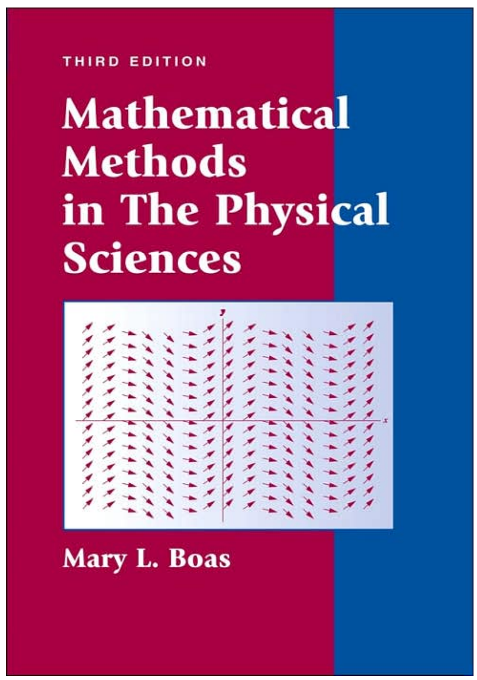
A Solution Manual For

Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006



Nasser M. Abbasi

May 15, 2024

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1 Chapter 8, Ordinary differential equations. Section 1. Introduction. page 394

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1.1 problem 1

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Internal problem ID [4748]
Internal file name [`OUTPUT/4241_Sunday_June_05_2022_12_46_09_PM_63695218/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 1. Introduction. page 394
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y = 0}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned}
\int \frac{1}{y} dy &= x + c_1 \\
\ln(y) &= x + c_1 \\
y &= \mathrm{e}^{x+c_1} \\
y &= c_1 \mathrm{e}^{x}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} \tag{1}$$

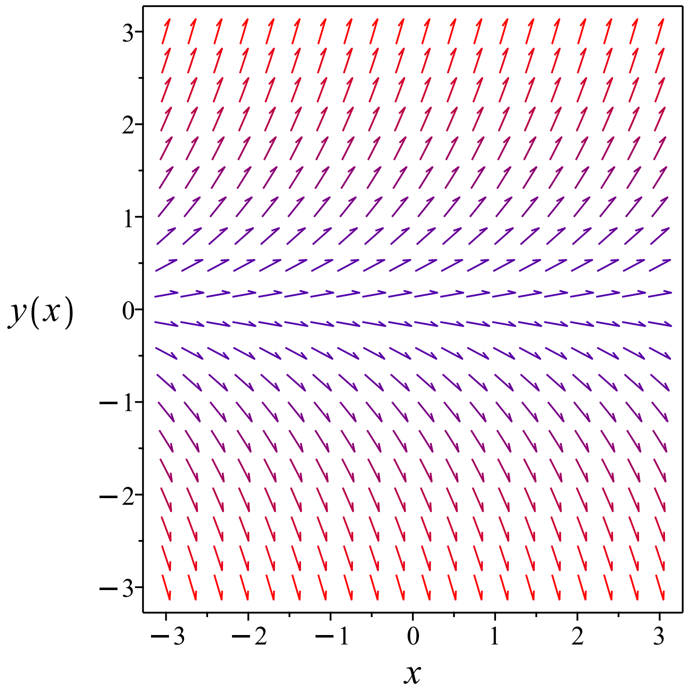

Figure 1: Slope field plot

Verification of solutions

$$y = c_1 e^x$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$y' - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln(y) = x + c_1$$

- Solve for y

$$y = e^{x+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 8

```
dsolve(diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^x$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x + c_2 e^{-x}$$

2 Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398

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2.1 problem 1

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Internal problem ID [4749]
Internal file name [`OUTPUT/4242_Sunday_June_05_2022_12_46_19_PM_96788878/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y + xy' = 0}$$

With initial conditions

$$[y(2) = 3]$$

2.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. Hence solution exists and is unique.

2.1.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\ \ln(y) &= \ln(x) + c_1 \\ y &= e^{\ln(x)+c_1} \\ &= c_1 x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = 2c_1$$

$$c_1 = \frac{3}{2}$$

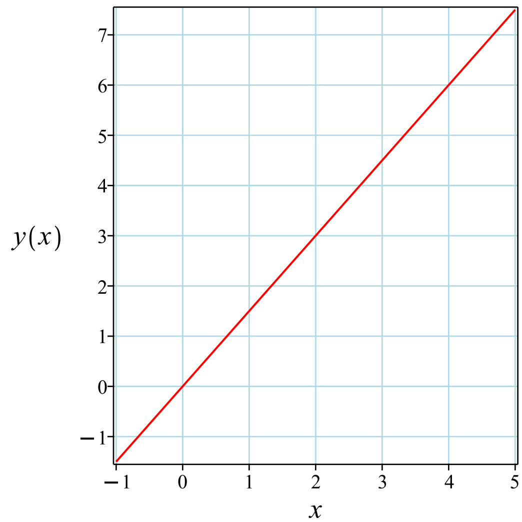
Substituting c_1 found above in the general solution gives

$$y = \frac{3x}{2}$$

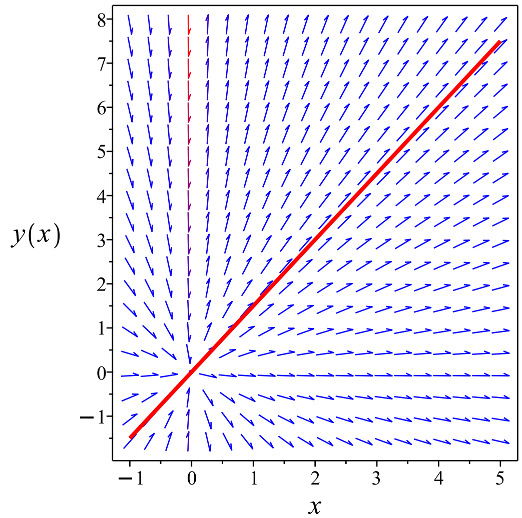
Summary

The solution(s) found are the following

$$y = \frac{3x}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3x}{2}$$

Verified OK.

2.1.3 Maple step by step solution

Let's solve

$[-y + xy' = 0, y(2) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

- Use initial condition $y(2) = 3$

  $3 = 2\,\mathrm{e}^{c_1}$
- Solve for c_1

  $c_1 = \ln\left(\frac{3}{2}\right)$
- Substitute $c_1 = \ln\left(\frac{3}{2}\right)$ into general solution and simplify

  $y = \frac{3x}{2}$
- Solution to the IVP

  $y = \frac{3x}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve([x*diff(y(x),x)=y(x),y(2) = 3],y(x), singsol=all)
```

$$y(x) = \frac{3x}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 10

```
DSolve[{x*y'[x]==y[x],{y[2]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x}{2}$$

2.2 problem 2

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Internal problem ID [4750]
Internal file name [`OUTPUT/4243_Sunday_June_05_2022_12_46_28_PM_67312242/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0}$$

With initial conditions

$$\left[y\left(\frac{1}{2}\right)=\frac{1}{2}\right]$$

2.2.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}} \end{aligned}$$

The x domain of $f(x,y)$ when $y=\frac{1}{2}$ is

$$\{-1<x<1\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{1}{2}$ is

$$\{-1 \leq y < 0, 0 < y \leq 1\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}\right)\\ &= \frac{x}{\sqrt{-y^2+1}\,\sqrt{-x^2+1}} + \frac{x\sqrt{-y^2+1}}{y^2\sqrt{-x^2+1}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{1}{2}$ is

$$\{-1 < x < 1\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{1}{2}$ is

$$\{-1 < y < 0, 0 < y < 1\}$$

And the point $y_0 = \frac{1}{2}$ is inside this domain. Therefore solution exists and is unique.

2.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{x\sqrt{-y^2+1}}{y\sqrt{-x^2+1}}\end{aligned}$$

Where $f(x) = -\frac{x}{\sqrt{-x^2+1}}$ and $g(y) = \frac{\sqrt{-y^2+1}}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy &= -\frac{x}{\sqrt{-x^2+1}}\,dx\\ \int \frac{1}{\frac{\sqrt{-y^2+1}}{y}}\,dy &= \int -\frac{x}{\sqrt{-x^2+1}}\,dx\\ -\sqrt{-y^2+1} &= \sqrt{-x^2+1} + c_1\end{aligned}$$

The solution is

$$-\sqrt{1-y^2}-\sqrt{-x^2+1}-c_1=0$$

Initial conditions are used to solve for c_1. Substituting $x=\frac{1}{2}$ and $y=\frac{1}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$-\sqrt{3}-c_1=0$$

$$c_1=-\sqrt{3}$$

Substituting c_1 found above in the general solution gives

$$-\sqrt{-y^2+1}-\sqrt{-x^2+1}+\sqrt{3}=0$$

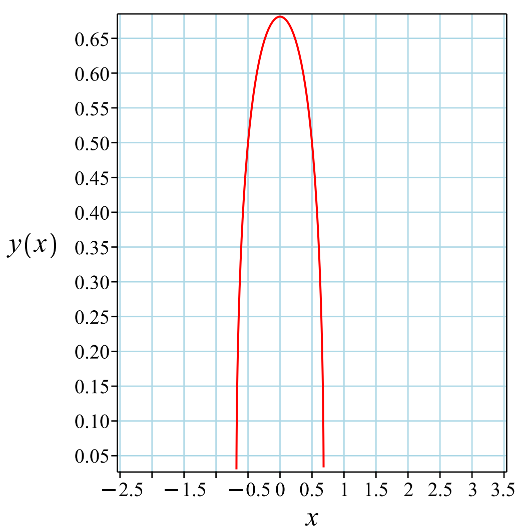
Solving for y from the above gives

$$y=\sqrt{2\sqrt{-x^2+1}\,\sqrt{3}+x^2-3}$$

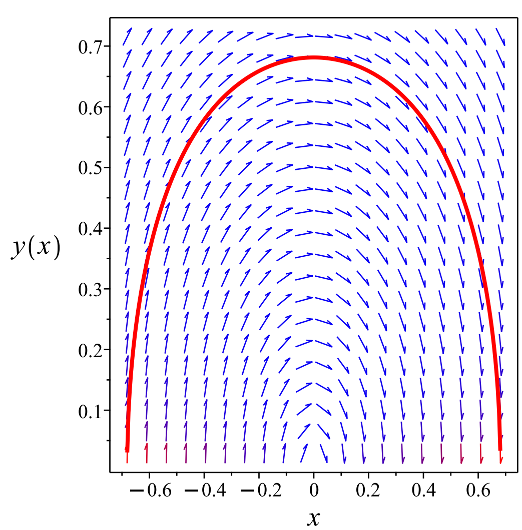
Summary

The solution(s) found are the following

$$y=\sqrt{2\sqrt{-x^2+1}\,\sqrt{3}+x^2-3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{2\sqrt{-x^2+1}\,\sqrt{3}+x^2-3}$$

Verified OK.

2.2.3 Maple step by step solution

Let's solve

$\left[x\sqrt{1-y^2}+y\sqrt{-x^2+1}\,y'=0, y\left(\frac{1}{2}\right)=\frac{1}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{1-y^2}}=-\frac{x}{\sqrt{-x^2+1}}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{1-y^2}}dx=\int -\frac{x}{\sqrt{-x^2+1}}dx+c_1$

- Evaluate integral

  $-\sqrt{1-y^2}=-\frac{(x-1)(x+1)}{\sqrt{-x^2+1}}+c_1$

- Solve for y

  $\left\{y=\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}, y=-\sqrt{-2c_1\sqrt{-x^2+1}-c_1^2+x^2}\right\}$

- Use initial condition $y\left(\frac{1}{2}\right)=\frac{1}{2}$

  $\frac{1}{2}=\sqrt{-\frac{c_1\sqrt{3}\,\sqrt{4}}{2}-c_1^2+\frac{1}{4}}$

- Solve for c_1

  $c_1=\left(0,-\sqrt{3}\right)$

- Substitute $c_1=\left(0,-\sqrt{3}\right)$ into general solution and simplify

  $y=\mathrm{csgn}(x)\,x$

- Use initial condition $y\left(\frac{1}{2}\right)=\frac{1}{2}$

  $\frac{1}{2}=-\sqrt{-\frac{c_1\sqrt{3}\,\sqrt{4}}{2}-c_1^2+\frac{1}{4}}$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \operatorname{csgn}(x)\, x$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 26

```
dsolve([x*sqrt(1-y(x)^2)+y(x)*sqrt(1-x^2)*diff(y(x),x)=0,y(1/2) = 1/2],y(x), singsol=all)
```

$$y(x) = \sqrt{2\sqrt{3}\,\sqrt{-x^2+1}+x^2-3}$$

✓ Solution by Mathematica

Time used: 3.578 (sec). Leaf size: 38

```
DSolve[{x*Sqrt[1-y[x]^2]+y[x]*Sqrt[1-x^2]*y'[x]==0,{y[1/2]==1/2}},y[x],x,IncludeSingularSolu
```

$$y(x) \to \sqrt{x^2}$$
$$y(x) \to \sqrt{x^2+2\sqrt{3-3x^2}-3}$$

2.3 problem 3

2.3.1 Existence and uniqueness analysis 16
2.3.2 Solving as separable ode 17
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Internal problem ID [4751]
Internal file name [`OUTPUT/4244_Sunday_June_05_2022_12_46_38_PM_35319905/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' \sin(x) - y \ln(y) = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{3}\right) = \mathrm{e}\right]$$

2.3.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{y \ln(y)}{\sin(x)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \mathrm{e}$ is

$$\{x < \pi\_Z60 \lor \pi\_Z60 < x\}$$

And the point $x_0 = \frac{\pi}{3}$ is inside this domain. The y domain of $f(x,y)$ when $x = \frac{\pi}{3}$ is

$$\{0 < y\}$$

And the point $y_0 = \mathrm{e}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y \ln(y)}{\sin(x)}\right) \\ &= \frac{\ln(y)}{\sin(x)} + \frac{1}{\sin(x)}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \mathrm{e}$ is

$$\{x < \pi\_Z60 \vee \pi\_Z60 < x\}$$

And the point $x_0 = \frac{\pi}{3}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{\pi}{3}$ is

$$\{0 < y\}$$

And the point $y_0 = \mathrm{e}$ is inside this domain. Therefore solution exists and is unique.

2.3.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y \ln(y)}{\sin(x)}\end{aligned}$$

Where $f(x) = \frac{1}{\sin(x)}$ and $g(y) = \ln(y)\, y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\ln(y)\, y}\, dy &= \frac{1}{\sin(x)}\, dx \\ \int \frac{1}{\ln(y)\, y}\, dy &= \int \frac{1}{\sin(x)}\, dx \\ \ln(\ln(y)) &= \ln(\csc(x) - \cot(x)) + c_1\end{aligned}$$

Raising both side to exponential gives

$$\ln(y) = \mathrm{e}^{\ln(\csc(x) - \cot(x)) + c_1}$$

Which simplifies to

$$\ln(y) = c_2(\csc(x) - \cot(x))$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{3}$ and $y = \mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$\mathrm{e} = \mathrm{e}^{\frac{\sqrt{3}\, c_2 \mathrm{e}^{c_1}}{3}}$$

$$c_1 = \frac{\ln\left(\frac{3}{c_2^2}\right)}{2}$$

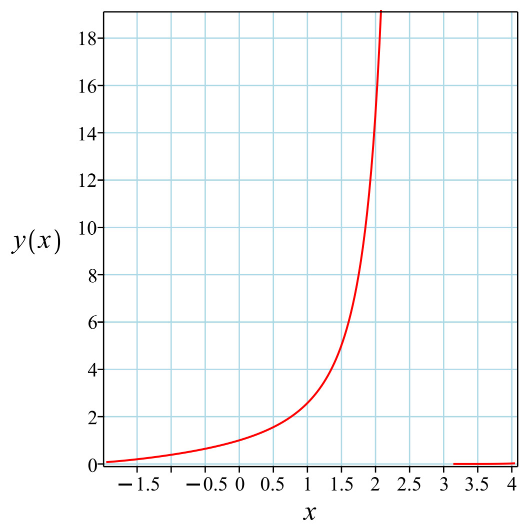
Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{-\sqrt{3}\,(-\csc(x)+\cot(x))}$$

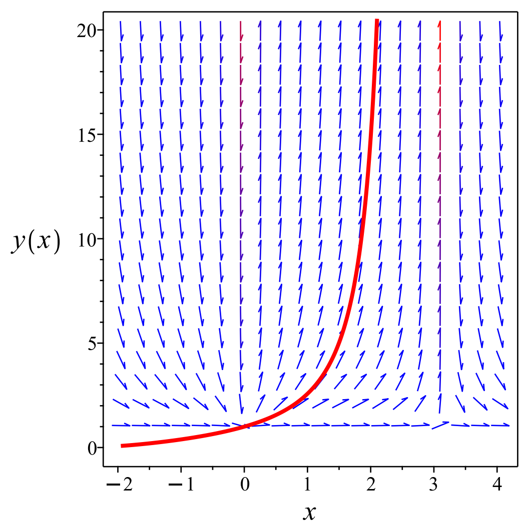
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\sqrt{3}\,(-\csc(x)+\cot(x))} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\sqrt{3}\,(-\csc(x)+\cot(x))}$$

Verified OK. `{positive}`

2.3.3 Maple step by step solution

Let's solve

$\left[y' \sin(x) - y \ln(y) = 0, y\left(\frac{\pi}{3}\right) = \mathrm{e}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y \ln(y)} = \frac{1}{\sin(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y \ln(y)} dx = \int \frac{1}{\sin(x)} dx + c_1$

- Evaluate integral

  $\ln(\ln(y)) = \ln(\csc(x) - \cot(x)) + c_1$

- Solve for y

  $y = \mathrm{e}^{-\frac{\mathrm{e}^{c_1} (\cos(x)-1)}{\sin(x)}}$

- Use initial condition $y\left(\frac{\pi}{3}\right) = \mathrm{e}$

  $\mathrm{e} = \mathrm{e}^{\frac{\mathrm{e}^{c_1} \sqrt{3}}{3}}$

- Solve for c_1

  $c_1 = \frac{\ln(3)}{2}$

- Substitute $c_1 = \frac{\ln(3)}{2}$ into general solution and simplify

  $y = \mathrm{e}^{\sqrt{3} (\csc(x)-\cot(x))}$

- Solution to the IVP

  $y = \mathrm{e}^{\sqrt{3} (\csc(x)-\cot(x))}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.516 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)*sin(x)=y(x)*ln(y(x)),y(1/3*Pi) = exp(1)],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-(\cot(x)-\csc(x))\sqrt{3}}$$

✓ Solution by Mathematica

Time used: 0.226 (sec). Leaf size: 19

```
DSolve[{y'[x]*Sin[x]==y[x]*Log[y[x]],{y[Pi/3]==Exp[1]}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{e^{\operatorname{arctanh}\left(\frac{1}{2}\right)-\operatorname{arctanh}(\cos(x))}}$$

2.4 problem 4

2.4.1 Existence and uniqueness analysis 21
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Internal problem ID [4752]
Internal file name [`OUTPUT/4245_Sunday_June_05_2022_12_47_05_PM_37881317/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y^2 + xyy' = -1}$$

With initial conditions

$$[y(5) = 0]$$

2.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= -\frac{y^2+1}{xy} \end{aligned}$$

$f(x,y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

2.4.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= -\frac{y^2+1}{xy}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y^2+1}{y}}\,dy &= -\frac{1}{x}\,dx \\
\int \frac{1}{\frac{y^2+1}{y}}\,dy &= \int -\frac{1}{x}\,dx \\
\frac{\ln\left(y^2+1\right)}{2} &= -\ln\left(x\right) + c_1
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = \mathrm{e}^{-\ln(x)+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = \frac{c_2}{x}$$

Which can be simplified to become

$$\sqrt{1+y^2} = \frac{c_2 \mathrm{e}^{c_1}}{x}$$

The solution is

$$\sqrt{1+y^2} = \frac{c_2 \mathrm{e}^{c_1}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2 \mathrm{e}^{c_1}}{5}$$

$$
c_1 = \ln\left(\frac{5}{c_2}\right)
$$

Substituting c_1 found above in the general solution gives

$$
\sqrt{y^2+1} = \frac{5}{x}
$$

The above simplifies to

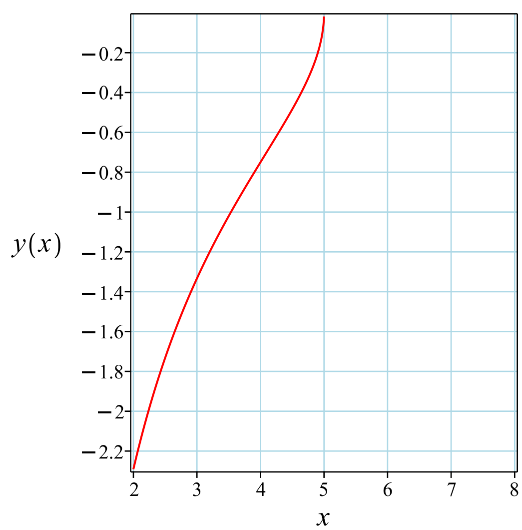
$$
\sqrt{y^2+1}\,x - 5 = 0
$$

Solving for y from the above gives

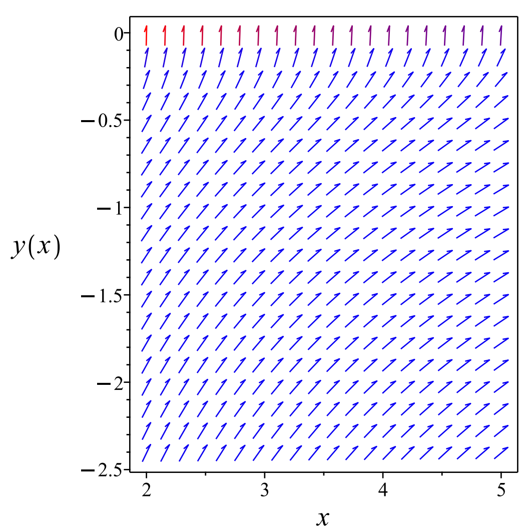
$$
y = \frac{\sqrt{-x^2+25}}{x}
$$
$$
y = -\frac{\sqrt{-x^2+25}}{x}
$$

Summary
The solution(s) found are the following

$$
y = \frac{\sqrt{-x^2+25}}{x} \tag{1}
$$
$$
y = -\frac{\sqrt{-x^2+25}}{x} \tag{2}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{-x^2+25}}{x}$$

Verified OK.

$$y = -\frac{\sqrt{-x^2+25}}{x}$$

Verified OK.

2.4.3 Maple step by step solution

Let's solve

$\left[y^2 + xyy' = -1, y(5) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{-1-y^2} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{-1-y^2}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $-\frac{\ln(1+y^2)}{2} = \ln(x) + c_1$

- Solve for y

  $\left\{y = \frac{\sqrt{1-(e^{c_1})^2 x^2}}{e^{c_1} x}, y = -\frac{\sqrt{1-(e^{c_1})^2 x^2}}{e^{c_1} x}\right\}$

- Use initial condition $y(5) = 0$

  $0 = \frac{\sqrt{1-25(e^{c_1})^2}}{5\,e^{c_1}}$

- Solve for c_1

  $c_1 = -\ln(5)$

- Substitute $c_1 = -\ln(5)$ into general solution and simplify

  $y = \frac{\sqrt{-x^2+25}}{x}$

- Use initial condition $y(5) = 0$

$$0 = -\frac{\sqrt{1-25\left(e^{c_1}\right)^2}}{5\,e^{c_1}}$$

- Solve for c_1

$$c_1 = -\ln(5)$$

- Substitute $c_1 = -\ln(5)$ into general solution and simplify

$$y = -\frac{\sqrt{-x^2+25}}{x}$$

- Solutions to the IVP

$$\left\{y = \frac{\sqrt{-x^2+25}}{x}, y = -\frac{\sqrt{-x^2+25}}{x}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 34

```
dsolve([(1+y(x)^2)+x*y(x)*diff(y(x),x)=0,y(5) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{-x^2+25}}{x}$$
$$y(x) = -\frac{\sqrt{-x^2+25}}{x}$$

✓ Solution by Mathematica

Time used: 0.329 (sec). Leaf size: 40

```
DSolve[{(1+y[x]^2)+x*y[x]*y'[x]==0,{y[5]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{25-x^2}}{x}$$
$$y(x) \to \frac{\sqrt{25-x^2}}{x}$$

2.5 problem 5

2.5.1 Existence and uniqueness analysis 26
2.5.2 Solving as separable ode 27
2.5.3 Maple step by step solution 28

Internal problem ID [4753]
Internal file name [`OUTPUT/4246_Sunday_June_05_2022_12_47_18_PM_40730792/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{xyy' - xy - y = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{x+1}{x}$$

Hence the ode is

$$y' = \frac{x+1}{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{x+1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.5.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x+1}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(y) = 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1}\, dy &= \frac{x+1}{x}\, dx \\ \int \frac{1}{1}\, dy &= \int \frac{x+1}{x}\, dx \\ y &= x + \ln(x) + c_1 \end{aligned}$$

Which results in

$$y = x + \ln(x) + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 + 1$$

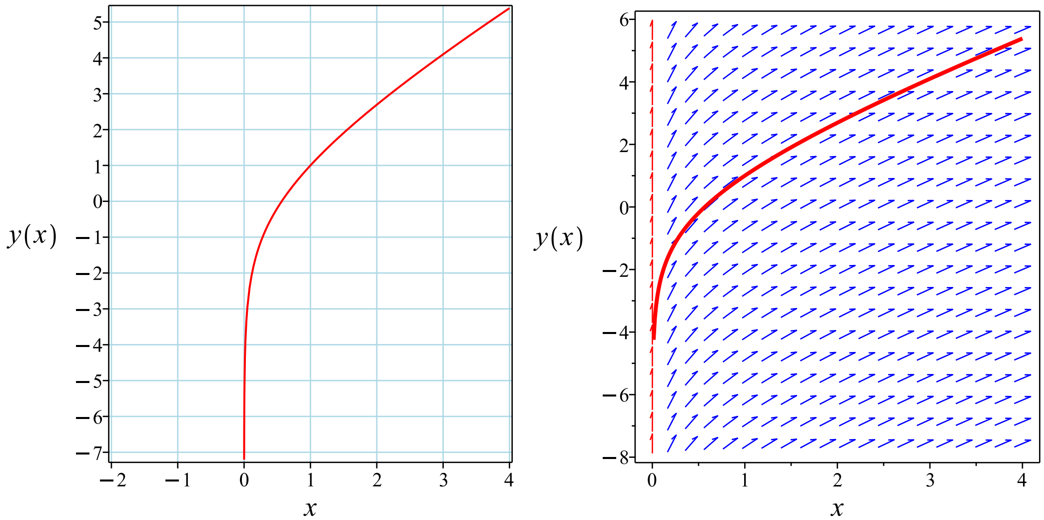
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = x + \ln(x)$$

Summary
The solution(s) found are the following

$$y = x + \ln(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x + \ln(x)$$

Verified OK.

