{4xy(x)-4x^2+1}}{2x} + y(x)}{x}, \frac{d}{dx}y(x) = \frac{-\frac{-1+\sqrt{4xy(x)-4x^2+1}}{2x} + y(x)}{x}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{\frac{1+\sqrt{4xy(x)-4x^2+1}}{2x} + y(x)}{x}$
- Solve the equation $\frac{d}{dx}y(x) = \frac{-\frac{-1+\sqrt{4xy(x)-4x^2+1}}{2x} + y(x)}{x}$
- Set of solutions
  $\{workingODE, workingODE\}$

✓ **Mathematica.** `Time used: 0.065 (sec). Leaf size: 27`

```
ode=y[x]==x*D[y[x],x]-Sqrt[D[y[x],x]-1];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow c_1x - \sqrt{-1 + c_1}$$
$$y(x) \rightarrow -i$$

✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*Derivative(y(x), x) + sqrt(Derivative(y(x), x) - 1) + y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
Timed Out
```

2.6.2 Problem (b)

Local contents

Solved using first_order_ode_clairaut 290
✓Maple 292
✓Mathematica 293
✓Sympy 293

Internal problem ID [20982]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:24:15 AM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _Clairaut]`

Solved using first_order_ode_clairaut

Time used: 0.053 (sec)

Solve

$$y = y'x + {y'}^2$$

This is Clairaut ODE. It has the form

$$y = y'x + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$y = p^2 + px$$

Solving for y from the above results in

$$y = p^2 + px \tag{1A}$$

The above ode is a Clairaut ode which is now solved.

Writing the equation (1A) as

$$y = px + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = px + g \tag{1}$$

Where

$$g = p^2$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1^2 + c_1 x$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = p^2$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + 2p \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = y = -\frac{x^2}{4}$$

Substituting the above back in (1) results in

$$y = -\frac{x^2}{4}$$

Simplifying the above gives

$$y = c_1(x + c_1)$$

$$y = -\frac{x^2}{4}$$

Summary of solutions found

$$y = c_1(x + c_1)$$

$$y = -\frac{x^2}{4}$$

✓ **Maple.** Time used: 0.017 (sec). Leaf size: 17

```
ode:=y(x) = diff(y(x),x)*x+diff(y(x),x)^2;
dsolve(ode,y(x), singsol=all);
```

$$y = -\frac{x^2}{4}$$
$$y = c_1(c_1 + x)$$

Maple trace

```
Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
```

Maple step by step

Let's solve
$y(x) = x\left(\frac{d}{dx}y(x)\right) + \left(\frac{d}{dx}y(x)\right)^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

$$\left[\frac{d}{dx}y(x) = -\frac{x}{2} - \frac{\sqrt{x^2+4y(x)}}{2}, \frac{d}{dx}y(x) = -\frac{x}{2} + \frac{\sqrt{x^2+4y(x)}}{2}\right]$$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{x}{2} - \frac{\sqrt{x^2+4y(x)}}{2}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{x}{2} + \frac{\sqrt{x^2+4y(x)}}{2}$
- Set of solutions
  $\{workingODE, workingODE\}$

✓ **Mathematica.** `Time used: 0.005 (sec). Leaf size: 23`

```
ode=y[x]==x*D[y[x],x]+D[y[x],x]^2;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1(x + c_1)$$
$$y(x) \to -\frac{x^2}{4}$$

✓ **Sympy.** `Time used: 1.143 (sec). Leaf size: 14`

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*Derivative(y(x), x) + y(x) + Derivative(y(x), x)**2,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \frac{x^2}{4} - \frac{(C_1 + x)^2}{4}$$

2.6.3 Problem (c)

Local contents

Solved using first_order_ode_linear 294
Solved using first_order_ode_separable 295
Solved using first_order_ode_exact 297
Solved using first_order_ode_homog_type_D2 301
Solved using first_order_ode_LIE 303
✓Maple 306
✓Mathematica 307
✓Sympy 307

Internal problem ID [20983]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:24:30 AM`
CAS classification : `[_separable]`

Solved using first_order_ode_linear

Time used: 0.096 (sec)

Solve

$$y = y'x + y'a + b$$

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{a+x} \\ p(x) &= -\frac{b}{a+x} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{1}{a+x}dx} \\ &= \frac{1}{a+x} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{b}{a+x}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{a+x}\right) = \left(\frac{1}{a+x}\right)\left(-\frac{b}{a+x}\right)$$

$$\mathrm{d}\left(\frac{y}{a+x}\right) = \left(-\frac{b}{(a+x)^2}\right)\mathrm{d}x$$

Integrating gives

$$\begin{aligned}\frac{y}{a+x} &= \int -\frac{b}{(a+x)^2}\,dx\\ &= \frac{b}{a+x} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{a+x}$ gives the final solution

$$y = b + c_1(a+x)$$

Summary of solutions found

$$y = b + c_1(a+x)$$

Solved using first_order_ode_separable

Time used: 0.167 (sec)

Solve

$$y = y'x + y'a + b$$

The ode

$$y' = \frac{y-b}{a+x} \tag{2.21}$$

is separable as it can be written as

$$\begin{aligned}y' &= \frac{y-b}{a+x}\\ &= f(x)g(y)\end{aligned}$$

Where

$$f(x) = \frac{1}{a+x}$$
$$g(y) = y - b$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{1}{y-b}\,dy = \int \frac{1}{a+x}\,dx$$

$$\ln(y-b) = \ln(a+x) + c_1$$

Taking the exponential of both sides the solution becomes

$$y - b = c_1(a+x)$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or

$$y - b = 0$$

for y gives

$$y = b$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$y - b = c_1(a+x)$$
$$y = b$$

Solving for y gives

$$y = b$$
$$y = c_1 a + c_1 x + b$$

Summary of solutions found

$$y = b$$
$$y = c_1 a + c_1 x + b$$

Solved using first_order_ode_exact

Time used: 0.160 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(-x-a)\,\mathrm{d}y=(-y+b)\,\mathrm{d}x$$
$$(y-b)\,\mathrm{d}x+(-x-a)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y)=y-b$$
$$N(x,y)=-x-a$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-b)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x-a)\\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= -\frac{1}{a+x}((1)-(-1))\\ &= -\frac{2}{a+x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{a+x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(a+x)}\\ &= \frac{1}{(a+x)^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(a+x)^2}(y-b)\\ &= \frac{y-b}{(a+x)^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{(a+x)^2}(-x-a) \\ &= -\frac{1}{a+x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y-b}{(a+x)^2}\right) + \left(-\frac{1}{a+x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int -\frac{1}{a+x}\,\mathrm{d}y \\ \phi &= -\frac{y}{a+x} + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = \frac{y}{(a+x)^2} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{y-b}{(a+x)^2}$. Therefore equation (4) becomes

$$\frac{y-b}{(a+x)^2} = \frac{y}{(a+x)^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{b}{(a+x)^2}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{b}{(a+x)^2}\right)\mathrm{d}x$$

$$f(x) = \frac{b}{a+x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{a+x} + \frac{b}{a+x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{y}{a+x} + \frac{b}{a+x}$$

Simplifying the above gives

$$\frac{-y+b}{a+x} = c_1$$

Solving for y gives

$$y = -c_1a - c_1x + b$$

Summary of solutions found

$$y = -c_1a - c_1x + b$$

Solved using first_order_ode_homog_type_D2

Time used: 0.466 (sec)

Solve

$$y = y'x + y'a + b$$

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$u(x)\, x = (u'(x)\, x + u(x))\, x + (u'(x)\, x + u(x))\, a + b$$

Which is now solved The ode

$$u'(x) = -\frac{u(x)\, a + b}{x\,(a + x)} \tag{2.22}$$

is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)\, a + b}{x\,(a + x)} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x\,(a + x)} \\ g(u) &= -ua - b \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$

$$\int \frac{1}{-ua - b}\, du = \int \frac{1}{x\,(a + x)}\, dx$$

$$-\frac{\ln\left(-u(x)\, a - b\right)}{a} = \frac{\ln\left(\frac{x}{a+x}\right)}{a} + c_1$$

Taking the exponential of both sides the solution becomes

$$(-u(x)\, a - b)^{-\frac{1}{a}} = c_1 \left(\frac{x}{a + x}\right)^{\frac{1}{a}}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or

$$-ua - b = 0$$

for $u(x)$ gives

$$u(x) = -\frac{b}{a}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$(-u(x)\, a - b)^{-\frac{1}{a}} = c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}$$

$$u(x) = -\frac{b}{a}$$

Converting $(-u(x)\, a - b)^{-\frac{1}{a}} = c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}$ back to y gives

$$\left(-\frac{ya}{x} - b\right)^{-\frac{1}{a}} = c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}$$

Converting $u(x) = -\frac{b}{a}$ back to y gives

$$y = -\frac{xb}{a}$$

Simplifying the above gives

$$\left(-\frac{ya + bx}{x}\right)^{-\frac{1}{a}} = c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}$$

$$y = -\frac{xb}{a}$$

Solving for y gives

$$y = -\frac{xb}{a}$$

$$y = -\frac{x\left(\left(c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}\right)^{-a} + b\right)}{a}$$

Summary of solutions found

$$y = -\frac{xb}{a}$$

$$y = -\frac{x\left(\left(c_1 \left(\frac{x}{a+x}\right)^{\frac{1}{a}}\right)^{-a} + b\right)}{a}$$

Solved using first_order_ode_LIE

Time used: 0.724 (sec)

Solve

$$y = y'x + y'a + b$$

Writing the ode as

$$y' = \frac{y-b}{a+x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(y-b)(b_3-a_2)}{a+x} - \frac{(y-b)^2 a_3}{(a+x)^2} + \frac{(y-b)(xa_2+ya_3+a_1)}{(a+x)^2} - \frac{xb_2+yb_3+b_1}{a+x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{a^2b_2 + aba_2 - abb_3 + axb_2 - aya_2 - b^2a_3 - bxb_3 + bya_3 - ab_1 - ba_1 - xb_1 + ya_1}{(a+x)^2} = 0$$

Setting the numerator to zero gives

$$a^2b_2 + aba_2 - abb_3 + axb_2 - aya_2 - b^2a_3 - bxb_3 + bya_3 - ab_1 - ba_1 - xb_1 + ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$a^2b_2 + aba_2 - abb_3 - aa_2v_2 + ab_2v_1 - b^2a_3 + ba_3v_2 - bb_3v_1 - ab_1 - ba_1 + a_1v_2 - b_1v_1 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(ab_2 - bb_3 - b_1)\, v_1 + (-aa_2 + ba_3 + a_1)\, v_2 + a^2b_2 + aba_2 - abb_3 - b^2a_3 - ab_1 - ba_1 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -aa_2 + ba_3 + a_1 &= 0 \\ ab_2 - bb_3 - b_1 &= 0 \\ a^2b_2 + aba_2 - abb_3 - b^2a_3 - ab_1 - ba_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= aa_2 - ba_3 \\ a_2 &= a_2 \\ a_3 &= a_3 \\ b_1 &= ab_2 - bb_3 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= a + x \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{a + x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{a + x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y - b}{a + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{(a + x)^2} \\ S_y &= \frac{1}{a + x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{b}{(a + x)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{b}{(a+R)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{b}{(a+R)^2}\, dR$$
$$S(R) = \frac{b}{a+R} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{y}{a+x} = \frac{b}{a+x} + c_2$$

Solving for y gives

$$y = c_2 a + c_2 x + b$$

Summary of solutions found

$$y = c_2 a + c_2 x + b$$

✓ **Maple.** `Time used: 0.001 (sec). Leaf size: 11`

```
ode:=y(x) = diff(y(x),x)*x+a*diff(y(x),x)+b;
dsolve(ode,y(x), singsol=all);
```

$$y = (a+x)\, c_1 + b$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful
```

Maple step by step

Let's solve

$y(x) = x\left(\frac{d}{dx}y(x)\right) + a\left(\frac{d}{dx}y(x)\right) + b$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-y(x)+b}{-a-x}$
- Separate variables

  $\frac{\frac{d}{dx}y(x)}{-y(x)+b} = \frac{1}{-a-x}$
- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{-y(x)+b}dx = \int \frac{1}{-a-x}dx + C1$
- Evaluate integral

  $-\ln(-y(x)+b) = -\ln(-a-x) + C1$
- Solve for $y(x)$

  $y(x) = \frac{e^{C1}b+a+x}{e^{C1}}$
- Simplify

  $y(x) = e^{-C1}(a+x) + b$
- Redefine the integration constant(s)

  $y(x) = C1(a+x) + b$

✓ **Mathematica.** Time used: 0.025 (sec). Leaf size: 18

```
ode=y[x]==x*D[y[x],x]+a*D[y[x],x]+b;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to b + c_1(a+x)$$
$$y(x) \to b$$

✓ **Sympy.** Time used: 0.153 (sec). Leaf size: 10

```
from sympy import *
x = symbols("x")
a = symbols("a")
b = symbols("b")
y = Function("y")
ode = Eq(-a*Derivative(y(x), x) - b - x*Derivative(y(x), x) + y(x),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = C_1a + C_1x + b$$

2.6.4 Problem (d)

Local contents

Solved using first_order_ode_dAlembert 308
✓Maple 310
✓Mathematica 311
✗Sympy 311

Internal problem ID [20984]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51
Problem number : (d)
Date solved : `Saturday, November 29, 2025 at 01:24:54 AM`
CAS classification : `[_dAlembert]`

Solved using first_order_ode_dAlembert

Time used: 0.307 (sec)

Solve

$$y = x{y'}^2 + \ln\left({y'}^2\right)$$

Let $p = y'$ the ode becomes

$$y = x\,p^2 + \ln\left(p^2\right)$$

Solving for y from the above results in

$$y = x\,p^2 + \ln\left(p^2\right) \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = p^2$$
$$g = \ln\left(p^2\right)$$

Hence (2) becomes

$$-p^2 + p = \left(2xp + \frac{2}{p}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving the above for p results in

$$p_1 = 0$$
$$p_2 = 1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = x$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\, x + \frac{2}{p(x)}} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{2x(p)\, p + \frac{2}{p}}{-p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\mu = \mathrm{e}^{\int \frac{2}{p-1} dp}$$
$$\mu = (p-1)^2$$
$$\mu = (p-1)^2 \tag{5}$$

Integrating gives

$$\begin{aligned} x(p) &= \frac{1}{\mu}\left(\int \mu\left(-\frac{2}{p^2(p-1)}\right) dp + c_1\right) \\ &= \frac{1}{\mu}\left(\frac{-2\ln(p) - \frac{2}{p} + c_1}{(p-1)^2} + c_1\right) \\ &= \frac{-2\ln(p) - \frac{2}{p} + c_1}{(p-1)^2} \end{aligned} \tag{5}$$

Now we need to eliminate p between the above solution and (1A). The first method is to solve for p from Eq. (1A) and substitute the result into Eq. (5). The Second method is to solve for p from Eq. (5) and substitute the result into (1A).

Eliminating p from the following two equations

$$x = \frac{-2\ln(p) - \frac{2}{p} + c_1}{(p-1)^2}$$

$$y = x\,p^2 + \ln(p^2)$$

results in

$$p = e^{\mathrm{RootOf}\left(-e^{3\_Z}x+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)}$$

Substituting the above into Eq (1A) and simplifying gives

$$y = x\,e^{2\,\mathrm{RootOf}\left(-e^{3\_Z}x+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)} + 2\,\mathrm{RootOf}\left(-e^{3\_Z}x+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)$$

The solution

$$y = x\,e^{2\,\mathrm{RootOf}\left(-e^{3\_Z}x+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)} + 2\,\mathrm{RootOf}\left(-e^{3\_Z}x+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = x$$

✓ **Maple.** `Time used: 0.009 (sec). Leaf size: 76`

```
ode:=y(x) = diff(y(x),x)^2*x+ln(diff(y(x),x)^2);
dsolve(ode,y(x), singsol=all);
```

$$y = x\,e^{2\,\mathrm{RootOf}\left(-x\,e^{3\_Z}+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)} + \ln\left(e^{2\,\mathrm{RootOf}\left(-x\,e^{3\_Z}+2x\,e^{2\_Z}+c_1\,e^{\_Z}-2\_Z e^{\_Z}-x\,e^{\_Z}-2\right)}\right)$$

Maple trace

```
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
```

```
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful
```

Maple step by step

Let's solve
$$y(x) = x\left(\frac{d}{dx}y(x)\right)^2 + \ln\left(\left(\frac{d}{dx}y(x)\right)^2\right)$$

- Highest derivative means the order of the ODE is 1
$$\frac{d}{dx}y(x)$$
- Solve for the highest derivative
$$\left[\frac{d}{dx}y(x) = \mathrm{e}^{-\frac{LambertW\left(x\,\mathrm{e}^{y(x)}\right)}{2}+\frac{y(x)}{2}}, \frac{d}{dx}y(x) = -\mathrm{e}^{-\frac{LambertW\left(x\,\mathrm{e}^{y(x)}\right)}{2}+\frac{y(x)}{2}}\right]$$
- Solve the equation $\frac{d}{dx}y(x) = \mathrm{e}^{-\frac{LambertW\left(x\mathrm{e}^{y(x)}\right)}{2}+\frac{y(x)}{2}}$
- Solve the equation $\frac{d}{dx}y(x) = -\mathrm{e}^{-\frac{LambertW\left(x\mathrm{e}^{y(x)}\right)}{2}+\frac{y(x)}{2}}$
- Set of solutions
$$\{workingODE, workingODE\}$$

✓ **Mathematica.** Time used: 0.236 (sec). Leaf size: 53

```
ode=y[x]==x*D[y[x],x]^2+Log[D[y[x],x]^2];
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\left\{x = \frac{-\frac{2}{K[1]} - 2\log(K[1])}{(K[1]-1)^2} + \frac{c_1}{(K[1]-1)^2}, y(x) = xK[1]^2 + \log\left(K[1]^2\right)\right\}, \{y(x), K[1]\}\right]$$

✗ **Sympy**

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*Derivative(y(x), x)**2 + y(x) - log(Derivative(y(x), x)**2),0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
Timed Out
```

2.6.5 Problem (e)

Local contents

Solved using first_order_ode_dAlembert 312
✓Maple 314
✓Mathematica 315
✗Sympy 315

Internal problem ID [20985]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 1. First order equations: Some integrable cases. Excercises VIII at page 51
Problem number : (e)
Date solved : `Saturday, November 29, 2025 at 01:26:05 AM`
CAS classification : `[_dAlembert]`

Solved using first_order_ode_dAlembert

Time used: 0.449 (sec)

