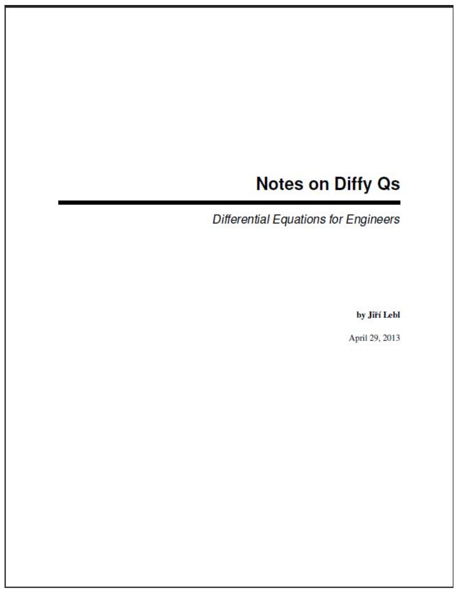
A Solution Manual For

Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.



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1.1 problem 7.2.1

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Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (2) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') &\quad (4)\\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f &\quad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 &\quad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \quad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y(t)}F_0 \\
&= -\frac{d}{dt}y(t) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y(t)}F_1 \\
&= y(t) \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y(t)}F_2 \\
&= \frac{d}{dt}y(t) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y(t)}F_3 \\
&= -y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6\right)y(0) + \left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5\right)y'(0) + O(t^6)
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = -\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^n\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{3}$$

For $0 \le n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{1}{2} a_0 t^2 - \frac{1}{6} a_1 t^3 + \frac{1}{24} a_0 t^4 + \frac{1}{120} a_1 t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2} t^2 + \frac{1}{24} t^4\right) a_0 + \left(t - \frac{1}{6} t^3 + \frac{1}{120} t^5\right) a_1 + O(t^6) \tag{3}$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2} t^2 + \frac{1}{24} t^4\right) c_1 + \left(t - \frac{1}{6} t^3 + \frac{1}{120} t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$using $t = x - 1$ results in

$$\begin{aligned} y = &\left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{24} - \frac{(x-1)^6}{720}\right) y(1) \\ &+ \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^5}{120}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1-\frac{(x-1)^{2}}{2}+\frac{(x-1)^{4}}{24}-\frac{(x-1)^{6}}{720}\right) y(1) \\
& + \left(x-1-\frac{(x-1)^{3}}{6}+\frac{(x-1)^{5}}{120}\right) y'(1)+O\left((x-1)^{6}\right)
\end{aligned} \tag{1}
$$

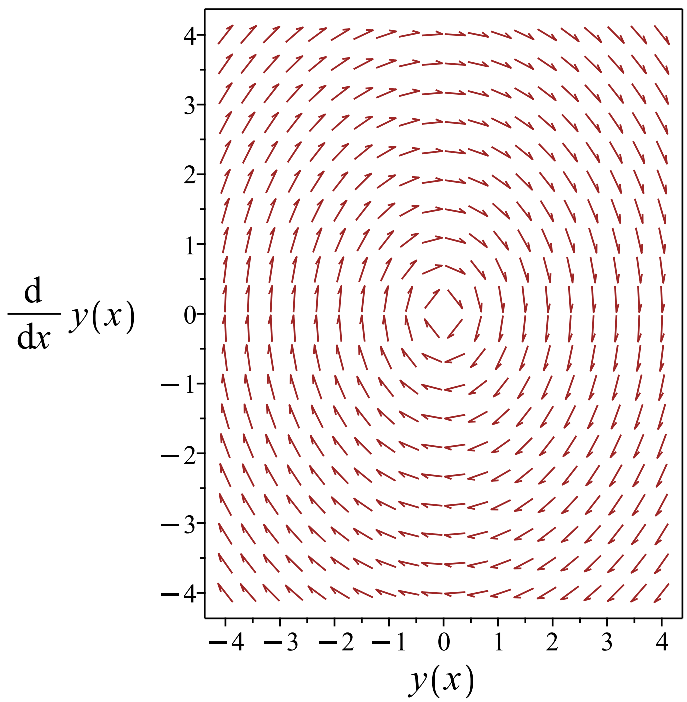

Figure 1: Slope field plot

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{(x-1)^{2}}{2}+\frac{(x-1)^{4}}{24}-\frac{(x-1)^{6}}{720}\right) y(1) \\
& + \left(x-1-\frac{(x-1)^{3}}{6}+\frac{(x-1)^{5}}{120}\right) y'(1)+O\left((x-1)^{6}\right)
\end{aligned}
$$

Verified OK.

1.1.1 Maple step by step solution

Let's solve

$y'' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^4}{24}\right) y(1) + \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^5}{120}\right) D(y)(1) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 51

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{1}{24}(x-1)^4 - \frac{1}{2}(x-1)^2 + 1\right) + c_2\left(\frac{1}{120}(x-1)^5 - \frac{1}{6}(x-1)^3 + x - 1\right)$$

1.2 problem 7.2.2

Internal problem ID [5504]
Internal file name [`OUTPUT/4752_Sunday_June_05_2022_03_04_58_PM_27593360/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + 4xy = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (4)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}F_n|_{x_0,y_0,y'_0} \qquad (7)$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= -4xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -4y - 4xy' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -8y' + 16yx^2 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16x(xy' + 4y) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -64yx^3 + 96xy' + 64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -4y(0) \\
F_2 &= -8y'(0) \\
F_3 &= 0 \\
F_4 &= 64y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{2}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O(x^6)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -4x\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 4x^{1+n} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\ \sum_{n=0}^{\infty} 4x^{1+n} a_n &= \sum_{n=1}^{\infty} 4a_{n-1} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=1}^{\infty} 4a_{n-1} x^n\right)=0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n)+4a_{n-1}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4a_{n-1}}{(n+2)(1+n)} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{2a_1}{63}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{2}{3} a_0 x^3 - \frac{1}{3} a_1 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{2x^3}{3}\right) a_0 + \left(x - \frac{1}{3}x^4\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{2x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{2}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{2x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{2}{3}x^3 + \frac{4}{45}x^6\right) y(0) + \left(x - \frac{1}{3}x^4\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{2x^3}{3}\right) c_1 + \left(x - \frac{1}{3}x^4\right) c_2 + O(x^6)$$

Verified OK.

1.2.1 Maple step by step solution

Let's solve

$y'' = -4xy$

- Highest derivative means the order of the ODE is 2

y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + 4xy = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

□ Rewrite ODE with series expansions

  ∘ Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$$

  ∘ Shift index using $k- >k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$$

  ∘ Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

  ∘ Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 4a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

$2a_2 = 0$

- Each term in the series must be 0, giving the recursion relation

$(k^2 + 3k + 2)\, a_{k+2} + 4a_{k-1} = 0$

- Shift index using $k- >k+1$

$$\left((k+1)^2+3k+5\right)a_{k+3}+4a_k=0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=-\frac{4a_k}{k^2+5k+6}, 2a_2=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+4*x*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=\left(1-\frac{2x^3}{3}\right)y(0)+\left(x-\frac{1}{3}x^4\right)D(y)(0)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+4*x*y[x]==0,y[x],{x,0,5}]
```

$$y(x)\to c_2\left(x-\frac{x^4}{3}\right)+c_1\left(1-\frac{2x^3}{3}\right)$$

1.3 problem 7.2.3

Internal problem ID [5505]
Internal file name [`OUTPUT/4753_Sunday_June_05_2022_03_04_59_PM_45143449/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' - xy = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$\frac{d^2}{dt^2}y(t) - (t + 1)\, y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x,y,y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\,y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (7)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (8)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3)\\
&\;\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= (t+1)\,y(t) \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_0}{\partial \frac{d}{dt}y\,(t)}F_0 \\
&= y(t) + (t+1)\left(\frac{d}{dt}y(t)\right) \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_1}{\partial \frac{d}{dt}y\,(t)}F_1 \\
&= 2\frac{d}{dt}y(t) + y(t)\,(t+1)^2 \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_2}{\partial \frac{d}{dt}y\,(t)}F_2 \\
&= (t+1)\left((t+1)\left(\frac{d}{dt}y(t)\right) + 4y(t)\right) \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y}\frac{d}{dt}y(t) + \frac{\partial F_3}{\partial \frac{d}{dt}y\,(t)}F_3 \\
&= (t+1)^3\,y(t) + (6t+6)\left(\frac{d}{dt}y(t)\right) + 4y(t)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y(0) + y'(0) \\
F_2 &= 2y'(0) + y(0) \\
F_3 &= y'(0) + 4y(0) \\
F_4 &= 5y(0) + 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(t) = &\left(1+\frac{1}{2}t^{2}+\frac{1}{6}t^{3}+\frac{1}{24}t^{4}+\frac{1}{30}t^{5}+\frac{1}{144}t^{6}\right) y(0) \\
&+\left(t+\frac{1}{6}t^{3}+\frac{1}{12}t^{4}+\frac{1}{120}t^{5}+\frac{1}{120}t^{6}\right) y'(0)+O\left(t^{6}\right)
\end{aligned}
$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y(t) = \sum_{n=0}^{\infty} a_n t^n
$$

Then

$$
\begin{aligned}
\frac{d}{dt}y(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} \\
\frac{d^2}{dt^2}y(t) &= \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = (t+1)\left(\sum_{n=0}^{\infty} a_n t^n\right) \tag{1}
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n t^n\right) = 0 \tag{2}
$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^n \\
\sum_{n=0}^{\infty} \left(-t^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} t^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, t^{n}\right) + \sum_{n=1}^{\infty} (-a_{n-1} t^{n}) + \sum_{n=0}^{\infty} (-a_{n} t^{n}) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(1+n) - a_{n-1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{a_{n-1} + a_n}{(n+2)\,(1+n)} \\ &= \frac{a_n}{(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6} + \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{30} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{144} + \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_4 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_1}{5040} + \frac{a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t + \frac{a_0 t^2}{2} + \left(\frac{a_0}{6} + \frac{a_1}{6}\right) t^3 + \left(\frac{a_1}{12} + \frac{a_0}{24}\right) t^4 + \left(\frac{a_0}{30} + \frac{a_1}{120}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{30}t^5\right) a_0 + \left(t + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \frac{1}{120}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{30}t^5\right) c_1 + \left(t + \frac{1}{6}t^3 + \frac{1}{12}t^4 + \frac{1}{120}t^5\right) c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x-1$ results in

$$\begin{aligned} y = &\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \frac{(x-1)^6}{144}\right) y(1) \\ &+ \left(x - 1 + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \frac{(x-1)^6}{120}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = &\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \frac{(x-1)^6}{144}\right) y(1) \\ &+ \left(x - 1 + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \frac{(x-1)^6}{120}\right) y'(1) + O\left((x-1)^6\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = &\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \frac{(x-1)^6}{144}\right) y(1) \\ &+ \left(x - 1 + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \frac{(x-1)^6}{120}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Verified OK.

1.3.1 Maple step by step solution

Let's solve

$y'' - xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y$ to series expansion

  $$x \cdot y = \sum_{k=0}^{\infty} a_k x^{k+1}$$

- Shift index using $k- >k-1$

  $$x \cdot y = \sum_{k=1}^{\infty} a_{k-1} x^k$$

- Convert y'' to series expansion

  $$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_{k-1}\right) x^k\right) = 0$$

- Each term must be 0

  $$2a_2 = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$\left(k^2+3k+2\right) a_{k+2} - a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$\left((k+1)^2 + 3k + 5\right) a_{k+3} - a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{k^2+5k+6}, 2a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(diff(y(x),x$2)-x*y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1+\frac{(x-1)^{2}}{2}+\frac{(x-1)^{3}}{6}+\frac{(x-1)^{4}}{24}+\frac{(x-1)^{5}}{30}\right) y(1) + \left(x-1+\frac{(x-1)^{3}}{6}+\frac{(x-1)^{4}}{12}+\frac{(x-1)^{5}}{120}\right) D(y)(1)+O(x^{6})$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[y''[x]-x*y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{1}{30}(x-1)^5+\frac{1}{24}(x-1)^4+\frac{1}{6}(x-1)^3+\frac{1}{2}(x-1)^2+1\right) + c_2\left(\frac{1}{120}(x-1)^5+\frac{1}{12}(x-1)^4+\frac{1}{6}(x-1)^3+x-1\right)$$

1.4 problem 7.2.4

Internal problem ID [5506]
Internal file name [`OUTPUT/4754_Sunday_June_05_2022_03_05_00_PM_32450824/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + y x^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (10)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (11)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -x(xy' + 2y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= yx^4 - 4xy' - 2y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y'x^4 + 8yx^3 - 6y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 12y'x^3 - x^2 y\left(x^4 - 30\right)
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= -2y(0) \\
F_3 &= -6y'(0) \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=0}^{\infty} x^{n+2} a_n &= \sum_{n=2}^{\infty} a_{n-2} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_{n-2} = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_{n-2}}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{12}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{20}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{12} a_0 x^4 - \frac{1}{20} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^4}{12}\right) a_0 + \left(x - \frac{1}{20} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20} x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^4}{12}\right) c_1 + \left(x - \frac{1}{20} x^5\right) c_2 + O(x^6)$$

Verified OK.

1.4.1 Maple step by step solution

Let's solve

$y'' = -y x^2$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + yx^2 = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- ○ Convert $x^2 \cdot y$ to series expansion

$$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+2}$$

- ○ Shift index using $k- >k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^k$$

- ○ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- ○ Shift index using $k- >k + 2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$6a_3 x + 2a_2 + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + a_{k-2} = 0$$

- Shift index using $k- >k + 2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{a_k}{k^2+7k+12}, a_2 = 0, a_3 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
Order:=6;
dsolve(diff(y(x),x$2)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^4}{12}\right) y(0) + \left(x - \frac{1}{20}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 28

```
AsymptoticDSolveValue[y''[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(x - \frac{x^5}{20}\right) + c_1\left(1 - \frac{x^4}{12}\right)$$

1.5 problem 7.2.5

Internal problem ID [5507]
Internal file name [`OUTPUT/4755_Sunday_June_05_2022_03_05_01_PM_36928345/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

1.5.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + xf + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0,y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0,y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f \tag{1}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$F_0 = f(x, y) \tag{4}$$

$$F_n = \frac{d}{dx}(F_{n-1})$$

$$= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 \tag{5}$$

For example, for $n = 1$ we see that

$$F_1 = \frac{d}{dx}(F_0)$$

$$= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

Which is (1). And when $n = 2$

$$F_2 = \frac{d}{dx}(F_1)$$

$$= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= xy \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(x^2+1\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= xy\left(x^2+3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= y\left(x^4+6x^2+3\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= xy\left(x^4+10x^2+15\right)
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 0 \\
F_3 &= 3y(0) \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (6) and simplifying gives the solution as

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O\left(x^6\right)$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - xy = 0$$

Where

$$q(x) = -x$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (1+n) a_{1+n} x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (1+n) a_{1+n} x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = 0 \tag{3}$$

For $1 \leq n$, the recurrence equation is

$$(1+n) a_{1+n} - a_{n-1} = 0 \tag{4}$$

Solving for a_{1+n}, gives

$$a_{1+n} = \frac{a_{n-1}}{1+n} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 2$ the recurrence equation gives

$$3a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = 0$$

For $n = 3$ the recurrence equation gives

$$4a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 4$ the recurrence equation gives

$$5a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{48}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + \frac{1}{2} a_0 x^2 + \frac{1}{8} a_0 x^4 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + O(x^6) \tag{2}$$

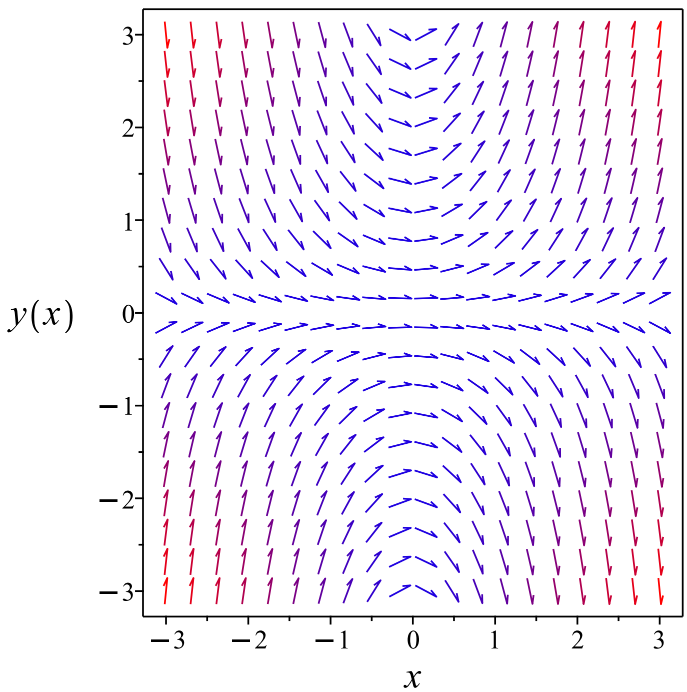

Figure 2: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + O(x^6)$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$y' - xy = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int x dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{x^2}{2} + c_1$