2.5.3 Maple step by step solution

Let's solve

$[xyy' - xy - y = 0, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{x+1}{x} dx + c_1$

- Evaluate integral

$$y = x + \ln(x) + c_1$$

- Solve for y

$$y = x + \ln(x) + c_1$$

- Use initial condition $y(1) = 1$

$$1 = c_1 + 1$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = x + \ln(x)$$

- Solution to the IVP

$$y = x + \ln(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 8

```
dsolve([x*y(x)*diff(y(x),x)-x*y(x)=y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = x + \ln(x)$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 9

```
DSolve[{x*y[x]*y'[x]-x*y[x]==y[x],{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \log(x)$$

2.6 problem 6

2.6.1 Existence and uniqueness analysis 30
2.6.2 Solving as separable ode 31
2.6.3 Maple step by step solution 33

Internal problem ID [4754]
Internal file name [`OUTPUT/4247_Sunday_June_05_2022_12_47_26_PM_51583505/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{2x y^2 + x}{y x^2 - y} = 0}$$

With initial conditions

$$\left[y\left(\sqrt{2}\right) = 0\right]$$

2.6.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x(2y^2 + 1)}{y\,(x^2 - 1)} \end{aligned}$$

$f(x, y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

2.6.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{x(2y^2+1)}{y\,(x^2-1)}\end{aligned}$$

Where $f(x)=\frac{x}{x^2-1}$ and $g(y)=\frac{2y^2+1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{2y^2+1}{y}}\,dy &= \frac{x}{x^2-1}\,dx\\ \int\frac{1}{\frac{2y^2+1}{y}}\,dy &= \int\frac{x}{x^2-1}\,dx\\ \frac{\ln\left(2y^2+1\right)}{4} &= \frac{\ln\left(x-1\right)}{2}+\frac{\ln\left(x+1\right)}{2}+c_1\end{aligned}$$

Raising both side to exponential gives

$$\left(2y^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$\left(2y^2+1\right)^{\frac{1}{4}} = c_2\mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which can be simplified to become

$$\left(2y^2+1\right)^{\frac{1}{4}} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}$$

The solution is

$$\left(2y^2+1\right)^{\frac{1}{4}} = c_2\sqrt{x-1}\,\sqrt{x+1}\,\mathrm{e}^{c_1}$$

Initial conditions are used to solve for c_1. Substituting $x=\sqrt{2}$ and $y=0$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{\sqrt{2}-1}\,\sqrt{1+\sqrt{2}}\,c_2\mathrm{e}^{c_1}$$

$$c_1 = -\ln\left(c_2\right)$$

Substituting c_1 found above in the general solution gives

$$\left(2y^2+1\right)^{\frac{1}{4}} = \sqrt{x-1}\,\sqrt{x+1}$$

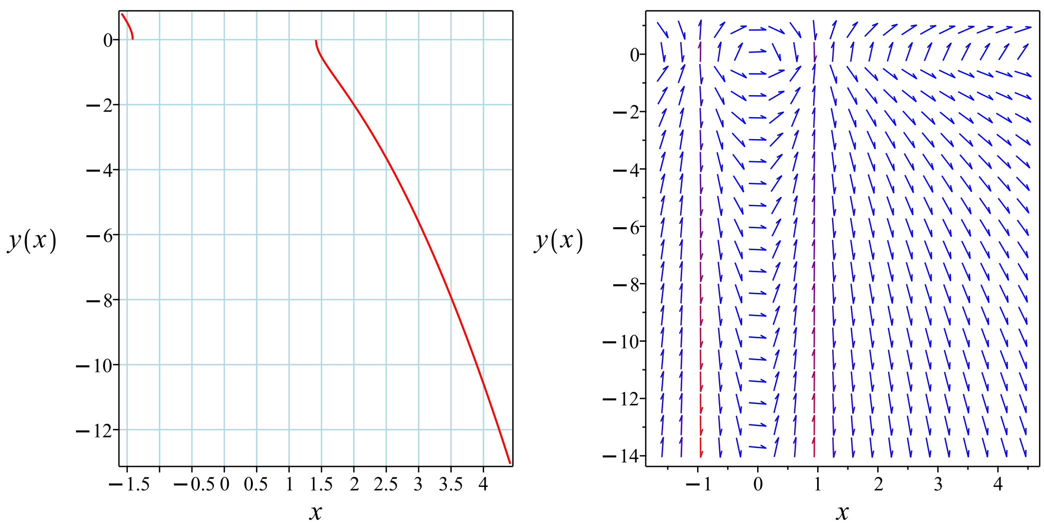
Solving for y from the above gives

$$y = \frac{\sqrt{2x^2-4}\,x}{2}$$
$$y = -\frac{\sqrt{2x^2-4}\,x}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{\sqrt{2x^2-4}\,x}{2} \tag{1}$$
$$y = -\frac{\sqrt{2x^2-4}\,x}{2} \tag{2}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sqrt{2x^2-4}\,x}{2}$$

Verified OK.

$$y = -\frac{\sqrt{2x^2-4}\,x}{2}$$

Verified OK.

2.6.3 Maple step by step solution

Let's solve

$\left[y' - \frac{2x y^2+x}{y x^2-y} = 0, y\left(\sqrt{2}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' y}{2y^2+1} = \frac{x}{(x-1)(x+1)}$

- Integrate both sides with respect to x

  $\int \frac{y' y}{2y^2+1}dx = \int \frac{x}{(x-1)(x+1)}dx + c_1$

- Evaluate integral

  $\frac{\ln(2y^2+1)}{4} = \frac{\ln((x-1)(x+1))}{2} + c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-2+2\,\mathrm{e}^{4c_1}x^4-4\,\mathrm{e}^{4c_1}x^2+2\,\mathrm{e}^{4c_1}}}{2}, y = \frac{\sqrt{-2+2\,\mathrm{e}^{4c_1}x^4-4\,\mathrm{e}^{4c_1}x^2+2\,\mathrm{e}^{4c_1}}}{2}\right\}$

- Use initial condition $y\left(\sqrt{2}\right) = 0$

  $0 = -\frac{\sqrt{-2+2\,\mathrm{e}^{4c_1}}}{2}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = -\frac{\sqrt{2}\,\sqrt{x^2(x^2-2)}}{2}$

- Use initial condition $y\left(\sqrt{2}\right) = 0$

  $0 = \frac{\sqrt{-2+2\,\mathrm{e}^{4c_1}}}{2}$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = \frac{\sqrt{2}\,\sqrt{x^2(x^2-2)}}{2}$

- Solutions to the IVP

  $\left\{y = -\frac{\sqrt{2}\,\sqrt{x^2(x^2-2)}}{2}, y = \frac{\sqrt{2}\,\sqrt{x^2(x^2-2)}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([diff(y(x),x)=(2*x*y(x)^2+x)/(x^2*y(x)-y(x)),y(sqrt(2)) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2x^2-4}\,x}{2}$$
$$y(x) = \frac{\sqrt{2x^2-4}\,x}{2}$$

✓ Solution by Mathematica

Time used: 3.88 (sec). Leaf size: 48

```
DSolve[{y'[x]==(2*x*y[x]^2+x)/(x^2*y[x]-y[x]),{y[Sqrt[2]]==0}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to -\frac{\sqrt{x^2(x^2-2)}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{x^2(x^2-2)}}{\sqrt{2}}$$

2.7 problem 7

2.7.1 Existence and uniqueness analysis 35
2.7.2 Solving as separable ode 36
2.7.3 Maple step by step solution 38

Internal problem ID [4755]
Internal file name [`OUTPUT/4248_Sunday_June_05_2022_12_47_39_PM_22574787/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'y + xy^2 = 8x}$$

With initial conditions

$$[y(1) = 3]$$

2.7.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{x(y^2 - 8)}{y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x,y)$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x(y^2-8)}{y}\right) \\ &= -2x + \frac{x(y^2-8)}{y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

2.7.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(y^2-8)}{y}\end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{y^2-8}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2-8}{y}}\,dy = -x\,dx$$

$$\int \frac{1}{\frac{y^2-8}{y}}\,dy = \int -x\,dx$$

$$\frac{\ln(y^2-8)}{2} = -\frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2-8} = e^{-\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2-8} = c_2 e^{-\frac{x^2}{2}}$$

The solution is

$$\sqrt{y^2-8} = c_2 e^{-\frac{x^2}{2}+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_2 e^{-\frac{1}{2}+c_1}$$

$$c_1 = \frac{1}{2} - \ln(c_2)$$

Substituting c_1 found above in the general solution gives

$$\sqrt{y^2-8} = e^{-\frac{(x-1)(x+1)}{2}}$$

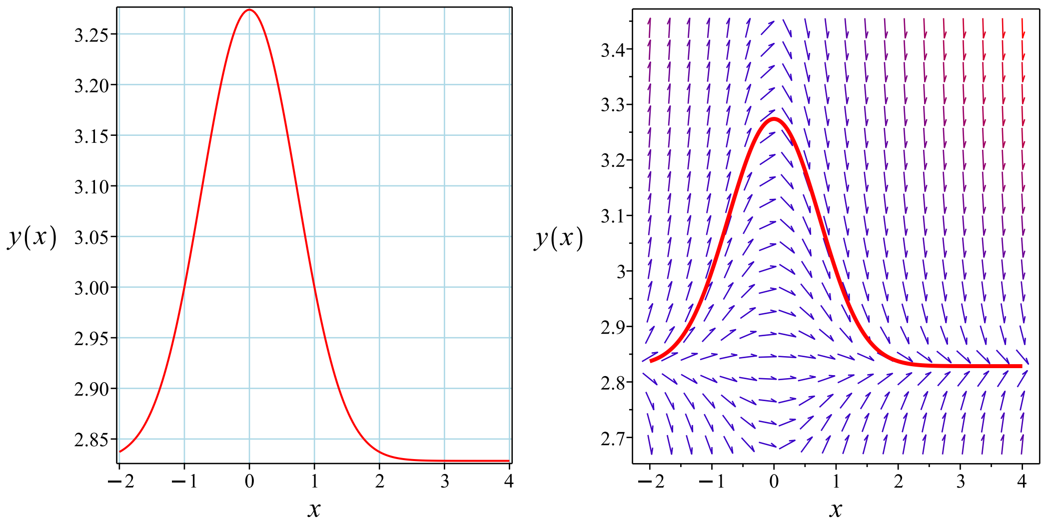
Solving for y from the above gives

$$y = \sqrt{8 + e^{-(x-1)(x+1)}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{8 + e^{-(x-1)(x+1)}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \sqrt{8 + e^{-(x-1)(x+1)}}$$

Verified OK.

2.7.3 Maple step by step solution

Let's solve

$[y'y + xy^2 = 8x, y(1) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{y^2-8} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'y}{y^2-8}dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{\ln(y^2-8)}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

$$\left\{y = \sqrt{8 + \mathrm{e}^{-x^2+2c_1}}, y = -\sqrt{8 + \mathrm{e}^{-x^2+2c_1}}\right\}$$

- Use initial condition $y(1) = 3$

  $3 = \sqrt{8 + \mathrm{e}^{-1+2c_1}}$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \sqrt{8 + \mathrm{e}^{-(x-1)(x+1)}}$

- Use initial condition $y(1) = 3$

  $3 = -\sqrt{8 + \mathrm{e}^{-1+2c_1}}$

- Solution does not satisfy initial condition

- Solution to the IVP

  $y = \sqrt{8 + \mathrm{e}^{-(x-1)(x+1)}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 17

```
dsolve([y(x)*diff(y(x),x)+(x*y(x)^2-8*x)=0,y(1) = 3],y(x), singsol=all)
```

$$y(x) = \sqrt{\mathrm{e}^{-(x-1)(1+x)} + 8}$$

✓ Solution by Mathematica

Time used: 1.924 (sec). Leaf size: 39

```
DSolve[{y[x]*y'[x]+(x*y[x]^2-8*x)==0,{y[1]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{e^{1-x^2}+8}$$
$$y(x) \to \sqrt{e^{1-x^2}+8}$$

2.8 problem 8

2.8.1 Existence and uniqueness analysis 41
2.8.2 Solving as separable ode 42
2.8.3 Maple step by step solution 44

Internal problem ID [4756]
Internal file name [`OUTPUT/4249_Sunday_June_05_2022_12_47_47_PM_42089755/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' + 2xy^2 = 0}$$

With initial conditions

$$[y(2) = 1]$$

2.8.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -2y^2x \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x, y)$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-2y^2x) \\ &= -4xy\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.8.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= -2y^2x\end{aligned}$$

Where $f(x) = -2x$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\,dy &= -2x\,dx \\ \int \frac{1}{y^2}\,dy &= \int -2x\,dx \\ -\frac{1}{y} &= -x^2 + c_1\end{aligned}$$

Which results in

$$y = -\frac{1}{-x^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\frac{1}{c_1 - 4}$$

$$c_1 = 3$$

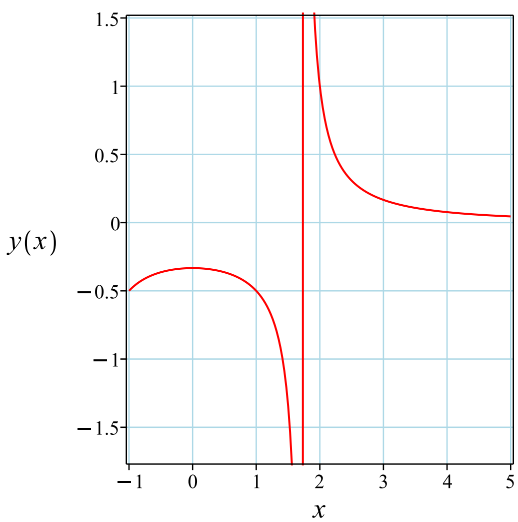
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{x^2 - 3}$$

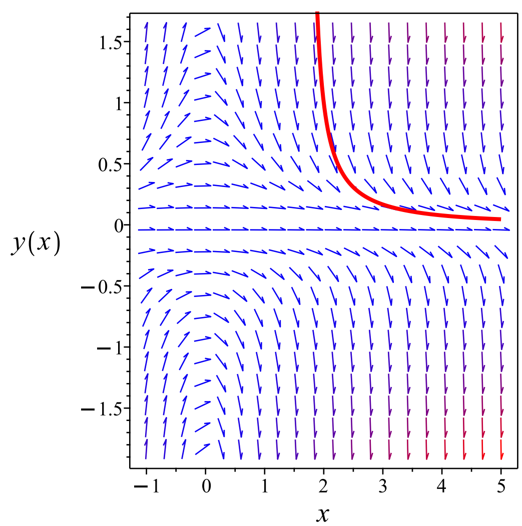
Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 - 3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 - 3}$$

Verified OK.

2.8.3 Maple step by step solution

Let's solve

$[y' + 2xy^2 = 0, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -2x dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -x^2 + c_1$

- Solve for y

  $y = -\frac{1}{-x^2+c_1}$

- Use initial condition $y(2) = 1$

  $1 = -\frac{1}{c_1-4}$

- Solve for c_1

  $c_1 = 3$

- Substitute $c_1 = 3$ into general solution and simplify

  $y = \frac{1}{x^2-3}$

- Solution to the IVP

  $y = \frac{1}{x^2-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 11

```
dsolve([diff(y(x),x)+2*x*y(x)^2=0,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{x^2 - 3}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 12

```
DSolve[{y'[x]+2*x*y[x]^2==0,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2 - 3}$$

2.9 problem 9

2.9.1 Existence and uniqueness analysis 46
2.9.2 Solving as separable ode 47
2.9.3 Maple step by step solution 48

Internal problem ID [4757]
Internal file name [`OUTPUT/4250_Sunday_June_05_2022_12_47_58_PM_60122639/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(1+y)\,y'-y=0}$$

With initial conditions

$$[y(1)=1]$$

2.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y}{1+y} \end{aligned}$$

The y domain of $f(x,y)$ when $x=1$ is

$$\{y<-1 \lor -1<y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{1+y}\right) \\
&= \frac{1}{1+y} - \frac{y}{(1+y)^2}
\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < -1 \lor -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= \frac{y}{1+y}
\end{aligned}$$

Where $f(x) = 1$ and $g(y) = \frac{y}{1+y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y}{1+y}}\, dy &= 1\, dx \\
\int \frac{1}{\frac{y}{1+y}}\, dy &= \int 1\, dx \\
y + \ln(y) &= x + c_1
\end{aligned}$$

Which results in

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(\mathrm{e}^{x+c_1}\right)+c_1+x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \operatorname{LambertW}\left(\mathrm{e}^{c_1+1}\right)$$

$$c_1 = 0$$

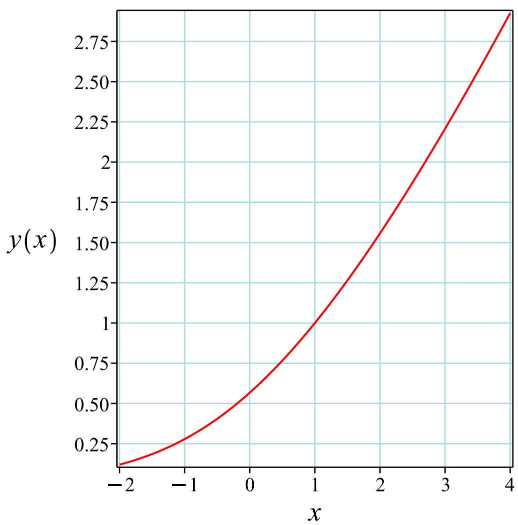
Substituting c_1 found above in the general solution gives

$$y = \mathrm{LambertW}(e^x)$$

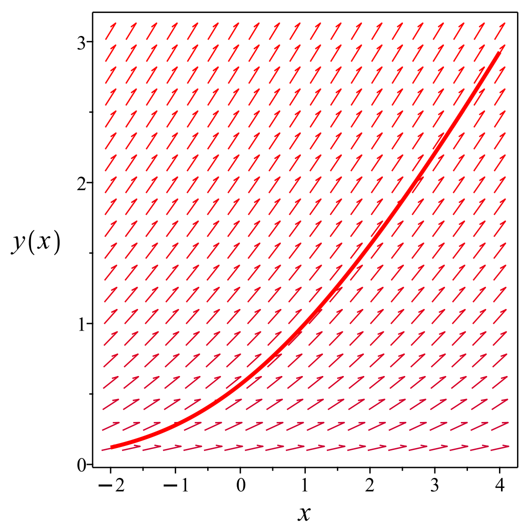
Summary

The solution(s) found are the following

$$y = \mathrm{LambertW}(e^x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \mathrm{LambertW}(e^x)$$

Verified OK.

2.9.3 Maple step by step solution

Let's solve

$[(1+y)\,y' - y = 0, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y)}{y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y)}{y} dx = \int 1 dx + c_1$
- Evaluate integral

  $y + \ln(y) = x + c_1$
- Solve for y

  $y = e^{-\mathit{LambertW}(e^{x+c_1})+c_1+x}$
- Use initial condition $y(1) = 1$

  $1 = e^{-\mathit{LambertW}(e^{c_1+1})+c_1+1}$
- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \mathit{LambertW}(e^x)$
- Solution to the IVP

  $y = \mathit{LambertW}(e^x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 7

```
dsolve([(1+y(x))*diff(y(x),x)=y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \operatorname{LambertW}(e^x)$$

✓ Solution by Mathematica

Time used: 2.162 (sec). Leaf size: 9

```
DSolve[{(1+y[x])*y'[x]==y[x],{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W(e^x)$$

2.10 problem 10

2.10.1 Existence and uniqueness analysis 51
2.10.2 Solving as separable ode 52
2.10.3 Maple step by step solution 53

Internal problem ID [4758]
Internal file name [`OUTPUT/4251_Sunday_June_05_2022_12_48_05_PM_44721005/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - xy = x}$$

With initial conditions

$$[y(0) = 1]$$

2.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -x \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$
y' - xy = x
$$

The domain of $p(x) = -x$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$
\{-\infty < x < \infty\}
$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

2.10.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= x(1 + y)
\end{aligned}
$$

Where $f(x) = x$ and $g(y) = 1 + y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{1+y}\, dy &= x\, dx \\
\int \frac{1}{1+y}\, dy &= \int x\, dx \\
\ln(1+y) &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
1 + y = e^{\frac{x^2}{2} + c_1}
$$

Which simplifies to

$$
1 + y = c_2 e^{\frac{x^2}{2}}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = c_2 e^{c_1} - 1
$$

$$
c_1 = \ln\left(\frac{2}{c_2}\right)
$$

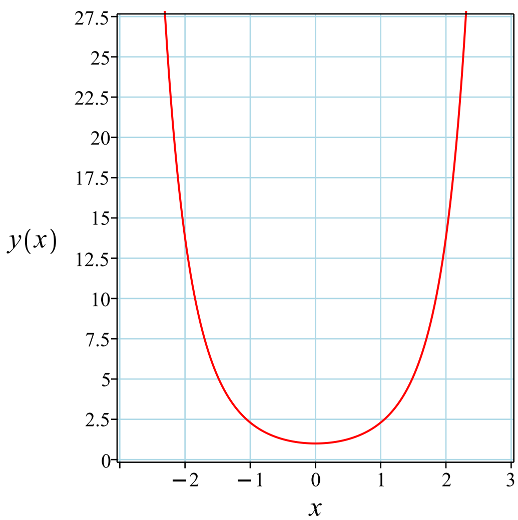
Substituting c_1 found above in the general solution gives

$$
y = 2\,\mathrm{e}^{\frac{x^2}{2}} - 1
$$

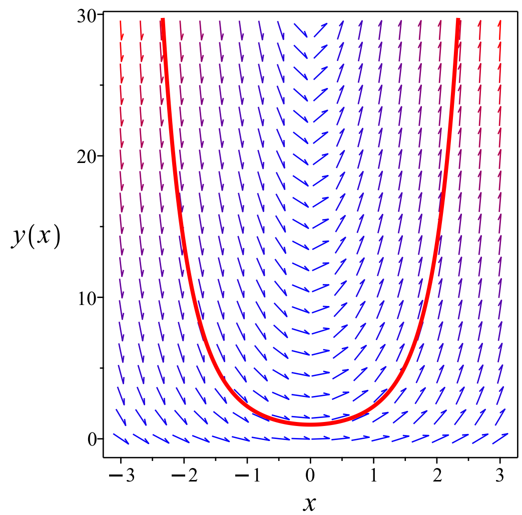
Summary

The solution(s) found are the following

$$
y = 2\,\mathrm{e}^{\frac{x^2}{2}} - 1 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = 2\,\mathrm{e}^{\frac{x^2}{2}} - 1
$$

Verified OK.