Solve

$$x = y\left(y' + \frac{1}{y'}\right) + {y'}^5$$

Let $p = y'$ the ode becomes

$$x = y\left(p + \frac{1}{p}\right) + p^5$$

Solving for y from the above results in

$$y = \frac{px}{p^2+1} - \frac{p^6}{p^2+1} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{p^2+1}$$
$$g = -\frac{p^6}{p^2+1}$$

Hence (2) becomes

$$p - \frac{p}{p^2+1} = \left(-\frac{2x\,p^2}{\left(p^2+1\right)^2} + \frac{x}{p^2+1} - \frac{6p^5}{p^2+1} + \frac{2p^7}{\left(p^2+1\right)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{p}{p^2+1} = 0$$

Solving the above for p results in

$$p_1 = 0$$
$$p_2 = 0$$
$$p_3 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$
$$y = 0$$
$$y = 0$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{p(x)}{p(x)^2+1}}{-\frac{2xp(x)^2}{\left(p(x)^2+1\right)^2} + \frac{x}{p(x)^2+1} - \frac{6p(x)^5}{p(x)^2+1} + \frac{2p(x)^7}{\left(p(x)^2+1\right)^2}} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-\frac{2x(p)p^2}{(p^2+1)^2} + \frac{x(p)}{p^2+1} - \frac{6p^5}{p^2+1} + \frac{2p^7}{(p^2+1)^2}}{p - \frac{p}{p^2+1}} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\mu = \mathrm{e}^{\int -\frac{-p^2+1}{p^3\left(p^2+1\right)}dp}$$
$$\mu = \mathrm{e}^{-\ln\left(p^2+1\right)+\frac{1}{2p^2}+2\ln(p)}$$
$$\mu = \frac{p^2\mathrm{e}^{\frac{1}{2p^2}}}{p^2+1} \tag{5}$$

Integrating gives

$$x(p) = \frac{(p^2+1)\, e^{-\frac{1}{2p^2}} \left(\int \frac{(-4p^4-6p^2)p^2 e^{\frac{1}{2p^2}}}{(p^2+1)^2} dp + c_1\right)}{p^2} \tag{5}$$

Unable to use this solution. skipping

Summary of solutions found

$$y = 0$$

✓ **Maple.** `Time used: 0.174 (sec). Leaf size: 110`

```
ode:=x = y(x)*(diff(y(x),x)+1/diff(y(x),x))+diff(y(x),x)^5;
dsolve(ode,y(x), singsol=all);
```

$$\left[x(\_T) = \frac{e^{-\frac{1}{2\_T^2}} (\_T^2+1) \left(-2\int \frac{\_T^4 (2\_T^2+3) e^{\frac{1}{2\_T^2}}}{(\_T^2+1)^2} d\_T + c_1\right)}{\_T^2}, y(\_T) = \frac{e^{-\frac{1}{2\_T^2}} \_T \left(1+\frac{1}{\_T^2}\right) \left(-2\int\right.}{}\right.$$

Maple trace

```
Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful
```

Maple step by step

Let's solve
$x = y(x)\left(\frac{d}{dx}y(x) + \frac{1}{\frac{d}{dx}y(x)}\right) + \left(\frac{d}{dx}y(x)\right)^5$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

$$\frac{d}{dx}y(x) = RootOf\left(\_Z^6 + y(x)\_Z^2 - x\_Z + y(x)\right)$$

✓ Mathematica. Time used: 2.788 (sec). Leaf size: 7957

```
ode=x==y[x]*(D[y[x],x]+1/D[y[x],x])+D[y[x],x]^5;
ic={};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Too large to display

✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(x - (Derivative(y(x), x) + 1/Derivative(y(x), x))*y(x) - Derivative(y(x),
    x)**5,0)
ics = {}
dsolve(ode,func=y(x),ics=ics)
```

```
NotImplementedError : The given ODE x - (Derivative(y(x), x) + 1/Derivative(y(x),
    x))*y(x) - Derivative(y(x), x)**5 cannot be solved by the lie group method
```

2.7 Chapter 2. Theory of First order differential equations. Excercises IV at page 89

Local contents

2.7.1 Problem (a) 317
2.7.2 Problem (b.1) 322
2.7.3 Problem (b.2) 327

2.7.1 Problem (a)

Local contents

✓Maple 320
✓Mathematica 321
✓Sympy 321

Internal problem ID [20986]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 2. Theory of First order differential equations. Excercises IV at page 89
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:26:21 AM`
CAS classification : `[‘y=_G(x,y’)‘]`

$$
\begin{aligned}
y' &= \mathrm{e}^{x}+\cos \left(y\right) x \\
y(0) &= 0
\end{aligned}
$$

Series expansion around $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$
y' = f(x,y)
$$

Where $f(x,y)$ is analytic at expansion point x_0. We can always shift to $x_0=0$ if x_0 is not zero. So from now we assume $x_0=0$. Assume also that $y(x_0)=y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$\begin{align}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \nonumber\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \nonumber\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}\\
&\vdots \nonumber
\end{align}$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$\begin{align}
F_0 &= f(x,y) \tag{4}\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}
\end{align}$$

For example, for $n=1$ we see that

$$\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}$$

Which is (1). And when $n=2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

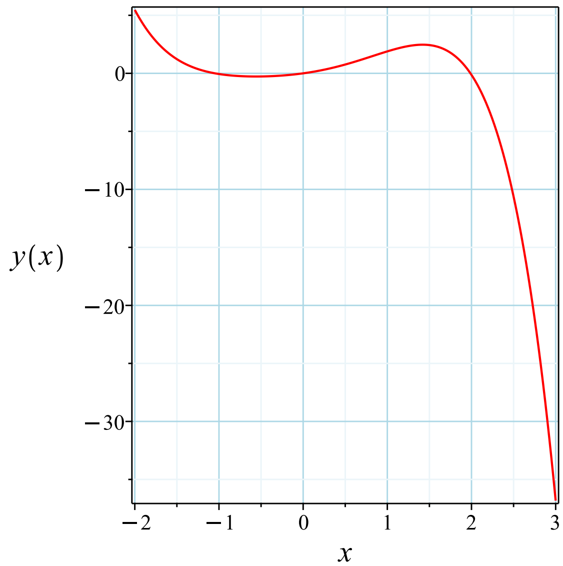
$$
\begin{aligned}
F_0 &= e^x + \cos(y)\, x \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(-x^2 \cos(y) - x\, e^x\right) \sin(y) + \cos(y) + e^x \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= -\cos(y)^3 x^3 - 2\, e^x \cos(y)^2 x^2 + x\left(x^2 \sin(y)^2 - e^{2x} - 3\sin(y)\right)\cos(y) + \left(1 + x^2 \sin(y)^2 + (-x-2)\sin \right. \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \left(\left(\sin(y)\, x^2 - 2x - 1\right)\cos(y) + e^x \sin(y)\, x\right) e^{2x} + 6\left(x^4 \sin(y) - 2x^2\right)\cos(y)^3 + 12\left(\sin(y)\, x^2 - \frac{x}{3} - \frac{7}{6}\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \left(14\cos(y)^3 x^3 + 14\, e^x \cos(y)^2 x^2 + \left(-9 + 17\left(x^2 + x\right)\sin(y) - 7x^3 - 6x\right)\cos(y) - 7\left(-\frac{3(x+1)\sin(y}{7}\right.\right.
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 0$ gives

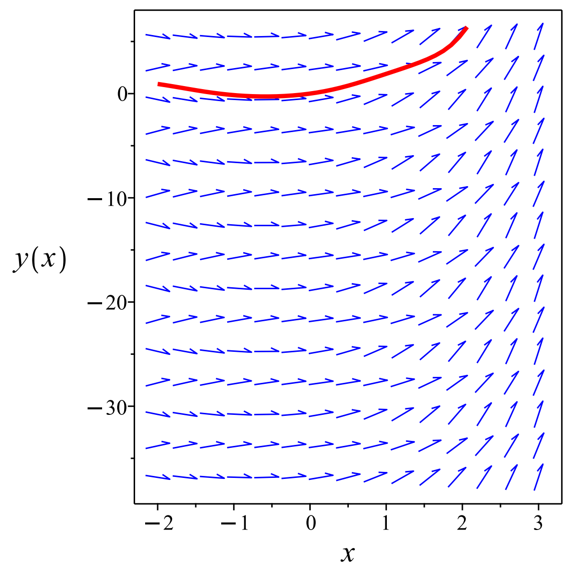
$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 2 \\
F_2 &= 1 \\
F_3 &= -2 \\
F_4 &= -23
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x^2 + x + \frac{x^3}{6} - \frac{x^4}{12} - \frac{23x^5}{120} + O\left(x^6\right)
$$



(a) Solution plot



(b) Slope field $y' = e^x + \cos(y) x$

✓ **Maple.** Time used: 0.003 (sec). Leaf size: 18

```
Order:=6;
ode:=diff(y(x),x) = exp(x)+x*cos(y(x));
ic:=[y(0) = 0];
dsolve([ode,op(ic)],y(x),type='series',x=0);
```

$$y = x + x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{23}{120}x^5 + O(x^6)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
```

```
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

✓ **Mathematica.** Time used: 0.081 (sec). Leaf size: 29

```
ode=D[y[x],x]==Exp[x]+x*Cos[y[x]];
ic={y[0]==0};
AsymptoticDSolveValue[{ode,ic},y[x],{x,0,5}]
```

$$y(x) \to -\frac{23x^5}{120} - \frac{x^4}{12} + \frac{x^3}{6} + x^2 + x$$

✓ **Sympy.** Time used: 0.399 (sec). Leaf size: 27

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x*cos(y(x)) - exp(x) + Derivative(y(x), x),0)
ics = {y(0): 0}
dsolve(ode,func=y(x),ics=ics,hint="1st_power_series",x0=0,n=6)
```

$$y(x) = x + x^2 + \frac{x^3}{6} - \frac{x^4}{12} - \frac{23x^5}{120} + O(x^6)$$

2.7.2 Problem (b.1)

Local contents

✓Maple 325
✓Mathematica 326
✓Sympy 326

Internal problem ID [20987]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 2. Theory of First order differential equations. Excercises IV at page 89
Problem number : (b.1)
Date solved : `Saturday, November 29, 2025 at 01:26:37 AM`
CAS classification : `[_Abel]`

$$
\begin{aligned}
y' &= x^3 + y^3 \\
y(0) &= 1
\end{aligned}
$$

Series expansion around $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \qquad (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \qquad (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \qquad (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y) \qquad (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \qquad (5)
\end{aligned}
$$

For example, for $n=1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x}F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n=2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0,y_0} \qquad (6)
$$

Hence

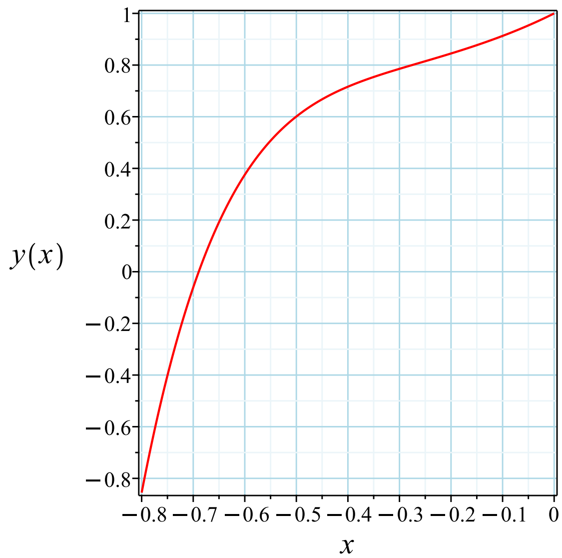
$$
\begin{aligned}
F_0 &= x^3 + y^3\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0\\
&= 3y^5 + 3y^2x^3 + 3x^2\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1\\
&= 15y^7 + 21y^4x^3 + 6yx^6 + 9y^2x^2 + 6x\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2\\
&= 105y^9 + 189x^3y^6 + 90y^3x^6 + 6x^9 + 81x^2y^4 + 54x^5y + 18xy^2 + 6\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3\\
&= 945y^{11} + 2079y^8x^3 + 891x^2y^6 + 1404y^5x^6 + 198y^4x + 918x^5y^3 + \left(270x^9 + 18\right) y^2 + 306x^4y + 108x^8
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = 1$ gives

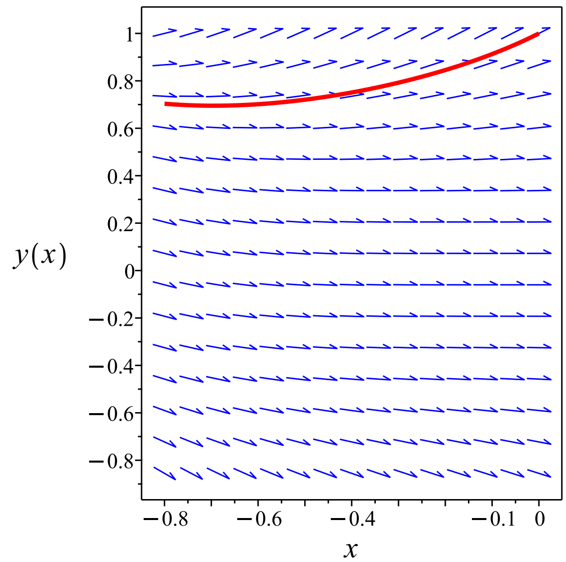
$$
\begin{aligned}
F_0 &= 1\\
F_1 &= 3\\
F_2 &= 15\\
F_3 &= 111\\
F_4 &= 963
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = x + 1 + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{37x^4}{8} + \frac{321x^5}{40} + O\left(x^6\right)
$$



(a) Solution plot



(b) Slope field $y' = x^3 + y^3$

✓ **Maple.** `Time used: 0.003 (sec). Leaf size: 20`

```
Order:=6;
ode:=diff(y(x),x) = x^3+y(x)^3;
ic:=[y(0) = 1];
dsolve([ode,op(ic)],y(x),type='series',x=0);
```

$$y = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \frac{37}{8}x^4 + \frac{321}{40}x^5 + \mathrm{O}\left(x^6\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
```

```
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

✓ **Mathematica.** Time used: 0.017 (sec). Leaf size: 34

```
ode=D[y[x],x]==x^3+y[x]^3;
ic={y[0]==1};
AsymptoticDSolveValue[{ode,ic},y[x],{x,0,5}]
```

$$y(x) \to \frac{321x^5}{40} + \frac{37x^4}{8} + \frac{5x^3}{2} + \frac{3x^2}{2} + x + 1$$

✓ **Sympy.** Time used: 0.193 (sec). Leaf size: 36

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x**3 - y(x)**3 + Derivative(y(x), x),0)
ics = {y(0): 1}
dsolve(ode,func=y(x),ics=ics,hint="1st_power_series",x0=0,n=6)
```

$$y(x) = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{37x^4}{8} + \frac{321x^5}{40} + O(x^6)$$

2.7.3 Problem (b.2)

Local contents

✓Maple 330
✓Mathematica 331
✓Sympy 332

Internal problem ID [20988]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 2. Theory of First order differential equations. Excercises IV at page 89
Problem number : (b.2)
Date solved : `Saturday, November 29, 2025 at 01:26:52 AM`
CAS classification : `[_quadrature]`

$$
\begin{aligned}
u' &= u^3 \\
u(0) &= 1
\end{aligned}
$$

Series expansion around $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$
y' = f(x, y)
$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4)\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0)\\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$
\begin{aligned}
F_2 &= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0\\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}
$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$
y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n\big|_{x_0, y_0} \tag{6}
$$

Hence

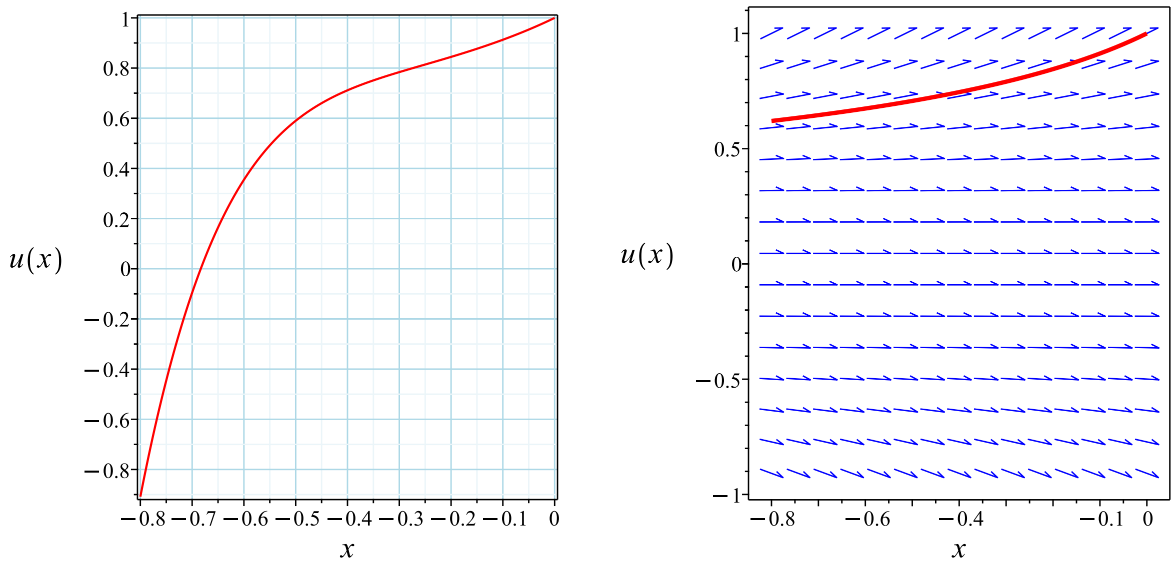
$$
\begin{aligned}
F_0 &= u^3 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial u} F_0 \\
&= 3u^5 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial u} F_1 \\
&= 15u^7 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial u} F_2 \\
&= 105u^9 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial u} F_3 \\
&= 945u^{11}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $u(0) = 1$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 3 \\
F_2 &= 15 \\
F_3 &= 105 \\
F_4 &= 945
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
u = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{35x^4}{8} + \frac{63x^5}{8} + O\left(x^6\right)
$$



(a) Solution plot

(b) Slope field $u' = u^3$

✓ **Maple.** `Time used: 0.002 (sec). Leaf size: 20`

```
Order:=6;
ode:=diff(u(x),x) = u(x)^3;
ic:=[u(0) = 1];
dsolve([ode,op(ic)],u(x),type='series',x=0);
```

$$u = 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \frac{35}{8}x^4 + \frac{63}{8}x^5 + \mathrm{O}\left(x^6\right)$$