- Solve for y

  $y = e^{\frac{x^2}{2} + c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
Order:=6;
dsolve(diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 22

```
AsymptoticDSolveValue[y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^4}{8} + \frac{x^2}{2} + 1 \right)$$

1.6 problem 7.2.6

Internal problem ID [5508]
Internal file name [`OUTPUT/4756_Sunday_June_05_2022_03_05_02_PM_29033608/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[_Gegenbauer, [_2nd_order, _linear, `_with_symmetry_[0,F(x)]`]]
```

$$\boxed{\left(-x^{2}+1\right) y''-x y'+p^{2} y=0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (14)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (15)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{p^2 y - x y'}{x^2 - 1}\\
F_1 &= \frac{dF_0}{dx}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
&= \frac{\left(\left(p^2+2\right)x^2 - p^2 + 1\right)y' - 3 y p^2 x}{\left(x^2-1\right)^2}\\
F_2 &= \frac{dF_1}{dx}\\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1\\
&= \frac{\left(-6p^2x^3 + 6x\,p^2 - 6x^3 - 9x\right)y' + \left(\left(p^2+11\right)x^2 - p^2 + 4\right) y p^2}{\left(x^2-1\right)^3}\\
F_3 &= \frac{dF_2}{dx}\\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2\\
&= \frac{\left(\left(p^4+35p^2+24\right)x^4 + \left(-2p^4 - 25p^2 + 72\right)x^2 + p^4 - 10p^2 + 9\right)y' - 10 y x\left(\left(p^2+5\right)x^2 - p^2 + \frac{11}{2}\right)p^2}{\left(x^2-1\right)^4}\\
F_4 &= \frac{dF_3}{dx}\\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3\\
&= \frac{(1+x)\left(-15x\left(\left(p^4+15p^2+8\right)x^4 + \left(-2p^4 - 2p^2 + 40\right)x^2 + p^4 - 13p^2 + 15\right)y' + y\left(\left(p^4+85p^2+274\right.\right.\right.}{\left(x^2-1\right)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0)\,p^2\\
F_1 &= -y'(0)\,p^2 + y'(0)\\
F_2 &= y(0)\,p^4 - 4y(0)\,p^2\\
F_3 &= y'(0)\,p^4 - 10y'(0)\,p^2 + 9y'(0)\\
F_4 &= -y(0)\,p^6 + 20y(0)\,p^4 - 64y(0)\,p^2
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = & \left(1-\frac{1}{2} x^{2} p^{2}+\frac{1}{24} p^{4} x^{4}-\frac{1}{6} p^{2} x^{4}-\frac{1}{720} x^{6} p^{6}+\frac{1}{36} x^{6} p^{4}-\frac{4}{45} x^{6} p^{2}\right) y(0) \\
& +\left(x-\frac{1}{6} p^{2} x^{3}+\frac{1}{6} x^{3}+\frac{1}{120} x^{5} p^{4}-\frac{1}{12} x^{5} p^{2}+\frac{3}{40} x^{5}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(-x^{2}+1\right) y''-x y'+p^{2} y=0
$$

Let the solution be represented as power series of the form

$$
y=\sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right)-x\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+p^{2}\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0 \quad (1)
$$

Which simplifies to

$$
\sum_{n=2}^{\infty}\left(-x^{n} a_{n} n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}\right)+\sum_{n=1}^{\infty}\left(-n a_{n} x^{n}\right)+\left(\sum_{n=0}^{\infty} p^{2} a_{n} x^{n}\right)=0 \quad (2)
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty} n(n-1) a_{n} x^{n-2}=\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^{n}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} \left(-n a_n x^n\right) + \left(\sum_{n=0}^{\infty} p^2 a_n x^n\right) = 0 \tag{3}$$

$n = 0$ gives

$$a_0 p^2 + 2a_2 = 0$$

$$a_2 = -\frac{a_0 p^2}{2}$$

$n = 1$ gives

$$a_1 p^2 - a_1 + 6a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{1}{6} a_1 p^2 + \frac{1}{6} a_1$$

For $2 \leq n$, the recurrence equation is

$$-n a_n(n-1) + (n+2)\, a_{n+2}(n+1) - n a_n + a_n p^2 = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n(n^2 - p^2)}{(n+2)\,(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$a_2 p^2 - 4a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24} p^4 a_0 - \frac{1}{6} a_0 p^2$$

For $n = 3$ the recurrence equation gives

$$a_3 p^2 - 9a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{1}{120} p^4 a_1 - \frac{1}{12} a_1 p^2 + \frac{3}{40} a_1$$

For $n = 4$ the recurrence equation gives

$$a_4 p^2 - 16a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{1}{720} p^6 a_0 + \frac{1}{36} p^4 a_0 - \frac{4}{45} a_0 p^2$$

For $n = 5$ the recurrence equation gives

$$a_5 p^2 - 25a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{1}{5040} p^6 a_1 + \frac{1}{144} p^4 a_1 - \frac{37}{720} a_1 p^2 + \frac{5}{112} a_1$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x - \frac{a_0 p^2 x^2}{2} + \left(-\frac{1}{6} a_1 p^2 + \frac{1}{6} a_1\right) x^3 \\ + \left(\frac{1}{24} p^4 a_0 - \frac{1}{6} a_0 p^2\right) x^4 + \left(\frac{1}{120} p^4 a_1 - \frac{1}{12} a_1 p^2 + \frac{3}{40} a_1\right) x^5 + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y = {} & \left(1-\frac{x^{2} p^{2}}{2}+\left(\frac{1}{24} p^{4}-\frac{1}{6} p^{2}\right) x^{4}\right) a_{0} \\
& +\left(x+\left(-\frac{p^{2}}{6}+\frac{1}{6}\right) x^{3}+\left(\frac{1}{120} p^{4}-\frac{1}{12} p^{2}+\frac{3}{40}\right) x^{5}\right) a_{1}+O\left(x^{6}\right)
\end{aligned} \tag{3}
$$

At $x = 0$ the solution above becomes

$$
\begin{aligned}
y = {} & \left(1-\frac{x^{2} p^{2}}{2}+\left(\frac{1}{24} p^{4}-\frac{1}{6} p^{2}\right) x^{4}\right) c_{1} \\
& +\left(x+\left(-\frac{p^{2}}{6}+\frac{1}{6}\right) x^{3}+\left(\frac{1}{120} p^{4}-\frac{1}{12} p^{2}+\frac{3}{40}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2} x^{2} p^{2}+\frac{1}{24} p^{4} x^{4}-\frac{1}{6} p^{2} x^{4}-\frac{1}{720} x^{6} p^{6}+\frac{1}{36} x^{6} p^{4}-\frac{4}{45} x^{6} p^{2}\right) y(0) \\
& +\left(x-\frac{1}{6} p^{2} x^{3}+\frac{1}{6} x^{3}+\frac{1}{120} x^{5} p^{4}-\frac{1}{12} x^{5} p^{2}+\frac{3}{40} x^{5}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
y = {} & \left(1-\frac{x^{2} p^{2}}{2}+\left(\frac{1}{24} p^{4}-\frac{1}{6} p^{2}\right) x^{4}\right) c_{1} \\
& +\left(x+\left(-\frac{p^{2}}{6}+\frac{1}{6}\right) x^{3}+\left(\frac{1}{120} p^{4}-\frac{1}{12} p^{2}+\frac{3}{40}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{aligned} \tag{2}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & \left(1-\frac{1}{2} x^{2} p^{2}+\frac{1}{24} p^{4} x^{4}-\frac{1}{6} p^{2} x^{4}-\frac{1}{720} x^{6} p^{6}+\frac{1}{36} x^{6} p^{4}-\frac{4}{45} x^{6} p^{2}\right) y(0) \\
& +\left(x-\frac{1}{6} p^{2} x^{3}+\frac{1}{6} x^{3}+\frac{1}{120} x^{5} p^{4}-\frac{1}{12} x^{5} p^{2}+\frac{3}{40} x^{5}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
\begin{aligned}
y = {} & \left(1-\frac{x^{2} p^{2}}{2}+\left(\frac{1}{24} p^{4}-\frac{1}{6} p^{2}\right) x^{4}\right) c_{1} \\
& +\left(x+\left(-\frac{p^{2}}{6}+\frac{1}{6}\right) x^{3}+\left(\frac{1}{120} p^{4}-\frac{1}{12} p^{2}+\frac{3}{40}\right) x^{5}\right) c_{2}+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

1.6.1 Maple step by step solution

Let's solve

$(-x^2+1)\,y''-xy'+p^2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{xy'}{x^2-1}+\frac{p^2y}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{xy'}{x^2-1}-\frac{p^2y}{x^2-1}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{x}{x^2-1}, P_3(x)=-\frac{p^2}{x^2-1}\right]$

  ◦ $(1+x)\cdot P_2(x)$ is analytic at $x=-1$

  $\left((1+x)\cdot P_2(x)\right)\Big|_{x=-1}=\frac{1}{2}$

  ◦ $(1+x)^2\cdot P_3(x)$ is analytic at $x=-1$

  $\left((1+x)^2\cdot P_3(x)\right)\Big|_{x=-1}=0$

  ◦ $x=-1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-1$

- Multiply by denominators

  $(x^2-1)\,y''+xy'-p^2y=0$

- Change variables using $x=u-1$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(u-1)\left(\frac{d}{du}y(u)\right)-p^2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

  $$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

  $$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,u^{k+r}$$

  Rewrite ODE with series expansions

  $$-a_0r(-1+2r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+1+r)\,(2k+1+2r) + a_k(k+p+r)\,(k-p+r)\right) u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$-r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

  $$-2\left(k+\frac{1}{2}+r\right)(k+1+r)\,a_{k+1} + a_k(k+p+r)\,(k-p+r) = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+1} = \frac{a_k(k+p+r)(k-p+r)}{(2k+1+2r)(k+1+r)}$$

- Recursion relation for $r = 0$

  $$a_{k+1} = \frac{a_k(k+p)(k-p)}{(2k+1)(k+1)}$$

- Solution for $r = 0$

  $$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+1} = \frac{a_k(k+p)(k-p)}{(2k+1)(k+1)}\right]$$

- Revert the change of variables $u = 1+x$

  $$\left[y = \sum_{k=0}^{\infty} a_k(1+x)^k, a_{k+1} = \frac{a_k(k+p)(k-p)}{(2k+1)(k+1)}\right]$$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = \frac{a_k \left(k+p+\frac{1}{2}\right) \left(k-p+\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}$

- Solution for $r = \frac{1}{2}$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+p+\frac{1}{2}\right) \left(k-p+\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Revert the change of variables $u = 1 + x$

  $\left[y = \sum_{k=0}^{\infty} a_k (1+x)^{k+\frac{1}{2}}, a_{k+1} = \frac{a_k \left(k+p+\frac{1}{2}\right) \left(k-p+\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k (1+x)^k\right) + \left(\sum_{k=0}^{\infty} b_k (1+x)^{k+\frac{1}{2}}\right), a_{k+1} = \frac{a_k (k+p)(k-p)}{(2k+1)(k+1)}, b_{k+1} = \frac{b_k \left(k+p+\frac{1}{2}\right) \left(k-p+\frac{1}{2}\right)}{(2k+2)\left(k+\frac{3}{2}\right)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 71

```
Order:=6;
dsolve((1-x^2)*diff(y(x),x$2)-x*diff(y(x),x)+p^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{p^2 x^2}{2} + \frac{p^2 (p^2 - 4) x^4}{24}\right) y(0) + \left(x - \frac{(p^2 - 1) x^3}{6} + \frac{(p^4 - 10p^2 + 9) x^5}{120}\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 88

```
AsymptoticDSolveValue[(1-x^2)*y''[x]-x*y'[x]+p^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{p^4x^5}{120} - \frac{p^2x^5}{12} - \frac{p^2x^3}{6} + \frac{3x^5}{40} + \frac{x^3}{6} + x\right) + c_1\left(\frac{p^4x^4}{24} - \frac{p^2x^4}{6} - \frac{p^2x^2}{2} + 1\right)$$

1.7 problem 7.2.7

Internal problem ID [5509]
Internal file name [`OUTPUT/4757_Sunday_June_05_2022_03_05_03_PM_97615417/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\, y''-2xy'+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\, y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{17}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{18}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n\big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$\begin{aligned}
F_0 &= \frac{2xy' - 2y}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 0 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(-x^2 + 1\right) y(0) + xy'(0) + O\left(x^6\right)$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$\left(x^2 + 1\right) y'' - 2xy' + 2y = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - 2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (2)$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=1}^{\infty} (-2n a_n x^n) + \left(\sum_{n=0}^{\infty} 2a_n x^n\right) = 0 \quad (3)$$

$n = 0$ gives

$$2a_2 + 2a_0 = 0$$

$$a_2 = -a_0$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2) a_{n+2}(n+1) - 2n a_n + 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - 3n + 2)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$2a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$6a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$12a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = -a_0 x^2 + a_1 x + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (-x^2 + 1)\, a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = (-x^2 + 1)\, c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = (-x^2 + 1)\, y(0) + x y'(0) + O(x^6) \tag{1}$$

$$y = (-x^2 + 1)\, c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = (-x^2 + 1)\, y(0) + x y'(0) + O(x^6)$$

Verified OK.

$$y = (-x^2 + 1)\, c_1 + c_2 x + O(x^6)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve((1+x^2)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x - x^2 y(0)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 18

```
AsymptoticDSolveValue[(1+x^2)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(1 - x^2) + c_2 x$$

1.8 problem 7.2.8 part(a)

Internal problem ID [5510]
Internal file name [`OUTPUT/4758_Sunday_June_05_2022_03_05_04_PM_60342664/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.8 part(a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{(x^2+1)\,y''+y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\ &= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\ &= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (20)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (21)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx} \nonumber\\
&= \frac{dF_0}{dx} \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \nonumber\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \nonumber\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \nonumber\\
&= \frac{d}{dx}(F_1) \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \nonumber\\
&\vdots \nonumber\\
F_n &= \frac{d}{dx}(F_{n-1}) \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \nonumber\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y}{x^2+1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{-x^2 y' + 2xy - y'}{(x^2+1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{4y' x^3 - 5yx^2 + 4xy' + 3y}{(x^2+1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(-17x^4 - 10x^2 + 7)\, y' + 16yx(x^2 - 2)}{(x^2+1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(84x^5 - 24x^3 - 108x)\, y' + (-63x^4 + 282x^2 - 39)\, y}{(x^2+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= 3y(0) \\
F_3 &= 7y'(0) \\
F_4 &= -39y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(x^2+1)\,y''+y=0$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\,a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty} (n+2)\,a_{n+2}(n+1)\,x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

$n = 1$ gives

$$6a_3 + a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_1}{6}$$

For $2 \leq n$, the recurrence equation is

$$n a_n(n-1) + (n+2) a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n(n^2 - n + 1)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3a_2 + 12a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$7a_3 + 20a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$13a_4 + 30a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{13a_0}{240}$$

For $n = 5$ the recurrence equation gives

$$21a_5 + 42a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{7a_1}{240}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{8} a_0 x^4 + \frac{7}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   <- hypergeometric successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 34

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[(x^2+1)*y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{8} - \frac{x^2}{2} + 1\right)$$

1.9 problem 7.2.8 part(b)

Internal problem ID [5511]
Internal file name [`OUTPUT/4759_Sunday_June_05_2022_03_05_05_PM_12112889/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.8 part(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{xy'' + y = 0}$$

With the expansion point for the power series method at $x = 1$.

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$t = x - 1$$

The ode is converted to be in terms of the new independent variable t. This results in

$$(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + y(t) = 0$$

With its expansion point and initial conditions now at $t = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (23) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (24) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{y(t)}{t+1} \\
F_1 &= \frac{dF_0}{dt} \\
&= \frac{\partial F_0}{\partial t} + \frac{\partial F_0}{\partial y} \frac{d}{dt} y(t) + \frac{\partial F_0}{\partial \frac{d}{dt} y(t)} F_0 \\
&= \frac{(-t-1)\left(\frac{d}{dt} y(t)\right) + y(t)}{(t+1)^2} \\
F_2 &= \frac{dF_1}{dt} \\
&= \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial y} \frac{d}{dt} y(t) + \frac{\partial F_1}{\partial \frac{d}{dt} y(t)} F_1 \\
&= \frac{(2t+2)\left(\frac{d}{dt} y(t)\right) + y(t)(t-1)}{(t+1)^3} \\
F_3 &= \frac{dF_2}{dt} \\
&= \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial y} \frac{d}{dt} y(t) + \frac{\partial F_2}{\partial \frac{d}{dt} y(t)} F_2 \\
&= \frac{(t^2-4t-5)\left(\frac{d}{dt} y(t)\right) + (-4t+2)\, y(t)}{(t+1)^4} \\
F_4 &= \frac{dF_3}{dt} \\
&= \frac{\partial F_3}{\partial t} + \frac{\partial F_3}{\partial y} \frac{d}{dt} y(t) + \frac{\partial F_3}{\partial \frac{d}{dt} y(t)} F_3 \\
&= \frac{(-6t^2+12t+18)\left(\frac{d}{dt} y(t)\right) - y(t)(t^2-16t+7)}{(t+1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $t = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= y(0) - y'(0) \\
F_2 &= 2y'(0) - y(0) \\
F_3 &= 2y(0) - 5y'(0) \\
F_4 &= -7y(0) + 18y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5 - \frac{7}{720}t^6\right) y(0)$$
$$+ \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5 + \frac{1}{40}t^6\right) y'(0) + O(t^6)$$

Since the expansion point $t = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$(t+1)\left(\frac{d^2}{dt^2}y(t)\right) + y(t) = 0$$

Let the solution be represented as power series of the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then

$$\frac{d}{dt}y(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$$
$$\frac{d^2}{dt^2}y(t) = \sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}$$

Substituting the above back into the ode gives

$$(t+1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n t^n\right) = 0 \tag{2}$$

The next step is to make all powers of t be n in each summation term. Going over each summation term above with power of t in it which is not already t^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n\, t^{n-1} a_n (n-1) = \sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^n$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n t^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (n+1)\, t^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of t are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} (n+1)\, a_{n+1} n\, t^{n}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, t^{n}\right) + \left(\sum_{n=0}^{\infty} a_n t^{n}\right) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2}$$

For $1 \le n$, the recurrence equation is

$$(n+1)\, a_{n+1} n + (n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= -\frac{n^2 a_{n+1} + n a_{n+1} + a_n}{(n+2)\,(n+1)} \\ &= -\frac{a_n}{(n+2)\,(n+1)} - \frac{(n^2+n)\, a_{n+1}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$2a_2 + 6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6} - \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$6a_3 + 12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_0}{24} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$12a_4 + 20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{60} - \frac{a_1}{24}$$

For $n = 4$ the recurrence equation gives

$$20a_5 + 30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{7a_0}{720} + \frac{a_1}{40}$$

For $n = 5$ the recurrence equation gives

$$30a_6 + 42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_0}{1680} - \frac{17a_1}{1008}$$

And so on. Therefore the solution is

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n \\ &= a_3 t^3 + a_2 t^2 + a_1 t + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y(t) = a_0 + a_1 t - \frac{a_0 t^2}{2} + \left(\frac{a_0}{6} - \frac{a_1}{6}\right) t^3 + \left(-\frac{a_0}{24} + \frac{a_1}{12}\right) t^4 + \left(\frac{a_0}{60} - \frac{a_1}{24}\right) t^5 + \dots$$

Collecting terms, the solution becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5\right) a_0 + \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5\right) a_1 + O(t^6) \quad (3)$$

At $t = 0$ the solution above becomes

$$y(t) = \left(1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{24}t^4 + \frac{1}{60}t^5\right)c_1 + \left(t - \frac{1}{6}t^3 + \frac{1}{12}t^4 - \frac{1}{24}t^5\right)c_2 + O(t^6)$$

Replacing t in the above with the original independent variable $x$susing $t = x - 1$ results in

$$\begin{aligned} y = {} & \left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right) y(1) \\ & + \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = {} & \left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right) y(1) \\ & + \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right) y'(1) + O\left((x-1)^6\right) \end{aligned} \tag{1}$$

Verification of solutions

$$\begin{aligned} y = {} & \left(1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} - \frac{(x-1)^4}{24} + \frac{(x-1)^5}{60} - \frac{7(x-1)^6}{720}\right) y(1) \\ & + \left(x - 1 - \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} - \frac{(x-1)^5}{24} + \frac{(x-1)^6}{40}\right) y'(1) + O\left((x-1)^6\right) \end{aligned}$$

Verified OK.

1.9.1 Maple step by step solution

Let's solve

$y''x + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x} = 0$

- □ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot y''$ to series expansion

    $x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)(k+r) + a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{(k+1+r)(k+r)}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{(k+1)k}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{(k+1)k}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{a_k}{(k+2)(k+1)}$

- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{a_k}{(k+2)(k+1)}\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{a_k}{(k+1)k}, b_{k+1} = -\frac{b_k}{(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 49