2.10.3 Maple step by step solution

Let's solve

$[y' - xy = x, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = x$
- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int x dx + c_1$
- Evaluate integral

  $\ln(1+y) = \frac{x^2}{2} + c_1$
- Solve for y

  $y = e^{\frac{x^2}{2}+c_1} - 1$
- Use initial condition $y(0) = 1$

  $1 = e^{c_1} - 1$
- Solve for c_1

  $c_1 = \ln(2)$
- Substitute $c_1 = \ln(2)$ into general solution and simplify

  $y = 2\,e^{\frac{x^2}{2}} - 1$
- Solution to the IVP

  $y = 2\,e^{\frac{x^2}{2}} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)-x*y(x)=x,y(0) = 1],y(x), singsol=all)
```

$$y(x) = -1 + 2\,e^{\frac{x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 20

```
DSolve[{y'[x]-x*y[x]==x,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{\frac{1}{2}(x^2-1)} - 1$$

2.11 problem 11

2.11.1 Existence and uniqueness analysis 56
2.11.2 Solving as separable ode 57
2.11.3 Maple step by step solution 59

Internal problem ID [4759]
Internal file name [`OUTPUT/4252_Sunday_June_05_2022_12_48_14_PM_13953597/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{2y' - 3(y-2)^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(1) = 3]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{3(y-2)^{\frac{1}{3}}}{2} \end{aligned}$$

The y domain of $f(x,y)$ when $x=1$ is

$$\{2 \leq y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{3(y-2)^{\frac{1}{3}}}{2}\right) \\ &= \frac{1}{2\,(y-2)^{\frac{2}{3}}}\end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

2.11.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{3(y-2)^{\frac{1}{3}}}{2}\end{aligned}$$

Where $f(x) = 1$ and $g(y) = \frac{3(y-2)^{\frac{1}{3}}}{2}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{3(y-2)^{\frac{1}{3}}}{2}}\,dy &= 1\,dx \\ \int \frac{1}{\frac{3(y-2)^{\frac{1}{3}}}{2}}\,dy &= \int 1\,dx \\ (y-2)^{\frac{2}{3}} &= x + c_1\end{aligned}$$

The solution is

$$(y-2)^{\frac{2}{3}} - x - c_1 = 0$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$-c_1 = 0$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

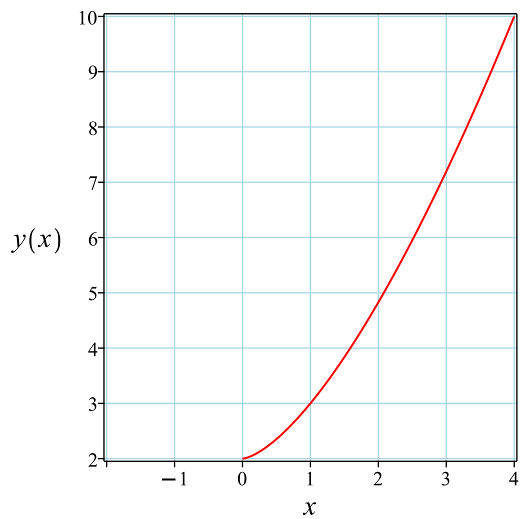
$$(y-2)^{\frac{2}{3}} - x = 0$$

Solving for y from the above gives

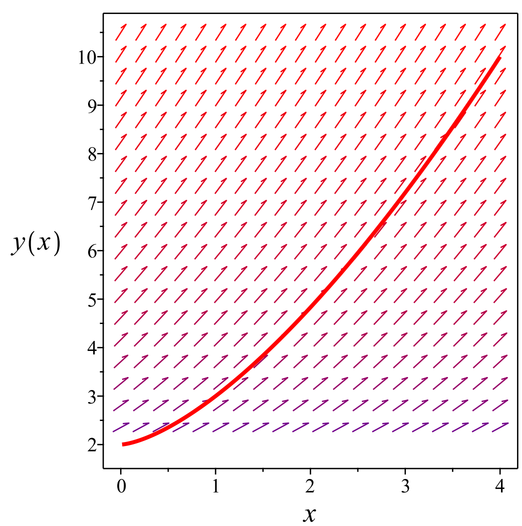
$$y = 2 + x^{\frac{3}{2}}$$

Summary
The solution(s) found are the following

$$y = 2 + x^{\frac{3}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2 + x^{\frac{3}{2}}$$

Verified OK.

2.11.3 Maple step by step solution

Let's solve

$\left[2y' - 3(y-2)^{\frac{1}{3}} = 0, y(1) = 3\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(y-2)^{\frac{1}{3}}} = \frac{3}{2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{(y-2)^{\frac{1}{3}}}dx = \int \frac{3}{2}dx + c_1$

- Evaluate integral

  $\frac{3(y-2)^{\frac{2}{3}}}{2} = \frac{3x}{2} + c_1$

- Solve for y

  $y = \frac{(6c_1+9x)^{\frac{3}{2}}}{27} + 2$

- Use initial condition $y(1) = 3$

  $3 = \frac{(6c_1+9)^{\frac{3}{2}}}{27} + 2$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = 2 + x^{\frac{3}{2}}$

- Solution to the IVP

  $y = 2 + x^{\frac{3}{2}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 9

```
dsolve([2*diff(y(x),x)=3*(y(x)-2)^(1/3),y(1) = 3],y(x), singsol=all)
```

$$y(x) = 2 + x^{\frac{3}{2}}$$

✓ Solution by Mathematica

Time used: 0.007 (sec). Leaf size: 12

```
DSolve[{2*y'[x]==3*(y[x]-2)^(1/3),{y[1]==3}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^{3/2} + 2$$

2.12 problem 12

2.12.1 Existence and uniqueness analysis 61
2.12.2 Solving as separable ode 62
2.12.3 Maple step by step solution 64

Internal problem ID [4760]
Internal file name [`OUTPUT/4253_Sunday_June_05_2022_12_48_20_PM_41611549/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 2. Separable equations. page 398
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(xy + x)\, y' + y = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{y}{x\,(1+y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x(1+y)}\right) \\
&= -\frac{1}{(1+y)x} + \frac{y}{x(1+y)^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < -1 \vee -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.12.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{y}{x(1+y)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(y) = \frac{y}{1+y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y}{1+y}}\, dy &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{y}{1+y}}\, dy &= \int -\frac{1}{x}\, dx \\
y + \ln(y) &= -\ln(x) + c_1
\end{aligned}$$

Which results in

$$y = \operatorname{LambertW}\left(\frac{e^{c_1}}{x}\right)$$

Since c_1 is constant, then exponential powers of this constant are constants also, and these can be simplified to just c_1 in the above solution. The solution becomes

$$y = \text{LambertW}\left(\frac{c_1}{x}\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \text{LambertW}(c_1)$$

$$c_1 = \mathrm{e}$$

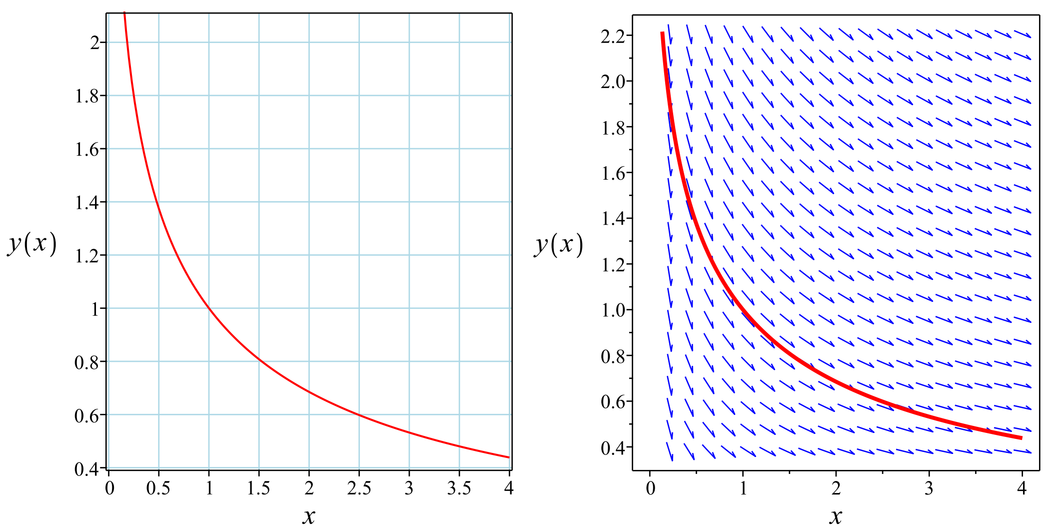
Substituting c_1 found above in the general solution gives

$$y = \text{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

Summary

The solution(s) found are the following

$$y = \text{LambertW}\left(\frac{\mathrm{e}}{x}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \text{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

Verified OK.

2.12.3 Maple step by step solution

Let's solve

$[(xy+x)\,y'+y=0, y(1)=1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(1+y)}{y}=-\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(1+y)}{y}dx=\int -\frac{1}{x}dx+c_1$

- Evaluate integral

  $y+\ln(y)=-\ln(x)+c_1$

- Solve for y

  $y=\mathit{LambertW}\left(\frac{e^{c_1}}{x}\right)$

- Use initial condition $y(1)=1$

  $1=\mathit{LambertW}(e^{c_1})$

- Solve for c_1

  $c_1=1$

- Substitute $c_1=1$ into general solution and simplify

  $y=\mathit{LambertW}\left(\frac{e}{x}\right)$

- Solution to the IVP

  $y=\mathit{LambertW}\left(\frac{e}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 11

```
dsolve([(x+x*y(x))*diff(y(x),x)+y(x)=0,y(1) = 1],y(x), singsol=all)
```

$$y(x) = \text{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

✓ Solution by Mathematica

Time used: 2.09 (sec). Leaf size: 11

```
DSolve[{(x+x*y[x])*y'[x]+y[x]==0,{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to W\left(\frac{e}{x}\right)$$

3 Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403

3.1 problem 1 67
3.2 problem 2 72
3.3 problem 3 76
3.4 problem 4 81
3.5 problem 5 85
3.6 problem 6 89
3.7 problem 7 94
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3.1 problem 1

3.1.1 Solving as linear ode 67
3.1.2 Maple step by step solution 69

Internal problem ID [4761]
Internal file name [`OUTPUT/4254_Sunday_June_05_2022_12_48_26_PM_95796451/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^{x}}$$

3.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= e^{x} \end{aligned}$$

Hence the ode is

$$y' + y = e^{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{x}\right) \\
\frac{d}{dx}\left(y\,e^{x}\right) &= \left(e^{x}\right)\left(e^{x}\right) \\
d\left(y\,e^{x}\right) &= e^{2x}\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,e^{x} &= \int e^{2x}\,dx \\
y\,e^{x} &= \frac{e^{2x}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = \frac{e^{-x}e^{2x}}{2} + c_1 e^{-x}
$$

which simplifies to

$$
y = c_1 e^{-x} + \frac{e^{x}}{2}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-x} + \frac{e^{x}}{2} \tag{1}
$$

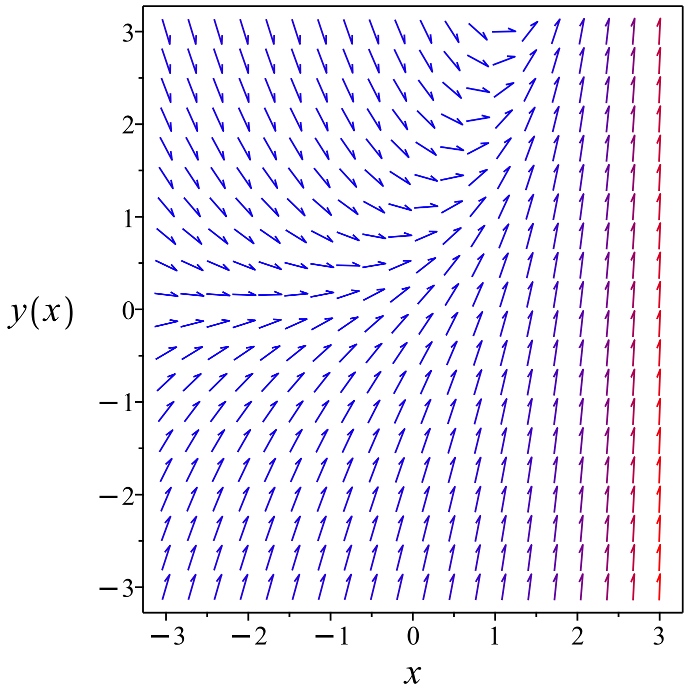

Figure 14: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{e^x}{2}$$

Verified OK.

3.1.2 Maple step by step solution

Let's solve

$y' + y = e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y + e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = e^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\,(y'+y)=\mu(x)\,\mathrm{e}^{x}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\,(y'+y)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\mu(x)$$

- Solve to find the integrating factor

$$\mu(x)=\mathrm{e}^{x}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,\mathrm{e}^{x}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\mu(x)\,\mathrm{e}^{x}dx+c_1$$

- Solve for y

$$y=\frac{\int\mu(x)\mathrm{e}^{x}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\mathrm{e}^{x}$

$$y=\frac{\int(\mathrm{e}^{x})^{2}dx+c_1}{\mathrm{e}^{x}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\frac{(\mathrm{e}^{x})^{2}}{2}+c_1}{\mathrm{e}^{x}}$$

- Simplify

$$y=c_1\mathrm{e}^{-x}+\frac{\mathrm{e}^{x}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)=exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^x}{2} + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 21

```
DSolve[y'[x]+y[x]==Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{2} + c_1 e^{-x}$$

3.2 problem 2

3.2.1 Solving as linear ode 72
3.2.2 Maple step by step solution 74

Internal problem ID [4762]
Internal file name [`OUTPUT/4255_Sunday_June_05_2022_12_48_35_PM_350268/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2y' + 3xy = 1}$$

3.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{3}{x}dx} \\ &= x^3 \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{d}{dx}\left(y\,x^3\right) &= \left(x^3\right)\left(\frac{1}{x^2}\right) \\
d\left(y\,x^3\right) &= x\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^3 &= \int x\,dx \\
y\,x^3 &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{1}{2x} + \frac{c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{2x} + \frac{c_1}{x^3} \tag{1}
$$

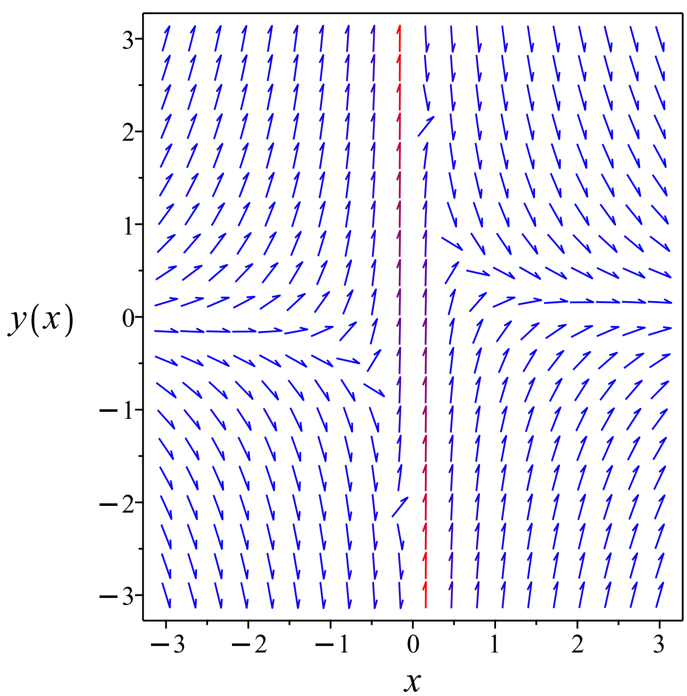

Figure 15: Slope field plot

Verification of solutions

$$y = \frac{1}{2x} + \frac{c_1}{x^3}$$

Verified OK.

3.2.2 Maple step by step solution

Let's solve

$x^2y' + 3xy = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + \frac{1}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^3$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \frac{\mu(x)}{x^2}dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{\mu(x)}{x^2}dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^2}dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

$y = \frac{\int x dx + c_1}{x^3}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{x^2}{2} + c_1}{x^3}$

- Simplify

$y = \frac{x^2 + 2c_1}{2x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x)+3*x*y(x)=1,y(x), singsol=all)
```

$$y(x) = \frac{x^2 + 2c_1}{2x^3}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[x^2*y'[x]+3*x*y[x]==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2 + 2c_1}{2x^3}$$

3.3 problem 3

3.3.1 Solving as linear ode 76
3.3.2 Maple step by step solution 78

Internal problem ID [4763]
Internal file name [`OUTPUT/4256_Sunday_June_05_2022_12_48_43_PM_84879189/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + 2xy = x\,\mathrm{e}^{-x^2}}$$

3.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= 2x\\ q(x) &= x\,\mathrm{e}^{-x^2}\end{aligned}$$

Hence the ode is

$$y' + 2xy = x\,\mathrm{e}^{-x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int 2x dx}\\ &= \mathrm{e}^{x^2}\end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x\,\mathrm{e}^{-x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} y\right) &= \left(\mathrm{e}^{x^2}\right)\left(x\,\mathrm{e}^{-x^2}\right) \\
\mathrm{d}\left(\mathrm{e}^{x^2} y\right) &= x\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x^2} y &= \int x\,\mathrm{d}x \\
\mathrm{e}^{x^2} y &= \frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$
y = \frac{x^2 \mathrm{e}^{-x^2}}{2} + c_1 \mathrm{e}^{-x^2}
$$

which simplifies to

$$
y = \mathrm{e}^{-x^2}\left(\frac{x^2}{2} + c_1\right)
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x^2}\left(\frac{x^2}{2} + c_1\right) \tag{1}
$$

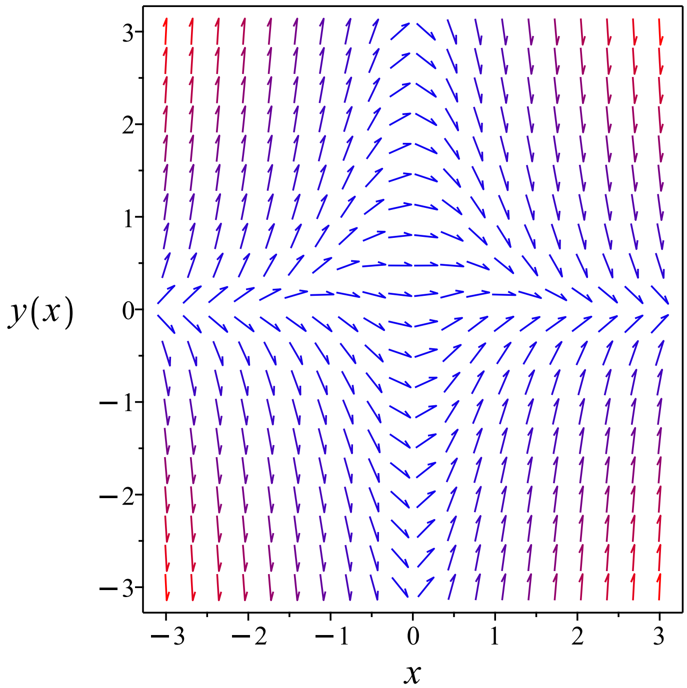

Figure 16: Slope field plot

Verification of solutions

$$y = e^{-x^2}\left(\frac{x^2}{2} + c_1\right)$$

Verified OK.

3.3.2 Maple step by step solution

Let's solve

$y' + 2xy = x\,e^{-x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2xy + x\,e^{-x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2xy = x\,e^{-x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+2xy\right)=\mu(x)\,x\,\mathrm{e}^{-x^{2}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+2xy\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=2\mu(x)\,x$$

- Solve to find the integrating factor

$$\mu(x)=\mathrm{e}^{x^{2}}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,x\,\mathrm{e}^{-x^{2}}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\mu(x)\,x\,\mathrm{e}^{-x^{2}}dx+c_1$$

- Solve for y

$$y=\frac{\int\mu(x)x\,\mathrm{e}^{-x^{2}}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\mathrm{e}^{x^{2}}$

$$y=\frac{\int x\,\mathrm{e}^{-x^{2}}\mathrm{e}^{x^{2}}dx+c_1}{\mathrm{e}^{x^{2}}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\frac{x^{2}}{2}+c_1}{\mathrm{e}^{x^{2}}}$$

- Simplify

$$y=\frac{\mathrm{e}^{-x^{2}}\left(x^{2}+2c_1\right)}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+2*x*y(x)-x*exp(-x^2)=0,y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 2c_1)\, \mathrm{e}^{-x^2}}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 24

```
DSolve[y'[x]+2*x*y[x]-x*Exp[-x^2]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-x^2} (x^2 + 2c_1)$$

3.4 problem 4

3.4.1 Solving as linear ode 81
3.4.2 Maple step by step solution 83

Internal problem ID [4764]
Internal file name [`OUTPUT/4257_Sunday_June_05_2022_12_48_52_PM_7362242/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2xy' + y = 2x^{\frac{5}{2}}}$$

3.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = x^{\frac{3}{2}}$$

Hence the ode is

$$y' + \frac{y}{2x} = x^{\frac{3}{2}}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2x}dx}$$
$$= \sqrt{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(x^{\frac{3}{2}}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(y\sqrt{x}) = (\sqrt{x})\left(x^{\frac{3}{2}}\right)$$
$$\mathrm{d}(y\sqrt{x}) = x^2\,\mathrm{d}x$$

Integrating gives

$$y\sqrt{x} = \int x^2\,\mathrm{d}x$$
$$y\sqrt{x} = \frac{x^3}{3} + c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$y = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}} \tag{1}$$

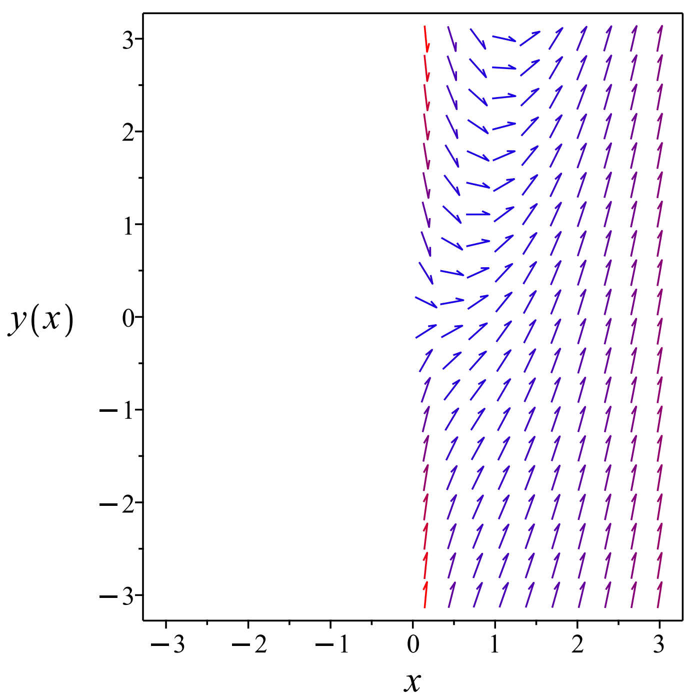

Figure 17: Slope field plot

Verification of solutions

$$y = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}}$$

Verified OK.

3.4.2 Maple step by step solution

Let's solve

$2xy' + y = 2x^{\frac{5}{2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{2x} + x^{\frac{3}{2}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2x} = x^{\frac{3}{2}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = \mu(x)\, x^{\frac{3}{2}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^{\frac{3}{2}} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^{\frac{3}{2}} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^{\frac{3}{2}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{x}$

$$y = \frac{\int x^2 dx + c_1}{\sqrt{x}}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{x^3}{3} + c_1}{\sqrt{x}}$$

- Simplify

$$y = \frac{x^3 + 3c_1}{3\sqrt{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(2*x*diff(y(x),x)+y(x)=2*x^(5/2),y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1}{3\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 22

```
DSolve[2*x*y'[x]+y[x]==2*x^(5/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 + 3c_1}{3\sqrt{x}}$$

3.5 problem 5

3.5.1 Solving as linear ode 85
3.5.2 Maple step by step solution 87

Internal problem ID [4765]
Internal file name [`OUTPUT/4258_Sunday_June_05_2022_12_49_00_PM_92327569/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\cos(x)\, y' + y = \cos(x)^2}$$

3.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \sec(x) \\ q(x) &= \cos(x) \end{aligned}$$

Hence the ode is

$$y' + \sec(x)\, y = \cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \sec(x)dx} \\ &= \sec(x) + \tan(x) \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\
\frac{d}{dx}((\sec(x)+\tan(x))\, y) &= (\sec(x)+\tan(x))(\cos(x)) \\
d((\sec(x)+\tan(x))\, y) &= (1+\sin(x))\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(\sec(x)+\tan(x))\, y &= \int 1+\sin(x)\, dx \\
(\sec(x)+\tan(x))\, y &= x-\cos(x)+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sec(x)+\tan(x)$ results in

$$
y = \frac{x-\cos(x)}{\sec(x)+\tan(x)} + \frac{c_1}{\sec(x)+\tan(x)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x-\cos(x)}{\sec(x)+\tan(x)} + \frac{c_1}{\sec(x)+\tan(x)} \tag{1}
$$

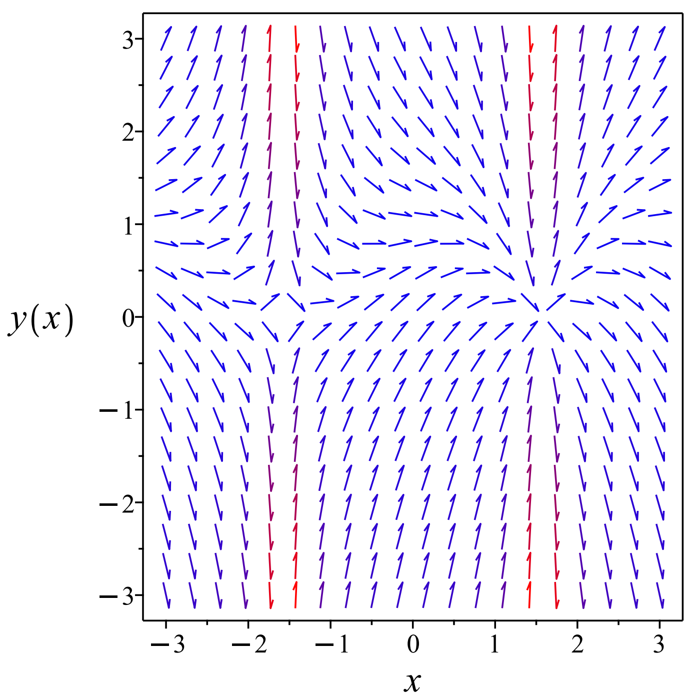

Figure 18: Slope field plot

Verification of solutions

$$y = \frac{x - \cos(x)}{\sec(x) + \tan(x)} + \frac{c_1}{\sec(x) + \tan(x)}$$

Verified OK.