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful
```

Maple step by step

Let's solve
$\left[\frac{d}{dx}u(x) = u(x)^3, u(0) = 1\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}u(x)$
- Solve for the highest derivative

$\frac{d}{dx}u(x) = u(x)^3$

- Separate variables

  $\frac{\frac{d}{dx}u(x)}{u(x)^3} = 1$
- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}u(x)}{u(x)^3}dx = \int 1dx + C1$
- Evaluate integral

  $-\frac{1}{2u(x)^2} = x + C1$
- Solve for $u(x)$

  $\left\{u(x) = \frac{1}{\sqrt{-2C1-2x}}, u(x) = -\frac{1}{\sqrt{-2C1-2x}}\right\}$
- Redefine the integration constant(s)

  $\left\{u(x) = \frac{1}{\sqrt{C1-2x}}, u(x) = -\frac{1}{\sqrt{C1-2x}}\right\}$
- Use initial condition $u(0) = 1$

  $1 = \frac{1}{\sqrt{C1}}$
- Solve for $\_C1$

  $C1 = 1$
- Substitute $\_C1 = 1$ into general solution and simplify

  $u(x) = \frac{1}{\sqrt{1-2x}}$
- Use initial condition $u(0) = 1$

  $1 = -\frac{1}{\sqrt{C1}}$
- Solve for $\_C1$

  No solution
- Solution does not satisfy initial condition
- Solution to the IVP

  $u(x) = \frac{1}{\sqrt{1-2x}}$

✓ **Mathematica.** `Time used: 0.018 (sec). Leaf size: 34`

```
ode=D[u[x],x]==u[x]^3;
ic={u[0]==1};
AsymptoticDSolveValue[{ode,ic},u[x],{x,0,5}]
```

$$u(x) \to \frac{63x^5}{8} + \frac{35x^4}{8} + \frac{5x^3}{2} + \frac{3x^2}{2} + x + 1$$

✓ **Sympy.** Time used: 0.138 (sec). Leaf size: 36

```
from sympy import *
x = symbols("x")
u = Function("u")
ode = Eq(-u(x)**3 + Derivative(u(x), x),0)
ics = {u(0): 1}
dsolve(ode,func=u(x),ics=ics,hint="1st_power_series",x0=0,n=6)
```

$$u(x) = 1 + x + \frac{3x^2}{2} + \frac{5x^3}{2} + \frac{35x^4}{8} + \frac{63x^5}{8} + O(x^6)$$

2.8 Chapter 2. Theory of First order differential equations. Excercises XII at page 98

Local contents

2.8.1 Problem (a) 334
2.8.2 Problem (b) 337

2.8.1 Problem (a)

Local contents

Existence and uniqueness analysis 334
✗Maple 335
✗Mathematica 336
✗Sympy 336

Internal problem ID [20989]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 2. Theory of First order differential equations. Excercises XII at page 98
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:27:07 AM`
CAS classification : `[_Abel]`

$$y' = x^3 + y^3$$
$$y(0) = 1$$

Existence and uniqueness analysis

$$y' = x^3 + y^3$$
$$y(0) = 1$$

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= x^3 + y^3 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^3 + y^3) \\ &= 3y^2 \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Unknown ode type.

✗ **Maple**

```
ode:=diff(y(x),x) = x^3+y(x)^3;
ic:=[y(0) = 1];
dsolve([ode,op(ic)],y(x), singsol=all);
```

No solution found

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 2
```

```
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type
```

✗ Mathematica

```
ode=D[y[x],x]==x^3+y[x]^3;
ic={y[0]==1};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

Not solved

✗ Sympy

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x**3 - y(x)**3 + Derivative(y(x), x),0)
ics = {y(0): 1}
dsolve(ode,func=y(x),ics=ics)
```

```
NotImplementedError : The given ODE -x**3 - y(x)**3 + Derivative(y(x), x) cannot
    be solved by the lie group method
```

2.8.2 Problem (b)

Local contents

Existence and uniqueness analysis . . . 337
✗Maple . . . 338
✓Mathematica . . . 339
✗Sympy . . . 339

Internal problem ID [20990]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter 2. Theory of First order differential equations. Excercises XII at page 98
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:27:52 AM`
CAS classification : `[‘y=_G(x,y’)‘]`

$$
\begin{aligned}
y' &= x+\sqrt{y^2+1} \\
y(0) &= 1
\end{aligned}
$$

Existence and uniqueness analysis

$$
\begin{aligned}
y' &= x+\sqrt{y^2+1} \\
y(0) &= 1
\end{aligned}
$$

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
y' &= f(x,y) \\
&= x+\sqrt{y^2+1}
\end{aligned}
$$

The x domain of $f(x,y)$ when $y=1$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0=0$ is inside this domain. The y domain of $f(x,y)$ when $x=0$ is

$$
\{-\infty < y < \infty\}
$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x + \sqrt{y^2+1}\right) \\ &= \frac{y}{\sqrt{y^2+1}}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Unknown ode type.

✗ **Maple**

```
ode:=diff(y(x),x) = x+(1+y(x)^2)^(1/2);
ic:=[y(0) = 1];
dsolve([ode,op(ic)],y(x), singsol=all);
```

No solution found

Maple trace

```
Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
```

```
   -> Computing symmetries using: way = 3
   -> Computing symmetries using: way = 4
   -> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of the form of conformal type
```

✓ **Mathematica.** Time used: 0.136 (sec). Leaf size: 93

```
ode=D[y[x],x]==x+Sqrt[1+y[x]];
ic={y[0]==1};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{6}\left(\frac{4\sqrt{y(x)+1}\arctan\left(\frac{x}{\sqrt{-y(x)-1}}\right)}{\sqrt{-y(x)-1}}+2\text{arctanh}\left(\frac{x}{2\sqrt{y(x)+1}}\right)\right.\right.$$
$$\left.\left.+2\log\left(-x^2+y(x)+1\right)+\log\left(-x^2+4y(x)+4\right)\right)=\frac{1}{6}(2\log(2)+\log(8)),y(x)\right]$$

✗ **Sympy**

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(-x - sqrt(y(x) + 1) + Derivative(y(x), x),0)
ics = {y(0): 1}
dsolve(ode,func=y(x),ics=ics)
```

```
NotImplementedError : The given ODE -x - sqrt(y(x) + 1) + Derivative(y(x), x)
    cannot be solved by the lie group method
```

2.9 Chapter IV. Linear Differential Equations. Excercises IV at page 172

Local contents

2.9.1 Problem (a) 341

2.9.1 Problem (a)

Local contents

✓Maple 341
✓Mathematica 341
✓Sympy 342

Internal problem ID [20991]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercises IV at page 172
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:33:37 AM`
CAS classification : `system_of_ODEs`

$$x' = x\cos(t) - y\sin(t)$$
$$y' = x\sin(t) + y\cos(t)$$

Does not currently support non autonomous system of first order linear differential equations.

✓ **Maple.** `Time used: 0.312 (sec). Leaf size: 57`

```
ode:=[diff(x(t),t) = x(t)*cos(t)-sin(t)*y(t), diff(y(t),t) = sin(t)*x(t)+cos(t)*y
    (t)];
dsolve(ode);
```

$$x(t) = c_1\,\mathrm{e}^{i\cos(t)+\sin(t)} + c_2\,\mathrm{e}^{-i\cos(t)+\sin(t)}$$
$$y(t) = i\left(c_1\,\mathrm{e}^{i\cos(t)+\sin(t)} - c_2\,\mathrm{e}^{-i\cos(t)+\sin(t)}\right)$$

✓ **Mathematica.** `Time used: 0.007 (sec). Leaf size: 45`

```
ode={D[x[t],t]==x[t]*Cos[t]-y[t]*Sin[t], D[y[t],t]==x[t]*Sin[t]+y[t]*Cos[t]};
ic={};
DSolve[{ode,ic},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{\sin(t)}(c_1\cos(\cos(t)) + c_2\sin(\cos(t)))$$
$$y(t) \to e^{\sin(t)}(c_2\cos(\cos(t)) - c_1\sin(\cos(t)))$$

✓ **Sympy.** Time used: 0.229 (sec). Leaf size: 207

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
ode=[Eq(-x(t)*cos(t) + y(t)*sin(t) + Derivative(x(t), t),0),Eq(-x(t)*sin(t) - y(t
    )*cos(t) + Derivative(y(t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t)],ics=ics)
```

$$\left[x(t) = -\frac{C_1 e^{-\sqrt{\sin^2(t)-1}+\sin(t)}\cos(t)}{2\sqrt{\sin^2(t)-1}} + \frac{C_1 e^{\sqrt{\sin^2(t)-1}+\sin(t)}\cos(t)}{2\sqrt{\sin^2(t)-1}} + \frac{C_2 e^{-\sqrt{\sin^2(t)-1}+\sin(t)}}{2} + \frac{C_2 e^{\sqrt{\sin^2(t)-1}+\sin(t)}}{2},\ y(t) = \frac{C_1 e^{-\sqrt{\sin^2(t)-1}+\sin(t)}}{2} + \frac{C_1 e^{\sqrt{\sin^2(t)-1}+\sin(t)}}{2} - \frac{C_2\sqrt{\sin^2(t)-1}\,e^{-\sqrt{\sin^2(t)-1}+\sin(t)}}{2\cos(t)} + \frac{C_2\sqrt{\sin^2(t)-1}\,e^{\sqrt{\sin^2(t)-1}+\sin(t)}}{2\cos(t)}\right]$$

2.10 Chapter IV. Linear Differential Equations. Excercises V at page 173

Local contents

2.10.1 Problem (a) 344

2.10.1 Problem (a)

Local contents

✓Maple 344
✓Mathematica 345
✗Sympy 345

Internal problem ID [20992]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercises V at page 173
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:33:51 AM`
CAS classification : `system_of_ODEs`

$$x' = (3t-1)\,x - (1-t)\,y + t\,e^{t^2}$$
$$y' = -(t+2)\,x + (t-2)\,y - e^{t^2}$$

Does not currently support non autonomous system of first order linear differential equations.

✓ **Maple.** `Time used: 0.145 (sec). Leaf size: 115`

```
ode:=[diff(x(t),t) = (3*t-1)*x(t)-(-t+1)*y(t)+t*exp(t^2), diff(y(t),t) = -(t+2)*x
   (t)+(t-2)*y(t)-exp(t^2)];
dsolve(ode);
```

$$x(t) = \left(c_2 + \frac{1}{9}t^3 + \frac{1}{9}t^2 + \frac{4}{9}t + \frac{4}{9}\right)e^{t^2} + \left(3\,e^{t^2-3t}t - 2\,e^{t^2-3t}\right)c_1$$
$$y(t) = -\frac{e^{t^2}t^3}{9} - \frac{t^2e^{t^2}}{9} - 3\,e^{t^2-3t}c_1 t - e^{t^2}c_2 - \frac{t\,e^{t^2}}{9} - 7\,e^{t^2-3t}c_1 - \frac{8\,e^{t^2}}{9}$$

✓ **Mathematica.** Time used: 0.475 (sec). Leaf size: 196

```
ode={D[x[t],t]==(3*t-1)*x[t]-(1-t)*y[t]+t*Exp[t^2], D[y[t],t]==-(t+2)*x[t]+(t-2)*
    y[t]-Exp[t^2]};
ic={};
DSolve[{ode,ic},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{c_1 e^{t^2-\frac{1}{2}}\sqrt{t-1}}{\sqrt{2-2t}} + \frac{\sqrt{2}c_2 e^{t^2-3t+\frac{5}{2}}\sqrt{t-1}(3t-2)}{9\sqrt{1-t}} + \frac{1}{81}e^{t^2}\left(9t^3+9t^2+36t-8\right)$$

$$y(t) \to \frac{1}{162}e^{t^2}\left(-2\left(9t^3+9t^2+9t+28\right) - \frac{81\sqrt{\frac{2}{e}}c_1(t-1)}{\sqrt{-(t-1)^2}} - \frac{18\sqrt{2}c_2 e^{\frac{5}{2}-3t}(t-1)(3t+7)}{\sqrt{-(t-1)^2}}\right)$$

✗ **Sympy**

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
ode=[Eq(-t*exp(t**2) + (1 - t)*y(t) - (3*t - 1)*x(t) + Derivative(x(t), t),0),Eq
    ((2 - t)*y(t) + (t + 2)*x(t) + exp(t**2) + Derivative(y(t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t)],ics=ics)
```

```
NotImplementedError :
```

2.11 Chapter IV. Linear Differential Equations. Excercise XII at page 189

Local contents

2.11.1 Problem (c) 347

2.11.1 Problem (c)

Local contents

Solution using Matrix exponential method 347
Solution using explicit Eigenvalue and Eigenvector method 348
✓Maple 352
✓Mathematica 354
✓Sympy 354

Internal problem ID [20993]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise XII at page 189
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:34:05 AM`
CAS classification : `system_of_ODEs`

$$x' = 2x - 4y$$
$$y' = -x + 2y$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{e^{4t}}{2} & -e^{4t} + 1 \\ -\frac{e^{4t}}{4} + \frac{1}{4} & \frac{1}{2} + \frac{e^{4t}}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{1}{2}+\frac{e^{4t}}{2} & -e^{4t}+1 \\ -\frac{e^{4t}}{4}+\frac{1}{4} & \frac{1}{2}+\frac{e^{4t}}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{1}{2}+\frac{e^{4t}}{2}\right)c_1 + \left(-e^{4t}+1\right)c_2 \\ \left(-\frac{e^{4t}}{4}+\frac{1}{4}\right)c_1 + \left(\frac{1}{2}+\frac{e^{4t}}{2}\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(c_1-2c_2)e^{4t}}{2} + \frac{c_1}{2} + c_2 \\ \frac{(-c_1+2c_2)e^{4t}}{4} + \frac{c_1}{4} + \frac{c_2}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A-\lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & -4 \\ -1 & 2-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda = 0$$

The roots of the above are the eigenvalues.

$$
\begin{aligned}
\lambda_1 &= 0 \\
\lambda_2 &= 4
\end{aligned}
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} - (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -4 & 0 \\ -1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} -2 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -2t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
0	1	1	No	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
4	1	1	No	$\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{0} \\ &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{0} \end{aligned}$$

Since eigenvalue 4 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{4t} \\ &= \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{4t} \end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2\, \mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2c_1 - 2c_2\, \mathrm{e}^{4t} \\ c_1 + c_2\, \mathrm{e}^{4t} \end{bmatrix}$$

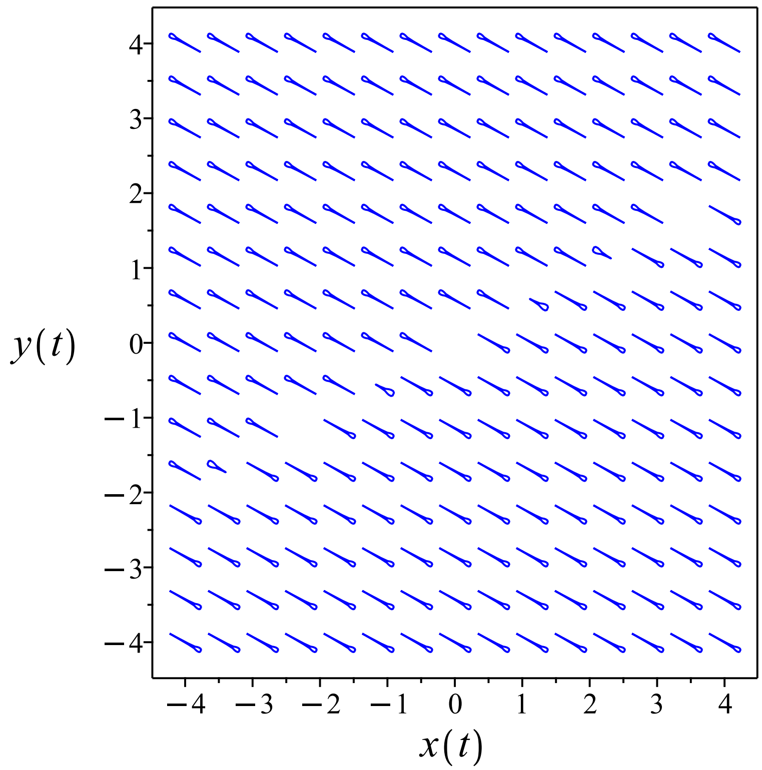


Figure 2.64: Phase plot

✓ **Maple.** `Time used: 0.063 (sec). Leaf size: 26`

```
ode:=[diff(x(t),t) = 2*x(t)-4*y(t), diff(y(t),t) = -x(t)+2*y(t)];
dsolve(ode);
```

$$x(t) = c_1 + c_2 \,\mathrm{e}^{4t}$$
$$y(t) = -\frac{c_2 \,\mathrm{e}^{4t}}{2} + \frac{c_1}{2}$$

Maple step by step

Let's solve
$\left[\frac{d}{dt}x(t) = 2x(t) - 4y(t), \frac{d}{dt}y(t) = -x(t) + 2y(t)\right]$

- Define vector
  $\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$
- Convert system into a vector equation
  $\frac{d}{dt}\vec{x}(t) = \left[\begin{array}{cc} 2 & -4 \\ -1 & 2 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$
- System to solve

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

  $A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}$

- Rewrite the system as

  $\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]\right]$

- Consider eigenpair

  $\left[0, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

  $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- Consider eigenpair

  $\left[4, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

  $\vec{x}_2 = \mathrm{e}^{4t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

- General solution to the system of ODEs

  $\vec{x} = C1\vec{x}_1 + C2\vec{x}_2$

- Substitute solutions into the general solution

  $\vec{x} = C2\,\mathrm{e}^{4t} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2C1 \\ C1 \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2C2\,\mathrm{e}^{4t} + 2C1 \\ C2\,\mathrm{e}^{4t} + C1 \end{bmatrix}$

- Solution to the system of ODEs

  $\{x(t) = -2C2\,\mathrm{e}^{4t} + 2C1, y(t) = C2\,\mathrm{e}^{4t} + C1\}$

✓ **Mathematica.** Time used: 0.002 (sec). Leaf size: 60

```
ode={D[x[t],t]==2*x[t]-4*y[t], D[y[t],t]==-x[t]+2*y[t]};
ic={};
DSolve[{ode,ic},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{2}c_1\left(e^{4t}+1\right) - c_2\left(e^{4t}-1\right)$$
$$y(t) \to \frac{1}{4}\left(2c_2\left(e^{4t}+1\right) - c_1\left(e^{4t}-1\right)\right)$$

✓ **Sympy.** Time used: 0.039 (sec). Leaf size: 24

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
ode=[Eq(-2*x(t) + 4*y(t) + Derivative(x(t), t),0),Eq(x(t) - 2*y(t) + Derivative(y
    (t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t)],ics=ics)
```

$$\left[x(t) = 2C_1 - 2C_2e^{4t},\ y(t) = C_1 + C_2e^{4t}\right]$$

2.12 Chapter IV. Linear Differential Equations. Excercise XIII at page 189

Local contents

2.12.1 Problem (a) 356
2.12.2 Problem (b) 364
2.12.3 Problem (c) 372
2.12.4 Problem (d) 383

2.12.1 Problem (a)

Local contents

Solution using Matrix exponential method 356
Solution using explicit Eigenvalue and Eigenvector method 357
✓Maple 361
✓Mathematica 363
✓Sympy 363