```
Order:=6;
dsolve(x*diff(y(x),x$2)+y(x)=0,y(x),type='series',x=1);
```

$$y(x) = \left(1-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{6}-\frac{(x-1)^4}{24}+\frac{(x-1)^5}{60}\right)y(1) + \left(x-1-\frac{(x-1)^3}{6}+\frac{(x-1)^4}{12}-\frac{(x-1)^5}{24}\right)D(y)(1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 78

```
AsymptoticDSolveValue[x*y''[x]+y[x]==0,y[x],{x,1,5}]
```

$$y(x) \to c_1\left(\frac{1}{60}(x-1)^5-\frac{1}{24}(x-1)^4+\frac{1}{6}(x-1)^3-\frac{1}{2}(x-1)^2+1\right) + c_2\left(-\frac{1}{24}(x-1)^5+\frac{1}{12}(x-1)^4-\frac{1}{6}(x-1)^3+x-1\right)$$

1.10 problem 7.2.101

Internal problem ID [5512]
Internal file name [`OUTPUT/4760_Sunday_June_05_2022_03_05_06_PM_24258052/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.101.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + 2yx^3 = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (26)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (27)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -2yx^3 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2x^2(xy' + 3y) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -12x^2 y' + 4xy\left(x^5 - 3\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 4\left(x^6 - 9x\right) y' + 12y\left(4x^5 - 1\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -8yx^9 + 72y'x^5 + 312yx^4 - 48y'
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= -12y(0) \\
F_4 &= -48y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{x^5}{10}\right) y(0) + \left(x - \frac{1}{15}x^6\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=-2\left(\sum_{n=0}^{\infty} a_n x^n\right) x^3 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} 2x^{n+3} a_n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$
$$\sum_{n=0}^{\infty} 2x^{n+3} a_n=\sum_{n=3}^{\infty} 2a_{n-3} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=3}^{\infty} 2a_{n-3} x^n\right)=0 \tag{3}$$

For $3\leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1)+2a_{n-3}=0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{2a_{n-3}}{(n+2)(n+1)} \tag{5}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 2a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 2a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_1}{15}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 2a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{10} a_0 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{x^5}{10}\right) a_0 + a_1 x + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{x^5}{10}\right) c_1 + c_2 x + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 - \frac{x^5}{10}\right) y(0) + \left(x - \frac{1}{15}x^6\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - \frac{x^5}{10}\right) c_1 + c_2 x + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1 - \frac{x^5}{10}\right) y(0) + \left(x - \frac{1}{15}x^6\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{x^5}{10}\right) c_1 + c_2 x + O(x^6)$$

Verified OK.

1.10.1 Maple step by step solution

Let's solve

$y'' = -2yx^3$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2yx^3 = 0$

- Assume series solution for y

$$y = \sum_{k=0}^{\infty} a_k x^k$$

☐ Rewrite ODE with series expansions

- Convert $x^3 \cdot y$ to series expansion

$$x^3 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+3}$$

- Shift index using $k- >k-3$

$$x^3 \cdot y = \sum_{k=3}^{\infty} a_{k-3} x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$12a_4x^2 + 6a_3x + 2a_2 + \left(\sum_{k=3}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_{k-3}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 = 0, 6a_3 = 0, 12a_4 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = 0, a_3 = 0, a_4 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_{k-3} = 0$$

- Shift index using $k- >k+3$

$$\left((k+3)^2 + 3k + 11\right) a_{k+5} + 2a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+5} = -\frac{2a_k}{k^2+9k+20}, a_2 = 0, a_3 = 0, a_4 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
Order:=6;
dsolve(diff(y(x),x$2)+2*x^3*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{x^5}{10}\right) y(0) + D(y)(0)\, x + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 20

```
AsymptoticDSolveValue[y''[x]+2*x^3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(1 - \frac{x^5}{10}\right) + c_2 x$$

1.11 problem 7.2.102

Internal problem ID [5513]
Internal file name [`OUTPUT/4761_Sunday_June_05_2022_03_05_07_PM_54975383/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.102.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_airy", "second_order_bessel_ode", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - xy = \frac{1}{1-x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 0]$$

With the expansion point for the power series method at $x = 0$.

1.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x \\ F &= \frac{1}{1-x} \end{aligned}$$

Hence the ode is

$$y'' - xy = \frac{1}{1-x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \frac{1}{1-x}$ is

$$\{x < 1 \lor 1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0) y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (29) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (30) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x=0$. Hence

$$
\begin{aligned}
F_0 &= \frac{yx^2 - xy - 1}{x-1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{x(x-1)^2 y' + 1 + (x-1)^2 y}{(x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{2(x-1)^3 y' + x^2(x-1)^3 y - x^3 + 2x^2 - x - 2}{(x-1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{x^2(x-1)^4 y' + 4x(x-1)^4 y - 2x^3 + 7x^2 - 8x + 9}{(x-1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{6x(x-1)^5 y' + (x^3+4)(x-1)^5 y - x^6 + 4x^5 - 6x^4 + 6x^3 - 9x^2 + 10x - 28}{(x-1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=0$ and $y'(0)=0$ gives

$$
\begin{aligned}
F_0 &= 1 \\
F_1 &= 1 \\
F_2 &= 2 \\
F_3 &= 9 \\
F_4 &= 28
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + \frac{7x^6}{180} + O(x^6)
$$

$$
y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + \frac{7x^6}{180} + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 - x\left(\sum_{n=0}^{\infty} a_n x^n\right) - 1}{x-1} \tag{1}
$$

Expanding $-\frac{1}{x-1}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
-\frac{1}{x-1} &= x^5 + x^4 + x^3 + x^2 + x + 1 + \dots \\
&= x^5 + x^4 + x^3 + x^2 + x + 1
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) - x\left(\sum_{n=0}^{\infty} a_n x^n\right) = x^5 + x^4 + x^3 + x^2 + x + 1
$$

Which simplifies to

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = x^5 + x^4 + x^3 + x^2 + x + 1 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n \\
\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)
\end{aligned}
$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$
\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) = x^5 + x^4 + x^3 + x^2 + x + 1 \quad (3)
$$

$n = 0$ gives

$$
\begin{aligned}
(2a_2)\, 1 &= 1 \\
2a_2 &= 1
\end{aligned}
$$

Or

$$
a_2 = \frac{1}{2}
$$

For $1 \leq n$, the recurrence equation is

$$
((n+2)\, a_{n+2}(1+n) - a_{n-1})\, x^n = x^5 + x^4 + x^3 + x^2 + x + 1 \quad (4)
$$

For $n = 1$ the recurrence equation gives

$$
\begin{aligned}
(6a_3 - a_0)\, x &= x \\
6a_3 - a_0 &= 1
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_3 = \frac{1}{6} + \frac{a_0}{6}
$$

For $n = 2$ the recurrence equation gives

$$
\begin{aligned}
(12a_4 - a_1)\, x^2 &= x^2 \\
12a_4 - a_1 &= 1
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{12} + \frac{a_1}{12}$$

For $n = 3$ the recurrence equation gives

$$\begin{aligned}(20a_5 - a_2)\, x^3 &= x^3 \\ 20a_5 - a_2 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{3}{40}$$

For $n = 4$ the recurrence equation gives

$$\begin{aligned}(30a_6 - a_3)\, x^4 &= x^4 \\ 30a_6 - a_3 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{7}{180} + \frac{a_0}{180}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(42a_7 - a_4)\, x^5 &= x^5 \\ 42a_7 - a_4 &= 1\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13}{504} + \frac{a_1}{504}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{x^2}{2} + \left(\frac{1}{6} + \frac{a_0}{6}\right) x^3 + \left(\frac{1}{12} + \frac{a_1}{12}\right) x^4 + \frac{3x^5}{40} + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{x^3}{6}\right) a_0 + \left(x + \frac{1}{12}x^4\right) a_1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{x^3}{6}\right) c_1 + \left(x + \frac{1}{12}x^4\right) c_2 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + O\left(x^6\right)$$

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + O\left(x^6\right)$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + \frac{7x^6}{180} + O\left(x^6\right) \tag{1}$$

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + O\left(x^6\right) \tag{2}$$

Verification of solutions

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + \frac{7x^6}{180} + O\left(x^6\right)$$

Verified OK.

$$y = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{3x^5}{40} + O\left(x^6\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
Order:=6;
dsolve([diff(y(x),x$2)-x*y(x)=1/(1-x),y(0) = 0, D(y)(0) = 0],y(x),type='series',x=0);
```

$$y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 56

```
AsymptoticDSolveValue[{y''[x]-x*y[x]==1/(1-x),{}},y[x],{x,0,5}]
```

$$y(x) \to \frac{3x^5}{40} + \frac{x^4}{12} + c_2\left(\frac{x^4}{12} + x\right) + \frac{x^3}{6} + c_1\left(\frac{x^3}{6} + 1\right) + \frac{x^2}{2}$$

1.12 problem 7.2.103

Internal problem ID [5514]
Internal file name [`OUTPUT/4762_Sunday_June_05_2022_03_05_09_PM_82704742/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.2.1 Exercises. page 290
Problem number: 7.2.103.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{x^2}$$

Table 9: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r} \right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r^2 - r - 1) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$r_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r^2 - r - 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)$$

Verified OK.

1.12.1 Maple step by step solution

Let's solve

$x^2y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm \left(\sqrt{5}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\sqrt{5}}{2}, \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)t}$
- 2nd solution of the ODE

  $y_2(t) = e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 e^{\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\ln(x)} + c_2 e^{\ln(x)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)}$
- Simplify

  $y = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}} c_1 + x^{\frac{\sqrt{5}}{2}} c_2\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}}c_1 + x^{\frac{\sqrt{5}}{2}}c_2\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x^2*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + c_2 x^{\frac{1}{2}\left(1-\sqrt{5}\right)}$$

2 Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300

2.1 problem 7.3.3

Internal problem ID [5515]
Internal file name [`OUTPUT/4763_Sunday_June_05_2022_03_05_10_PM_89001034/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Complex roots"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2y'' + xy' + (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + (1+x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{1+x}{x^2}$$

Table 11: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1+x}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + (1 + x) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + (1 + x) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2+1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2+1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = i$$
$$r_2 = -i$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+i}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-i}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + a_n + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n^2 + 2nr + r^2 + 1} \tag{4}$$

Which for the root $r = i$ becomes

$$a_n = -\frac{a_{n-1}}{n (2i + n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = i$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{r^2 + 2r + 2}$$

Which for the root $r = i$ becomes

$$a_1 = -\frac{1}{5} + \frac{2i}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+2}$	$-\frac{1}{5} + \frac{2i}{5}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r^2 + 2r + 2)(r^2 + 4r + 5)}$$

Which for the root $r = i$ becomes

$$a_2 = -\frac{1}{40} - \frac{3i}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+2}$	$-\frac{1}{5} + \frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40} - \frac{3i}{40}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(r^2 + 2r + 2)(r^2 + 4r + 5)(r^2 + 6r + 10)}$$

Which for the root $r = i$ becomes

$$a_3 = \frac{3}{520} + \frac{7i}{1560}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+2}$	$-\frac{1}{5}+\frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40}-\frac{3i}{40}$
a_3	$-\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$\frac{3}{520}+\frac{7i}{1560}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$$

Which for the root $r = i$ becomes

$$a_4 = -\frac{1}{2496} - \frac{i}{12480}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+2}$	$-\frac{1}{5}+\frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40}-\frac{3i}{40}$
a_3	$-\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$\frac{3}{520}+\frac{7i}{1560}$
a_4	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$	$-\frac{1}{2496}-\frac{i}{12480}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)(r^2+10r+26)}$$

Which for the root $r = i$ becomes

$$a_5 = \frac{9}{603200} - \frac{i}{361920}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r^2+2r+2}$	$-\frac{1}{5}+\frac{2i}{5}$
a_2	$\frac{1}{(r^2+2r+2)(r^2+4r+5)}$	$-\frac{1}{40}-\frac{3i}{40}$
a_3	$-\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)}$	$\frac{3}{520}+\frac{7i}{1560}$
a_4	$\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)}$	$-\frac{1}{2496}-\frac{i}{12480}$
a_5	$-\frac{1}{(r^2+2r+2)(r^2+4r+5)(r^2+6r+10)(r^2+8r+17)(r^2+10r+26)}$	$\frac{9}{603200}-\frac{i}{361920}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^i\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6\ldots\right)\\
&= x^i\left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3\right.\\
&\quad\left.+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
\begin{aligned}
y_2(x) = x^{-i}&\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(\frac{3}{520}-\frac{7i}{1560}\right)x^3\right.\\
&\left.+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x)\\
&= c_1x^i\left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4\right.\\
&\quad\left.+\left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)+c_2x^{-i}\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2\right.\\
&\quad\left.+\left(\frac{3}{520}-\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$y = y_h$$
$$= c_1 x^i \left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4 + \left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)+c_2 x^{-i}\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2 + \left(\frac{3}{520}-\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)$$

Summary

The solution(s) found are the following

$$y = c_1 x^i \left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3 + \left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right) + c_2 x^{-i}\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(\frac{3}{520}-\frac{7i}{1560}\right)x^3 + \left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^i \left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4 + \left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+O(x^6)\right)+c_2 x^{-i}\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2 + \left(\frac{3}{520}-\frac{7i}{1560}\right)x^3+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+O(x^6)\right)$$

Verified OK.

2.1.1 Maple step by step solution

Let's solve

$x^2 y'' + x y' + (1+x) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{x} - \frac{(1+x)y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{x} + \frac{(1+x)y}{x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

- ○ Define functions

$\left[P_2(x) = \frac{1}{x}, P_3(x) = \frac{1+x}{x^2}\right]$

- ○ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 1$

- ○ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = 1$

- ○ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$x^2y'' + xy' + (1+x)\,y = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ○ Shift index using $k- >k - m$

$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- ○ Convert $x \cdot y'$ to series expansion

$x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- ○ Convert $x^2 \cdot y''$ to series expansion

$$x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(r^2+1) x^r + \left(\sum_{k=1}^{\infty} (a_k(k^2+2kr+r^2+1)+a_{k-1}) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2+1=0$

- Values of r that satisfy the indicial equation

  $r \in \{-\mathrm{I}, \mathrm{I}\}$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k^2+2kr+r^2+1)+a_{k-1}=0$

- Shift index using $k->k+1$

  $a_{k+1}\left((k+1)^2+2(k+1)r+r^2+1\right)+a_k=0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{k^2+2kr+r^2+2k+2r+2}$

- Recursion relation for $r=-\mathrm{I}$

  $a_{k+1} = -\frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}$

- Solution for $r=-\mathrm{I}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}, a_{k+1} = -\frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}\right]$

- Recursion relation for $r=\mathrm{I}$

  $a_{k+1} = -\frac{a_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}$

- Solution for $r=\mathrm{I}$

  $\left[y=\sum_{k=0}^{\infty} a_k x^{k+\mathrm{I}}, a_{k+1} = -\frac{a_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}\right]$

- Combine solutions and rename parameters

  $\left[y=\left(\sum_{k=0}^{\infty} a_k x^{k-\mathrm{I}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\mathrm{I}}\right), a_{k+1} = -\frac{a_k}{k^2-2\,\mathrm{I}k+1-2\,\mathrm{I}+2k}, b_{k+1} = -\frac{b_k}{k^2+2\,\mathrm{I}k+1+2\,\mathrm{I}+2k}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 69

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

$$
\begin{aligned}
y(x) = {} & c_1 x^{-i}\left(1+\left(-\frac{1}{5}-\frac{2i}{5}\right)x+\left(-\frac{1}{40}+\frac{3i}{40}\right)x^2+\left(\frac{3}{520}-\frac{7i}{1560}\right)x^3\right. \\
& \left.+\left(-\frac{1}{2496}+\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}+\frac{i}{361920}\right)x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x^{i}\left(1+\left(-\frac{1}{5}+\frac{2i}{5}\right)x+\left(-\frac{1}{40}-\frac{3i}{40}\right)x^2+\left(\frac{3}{520}+\frac{7i}{1560}\right)x^3\right. \\
& \left.+\left(-\frac{1}{2496}-\frac{i}{12480}\right)x^4+\left(\frac{9}{603200}-\frac{i}{361920}\right)x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 90

```
AsymptoticDSolveValue[x^2*y''[x]+x*y'[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$
y(x) \to \left(\frac{1}{12480}+\frac{i}{2496}\right) c_2 x^{-i}\left(i x^4-(8+16 i) x^3+(168+96 i) x^2-(1056-288 i) x+(480-2400 i)\right)-\left(\frac{1}{2496}+\frac{i}{12480}\right) c_1 x^i\left(x^4-(16+8 i) x^3+(96+168 i) x^2+(288-1056 i) x-(2400-480 i)\right)
$$

2.2 problem 7.3.4

Internal problem ID [5516]
Internal file name [OUTPUT/4764_Sunday_June_05_2022_03_05_15_PM_79439406/index.tex]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{x^2}$$

Table 13: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$r_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)$$

Verified OK.