3.5.2 Maple step by step solution

Let's solve

$\cos(x) y' + y = \cos(x)^2$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = -\frac{y}{\cos(x)} + \cos(x)$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\cos(x)} = \cos(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{\cos(x)}\right) = \mu(x)\cos(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)\left(y' + \frac{y}{\cos(x)}\right) = \mu'(x) y + \mu(x) y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{\cos(x)}$
- Solve to find the integrating factor

  $\mu(x) = \sec(x) + \tan(x)$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x)\cos(x) dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x)\cos(x) dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\cos(x)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \sec(x) + \tan(x)$

$y = \frac{\int \cos(x)(\sec(x)+\tan(x))dx+c_1}{\sec(x)+\tan(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{x-\cos(x)+c_1}{\sec(x)+\tan(x)}$

- Simplify

  $y = \frac{(x-\cos(x)+c_1)(\cos(x)-\sin(x)+1)}{\cos(x)+1+\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x)*cos(x)+y(x)=cos(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(x - \cos(x) + c_1)(\cos(x) - \sin(x) + 1)}{\sin(x) + \cos(x) + 1}$$

✓ Solution by Mathematica

Time used: 0.079 (sec). Leaf size: 25

```
DSolve[y'[x]*Cos[x]+y[x]==Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2\operatorname{arctanh}\left(\tan\left(\frac{x}{2}\right)\right)}(x - \cos(x) + c_1)$$

3.6 problem 6

3.6.1 Solving as linear ode 89
3.6.2 Maple step by step solution 91

Internal problem ID [4766]
Internal file name [`OUTPUT/4259_Sunday_June_05_2022_12_49_09_PM_4922600/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}}}$$

3.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{\sqrt{x^2+1}} \\ q(x) &= \frac{1}{x+\sqrt{x^2+1}} \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{\sqrt{x^2+1}}dx} \\
&= x + \sqrt{x^2+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x+\sqrt{x^2+1}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x+\sqrt{x^2+1}\right)y\right) &= \left(x+\sqrt{x^2+1}\right)\left(\frac{1}{x+\sqrt{x^2+1}}\right) \\
\mathrm{d}\left(\left(x+\sqrt{x^2+1}\right)y\right) &= \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(x+\sqrt{x^2+1}\right)y &= \int \mathrm{d}x \\
\left(x+\sqrt{x^2+1}\right)y &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x + \sqrt{x^2+1}$ results in

$$
y = \frac{x}{x+\sqrt{x^2+1}} + \frac{c_1}{x+\sqrt{x^2+1}}
$$

which simplifies to

$$
y = \frac{x+c_1}{x+\sqrt{x^2+1}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x+c_1}{x+\sqrt{x^2+1}} \tag{1}
$$

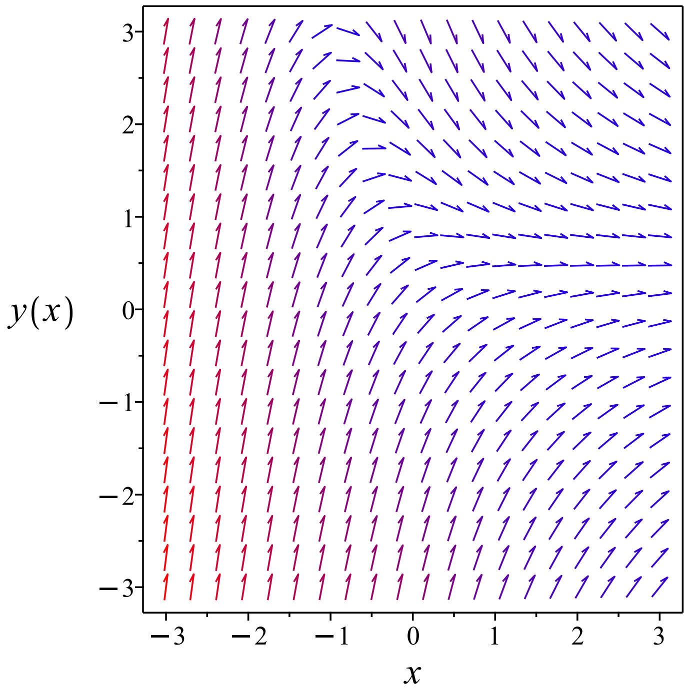

Figure 19: Slope field plot

Verification of solutions

$$y = \frac{x + c_1}{x + \sqrt{x^2 + 1}}$$

Verified OK.

3.6.2 Maple step by step solution

Let's solve

$y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{\sqrt{x^2+1}} + \frac{1}{x+\sqrt{x^2+1}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{\sqrt{x^2+1}} = \frac{1}{x+\sqrt{x^2+1}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{y}{\sqrt{x^2+1}}\right) = \frac{\mu(x)}{x+\sqrt{x^2+1}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y' + \frac{y}{\sqrt{x^2+1}}\right) = \mu'(x)\,y + \mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)}{\sqrt{x^2+1}}$$

- Solve to find the integrating factor

$$\mu(x) = \frac{\sqrt{x^2+1}\,x + x^2 + 1}{\sqrt{x^2+1}}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{\mu(x)}{x+\sqrt{x^2+1}} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y = \int \frac{\mu(x)}{x+\sqrt{x^2+1}} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)}{x+\sqrt{x^2+1}} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{\sqrt{x^2+1}\,x + x^2 + 1}{\sqrt{x^2+1}}$

$$y = \frac{\sqrt{x^2+1}\left(\int \frac{\sqrt{x^2+1}\,x + x^2 + 1}{\sqrt{x^2+1}\left(x+\sqrt{x^2+1}\right)} dx + c_1\right)}{\sqrt{x^2+1}\,x + x^2 + 1}$$

- Evaluate the integrals on the rhs

$$y = \frac{\sqrt{x^2+1}\,(x + c_1)}{\sqrt{x^2+1}\,x + x^2 + 1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+y(x)/sqrt(x^2+1)=1/(x+sqrt(x^2+1)),y(x), singsol=all)
```

$$y(x) = \frac{x + c_1}{x + \sqrt{x^2 + 1}}$$

✓ Solution by Mathematica

Time used: 0.121 (sec). Leaf size: 23

```
DSolve[y'[x]+y[x]/Sqrt[x^2+1]==1/(x+Sqrt[x^2+1]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\sqrt{x^2 + 1} - x\right)(x + c_1)$$

3.7 problem 7

3.7.1 Solving as linear ode 94
3.7.2 Maple step by step solution 96

Internal problem ID [4767]
Internal file name [`OUTPUT/4260_Sunday_June_05_2022_12_49_18_PM_82020420/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(1+\mathrm{e}^{x})\,y'+2\,\mathrm{e}^{x}y=(1+\mathrm{e}^{x})\,\mathrm{e}^{x}}$$

3.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{2\,\mathrm{e}^{x}}{1+\mathrm{e}^{x}}$$
$$q(x)=\mathrm{e}^{x}$$

Hence the ode is

$$y'+\frac{2\,\mathrm{e}^{x}y}{1+\mathrm{e}^{x}}=\mathrm{e}^{x}$$

The integrating factor μ is

$$\mu=\mathrm{e}^{\int \frac{2\,\mathrm{e}^{x}}{1+\mathrm{e}^{x}}dx}$$
$$=(1+\mathrm{e}^{x})^{2}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{x}\right) \\
\frac{d}{dx}\left(\left(1+e^{x}\right)^{2} y\right) &= \left(\left(1+e^{x}\right)^{2}\right)\left(e^{x}\right) \\
d\left(\left(1+e^{x}\right)^{2} y\right) &= \left(e^{x}\left(1+e^{x}\right)^{2}\right)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(1+e^{x}\right)^{2} y &= \int e^{x}\left(1+e^{x}\right)^{2}\, dx \\
\left(1+e^{x}\right)^{2} y &= \frac{\left(1+e^{x}\right)^{3}}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \left(1+e^{x}\right)^{2}$ results in

$$
y = \frac{1}{3} + \frac{e^{x}}{3} + \frac{c_1}{\left(1+e^{x}\right)^{2}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{3} + \frac{e^{x}}{3} + \frac{c_1}{\left(1+e^{x}\right)^{2}} \tag{1}
$$

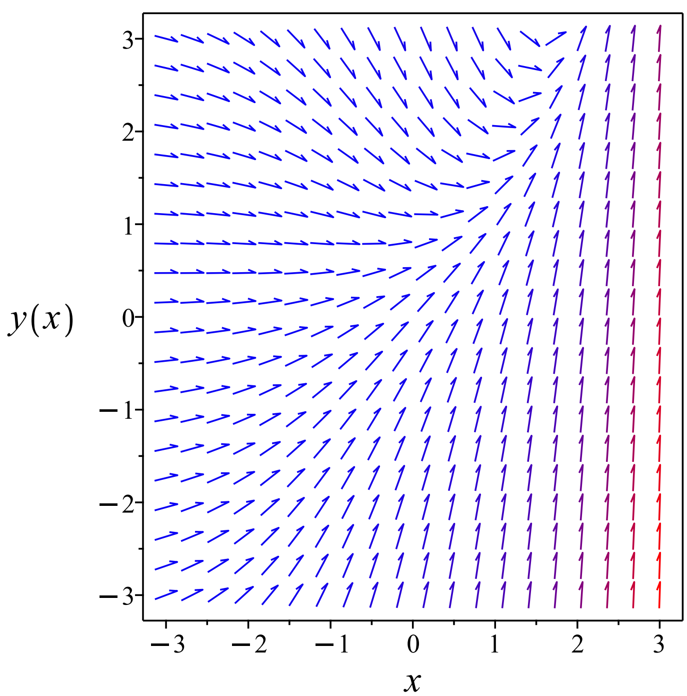

Figure 20: Slope field plot

Verification of solutions

$$y = \frac{1}{3} + \frac{e^x}{3} + \frac{c_1}{(1+e^x)^2}$$

Verified OK.

3.7.2 Maple step by step solution

Let's solve

$(1+e^x) y' + 2 e^x y = (1+e^x) e^x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2 e^x y}{1+e^x} + e^x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2 e^x y}{1+e^x} = e^x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{2 e^x y}{1+e^x}\right) = \mu(x) e^x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y' + \frac{2 e^x y}{1+e^x}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x) e^x}{1+e^x}$

- Solve to find the integrating factor

  $\mu(x) = (1+e^x)^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \mu(x) e^x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \mu(x) e^x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (1+e^x)^2$

$$y = \frac{\int e^x(1+e^x)^2 dx + c_1}{(1+e^x)^2}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{(1+e^x)^3}{3} + c_1}{(1+e^x)^2}$$

- Simplify

$$y = \frac{e^{3x} + 3e^{2x} + 3e^x + 3c_1 + 1}{3(1+e^x)^2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((1+exp(x))*diff(y(x),x)+2*exp(x)*y(x)=(1+exp(x))*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{e^{3x} + 3e^{2x} + 3e^x + 3c_1}{3(1+e^x)^2}$$

✓ Solution by Mathematica

Time used: 0.08 (sec). Leaf size: 25

```
DSolve[(1+Exp[x])*y'[x]+2*Exp[x]*y[x]==(1+Exp[x])*Exp[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{3}(e^x + 1) + \frac{c_1}{(e^x + 1)^2}$$

3.8 problem 8

3.8.1 Solving as linear ode 98
3.8.2 Maple step by step solution 100

Internal problem ID [4768]
Internal file name [`OUTPUT/4261_Sunday_June_05_2022_12_49_26_PM_26698296/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x \ln (x) y' + y = \ln (x)}$$

3.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x \ln (x)}$$
$$q(x) = \frac{1}{x}$$

Hence the ode is

$$y' + \frac{y}{x \ln (x)} = \frac{1}{x}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{x \ln(x)} dx}$$
$$= \ln (x)$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{1}{x}\right)$$

$$\frac{d}{dx}(\ln(x)\, y) = (\ln(x))\left(\frac{1}{x}\right)$$

$$d(\ln(x)\, y) = \left(\frac{\ln(x)}{x}\right) dx$$

Integrating gives

$$\ln(x)\, y = \int \frac{\ln(x)}{x}\, dx$$

$$\ln(x)\, y = \frac{\ln(x)^2}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = \ln(x)$ results in

$$y = \frac{\ln(x)}{2} + \frac{c_1}{\ln(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\ln(x)}{2} + \frac{c_1}{\ln(x)} \tag{1}$$

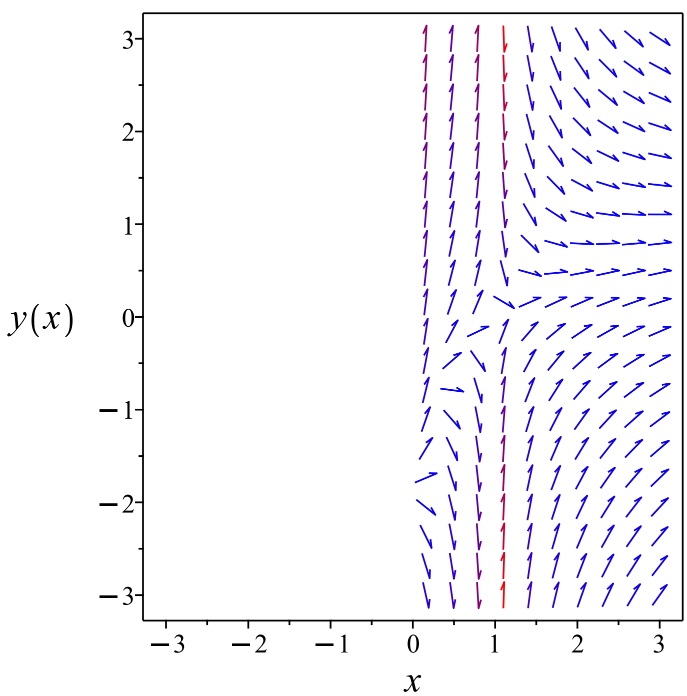

Figure 21: Slope field plot

Verification of solutions

$$y = \frac{\ln(x)}{2} + \frac{c_1}{\ln(x)}$$

Verified OK.

3.8.2 Maple step by step solution

Let's solve

$x \ln(x) y' + y = \ln(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x \ln(x)} + \frac{1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x \ln(x)} = \frac{1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{y}{x \ln(x)}\right) = \frac{\mu(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y' + \frac{y}{x \ln(x)}\right) = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x \ln(x)}$

- Solve to find the integrating factor

  $\mu(x) = \ln(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \frac{\mu(x)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \frac{\mu(x)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \ln(x)$

  $y = \frac{\int \frac{\ln(x)}{x} dx + c_1}{\ln(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\ln(x)^2}{2} + c_1}{\ln(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((x*ln(x))*diff(y(x),x)+y(x)=ln(x),y(x), singsol=all)
```

$$y(x) = \frac{\ln(x)}{2} + \frac{c_1}{\ln(x)}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 19

```
DSolve[(x*Log[x])*y'[x]+y[x]==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x)}{2} + \frac{c_1}{\log(x)}$$

3.9 problem 9

3.9.1 Solving as linear ode 102
3.9.2 Maple step by step solution 104

Internal problem ID [4769]
Internal file name [`OUTPUT/4262_Sunday_June_05_2022_12_49_35_PM_96920737/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\left(-x^{2}+1\right) y^{\prime}-x y=2 \sqrt{-x^{2}+1}\, x}$$

3.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y^{\prime}+p(x) y=q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{x}{x^{2}-1} \\ q(x) &= \frac{2 x}{\sqrt{-x^{2}+1}} \end{aligned}$$

Hence the ode is

$$y^{\prime}+\frac{x y}{x^{2}-1}=\frac{2 x}{\sqrt{-x^{2}+1}}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{x}{x^{2}-1} d x} \\ &= \mathrm{e}^{\frac{\ln (x-1)}{2}+\frac{\ln (x+1)}{2}} \end{aligned}$$

Which simplifies to

$$\mu = \sqrt{x-1}\,\sqrt{x+1}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{2x}{\sqrt{-x^2+1}}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x-1}\,\sqrt{x+1}\,y\right) &= \left(\sqrt{x-1}\,\sqrt{x+1}\right)\left(\frac{2x}{\sqrt{-x^2+1}}\right) \\
\mathrm{d}\left(\sqrt{x-1}\,\sqrt{x+1}\,y\right) &= \left(\frac{2x\sqrt{x-1}\,\sqrt{x+1}}{\sqrt{-x^2+1}}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sqrt{x-1}\,\sqrt{x+1}\,y &= \int \frac{2x\sqrt{x-1}\,\sqrt{x+1}}{\sqrt{-x^2+1}}\,\mathrm{d}x \\
\sqrt{x-1}\,\sqrt{x+1}\,y &= \frac{x^2\sqrt{x-1}\,\sqrt{x+1}}{\sqrt{-x^2+1}} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x-1}\,\sqrt{x+1}$ results in

$$y = \frac{x^2}{\sqrt{-x^2+1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{\sqrt{-x^2+1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}} \tag{1}$$

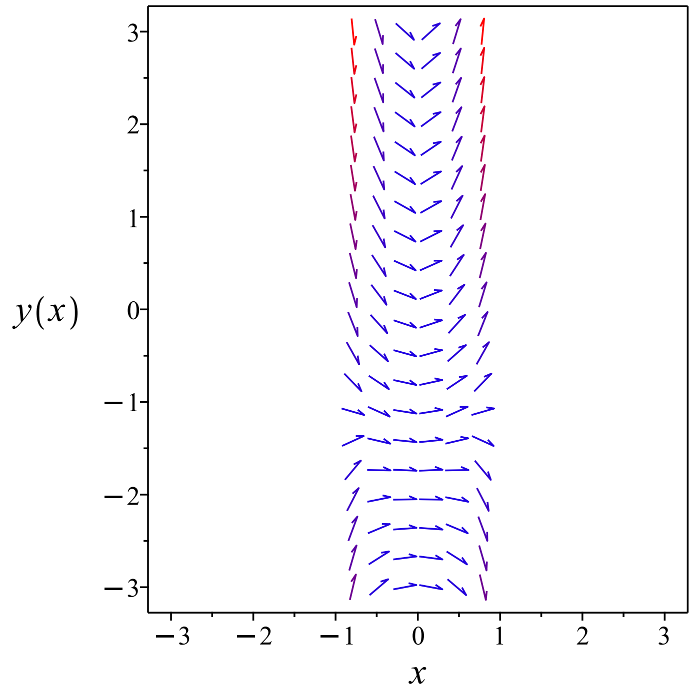

Figure 22: Slope field plot

Verification of solutions

$$y = \frac{x^2}{\sqrt{-x^2+1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

Verified OK.

3.9.2 Maple step by step solution

Let's solve

$(-x^2+1)\,y' - xy = 2\sqrt{-x^2+1}\,x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{xy}{x^2-1} + \frac{2x}{\sqrt{-x^2+1}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{xy}{x^2-1} = \frac{2x}{\sqrt{-x^2+1}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{xy}{x^2-1}\right)=\frac{2\mu(x)x}{\sqrt{-x^2+1}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{xy}{x^2-1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)x}{x^2-1}$$

- Solve to find the integrating factor

$$\mu(x)=\sqrt{x-1}\,\sqrt{x+1}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{2\mu(x)x}{\sqrt{-x^2+1}}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{2\mu(x)x}{\sqrt{-x^2+1}}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{2\mu(x)x}{\sqrt{-x^2+1}}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\sqrt{x-1}\,\sqrt{x+1}$

$$y=\frac{\int\frac{2x\sqrt{x-1}\,\sqrt{x+1}}{\sqrt{-x^2+1}}dx+c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

- Evaluate the integrals on the rhs

$$y=\frac{\frac{x^2\sqrt{x-1}\,\sqrt{x+1}}{\sqrt{-x^2+1}}+c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve((1-x^2)*diff(y(x),x)=x*y(x)+2*x*sqrt(1-x^2),y(x), singsol=all)
```

$$y(x) = \frac{x^2}{\sqrt{-x^2+1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{1+x}}$$

✓ Solution by Mathematica

Time used: 0.062 (sec). Leaf size: 33

```
DSolve[(1-x^2)*y'[x]==x*y[x]+2*x*Sqrt[1-x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{\sqrt{1-x^2}} + \frac{c_1}{\sqrt{x^2-1}}$$

3.10 problem 10

3.10.1 Solving as linear ode 107
3.10.2 Maple step by step solution 109

Internal problem ID [4770]
Internal file name [`OUTPUT/4263_Sunday_June_05_2022_12_49_44_PM_29245501/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\tanh(x) = 2\,\mathrm{e}^{x}}$$

3.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tanh(x) \\ q(x) &= 2\,\mathrm{e}^{x} \end{aligned}$$

Hence the ode is

$$y' + y\tanh(x) = 2\,\mathrm{e}^{x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \tanh(x)dx} \\ &= \cosh(x) \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2\,e^{x}\right)$$
$$\frac{d}{dx}(\cosh(x)\,y) = (\cosh(x))\left(2\,e^{x}\right)$$
$$d(\cosh(x)\,y) = \left(2\,e^{x}\cosh(x)\right)\,dx$$

Integrating gives

$$\cosh(x)\,y = \int 2\,e^{x}\cosh(x)\,dx$$
$$\cosh(x)\,y = \cosh(x)^{2} + \cosh(x)\sinh(x) + x + c_1$$

Dividing both sides by the integrating factor $\mu = \cosh(x)$ results in

$$y = \operatorname{sech}(x)\left(\cosh(x)^{2} + \cosh(x)\sinh(x) + x\right) + c_1 \operatorname{sech}(x)$$

which simplifies to

$$y = (x + c_1)\operatorname{sech}(x) + \cosh(x) + \sinh(x)$$

Summary

The solution(s) found are the following

$$y = (x + c_1)\operatorname{sech}(x) + \cosh(x) + \sinh(x) \tag{1}$$

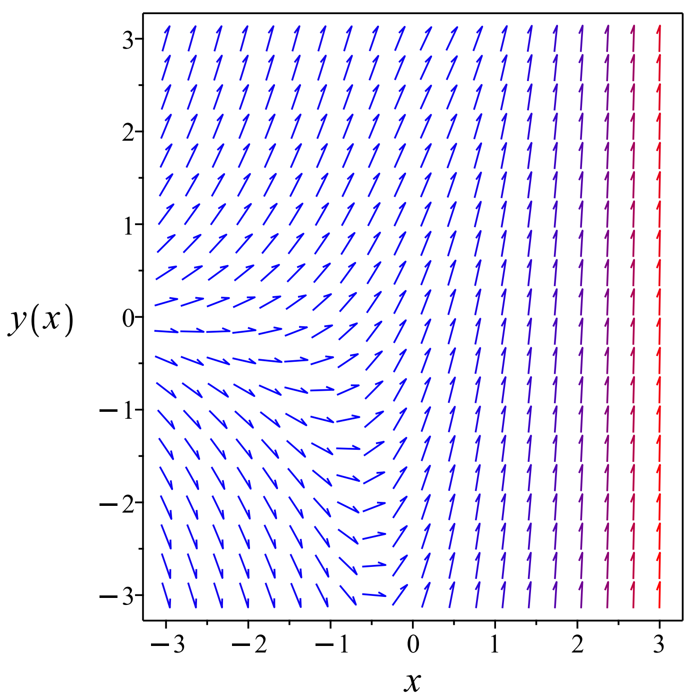

Figure 23: Slope field plot

Verification of solutions

$$y = (x + c_1) \operatorname{sech}(x) + \cosh(x) + \sinh(x)$$

Verified OK.