Internal problem ID [20994]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise XIII at page 189
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:34:20 AM`
CAS classification : `system_of_ODEs`

$$x' = 3x + 6y$$
$$y' = -2x - 3y$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \cos\left(\sqrt{3}\,t\right) + \sqrt{3}\,\sin\left(\sqrt{3}\,t\right) & 2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right) \\ -\frac{2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)}{3} & \cos\left(\sqrt{3}\,t\right) - \sqrt{3}\,\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \cos\left(\sqrt{3}\,t\right)+\sqrt{3}\,\sin\left(\sqrt{3}\,t\right) & 2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right) \\ -\frac{2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)}{3} & \cos\left(\sqrt{3}\,t\right)-\sqrt{3}\,\sin\left(\sqrt{3}\,t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \left(\cos\left(\sqrt{3}\,t\right)+\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)\right)c_1 + 2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)c_2 \\ -\frac{2\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)c_1}{3} + \left(\cos\left(\sqrt{3}\,t\right)-\sqrt{3}\,\sin\left(\sqrt{3}\,t\right)\right)c_2 \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{3}\,(c_1+2c_2)\sin\left(\sqrt{3}\,t\right)+\cos\left(\sqrt{3}\,t\right)c_1 \\ -\frac{2\sqrt{3}\left(c_1+\frac{3c_2}{2}\right)\sin\left(\sqrt{3}\,t\right)}{3}+\cos\left(\sqrt{3}\,t\right)c_2 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I)=0$$

Expanding gives

$$\det\left(\begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)=0$$

Therefore

$$\det\left(\begin{bmatrix} 3-\lambda & 6 \\ -2 & -3-\lambda \end{bmatrix}\right)=0$$

Which gives the characteristic equation

$$\lambda^2+3=0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}
\lambda_1 &= i\sqrt{3} \\
\lambda_2 &= -i\sqrt{3}
\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$i\sqrt{3}$	1	complex eigenvalue
$-i\sqrt{3}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -i\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} - \left(-i\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i\sqrt{3}+3 & 6 \\ -2 & i\sqrt{3}-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} i\sqrt{3}+3 & 6 & 0 \\ -2 & i\sqrt{3}-3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{i\sqrt{3}+3} \Longrightarrow \left[\begin{array}{cc|c} i\sqrt{3}+3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i\sqrt{3}+3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{6t}{i\sqrt{3}+3}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6t}{i\sqrt{3}+3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{6}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{6}{i\sqrt{3}+3} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = i\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} - \left(i\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i\sqrt{3}+3 & 6 \\ -2 & -i\sqrt{3}-3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -i\sqrt{3}+3 & 6 & 0 \\ -2 & -i\sqrt{3}-3 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{2R_1}{-i\sqrt{3}+3} \Longrightarrow \left[\begin{array}{cc|c} -i\sqrt{3}+3 & 6 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i\sqrt{3}+3 & 6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{6t}{i\sqrt{3}-3}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{6t}{i\sqrt{3}-3} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{6}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{6}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$i\sqrt{3}$	1	1	No	$\begin{bmatrix} \frac{6}{i\sqrt{3}-3} \\ 1 \end{bmatrix}$
$-i\sqrt{3}$	1	1	No	$\begin{bmatrix} \frac{6}{-i\sqrt{3}-3} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} \frac{6\,\mathrm{e}^{i\sqrt{3}\,t}}{i\sqrt{3}-3} \\ \mathrm{e}^{i\sqrt{3}\,t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{6\,\mathrm{e}^{-i\sqrt{3}\,t}}{-i\sqrt{3}-3} \\ \mathrm{e}^{-i\sqrt{3}\,t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3i\left(i+\frac{\sqrt{3}}{3}\right)c_2\,\mathrm{e}^{-i\sqrt{3}\,t}}{2} + \frac{3i\mathrm{e}^{i\sqrt{3}\,t}\left(i-\frac{\sqrt{3}}{3}\right)c_1}{2} \\ c_1\,\mathrm{e}^{i\sqrt{3}\,t} + c_2\,\mathrm{e}^{-i\sqrt{3}\,t} \end{bmatrix}$$

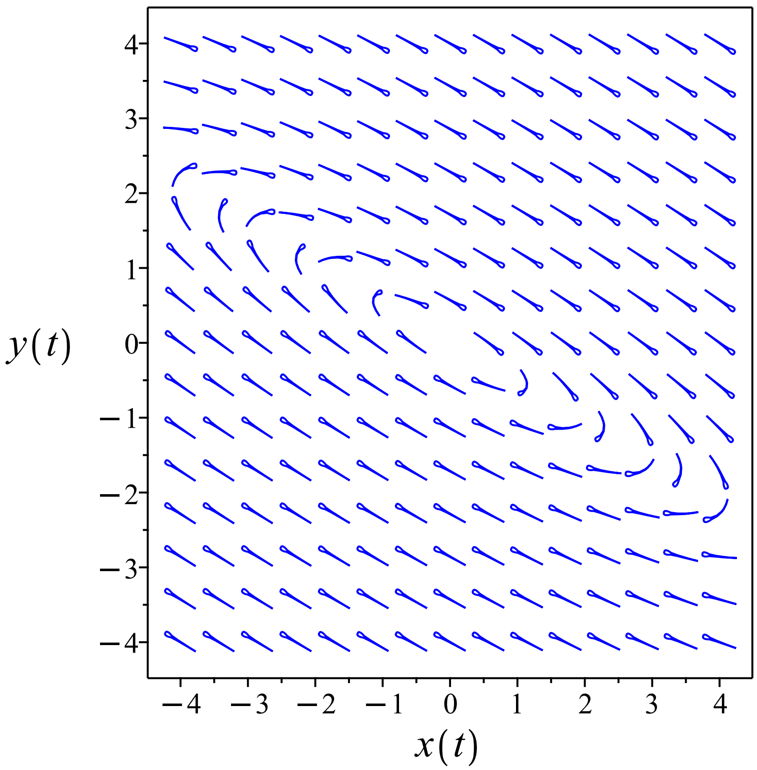


Figure 2.65: Phase plot

✓ **Maple.** `Time used: 0.078 (sec). Leaf size: 67`

```
ode:=[diff(x(t),t) = 3*x(t)+6*y(t), diff(y(t),t) = -2*x(t)-3*y(t)];
dsolve(ode);
```

$$x(t) = c_1 \sin\left(\sqrt{3}\, t\right) + c_2 \cos\left(\sqrt{3}\, t\right)$$
$$y(t) = \frac{c_1\sqrt{3}\, \cos\left(\sqrt{3}\, t\right)}{6} - \frac{c_2\sqrt{3}\, \sin\left(\sqrt{3}\, t\right)}{6} - \frac{c_1 \sin\left(\sqrt{3}\, t\right)}{2} - \frac{c_2 \cos\left(\sqrt{3}\, t\right)}{2}$$

Maple step by step

Let's solve
$\left[\frac{d}{dt}x(t) = 3x(t) + 6y(t), \frac{d}{dt}y(t) = -2x(t) - 3y(t)\right]$

- Define vector
$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \end{array}\right]$
- Convert system into a vector equation
$\frac{d}{dt}\vec{x}(t) = \left[\begin{array}{cc} 3 & 6 \\ -2 & -3 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \end{array}\right]$
- System to solve

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

  $A = \begin{bmatrix} 3 & 6 \\ -2 & -3 \end{bmatrix}$

- Rewrite the system as

  $\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[-\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{6}{-\mathrm{I}\sqrt{3}-3} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{6}{\mathrm{I}\sqrt{3}-3} \\ 1 \end{bmatrix}\right]\right]$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

  $\left[-\mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{6}{-\mathrm{I}\sqrt{3}-3} \\ 1 \end{bmatrix}\right]$

- Solution from eigenpair

  $\mathrm{e}^{-\mathrm{I}\sqrt{3}\,t} \cdot \begin{bmatrix} \frac{6}{-\mathrm{I}\sqrt{3}-3} \\ 1 \end{bmatrix}$

- Use Euler identity to write solution in terms of sin and cos

  $\left(\cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right)\right) \cdot \begin{bmatrix} \frac{6}{-\mathrm{I}\sqrt{3}-3} \\ 1 \end{bmatrix}$

- Simplify expression

  $\begin{bmatrix} \frac{6\left(\cos\left(\sqrt{3}\,t\right)-\mathrm{I}\sin\left(\sqrt{3}\,t\right)\right)}{-\mathrm{I}\sqrt{3}-3} \\ \cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$

- Both real and imaginary parts are solutions to the homogeneous system

  $\left[\vec{x}_1(t) = \begin{bmatrix} -\frac{3\cos\left(\sqrt{3}\,t\right)}{2} + \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{2} \\ \cos\left(\sqrt{3}\,t\right) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} \frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{2} + \frac{3\sin\left(\sqrt{3}\,t\right)}{2} \\ -\sin\left(\sqrt{3}\,t\right) \end{bmatrix}\right]$

- General solution to the system of ODEs

  $\vec{x} = C1\vec{x}_1(t) + C2\vec{x}_2(t)$

- Substitute solutions into the general solution

  $\vec{x} = \begin{bmatrix} C2\left(\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{2} + \frac{3\sin\left(\sqrt{3}\,t\right)}{2}\right) + C1\left(-\frac{3\cos\left(\sqrt{3}\,t\right)}{2} + \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{2}\right) \\ -C2\sin\left(\sqrt{3}\,t\right) + C1\cos\left(\sqrt{3}\,t\right) \end{bmatrix}$

- Substitute in vector of dependent variables

  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{\left(C2\sqrt{3}-3C1\right)\cos\left(\sqrt{3}\,t\right)}{2} + \frac{\sin\left(\sqrt{3}\,t\right)\left(C1\sqrt{3}+3C2\right)}{2} \\ -C2\sin\left(\sqrt{3}\,t\right) + C1\cos\left(\sqrt{3}\,t\right) \end{bmatrix}$

- Solution to the system of ODEs

  $\left\{x(t) = \frac{\left(C2\sqrt{3}-3C1\right)\cos\left(\sqrt{3}\,t\right)}{2} + \frac{\sin\left(\sqrt{3}\,t\right)\left(C1\sqrt{3}+3C2\right)}{2}, y(t) = -C2\sin\left(\sqrt{3}\,t\right) + C1\cos\left(\sqrt{3}\,t\right)\right\}$

✓ **Mathematica.** Time used: 0.008 (sec). Leaf size: 77

```
ode={D[x[t],t]==3*x[t]+6*y[t], D[y[t],t]==-2*x[t]-3*y[t]};
ic={};
DSolve[{ode,ic},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to c_1 \cos\left(\sqrt{3}t\right) + \sqrt{3}(c_1 + 2c_2) \sin\left(\sqrt{3}t\right)$$
$$y(t) \to c_2 \cos\left(\sqrt{3}t\right) - \frac{(2c_1 + 3c_2) \sin\left(\sqrt{3}t\right)}{\sqrt{3}}$$

✓ **Sympy.** Time used: 0.108 (sec). Leaf size: 61

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
ode=[Eq(-3*x(t) - 6*y(t) + Derivative(x(t), t),0),Eq(-2*x(t) + 3*y(t) +
    Derivative(y(t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t)],ics=ics)
```

$$\left[x(t) = \frac{C_1 \left(3 - \sqrt{21}\right) e^{-\sqrt{21}t}}{2} + \frac{C_2 \left(3 + \sqrt{21}\right) e^{\sqrt{21}t}}{2},\ y(t) = C_1 e^{-\sqrt{21}t} + C_2 e^{\sqrt{21}t}\right]$$

2.12.2 Problem (b)

Local contents

Solution using Matrix exponential method 364
Solution using explicit Eigenvalue and Eigenvector method 365
✓Maple 369
✓Mathematica 371
✓Sympy 371

Internal problem ID [20995]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise XIII at page 189
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:34:36 AM`
CAS classification : `system_of_ODEs`

$$x' = 8x + y$$

$$y' = -4x + 4y$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{6t}(2t+1) & \mathrm{e}^{6t}t \\ -4\,\mathrm{e}^{6t}t & \mathrm{e}^{6t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{6t}(2t+1) & e^{6t}t \\ -4\,e^{6t}t & e^{6t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(2t+1)\,c_1 + e^{6t}tc_2 \\ -4\,e^{6t}tc_1 + e^{6t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{6t}(2c_1t + tc_2 + c_1) \\ e^{6t}(-4c_1t + (1-2t)\,c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 8-\lambda & 1 \\ -4 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 12\lambda + 36 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 6$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
6	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 6$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -\frac{t}{2}\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
6	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 6 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

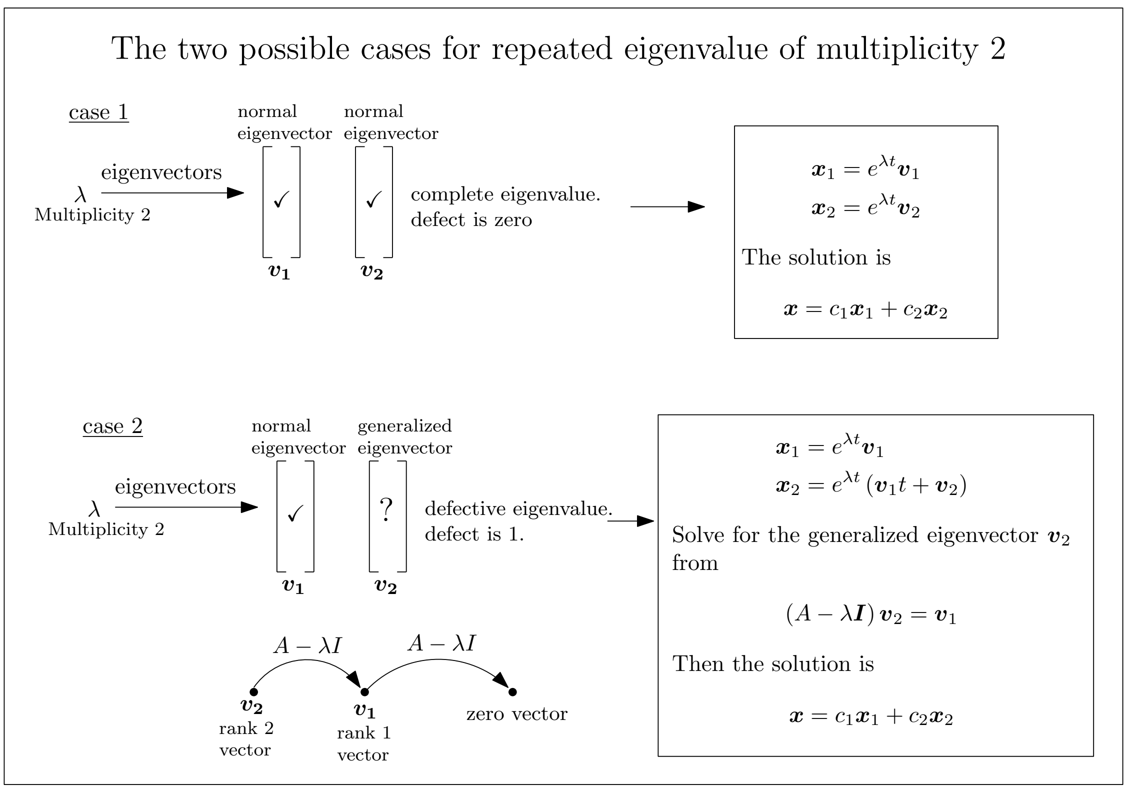


Figure 2.66: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} - (6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 6. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{6t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{6t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}\right)\mathrm{e}^{6t} \\ &= \begin{bmatrix} \mathrm{e}^{6t}\left(-\frac{t}{2}+1\right) \\ \mathrm{e}^{6t}\left(t-\frac{5}{2}\right) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1\begin{bmatrix} -\frac{\mathrm{e}^{6t}}{2} \\ \mathrm{e}^{6t} \end{bmatrix} + c_2\begin{bmatrix} \mathrm{e}^{6t}\left(-\frac{t}{2}+1\right) \\ \mathrm{e}^{6t}\left(t-\frac{5}{2}\right) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{((t-2)c_2+c_1)\mathrm{e}^{6t}}{2} \\ \frac{\mathrm{e}^{6t}(2c_1+c_2(2t-5))}{2} \end{bmatrix}$$

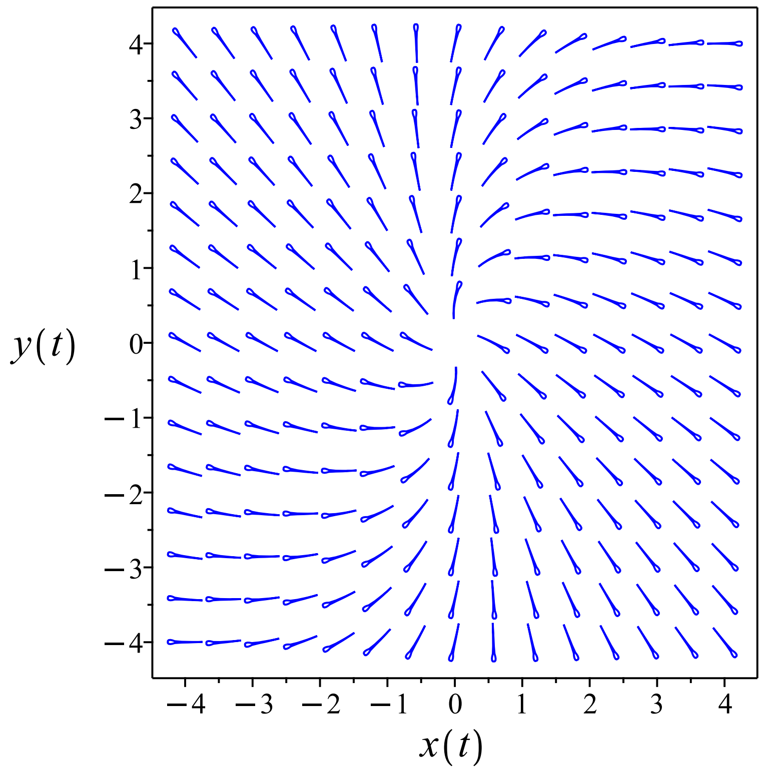


Figure 2.67: Phase plot

✓ **Maple.** `Time used: 0.100 (sec). Leaf size: 34`

```
ode:=[diff(x(t),t) = 8*x(t)+y(t), diff(y(t),t) = -4*x(t)+4*y(t)];
dsolve(ode);
```

$$x(t) = e^{6t}(c_2 t + c_1)$$
$$y(t) = -e^{6t}(2c_2 t + 2c_1 - c_2)$$

Maple step by step

Let's solve
$\left[\frac{d}{dt}x(t) = 8x(t) + y(t), \frac{d}{dt}y(t) = -4x(t) + 4y(t)\right]$