2.2.1 Maple step by step solution

Let's solve

$x^2y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1\pm\left(\sqrt{5}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{1}{2}+\frac{\sqrt{5}}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\ln(x)} + c_2 \mathrm{e}^{\ln(x)\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}$
- Simplify

  $y = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}} c_1 + x^{\frac{\sqrt{5}}{2}} c_2\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}}c_1 + x^{\frac{\sqrt{5}}{2}}c_2\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x^2*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + c_2 x^{\frac{1}{2}\left(1-\sqrt{5}\right)}$$

2.3 problem 7.3.5

Internal problem ID [5517]
Internal file name [`OUTPUT/4765_Sunday_June_05_2022_03_05_17_PM_81539624/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{y'' + \frac{y'}{x} - xy = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + \frac{y'}{x} - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -x$$

Table 15: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -x$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-yx^2 + y''x + y' = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^2 + \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{2+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{2+n+r} a_n\right) = \sum_{n=3}^{\infty} \left(-a_{n-3} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=3}^{\infty} \left(-a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-3}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-3}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{(r+3)^2}$	$\frac{1}{9}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{(r+3)^2}$	$\frac{1}{9}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{(r+3)^2}$	$\frac{1}{9}$
a_4	0	0
a_5	0	0

Using the above table, then the first solution $y_1(x)$ becomes

$$y_1(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots$$
$$= 1 + \frac{x^3}{9} + O(x^6)$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	0	0	0	0
b_3	$\frac{1}{(r+3)^2}$	$\frac{1}{9}$	$-\frac{2}{(r+3)^3}$	$-\frac{2}{27}$
b_4	0	0	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + \frac{x^3}{9} + O(x^6)\right)\ln(x) - \frac{2x^3}{27} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{x^3}{9} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^3}{9} + O(x^6)\right)\ln(x) - \frac{2x^3}{27} + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 + \frac{x^3}{9} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^3}{9} + O(x^6)\right)\ln(x) - \frac{2x^3}{27} + O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\left(1 + \frac{x^3}{9} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^3}{9} + O(x^6)\right)\ln(x) - \frac{2x^3}{27} + O(x^6)\right) \tag{1}
$$

Verification of solutions

$$
y = c_1\left(1 + \frac{x^3}{9} + O(x^6)\right) + c_2\left(\left(1 + \frac{x^3}{9} + O(x^6)\right)\ln(x) - \frac{2x^3}{27} + O(x^6)\right)
$$

Verified OK.

2.3.1 Maple step by step solution

Let's solve

$-yx^2 + y''x + y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y'}{x} + xy$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' + \frac{y'}{x} - xy = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

◦ Define functions

$\left[P_2(x) = \frac{1}{x}, P_3(x) = -x\right]$

◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$(x \cdot P_2(x))\bigg|_{x=0} = 1$

◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$

◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$x_0 = 0$

- Multiply by denominators

$-yx^2 + y''x + y' = 0$

- Assume series solution for y

$y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

◦ Convert $x^2 \cdot y$ to series expansion

$x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$

◦ Shift index using $k- >k - 2$

$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$

◦ Convert y' to series expansion

$y' = \sum_{k=0}^{\infty} a_k (k + r) x^{k+r-1}$

◦ Shift index using $k- >k + 1$

$$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r^2 x^{-1+r} + a_1(1+r)^2 x^r + a_2(2+r)^2 x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$r^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 0$$

- The coefficients of each power of x must be 0

$$\left[a_1(1+r)^2 = 0, a_2(2+r)^2 = 0\right]$$

- Solve for the dependent coefficient(s)

$$\{a_1 = 0, a_2 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

$$a_{k+1}(k+1)^2 - a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$a_{k+3}(k+3)^2 - a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{a_k}{(k+3)^2}$$

- Recursion relation for $r = 0$

$$a_{k+3} = \frac{a_k}{(k+3)^2}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{(k+3)^2}, a_1 = 0, a_2 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
Order:=6;
dsolve(diff(y(x),x$2)+1/x*diff(y(x),x)-x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = (c_2 \ln(x) + c_1)\left(1 + \frac{1}{9}x^3 + \mathrm{O}\left(x^6\right)\right) + \left(-\frac{2}{27}x^3 + \mathrm{O}\left(x^6\right)\right)c_2$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 39

```
AsymptoticDSolveValue[y''[x]+1/x*y'[x]-x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^3}{9} + 1\right) + c_2\left(\left(\frac{x^3}{9} + 1\right)\log(x) - \frac{2x^3}{27}\right)$$

2.4 problem 7.3.6

Internal problem ID [5518]
Internal file name [`OUTPUT/4766_Sunday_June_05_2022_03_05_18_PM_92144177/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{2y''x + y' - yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2y''x + y' - yx^2 = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = -\frac{x}{2}$$

Table 17: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x}{2}$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2y''x + y' - yx^2 = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) x^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{2+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{2+n+r} a_n\right) = \sum_{n=3}^{\infty} \left(-a_{n-3} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=3}^{\infty} \left(-a_{n-3} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(2x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(-1+2r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(-1+2r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

For $3 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-3} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-3}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{a_{n-3}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{2r^2 + 11r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = \frac{1}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{21}$
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{21}$
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^3}{21} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = 0$$

For $3 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + (n+r)\,b_n - b_{n-3} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-3}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{b_{n-3}}{n\,(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{2r^2 + 11r + 15}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{1}{15}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{15}$

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{15}$
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	$\frac{1}{2r^2+11r+15}$	$\frac{1}{15}$
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 + \frac{x^3}{15} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x} \left(1 + \frac{x^3}{21} + O(x^6)\right) + c_2 \left(1 + \frac{x^3}{15} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x} \left(1 + \frac{x^3}{21} + O(x^6)\right) + c_2 \left(1 + \frac{x^3}{15} + O(x^6)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \sqrt{x} \left(1 + \frac{x^3}{21} + O(x^6)\right) + c_2 \left(1 + \frac{x^3}{15} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 \sqrt{x} \left(1 + \frac{x^3}{21} + O(x^6)\right) + c_2 \left(1 + \frac{x^3}{15} + O(x^6)\right)$$

Verified OK.

2.4.1 Maple step by step solution

Let's solve

$2y''x + y' - yx^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2x} + \frac{xy}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2x} - \frac{xy}{2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = -\frac{x}{2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left.(x \cdot P_2(x))\right|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2y''x + y' - yx^2 = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^2 \cdot y$ to series expansion

    $x^2 \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+2}$

  - Shift index using $k- > k - 2$

$$x^2 \cdot y = \sum_{k=2}^{\infty} a_{k-2} x^{k+r}$$

- Convert y' to series expansion

  $$y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$y' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+2r)\, x^{-1+r} + a_1(1+r)\,(1+2r)\, x^r + a_2(2+r)\,(3+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_{k+1}(k+1+r)\,(2k$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-1+2r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \left\{0, \frac{1}{2}\right\}$$

- The coefficients of each power of x must be 0

  $$[a_1(1+r)\,(1+2r) = 0, a_2(2+r)\,(3+2r) = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_1 = 0, a_2 = 0\}$$

- Each term in the series must be 0, giving the recursion relation

  $$2\left(k+\frac{1}{2}+r\right)(k+1+r)\, a_{k+1} - a_{k-2} = 0$$

- Shift index using $k->k+2$

  $$2\left(k+\frac{5}{2}+r\right)(k+3+r)\, a_{k+3} - a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+3} = \frac{a_k}{(2k+5+2r)(k+3+r)}$$

- Recursion relation for $r = 0$

$a_{k+3} = \frac{a_k}{(2k+5)(k+3)}$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{a_k}{(2k+5)(k+3)}, a_1 = 0, a_2 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$a_{k+3} = \frac{a_k}{(2k+6)\left(k+\frac{7}{2}\right)}$

- Solution for $r = \frac{1}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+3} = \frac{a_k}{(2k+6)\left(k+\frac{7}{2}\right)}, a_1 = 0, a_2 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+3} = \frac{a_k}{(2k+5)(k+3)}, a_1 = 0, a_2 = 0, b_{k+3} = \frac{b_k}{(2k+6)\left(k+\frac{7}{2}\right)}, b_1 = 0, b$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
Order:=6;
dsolve(2*x*diff(y(x),x$2)+diff(y(x),x)-x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1\sqrt{x}\left(1 + \frac{1}{21}x^3 + \mathrm{O}\left(x^6\right)\right) + c_2\left(1 + \frac{1}{15}x^3 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 33

```
AsymptoticDSolveValue[2*x*y''[x]+y'[x]-x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^3}{21} + 1\right) + c_2\left(\frac{x^3}{15} + 1\right)$$

2.5 problem 7.3.7

Internal problem ID [5519]
Internal file name [`OUTPUT/4767_Sunday_June_05_2022_03_05_20_PM_2529632/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Table 19: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - x^r a_0 r - a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - x^r r - x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1 + \sqrt{2}$$
$$r_2 = 1 - \sqrt{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2\sqrt{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1+\sqrt{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+1-\sqrt{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = 1 + \sqrt{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1 + \sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{1+\sqrt{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{1+\sqrt{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = 1 - \sqrt{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1 - \sqrt{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{1+\sqrt{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= x^{1-\sqrt{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{1+\sqrt{2}}\left(1 + O(x^6)\right) + c_2 x^{1-\sqrt{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{1+\sqrt{2}}\left(1 + O(x^6)\right) + c_2 x^{1-\sqrt{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{1+\sqrt{2}}\left(1 + O(x^6)\right) + c_2 x^{1-\sqrt{2}}\left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{1+\sqrt{2}}\left(1 + O(x^6)\right) + c_2 x^{1-\sqrt{2}}\left(1 + O(x^6)\right)$$

Verified OK.

2.5.1 Maple step by step solution

Let's solve

$x^2y'' - xy' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - xy' - y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ○ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  ○ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ○ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm\left(\sqrt{8}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(1-\sqrt{2}, 1+\sqrt{2}\right)$

- 1st solution of the ODE

  $y_1(t) = e^{\left(1-\sqrt{2}\right)t}$

- 2nd solution of the ODE

  $y_2(t) = e^{\left(1+\sqrt{2}\right)t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^{\left(1-\sqrt{2}\right)t} + c_2 e^{\left(1+\sqrt{2}\right)t}$

- Change variables back using $t = \ln(x)$

  $y = c_1 e^{\left(1-\sqrt{2}\right)\ln(x)} + c_2 e^{\left(1+\sqrt{2}\right)\ln(x)}$

- Simplify

  $y = x\left(x^{\sqrt{2}} c_2 + x^{-\sqrt{2}} c_1\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = x\left(x^{-\sqrt{2}}c_1 + x^{\sqrt{2}}c_2\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 30

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{1+\sqrt{2}} + c_2 x^{1-\sqrt{2}}$$

2.6 problem 7.3.8 (a)

Internal problem ID [5520]
Internal file name [`OUTPUT/4768_Sunday_June_05_2022_03_05_21_PM_17524057/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.8 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{x^2(x^2+1)\,y''+xy=0}$$

With the expansion point for the power series method at $x=0$.

The ODE is

$$\left(x^4+x^2\right)y''+xy=0$$

Or

$$x\left(y''x^3+y''x+y\right)=0$$

For $x\neq 0$ the above simplifies to

$$\left(x^3+x\right)y''+y=0$$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4+x^2\right)y''+xy=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$
\begin{aligned}
p(x) &= 0 \\
q(x) &= \frac{1}{x\,(x^2+1)}
\end{aligned}
$$

Table 21: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x(x^2+1)}$	
singularity	type
$x = 0$	"regular"
$x = -i$	"regular"
$x = i$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, -i, i, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
x^2\left(x^2+1\right) y'' + xy = 0
$$

Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$x^2(x^2+1)\left(\sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}\right)+x\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)(n+r-1)=\sum_{n=2}^{\infty}a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}$$

$$\sum_{n=0}^{\infty}x^{1+n+r}a_n=\sum_{n=1}^{\infty}a_{n-1}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=2}^{\infty}a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)(n+r-1)\right)+\left(\sum_{n=1}^{\infty}a_{n-1}x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1)=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)=0$$

Or

$$x^r a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

For $2 \leq n$ the recursive equation is

$$a_{n-2}(n+r-2)(n-3+r) + a_n(n+r)(n+r-1) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-2} + 2nr a_{n-2} + r^2 a_{n-2} - 5n a_{n-2} - 5r a_{n-2} + 6a_{n-2} + a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n^2 a_{n-2} + 3n a_{n-2} - 2a_{n-2} - a_{n-1}}{(1+n)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{-r^4 + r^2 + 1}{r(1+r)^2(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{2r^4 + 4r^3 + 4r^2 + 2r - 1}{r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{11}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{2r^4+4r^3+4r^2+2r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{11}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^8 + 8r^7 + 22r^6 + 20r^5 - 14r^4 - 40r^3 - 39r^2 - 30r - 11}{r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)}$$

Which for the root $r = 1$ becomes

$$a_4 = -\frac{83}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{2r^4+4r^3+4r^2+2r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{11}{144}$
a_4	$\frac{r^8+8r^7+22r^6+20r^5-14r^4-40r^3-39r^2-30r-11}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{83}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-3r^8 - 36r^7 - 180r^6 - 486r^5 - 773r^4 - 750r^3 - 400r^2 - 12r + 83}{r\left(1+r\right)^2\left(2+r\right)^2\left(3+r\right)^2\left(4+r\right)^2\left(5+r\right)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{2557}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{-r^4+r^2+1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{2r^4+4r^3+4r^2+2r-1}{r(1+r)^2(2+r)^2(3+r)}$	$\frac{11}{144}$
a_4	$\frac{r^8+8r^7+22r^6+20r^5-14r^4-40r^3-39r^2-30r-11}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$-\frac{83}{2880}$
a_5	$\frac{-3r^8-36r^7-180r^6-486r^5-773r^4-750r^3-400r^2-12r+83}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{2557}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{r(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r(1+r)} &= \lim_{r\to 0} -\frac{1}{r(1+r)} \\ &= \mathit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2(x^2+1)y'' + xy = 0$ gives

$$\begin{aligned}
&x^2(x^2+1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}$$

Which can be written as

$$\begin{aligned}
&\left(\left((x^2+1)x^2y_1''(x) + y_1(x)x\right)\ln(x) + (x^2+1)x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&+ (x^2+1)x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \qquad (7) \\
&+ x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

But since $y_1(x)$ is a solution to the ode, then

$$\left(x^2+1\right)x^2 y_1''(x) + y_1(x)\,x = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x^2+1\right)x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)C \\
&+ \left(x^2+1\right)x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2x\left(x^2+1\right)\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) + \left(-x^2-1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C \\
&+ \left(x^4+x^2\right)\left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) + x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\begin{aligned}
&\left(2x\left(x^2+1\right)\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right) + \left(-x^2-1\right)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right)C \\
&+ \left(x^4+x^2\right)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) + x\left(\sum_{n=0}^{\infty} b_n x^n\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\,x^{n+3} a_n(1+n)\right) + \left(\sum_{n=0}^{\infty} 2C\,x^{1+n} a_n(1+n)\right) \\
&+ \sum_{n=0}^{\infty}\left(-C\,x^{n+3} a_n\right) + \sum_{n=0}^{\infty}\left(-C\,x^{1+n} a_n\right) + \left(\sum_{n=0}^{\infty} n\,x^{n+2} b_n(n-1)\right) \\
&+ \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n} b_n\right) = 0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{n+3} a_n (1+n) = \sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^n$$

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n (1+n) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{n+3} a_n\right) = \sum_{n=3}^{\infty} \left(-C a_{n-3} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-C\, x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} n\, x^{n+2} b_n (n-1) = \sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^n$$

$$\sum_{n=0}^{\infty} x^{1+n} b_n = \sum_{n=1}^{\infty} b_{n-1} x^n$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\left(\sum_{n=3}^{\infty} 2C a_{n-3} (n-2)\, x^n\right) + \left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n\right) + \sum_{n=3}^{\infty} \left(-C a_{n-3} x^n\right) \\ &+ \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right) + \left(\sum_{n=2}^{\infty} (n-2)\, b_{n-2} (n-3)\, x^n\right) \\ &+ \left(\sum_{n=0}^{\infty} x^n b_n n (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^n\right) = 0 \end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 + b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$(a_0 + 5a_2)\, C + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{13}{6} + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{13}{36}$$

For $n = 4$, Eq (2B) gives

$$(3a_1 + 7a_3)\, C + 2b_2 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{25}{144} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{1728}$$

For $n = 5$, Eq (2B) gives

$$(5a_2 + 9a_4)\, C + 6b_3 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{8743}{4320} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{8743}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1) \left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right)\right) \ln(x) \\ + 1 - \frac{3x^2}{4} + \frac{13x^3}{36} + \frac{25x^4}{1728} - \frac{8743x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \\ &\quad + c_2 \left((-1) \left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right)\right) \ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{13x^3}{36} + \frac{25x^4}{1728} - \frac{8743x^5}{86400} + O(x^6) \right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \\ &\quad + c_2 \left(-x\left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} + \frac{13x^3}{36} \right. \\ &\quad \left. + \frac{25x^4}{1728} - \frac{8743x^5}{86400} + O(x^6) \right) \end{aligned}$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \\
& + c_2 \left(-x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} \right. \\
& \left. + \frac{13x^3}{36} + \frac{25x^4}{1728} - \frac{8743x^5}{86400} + O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y = {} & c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \\
& + c_2 \left(-x \left(1 - \frac{x}{2} + \frac{x^2}{12} + \frac{11x^3}{144} - \frac{83x^4}{2880} - \frac{2557x^5}{86400} + O(x^6)\right) \ln(x) + 1 - \frac{3x^2}{4} \right. \\
& \left. + \frac{13x^3}{36} + \frac{25x^4}{1728} - \frac{8743x^5}{86400} + O(x^6)\right)
\end{aligned}
$$

Verified OK.