3.10.2 Maple step by step solution

Let's solve

$y' + y \tanh(x) = 2\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y \tanh(x) + 2\,\mathrm{e}^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y \tanh(x) = 2\,\mathrm{e}^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y \tanh(x)) = 2\mu(x)\,\mathrm{e}^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y \tanh(x)) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x) \tanh(x)$

- Solve to find the integrating factor

  $\mu(x) = \cosh(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,\mathrm{e}^{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\,\mathrm{e}^{x} dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)\mathrm{e}^{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cosh(x)$

  $y = \frac{\int 2\,\mathrm{e}^{x} \cosh(x) dx + c_1}{\cosh(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\cosh(x)^2+\cosh(x)\sinh(x)+x+c_1}{\cosh(x)}$

- Simplify

  $y = (x + c_1)\operatorname{sech}(x) + \cosh(x) + \sinh(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)*tanh(x)=2*exp(x),y(x), singsol=all)
```

$$y(x) = (x + c_1)\operatorname{sech}(x) + \cosh(x) + \sinh(x)$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 29

```
DSolve[y'[x]+y[x]*Tanh[x]==2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x(2x + e^{2x} + c_1)}{e^{2x} + 1}$$

3.11 problem 11

3.11.1 Solving as linear ode 111
3.11.2 Maple step by step solution 113

Internal problem ID [4771]
Internal file name [`OUTPUT/4264_Sunday_June_05_2022_12_49_53_PM_71133227/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cos(x) = \sin(2x)}$$

3.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = \sin(2x)$$

Hence the ode is

$$y' + y\cos(x) = \sin(2x)$$

The integrating factor μ is

$$\mu = e^{\int \cos(x)dx}$$
$$= e^{\sin(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\sin(2x))$$
$$\frac{d}{dx}\left(e^{\sin(x)} y\right) = \left(e^{\sin(x)}\right)(\sin(2x))$$
$$d\left(e^{\sin(x)} y\right) = \left(\sin(2x)\, e^{\sin(x)}\right)\, dx$$

Integrating gives

$$e^{\sin(x)} y = \int \sin(2x)\, e^{\sin(x)}\, dx$$
$$e^{\sin(x)} y = 2\sin(x)\, e^{\sin(x)} - 2\, e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}\left(2\sin(x)\, e^{\sin(x)} - 2\, e^{\sin(x)}\right) + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)}$$

Summary

The solution(s) found are the following

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)} \tag{1}$$

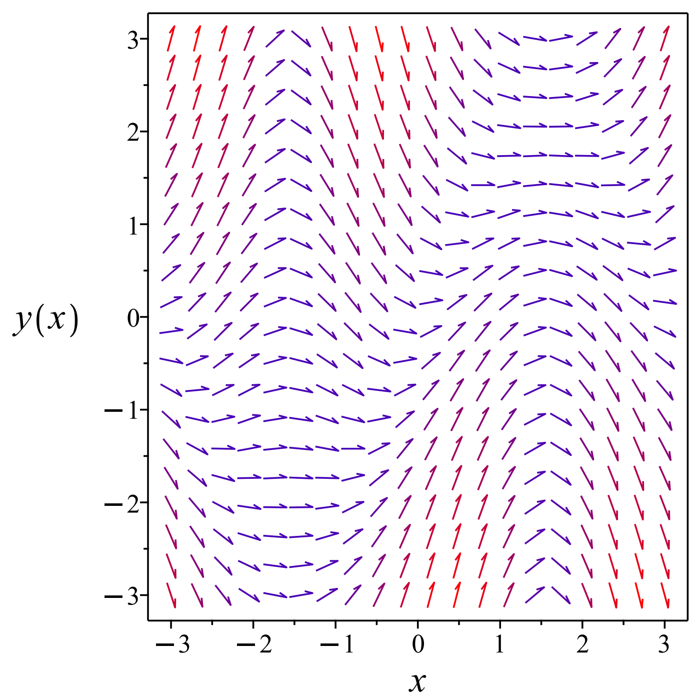

Figure 24: Slope field plot

Verification of solutions

$$y = 2\sin(x) - 2 + c_1 e^{-\sin(x)}$$

Verified OK.

3.11.2 Maple step by step solution

Let's solve

$y' + y\cos(x) = \sin(2x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\cos(x) + \sin(2x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\cos(x) = \sin(2x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(y' + y\cos(x)) = \mu(x)\sin(2x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)(y' + y\cos(x)) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\cos(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{\sin(x)}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\sin(2x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\sin(2x)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(2x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{\sin(x)}$

  $y = \frac{\int \sin(2x)e^{\sin(x)}dx + c_1}{e^{\sin(x)}}$

- Evaluate the integrals on the rhs

  $y = \frac{2\sin(x)\mathrm{e}^{\sin(x)} - 2\,\mathrm{e}^{\sin(x)} + c_1}{\mathrm{e}^{\sin(x)}}$

- Simplify

  $y = 2\sin(x) - 2 + c_1\mathrm{e}^{-\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x)+y(x)*cos(x)=sin(2*x),y(x), singsol=all)
```

$$y(x) = 2\sin(x) - 2 + \mathrm{e}^{-\sin(x)}c_1$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]*Cos[x]==Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sin(x) + c_1e^{-\sin(x)} - 2$$

3.12 problem 12

3.12.1 Solving as linear ode 115
3.12.2 Maple step by step solution 117

Internal problem ID [4772]
Internal file name [`OUTPUT/4265_Sunday_June_05_2022_12_50_02_PM_77883974/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' + x\tan(y) = \cos(y)}$$

3.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$\begin{aligned} p(y) &= \tan(y) \\ q(y) &= \cos(y) \end{aligned}$$

Hence the ode is

$$x' + x\tan(y) = \cos(y)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \tan(y)dy} \\ &= \frac{1}{\cos(y)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(y)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dy}(\mu x) &= (\mu)(\cos(y)) \\
\frac{d}{dy}(\sec(y)\,x) &= (\sec(y))(\cos(y)) \\
d(\sec(y)\,x) &= dy
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(y)\,x &= \int dy \\
\sec(y)\,x &= y + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(y)$ results in

$$x = \cos(y)\,y + c_1\cos(y)$$

which simplifies to

$$x = \cos(y)\,(y + c_1)$$

Summary

The solution(s) found are the following

$$x = \cos(y)\,(y + c_1) \tag{1}$$

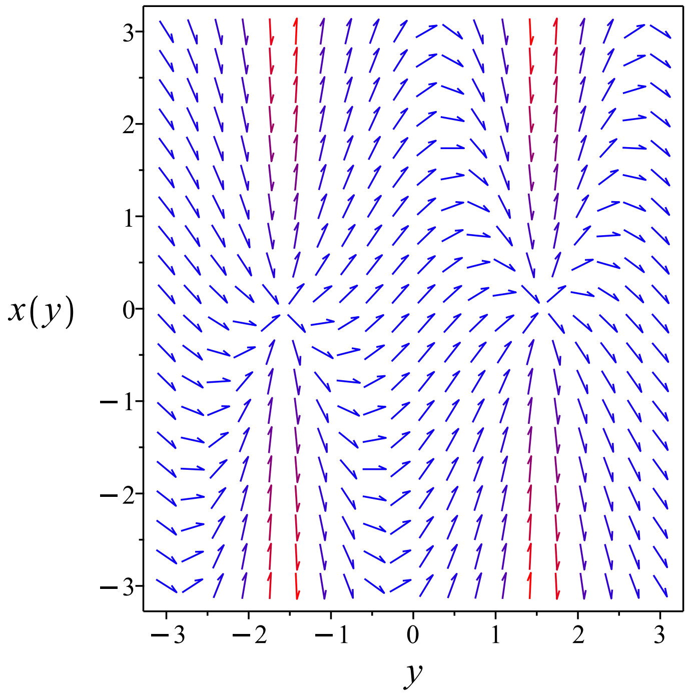

Figure 25: Slope field plot

Verification of solutions

$$x = \cos(y)(y + c_1)$$

Verified OK.

3.12.2 Maple step by step solution

Let's solve

$x' + x\tan(y) = \cos(y)$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = \cos(y) - x\tan(y)$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x\tan(y) = \cos(y)$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

$$\mu(y)\left(x' + x\tan(y)\right) = \mu(y)\cos(y)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\, x)$

$$\mu(y)\left(x' + x\tan(y)\right) = \mu'(y)\, x + \mu(y)\, x'$$

- Isolate $\mu'(y)$

$$\mu'(y) = \mu(y)\tan(y)$$

- Solve to find the integrating factor

$$\mu(y) = \frac{1}{\cos(y)}$$

- Integrate both sides with respect to y

$$\int \left(\frac{d}{dy}(\mu(y)\, x)\right) dy = \int \mu(y)\cos(y)\, dy + c_1$$

- Evaluate the integral on the lhs

$$\mu(y)\, x = \int \mu(y)\cos(y)\, dy + c_1$$

- Solve for x

$$x = \frac{\int \mu(y)\cos(y)dy + c_1}{\mu(y)}$$

- Substitute $\mu(y) = \frac{1}{\cos(y)}$

$$x = \cos(y)\left(\int 1dy + c_1\right)$$

- Evaluate the integrals on the rhs

$$x = \cos(y)\,(y + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(x(y),y)=cos(y)-x(y)*tan(y),x(y), singsol=all)
```

$$x(y) = (y + c_1)\cos(y)$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 12

```
DSolve[x'[y]==Cos[y]-x[y]*Tan[y],x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to (y + c_1)\cos(y)$$

3.13 problem 13

3.13.1 Solving as linear ode 120
3.13.2 Maple step by step solution 122

Internal problem ID [4773]
Internal file name [`OUTPUT/4266_Sunday_June_05_2022_12_50_10_PM_77201851/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{x' + x = e^{y}}$$

3.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = 1$$
$$q(y) = e^{y}$$

Hence the ode is

$$x' + x = e^{y}$$

The integrating factor μ is

$$\mu = e^{\int 1 dy}$$
$$= e^{y}$$

The ode becomes

$$\frac{d}{dy}(\mu x) = (\mu)\left(e^{y}\right)$$
$$\frac{d}{dy}\left(x\,e^{y}\right) = \left(e^{y}\right)\left(e^{y}\right)$$
$$d\left(x\,e^{y}\right) = e^{2y}\,dy$$

Integrating gives

$$x\,e^{y} = \int e^{2y}\,dy$$
$$x\,e^{y} = \frac{e^{2y}}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{y}$ results in

$$x = \frac{e^{-y}e^{2y}}{2} + c_1 e^{-y}$$

which simplifies to

$$x = \frac{e^{y}}{2} + c_1 e^{-y}$$

Summary
The solution(s) found are the following

$$x = \frac{e^{y}}{2} + c_1 e^{-y} \tag{1}$$

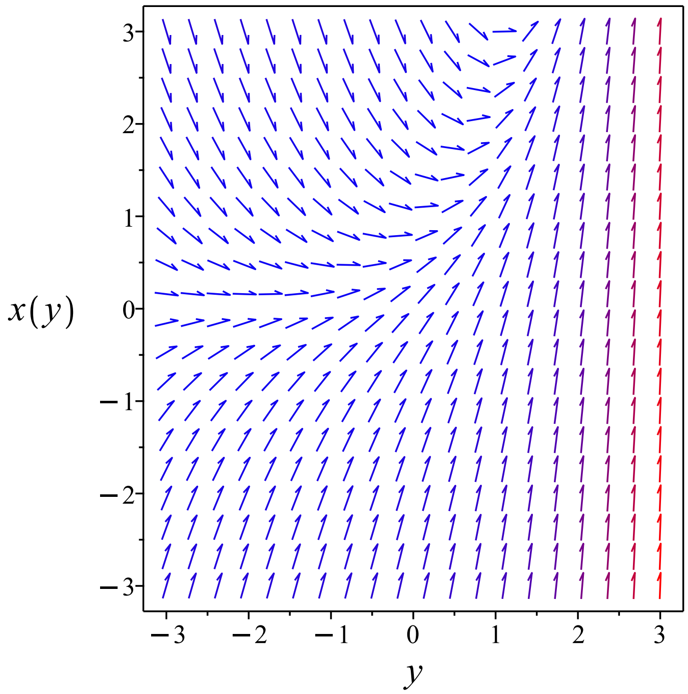

Figure 26: Slope field plot

Verification of solutions

$$x = \frac{e^y}{2} + c_1 e^{-y}$$

Verified OK.

3.13.2 Maple step by step solution

Let's solve

$x' + x = e^y$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -x + e^y$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + x = e^y$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

$$\mu(y)\left(x'+x\right)=\mu(y)\,\mathrm{e}^{y}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

$$\mu(y)\left(x'+x\right)=\mu'(y)\,x+\mu(y)\,x'$$

- Isolate $\mu'(y)$

$$\mu'(y)=\mu(y)$$

- Solve to find the integrating factor

$$\mu(y)=\mathrm{e}^{y}$$

- Integrate both sides with respect to y

$$\int\left(\frac{d}{dy}(\mu(y)\,x)\right)dy=\int\mu(y)\,\mathrm{e}^{y}dy+c_{1}$$

- Evaluate the integral on the lhs

$$\mu(y)\,x=\int\mu(y)\,\mathrm{e}^{y}dy+c_{1}$$

- Solve for x

$$x=\frac{\int\mu(y)\mathrm{e}^{y}dy+c_{1}}{\mu(y)}$$

- Substitute $\mu(y)=\mathrm{e}^{y}$

$$x=\frac{\int(\mathrm{e}^{y})^{2}dy+c_{1}}{\mathrm{e}^{y}}$$

- Evaluate the integrals on the rhs

$$x=\frac{\frac{(\mathrm{e}^{y})^{2}}{2}+c_{1}}{\mathrm{e}^{y}}$$

- Simplify

$$x=\frac{\mathrm{e}^{y}}{2}+c_{1}\mathrm{e}^{-y}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(x(y),y)+(x(y)-exp(y))=0,x(y), singsol=all)
```

$$x(y) = \frac{e^{y}}{2} + e^{-y} c_{1}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 21

```
DSolve[x'[y]+(x[y]-Exp[y])==0,x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to \frac{e^{y}}{2} + c_{1} e^{-y}$$

3.14 problem 14

3.14.1 Solving as linear ode 125
3.14.2 Maple step by step solution 127

Internal problem ID [4774]
Internal file name [`OUTPUT/4267_Sunday_June_05_2022_12_50_22_PM_1604250/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 3. Linear First-Order Equations. page 403
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x' - \frac{3y^{\frac{2}{3}} - x}{3y} = 0}$$

3.14.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$x' + p(y)x = q(y)$$

Where here

$$p(y) = \frac{1}{3y}$$
$$q(y) = \frac{1}{y^{\frac{1}{3}}}$$

Hence the ode is

$$x' + \frac{x}{3y} = \frac{1}{y^{\frac{1}{3}}}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{3y}dy}$$
$$= y^{\frac{1}{3}}$$

The ode becomes

$$\frac{d}{dy}(\mu x) = (\mu)\left(\frac{1}{y^{\frac{1}{3}}}\right)$$
$$\frac{d}{dy}\left(x\,y^{\frac{1}{3}}\right) = \left(y^{\frac{1}{3}}\right)\left(\frac{1}{y^{\frac{1}{3}}}\right)$$
$$d\left(x\,y^{\frac{1}{3}}\right) = dy$$

Integrating gives

$$x\,y^{\frac{1}{3}} = \int dy$$
$$x\,y^{\frac{1}{3}} = y + c_1$$

Dividing both sides by the integrating factor $\mu = y^{\frac{1}{3}}$ results in

$$x = y^{\frac{2}{3}} + \frac{c_1}{y^{\frac{1}{3}}}$$

which simplifies to

$$x = \frac{y + c_1}{y^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$x = \frac{y + c_1}{y^{\frac{1}{3}}} \tag{1}$$

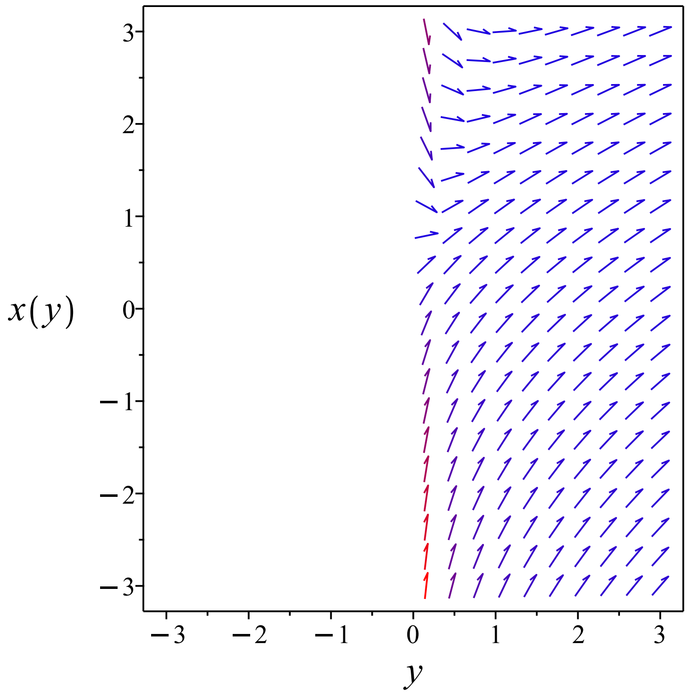

Figure 27: Slope field plot

Verification of solutions

$$x = \frac{y + c_1}{y^{\frac{1}{3}}}$$

Verified OK.

3.14.2 Maple step by step solution

Let's solve

$x' - \frac{3y^{\frac{2}{3}} - x}{3y} = 0$

- Highest derivative means the order of the ODE is 1

  x'

- Isolate the derivative

  $x' = -\frac{x}{3y} + \frac{1}{y^{\frac{1}{3}}}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE

  $x' + \frac{x}{3y} = \frac{1}{y^{\frac{1}{3}}}$

- The ODE is linear; multiply by an integrating factor $\mu(y)$

  $\mu(y)\left(x'+\frac{x}{3y}\right)=\frac{\mu(y)}{y^{\frac{1}{3}}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dy}(\mu(y)\,x)$

  $\mu(y)\left(x'+\frac{x}{3y}\right)=\mu'(y)\,x+\mu(y)\,x'$

- Isolate $\mu'(y)$

  $\mu'(y)=\frac{\mu(y)}{3y}$

- Solve to find the integrating factor

  $\mu(y)=y^{\frac{1}{3}}$

- Integrate both sides with respect to y

  $\int\left(\frac{d}{dy}(\mu(y)\,x)\right)dy=\int\frac{\mu(y)}{y^{\frac{1}{3}}}dy+c_1$

- Evaluate the integral on the lhs

  $\mu(y)\,x=\int\frac{\mu(y)}{y^{\frac{1}{3}}}dy+c_1$

- Solve for x

  $x=\frac{\int\frac{\mu(y)}{y^{\frac{1}{3}}}dy+c_1}{\mu(y)}$

- Substitute $\mu(y)=y^{\frac{1}{3}}$

  $x=\frac{\int 1dy+c_1}{y^{\frac{1}{3}}}$

- Evaluate the integrals on the rhs

  $x=\frac{y+c_1}{y^{\frac{1}{3}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(x(y),y)=(3*y^(2/3)-x(y))/(3*y),x(y), singsol=all)
```

$$x(y) = \frac{y + c_1}{y^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 15

```
DSolve[x'[y]==(3*y^(2/3)-x[y])/(3*y),x[y],y,IncludeSingularSolutions -> True]
```

$$x(y) \to \frac{y + c_1}{\sqrt[3]{y}}$$

4 Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406

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4.1 problem 1

4.1.1 Solving as first order ode lie symmetry lookup ode 131
4.1.2 Solving as bernoulli ode 135

Internal problem ID [4775]
Internal file name [`OUTPUT/4268_Sunday_June_05_2022_12_50_30_PM_40268562/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + y - x y^{\frac{2}{3}} = 0}$$

4.1.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y + x\, y^{\frac{2}{3}} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 28: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= y^{\frac{2}{3}} \mathrm{e}^{-\frac{x}{3}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^{\frac{2}{3}} \mathrm{e}^{-\frac{x}{3}}} dy \end{aligned}$$

Which results in

$$S = 3y^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y + x\, y^{\frac{2}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^{\frac{1}{3}} \mathrm{e}^{\frac{x}{3}} \\ S_y &= \frac{\mathrm{e}^{\frac{x}{3}}}{y^{\frac{2}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \mathrm{e}^{\frac{x}{3}} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \mathrm{e}^{\frac{R}{3}} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3(R-3)\,\mathrm{e}^{\frac{R}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3y^{\frac{1}{3}}\mathrm{e}^{\frac{x}{3}} = 3(x-3)\,\mathrm{e}^{\frac{x}{3}} + c_1$$

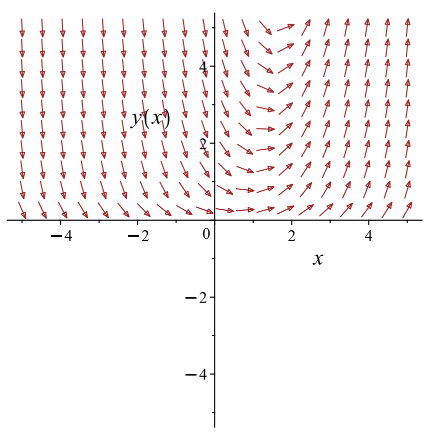
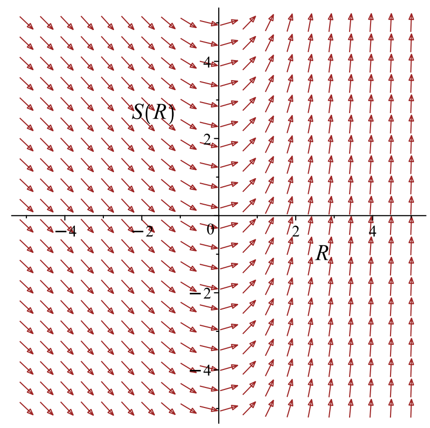
Which simplifies to

$$3y^{\frac{1}{3}}\mathrm{e}^{\frac{x}{3}} = 3(x-3)\,\mathrm{e}^{\frac{x}{3}} + c_1$$

Which gives

$$y = \frac{\left(3\,\mathrm{e}^{\frac{x}{3}}x - 9\,\mathrm{e}^{\frac{x}{3}} + c_1\right)^3 \mathrm{e}^{-x}}{27}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -y + x\,y^{\frac{2}{3}}$

</td><td>$R = x$
$S = 3y^{\frac{1}{3}}\mathrm{e}^{\frac{x}{3}}$</td><td>$\frac{dS}{dR} = \mathrm{e}^{\frac{R}{3}}R$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{\left(3\,\mathrm{e}^{\frac{x}{3}}x - 9\,\mathrm{e}^{\frac{x}{3}} + c_1\right)^3 \mathrm{e}^{-x}}{27} \tag{1}$$

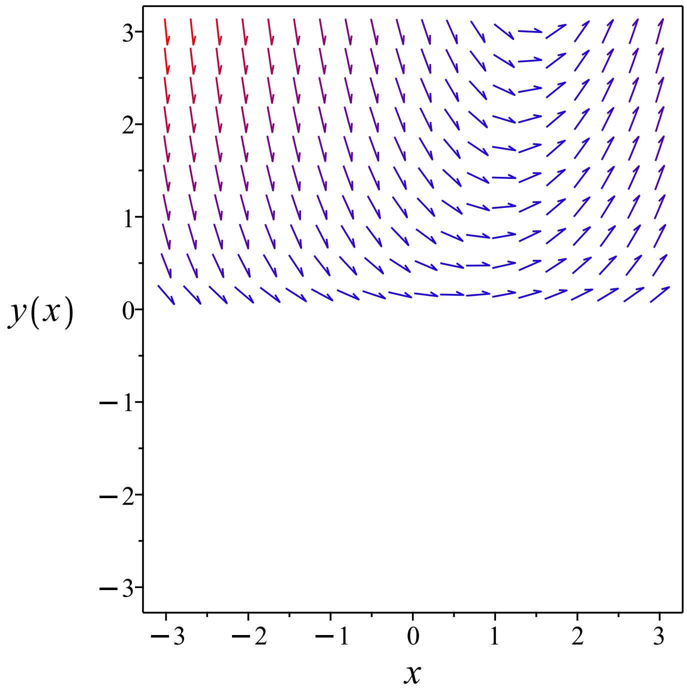


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{\left(3\,\mathrm{e}^{\frac{x}{3}} x - 9\,\mathrm{e}^{\frac{x}{3}} + c_1\right)^3 \mathrm{e}^{-x}}{27}$$

Verified OK.