- Define vector
$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$
- Convert system into a vector equation
$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- System to solve
$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix
$A = \begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix}$
- Rewrite the system as
$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
$\left[\left[6, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right], \left[6, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2
$\left[6, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right]$
- First solution from eigenvalue 6
$\vec{x}_1(t) = \mathrm{e}^{6t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 6$ is the eigenvalue, and $\vec{v}$ i
$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtained
- Substitute $\vec{x}_2(t)$ into the homogeneous system
$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A
$\lambda \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation
$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I
$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system
$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 6
$\left(\begin{bmatrix} 8 & 1 \\ -4 & 4 \end{bmatrix} - 6 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$
- Choice of $\vec{p}$
$\vec{p} = \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}$
- Second solution from eigenvalue 6
$\vec{x}_2(t) = \mathrm{e}^{6t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right)$
- General solution to the system of ODEs
$\vec{x} = C1\vec{x}_1(t) + C2\vec{x}_2(t)$

- Substitute solutions into the general solution
  $\overrightarrow{x} = C1\,\mathrm{e}^{6t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C2\,\mathrm{e}^{6t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right)$
- Substitute in vector of dependent variables
  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{6t}(2C2t+2C1+C2)}{4} \\ \mathrm{e}^{6t}(C2t + C1) \end{bmatrix}$
- Solution to the system of ODEs
  $\left\{x(t) = -\frac{\mathrm{e}^{6t}(2C2t+2C1+C2)}{4}, y(t) = \mathrm{e}^{6t}(C2t + C1)\right\}$

✓ **Mathematica.** `Time used: 0.002 (sec). Leaf size: 45`

```
ode={D[x[t],t]==8*x[t]+y[t], D[y[t],t]==-4*x[t]+4*y[t]};
ic={};
DSolve[{ode,ic},{x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{6t}(2c_1t + c_2t + c_1)$$
$$y(t) \to e^{6t}(c_2 - 2(2c_1 + c_2)t)$$

✓ **Sympy.** `Time used: 0.059 (sec). Leaf size: 44`

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
ode=[Eq(-8*x(t) - y(t) + Derivative(x(t), t),0),Eq(4*x(t) - 4*y(t) + Derivative(y
    (t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t)],ics=ics)
```

$$\left[x(t) = 2C_1te^{6t} + (C_1 + 2C_2)\,e^{6t},\ \ y(t) = -4C_1te^{6t} - 4C_2e^{6t}\right]$$

2.12.3 Problem (c)

Local contents

Solution using Matrix exponential method 372
Solution using explicit Eigenvalue and Eigenvector method 373
✓Maple 379
✓Mathematica 381
✓Sympy 382

Internal problem ID [20996]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise XIII at page 189
Problem number : (c)
Date solved : `Saturday, November 29, 2025 at 01:34:51 AM`
CAS classification : `system_of_ODEs`

$$x' = x - y + 2z$$
$$y' = -x + y + 2z$$
$$z' = x + y$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4} & -\frac{\sinh(2t)}{2} & \sinh(2t) \\ -\frac{\sinh(2t)}{2} & \frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4} & \sinh(2t) \\ \frac{\sinh(2t)}{2} & \frac{\sinh(2t)}{2} & \cosh(2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4} & -\frac{\sinh(2t)}{2} & \sinh(2t) \\ -\frac{\sinh(2t)}{2} & \frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4} & \sinh(2t) \\ \frac{\sinh(2t)}{2} & \frac{\sinh(2t)}{2} & \cosh(2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4}\right) c_1 - \frac{\sinh(2t)c_2}{2} + \sinh(2t)\,c_3 \\ -\frac{\sinh(2t)c_1}{2} + \left(\frac{e^{-2t}}{4} + \frac{3\,e^{2t}}{4}\right) c_2 + \sinh(2t)\,c_3 \\ \frac{\sinh(2t)c_1}{2} + \frac{\sinh(2t)c_2}{2} + \cosh(2t)\,c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-2t}((3c_1-c_2+2c_3)e^{4t}+c_1+c_2-2c_3)}{4} \\ -\frac{e^{-2t}((c_1-3c_2-2c_3)e^{4t}-c_1-c_2+2c_3)}{4} \\ \frac{(c_1+c_2)\sinh(2t)}{2} + \cosh(2t)\,c_3 \end{bmatrix}
\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -1 & 2 \\ -1 & 1-\lambda & 2 \\ 1 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -2$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-2	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} - (-2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3 & -1 & 2 \\ -1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ -1 & 3 & 2 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & \frac{8}{3} & \frac{8}{3} & 0 \\ 1 & 1 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_1}{3} \implies \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & \frac{8}{3} & \frac{8}{3} & 0 \\ 0 & \frac{4}{3} & \frac{4}{3} & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{R_2}{2} \implies \left[\begin{array}{ccc|c} 3 & -1 & 2 & 0 \\ 0 & \frac{8}{3} & \frac{8}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & \frac{8}{3} & \frac{8}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} - (2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & -1 & 2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ -1 & -1 & 2 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t + 2s\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t + 2s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 2s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-2	1	1	No	$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$
2	2	2	No	$\begin{bmatrix} -1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -2 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-2t} \\ &= \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{-2t} \end{aligned}$$

eigenvalue 2 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

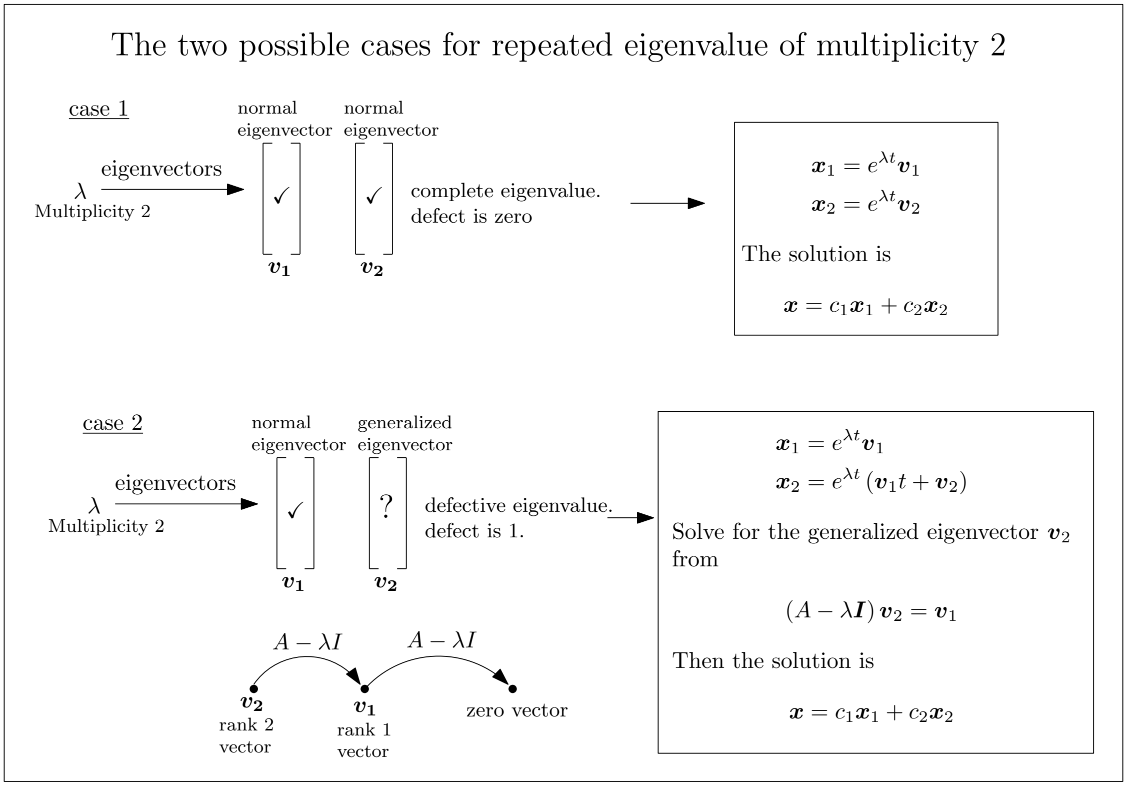


Figure 2.68: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{2t} \\
&= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{2t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{2t} \\
&= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{2t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-2t} \\ -\mathrm{e}^{-2t} \\ \mathrm{e}^{-2t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2\,\mathrm{e}^{2t} \\ 0 \\ \mathrm{e}^{2t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -c_1 e^{-2t} + (-c_2 + 2c_3) e^{2t} \\ -c_1 e^{-2t} + c_2 e^{2t} \\ c_1 e^{-2t} + c_3 e^{2t} \end{bmatrix}$$

✓ **Maple.** `Time used: 0.095 (sec). Leaf size: 66`

```
ode:=[diff(x(t),t) = x(t)-y(t)+2*z(t), diff(y(t),t) = -x(t)+y(t)+2*z(t), diff(z(t
   ),t) = x(t)+y(t)];
dsolve(ode);
```

$$x(t) = -c_2 e^{-2t} - c_3 e^{2t} + e^{2t} c_1$$
$$y(t) = -c_2 e^{-2t} + 3c_3 e^{2t} - e^{2t} c_1$$
$$z(t) = c_2 e^{-2t} + c_3 e^{2t}$$

Maple step by step

Let's solve
$\left[\frac{d}{dt}x(t) = x(t) - y(t) + 2z(t), \frac{d}{dt}y(t) = -x(t) + y(t) + 2z(t), \frac{d}{dt}z(t) = x(t) + y(t)\right]$

- Define vector
$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$
- Convert system into a vector equation
$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- System to solve
$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$
- Define the coefficient matrix
$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$
- Rewrite the system as
$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair
$$\left[-2, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair
$$\vec{x}_1 = e^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2
$$\left[2, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right]$$
- First solution from eigenvalue 2
$$\vec{x}_2(t) = e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 2$ is the eigenvalue, and $\vec{v}$ i
$$\vec{x}_3(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtained
- Substitute $\vec{x}_3(t)$ into the homogeneous system
$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$
- Use the fact that $\vec{v}$ is an eigenvector of A
$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$
- Simplify equation
$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$
- Make use of the identity matrix I
$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$
- Condition $\vec{p}$ must meet for $\vec{x}_3(t)$ to be a solution to the homogeneous system
$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 2
$$\left(\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
- Choice of $\vec{p}$
$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue 2
$$\vec{x}_3(t) = e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs
$$\vec{x} = C1\vec{x}_1 + C2\vec{x}_2(t) + C3\vec{x}_3(t)$$
- Substitute solutions into the general solution
$$\vec{x} = C1\,e^{-2t} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} + C2\,e^{2t} \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + C3\,e^{2t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} -C1\,e^{-2t} + (-C2 + C3(-t+1))\,e^{2t} \\ -C1\,e^{-2t} + e^{2t}(C3t + C2) \\ C1\,e^{-2t} \end{bmatrix}$$
- Solution to the system of ODEs
$$\{x(t) = -C1\,e^{-2t} + (-C2 + C3(-t+1))\,e^{2t}, y(t) = -C1\,e^{-2t} + e^{2t}(C3t + C2), z(t) = C1\,e^{-2t}\}$$

✓ **Mathematica.** `Time used: 0.004 (sec). Leaf size: 134`

```
ode={D[x[t],t]==x[t]-y[t]+2*z[t], D[y[t],t]==-x[t]+y[t]+2*z[t],D[z[t],t]==x[t]+y[
    t]};
ic={};
DSolve[{ode,ic},{x[t],y[t],z[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{4}e^{-2t}\left(c_1\left(3e^{4t}+1\right) - (c_2 - 2c_3)\left(e^{4t}-1\right)\right)$$
$$y(t) \to \frac{1}{4}e^{-2t}\left(-\left(c_1\left(e^{4t}-1\right)\right) + c_2\left(3e^{4t}+1\right) + 2c_3\left(e^{4t}-1\right)\right)$$
$$z(t) \to \frac{1}{4}e^{-2t}\left(c_1\left(e^{4t}-1\right) + c_2\left(e^{4t}-1\right) + 2c_3\left(e^{4t}+1\right)\right)$$

✓ **Sympy.** `Time used: 0.084 (sec). Leaf size: 51`

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
z = Function("z")
ode=[Eq(-x(t) + y(t) - 2*z(t) + Derivative(x(t), t),0),Eq(x(t) - y(t) - 2*z(t) +
    Derivative(y(t), t),0),Eq(-x(t) - y(t) + Derivative(z(t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t),z(t)],ics=ics)
```

$$\left[x(t) = -C_1 e^{-2t} - (C_2 - 2C_3)\, e^{2t},\ y(t) = -C_1 e^{-2t} + C_2 e^{2t},\ z(t) = C_1 e^{-2t} + C_3 e^{2t}\right]$$

2.12.4 Problem (d)

Local contents

Solution using Matrix exponential method 383
Solution using explicit Eigenvalue and Eigenvector method 384
✓Maple 390
✓Mathematica 392
✓Sympy 393

Internal problem ID [20997]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise XIII at page 189
Problem number : (d)
Date solved : `Saturday, November 29, 2025 at 01:35:06 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= -x + y - z \\
y' &= 2x - y + 2z \\
z' &= 2x + 2y - z
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{-t} & \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2} & -\frac{\mathrm{e}^{-t}}{2} + \frac{\mathrm{e}^{-3t}}{2} \\ 2\sinh(t) & -\frac{\mathrm{e}^{-t}}{2} + \frac{3\,\mathrm{e}^{t}}{4} + \frac{3\,\mathrm{e}^{-3t}}{4} & \frac{\mathrm{e}^{t}}{4} + \frac{\mathrm{e}^{-t}}{2} - \frac{3\,\mathrm{e}^{-3t}}{4} \\ 2\sinh(t) & \frac{3\,\mathrm{e}^{t}}{4} - \frac{\mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{4} & \cosh(t)^2\,\mathrm{e}^{-t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{-t} & \frac{e^{-t}}{2}-\frac{e^{-3t}}{2} & -\frac{e^{-t}}{2}+\frac{e^{-3t}}{2} \\ 2\sinh(t) & -\frac{e^{-t}}{2}+\frac{3e^{t}}{4}+\frac{3e^{-3t}}{4} & \frac{e^{t}}{4}+\frac{e^{-t}}{2}-\frac{3e^{-3t}}{4} \\ 2\sinh(t) & \frac{3e^{t}}{4}-\frac{e^{-t}}{2}-\frac{e^{-3t}}{4} & \cosh(t)^2 e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t}c_1+\left(\frac{e^{-t}}{2}-\frac{e^{-3t}}{2}\right)c_2+\left(-\frac{e^{-t}}{2}+\frac{e^{-3t}}{2}\right)c_3 \\ 2\sinh(t)\,c_1+\left(-\frac{e^{-t}}{2}+\frac{3e^{t}}{4}+\frac{3e^{-3t}}{4}\right)c_2+\left(\frac{e^{t}}{4}+\frac{e^{-t}}{2}-\frac{3e^{-3t}}{4}\right)c_3 \\ 2\sinh(t)\,c_1+\left(\frac{3e^{t}}{4}-\frac{e^{-t}}{2}-\frac{e^{-3t}}{4}\right)c_2+\cosh(t)^2 e^{-t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-3t}\left((2c_1+c_2-c_3)e^{2t}-c_2+c_3\right)}{2} \\ \frac{e^{-3t}\left((4c_1+3c_2+c_3)e^{4t}+2(-2c_1-c_2+c_3)e^{2t}+3c_2-3c_3\right)}{4} \\ \frac{e^{-3t}\left((4c_1+3c_2+c_3)e^{4t}+2(-2c_1-c_2+c_3)e^{2t}-c_2+c_3\right)}{4} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & 1 & -1 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 + 3\lambda^2 - \lambda - 3 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -3$$
$$\lambda_2 = -1$$
$$\lambda_3 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
1	1	real eigenvalue
-3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} - (-3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 & -1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 2t, v_2 = -3t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ -3t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} - (-1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - 2R_2 \Longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} - (1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 2 & -2 & 2 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$R_3 = R_3 + 3R_2 \Longrightarrow \left[\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
-3	1	1	No	$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$
-1	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$
1	1	1	No	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue -3 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{-3t} \\ &= \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} e^{-3t} \end{aligned}$$

Since eigenvalue -1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_2(t) &= \vec{v}_2 e^{-t} \\ &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{-t}\end{aligned}$$

Since eigenvalue 1 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}\vec{x}_3(t) &= \vec{v}_3 e^{t} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{t}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 2\,\mathrm{e}^{-3t} \\ -3\,\mathrm{e}^{-3t} \\ \mathrm{e}^{-3t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ \mathrm{e}^{-t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ \mathrm{e}^{t} \\ \mathrm{e}^{t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2c_1\,\mathrm{e}^{-3t} - c_2\,\mathrm{e}^{-t} \\ (c_3\,\mathrm{e}^{4t} + c_2\,\mathrm{e}^{2t} - 3c_1)\,\mathrm{e}^{-3t} \\ (c_3\,\mathrm{e}^{4t} + c_2\,\mathrm{e}^{2t} + c_1)\,\mathrm{e}^{-3t} \end{bmatrix}$$

✓ **Maple.** `Time used: 0.132 (sec). Leaf size: 61`

```
ode:=[diff(x(t),t) = -x(t)+y(t)-z(t), diff(y(t),t) = 2*x(t)-y(t)+2*z(t), diff(z(t
   ),t) = 2*x(t)+2*y(t)-z(t)];
dsolve(ode);
```

$$\begin{aligned}x(t) &= c_2\,\mathrm{e}^{-t} + c_3\,\mathrm{e}^{-3t} \\ y(t) &= \mathrm{e}^{t} c_1 - c_2\,\mathrm{e}^{-t} - \frac{3c_3\,\mathrm{e}^{-3t}}{2} \\ z(t) &= \mathrm{e}^{t} c_1 + \frac{c_3\,\mathrm{e}^{-3t}}{2} - c_2\,\mathrm{e}^{-t}\end{aligned}$$