2.6.1 Maple step by step solution

Let's solve

$x^2(x^2+1)\, y'' + xy = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y}{x(x^2+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y}{x(x^2+1)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = 0, P_3(x) = \frac{1}{x(x^2+1)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $y'' x(x^2 + 1) + y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot y''$ to series expansion for $m = 1..3$

  $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1) x^{k+r-2+m}$

- Shift index using $k- > k + 2 - m$

  $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r) x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-1+r) x^{-1+r} + (a_1(1+r) r + a_0) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k+r) + a_k + a_{k-1}(k+r-1$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 1\}$

- Each term must be 0

  $a_1(1+r) r + a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r) + a_k + a_{k-1}(k+r-1)(k-2+r) = 0$

- Shift index using $k- > k + 1$

  $a_{k+2}(k+2+r)(k+1+r) + a_{k+1} + a_k(k+r)(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{k^2 a_k + 2kr a_k + r^2 a_k - k a_k - r a_k + a_{k+1}}{(k+2+r)(k+1+r)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{k^2 a_k - k a_k + a_{k+1}}{(k+2)(k+1)}$$

- Solution for $r = 0$

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{k^2 a_k - k a_k + a_{k+1}}{(k+2)(k+1)}, a_0 = 0\right]$$

- Recursion relation for $r = 1$

$$a_{k+2} = -\frac{k^2 a_k + k a_k + a_{k+1}}{(k+3)(k+2)}$$

- Solution for $r = 1$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2} = -\frac{k^2 a_k + k a_k + a_{k+1}}{(k+3)(k+2)}, 2a_1 + a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+2} = -\frac{k^2 a_k - k a_k + a_{k+1}}{(k+2)(k+1)}, a_0 = 0, b_{k+2} = -\frac{k^2 b_k + k b_k + b_{k+1}}{(k+3)(k+2)}, 2b_1 + b\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 58

```
Order:=6;
dsolve(x^2*(1+x^2)*diff(y(x),x$2)+x*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 + \frac{11}{144}x^3 - \frac{83}{2880}x^4 - \frac{2557}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right) \left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 - \frac{11}{144}x^4 + \frac{83}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{13}{36}x^3 + \frac{25}{1728}x^4 - \frac{8743}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x^2*(1+x^2)*y''[x]+x*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{157x^4 + 768x^3 - 2160x^2 + 1728x + 1728}{1728} - \frac{1}{144}x\left(11x^3 + 12x^2 - 72x + 144\right)\log(x)\right) + c_2 \left(-\frac{83x^5}{2880} + \frac{11x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

2.7 problem 7.3.8 (b)

Internal problem ID [5521]
Internal file name [`OUTPUT/4769_Sunday_June_05_2022_03_05_23_PM_14949691/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.8 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^2y'' + y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x^2}$$
$$q(x) = \frac{1}{x^2}$$

Table 23: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"irregular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   <- Kummer successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 84

```
AsymptoticDSolveValue[x^2*y''[x]+y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 e^{\frac{1}{x}} \left(\frac{59241x^5}{40} + \frac{1911x^4}{8} + \frac{91x^3}{2} + \frac{21x^2}{2} + 3x + 1\right) x^2 + c_1 \left(-\frac{91x^5}{40} + \frac{7x^4}{8} - \frac{x^3}{2} + \frac{x^2}{2} - x + 1\right)$$

2.8 problem 7.3.8 (c)

Internal problem ID [5522]
Internal file name [OUTPUT/4770_Sunday_June_05_2022_03_05_24_PM_67992485/index.tex]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.8 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x + y'x^3 + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + y'x^3 + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = x^2$$
$$q(x) = \frac{1}{x}$$

Table 24: Table $p(x), q(x)$ singularites.

$p(x) = x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + y'x^3 + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) x^3 + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{2+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{2+n+r} a_n(n+r) = \sum_{n=3}^{\infty} a_{n-3}(n+r-3)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=3}^{\infty} a_{n-3}(n+r-3)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{r(1+r)^2(2+r)}$$

For $3 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-3}(n+r-3) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-3} + r a_{n-3} - 3a_{n-3} + a_{n-1}}{(n+r)(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-3} + 2a_{n-3} - a_{n-1}}{(n+1)\, n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^5 - 4r^4 - 5r^3 - 2r^2 - 1}{r(1+r)^2(2+r)^2(3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{13}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{-r^5-4r^4-5r^3-2r^2-1}{r(1+r)^2(2+r)^2(3+r)}$	$-\frac{13}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{2r^5 + 13r^4 + 36r^3 + 53r^2 + 40r + 13}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{157}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{-r^5-4r^4-5r^3-2r^2-1}{r(1+r)^2(2+r)^2(3+r)}$	$-\frac{13}{144}$
a_4	$\frac{2r^5+13r^4+36r^3+53r^2+40r+13}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{157}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-3r^5 - 27r^4 - 113r^3 - 261r^2 - 316r - 157}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{877}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{-r^5-4r^4-5r^3-2r^2-1}{r(1+r)^2(2+r)^2(3+r)}$	$-\frac{13}{144}$
a_4	$\frac{2r^5+13r^4+36r^3+53r^2+40r+13}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{157}{2880}$
a_5	$\frac{-3r^5-27r^4-113r^3-261r^2-316r-157}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{877}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\big(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\big) \\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{13x^3}{144} + \frac{157x^4}{2880} - \frac{877x^5}{86400} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= -\frac{1}{r(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} -\frac{1}{r(1+r)} &= \lim_{r\to 0} -\frac{1}{r(1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $y''x + y'x^3 + y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right)x^3 + Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0$$

Which can be written as

$$\left(\left(y_1'(x)\,x^3 + y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\,x^2\right)C + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^3 + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1'(x)\,x^3 + y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\,x^2\right)C + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^3 + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x^3-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^4 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n (n+1)\right) x + (x^3-1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right) C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^4 + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right) x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right) x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1)\right) + \left(\sum_{n=0}^{\infty} C\, x^{n+3} a_n\right) + \sum_{n=0}^{\infty} (-C a_n x^n) + \left(\sum_{n=0}^{\infty} n\, x^{2+n} b_n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (n+1) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{n+3} a_n = \sum_{n=4}^{\infty} C a_{n-4} x^{n-1}$$

$$\sum_{n=0}^{\infty} (-C a_n x^n) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right)$$

$$\sum_{n=0}^{\infty} n\, x^{2+n} b_n = \sum_{n=3}^{\infty} (n-3)\, b_{n-3} x^{n-1}$$

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right) + \left(\sum_{n=4}^{\infty} C a_{n-4} x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=3}^{\infty} (n-3)\, b_{n-3} x^{n-1}\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 + b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 - \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$(a_0 + 7a_3)\, C + b_1 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{25}{144} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{1728}$$

For $n = 5$, Eq (2B) gives

$$(a_1 + 9a_4)\, C + 2b_2 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{6377}{4320} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{6377}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = (-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{13x^3}{144} + \frac{157x^4}{2880} - \frac{877x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} + \frac{25x^4}{1728} + \frac{6377x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{13x^3}{144} + \frac{157x^4}{2880} - \frac{877x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{13x^3}{144} + \frac{157x^4}{2880} - \frac{877x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 \right. \\ &\quad \left. - \frac{3x^2}{4} + \frac{7x^3}{36} + \frac{25x^4}{1728} + \frac{6377x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}\right. \\
&\quad \left.+\frac{25x^4}{1728}+\frac{6377x^5}{86400}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}\right. \\
&\quad \left.+\frac{7x^3}{36}+\frac{25x^4}{1728}+\frac{6377x^5}{86400}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x \left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right) \\
&\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{13x^3}{144}+\frac{157x^4}{2880}-\frac{877x^5}{86400}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}+\frac{7x^3}{36}\right. \\
&\quad \left.+\frac{25x^4}{1728}+\frac{6377x^5}{86400}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+x^3*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{13}{144}x^3 + \frac{157}{2880}x^4 - \frac{877}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{13}{144}x^4 - \frac{157}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 + \frac{25}{1728}x^4 + \frac{6377}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 87

```
AsymptoticDSolveValue[x*y''[x]+x^3*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{1}{144}x\left(13x^3 - 12x^2 + 72x - 144\right)\log(x) + \frac{-131x^4 + 480x^3 - 2160x^2 + 1728x + 1728}{1728}\right) + c_2\left(\frac{157x^5}{2880} - \frac{13x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

2.9 problem 7.3.8 (d)

Internal problem ID [5523]
Internal file name [`OUTPUT/4771_Sunday_June_05_2022_03_05_27_PM_18805796/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.8 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x + xy' - e^{x}y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + xy' - e^{x}y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -\frac{e^{x}}{x} \end{aligned}$$

Table 25: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = -\frac{e^x}{x}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + xy' - e^x y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) x + x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - e^x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $-e^x$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-e^x &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\
&= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6
\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+2} a_n}{2}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+3} a_n}{6}\right) \\
&+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+4} a_n}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+5} a_n}{120}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+6} a_n}{720}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) &= \sum_{n=1}^{\infty} a_{n-1} (n+r-1) x^{n+r-1} \\
\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) &= \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) &= \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+2} a_n}{2}\right) &= \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^{n+r-1}}{2}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+3} a_n}{6}\right) &= \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^{n+r-1}}{6}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+4} a_n}{24}\right) &= \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^{n+r-1}}{24}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+5} a_n}{120}\right) &= \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^{n+r-1}}{120}\right) \\
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+6} a_n}{720}\right) &= \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^{n+r-1}}{720}\right)
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\
&+ \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r-1}\right) + \sum_{n=3}^{\infty} \left(-\frac{a_{n-3} x^{n+r-1}}{2}\right) \\
&+ \sum_{n=4}^{\infty} \left(-\frac{a_{n-4} x^{n+r-1}}{6}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^{n+r-1}}{24}\right) \\
&+ \sum_{n=6}^{\infty} \left(-\frac{a_{n-6} x^{n+r-1}}{120}\right) + \sum_{n=7}^{\infty} \left(-\frac{a_{n-7} x^{n+r-1}}{720}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{1 - r}{r(1 + r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{2r}{(1 + r)^2 (2 + r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{(r - 2)^2}{2r(2 + r)^2 (3 + r)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{r^4 - r^3 + 22r^2 + 3r + 3}{6(1 + r)^2 r(3 + r)^2 (4 + r)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{r^6 + 2r^5 + 41r^4 + 32r^3 + 32r^2 + 128r + 112}{24\left(2+r\right)^2 r\left(4+r\right)^2 \left(1+r\right)^2 \left(5+r\right)}$$

Substituting $n = 6$ in Eq. (2B) gives

$$a_6 = \frac{r^8 + 7r^7 + 113r^6 + 554r^5 + 1943r^4 + 4807r^3 + 5931r^2 + 3240r + 1740}{120\left(3+r\right)^2 r\left(5+r\right)^2 \left(1+r\right)^2 \left(2+r\right)^2 \left(6+r\right)}$$

For $7 \leq n$ the recursive equation is

$$\begin{aligned} &a_n(n+r)\left(n+r-1\right) + a_{n-1}(n+r-1) - a_{n-1} \\ &- a_{n-2} - \frac{a_{n-3}}{2} - \frac{a_{n-4}}{6} - \frac{a_{n-5}}{24} - \frac{a_{n-6}}{120} - \frac{a_{n-7}}{720} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{720na_{n-1} + 720ra_{n-1} - a_{n-7} - 6a_{n-6} - 30a_{n-5} - 120a_{n-4} - 360a_{n-3} - 720a_{n-2} - 1440a_{n-1}}{720\left(n+r\right)\left(n+r-1\right)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-720na_{n-1} + a_{n-7} + 6a_{n-6} + 30a_{n-5} + 120a_{n-4} + 360a_{n-3} + 720a_{n-2} + 720a_{n-1}}{720\left(1+n\right)n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1-r}{r(1+r)}$	0
a_2	$\frac{2r}{(1+r)^2(2+r)}$	$\frac{1}{6}$
a_3	$\frac{(r-2)^2}{2r(2+r)^2(3+r)}$	$\frac{1}{72}$
a_4	$\frac{r^4-r^3+22r^2+3r+3}{6(1+r)^2r(3+r)^2(4+r)}$	$\frac{7}{480}$
a_5	$\frac{r^6+2r^5+41r^4+32r^3+32r^2+128r+112}{24(2+r)^2r(4+r)^2(1+r)^2(5+r)}$	$\frac{29}{10800}$
a_6	$\frac{r^8+7r^7+113r^6+554r^5+1943r^4+4807r^3+5931r^2+3240r+1740}{120(3+r)^2r(5+r)^2(1+r)^2(2+r)^2(6+r)}$	$\frac{191}{181440}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 + \frac{x^2}{6} + \frac{x^3}{72} + \frac{7x^4}{480} + \frac{29x^5}{10800} + \frac{191x^6}{181440} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{1-r}{r\,(1+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r\to r_2} \frac{1-r}{r\,(1+r)} &= \lim_{r\to 0} \frac{1-r}{r\,(1+r)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx} y_2(x) &= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= C y_1'(x) \ln(x) + \frac{C y_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2} y_2(x) &= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $y''x + xy' - e^x y = 0$ gives

$$\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) - e^x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(-e^x y_1(x) + y_1'(x)\,x + y_1''(x)\,x\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\right)C - e^x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$-e^x y_1(x) + y_1'(x)\,x + y_1''(x)\,x = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\right)C - e^x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) + x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1)\right) x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C}{x} + \frac{-\mathrm{e}^{x}\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right) x^2}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n} a_n (1+n)\right) x + (x-1)\left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C}{x} + \frac{-\mathrm{e}^{x}\left(\sum_{n=0}^{\infty} b_n x^{n}\right) x + \left(\sum_{n=0}^{\infty} x^{n-1} b_n n\right) x^2 + \left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) x^2}{x} = 0 \tag{10}$$

Expanding $-\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\mathrm{e}^{x} &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots \\ &= -1 - x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{120}x^5 - \frac{1}{720}x^6 \end{aligned}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 2C\, x^n a_n (1+n)\right) + \left(\sum_{n=0}^{\infty} C\, x^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-C a_n x^n) \\ &+ \sum_{n=0}^{\infty} (-b_n x^n) + \sum_{n=0}^{\infty} (-x^{1+n} b_n) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+2} b_n}{2}\right) \\ &+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+3} b_n}{6}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+4} b_n}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+5} b_n}{120}\right) \\ &+ \sum_{n=0}^{\infty} \left(-\frac{x^{n+6} b_n}{720}\right) + \left(\sum_{n=0}^{\infty} x^n b_n n\right) + \left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^n a_n (1+n) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}$$

$$\sum_{n=0}^{\infty} C\, x^{1+n} a_n = \sum_{n=2}^{\infty} C a_{n-2} x^{n-1}$$

$$\sum_{n=0}^{\infty} (-C a_n x^n) = \sum_{n=1}^{\infty} (-C a_{n-1} x^{n-1})$$

$$\sum_{n=0}^{\infty} (-b_n x^n) = \sum_{n=1}^{\infty} (-b_{n-1} x^{n-1})$$

$$\sum_{n=0}^{\infty} (-x^{1+n} b_n) = \sum_{n=2}^{\infty} (-b_{n-2} x^{n-1})$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+2} b_n}{2}\right) = \sum_{n=3}^{\infty} \left(-\frac{b_{n-3} x^{n-1}}{2}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+3} b_n}{6}\right) = \sum_{n=4}^{\infty} \left(-\frac{b_{n-4} x^{n-1}}{6}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+4} b_n}{24}\right) = \sum_{n=5}^{\infty} \left(-\frac{b_{n-5} x^{n-1}}{24}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+5} b_n}{120}\right) = \sum_{n=6}^{\infty} \left(-\frac{b_{n-6} x^{n-1}}{120}\right)$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+6} b_n}{720}\right) = \sum_{n=7}^{\infty} \left(-\frac{b_{n-7} x^{n-1}}{720}\right)$$

$$\sum_{n=0}^{\infty} x^n b_n n = \sum_{n=1}^{\infty} (n-1)\, b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^{n-1}\right)+\left(\sum_{n=2}^{\infty} C a_{n-2} x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-C a_{n-1} x^{n-1}\right)$$
$$+\sum_{n=1}^{\infty}\left(-b_{n-1} x^{n-1}\right)+\sum_{n=2}^{\infty}\left(-b_{n-2} x^{n-1}\right)+\sum_{n=3}^{\infty}\left(-\frac{b_{n-3} x^{n-1}}{2}\right)$$
$$+\sum_{n=4}^{\infty}\left(-\frac{b_{n-4} x^{n-1}}{6}\right)+\sum_{n=5}^{\infty}\left(-\frac{b_{n-5} x^{n-1}}{24}\right) \tag{2B}$$
$$+\sum_{n=6}^{\infty}\left(-\frac{b_{n-6} x^{n-1}}{120}\right)+\sum_{n=7}^{\infty}\left(-\frac{b_{n-7} x^{n-1}}{720}\right)$$
$$+\left(\sum_{n=1}^{\infty}(n-1)\, b_{n-1} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} n\, x^{n-1} b_n (n-1)\right)=0$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$C-1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$\left(a_0+3a_1\right) C-b_0+2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2=0$$

Solving the above for b_2 gives

$$b_2=0$$

For $n=3$, Eq (2B) gives

$$\left(a_1+5a_2\right) C-\frac{b_0}{2}-b_1+b_2+6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{3}+6b_3=0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{18}$$

For $n = 4$, Eq (2B) gives

$$(a_2 + 7a_3)\, C - \frac{b_0}{6} - \frac{b_1}{2} - b_2 + 2b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-\frac{1}{72} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{864}$$

For $n = 5$, Eq (2B) gives

$$(a_3 + 9a_4)\, C - \frac{b_0}{24} - \frac{b_1}{6} - \frac{b_2}{2} - b_3 + 3b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{13}{80} + 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{13}{1600}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = 1\left(x\left(1 + \frac{x^2}{6} + \frac{x^3}{72} + \frac{7x^4}{480} + \frac{29x^5}{10800} + \frac{191x^6}{181440} + O(x^6)\right)\right) \ln(x) \\ + 1 - \frac{x^3}{18} + \frac{x^4}{864} - \frac{13x^5}{1600} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{6} + \frac{x^3}{72} + \frac{7x^4}{480} + \frac{29x^5}{10800} + \frac{191x^6}{181440} + O(x^6)\right) \\ &\quad + c_2\left(1\left(x\left(1 + \frac{x^2}{6} + \frac{x^3}{72} + \frac{7x^4}{480} + \frac{29x^5}{10800} + \frac{191x^6}{181440} + O(x^6)\right)\right) \ln(x) + 1 - \frac{x^3}{18} \right. \\ &\quad \left. + \frac{x^4}{864} - \frac{13x^5}{1600} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right)\ln(x)+1-\frac{x^3}{18}+\frac{x^4}{864}\right. \\
&\quad \left. -\frac{13x^5}{1600}+O(x^6)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right)\ln(x)+1-\frac{x^3}{18}\right. \\
&\quad \left. +\frac{x^4}{864}-\frac{13x^5}{1600}+O(x^6)\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$
\begin{aligned}
y &= c_1 x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right) \\
&\quad + c_2\left(x\left(1+\frac{x^2}{6}+\frac{x^3}{72}+\frac{7x^4}{480}+\frac{29x^5}{10800}+\frac{191x^6}{181440}+O(x^6)\right)\ln(x)+1-\frac{x^3}{18}+\frac{x^4}{864}\right. \\
&\quad \left. -\frac{13x^5}{1600}+O(x^6)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
   <- unable to find a useful change of variables
      trying a symmetry of the form [xi=0, eta=F(x)]
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 54