4.1.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + x\,y^{\frac{2}{3}} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + xy^{\frac{2}{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -1 \\
f_1(x) &= x \\
n &= \frac{2}{3}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{2}{3}}$ gives

$$
y' \frac{1}{y^{\frac{2}{3}}} = -y^{\frac{1}{3}} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^{\frac{1}{3}} \tag{5}
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = \frac{1}{3y^{\frac{2}{3}}} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
3w'(x) &= -w(x) + x \\
w' &= -\frac{w}{3} + \frac{x}{3} \tag{7}
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{3} \\
q(x) &= \frac{x}{3}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{3} = \frac{x}{3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{3}dx} \\ &= e^{\frac{x}{3}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{x}{3}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{3}}w\right) &= \left(e^{\frac{x}{3}}\right)\left(\frac{x}{3}\right) \\ d\left(e^{\frac{x}{3}}w\right) &= \left(\frac{e^{\frac{x}{3}}x}{3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{\frac{x}{3}}w &= \int \frac{e^{\frac{x}{3}}x}{3} dx \\ e^{\frac{x}{3}}w &= (x-3)e^{\frac{x}{3}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{3}}$ results in

$$w(x) = e^{-\frac{x}{3}}(x-3)e^{\frac{x}{3}} + c_1 e^{-\frac{x}{3}}$$

which simplifies to

$$w(x) = x - 3 + c_1 e^{-\frac{x}{3}}$$

Replacing w in the above by $y^{\frac{1}{3}}$ using equation (5) gives the final solution.

$$y^{\frac{1}{3}} = x - 3 + c_1 e^{-\frac{x}{3}}$$

Summary
The solution(s) found are the following

$$y^{\frac{1}{3}} = x - 3 + c_1 e^{-\frac{x}{3}} \tag{1}$$

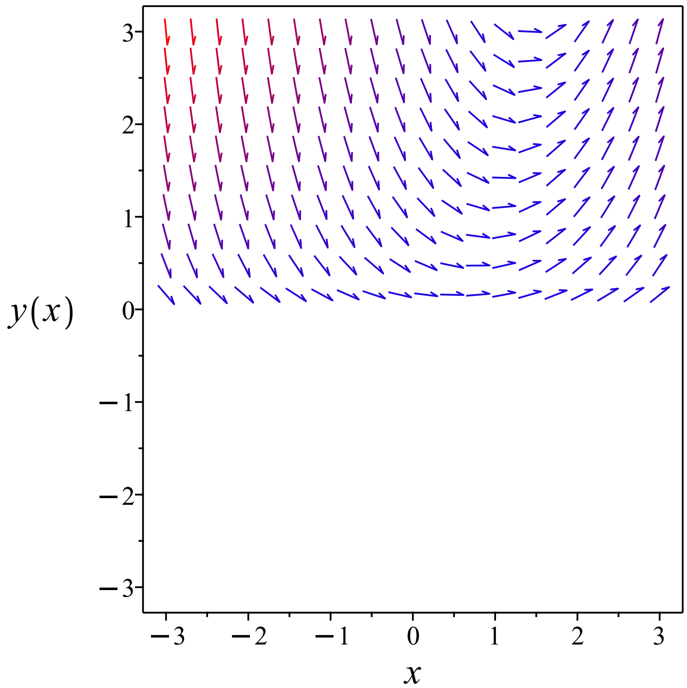


Figure 29: Slope field plot

Verification of solutions

$$y^{\frac{1}{3}} = x - 3 + c_1 e^{-\frac{x}{3}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x)+y(x)=x*y(x)^(2/3),y(x), singsol=all)
```

$$-x+3-\mathrm{e}^{-\frac{x}{3}} c_1+y(x)^{\frac{1}{3}}=0$$

✓ Solution by Mathematica

Time used: 0.167 (sec). Leaf size: 27

```
DSolve[y'[x]+y[x]==x*y[x]^(2/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to e^{-x}\left(e^{x/3}(x-3)+c_1\right)^3$$

4.2 problem 2

4.2.1 Solving as first order ode lie symmetry lookup ode 140
4.2.2 Solving as bernoulli ode 144
4.2.3 Solving as exact ode 147

Internal problem ID [4776]
Internal file name [`OUTPUT/4269_Sunday_June_05_2022_12_50_41_PM_62636223/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{y}{x} - 2x^{\frac{3}{2}}\sqrt{y} = 0}$$

4.2.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x^{\frac{5}{2}}\sqrt{y} - y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 30: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\sqrt{y}}{\sqrt{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{y}}{\sqrt{x}}} dy \end{aligned}$$

Which results in

$$S = 2\sqrt{x}\,\sqrt{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2x^{\frac{5}{2}}\sqrt{y} - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sqrt{y}}{\sqrt{x}} \\ S_y &= \frac{\sqrt{x}}{\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{2R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2\sqrt{x}\,\sqrt{y} = \frac{2x^3}{3} + c_1$$

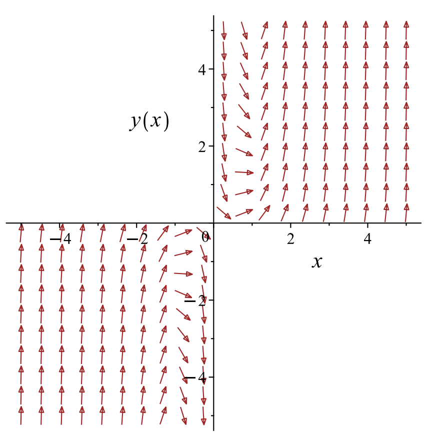
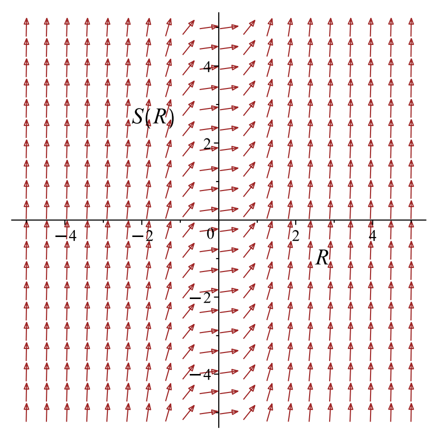
Which simplifies to

$$2\sqrt{x}\,\sqrt{y} = \frac{2x^3}{3} + c_1$$

Which gives

$$y = \frac{(2x^3 + 3c_1)^2}{36x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x^{\frac{5}{2}}\sqrt{y}-y}{x}$ 	$R = x$ $S = 2\sqrt{x}\,\sqrt{y}$	$\frac{dS}{dR} = 2R^2$ 

Summary
The solution(s) found are the following

$$y = \frac{(2x^3 + 3c_1)^2}{36x} \tag{1}$$

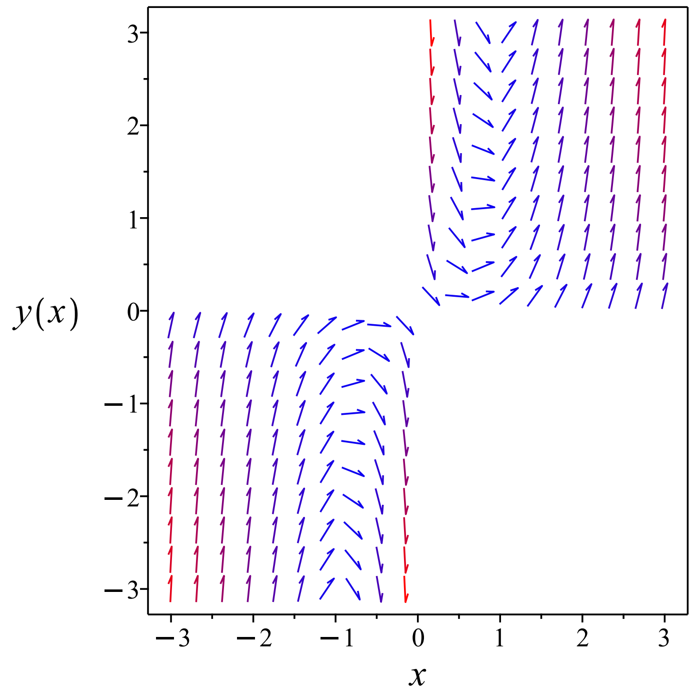


Figure 30: Slope field plot

Verification of solutions

$$y = \frac{(2x^3 + 3c_1)^2}{36x}$$

Verified OK.

4.2.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2x^{\frac{5}{2}}\sqrt{y} - y}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + 2x^{\frac{3}{2}}\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= 2x^{\frac{3}{2}} \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y'\frac{1}{\sqrt{y}} = -\frac{\sqrt{y}}{x} + 2x^{\frac{3}{2}} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= -\frac{w(x)}{x} + 2x^{\frac{3}{2}} \\ w' &= -\frac{w}{2x} + x^{\frac{3}{2}} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{2x} \\ q(x) &= x^{\frac{3}{2}} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{2x} = x^{\frac{3}{2}}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(x^{\frac{3}{2}}\right) \\ \frac{d}{dx}\left(\sqrt{x}\, w\right) &= \left(\sqrt{x}\right)\left(x^{\frac{3}{2}}\right) \\ d\left(\sqrt{x}\, w\right) &= x^2\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \sqrt{x}\, w &= \int x^2\, dx \\ \sqrt{x}\, w &= \frac{x^3}{3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$w(x) = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}} \tag{1}$$

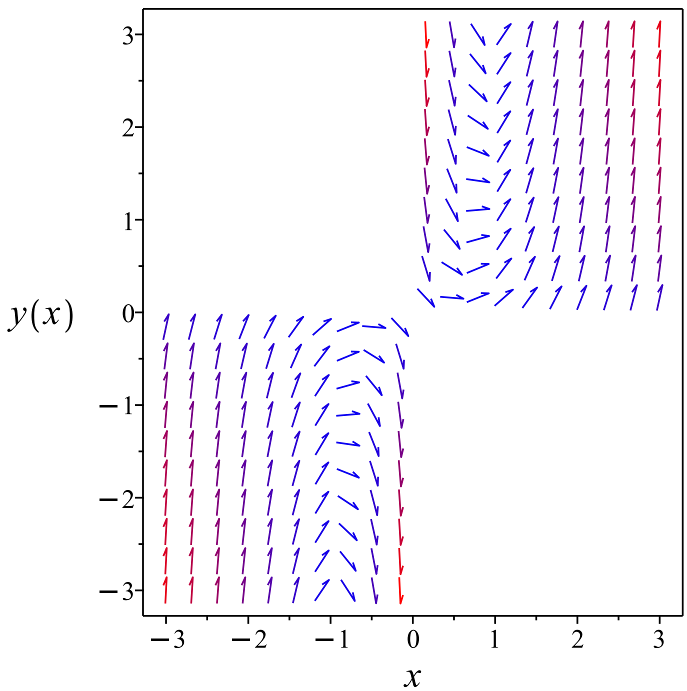


Figure 31: Slope field plot

Verification of solutions

$$\sqrt{y} = \frac{x^{\frac{5}{2}}}{3} + \frac{c_1}{\sqrt{x}}$$

Verified OK.

4.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(2x^{\frac{5}{2}}\sqrt{y} - y\right)\mathrm{d}x \\ \left(-2x^{\frac{5}{2}}\sqrt{y} + y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2x^{\frac{5}{2}}\sqrt{y} + y \\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^{\frac{5}{2}}\sqrt{y}+y\right) \\
&= -\frac{x^{\frac{5}{2}}}{\sqrt{y}}+1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(-\frac{x^{\frac{5}{2}}}{\sqrt{y}}+1\right)-(1)\right) \\
&= -\frac{x^{\frac{3}{2}}}{\sqrt{y}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \\
&= \frac{1}{-2x^{\frac{5}{2}}\sqrt{y}+y}\left((1)-\left(-\frac{x^{\frac{5}{2}}}{\sqrt{y}}+1\right)\right) \\
&= \frac{x^{\frac{5}{2}}}{\left(-2x^{\frac{5}{2}}\sqrt{y}+y\right)\sqrt{y}}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{xM-yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - \left(-\frac{x^{\frac{5}{2}}}{\sqrt{y}} + 1\right)}{x\left(-2x^{\frac{5}{2}}\sqrt{y} + y\right) - y\,(x)} \\
&= -\frac{1}{2yx}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{1}{2t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \, \mathrm{d}t} \\
&= e^{\int \left(-\frac{1}{2t}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\frac{\ln(t)}{2}} \\
&= \frac{1}{\sqrt{t}}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{\sqrt{xy}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{xy}}\left(-2x^{\frac{5}{2}}\sqrt{y} + y\right) \\
&= \frac{-2x^{\frac{5}{2}}\sqrt{y} + y}{\sqrt{xy}}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{xy}}(x) \\
&= \frac{x}{\sqrt{xy}}
\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x}&=0\\
\left(\frac{-2x^{\frac{5}{2}}\sqrt{y}+y}{\sqrt{xy}}\right)+\left(\frac{x}{\sqrt{xy}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}&=0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\overline{M}\,\mathrm{d}x\\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int\frac{-2x^{\frac{5}{2}}\sqrt{y}+y}{\sqrt{xy}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi=\frac{2\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3}+y\right)x}{\sqrt{xy}}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y}=\frac{2\left(-\frac{x^{\frac{5}{2}}}{6\sqrt{y}}+1\right)x}{\sqrt{xy}}-\frac{\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3}+y\right)x^2}{(xy)^{\frac{3}{2}}}+f'(y) \tag{4}
$$

$$
=\frac{x}{\sqrt{xy}}+f'(y)
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{x}{\sqrt{xy}}$. Therefore equation (4) becomes

$$
\frac{x}{\sqrt{xy}}=\frac{x}{\sqrt{xy}}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3} + y\right)x}{\sqrt{xy}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3} + y\right)x}{\sqrt{xy}}$$

Summary

The solution(s) found are the following

$$\frac{2\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3} + y\right)x}{\sqrt{xy}} = c_1 \tag{1}$$

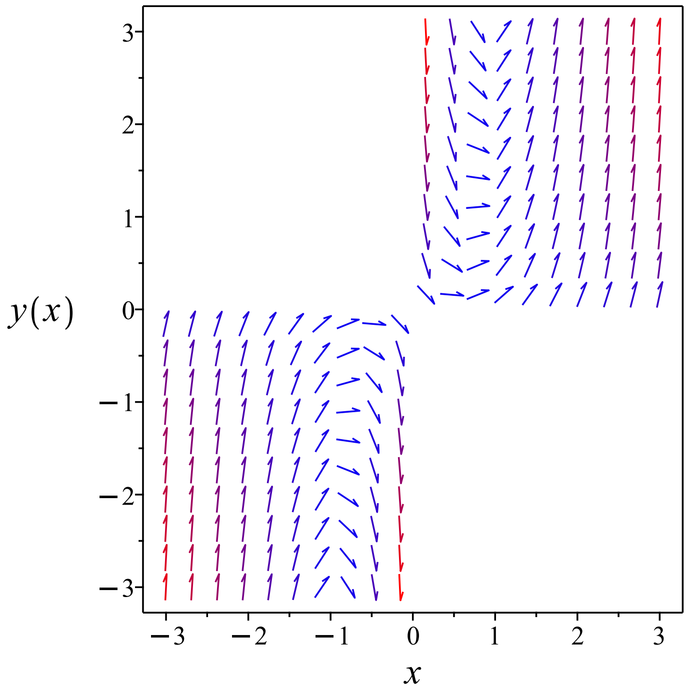


Figure 32: Slope field plot

Verification of solutions

$$\frac{2\left(-\frac{x^{\frac{5}{2}}\sqrt{y}}{3}+y\right)x}{\sqrt{xy}} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)+1/x*y(x)=2*x^(3/2)*y(x)^(1/2),y(x), singsol=all)
```

$$\sqrt{y(x)} - \frac{x^3 + 3c_1}{3\sqrt{x}} = 0$$

✓ Solution by Mathematica

Time used: 0.162 (sec). Leaf size: 22

```
DSolve[y'[x]+1/x*y[x]==2*x^(3/2)*y[x]^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(x^3 + 3c_1)^2}{9x}$$

4.3 problem 3

4.3.1 Solving as separable ode 155
4.3.2 Solving as first order ode lie symmetry lookup ode 157
4.3.3 Solving as bernoulli ode 161
4.3.4 Solving as exact ode 165
4.3.5 Maple step by step solution 168

Internal problem ID [4777]
Internal file name [`OUTPUT/4270_Sunday_June_05_2022_12_50_51_PM_80748334/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3xy^2y' + 3y^3 = 1}$$

4.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{3y^3-1}{3y^2x} \end{aligned}$$

Where $f(x) = -\frac{1}{3x}$ and $g(y) = \frac{3y^3-1}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{3y^3-1}{y^2}}\,dy = -\frac{1}{3x}\,dx$$

$$\int \frac{1}{\frac{3y^3-1}{y^2}}\, dy = \int -\frac{1}{3x}\, dx$$

$$\frac{\ln\left(3y^3-1\right)}{9} = -\frac{\ln\left(x\right)}{3} + c_1$$

Raising both side to exponential gives

$$\left(3y^3-1\right)^{\frac{1}{9}} = \mathrm{e}^{-\frac{\ln(x)}{3}+c_1}$$

Which simplifies to

$$\left(3y^3-1\right)^{\frac{1}{9}} = \frac{c_2}{x^{\frac{1}{3}}}$$

Which simplifies to

$$\left(3y^3-1\right)^{\frac{1}{9}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}}$$

The solution is

$$\left(3y^3-1\right)^{\frac{1}{9}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\left(3y^3-1\right)^{\frac{1}{9}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}} \tag{1}$$

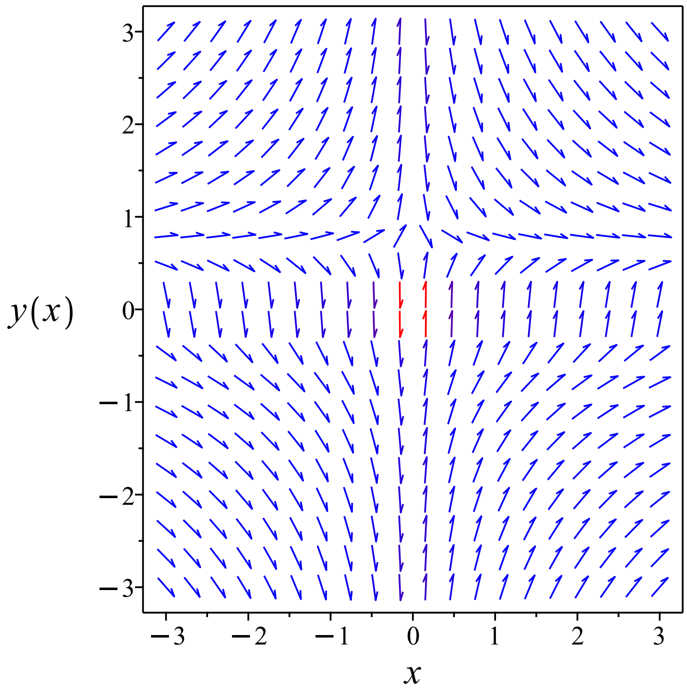


Figure 33: Slope field plot

<u>Verification of solutions</u>

$$\left(3y^3-1\right)^{\frac{1}{9}} = \frac{c_2 \mathrm{e}^{c_1}}{x^{\frac{1}{3}}}$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y^3-1}{3y^2x}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 32: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= -3x \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-3x} dx
\end{aligned}$$

Which results in

$$S = -\frac{\ln(x)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y^3 - 1}{3y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -\frac{1}{3x} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y^2}{3y^3 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{3R^3 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(3R^3-1)}{9} + c_1 \tag{4}$$

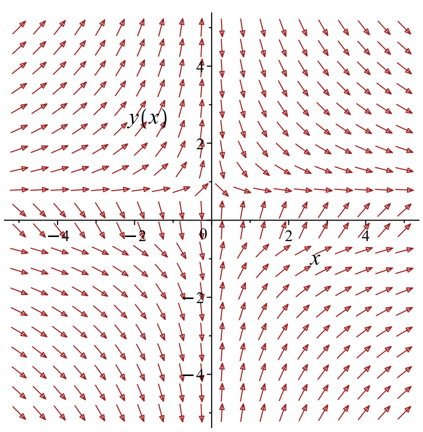
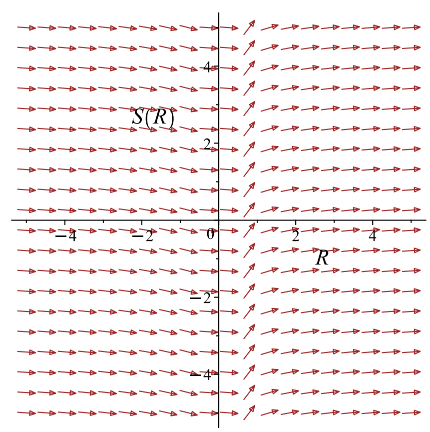
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(x)}{3} = \frac{\ln(3y^3-1)}{9} + c_1$$

Which simplifies to

$$-\frac{\ln(x)}{3} = \frac{\ln(3y^3-1)}{9} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y^3-1}{3y^2x}$ 	$R = y$ $S = -\frac{\ln(x)}{3}$	$\frac{dS}{dR} = \frac{R^2}{3R^3-1}$ 

Summary
The solution(s) found are the following

$$-\frac{\ln(x)}{3} = \frac{\ln(3y^3-1)}{9} + c_1 \tag{1}$$

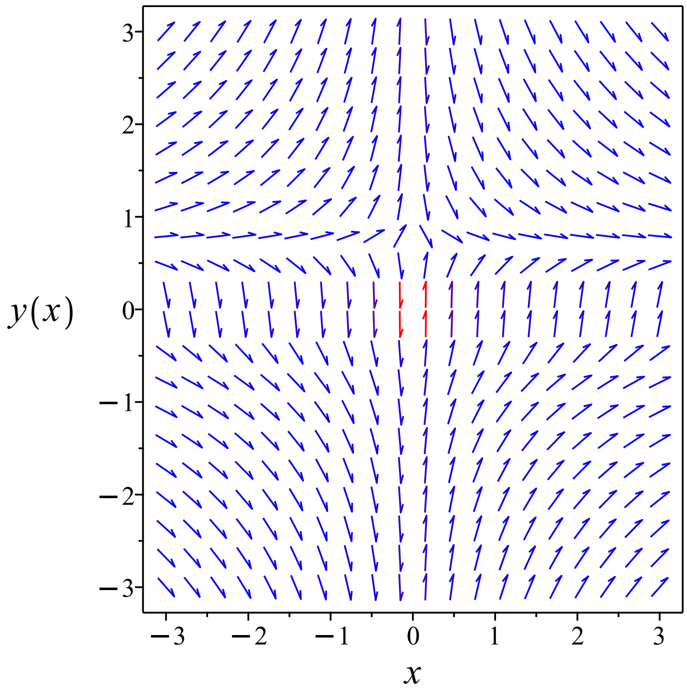


Figure 34: Slope field plot

Verification of solutions

$$-\frac{\ln(x)}{3} = \frac{\ln(3y^3 - 1)}{9} + c_1$$

Verified OK.

4.3.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{3y^3 - 1}{3y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{3x}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{x} \\
f_1(x) &= \frac{1}{3x} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = -\frac{y^3}{x} + \frac{1}{3x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= -\frac{w(x)}{x} + \frac{1}{3x} \\
w' &= -\frac{3w}{x} + \frac{1}{x} \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{1}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{3w(x)}{x} = \frac{1}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{1}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^3 w) &= (x^3)\left(\frac{1}{x}\right) \\ \mathrm{d}(x^3 w) &= x^2\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}x^3 w &= \int x^2\,\mathrm{d}x \\ x^3 w &= \frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$w(x) = \frac{1}{3} + \frac{c_1}{x^3}$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = \frac{1}{3} + \frac{c_1}{x^3}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{(9x^3 + 27c_1)^{\frac{1}{3}}}{3x} \\ y(x) &= \frac{(9x^3 + 27c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{6x} \\ y(x) &= -\frac{(9x^3 + 27c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{6x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(9x^{3}+27c_{1}\right)^{\frac{1}{3}}}{3x} \tag{1}$$

$$y = \frac{\left(9x^{3}+27c_{1}\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right)}{6x} \tag{2}$$

$$y = -\frac{\left(9x^{3}+27c_{1}\right)^{\frac{1}{3}} \left(1+i\sqrt{3}\right)}{6x} \tag{3}$$

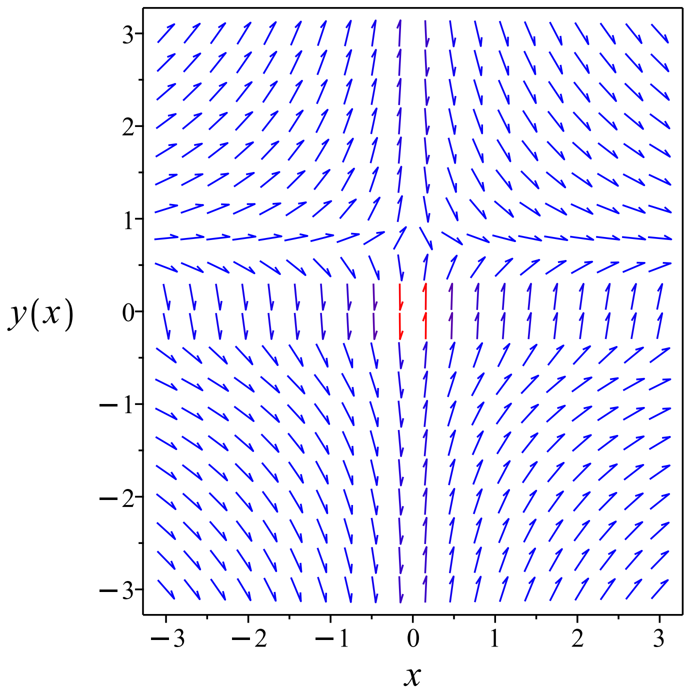


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{(9x^3 + 27c_1)^{\frac{1}{3}}}{3x}$$

Verified OK.

$$y = \frac{(9x^3 + 27c_1)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{6x}$$

Verified OK.

$$y = -\frac{(9x^3 + 27c_1)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{6x}$$

Verified OK.

4.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{3y^2}{3y^3-1}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(-\frac{3y^2}{3y^3-1}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{1}{x}\\ N(x,y) &= -\frac{3y^2}{3y^3-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{3y^2}{3y^3-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{1}{x} \mathrm{d}x$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{3y^2}{3y^3-1}$. Therefore equation (4) becomes

$$-\frac{3y^2}{3y^3-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{3y^2}{3y^3-1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(-\frac{3y^2}{3y^3-1}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(3y^3-1)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(3y^3-1)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(3y^3-1)}{3}$$

Summary
The solution(s) found are the following

$$
-\ln(x) - \frac{\ln(3y^3 - 1)}{3} = c_1 \tag{1}
$$

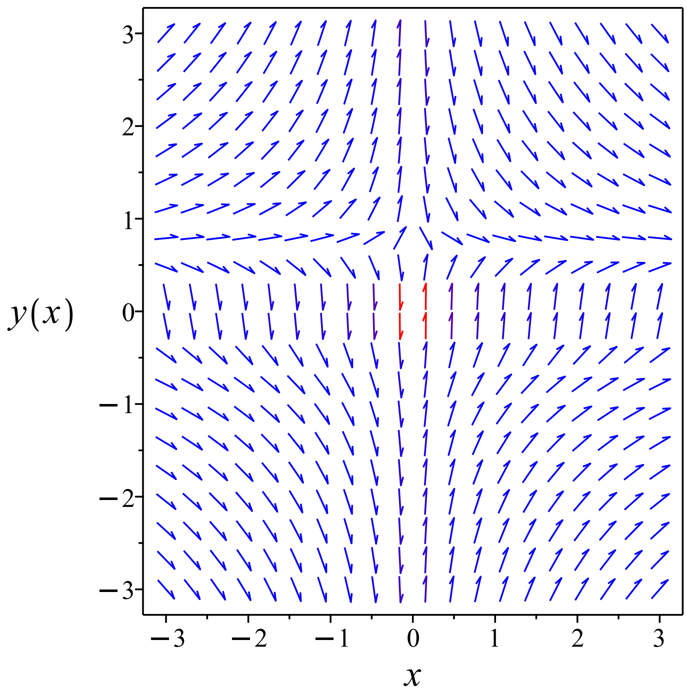


Figure 36: Slope field plot

Verification of solutions

$$
-\ln(x) - \frac{\ln(3y^3 - 1)}{3} = c_1
$$

Verified OK.