Maple step by step

Let's solve
$\left[\frac{d}{dt}x(t) = -x(t) + y(t) - z(t), \frac{d}{dt}y(t) = 2x(t) - y(t) + 2z(t), \frac{d}{dt}z(t) = 2x(t) + 2y(t) - z(t)\right]$

- Define vector
$\overrightarrow{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right]$
- Convert system into a vector equation
$\frac{d}{dt}\overrightarrow{x}(t) = \left[\begin{array}{ccc} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{array}\right] \cdot \overrightarrow{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$
- System to solve
$\frac{d}{dt}\overrightarrow{x}(t) = \left[\begin{array}{ccc} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{array}\right] \cdot \overrightarrow{x}(t)$
- Define the coefficient matrix
$A = \left[\begin{array}{ccc} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{array}\right]$
- Rewrite the system as
$\frac{d}{dt}\overrightarrow{x}(t) = A \cdot \overrightarrow{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
$\left[\left[-3, \left[\begin{array}{c} 2 \\ -3 \\ 1 \end{array}\right]\right], \left[-1, \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]\right], \left[1, \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]\right]\right]$
- Consider eigenpair
$\left[-3, \left[\begin{array}{c} 2 \\ -3 \\ 1 \end{array}\right]\right]$
- Solution to homogeneous system from eigenpair
$\overrightarrow{x}_1 = \mathrm{e}^{-3t} \cdot \left[\begin{array}{c} 2 \\ -3 \\ 1 \end{array}\right]$
- Consider eigenpair
$\left[-1, \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]\right]$
- Solution to homogeneous system from eigenpair
$\overrightarrow{x}_2 = \mathrm{e}^{-t} \cdot \left[\begin{array}{c} -1 \\ 1 \\ 1 \end{array}\right]$

- Consider eigenpair

$$\left[1, \left[\begin{array}{c}0\\1\\1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = \mathrm{e}^{t} \cdot \left[\begin{array}{c}0\\1\\1\end{array}\right]$$

- General solution to the system of ODEs

$$\vec{x} = C1\vec{x}_1 + C2\vec{x}_2 + C3\vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = C1\,\mathrm{e}^{-3t} \cdot \left[\begin{array}{c}2\\-3\\1\end{array}\right] + C2\,\mathrm{e}^{-t} \cdot \left[\begin{array}{c}-1\\1\\1\end{array}\right] + C3\,\mathrm{e}^{t} \cdot \left[\begin{array}{c}0\\1\\1\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\y(t)\\z(t)\end{array}\right] = \left[\begin{array}{c}2C1\,\mathrm{e}^{-3t} - C2\,\mathrm{e}^{-t}\\(C3\,\mathrm{e}^{4t} + C2\,\mathrm{e}^{2t} - 3C1)\,\mathrm{e}^{-3t}\\(C3\,\mathrm{e}^{4t} + C2\,\mathrm{e}^{2t} + C1)\,\mathrm{e}^{-3t}\end{array}\right]$$

- Solution to the system of ODEs

$$\{x(t) = 2C1\,\mathrm{e}^{-3t} - C2\,\mathrm{e}^{-t}, y(t) = (C3\,\mathrm{e}^{4t} + C2\,\mathrm{e}^{2t} - 3C1)\,\mathrm{e}^{-3t}, z(t) = (C3\,\mathrm{e}^{4t} + C2\,\mathrm{e}^{2t} + C1)\,\mathrm{e}^{-3t}$$

✓ **Mathematica.** `Time used: 0.013 (sec). Leaf size: 149`

```
ode={D[x[t],t]==-x[t]+y[t]-z[t], D[y[t],t]==2*x[t]-y[t]+2*z[t],D[z[t],t]==2*x[t
   ]+2*y[t]-z[t]};
ic={};
DSolve[{ode,ic},{x[t],y[t],z[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{2}e^{-3t}\left((2c_1 + c_2 - c_3)e^{2t} - c_2 + c_3\right)$$
$$y(t) \to \frac{1}{4}e^{-3t}\left(-2(2c_1 + c_2 - c_3)e^{2t} + (4c_1 + 3c_2 + c_3)e^{4t} + 3(c_2 - c_3)\right)$$
$$z(t) \to \frac{1}{4}e^{-3t}\left(-2(2c_1 + c_2 - c_3)e^{2t} + (4c_1 + 3c_2 + c_3)e^{4t} - c_2 + c_3\right)$$

✓ **Sympy.** Time used: 0.078 (sec). Leaf size: 54

```
from sympy import *
t = symbols("t")
x = Function("x")
y = Function("y")
z = Function("z")
ode=[Eq(x(t) - y(t) + z(t) + Derivative(x(t), t),0),Eq(-2*x(t) + y(t) - 2*z(t) +
   Derivative(y(t), t),0),Eq(-2*x(t) - 2*y(t) + z(t) + Derivative(z(t), t),0)]
ics = {}
dsolve(ode,func=[x(t),y(t),z(t)],ics=ics)
```

$$\left[x(t) = 2C_1e^{-3t} - C_2e^{-t},\ y(t) = -3C_1e^{-3t} + C_2e^{-t} + C_3e^t,\ z(t) = C_1e^{-3t} + C_2e^{-t} + C_3e^t\right]$$

2.13 Chapter IV. Linear Differential Equations. Excercise VI at page 209

Local contents

2.13.1 Problem (a) 395
2.13.2 Problem (b) 407

2.13.1 Problem (a)

Local contents

Existence and uniqueness analysis 395
Solved as second order linear constant coeff ode 396
Solved as second order solved by an integrating factor 399
Solved as second order ode using Kovacic algorithm 400
✓ Maple 404
✓ Mathematica 406
✓ Sympy 406

Internal problem ID [20998]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise VI at page 209
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:35:21 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Existence and uniqueness analysis

$$
\begin{aligned}
y''+4y'+4y &= \mathrm{e}^{x} \\
y(0) &= 1 \\
y'(0) &= 0
\end{aligned}
$$

This is a linear ODE. In canonical form it is written as

$$
y''+p(x)y'+q(x)y = F
$$

Where here

$$
\begin{aligned}
p(x) &= 4 \\
q(x) &= 4 \\
F &= \mathrm{e}^{x}
\end{aligned}
$$

Hence the ode is

$$
y''+4y'+4y = \mathrm{e}^{x}
$$

The domain of $p(x)=4$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.167 (sec)

Solve

$$y'' + 4y' + 4y = e^x$$
$$y(0) = 1$$
$$y'(0) = 0$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + 4\lambda\, e^{x\lambda} + 4\, e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 e^{-2x} + c_2 e^{-2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \,e^{-2x} + x \,e^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{9}$$

Therefore the general solution is

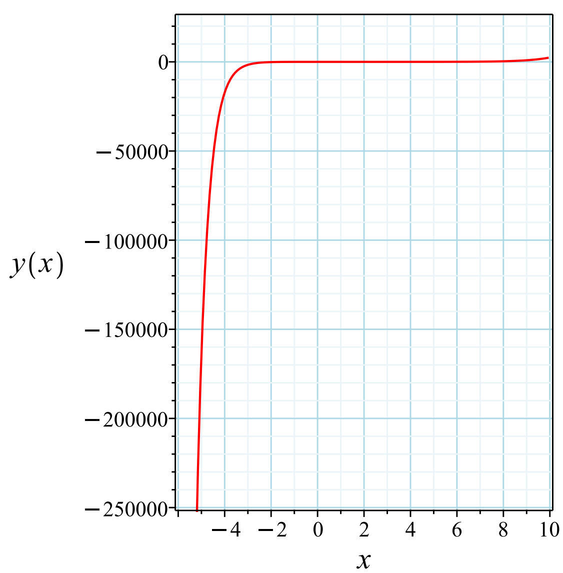
$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x e^{-2x} c_2\right) + \left(\frac{e^x}{9}\right) \end{aligned}$$

Solving for initial conditions the solution is

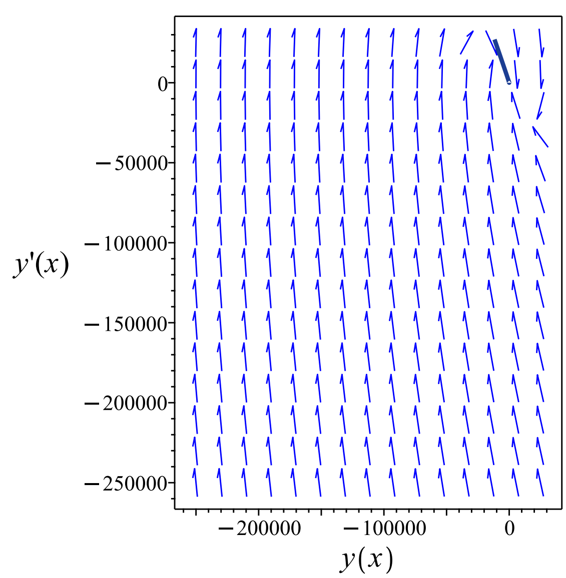
$$y = \frac{(e^{3x} + 15x + 8) e^{-2x}}{9}$$

Summary of solutions found

$$y = \frac{(e^{3x} + 15x + 8) e^{-2x}}{9}$$



(a) Solution plot



(b) Slope field $y'' + 4y' + 4y = e^x$

Solved as second order solved by an integrating factor

Time used: 0.094 (sec)

Solve

$$y'' + 4y' + 4y = \mathrm{e}^{x}$$
$$y(0) = 1$$
$$y'(0) = 0$$

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 4\, dx} \\ &= \mathrm{e}^{2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \mathrm{e}^{x}\mathrm{e}^{2x}$$
$$\left(\mathrm{e}^{2x}y\right)'' = \mathrm{e}^{x}\mathrm{e}^{2x}$$

Integrating once gives

$$\left(\mathrm{e}^{2x}y\right)' = \frac{\mathrm{e}^{3x}}{3} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{2x}y\right) = c_1 x + \frac{\mathrm{e}^{3x}}{9} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + \frac{\mathrm{e}^{3x}}{9} + c_2}{\mathrm{e}^{2x}}$$

Or

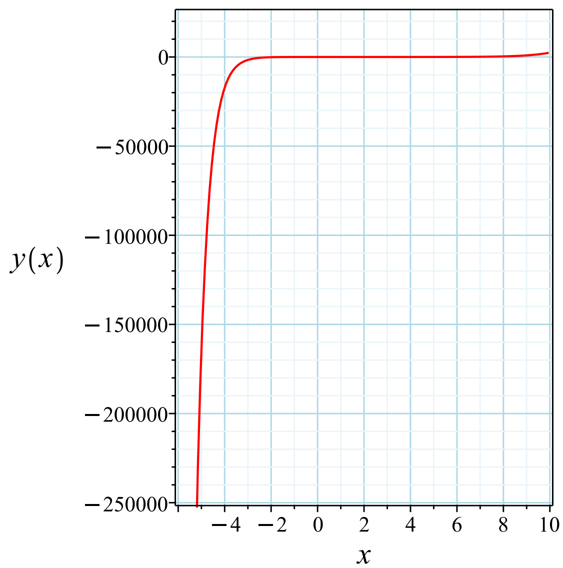
$$y = \frac{\mathrm{e}^{x}}{9} + c_1\, \mathrm{e}^{-2x} x + c_2\, \mathrm{e}^{-2x}$$

Solving for initial conditions the solution is

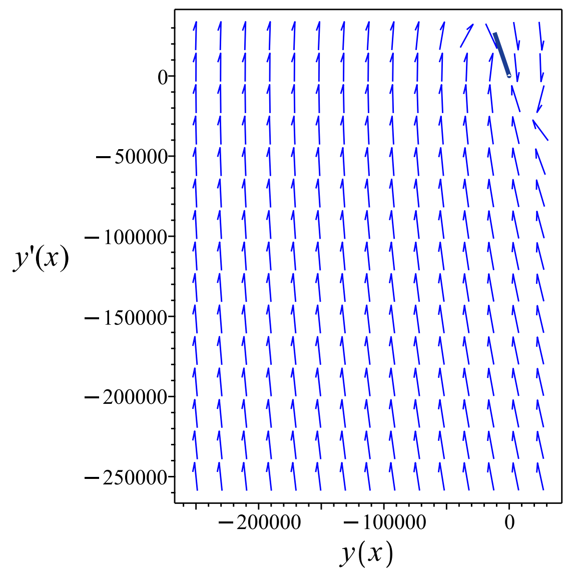
$$y = \frac{\left(\mathrm{e}^{3x} + 15x + 8\right) \mathrm{e}^{-2x}}{9}$$

Summary of solutions found

$$y = \frac{(\mathrm{e}^{3x} + 15x + 8)\,\mathrm{e}^{-2x}}{9}$$



(a) Solution plot



(b) Slope field $y'' + 4y' + 4y = \mathrm{e}^{x}$

Solved as second order ode using Kovacic algorithm

Time used: 0.119 (sec)

Solve

$$y'' + 4y' + 4y = \mathrm{e}^{x}$$
$$y(0) = 1$$
$$y'(0) = 0$$

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.41: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1 \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{\left(y_1\right)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{\left(y_1\right)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\,\mathrm{e}^{-2x} + x\,\mathrm{e}^{-2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}x, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1\mathrm{e}^{x} = \mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{9}$$

Therefore the general solution is

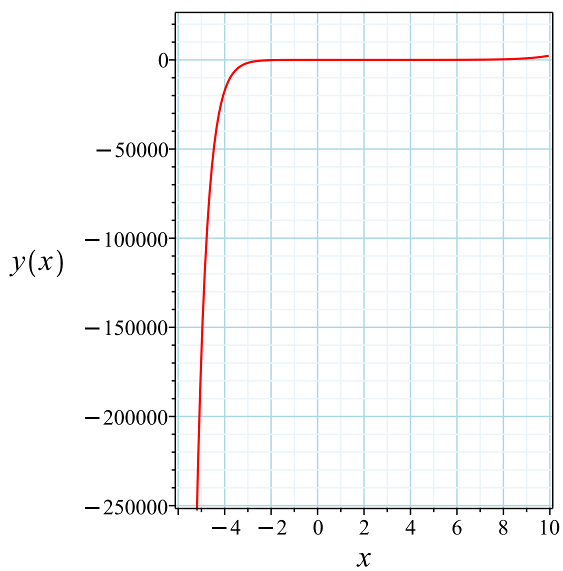
$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + x e^{-2x} c_2\right) + \left(\frac{e^x}{9}\right) \end{aligned}$$

Solving for initial conditions the solution is

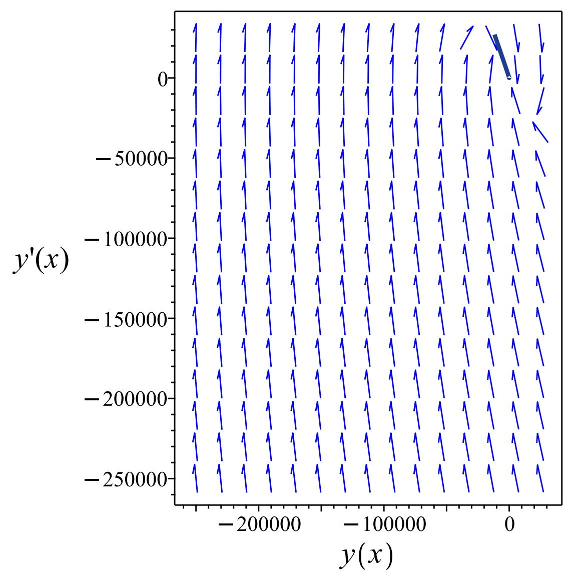
$$y = \frac{(e^{3x} + 15x + 8) e^{-2x}}{9}$$

Summary of solutions found

$$y = \frac{(e^{3x} + 15x + 8) e^{-2x}}{9}$$



(a) Solution plot



(b) Slope field $y'' + 4y' + 4y = e^x$

✓ **Maple.** `Time used: 0.017 (sec). Leaf size: 19`

```
ode:=diff(diff(y(x),x),x)+4*diff(y(x),x)+4*y(x) = exp(x);
ic:=[y(0) = 1, D(y)(0) = 0];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$y = \frac{(e^{3x} + 15x + 8) e^{-2x}}{9}$$

Maple trace

```
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful
```

Maple step by step

Let's solve
$\left[\frac{d^2}{dx^2}y(x)+4\frac{d}{dx}y(x)+4y(x)=\mathrm{e}^x, y(0)=1, \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2+4r+4=0$
- Factor the characteristic polynomial
  $(r+2)^2=0$
- Root of the characteristic polynomial
  $r=-2$
- 1st solution of the homogeneous ODE
  $y_1(x)=\mathrm{e}^{-2x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x)=x\,\mathrm{e}^{-2x}$
- General solution of the ODE
  $y(x)=C1y_1(x)+C2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x)=C1\,\mathrm{e}^{-2x}+C2x\,\mathrm{e}^{-2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x)=-y_1(x)\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx+y_2(x)\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx, f(x)=\mathrm{e}^x\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x))=\begin{bmatrix}\mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x}\\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}-2x\,\mathrm{e}^{-2x}\end{bmatrix}$
  - Compute Wronskian

$W(y_1(x), y_2(x)) = e^{-4x}$

- ○ Substitute functions into equation for $y_p(x)$
  $y_p(x) = e^{-2x}\left(x\int e^{3x}dx - \int x\,e^{3x}dx\right)$
- ○ Compute integrals
  $y_p(x) = \frac{e^x}{9}$

- • Substitute particular solution into general solution to ODE
  $y(x) = C1\,e^{-2x} + C2x\,e^{-2x} + \frac{e^x}{9}$

- □ Check validity of solution $y(x) = \_C1e^{-2x} + \_C2xe^{-2x} + \frac{e^x}{9}$
  - ○ Use initial condition $y(0) = 1$
    $1 = \_C1 + \frac{1}{9}$
  - ○ Compute derivative of the solution
    $\frac{d}{dx}y(x) = -2\_C1\,e^{-2x} + \_C2\,e^{-2x} - 2\_C2x\,e^{-2x} + \frac{e^x}{9}$
  - ○ Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 0$
    $0 = -2\_C1 + \_C2 + \frac{1}{9}$
  - ○ Solve for $\_C1$ and $\_C2$
    $\left\{\_C1 = \frac{8}{9}, \_C2 = \frac{5}{3}\right\}$
  - ○ Substitute constant values into general solution and simplify
    $y(x) = \frac{(e^{3x}+15x+8)e^{-2x}}{9}$