```
Order:=6;
dsolve(x*diff(y(x),x$2)+x*diff(y(x),x)-exp(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x\left(1+\frac{1}{6}x^2+\frac{1}{72}x^3+\frac{7}{480}x^4+\frac{29}{10800}x^5+\mathrm{O}\left(x^6\right)\right) + c_2\left(\ln\left(x\right)\left(x+\frac{1}{6}x^3+\frac{1}{72}x^4+\frac{7}{480}x^5+\mathrm{O}\left(x^6\right)\right) + \left(1-x-\frac{2}{9}x^3-\frac{11}{864}x^4-\frac{109}{4800}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 70

```
AsymptoticDSolveValue[x*y''[x]+x*y'[x]-Exp[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{7x^5}{480}+\frac{x^4}{72}+\frac{x^3}{6}+x\right) + c_1\left(\frac{1}{864}\left(-23x^4-336x^3-1728x+864\right)+\frac{1}{72}x\left(x^3+12x^2+72\right)\log(x)\right)$$

2.10 problem 7.3.8 (e)

Internal problem ID [5524]
Internal file name [`OUTPUT/4772_Sunday_June_05_2022_03_05_29_PM_46121535/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.8 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x^2y'' + x^2y' + yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

Simplyfing the ode gives

$$y'' + y' + y = 0$$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0, y_0, y_0'} + \frac{x^3}{3!}\, f'|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (41) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (42) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x=0$. Hence

$$\begin{aligned}
F_0 &= -y' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y' \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -y' - y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x=0$ and $y(0)=y(0)$ and $y'(0)=y'(0)$ gives

$$\begin{aligned}
F_0 &= -y'(0) - y(0) \\
F_1 &= y(0) \\
F_2 &= y'(0) \\
F_3 &= -y'(0) - y(0) \\
F_4 &= y(0)
\end{aligned}$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right)$$

Since the expansion point $x=0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n \\ \sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n \end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + (n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= -\frac{n a_{n+1} + a_n + a_{n+1}}{(n+2)(n+1)} \\
&= -\frac{a_n}{(n+2)(n+1)} - \frac{a_{n+1}}{n+2}
\end{aligned} \tag{5}
$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_1}{2} - \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 2a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 3a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_1}{120} - \frac{a_0}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 5a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 6a_6 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(-\frac{a_1}{2} - \frac{a_0}{2}\right) x^2 + \frac{a_0 x^3}{6} + \frac{a_1 x^4}{24} + \left(-\frac{a_1}{120} - \frac{a_0}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) a_0 + \left(x - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{120}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) c_1 + \left(x - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{720}x^{6}\right)y(0) + \left(x-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{120}x^{5}\right)y'(0)+O\left(x^{6}\right) \tag{1}$$

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}\right)c_{1}+\left(x-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{120}x^{5}\right)c_{2}+O\left(x^{6}\right) \tag{2}$$

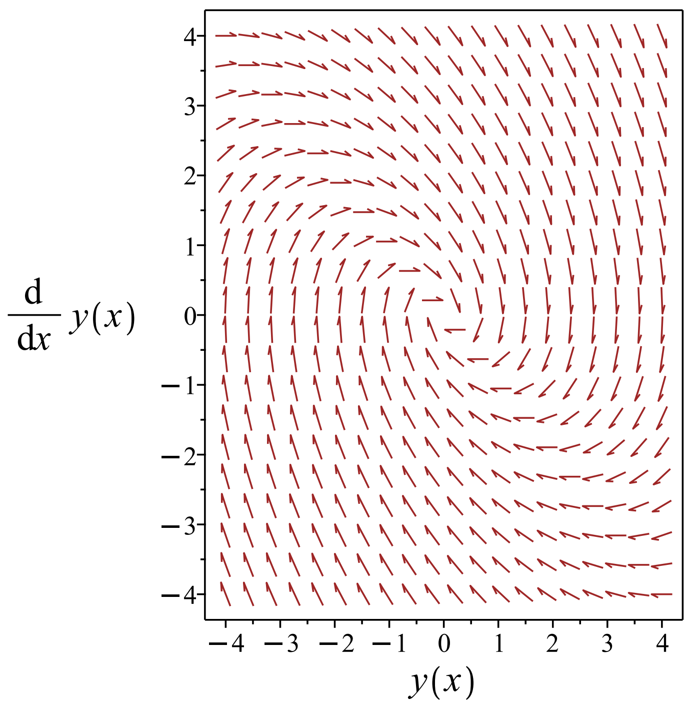

Figure 3: Slope field plot

Verification of solutions

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{720}x^{6}\right)y(0)+\left(x-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{120}x^{5}\right)y'(0)+O\left(x^{6}\right)$$

Verified OK.

$$y = \left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}\right)c_{1}+\left(x-\frac{1}{2}x^{2}+\frac{1}{24}x^{4}-\frac{1}{120}x^{5}\right)c_{2}+O\left(x^{6}\right)$$

Verified OK.

2.10.1 Maple step by step solution

Let's solve

$y'' = -y' - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I\sqrt{3}}{2}, -\frac{1}{2} + \frac{I\sqrt{3}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_1 + e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+x^2*diff(y(x),x)+x^2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{120}x^5\right) y(0) + \left(x - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 56

```
AsymptoticDSolveValue[x^2*y''[x]+x^2*y'[x]+x^2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(-\frac{x^5}{120} + \frac{x^4}{24} - \frac{x^2}{2} + x\right) + c_1\left(-\frac{x^5}{120} + \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

2.11 problem 7.3.101 (a)

Internal problem ID [5525]
Internal file name [`OUTPUT/4773_Sunday_June_05_2022_03_05_30_PM_90207999/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.101 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\ &= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0} \end{aligned}$$

But

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{44}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{45}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}$$

$$\frac{d^3 f}{dx^3} = \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}$$

$$\vdots$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y(0) \\
F_1 &= -y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= -y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard

power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -\frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - \frac{1}{2} a_0 x^2 - \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) a_1 + O\left(x^6\right) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O\left(x^6\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O\left(x^6\right) \tag{1}$$

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O\left(x^6\right) \tag{2}$$

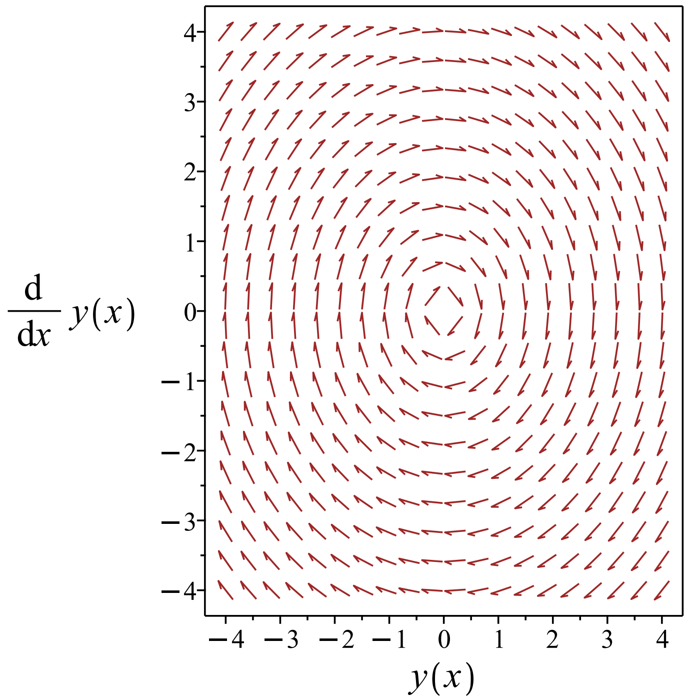

Figure 4: Slope field plot

Verification of solutions

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

2.11.1 Maple step by step solution

Let's solve

$y'' = -y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y = 0$

- Characteristic polynomial of ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = \cos(x)\, c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(\frac{x^4}{24} - \frac{x^2}{2} + 1\right)$$

2.12 problem 7.3.101 (b)

Internal problem ID [5526]
Internal file name [`OUTPUT/4774_Sunday_June_05_2022_03_05_31_PM_62980122/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.101 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second__order__bessel__ode", "second order series method. Irregular singular point"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

Unable to solve or complete the solution.

$$\boxed{x^3y'' + (1+x)\,y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^3y'' + (1+x)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1+x}{x^3}$$

Table 28: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1+x}{x^3}$	
singularity	type
$x = 0$	"irregular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : $[0]$

Since $x = 0$ is not an ordinary point, then we will now check if it is a regular singular point. Unable to solve since $x = 0$ is not regular singular point. Terminating.

Verification of solutions N/A

2.12.1 Maple step by step solution

Let's solve

$y''x^3 + (1 + x)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(1+x)y}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(1+x)y}{x^3} = 0$

- Multiply by denominators of the ODE

  $y''x^3 + (1 + x)\, y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) x^3 + (1+x)\, y(t) = 0$

- Simplify

  $x\left(\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t)\right) + (1+x)\, y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{(1+x)y(t)}{x} + \frac{d}{dt}y(t)$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{(1+x)y(t)}{x} - \frac{d}{dt}y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 + \frac{1+x}{x} - r = 0$
- Factor the characteristic polynomial

  $\frac{r^2x - rx + x + 1}{x} = 0$
- Roots of the characteristic polynomial

  $r = \left(\frac{x+\sqrt{-3x^2-4x}}{2x}, -\frac{-x+\sqrt{-3x^2-4x}}{2x}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\frac{\left(x+\sqrt{-3x^2-4x}\right)t}{2x}}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-\frac{\left(-x+\sqrt{-3x^2-4x}\right)t}{2x}}$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

$$y(t) = c_1 \mathrm{e}^{\frac{\left(x+\sqrt{-3x^2-4x}\right)t}{2x}} + c_2 \mathrm{e}^{-\frac{\left(-x+\sqrt{-3x^2-4x}\right)t}{2x}}$$

- Change variables back using $t = \ln(x)$

$$y = c_1 \mathrm{e}^{\frac{\left(x+\sqrt{-3x^2-4x}\right)\ln(x)}{2x}} + c_2 \mathrm{e}^{-\frac{\left(-x+\sqrt{-3x^2-4x}\right)\ln(x)}{2x}}$$

- Simplify

$$y = c_1 x^{\frac{x+\sqrt{-3x^2-4x}}{2x}} + c_2 x^{-\frac{-x+\sqrt{-3x^2-4x}}{2x}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✗ Solution by Maple

```
Order:=6;
dsolve(x^3*diff(y(x),x$2)+(1+x)*y(x)=0,y(x),type='series',x=0);
```

No solution found

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 222

```
AsymptoticDSolveValue[x^3*y''[x]+(1+x)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 e^{-\frac{2i}{\sqrt{x}}} x^{3/4} \left(\frac{520667425699057ix^{9/2}}{131941395333120} - \frac{21896102683ix^{7/2}}{21474836480} + \frac{19100991ix^{5/2}}{41943040} - \frac{3367ix^{3/2}}{8192} - \frac{194208949785748261x^5}{21110623253299200} + \frac{5189376335871x^4}{2748779069440} - \frac{846810601x^3}{1342177280} + \frac{205387x^2}{524288} - \frac{273x}{512} + \frac{13i\sqrt{x}}{16} + 1 \right) + c_2 e^{\frac{2i}{\sqrt{x}}} x^{3/4} \left(-\frac{520667425699057ix^{9/2}}{131941395333120} + \frac{21896102683ix^{7/2}}{21474836480} - \frac{19100991ix^{5/2}}{41943040} + \frac{3367ix^{3/2}}{8192} - \frac{1942089}{21110}\right.$$

2.13 problem 7.3.101 (c)

Internal problem ID [5527]
Internal file name [`OUTPUT/4775_Sunday_June_05_2022_03_05_32_PM_51279556/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.101 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y''x + y'x^5 + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y''x + y'x^5 + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = x^4$$
$$q(x) = \frac{1}{x}$$

Table 30: Table $p(x), q(x)$ singularites.

$p(x) = x^4$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty, -\infty, 0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x + y'x^5 + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) x + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) x^5 + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{4+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{4+n+r} a_n(n+r) = \sum_{n=5}^{\infty} a_{n-5}(n+r-5)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=5}^{\infty} a_{n-5}(n+r-5)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = -\frac{1}{r(1+r)}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{1}{r(1+r)^2(2+r)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = -\frac{1}{r(1+r)^2(2+r)^2(3+r)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{1}{r\,(1+r)^2\,(2+r)^2\,(3+r)^2\,(4+r)}$$

For $5 \leq n$ the recursive equation is

$$a_n(n+r)\,(n+r-1) + a_{n-5}(n+r-5) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n a_{n-5} + r a_{n-5} - 5a_{n-5} + a_{n-1}}{(n+r)\,(n+r-1)} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{-n a_{n-5} + 4a_{n-5} - a_{n-1}}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{r(1+r)^2(2+r)^2(3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^9 - 16r^8 - 106r^7 - 376r^6 - 769r^5 - 904r^4 - 564r^3 - 144r^2 - 1}{r\,(1+r)^2\,(2+r)^2\,(3+r)^2\,(4+r)^2\,(5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{2881}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{r(1+r)}$	$-\frac{1}{2}$
a_2	$\frac{1}{r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$-\frac{1}{r(1+r)^2(2+r)^2(3+r)}$	$-\frac{1}{144}$
a_4	$\frac{1}{r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$
a_5	$\frac{-r^9-16r^8-106r^7-376r^6-769r^5-904r^4-564r^3-144r^2-1}{r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$-\frac{2881}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)\\ &= x\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{2881 x^5}{86400} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}a_N &= a_1\\ &= -\frac{1}{r(1+r)}\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{r \to r_2} -\frac{1}{r(1+r)} &= \lim_{r \to 0} -\frac{1}{r(1+r)}\\ &= \textit{undefined}\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $y''x + y'x^5 + y = 0$ gives

$$
\begin{aligned}
&\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left.+ \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x \\
&+ \left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) x^5 \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(y_1'(x)\,x^5 + y_1''(x)\,x + y_1(x)\right)\ln(x) + \left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\,x^4\right)C \\
&+ \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) x^5 \\
&+ \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1'(x)\,x^5 + y_1''(x)\,x + y_1(x) = 0$$

Eq (7) simplifes to

$$\left(\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)x + y_1(x)\,x^4\right)C + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)x^5 + \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)x + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1}a_n(n+r_1)\right)x + (x^5-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{-1+n+r_2}b_n(n+r_2)\right)x^6 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2}b_n(n+r_2)(-1+n+r_2)\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)x}{x} = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\left(2\left(\sum_{n=0}^{\infty} x^{n}a_n(n+1)\right)x + (x^5-1)\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\right)C}{x} + \frac{\left(\sum_{n=0}^{\infty} x^{n-1}b_n n\right)x^6 + \left(\sum_{n=0}^{\infty} x^{-2+n}b_n n(n-1)\right)x^2 + \left(\sum_{n=0}^{\infty} b_n x^{n}\right)x}{x} = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\,x^n a_n(n+1)\right) + \left(\sum_{n=0}^{\infty} C\,x^{n+5}a_n\right) + \sum_{n=0}^{\infty}(-Ca_n x^n) + \left(\sum_{n=0}^{\infty} n\,x^{4+n}b_n\right) + \left(\sum_{n=0}^{\infty} n\,x^{n-1}b_n(n-1)\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\,x^{n} a_{n}(n+1) &= \sum_{n=1}^{\infty} 2C a_{n-1} n\,x^{n-1} \\
\sum_{n=0}^{\infty} C\,x^{n+5} a_{n} &= \sum_{n=6}^{\infty} C a_{n-6} x^{n-1} \\
\sum_{n=0}^{\infty} \left(-C a_{n} x^{n}\right) &= \sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} n\,x^{4+n} b_{n} &= \sum_{n=5}^{\infty} (n-5)\, b_{n-5} x^{n-1} \\
\sum_{n=0}^{\infty} b_{n} x^{n} &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$
\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\,x^{n-1}\right)+\left(\sum_{n=6}^{\infty} C a_{n-6} x^{n-1}\right)+\sum_{n=1}^{\infty} \left(-C a_{n-1} x^{n-1}\right) + \left(\sum_{n=5}^{\infty} (n-5)\, b_{n-5} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} n\,x^{n-1} b_{n}(n-1)\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0 \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$
C+1=0
$$

Which is solved for C. Solving for C gives

$$
C=-1
$$

For $n=2$, Eq (2B) gives

$$
3C a_1 + b_1 + 2b_2 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 + b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 - \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = \frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 + b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{35}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 + b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 - \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = \frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -1$ and all b_n, then the second solution becomes

$$y_2(x) = (-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right) \\ &\quad + c_2\left((-1)\left(x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right) \\ &\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y &= c_1 x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right) \\ &\quad + c_2\left(-x\left(1-\frac{x}{2}+\frac{x^2}{12}-\frac{x^3}{144}+\frac{x^4}{2880}-\frac{2881x^5}{86400}+O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} - \frac{35x^4}{1728} + \frac{101x^5}{86400} + O(x^6)\right) \end{aligned} \tag{1}$$

Verification of solutions

$$
\begin{aligned}
y = c_1 x \left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{2881 x^5}{86400} + O\left(x^6\right)\right) \\
+ c_2 \left(-x \left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^3}{144} + \frac{x^4}{2880} - \frac{2881 x^5}{86400} + O\left(x^6\right)\right) \ln\left(x\right) + 1 - \frac{3x^2}{4} + \frac{7x^3}{36} \right. \\
\left. - \frac{35 x^4}{1728} + \frac{101 x^5}{86400} + O\left(x^6\right)\right)
\end{aligned}
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying to convert to an ODE of Bessel type
   trying to convert to an ODE of Bessel type
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
Order:=6;
dsolve(x*diff(y(x),x$2)+x^5*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \frac{2881}{86400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(-x + \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{144}x^4 - \frac{1}{2880}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \frac{101}{86400}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 85

```
AsymptoticDSolveValue[x*y''[x]+x^5*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{144}x\left(x^3 - 12x^2 + 72x - 144\right)\log(x) + \frac{-47x^4 + 480x^3 - 2160x^2 + 1728x + 1728}{1728}\right) + c_2\left(\frac{x^5}{2880} - \frac{x^4}{144} + \frac{x^3}{12} - \frac{x^2}{2} + x\right)$$

2.14 problem 7.3.101 (d)

Internal problem ID [5528]
Internal file name [`OUTPUT/4776_Sunday_June_05_2022_03_05_34_PM_16023406/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.101 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\sin(x)\, y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\sin(x)\, y'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -\frac{1}{\sin(x)} \end{aligned}$$