4.3.5 Maple step by step solution

Let's solve

$3xy^2y' + 3y^3 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'y^2}{-3y^3+1} = \frac{1}{3x}$

- Integrate both sides with respect to x

  $\int \frac{y'y^2}{-3y^3+1}dx = \int \frac{1}{3x}dx + c_1$

- Evaluate integral

  $-\frac{\ln(3y^3-1)}{9} = \frac{\ln(x)}{3} + c_1$

- Solve for y

  $y = \frac{\left(9+9x^3\left(\mathrm{e}^{3c_1}\right)^3\right)^{\frac{1}{3}}}{3\,\mathrm{e}^{3c_1}x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve(3*x*y(x)^2*diff(y(x),x)+3*y(x)^3=1,y(x), singsol=all)
```

$$y(x) = \frac{(9x^3+27c_1)^{\frac{1}{3}}}{3x}$$
$$y(x) = -\frac{(9x^3+27c_1)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{6x}$$
$$y(x) = \frac{(9x^3+27c_1)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{6x}$$

✓ Solution by Mathematica

Time used: 0.282 (sec). Leaf size: 195

```
DSolve[3*x*y[x]^2*y'[x]+3*y[x]^3==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{x^3 + e^{9c_1}}}{x}$$
$$y(x) \to \frac{\sqrt[3]{x^3 + e^{9c_1}}}{\sqrt[3]{3}x}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{x^3 + e^{9c_1}}}{\sqrt[3]{3}x}$$
$$y(x) \to -\sqrt[3]{-\frac{1}{3}}$$
$$y(x) \to \frac{1}{\sqrt[3]{3}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{3}}$$
$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{x^3}}{x}$$
$$y(x) \to \frac{\sqrt[3]{x^3}}{\sqrt[3]{3}x}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{x^3}}{\sqrt[3]{3}x}$$

4.4 problem 4

4.4.1 Solving as exact ode 171
4.4.2 Maple step by step solution 175

Internal problem ID [4778]
Internal file name [`OUTPUT/4271_Sunday_June_05_2022_12_51_00_PM_94149453/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2x\,\mathrm{e}^{3y}+\left(3x^{2}\mathrm{e}^{3y}-y^{2}\right)y'=-\mathrm{e}^{x}}$$

4.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(3x^2 \mathrm{e}^{3y} - y^2\right) \mathrm{d}y &= \left(-2x\, \mathrm{e}^{3y} - \mathrm{e}^x\right) \mathrm{d}x \\
\left(2x\, \mathrm{e}^{3y} + \mathrm{e}^x\right) \mathrm{d}x + \left(3x^2 \mathrm{e}^{3y} - y^2\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2x\, \mathrm{e}^{3y} + \mathrm{e}^x \\
N(x, y) &= 3x^2 \mathrm{e}^{3y} - y^2
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\, \mathrm{e}^{3y} + \mathrm{e}^x\right) \\
&= 6x\, \mathrm{e}^{3y}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x^2 \mathrm{e}^{3y} - y^2\right) \\
&= 6x\, \mathrm{e}^{3y}
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x \, \mathrm{e}^{3y} + \mathrm{e}^{x} \, \mathrm{d}x$$

$$\phi = x^2 \mathrm{e}^{3y} + \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x^2 \mathrm{e}^{3y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^2 \mathrm{e}^{3y} - y^2$. Therefore equation (4) becomes

$$3x^2 \mathrm{e}^{3y} - y^2 = 3x^2 \mathrm{e}^{3y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -y^2$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (-y^2) \, \mathrm{d}y$$

$$f(y) = -\frac{y^3}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 \mathrm{e}^{3y} + \mathrm{e}^{x} - \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 e^{3y} + e^x - \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$x^2 e^{3y} + e^x - \frac{y^3}{3} = c_1 \tag{1}$$

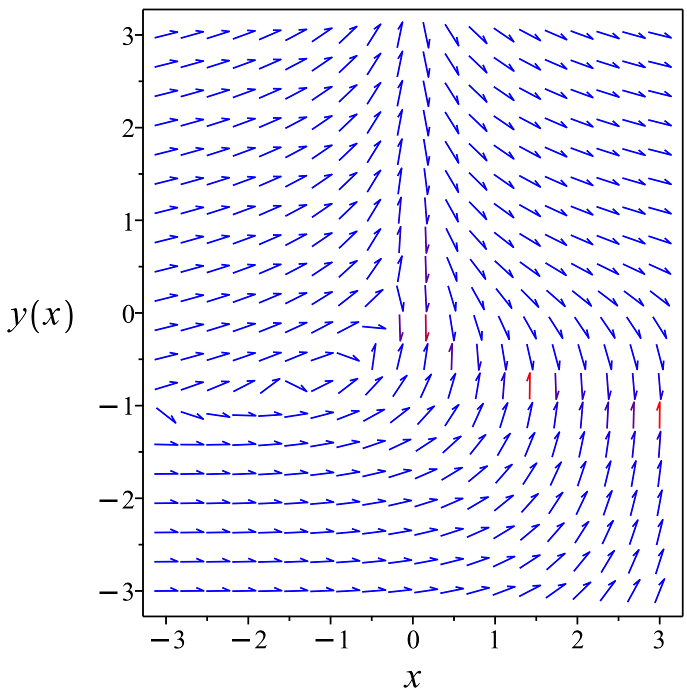


Figure 37: Slope field plot

Verification of solutions

$$x^2 e^{3y} + e^x - \frac{y^3}{3} = c_1$$

Verified OK.

4.4.2 Maple step by step solution

Let's solve

$2x\,\mathrm{e}^{3y}+\left(3x^{2}\mathrm{e}^{3y}-y^{2}\right)y'=-\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- ☐ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $6x\,\mathrm{e}^{3y}=6x\,\mathrm{e}^{3y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_{1},M(x,y)=F'(x,y),N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(2x\,\mathrm{e}^{3y}+\mathrm{e}^{x}\right)dx+f_{1}(y)$

- Evaluate integral

  $F(x,y)=x^{2}\mathrm{e}^{3y}+\mathrm{e}^{x}+f_{1}(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $3x^{2}\mathrm{e}^{3y}-y^{2}=3x^{2}\mathrm{e}^{3y}+\frac{d}{dy}f_{1}(y)$

- Isolate for $\frac{d}{dy}f_{1}(y)$

  $\frac{d}{dy}f_{1}(y)=-y^{2}$

- Solve for $f_{1}(y)$

  $f_{1}(y)=-\frac{y^{3}}{3}$

- Substitute $f_{1}(y)$ into equation for $F(x,y)$

$$F(x, y) = x^2 e^{3y} + e^x - \frac{y^3}{3}$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x^2 e^{3y} + e^x - \frac{y^3}{3} = c_1$$

- Solve for y

$$y = \mathit{RootOf}\left(-3x^2 e^{3\_Z} + \_Z^3 - 3\,e^x + 3c_1\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve((2*x*exp(3*y(x))+exp(x))+(3*x^2*exp(3*y(x))-y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$x^2 e^{3y(x)} + e^x - \frac{y(x)^3}{3} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.262 (sec). Leaf size: 28

```
DSolve[(2*x*Exp[3*y[x]]+Exp[x])+(3*x^2*Exp[3*y[x]]-y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSo
```

$$\text{Solve}\left[x^2 e^{3y(x)} - \frac{1}{3}y(x)^3 + e^x = c_1, y(x)\right]$$

4.5 problem 5

4.5.1 Solving as differentialType ode 177
4.5.2 Solving as homogeneousTypeMapleC ode 179
4.5.3 Solving as first order ode lie symmetry calculated ode 182
4.5.4 Solving as exact ode 187
4.5.5 Maple step by step solution 191

Internal problem ID [4779]
Internal file name [`OUTPUT/4272_Sunday_June_05_2022_12_51_09_PM_30912130/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(x-y)\,y'+y=-1-x}$$

4.5.1 Solving as differentialType ode

Writing the ode as

$$y'=\frac{-y-x-1}{x-y} \tag{1}$$

Which becomes

$$(-y)\,dy=(-x)\,dy+(-y-x-1)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy+(-y-x-1)\,dx=d\left(-\frac{1}{2}x^2-xy-x\right)$$

Hence (2) becomes

$$(-y)\,dy = d\left(-\frac{1}{2}x^2 - xy - x\right)$$

Integrating both sides gives gives these solutions

$$y = x + \sqrt{2x^2 - 2c_1 + 2x} + c_1$$
$$y = x - \sqrt{2x^2 - 2c_1 + 2x} + c_1$$

Summary

The solution(s) found are the following

$$y = x + \sqrt{2x^2 - 2c_1 + 2x} + c_1 \tag{1}$$
$$y = x - \sqrt{2x^2 - 2c_1 + 2x} + c_1 \tag{2}$$

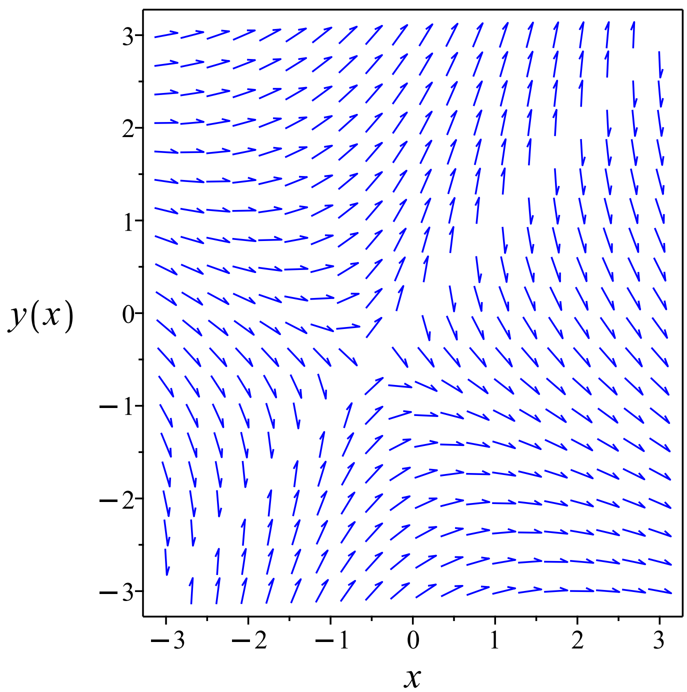


Figure 38: Slope field plot

Verification of solutions

$$y = x + \sqrt{2x^2 - 2c_1 + 2x} + c_1$$

Verified OK.

$$y = x - \sqrt{2x^2 - 2c_1 + 2x} + c_1$$

Verified OK.

4.5.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 + X + x_0 + 1}{-X - x_0 + Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= -\frac{1}{2} \\ y_0 &= -\frac{1}{2} \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X) + X}{-X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{Y + X}{-X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -Y - X$ and $N = X - Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u + 1}{u - 1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)+1}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) - \left(\frac{d}{dX}u(X)\right) X + u(X)^2 - 2u(X) - 1 = 0$$

Or

$$X(u(X) - 1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 - 2u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= -\frac{u^2 - 2u - 1}{X\,(u - 1)}
\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-2u-1}{u-1}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2-2u-1}{u-1}}\,du &= -\frac{1}{X}\,dX \\
\int \frac{1}{\frac{u^2-2u-1}{u-1}}\,du &= \int -\frac{1}{X}\,dX \\
\frac{\ln\left(u^2 - 2u - 1\right)}{2} &= -\ln\left(X\right) + c_2
\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{u^2 - 2u - 1} = \mathrm{e}^{-\ln(X)+c_2}$$

Which simplifies to

$$\sqrt{u^2 - 2u - 1} = \frac{c_3}{X}$$

Which simplifies to

$$\sqrt{u\left(X\right)^2 - 2u\left(X\right) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

The solution is

$$\sqrt{u\left(X\right)^2 - 2u\left(X\right) - 1} = \frac{c_3 \mathrm{e}^{c_2}}{X}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\sqrt{\frac{Y(X)^2}{X^2} - \frac{2Y(X)}{X} - 1} = \frac{c_3 e^{c_2}}{X}$$

Using the solution for $Y(X)$

$$\sqrt{\frac{Y(X)^2 - 2Y(X)X - X^2}{X^2}} = \frac{c_3 e^{c_2}}{X}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - \frac{1}{2}$$
$$X = x - \frac{1}{2}$$

Then the solution in y becomes

$$\sqrt{\frac{\left(y+\frac{1}{2}\right)^2 - 2\left(y+\frac{1}{2}\right)\left(\frac{1}{2}+x\right) - \left(\frac{1}{2}+x\right)^2}{\left(\frac{1}{2}+x\right)^2}} = \frac{c_3 e^{c_2}}{\frac{1}{2}+x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{\left(y+\frac{1}{2}\right)^2 - 2\left(y+\frac{1}{2}\right)\left(\frac{1}{2}+x\right) - \left(\frac{1}{2}+x\right)^2}{\left(\frac{1}{2}+x\right)^2}} = \frac{c_3 e^{c_2}}{\frac{1}{2}+x} \tag{1}$$

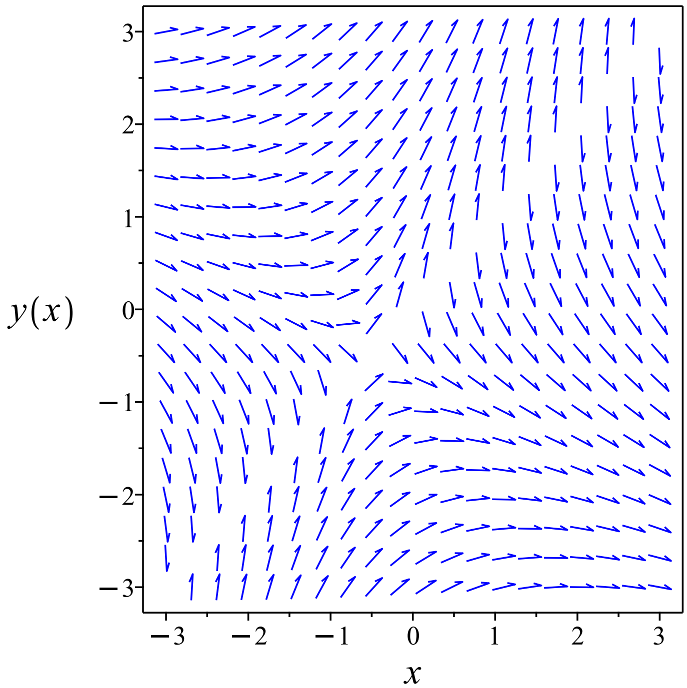


Figure 39: Slope field plot

Verification of solutions

$$\sqrt{\frac{\left(y+\frac{1}{2}\right)^{2}-2\left(y+\frac{1}{2}\right)\left(\frac{1}{2}+x\right)-\left(\frac{1}{2}+x\right)^{2}}{\left(\frac{1}{2}+x\right)^{2}}}=\frac{c_{3}\mathrm{e}^{c_{2}}}{\frac{1}{2}+x}$$

Verified OK.

4.5.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{x+y+1}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(x+y+1)(b_3 - a_2)}{-x+y} - \frac{(x+y+1)^2 a_3}{(-x+y)^2} \\ &- \left(\frac{1}{-x+y} + \frac{x+y+1}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{1}{-x+y} - \frac{x+y+1}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 - y^2a_2 - 3y^2a_3 + y^2b_2 + y^2b_3 - 2xa_3 + 2x}{(x-y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &x^2a_2 - x^2a_3 + 3x^2b_2 - x^2b_3 - 2xya_2 - 2xya_3 - 2xyb_2 + 2xyb_3 - y^2a_2 - 3y^2a_3 \\ &+ y^2b_2 + y^2b_3 - 2xa_3 + 2xb_1 + xb_2 - xb_3 - 2ya_1 - ya_2 - 3ya_3 + 2yb_3 - a_1 - a_3 + b_1 \\ &= 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &a_2v_1^2 - 2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 \\ &- 2b_2v_1v_2 + b_2v_2^2 - b_3v_1^2 + 2b_3v_1v_2 + b_3v_2^2 - 2a_1v_2 - a_2v_2 - 2a_3v_1 \\ &- 3a_3v_2 + 2b_1v_1 + b_2v_1 - b_3v_1 + 2b_3v_2 - a_1 - a_3 + b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(a_2 - a_3 + 3b_2 - b_3) v_1^2 + (-2a_2 - 2a_3 - 2b_2 + 2b_3) v_1 v_2 + (-2a_3 + 2b_1 + b_2 - b_3) v_1 \\ + (-a_2 - 3a_3 + b_2 + b_3) v_2^2 + (-2a_1 - a_2 - 3a_3 + 2b_3) v_2 - a_1 - a_3 + b_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-a_1 - a_3 + b_1 &= 0 \\ -2a_1 - a_2 - 3a_3 + 2b_3 &= 0 \\ -2a_2 - 2a_3 - 2b_2 + 2b_3 &= 0 \\ -a_2 - 3a_3 + b_2 + b_3 &= 0 \\ a_2 - a_3 + 3b_2 - b_3 &= 0 \\ -2a_3 + 2b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_1 + b_3 \\ a_2 &= -4b_1 + 3b_3 \\ a_3 &= 2b_1 - b_3 \\ b_1 &= b_1 \\ b_2 &= 2b_1 - b_3 \\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -4x + 2y - 1 \\ \eta &= 1 + 2x\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\, \xi \\ &= 1 + 2x - \left(\frac{x + y + 1}{-x + y}\right)(-4x + 2y - 1) \\ &= \frac{-2x^2 - 4xy + 2y^2 - 4x - 1}{x - y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x^2-4xy+2y^2-4x-1}{x-y}} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln\left(-2x^2 - 4xy + 2y^2 - 4x - 1\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + y + 1}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-y - x - 1}{2x^2 + (4y + 4)x - 2y^2 + 1} \\ S_y &= \frac{-x + y}{2x^2 + (4y + 4)x - 2y^2 + 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

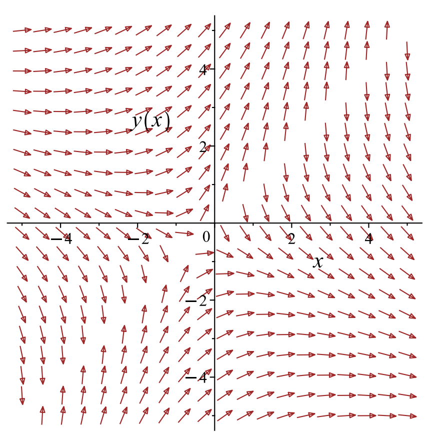
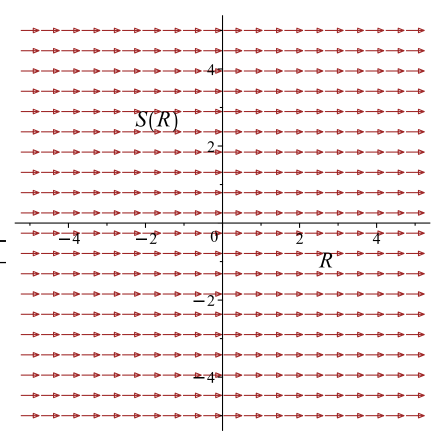
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(-2x^2 + (-4y - 4)x + 2y^2 - 1)}{4} = c_1$$

Which simplifies to

$$-\frac{\ln(-2x^2 + (-4y - 4)x + 2y^2 - 1)}{4} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x+y+1}{-x+y}$

</td><td>$R = x$
$S = -\frac{\ln(-2x^2 + (-4y -}{4}$</td><td>$\frac{dS}{dR} = 0$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$-\frac{\ln\left(-2x^2+(-4y-4)\,x+2y^2-1\right)}{4}=c_1 \tag{1}$$

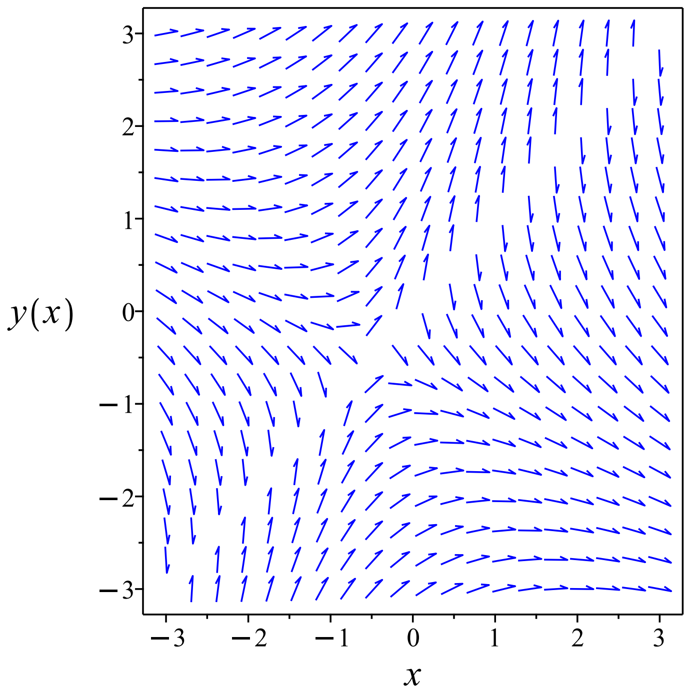


Figure 40: Slope field plot

Verification of solutions

$$-\frac{\ln\left(-2x^2+(-4y-4)\,x+2y^2-1\right)}{4}=c_1$$

Verified OK.

4.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\\frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x-y)\,\mathrm{d}y&=(-y-x-1)\,\mathrm{d}x\\(x+y+1)\,\mathrm{d}x+(x-y)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=x+y+1\\N(x,y)&=x-y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(x+y+1)\\&=1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x - y) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int x + y + 1 \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x(x + 2y + 2)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - y$. Therefore equation (4) becomes

$$
x - y = x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -y
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \mathrm{d}y = \int (-y) \mathrm{d}y
$$

$$
f(y) = -\frac{y^2}{2} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x+2y+2)}{2} - \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x+2y+2)}{2} - \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(x+2y+2)}{2} - \frac{y^2}{2} = c_1 \tag{1}$$

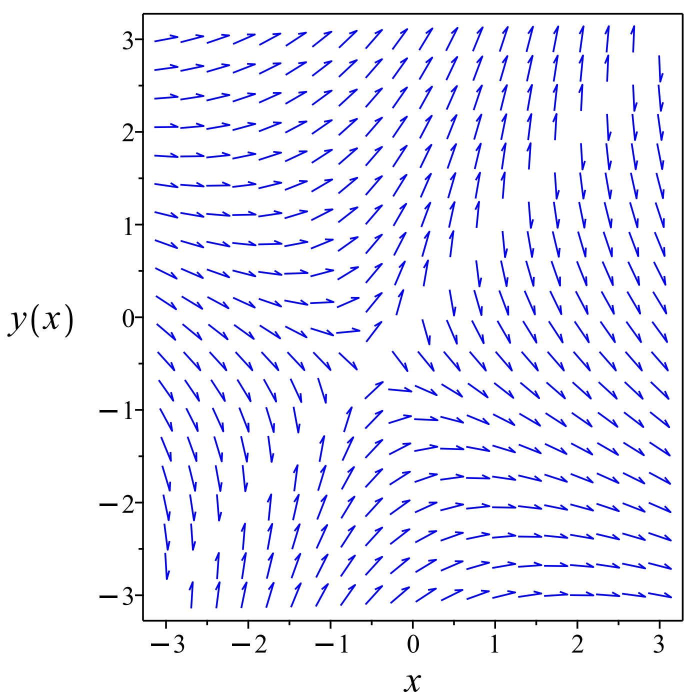


Figure 41: Slope field plot

Verification of solutions

$$\frac{x(x+2y+2)}{2} - \frac{y^2}{2} = c_1$$

Verified OK.