- • Solution to the IVP
  $y(x) = \frac{(e^{3x}+15x+8)e^{-2x}}{9}$

✓ **Mathematica.** Time used: 0.025 (sec). Leaf size: 24

```
ode=D[y[x],{x,2}]+4*D[y[x],x]+4*y[x]==Exp[x];
ic={y[0]==1,Derivative[1][y][0] ==0};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{9}e^{-2x}\left(15x + e^{3x} + 8\right)$$

✓ **Sympy.** Time used: 0.121 (sec). Leaf size: 20

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(4*y(x) - exp(x) + 4*Derivative(y(x), x) + Derivative(y(x), (x, 2)),0)
ics = {y(0): 1, Subs(Derivative(y(x), x), x, 0): 0}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \left(\frac{5x}{3} + \frac{8}{9}\right)e^{-2x} + \frac{e^x}{9}$$

2.13.2 Problem (b)

Local contents

Existence and uniqueness analysis 407
Solved as second order linear constant coeff ode 408
Solved as second order ode using Kovacic algorithm 411
✓ Maple 415
✓ Mathematica 417
✓ Sympy 417

Internal problem ID [20999]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise VI at page 209
Problem number : (b)
Date solved : `Saturday, November 29, 2025 at 01:35:36 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Existence and uniqueness analysis

$$
\begin{aligned}
y'' - 2y' + 5y &= e^{x} \\
y(0) &= 1 \\
y'(0) &= 0
\end{aligned}
$$

This is a linear ODE. In canonical form it is written as

$$
y'' + p(x)y' + q(x)y = F
$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= 5 \\
F &= e^{x}
\end{aligned}
$$

Hence the ode is

$$
y'' - 2y' + 5y = e^{x}
$$

The domain of $p(x) = -2$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.357 (sec)

Solve

$$y'' - 2y' + 5y = e^x$$
$$y(0) = 1$$
$$y'(0) = 0$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 5, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} - 2\lambda e^{x\lambda} + 5 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Which simplifies to

$$\lambda_1 = 1 + 2i$$
$$\lambda_2 = 1 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x \cos(2x), e^x \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{4}$$

Therefore the general solution is

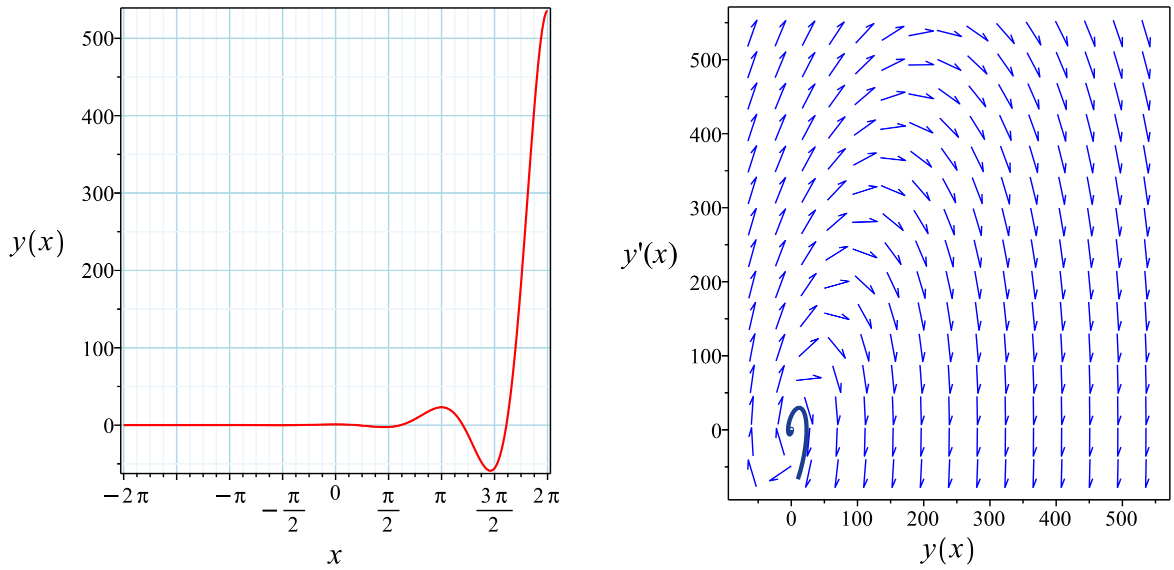
$$\begin{aligned} y &= y_h + y_p \\ &= (e^x(c_1 \cos(2x) + c_2 \sin(2x))) + \left(\frac{e^x}{4}\right) \end{aligned}$$

Solving for initial conditions the solution is

$$y = \frac{e^x(1 + 3\cos(2x) - 2\sin(2x))}{4}$$

Summary of solutions found

$$y = \frac{e^x(1 + 3\cos(2x) - 2\sin(2x))}{4}$$



(a) Solution plot (b) Slope field $y'' - 2y' + 5y = e^x$

Solved as second order ode using Kovacic algorithm

Time used: 0.184 (sec)

Solve

$$y'' - 2y' + 5y = e^x$$
$$y(0) = 1$$
$$y'(0) = 0$$

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.43: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1(\mathrm{e}^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\mathrm{e}^{x}\cos(2x)) + c_2\left(\mathrm{e}^{x}\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \, e^{x} \cos(2x) + \frac{c_2 \, e^{x} \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{x} \cos(2x), \frac{e^{x} \sin(2x)}{2}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{x} = e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{x}}{4}$$

Therefore the general solution is

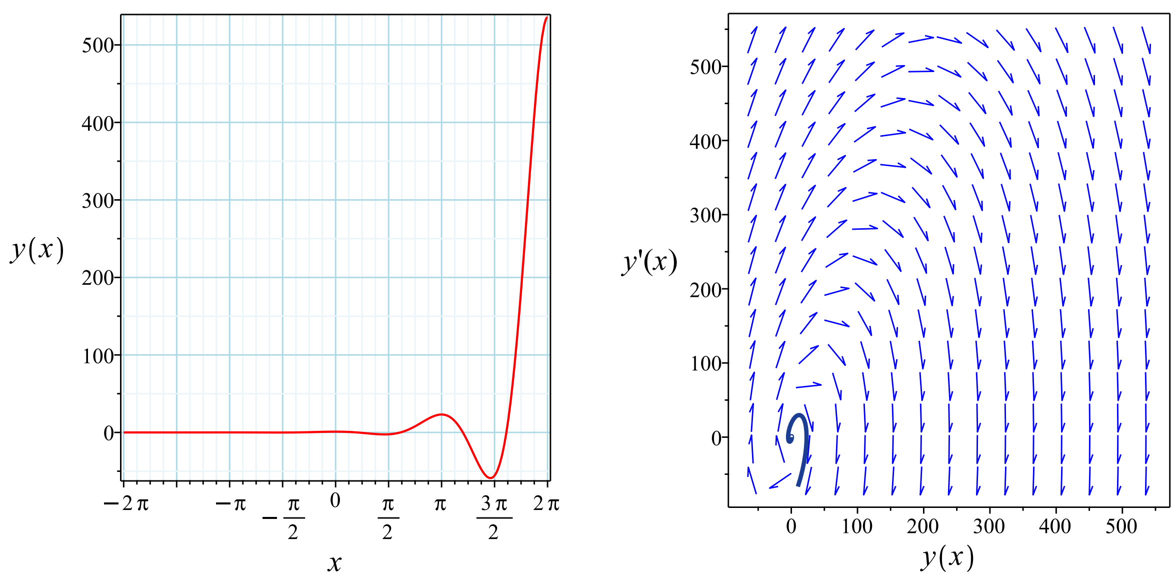
$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^x \cos(2x) + \frac{c_2 \mathrm{e}^x \sin(2x)}{2}\right) + \left(\frac{\mathrm{e}^x}{4}\right)
\end{aligned}
$$

Solving for initial conditions the solution is

$$
y = \frac{\mathrm{e}^x (1 + 3\cos(2x) - 2\sin(2x))}{4}
$$

Summary of solutions found

$$
y = \frac{\mathrm{e}^x (1 + 3\cos(2x) - 2\sin(2x))}{4}
$$



(a) Solution plot

(b) Slope field $y'' - 2y' + 5y = \mathrm{e}^x$

✓ **Maple.** `Time used: 0.024 (sec). Leaf size: 22`

```
ode:=diff(diff(y(x),x),x)-2*diff(y(x),x)+5*y(x) = exp(x);
ic:=[y(0) = 1, D(y)(0) = 0];
dsolve([ode,op(ic)],y(x), singsol=all);
```

$$
y = -\frac{\mathrm{e}^x (2\sin(2x) - 3\cos(2x) - 1)}{4}
$$

Maple trace

```
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful
```

Maple step by step

Let's solve
$\left[\frac{d^2}{dx^2}y(x) - 2\frac{d}{dx}y(x) + 5y(x) = e^x, y(0) = 1, \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 - 2r + 5 = 0$
- Use quadratic formula to solve for r
  $r = \frac{2\pm(\sqrt{-16})}{2}$
- Roots of the characteristic polynomial
  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = e^x \cos(2x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = e^x \sin(2x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\, e^x \cos(2x) + C2\, e^x \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx, f(x) = e^x\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x \cos(2x) & e^x \sin(2x) \\ e^x \cos(2x) - 2e^x \sin(2x) & e^x \sin(2x) + 2e^x \cos(2x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 2\,e^{2x}$

- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\frac{e^x\left(\int \sin(2x)dx\cos(2x)-\int \cos(2x)dx\sin(2x)\right)}{2}$
- Compute integrals
  $y_p(x) = \frac{e^x}{4}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\,e^x\cos(2x) + C2\,e^x\sin(2x) + \frac{e^x}{4}$

☐ Check validity of solution $y(x) = \_C1e^x\cos(2x) + \_C2e^x\sin(2x) + \frac{e^x}{4}$

- Use initial condition $y(0) = 1$
  $1 = \_C1 + \frac{1}{4}$
- Compute derivative of the solution
  $\frac{d}{dx}y(x) = \_C1\,e^x\cos(2x) - 2\_C1\,e^x\sin(2x) + \_C2\,e^x\sin(2x) + 2\_C2\,e^x\cos(2x) + \frac{e^x}{4}$
- Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 0$
  $0 = \_C1 + \frac{1}{4} + 2\_C2$
- Solve for $\_C1$ and $\_C2$
  $\left\{\_C1 = \frac{3}{4}, \_C2 = -\frac{1}{2}\right\}$
- Substitute constant values into general solution and simplify
  $y(x) = \frac{e^x(1+3\cos(2x)-2\sin(2x))}{4}$

- Solution to the IVP
  $y(x) = \frac{e^x(1+3\cos(2x)-2\sin(2x))}{4}$

✓ **Mathematica.** `Time used: 0.02 (sec). Leaf size: 26`

```
ode=D[y[x],{x,2}]-2*D[y[x],x]+5*y[x]==Exp[x];
ic={y[0]==1,Derivative[1][y][0] ==0};
DSolve[{ode,ic},y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}e^x(-2\sin(2x) + 3\cos(2x) + 1)$$

✓ **Sympy.** `Time used: 0.145 (sec). Leaf size: 24`

```
from sympy import *
x = symbols("x")
y = Function("y")
ode = Eq(5*y(x) - exp(x) - 2*Derivative(y(x), x) + Derivative(y(x), (x, 2)),0)
ics = {y(0): 1, Subs(Derivative(y(x), x), x, 0): 0}
dsolve(ode,func=y(x),ics=ics)
```

$$y(x) = \left(-\frac{\sin(2x)}{2} + \frac{3\cos(2x)}{4} + \frac{1}{4}\right)e^x$$

2.14 Chapter IV. Linear Differential Equations. Excercise VIII at page 210

Local contents

2.14.1 Problem (a) 419

2.14.1 Problem (a)

Local contents

Solved as second order linear constant coeff ode 419
Solved as second order ode using Kovacic algorithm 422
✓Maple 426
✓Mathematica 427
✓Sympy 427

Internal problem ID [21000]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter IV. Linear Differential Equations. Excercise VIII at page 210
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:35:51 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solved as second order linear constant coeff ode

Time used: 0.598 (sec)

Solve

$$u'' + 2au' + \omega^2 u = c\cos(\omega t)$$

This is second order non-homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = f(t)$$

Where $A = 1, B = 2a, C = \omega^2, f(t) = c\cos(\omega t)$. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = 0$, and u_p is a particular solution to the non-homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + 2au' + \omega^2 u = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(t) + Bu'(t) + Cu(t) = 0$$

Where in the above $A = 1, B = 2a, C = \omega^2$. Let the solution be $u = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} + 2a\lambda\, e^{t\lambda} + \omega^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2a\lambda + \lambda^2 + \omega^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2a, C = \omega^2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2a}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2a^2 - (4)(1)(\omega^2)} \\ &= -a \pm \sqrt{a^2 - \omega^2}\end{aligned}$$

Hence

$$\lambda_1 = -a + \sqrt{a^2 - \omega^2}$$
$$\lambda_2 = -a - \sqrt{a^2 - \omega^2}$$

Which simplifies to

$$\lambda_1 = -a + \sqrt{a^2 - \omega^2}$$
$$\lambda_2 = -a - \sqrt{a^2 - \omega^2}$$

Since roots are distinct, then the solution is

$$u = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$u = c_1 e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + c_2 e^{\left(-a-\sqrt{a^2-\omega^2}\right)t}$$

Or

$$u = c_1 \,\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + c_2 \,\mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t}$$

Therefore the homogeneous solution u_h is

$$u_h = c_1 \,\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + c_2 \,\mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$c\cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(-a-\sqrt{a^2-\omega^2}\right)t}, e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t) + 2a(-A_1\omega \sin(\omega t) + A_2\omega \cos(\omega t)) \\ + \omega^2(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = c\cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{c}{2a\omega}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = \frac{c\sin(\omega t)}{2a\omega}$$

Therefore the general solution is

$$\begin{aligned} u &= u_h + u_p \\ &= \left(c_1 e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + c_2 e^{\left(-a-\sqrt{a^2-\omega^2}\right)t}\right) + \left(\frac{c\sin(\omega t)}{2a\omega}\right) \end{aligned}$$

Summary of solutions found

$$u = \frac{c\sin(\omega t)}{2a\omega} + c_1 e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + c_2 e^{\left(-a-\sqrt{a^2-\omega^2}\right)t}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Solve

$$u'' + 2au' + \omega^2 u = c\cos(\omega t)$$

Writing the ode as

$$u'' + 2au' + \omega^2 u = 0 \tag{1}$$
$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2a \\ C &= \omega^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = u e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{a^2 - \omega^2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= a^2 - \omega^2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(a^2 - \omega^2\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then u is found using the inverse transformation

$$u = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.45: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = a^2 - \omega^2$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{\sqrt{a^2-\omega^2}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2a}{1}\, dt} \\ &= z_1 e^{-at} \\ &= z_1 \left(\mathrm{e}^{-at}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{u_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{2a}{1}\,dt}}{(u_1)^2}\,dt \\
&= u_1 \int \frac{e^{-2at}}{(u_1)^2}\,dt \\
&= u_1 \left(-\frac{\mathrm{e}^{-2at}\mathrm{e}^{-2\left(-a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t}\right) + c_2 \left(\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} \left(-\frac{\mathrm{e}^{-2at}\mathrm{e}^{-2\left(-a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$u = u_h + u_p$$

Where u_h is the solution to the homogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = 0$, and u_p is a particular solution to the nonhomogeneous ODE $Au''(t) + Bu'(t) + Cu(t) = f(t)$. u_h is the solution to

$$u'' + 2au' + \omega^2 u = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$u_h = c_1 \,\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} - \frac{c_2 \,\mathrm{e}^{-\left(a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$c\cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{-\frac{e^{-\left(a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}, e^{\left(-a+\sqrt{a^2-\omega^2}\right)t}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$u_p = A_1\cos(\omega t) + A_2\sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution u_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-A_1\omega^2\cos(\omega t) - A_2\omega^2\sin(\omega t) + 2a(-A_1\omega\sin(\omega t) + A_2\omega\cos(\omega t))\\&+\omega^2(A_1\cos(\omega t) + A_2\sin(\omega t)) = c\cos(\omega t)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{c}{2a\omega}\right]$$

Substituting the above back in the above trial solution u_p, gives the particular solution

$$u_p = \frac{c\sin(\omega t)}{2a\omega}$$

Therefore the general solution is

$$\begin{aligned}u &= u_h + u_p\\&= \left(c_1 e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} - \frac{c_2 e^{-\left(a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}\right) + \left(\frac{c\sin(\omega t)}{2a\omega}\right)\end{aligned}$$

Summary of solutions found

$$u = \frac{c\sin(\omega t)}{2a\omega} + c_1 e^{\left(-a+\sqrt{a^2-\omega^2}\right)t} - \frac{c_2 e^{-\left(a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}$$

✓ **Maple.** `Time used: 0.007 (sec). Leaf size: 64`

```
ode:=diff(diff(u(t),t),t)+2*a*diff(u(t),t)+omega^2*u(t) = c*cos(omega*t);
dsolve(ode,u(t), singsol=all);
```

$$u = \frac{2\,\mathrm{e}^{\left(-a+\sqrt{a^{2}-\omega^{2}}\right)t} c_{2} a \omega + 2\,\mathrm{e}^{-\left(a+\sqrt{a^{2}-\omega^{2}}\right)t} c_{1} a \omega + \sin\left(\omega t\right) c}{2 a \omega}$$

Maple trace

```
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful
```

Maple step by step

Let's solve
$\frac{d^{2}}{dt^{2}}u(t) + 2a\left(\frac{d}{dt}u(t)\right) + \omega^{2}u(t) = c\cos\left(\omega t\right)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^{2}}{dt^{2}}u(t)$
- Characteristic polynomial of homogeneous ODE
  $2ar + \omega^{2} + r^{2} = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-2a)\pm\left(\sqrt{4a^{2}-4\omega^{2}}\right)}{2}$
- Roots of the characteristic polynomial
  $r = \left(-a-\sqrt{a^{2}-\omega^{2}}, -a+\sqrt{a^{2}-\omega^{2}}\right)$
- 1st solution of the homogeneous ODE
  $u_{1}(t) = \mathrm{e}^{\left(-a-\sqrt{a^{2}-\omega^{2}}\right)t}$
- 2nd solution of the homogeneous ODE
  $u_{2}(t) = \mathrm{e}^{\left(-a+\sqrt{a^{2}-\omega^{2}}\right)t}$
- General solution of the ODE
  $u(t) = C1 u_{1}(t) + C2 u_{2}(t) + u_{p}(t)$
- Substitute in solutions of the homogeneous ODE

$u(t) = C1\,\mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t} + C2\,\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + u_p(t)$