Table 31: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{\sin(x)}$	
singularity	type
$x = \pi Z$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\pi Z]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\sin(x) \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\end{aligned}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5} a_n (n+r)(n+r-1)}{5040}\right) + \left(\sum_{n=0}^{\infty}\frac{x^{n+r+3} a_n (n+r)(n+r-1)}{120}\right) + \sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r} a_n (n+r)(n+r-1)}{6}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{x^{n+r+5} a_n (n+r)(n+r-1)}{5040}\right) = \sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)(n-7+r)\, x^{n+r-1}}{5040}\right)$$

$$\sum_{n=0}^{\infty}\frac{x^{n+r+3} a_n (n+r)(n+r-1)}{120} = \sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)(n-5+r)\, x^{n+r-1}}{120}$$

$$\sum_{n=0}^{\infty}\left(-\frac{x^{1+n+r} a_n (n+r)(n+r-1)}{6}\right) = \sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)(n-3+r)\, x^{n+r-1}}{6}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=6}^{\infty}\left(-\frac{a_{n-6}(n+r-6)\,(n-7+r)\,x^{n+r-1}}{5040}\right) + \left(\sum_{n=4}^{\infty}\frac{a_{n-4}(-4+n+r)\,(n-5+r)\,x^{n+r-1}}{120}\right) + \sum_{n=2}^{\infty}\left(-\frac{a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r-1}}{6}\right) + \left(\sum_{n=0}^{\infty}x^{n+r-1}a_n(n+r)\,(n+r-1)\right) + \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1}a_n(n+r)\,(n+r-1) = 0$$

When $n=0$ the above becomes

$$x^{-1+r}a_0r(-1+r) = 0$$

Or

$$x^{-1+r}a_0r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r}r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r}r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = \frac{1}{r(1+r)}
$$

Substituting $n = 2$ in Eq. (2B) gives

$$
a_2 = \frac{r^4 - r^2 + 6}{6r(1+r)^2(2+r)}
$$

Substituting $n = 3$ in Eq. (2B) gives

$$
a_3 = \frac{r^4 + 2r^3 + 2r^2 + r + 3}{3r(1+r)^2(2+r)^2(3+r)}
$$

Substituting $n = 4$ in Eq. (2B) gives

$$
a_4 = \frac{7r^8 + 56r^7 + 154r^6 + 140r^5 + 103r^4 + 524r^3 + 1536r^2 + 1800r + 1080}{360r(1+r)^2(2+r)^2(3+r)^2(4+r)}
$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{r^8 + 12r^7 + 59r^6 + 153r^5 + 239r^4 + 273r^3 + 331r^2 + 372r + 225}{15r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$$

For $6 \leq n$ the recursive equation is

$$-\frac{a_{n-6}(n+r-6)(n-7+r)}{5040} + \frac{a_{n-4}(-4+n+r)(n-5+r)}{120} - \frac{a_{n-2}(n+r-2)(n-3+r)}{6} + a_n(n+r)(n+r-1) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-6} - 42n^2 a_{n-4} + 840n^2 a_{n-2} + 2nra_{n-6} - 84nra_{n-4} + 1680nra_{n-2} + r^2 a_{n-6} - 42r^2 a_{n-4} + 840r^2}{} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(a_{n-6} - 42a_{n-4} + 840a_{n-2})n^2 + (-11a_{n-6} + 294a_{n-4} - 2520a_{n-2})n + 30a_{n-6} - 504a_{n-4} + 1680a_{n-}}{5040n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{r(1+r)}$	$\frac{1}{2}$
a_2	$\frac{r^4-r^2+6}{6r(1+r)^2(2+r)}$	$\frac{1}{12}$
a_3	$\frac{r^4+2r^3+2r^2+r+3}{3r(1+r)^2(2+r)^2(3+r)}$	$\frac{1}{48}$
a_4	$\frac{7r^8+56r^7+154r^6+140r^5+103r^4+524r^3+1536r^2+1800r+1080}{360r(1+r)^2(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{192}$
a_5	$\frac{r^8+12r^7+59r^6+153r^5+239r^4+273r^3+331r^2+372r+225}{15r(1+r)^2(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$\frac{37}{28800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{48} + \frac{x^4}{192} + \frac{37x^5}{28800} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{1}{r\,(1+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{1}{r\,(1+r)} &= \lim_{r\to 0} \frac{1}{r\,(1+r)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned} \frac{d}{dx} y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\ &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\ \frac{d^2}{dx^2} y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\ &\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\ &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right) \end{aligned}$$

Substituting these back into the given ode $\sin(x)\, y'' - y = 0$ gives

$$\sin(x) \left(C y_1''(x) \ln(x) + \frac{2C y_1'(x)}{x} - \frac{C y_1(x)}{x^2} + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) - C y_1(x) \ln(x) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0$$

Which can be written as

$$\left(\left(\sin(x)\, y_1''(x) - y_1(x)\right) \ln(x) + \sin(x) \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) \right) C + \sin(x) \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$\sin(x)\, y_1''(x) - y_1(x) = 0$$

Eq (7) simplifes to

$$\sin(x) \left(\frac{2 y_1'(x)}{x} - \frac{y_1(x)}{x^2} \right) C + \sin(x) \left(\sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2} (n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2} (n+r_2)}{x^2} \right) \right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\frac{\sin(x) \left(2 \left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n (n+r_1) \right) x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1} \right) \right) C}{x^2} + \sin(x) \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2) \right) - \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\frac{\sin(x)\left(2\left(\sum_{n=0}^{\infty} x^n a_n (1+n)\right) x - \left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right) C}{x^2} + \sin(x)\left(\sum_{n=0}^{\infty} x^{n-2} b_n n(n-1)\right) - \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{10}$$

Expanding $\frac{2\sin(x)C}{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
\frac{2\sin(x)\,C}{x} &= 2C - \frac{1}{3}C\,x^2 + \frac{1}{60}C\,x^4 - \frac{1}{2520}C\,x^6 + \dots \\
&= 2C - \frac{1}{3}C\,x^2 + \frac{1}{60}C\,x^4 - \frac{1}{2520}C\,x^6
\end{aligned}$$

Expanding $-\frac{\sin(x)C}{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
-\frac{\sin(x)\,C}{x} &= -C + \frac{1}{6}C\,x^2 - \frac{1}{120}C\,x^4 + \frac{1}{5040}C\,x^6 + \dots \\
&= -C + \frac{1}{6}C\,x^2 - \frac{1}{120}C\,x^4 + \frac{1}{5040}C\,x^6
\end{aligned}$$

Expanding $\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}
\sin(x) &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \\
&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+6}a_n(1+n)}{2520}\right)+\left(\sum_{n=0}^{\infty}\frac{C\,x^{n+4}a_n(1+n)}{60}\right)\\
&+\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+2}a_n(1+n)}{3}\right)+\left(\sum_{n=0}^{\infty}2a_nx^nC(1+n)\right)\\
&+\sum_{n=0}^{\infty}(-a_nx^nC)+\left(\sum_{n=0}^{\infty}\frac{C\,x^{n+2}a_n}{6}\right)+\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+4}a_n}{120}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{C\,x^{n+6}a_n}{5040}\right)+\sum_{n=0}^{\infty}\left(-\frac{n\,x^{n+5}b_n(n-1)}{5040}\right)\\
&+\left(\sum_{n=0}^{\infty}\frac{n\,x^{n+3}b_n(n-1)}{120}\right)+\sum_{n=0}^{\infty}\left(-\frac{n\,x^{1+n}b_n(n-1)}{6}\right)\\
&+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=0}^{\infty}(-b_nx^n)=0
\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+6}a_n(1+n)}{2520}\right)=\sum_{n=7}^{\infty}\left(-\frac{Ca_{n-7}(n-6)\,x^{n-1}}{2520}\right)$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+4}a_n(1+n)}{60}=\sum_{n=5}^{\infty}\frac{Ca_{-5+n}(n-4)\,x^{n-1}}{60}$$

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+2}a_n(1+n)}{3}\right)=\sum_{n=3}^{\infty}\left(-\frac{Ca_{n-3}(n-2)\,x^{n-1}}{3}\right)$$

$$\sum_{n=0}^{\infty}2a_nx^nC(1+n)=\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}$$

$$\sum_{n=0}^{\infty}(-a_nx^nC)=\sum_{n=1}^{\infty}\left(-Ca_{n-1}x^{n-1}\right)$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+2}a_n}{6}=\sum_{n=3}^{\infty}\frac{Ca_{n-3}x^{n-1}}{6}$$

$$\sum_{n=0}^{\infty}\left(-\frac{C\,x^{n+4}a_n}{120}\right)=\sum_{n=5}^{\infty}\left(-\frac{Ca_{-5+n}x^{n-1}}{120}\right)$$

$$\sum_{n=0}^{\infty}\frac{C\,x^{n+6}a_n}{5040}=\sum_{n=7}^{\infty}\frac{Ca_{n-7}x^{n-1}}{5040}$$

$$\sum_{n=0}^{\infty}\left(-\frac{n\,x^{n+5}b_n(n-1)}{5040}\right)=\sum_{n=6}^{\infty}\left(-\frac{(n-6)\,b_{n-6}(n-7)\,x^{n-1}}{5040}\right)$$

$$\sum_{n=0}^{\infty}\frac{n\,x^{n+3}b_n(n-1)}{120}=\sum_{n=4}^{\infty}\frac{(n-4)\,b_{n-4}(-5+n)\,x^{n-1}}{120}$$

$$\sum_{n=0}^{\infty}\left(-\frac{n\,x^{1+n}b_n(n-1)}{6}\right)=\sum_{n=2}^{\infty}\left(-\frac{(n-2)\,b_{n-2}(n-3)\,x^{n-1}}{6}\right)$$

$$\sum_{n=0}^{\infty}(-b_n x^n)=\sum_{n=1}^{\infty}(-b_{n-1}x^{n-1})$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\sum_{n=7}^{\infty}\left(-\frac{Ca_{n-7}(n-6)\,x^{n-1}}{2520}\right)+\left(\sum_{n=5}^{\infty}\frac{Ca_{-5+n}(n-4)\,x^{n-1}}{60}\right)\\
&+\sum_{n=3}^{\infty}\left(-\frac{Ca_{n-3}(n-2)\,x^{n-1}}{3}\right)+\left(\sum_{n=1}^{\infty}2Ca_{n-1}n\,x^{n-1}\right)\\
&+\sum_{n=1}^{\infty}(-Ca_{n-1}x^{n-1})+\left(\sum_{n=3}^{\infty}\frac{Ca_{n-3}x^{n-1}}{6}\right)+\sum_{n=5}^{\infty}\left(-\frac{Ca_{-5+n}x^{n-1}}{120}\right)\\
&+\left(\sum_{n=7}^{\infty}\frac{Ca_{n-7}x^{n-1}}{5040}\right)+\sum_{n=6}^{\infty}\left(-\frac{(n-6)\,b_{n-6}(n-7)\,x^{n-1}}{5040}\right)\\
&+\left(\sum_{n=4}^{\infty}\frac{(n-4)\,b_{n-4}(-5+n)\,x^{n-1}}{120}\right)\\
&+\sum_{n=2}^{\infty}\left(-\frac{(n-2)\,b_{n-2}(n-3)\,x^{n-1}}{6}\right)\\
&+\left(\sum_{n=0}^{\infty}n\,x^{n-1}b_n(n-1)\right)+\sum_{n=1}^{\infty}(-b_{n-1}x^{n-1})=0
\end{aligned}\tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 - b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$\frac{(-a_0 + 30a_2)\,C}{6} - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 + 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{1}{6}$$

For $n = 4$, Eq (2B) gives

$$\frac{(-3a_1 + 42a_3)\,C}{6} - \frac{b_2}{3} - b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{5}{16} + 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{5}{192}$$

For $n = 5$, Eq (2B) gives

$$\frac{(a_0 - 100a_2 + 1080a_4)\,C}{120} - b_3 - b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{257}{1440}+20b_5=0$$

Solving the above for b_5 gives

$$b_5=-\frac{257}{28800}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x)=Cy_1(x)\ln(x)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)$$

Using the above value found for $C=1$ and all b_n, then the second solution becomes

$$\begin{aligned}y_2(x)=1&\left(x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\right)\ln(x)\\&+1-\frac{3x^2}{4}-\frac{x^3}{6}-\frac{5x^4}{192}-\frac{257x^5}{28800}+O(x^6)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x)&=c_1y_1(x)+c_2y_2(x)\\&=c_1x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\\&\quad+c_2\left(1\left(x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\right)\ln(x)+1-\frac{3x^2}{4}-\frac{x^3}{6}\right.\\&\quad\left.-\frac{5x^4}{192}-\frac{257x^5}{28800}+O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y&=y_h\\&=c_1x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\\&\quad+c_2\left(x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}-\frac{x^3}{6}-\frac{5x^4}{192}\right.\\&\quad\left.-\frac{257x^5}{28800}+O(x^6)\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x \left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right) + c_2 \left(x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}-\frac{x^3}{6}-\frac{5x^4}{192}-\frac{257x^5}{28800}+O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x \left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right) + c_2 \left(x\left(1+\frac{x}{2}+\frac{x^2}{12}+\frac{x^3}{48}+\frac{x^4}{192}+\frac{37x^5}{28800}+O(x^6)\right)\ln(x)+1-\frac{3x^2}{4}-\frac{x^3}{6}-\frac{5x^4}{192}-\frac{257x^5}{28800}+O(x^6)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying an equivalence, under non-integer power transformations,
      to LODEs admitting Liouvillian solutions.
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Whittaker
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0, e <> 0,
   Change of variables used:
      [x = arccos(t)]
   Linear ODE actually solved:
      -(-t^2+1)^(1/2)*u(t)+(t^3-t)*diff(u(t),t)+(t^4-2*t^2+1)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 58

```
Order:=6;
dsolve(sin(x)*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{48}x^3 + \frac{1}{192}x^4 + \frac{37}{28800}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right) \left(x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{48}x^4 + \frac{1}{192}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(1 - \frac{3}{4}x^2 - \frac{1}{6}x^3 - \frac{5}{192}x^4 - \frac{257}{28800}x^5 + \mathrm{O}\left(x^6\right)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 85

```
AsymptoticDSolveValue[Sin[x]*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{1}{48}x\left(x^3 + 4x^2 + 24x + 48\right)\log(x) + \frac{1}{64}\left(-3x^4 - 16x^3 - 80x^2 - 64x + 64\right)\right) + c_2 \left(\frac{x^5}{192} + \frac{x^4}{48} + \frac{x^3}{12} + \frac{x^2}{2} + x\right)$$

2.15 problem 7.3.101 (e)

Internal problem ID [5529]
Internal file name [`OUTPUT/4777_Sunday_June_05_2022_03_05_37_PM_75403330/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.101 (e).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\cos(x)\, y'' - \sin(x)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (49)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (50)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{\sin(x)\,y}{\cos(x)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \tan(x)\,y' + \sec(x)^2\,y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y\tan(x)^2 + 2\sec(x)^2\left(y\tan(x) + y'\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \left(\tan(x)^2 + 6\sec(x)^2\tan(x)\right) y' + 6\sec(x)^2\left(\sec(x)^2 + \frac{2\tan(x)}{3} - \frac{2}{3}\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(24\sec(x)^4 + (6\tan(x) - 16)\sec(x)^2\right) y' + 24y\left(\left(\tan(x) + \frac{3}{4}\right)\sec(x)^4 + \left(-\frac{7\tan(x)}{24} - \frac{7}{12}\right)\sec(
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= y(0) \\
F_2 &= 2y'(0) \\
F_3 &= 2y(0) \\
F_4 &= 4y(0) + 8y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \frac{1}{180}x^6\right) y(0) + \left(x + \frac{1}{12}x^4 + \frac{1}{90}x^6\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{\sin(x) \left(\sum_{n=0}^{\infty} a_n x^n\right)}{\cos(x)} \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \end{aligned}$$

Expanding $-\sin(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} -\sin(x) &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots \\ &= -x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(-x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{5040}x^7\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0$$

Expanding the first term in (1) gives

$$
1\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)-\frac{x^{2}}{2}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\frac{x^{4}}{24}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)
-\frac{x^{6}}{720}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(-x+\frac{1}{6}x^{3}-\frac{1}{120}x^{5}+\frac{1}{5040}x^{7}\right)\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0
$$

Expanding the second term in (1) gives

$$
1\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)-\frac{x^{2}}{2}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\frac{x^{4}}{24}
\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)-\frac{x^{6}}{720}\cdot\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+-x\cdot\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)
+\frac{x^{3}}{6}\cdot\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)-\frac{x^{5}}{120}\cdot\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)+\frac{x^{7}}{5040}\cdot\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right)=0
$$

Which simplifies to

$$
\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_{n}(n-1)}{720}\right)+\left(\sum_{n=2}^{\infty}\frac{n\, x^{n+2} a_{n}(n-1)}{24}\right)
+\sum_{n=2}^{\infty}\left(-\frac{n a_{n} x^{n}(n-1)}{2}\right)+\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\sum_{n=0}^{\infty}\left(-x^{1+n} a_{n}\right)
+\left(\sum_{n=0}^{\infty}\frac{x^{n+3} a_{n}}{6}\right)+\sum_{n=0}^{\infty}\left(-\frac{x^{n+5} a_{n}}{120}\right)+\left(\sum_{n=0}^{\infty}\frac{x^{n+7} a_{n}}{5040}\right)=0 \tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_{n}(n-1)}{720}\right)=\sum_{n=6}^{\infty}\left(-\frac{(n-4)\, a_{n-4}(n-5)\, x^{n}}{720}\right)
$$

$$
\sum_{n=2}^{\infty}\frac{n\, x^{n+2} a_{n}(n-1)}{24}=\sum_{n=4}^{\infty}\frac{(n-2)\, a_{n-2}(n-3)\, x^{n}}{24}
$$

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+3} a_n}{6} = \sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}$$

$$\sum_{n=0}^{\infty} \left(-\frac{x^{n+5} a_n}{120}\right) = \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right)$$

$$\sum_{n=0}^{\infty} \frac{x^{n+7} a_n}{5040} = \sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned} &\sum_{n=6}^{\infty} \left(-\frac{(n-4)\, a_{n-4}(n-5)\, x^n}{720}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-2)\, a_{n-2}(n-3)\, x^n}{24}\right) \\ &+ \sum_{n=2}^{\infty} \left(-\frac{n a_n x^n (n-1)}{2}\right) + \left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(1+n)\, x^n\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^n\right) \\ &+ \left(\sum_{n=3}^{\infty} \frac{a_{n-3} x^n}{6}\right) + \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} x^n}{120}\right) + \left(\sum_{n=7}^{\infty} \frac{a_{n-7} x^n}{5040}\right) = 0 \end{aligned} \tag{3}$$

$n = 1$ gives

$$6a_3 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6}$$

$n = 2$ gives

$$-a_2 + 12a_4 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_1}{12}$$

$n = 3$ gives

$$-3a_3 + 20a_5 - a_2 + \frac{a_0}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_5 = \frac{a_0}{60}$$

$n = 4$ gives

$$\frac{a_2}{12} - 6a_4 + 30a_6 - a_3 + \frac{a_1}{6} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = \frac{a_0}{180} + \frac{a_1}{90}$$

$n = 5$ gives

$$\frac{a_3}{4} - 10a_5 + 42a_7 - a_4 + \frac{a_2}{6} - \frac{a_0}{120} = 0$$

Which after substituting earlier equations, simplifies to

$$a_7 = \frac{a_0}{315} + \frac{a_1}{504}$$

For $7 \leq n$, the recurrence equation is

$$\begin{aligned} &-\frac{(n-4)\, a_{n-4}(n-5)}{720} + \frac{(n-2)\, a_{n-2}(n-3)}{24} - \frac{n a_n (n-1)}{2} \\ &+ (n+2)\, a_{n+2}(1+n) - a_{n-1} + \frac{a_{n-3}}{6} - \frac{a_{n-5}}{120} + \frac{a_{n-7}}{5040} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &\\ = \frac{2520n^2a_n + 7n^2a_{n-4} - 210n^2a_{n-2} - 2520na_n - 63na_{n-4} + 1050na_{n-2} - a_{n-7} + 42a_{n-5} + 140a_{n-4} - 84}{5040\,(n+2)\,(1+n)} \end{aligned}$$

$$\begin{aligned} &= \frac{(2520n^2 - 2520n)\, a_n}{5040\,(n+2)\,(1+n)} - \frac{a_{n-7}}{5040\,(n+2)\,(1+n)} + \frac{a_{n-5}}{120\,(n+2)\,(1+n)} \\ &+ \frac{(7n^2 - 63n + 140)\, a_{n-4}}{5040\,(n+2)\,(1+n)} - \frac{a_{n-3}}{6\,(n+2)\,(1+n)} \\ &+ \frac{(-210n^2 + 1050n - 1260)\, a_{n-2}}{5040\,(n+2)\,(1+n)} + \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1 x + \frac{1}{6} a_0 x^3 + \frac{1}{12} a_1 x^4 + \frac{1}{60} a_0 x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) a_0 + \left(x + \frac{1}{12} x^4\right) a_1 + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O\left(x^6\right)
$$

Summary

The solution(s) found are the following

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4 + \frac{1}{90} x^6\right) y'(0) + O\left(x^6\right) \tag{1}
$$

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O\left(x^6\right) \tag{2}
$$

Verification of solutions

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5 + \frac{1}{180} x^6\right) y(0) + \left(x + \frac{1}{12} x^4 + \frac{1}{90} x^6\right) y'(0) + O\left(x^6\right)
$$

Verified OK.

$$
y = \left(1 + \frac{1}{6} x^3 + \frac{1}{60} x^5\right) c_1 + \left(x + \frac{1}{12} x^4\right) c_2 + O\left(x^6\right)
$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1([a
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebius
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) * 2F1
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
Order:=6;
dsolve(cos(x)*diff(y(x),x$2)-sin(x)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{6}x^3 + \frac{1}{60}x^5\right) y(0) + \left(x + \frac{1}{12}x^4\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 35

```
AsymptoticDSolveValue[Cos[x]*y''[x]-Sin[x]*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{12} + x\right) + c_1\left(\frac{x^5}{60} + \frac{x^3}{6} + 1\right)$$

2.16 problem 7.3.102

Internal problem ID [5530]
Internal file name [`OUTPUT/4778_Sunday_June_05_2022_03_05_40_PM_71362295/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.102.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference not integer"**

Maple gives the following as the ode type

`[[_Emden, _Fowler]]`

$$\boxed{x^2y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{x^2}$$

Table 32: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) \right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r} \right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$r_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = \sqrt{5}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{\sqrt{5}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{\sqrt{5}}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2} - \frac{\sqrt{5}}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{1}{2}+\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right) + c_2 x^{\frac{1}{2}-\frac{\sqrt{5}}{2}}\left(1 + O(x^6)\right)$$

Verified OK.

2.16.1 Maple step by step solution

Let's solve

$x^2y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - y = 0$

- Make a change of variables

  $t = \ln(x)$

- ☐ Substitute the change of variables back into the ODE

  - ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) - y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - r - 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1\pm\left(\sqrt{5}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2}-\frac{\sqrt{5}}{2}, \frac{1}{2}+\frac{\sqrt{5}}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t}$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t}$
- Change variables back using $t = \ln(x)$

  $y = c_1\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)\ln(x)} + c_2\mathrm{e}^{\ln(x)\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}$
- Simplify

  $y = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}}c_1 + x^{\frac{\sqrt{5}}{2}}c_2\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 39

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \sqrt{x}\left(x^{-\frac{\sqrt{5}}{2}}c_1 + x^{\frac{\sqrt{5}}{2}}c_2\right) + O\left(x^6\right)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x^2*y''[x]-y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{\frac{1}{2}\left(1+\sqrt{5}\right)} + c_2 x^{\frac{1}{2}\left(1-\sqrt{5}\right)}$$

2.17 problem 7.3.103

Internal problem ID [5531]
Internal file name [`OUTPUT/4779_Sunday_June_05_2022_03_05_41_PM_82105689/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.103.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + \left(x - \frac{3}{4}\right) y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x - \frac{3}{4}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{-3 + 4x}{4x^2}$$

Table 34: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{-3+4x}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + \left(x - \frac{3}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + \left(x - \frac{3}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{3 a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{3a_n x^{n+r}}{4}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)\,(n+r-1) - \frac{3a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - \frac{3a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) - \frac{3x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2 - 4r - 3)\,x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r - \frac{3}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 4r - 3)\,x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1} - \frac{3a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-1}}{4n^2 + 8nr + 4r^2 - 4n - 4r - 3} \tag{4}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{4}{4r^2 + 4r - 3}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_1 = -\frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{4r^2+4r-3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_2 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{4r^2+4r-3}$	$-\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{64}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)(4r^2 + 20r + 21)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = -\frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{4r^2+4r-3}$	$-\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$-\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{1}{360}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{256}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)(4r^2 + 20r + 21)(4r^2 + 28r + 45)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{1}{8640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{4r^2+4r-3}$	$-\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$-\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{1}{360}$
a_4	$\frac{256}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{1}{8640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1024}{(2r + 11)(2r + 5)^2 (2r + 3)^2 (2r - 1)(2r + 9)(2r + 7)^2 (2r + 1)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = -\frac{1}{302400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{4}{4r^2+4r-3}$	$-\frac{1}{3}$
a_2	$\frac{16}{(4r^2+4r-3)(4r^2+12r+5)}$	$\frac{1}{24}$
a_3	$-\frac{64}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)}$	$-\frac{1}{360}$
a_4	$\frac{256}{(4r^2+4r-3)(4r^2+12r+5)(4r^2+20r+21)(4r^2+28r+45)}$	$\frac{1}{8640}$
a_5	$-\frac{1024}{(2r+11)(2r+5)^2(2r+3)^2(2r-1)(2r+9)(2r+7)^2(2r+1)}$	$-\frac{1}{302400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{\frac{3}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{\frac{3}{2}}\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)} &= \lim_{r\to -\frac{1}{2}} \frac{16}{(4r^2 + 4r - 3)(4r^2 + 12r + 5)} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' + \left(x - \frac{3}{4}\right)y = 0$ gives

$$
\begin{aligned}
&x^2\Bigg(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&+ \sum_{n=0}^{\infty} \left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\Bigg) \\
&+ \left(x - \frac{3}{4}\right)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2y_1''(x) + \left(x - \frac{3}{4}\right)y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C \\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ \left(x - \frac{3}{4}\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$
x^2y_1''(x) + \left(x - \frac{3}{4}\right)y_1(x) = 0
$$

Eq (7) simplifes to

$$x^{2}\left(\frac{2y_{1}'(x)}{x}-\frac{y_{1}(x)}{x^{2}}\right)C+x^{2}\left(\sum_{n=0}^{\infty}\left(\frac{b_{n}x^{n+r_{2}}(n+r_{2})^{2}}{x^{2}}-\frac{b_{n}x^{n+r_{2}}(n+r_{2})}{x^{2}}\right)\right) + \left(x-\frac{3}{4}\right)\left(\sum_{n=0}^{\infty}b_{n}x^{n+r_{2}}\right)=0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty}x^{-1+n+r_{1}}a_{n}(n+r_{1})\right)x-\left(\sum_{n=0}^{\infty}a_{n}x^{n+r_{1}}\right)\right)C + \left(\sum_{n=0}^{\infty}x^{-2+n+r_{2}}b_{n}(n+r_{2})\left(-1+n+r_{2}\right)\right)x^{2} + \left(\sum_{n=0}^{\infty}b_{n}x^{n+r_{2}}\right)x-\frac{3\left(\sum_{n=0}^{\infty}b_{n}x^{n+r_{2}}\right)}{4}=0 \tag{9}$$

Since $r_1 = \frac{3}{2}$ and $r_2 = -\frac{1}{2}$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty}x^{\frac{1}{2}+n}a_{n}\left(n+\frac{3}{2}\right)\right)x-\left(\sum_{n=0}^{\infty}a_{n}x^{n+\frac{3}{2}}\right)\right)C + \left(\sum_{n=0}^{\infty}x^{-\frac{5}{2}+n}b_{n}\left(n-\frac{1}{2}\right)\left(-\frac{3}{2}+n\right)\right)x^{2} + \left(\sum_{n=0}^{\infty}b_{n}x^{n-\frac{1}{2}}\right)x-\frac{3\left(\sum_{n=0}^{\infty}b_{n}x^{n-\frac{1}{2}}\right)}{4}=0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty}C\,x^{n+\frac{3}{2}}a_{n}(2n+3)\right)+\sum_{n=0}^{\infty}\left(-Ca_{n}x^{n+\frac{3}{2}}\right) + \left(\sum_{n=0}^{\infty}\frac{x^{n-\frac{1}{2}}b_{n}(4n^{2}-8n+3)}{4}\right)+\left(\sum_{n=0}^{\infty}x^{\frac{1}{2}+n}b_{n}\right)+\sum_{n=0}^{\infty}\left(-\frac{3b_{n}x^{n-\frac{1}{2}}}{4}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n-\frac{1}{2}$ in each summation term. Going over each summation term above with power of x in it which is not already $x^{n-\frac{1}{2}}$ and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} C\,x^{n+\frac{3}{2}} a_n(2n+3) = \sum_{n=2}^{\infty} C a_{n-2}(2n-1)\,x^{n-\frac{1}{2}}$$

$$\sum_{n=0}^{\infty} \left(-C a_n x^{n+\frac{3}{2}}\right) = \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-\frac{1}{2}}\right)$$

$$\sum_{n=0}^{\infty} x^{\frac{1}{2}+n} b_n = \sum_{n=1}^{\infty} b_{n-1} x^{n-\frac{1}{2}}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-\frac{1}{2}$.

$$\begin{aligned}\left(\sum_{n=2}^{\infty} C a_{n-2}(2n-1)\,x^{n-\frac{1}{2}}\right) + \sum_{n=2}^{\infty} \left(-C a_{n-2} x^{n-\frac{1}{2}}\right) \\ + \left(\sum_{n=0}^{\infty} \frac{x^{n-\frac{1}{2}} b_n(4n^2-8n+3)}{4}\right) \\ + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-\frac{1}{2}}\right) + \sum_{n=0}^{\infty} \left(-\frac{3 b_n x^{n-\frac{1}{2}}}{4}\right) = 0\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$-b_1 + b_0 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-b_1 + 1 = 0$$

Solving the above for b_1 gives

$$b_1 = 1$$

For $n=N$, where $N=2$ which is the difference between the two roots, we are free to choose $b_2=0$. Hence for $n=2$, Eq (2B) gives

$$2C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{2}$$

For $n = 3$, Eq (2B) gives

$$4Ca_1 + b_2 + 3b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$3b_3 + \frac{2}{3} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{2}{9}$$

For $n = 4$, Eq (2B) gives

$$6Ca_2 + b_3 + 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$8b_4 - \frac{25}{72} = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{25}{576}$$

For $n = 5$, Eq (2B) gives

$$8Ca_3 + b_4 + 15b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$15b_5 + \frac{157}{2880} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{157}{43200}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{2}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{2}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right)\right)\ln(x) + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{\sqrt{x}}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{1}{2}\left(x^{\frac{3}{2}}\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right)\right) \ln(x) \right. \\
&\quad \left. + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^{\frac{3}{2}}\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \ln(x)}{2} \right. \\
&\quad \left. + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{\sqrt{x}}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\begin{aligned}
y &= c_1 x^{\frac{3}{2}} \left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \\
&\quad + c_2 \left(-\frac{x^{\frac{3}{2}}\left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \frac{x^5}{302400} + O(x^6)\right) \ln(x)}{2} \right. \\
&\quad \left. + \frac{1 + x - \frac{2x^3}{9} + \frac{25x^4}{576} - \frac{157x^5}{43200} + O(x^6)}{\sqrt{x}}\right)
\end{aligned} \tag{1}
$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} \left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O(x^6)\right) + c_2 \left(-\frac{x^{\frac{3}{2}}\left(1-\frac{x}{3}+\frac{x^2}{24}-\frac{x^3}{360}+\frac{x^4}{8640}-\frac{x^5}{302400}+O(x^6)\right)\ln(x)}{2} + \frac{1+x-\frac{2x^3}{9}+\frac{25x^4}{576}-\frac{157x^5}{43200}+O(x^6)}{\sqrt{x}}\right)$$

Verified OK.

2.17.1 Maple step by step solution

Let's solve

$x^2 y'' + \left(x-\frac{3}{4}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(-3+4x)y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{(-3+4x)y}{4x^2} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = \frac{-3+4x}{4x^2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = -\frac{3}{4}$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $4x^2y'' + (-3+4x)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - Convert $x^2 \cdot y''$ to series expansion

    $x^2 \cdot y'' = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\,x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-3+2r)\,x^r + \left(\sum_{k=1}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-3) + 4a_{k-1}\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-3+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{3}{2}\right\}$

- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r+\frac{1}{2}\right)\left(k+r-\frac{3}{2}\right) a_k + 4a_{k-1} = 0$

- Shift index using $k->k+1$

  $4\left(k+\frac{3}{2}+r\right)\left(k-\frac{1}{2}+r\right) a_{k+1} + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{4a_k}{(2k+3+2r)(2k-1+2r)}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+1} = -\frac{4a_k}{(2k+2)(2k-2)}$

- Series not valid for $r = -\frac{1}{2}$, division by 0 in the recursion relation at $k = 1$

$$a_{k+1} = -\frac{4a_k}{(2k+2)(2k-2)}$$

- Recursion relation for $r = \frac{3}{2}$

$$a_{k+1} = -\frac{4a_k}{(2k+6)(2k+2)}$$

- Solution for $r = \frac{3}{2}$

$$\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+1} = -\frac{4a_k}{(2k+6)(2k+2)}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 65

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)+(x-3/4)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \frac{c_1 x^2 \left(1 - \frac{1}{3}x + \frac{1}{24}x^2 - \frac{1}{360}x^3 + \frac{1}{8640}x^4 - \frac{1}{302400}x^5 + \mathrm{O}\left(x^6\right)\right) + c_2 \left(\ln\left(x\right)\left(x^2 - \frac{1}{3}x^3 + \frac{1}{24}x^4 - \frac{1}{360}x^5 + \mathrm{O}\left(x^6\right.\right.\right.}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 101

```
AsymptoticDSolveValue[x^2*y''[x]+(x-3/4)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^{11/2}}{8640}-\frac{x^{9/2}}{360}+\frac{x^{7/2}}{24}-\frac{x^{5/2}}{3}+x^{3/2}\right)+c_1\left(\frac{31x^4-176x^3+144x^2+576x+576}{576\sqrt{x}}-\frac{1}{48}x^{3/2}\left(x^2-8x+24\right)\log(x)\right)$$

2.18 problem 7.3.104 (d)

Internal problem ID [5532]
Internal file name [`OUTPUT/4780_Sunday_June_05_2022_03_05_44_PM_5666023/index.tex`]

Book: Notes on Diffy Qs. Differential Equations for Engineers. By by Jiri Lebl, 2013.
Section: Chapter 7. POWER SERIES METHODS. 7.3.2 The method of Frobenius. Exercises. page 300
Problem number: 7.3.104 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Repeated root"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Table 36: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)-x^{n+r} a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-x^r r+x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(-1+r)^2 x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$(-1+r)^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=1$$
$$r_2=1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)^2 x^r=0$$

Solving for r gives the roots of the indicial equation as Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{(1A)}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 1$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+1}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 1$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=1)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	0	0	0	0
b_3	0	0	0	0
b_4	0	0	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= x\left(1 + O(x^6)\right) \ln(x) + x O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + O(x^6)\right) + c_2\left(x\left(1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + O(x^6)\right) + c_2\left(x\left(1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x\left(1 + O(x^6)\right) + c_2\left(x\left(1 + O(x^6)\right) \ln(x) + x O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x\left(1 + O(x^6)\right) + c_2\left(x\left(1 + O(x^6)\right) \ln(x) + x O(x^6)\right)$$

Verified OK.

2.18.1 Maple step by step solution

Let's solve

$x^2y'' - xy' + y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

$x^2y'' - xy' + y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

 ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

 $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

 ◦ Compute derivative

 $y' = \frac{\frac{d}{dt}y(t)}{x}$

 ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

 $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

 ◦ Compute derivative

 $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - \frac{d}{dt}y(t) + y(t) = 0$

- Simplify

$\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2=0$
- Root of the characteristic polynomial

  $r=1$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t)=t\,\mathrm{e}^t$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^t+c_2t\,\mathrm{e}^t$
- Change variables back using $t=\ln(x)$

  $y=c_2\ln(x)\,x+c_1x$
- Simplify

  $y=x(c_1+c_2\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x)=x(c_2\ln(x)+c_1)+O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 14

```
AsymptoticDSolveValue[x^2*y''[x]-x*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1 x + c_2 x \log(x)$$