- ☐ Find a particular solution $u_p(t)$ of the ODE
  - Use variation of parameters to find u_p here $f(t)$ is the forcing function
    $\left[u_p(t) = -u_1(t)\int \frac{u_2(t)f(t)}{W(u_1(t),u_2(t))}dt + u_2(t)\int \frac{u_1(t)f(t)}{W(u_1(t),u_2(t))}dt, f(t) = c\cos(\omega t)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(u_1(t), u_2(t)) = \begin{bmatrix} \mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t} & \mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} \\ \left(-a-\sqrt{a^2-\omega^2}\right)\mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t} & \left(-a+\sqrt{a^2-\omega^2}\right)\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} \end{bmatrix}$
  - Compute Wronskian
    $W(u_1(t), u_2(t)) = 2\sqrt{a^2-\omega^2}\,\mathrm{e}^{-2at}$
  - Substitute functions into equation for $u_p(t)$
    $u_p(t) = \frac{c\left(\int \cos(\omega t)\mathrm{e}^{\left(a-\sqrt{a^2-\omega^2}\right)t}dt\mathrm{e}^{2t\sqrt{a^2-\omega^2}} - \int \cos(\omega t)\mathrm{e}^{\left(a+\sqrt{a^2-\omega^2}\right)t}dt\right)\mathrm{e}^{-\left(a+\sqrt{a^2-\omega^2}\right)t}}{2\sqrt{a^2-\omega^2}}$
  - Compute integrals
    $u_p(t) = \frac{c\sin(\omega t)}{2\omega a}$
- Substitute particular solution into general solution to ODE
  $u(t) = C1\,\mathrm{e}^{\left(-a-\sqrt{a^2-\omega^2}\right)t} + C2\,\mathrm{e}^{\left(-a+\sqrt{a^2-\omega^2}\right)t} + \frac{c\sin(\omega t)}{2\omega a}$

✓ **Mathematica.** `Time used: 0.022 (sec). Leaf size: 66`

```
ode=D[u[t],{t,2}]+2*a*D[u[t],t]+\[Omega]^2*u[t]==c*Cos[\[Omega]*t];
ic={};
DSolve[{ode,ic},u[t],t,IncludeSingularSolutions->True]
```

$$u(t) \to \frac{c\sin(t\omega)}{2a\omega} + e^{-t\left(\sqrt{a^2-\omega^2}+a\right)}\left(c_2e^{2t\sqrt{a^2-\omega^2}} + c_1\right)$$

✓ **Sympy.** `Time used: 0.198 (sec). Leaf size: 48`

```
from sympy import *
t = symbols("t")
a = symbols("a")
w = symbols("w")
c = symbols("c")
u = Function("u")
ode = Eq(2*a*Derivative(u(t), t) - c*cos(t*w) + w**2*u(t) + Derivative(u(t), (t,
    2)),0)
ics = {}
dsolve(ode,func=u(t),ics=ics)
```

$$u(t) = C_1e^{t\left(-a+\sqrt{a^2-w^2}\right)} + C_2e^{-t\left(a+\sqrt{a^2-w^2}\right)} + \frac{c\sin(tw)}{2aw}$$

2.15 Chapter V. Complex Linear Systems. Excercise VIII at page 221

Local contents

2.15.1 Problem (a) 429

2.15.1 Problem (a)

Local contents

✓Maple 429
✓Mathematica 429
✗Sympy 430

Internal problem ID [21001]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter V. Complex Linear Systems. Excercise VIII at page 221
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:36:06 AM`
CAS classification : `system_of_ODEs`

$$w_1' = w_2$$
$$w_2' = \frac{a w_1}{z^2}$$

Does not currently support non autonomous system of first order linear differential equations.

✓ **Maple.** `Time used: 0.057 (sec). Leaf size: 94`

```
ode:=[diff(w__1(z),z) = w__2(z), diff(w__2(z),z) = a*w__1(z)/z^2];
dsolve(ode);
```

$$w_1(z) = c_1\, z^{\frac{1}{2}+\frac{\sqrt{1+4a}}{2}} + c_2\, z^{\frac{1}{2}-\frac{\sqrt{1+4a}}{2}}$$
$$w_2(z) = \frac{c_1\, z^{\frac{1}{2}+\frac{\sqrt{1+4a}}{2}}\left(1+\sqrt{1+4a}\right) + c_2\, z^{\frac{1}{2}-\frac{\sqrt{1+4a}}{2}}\left(1-\sqrt{1+4a}\right)}{2z}$$

✓ **Mathematica.** `Time used: 0.004 (sec). Leaf size: 128`

```
ode={D[w1[z],z]==w2[z],D[w2[z],z]==a*w1[z]/z^2};
ic={};
DSolve[{ode,ic},{w1[z],w2[z]},z,IncludeSingularSolutions->True]
```

$$\mathrm{w1}(z) \to z^{\frac{1}{2}-\frac{1}{2}i\sqrt{-4a-1}}\left(c_2 z^{i\sqrt{-4a-1}} + c_1\right)$$
$$\mathrm{w2}(z) \to \frac{1}{2} z^{-\frac{1}{2}-\frac{1}{2}i\sqrt{-4a-1}}\left(\left(1+i\sqrt{-4a-1}\right) c_2 z^{i\sqrt{-4a-1}} + \left(1-i\sqrt{-4a-1}\right) c_1\right)$$

✗ Sympy

```
from sympy import *
z = symbols("z")
a = symbols("a")
w1 = Function("w1")
w2 = Function("w2")
ode=[Eq(-w2(z) + Derivative(w1(z), z),0),Eq(-a*w1(z)/z**2 + Derivative(w2(z), z)
    ,0)]
ics = {}
dsolve(ode,func=[w1(z),w2(z)],ics=ics)
```

```
ValueError : The function cannot be automatically detected for nan.
```

2.16 Chapter V. Complex Linear Systems. Excercise XII at page 244

Local contents

2.16.1 Problem (a) 432

2.16.1 Problem (a)

Local contents

Solved as second order linear exact ode 432
Solved as second order integrable as is ode 434
Solved as second order integrable as is ode (ABC method) 435
Solved as second order ode using Kovacic algorithm 437
✓ Maple 443
✓ Mathematica 443
✗ Sympy 444

Internal problem ID [21002]
Book : Ordinary Differential Equations. By Wolfgang Walter. Graduate texts in Mathematics. Springer. NY. QA372.W224 1998
Section : Chapter V. Complex Linear Systems. Excercise XII at page 244
Problem number : (a)
Date solved : `Saturday, November 29, 2025 at 01:36:21 AM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solved as second order linear exact ode

Time used: 0.331 (sec)

Solve

$$z^2u'' + (3z + 1)\,u' + u = 0$$

An ode of the form

$$p(z)\,u'' + q(z)\,u' + r(z)\,u = s(z)$$

is exact if

$$p''(z) - q'(z) + r(z) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= z^2\\ q(x) &= 3z + 1\\ r(x) &= 1\\ s(x) &= 0\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 2\\ q'(x) &= 3\end{aligned}$$

Therefore (1) becomes

$$2-(3)+(1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(z)\,u'+(q(z)-p'(z))\,u)'=s(x)$$

Integrating gives

$$p(z)\,u'+(q(z)-p'(z))\,u=\int s(z)\;dz$$

Substituting the above values for p, q, r, s gives

$$z^2u'+(z+1)\,u=c_1$$

We now have a first order ode to solve which is

$$z^2u'+(z+1)\,u=c_1$$

Solve In canonical form a linear first order is

$$u'+q(z)u=p(z)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(z)&=-\frac{-z-1}{z^2}\\p(z)&=\frac{c_1}{z^2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int q\,dz}\\&=\mathrm{e}^{\int -\frac{-z-1}{z^2}dz}\\&=z\,\mathrm{e}^{-\frac{1}{z}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(\mu u)&=\mu p\\\frac{\mathrm{d}}{\mathrm{d}z}(\mu u)&=(\mu)\left(\frac{c_1}{z^2}\right)\\\frac{\mathrm{d}}{\mathrm{d}z}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right)&=\left(z\,\mathrm{e}^{-\frac{1}{z}}\right)\left(\frac{c_1}{z^2}\right)\\\mathrm{d}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right)&=\left(\frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\right)\,\mathrm{d}z\end{aligned}$$

Integrating gives

$$
\begin{aligned}
u z\,\mathrm{e}^{-\frac{1}{z}} &= \int \frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\,dz \\
&= c_1\ \mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor $z\,\mathrm{e}^{-\frac{1}{z}}$ gives the final solution

$$
u = \frac{\mathrm{e}^{\frac{1}{z}}\left(c_1\ \mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}
$$

Summary of solutions found

$$
u = \frac{\mathrm{e}^{\frac{1}{z}}\left(c_1\ \mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}
$$

Solved as second order integrable as is ode

Time used: 0.122 (sec)

Solve

$$
z^2 u'' + (3z+1)\,u' + u = 0
$$

Integrating both sides of the ODE w.r.t z gives

$$
\begin{aligned}
\int \left(z^2 u'' + (3z+1)\,u' + u\right) dz &= 0 \\
z^2 u' + (z+1)\,u &= c_1
\end{aligned}
$$

Which is now solved for u. Solve In canonical form a linear first order is

$$
u' + q(z)u = p(z)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(z) &= -\frac{-z-1}{z^2} \\
p(z) &= \frac{c_1}{z^2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dz} \\
&= \mathrm{e}^{\int -\frac{-z-1}{z^2}dz} \\
&= z\,\mathrm{e}^{-\frac{1}{z}}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dz}(\mu u) = \mu p$$

$$\frac{d}{dz}(\mu u) = (\mu)\left(\frac{c_1}{z^2}\right)$$

$$\frac{d}{dz}\left(uz\,e^{-\frac{1}{z}}\right) = \left(z\,e^{-\frac{1}{z}}\right)\left(\frac{c_1}{z^2}\right)$$

$$d\left(uz\,e^{-\frac{1}{z}}\right) = \left(\frac{c_1\,e^{-\frac{1}{z}}}{z}\right) dz$$

Integrating gives

$$\begin{aligned} uz\,e^{-\frac{1}{z}} &= \int \frac{c_1\,e^{-\frac{1}{z}}}{z}\,dz \\ &= c_1 \operatorname{Ei}_1\left(\frac{1}{z}\right) + c_2 \end{aligned}$$

Dividing throughout by the integrating factor $z\,e^{-\frac{1}{z}}$ gives the final solution

$$u = \frac{e^{\frac{1}{z}}\left(c_1 \operatorname{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}$$

Summary of solutions found

$$u = \frac{e^{\frac{1}{z}}\left(c_1 \operatorname{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}$$

Solved as second order integrable as is ode (ABC method)

Time used: 0.088 (sec)

Solve

$$z^2u'' + (3z+1)\,u' + u = 0$$

Writing the ode as

$$z^2u'' + (3z+1)\,u' + u = 0$$

Integrating both sides of the ODE w.r.t z gives

$$\begin{aligned} \int \left(z^2u'' + (3z+1)\,u' + u\right) dz &= 0 \\ z^2u' + (z+1)\,u &= c_1 \end{aligned}$$

Which is now solved for u. Solve In canonical form a linear first order is

$$u' + q(z)u = p(z)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(z) &= -\frac{-z-1}{z^2} \\ p(z) &= \frac{c_1}{z^2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dz} \\ &= \mathrm{e}^{\int -\frac{-z-1}{z^2}dz} \\ &= z\,\mathrm{e}^{-\frac{1}{z}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}(\mu u) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}z}(\mu u) &= (\mu)\left(\frac{c_1}{z^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}z}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right) &= \left(z\,\mathrm{e}^{-\frac{1}{z}}\right)\left(\frac{c_1}{z^2}\right) \\ \mathrm{d}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right) &= \left(\frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\right)\mathrm{d}z \end{aligned}$$

Integrating gives

$$\begin{aligned} uz\,\mathrm{e}^{-\frac{1}{z}} &= \int \frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\,dz \\ &= c_1\ \mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2 \end{aligned}$$

Dividing throughout by the integrating factor $z\,\mathrm{e}^{-\frac{1}{z}}$ gives the final solution

$$u = \frac{\mathrm{e}^{\frac{1}{z}}\left(c_1\ \mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}$$

Solve In canonical form a linear first order is

$$u' + q(z)u = p(z)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(z) &= -\frac{-z-1}{z^2}\\ p(z) &= \frac{c_1}{z^2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dz}\\ &= \mathrm{e}^{\int -\frac{-z-1}{z^2}dz}\\ &= z\,\mathrm{e}^{-\frac{1}{z}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}(\mu u) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}z}(\mu u) &= (\mu)\left(\frac{c_1}{z^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}z}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right) &= \left(z\,\mathrm{e}^{-\frac{1}{z}}\right)\left(\frac{c_1}{z^2}\right)\\ \mathrm{d}\left(uz\,\mathrm{e}^{-\frac{1}{z}}\right) &= \left(\frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\right)\mathrm{d}z\end{aligned}$$

Integrating gives

$$\begin{aligned}uz\,\mathrm{e}^{-\frac{1}{z}} &= \int \frac{c_1\,\mathrm{e}^{-\frac{1}{z}}}{z}\,dz\\ &= c_1\,\mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2\end{aligned}$$

Dividing throughout by the integrating factor $z\,\mathrm{e}^{-\frac{1}{z}}$ gives the final solution

$$u = \frac{\mathrm{e}^{\frac{1}{z}}\left(c_1\,\mathrm{Ei}_1\left(\frac{1}{z}\right) + c_2\right)}{z}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.321 (sec)

Solve

$$z^2u'' + (3z+1)\,u' + u = 0$$

Writing the ode as

$$z^2u'' + (3z+1)\,u' + u = 0 \tag{1}$$

$$Au'' + Bu' + Cu = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= z^2 \\ B &= 3z + 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(z) = ue^{\int \frac{B}{2A}\, dz}$$

Then (2) becomes

$$z''(z) = rz(z) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-z^2 + 2z + 1}{4z^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -z^2 + 2z + 1 \\ t &= 4z^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(z) = \left(\frac{-z^2 + 2z + 1}{4z^4} \right) z(z) \tag{7}$$

Equation (7) is now solved. After finding $z(z)$ then u is found using the inverse transformation

$$u = z(z)\, e^{-\int \frac{B}{2A}\, dz}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.47: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4z^4$. There is a pole at $z = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(z-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(z-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(z-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(z-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(z-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{2z^3} + \frac{1}{4z^4} - \frac{1}{4z^2}$$

There is pole in r at $z = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{2} + \frac{z}{2} - \frac{3z^2}{4} + \frac{5z^3}{4} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{2z^2} \tag{3V}$$

The above shows that the coefficient of $\frac{1}{(z-0)^2}$ is

$$a = \frac{1}{2}$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(z-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{z^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be $\frac{1}{2}$. Therefore

$$\begin{aligned} b &= \left(\frac{1}{2}\right) - (0) \\ &= \frac{1}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{2z^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{\frac{1}{2}}{\frac{1}{2}} + 2\right) &&= \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{\frac{1}{2}}{\frac{1}{2}} + 2\right) &&= \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{z^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{-z^2 + 2z + 1}{4z^4}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-z^2 + 2z + 1}{4z^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{2z^2}$	$\frac{3}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{z-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{z - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2z^2} + \frac{1}{2z} + (-)(0) \\ &= -\frac{1}{2z^2} + \frac{1}{2z} \\ &= \frac{z-1}{2z^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(z)$ of degree $d = 0$ to solve the ode. The polynomial $p(z)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(z) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2z^2} + \frac{1}{2z}\right)(0) + \left(\left(\frac{1}{z^3} - \frac{1}{2z^2}\right) + \left(-\frac{1}{2z^2} + \frac{1}{2z}\right)^2 - \left(\frac{-z^2 + 2z + 1}{4z^4}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(z) &= pe^{\int \omega \, dz} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2z^2} + \frac{1}{2z}\right) dz} \\ &= \sqrt{z}\, \mathrm{e}^{\frac{1}{2z}} \end{aligned}$$

The first solution to the original ode in u is found from

$$\begin{aligned} u_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dz} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3z+1}{z^2} \, dz} \\ &= z_1 e^{\frac{1}{2z} - \frac{3\ln(z)}{2}} \\ &= z_1 \left(\frac{\mathrm{e}^{\frac{1}{2z}}}{z^{3/2}}\right) \end{aligned}$$

Which simplifies to

$$u_1 = \frac{\mathrm{e}^{\frac{1}{z}}}{z}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A} \, dz}}{u_1^2} \, dz$$

Substituting gives

$$\begin{aligned} u_2 &= u_1 \int \frac{e^{\int -\frac{3z+1}{z^2} \, dz}}{(u_1)^2} \, dz \\ &= u_1 \int \frac{e^{\frac{1}{z} - 3\ln(z)}}{(u_1)^2} \, dz \\ &= u_1 \left(\mathrm{e}^{-3\ln\left(\frac{1}{z}\right) - 3\ln(z)} \operatorname{Ei}_1\left(\frac{1}{z}\right)\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\frac{e^{\frac{1}{z}}}{z}\right) + c_2 \left(\frac{e^{\frac{1}{z}}}{z} \left(e^{-3\ln\left(\frac{1}{z}\right)-3\ln(z)} \operatorname{Ei}_1\left(\frac{1}{z}\right)\right)\right)
\end{aligned}
$$

Summary of solutions found

$$
u = \frac{c_1 e^{\frac{1}{z}}}{z} + \frac{c_2 e^{\frac{1}{z}} \operatorname{Ei}_1\left(\frac{1}{z}\right)}{z}
$$

✓ **Maple.** Time used: 0.000 (sec). Leaf size: 21

```
ode:=z^2*diff(diff(u(z),z),z)+(3*z+1)*diff(u(z),z)+u(z) = 0;
dsolve(ode,u(z), singsol=all);
```

$$
u = \frac{\left(c_1 \operatorname{Ei}_1\left(\frac{1}{z}\right) + c_2\right) e^{\frac{1}{z}}}{z}
$$

Maple trace

```
Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful
```

✓ **Mathematica.** Time used: 0.051 (sec). Leaf size: 27

```
ode=z^2*D[u[z],{z,2}]+(3*z+1)*D[u[z],z]+u[z]==0;
ic={};
DSolve[{ode,ic},u[z],z,IncludeSingularSolutions->True]
```

$$
u(z) \rightarrow \frac{e^{\frac{1}{z}}\left(c_1 - c_2 \operatorname{ExpIntegralEi}\left(-\frac{1}{z}\right)\right)}{z}
$$

✗ Sympy

```
from sympy import *
z = symbols("z")
u = Function("u")
ode = Eq(z**2*Derivative(u(z), (z, 2)) + (3*z + 1)*Derivative(u(z), z) + u(z),0)
ics = {}
dsolve(ode,func=u(z),ics=ics)
```

```
NotImplementedError : The given ODE Derivative(u(z), z) - (-z**2*Derivative(u(z),
    (z, 2)) - u(z))/(3*z + 1) cannot be solved by the factorable group method
```