4.5.5 Maple step by step solution

Let's solve

$(x-y)\,y'+y=-1-x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - Evaluate derivatives

    $1=1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\,, N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (x+y+1)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2}{2}+xy+x+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x-y=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=-y$

- Solve for $f_1(y)$

  $f_1(y)=-\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x, y) = \frac{1}{2}x^2 + xy + x - \frac{1}{2}y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{1}{2}x^2 + xy + x - \frac{1}{2}y^2 = c_1$

- Solve for y

  $\left\{y = x - \sqrt{2x^2 - 2c_1 + 2x}, y = x + \sqrt{2x^2 - 2c_1 + 2x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.11 (sec). Leaf size: 30

```
dsolve((x-y(x))*diff(y(x),x)+(y(x)+x+1)=0,y(x), singsol=all)
```

$$y(x) = \frac{2c_1x - \sqrt{1 + 8\left(x + \frac{1}{2}\right)^2 c_1^2}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.112 (sec). Leaf size: 55

```
DSolve[(x-y[x])*y'[x]+(y[x]+x+1)==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x - i\sqrt{-2x^2 - 2x - c_1}$$
$$y(x) \to x + i\sqrt{-2x^2 - 2x - c_1}$$

4.6 problem 6

4.6.1 Solving as exact ode 194
4.6.2 Maple step by step solution 198

Internal problem ID [4780]
Internal file name [`OUTPUT/4273_Sunday_June_05_2022_12_51_20_PM_91799758/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`unknown`

$$\boxed{\cos(x)\cos(y) - \left(\sin(x)\sin(y) + \cos(y)^2\right) y' = -\sin(x)^2}$$

4.6.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\sin(x)\sin(y) - \cos(y)^2\right)\mathrm{d}y = \left(-\cos(x)\cos(y) - \sin(x)^2\right)\mathrm{d}x$$
$$\left(\cos(x)\cos(y) + \sin(x)^2\right)\mathrm{d}x + \left(-\sin(x)\sin(y) - \cos(y)^2\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \cos(x)\cos(y) + \sin(x)^2$$
$$N(x, y) = -\sin(x)\sin(y) - \cos(y)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\cos(x)\cos(y) + \sin(x)^2\right) \\ &= -\cos(x)\sin(y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\sin(x)\sin(y) - \cos(y)^2\right) \\ &= -\cos(x)\sin(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1)\\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \cos(x)\cos(y)+\sin(x)^2\,\mathrm{d}x\\
\phi &= \frac{\sin(x)\left(-\cos(x)+2\cos(y)\right)}{2}+\frac{x}{2}+f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\sin(x)\sin(y)+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\sin(x)\sin(y)-\cos(y)^2$. Therefore equation (4) becomes

$$
-\sin(x)\sin(y)-\cos(y)^2 = -\sin(x)\sin(y)+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\cos(y)^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(-\cos(y)^2\right)\mathrm{d}y\\
f(y) &= -\frac{\cos(y)\sin(y)}{2}-\frac{y}{2}+c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\sin(x)(-\cos(x)+2\cos(y))}{2} + \frac{x}{2} - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\sin(x)(-\cos(x)+2\cos(y))}{2} + \frac{x}{2} - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2}$$

Summary

The solution(s) found are the following

$$\frac{\sin(x)(-\cos(x)+2\cos(y))}{2} + \frac{x}{2} - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = c_1 \tag{1}$$

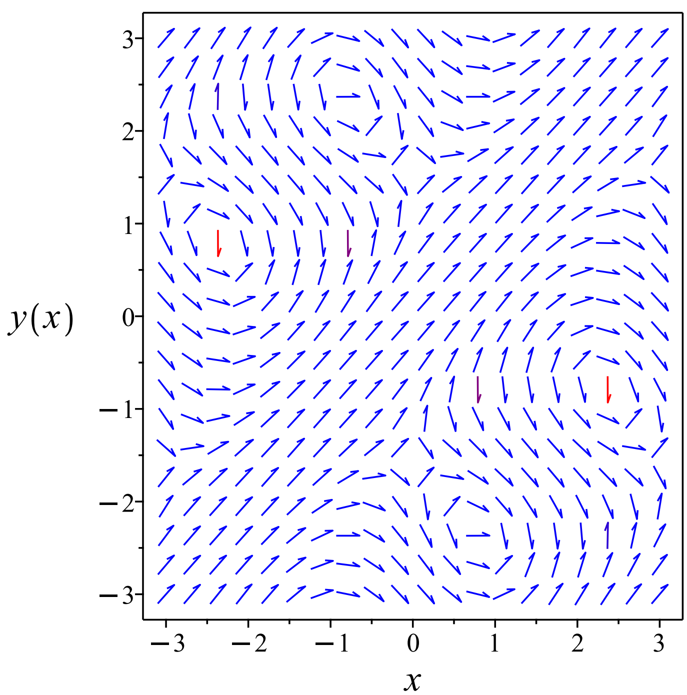


Figure 42: Slope field plot

Verification of solutions

$$\frac{\sin(x)(-\cos(x)+2\cos(y))}{2} + \frac{x}{2} - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = c_1$$

Verified OK.

4.6.2 Maple step by step solution

Let's solve

$\cos(x)\cos(y) - \left(\sin(x)\sin(y) + \cos(y)^2\right) y' = -\sin(x)^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $-\cos(x)\sin(y) = -\cos(x)\sin(y)$
  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(\cos(x)\cos(y) + \sin(x)^2\right) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + \sin(x)\cos(y) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $-\sin(x)\sin(y) - \cos(y)^2 = -\sin(x)\sin(y) + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = -\cos(y)^2$

- Solve for $f_1(y)$

  $f_1(y) = -\frac{\cos(y)\sin(y)}{2} - \frac{y}{2}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = -\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + \sin(x)\cos(y) - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2}$$

- Substitute $F(x,y)$ into the solution of the ODE
$$-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2} + \sin(x)\cos(y) - \frac{\cos(y)\sin(y)}{2} - \frac{y}{2} = c_1$$
- Solve for y
$$y = \mathit{RootOf}(\cos(x)\sin(x) - 2\sin(x)\cos(\_Z) + \sin(\_Z)\cos(\_Z) + 2c_1 - x + \_Z)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.312 (sec). Leaf size: 35

```
dsolve((cos(x)*cos(y(x))+sin(x)^2)-(sin(x)*sin(y(x))+cos(y(x))^2)*diff(y(x),x)=0,y(x), sings
```

$$c_1 + x - y(x) - \frac{\sin(2x)}{2} + \sin(y(x) + x) + \sin(-y(x) + x) - \frac{\sin(2y(x))}{2} = 0$$

✓ Solution by Mathematica

Time used: 0.375 (sec). Leaf size: 43

```
DSolve[(Cos[x]*Cos[y[x]]+Sin[x]^2)-(Sin[x]*Sin[y[x]]+Cos[y[x]]^2)*y'[x]==0,y[x],x,IncludeSin
```

$$\text{Solve}\left[2\left(\frac{y(x)}{2} + \frac{1}{4}\sin(2y(x))\right) - 2\sin(x)\cos(y(x)) - x + \frac{1}{2}\sin(2x) = c_1, y(x)\right]$$

4.7 problem 7

4.7.1 Solving as homogeneousTypeD2 ode 200
4.7.2 Solving as first order ode lie symmetry lookup ode 202
4.7.3 Solving as bernoulli ode 206
4.7.4 Solving as exact ode 210
4.7.5 Solving as riccati ode 215

Internal problem ID [4781]
Internal file name [`OUTPUT/4274_Sunday_June_05_2022_12_52_37_PM_26068840/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{x^2y' + y^2 - xy = 0}$$

4.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) + u(x)^2\,x^2 - x^2u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u^2}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u^2}\,du &= \int -\frac{1}{x}\,dx \\
-\frac{1}{u} &= -\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
-\frac{1}{u(x)} + \ln(x) - c_2 = 0
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
-\frac{x}{y} + \ln(x) - c_2 &= 0 \\
-\frac{x}{y} + \ln(x) - c_2 &= 0
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
-\frac{x}{y} + \ln(x) - c_2 = 0 \tag{1}
$$

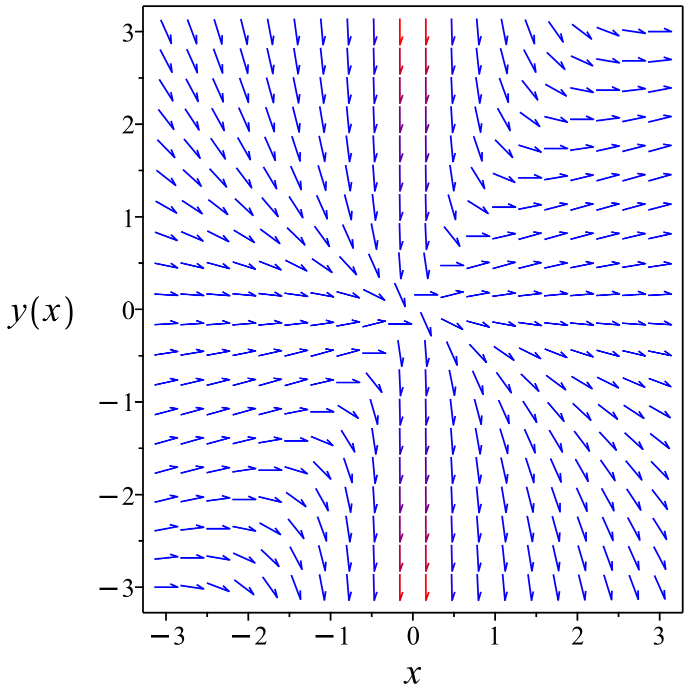


Figure 43: Slope field plot

Verification of solutions

$$-\frac{x}{y}+\ln(x)-c_2=0$$

Verified OK.

4.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}y' &= -\frac{y(-x+y)}{x^2}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 38: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x + y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = -\ln(x) + c_1$$

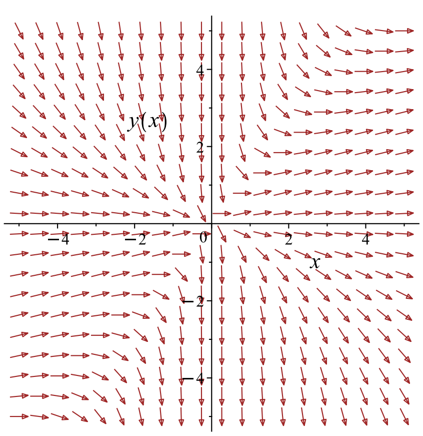
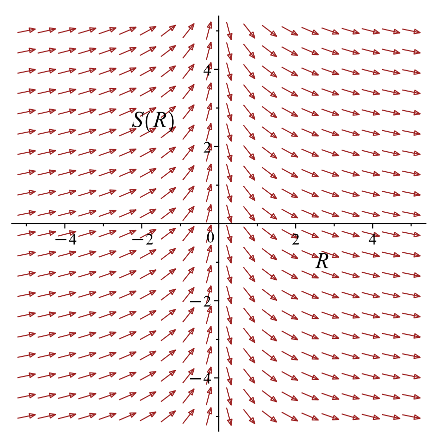
Which simplifies to

$$-\frac{x}{y} = -\ln(x) + c_1$$

Which gives

$$y = \frac{x}{\ln(x) - c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

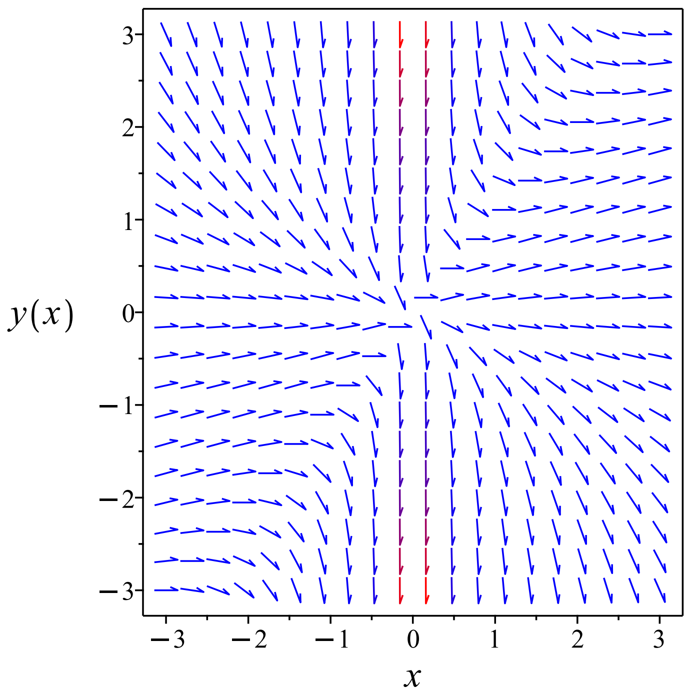


Figure 44: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

4.7.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(-x+y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} - \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} - \frac{1}{x^2} \\ w' &= -\frac{w}{x} + \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = \frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{x} dx} \\
&= x
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^2}\right) \\
\frac{d}{dx}(xw) &= (x)\left(\frac{1}{x^2}\right) \\
d(xw) &= \frac{1}{x}\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
xw &= \int \frac{1}{x}\, dx \\
xw &= \ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{\ln(x) + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_1} \tag{1}$$

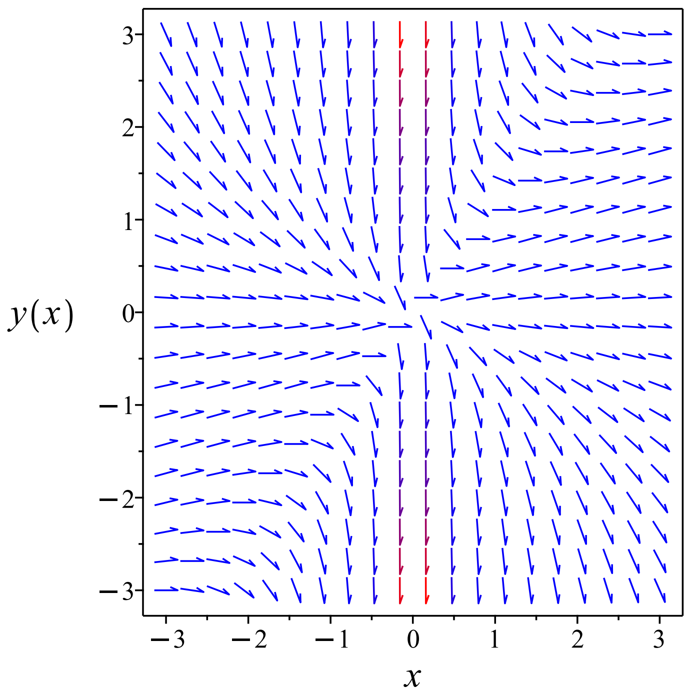


Figure 45: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_1}$$

Verified OK.

4.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (xy - y^2)\,\mathrm{d}x\\ (-xy + y^2)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -xy + y^2\\ N(x,y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-xy + y^2\right) \\
&= -x + 2y
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right) \\
&= 2x
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - xy$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}
M &= \frac{y^2 - xy}{xy^2} \\
N &= \frac{x}{y^2}
\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{x}{y^2}\right) \mathrm{d}y &= \left(-\frac{-xy + y^2}{x\,y^2}\right) \mathrm{d}x \\
\left(\frac{-xy + y^2}{x\,y^2}\right) \mathrm{d}x + \left(\frac{x}{y^2}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \frac{-xy + y^2}{x\,y^2} \\
N(x, y) &= \frac{x}{y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-xy + y^2}{x\,y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-xy + y^2}{x\, y^2} \,\mathrm{d}x$$
$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y}$$

The solution becomes

$$y = \frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\ln(x) - c_1} \tag{1}$$

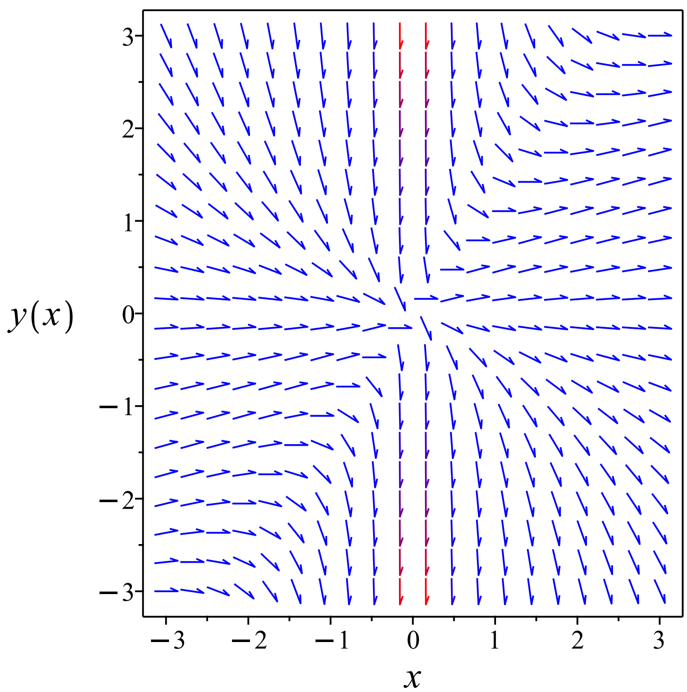


Figure 46: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) - c_1}$$

Verified OK.

4.7.5 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{y(-x+y)}{x^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{y}{x} - \frac{y^2}{x^2}
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{u}{x^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{2}{x^3} \\
f_1 f_2 &= -\frac{1}{x^3} \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{u''(x)}{x^2} - \frac{u'(x)}{x^3} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_2 \ln(x) + c_1
$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = \frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x}{\ln(x) + c_3}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{x}{\ln(x) + c_3} \tag{1}$$

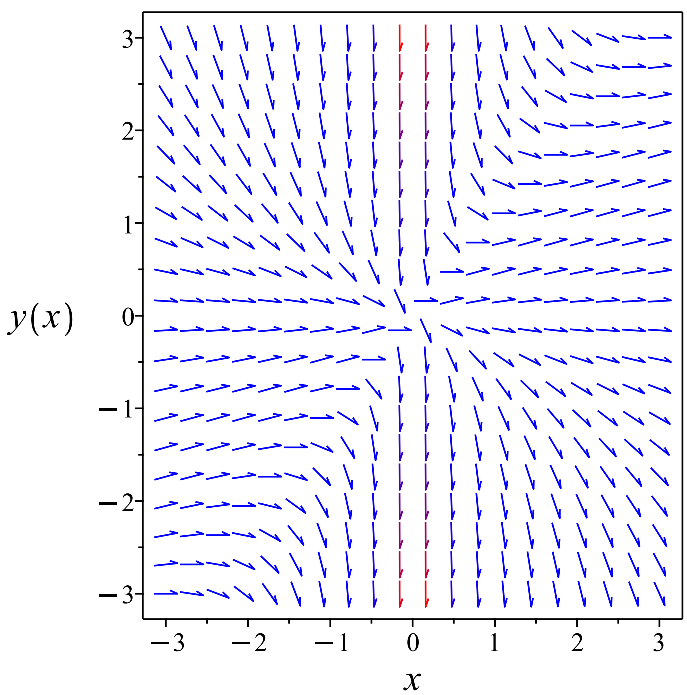


Figure 47: Slope field plot

Verification of solutions

$$y = \frac{x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(x^2*diff(y(x),x)+(y(x)^2-x*y(x))=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\ln(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 19

```
DSolve[x^2*y'[x]+(y[x]^2-x*y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{\log(x) + c_1}$$
$$y(x) \to 0$$

4.8 problem 8

4.8.1 Solving as first order ode lie symmetry calculated ode 218

Internal problem ID [4782]
Internal file name [`OUTPUT/4275_Sunday_June_05_2022_12_52_47_PM_16640539/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'y-\sqrt{x^2+y^2}=-x}$$

4.8.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{-x+\sqrt{x^2+y^2}}{y}\\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(-x+\sqrt{x^2+y^2}\right)(b_3-a_2)}{y} - \frac{\left(-x+\sqrt{x^2+y^2}\right)^2 a_3}{y^2} \\
&- \frac{\left(-1+\frac{x}{\sqrt{x^2+y^2}}\right)(xa_2+ya_3+a_1)}{y} \\
&- \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{-x+\sqrt{x^2+y^2}}{y^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(x^2+y^2)^{\frac{3}{2}} a_3 + \sqrt{x^2+y^2}\, x^2 a_3 + \sqrt{x^2+y^2}\, x^2 b_2 - 2\sqrt{x^2+y^2}\, xya_2 + 2\sqrt{x^2+y^2}\, xyb_3 - \sqrt{x^2+y^2}\, y^2 a_3 -}{}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-(x^2+y^2)^{\frac{3}{2}} a_3 - \sqrt{x^2+y^2}\, x^2 a_3 - \sqrt{x^2+y^2}\, x^2 b_2 + 2\sqrt{x^2+y^2}\, xya_2 \\
&- 2\sqrt{x^2+y^2}\, xyb_3 + \sqrt{x^2+y^2}\, y^2 a_3 + b_2\sqrt{x^2+y^2}\, y^2 \\
&+ 2x^3 a_3 + x^3 b_2 - 2x^2 ya_2 + 2yb_3 x^2 + x\, y^2 a_3 - y^3 a_2 + y^3 b_3 \\
&- \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 + b_1 x^2 - xya_1 = 0
\end{aligned}
\tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-(x^2+y^2)^{\frac{3}{2}} a_3 + 2(x^2+y^2)\, xa_3 + (x^2+y^2)\, xb_2 - (x^2+y^2)\, ya_2 \\
&+ 2(x^2+y^2)\, yb_3 - \sqrt{x^2+y^2}\, x^2 a_3 - \sqrt{x^2+y^2}\, x^2 b_2 \\
&+ 2\sqrt{x^2+y^2}\, xya_2 - 2\sqrt{x^2+y^2}\, xyb_3 + \sqrt{x^2+y^2}\, y^2 a_3 \\
&+ b_2\sqrt{x^2+y^2}\, y^2 - x^2 ya_2 - x\, y^2 a_3 - x\, y^2 b_2 - y^3 b_3 + (x^2+y^2)\, b_1 \\
&- \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - xya_1 - y^2 b_1 = 0
\end{aligned}
\tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&2x^3 a_3 + x^3 b_2 - 2\sqrt{x^2+y^2}\, x^2 a_3 - \sqrt{x^2+y^2}\, x^2 b_2 - 2x^2 y a_2 + 2y b_3 x^2 \\
&+ 2\sqrt{x^2+y^2}\, x y a_2 - 2\sqrt{x^2+y^2}\, x y b_3 + x\, y^2 a_3 + b_2 \sqrt{x^2+y^2}\, y^2 \\
&- y^3 a_2 + y^3 b_3 + b_1 x^2 - \sqrt{x^2+y^2}\, x b_1 - x y a_1 + \sqrt{x^2+y^2}\, y a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2v_1^2 v_2 a_2 + 2v_3 v_1 v_2 a_2 - v_2^3 a_2 + 2v_1^3 a_3 - 2v_3 v_1^2 a_3 + v_1 v_2^2 a_3 + v_1^3 b_2 - v_3 v_1^2 b_2 \\
&+ b_2 v_3 v_2^2 + 2v_2 b_3 v_1^2 - 2v_3 v_1 v_2 b_3 + v_2^3 b_3 - v_1 v_2 a_1 + v_3 v_2 a_1 + b_1 v_1^2 - v_3 v_1 b_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&(2a_3 + b_2)\, v_1^3 + (-2a_2 + 2b_3)\, v_1^2 v_2 + (-2a_3 - b_2)\, v_1^2 v_3 + b_1 v_1^2 + v_1 v_2^2 a_3 \\
&+ (2a_2 - 2b_3)\, v_1 v_2 v_3 - v_1 v_2 a_1 - v_3 v_1 b_1 + (b_3 - a_2)\, v_2^3 + b_2 v_3 v_2^2 + v_3 v_2 a_1 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-b_1 &= 0 \\
-2a_2 + 2b_3 &= 0 \\
2a_2 - 2b_3 &= 0 \\
-2a_3 - b_2 &= 0 \\
2a_3 + b_2 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{-x + \sqrt{x^2 + y^2}}{y}\right)(x) \\
&= \frac{x^2 - x\sqrt{x^2 + y^2} + y^2}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2 - x\sqrt{x^2+y^2} + y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + \sqrt{x^2 + y^2}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x + \sqrt{x^2 + y^2}}{x\sqrt{x^2 + y^2}} \\ S_y &= \frac{2x^2 + y^2 + 2x\sqrt{x^2 + y^2}}{y\sqrt{x^2 + y^2}\,\left(x + \sqrt{x^2 + y^2}\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x\sqrt{x^2+y^2}+x^2+y^2}{x\sqrt{x^2+y^2}\,\left(x+\sqrt{x^2+y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

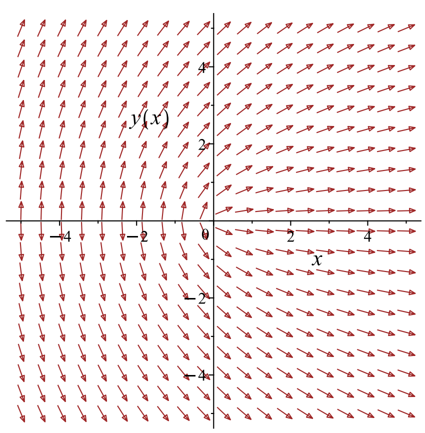
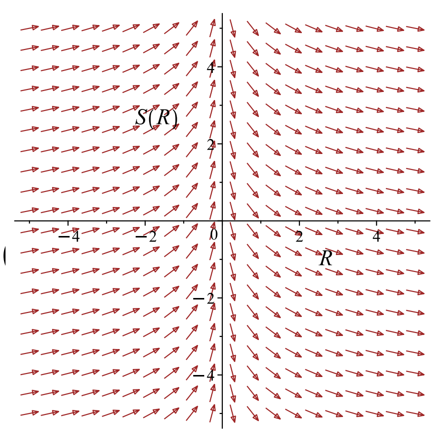
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

Which gives

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln\left(2x+2\,\mathrm{e}^{c_1}\right)}{2}+\frac{c_1}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y}$

</td><td>$R = x$
$S = 2\ln(y) - \ln(2) - \ln($</td><td>$\frac{dS}{dR} = -\frac{1}{R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln\left(2x+2\,\mathrm{e}^{c_1}\right)}{2}+\frac{c_1}{2}} \tag{1}$$

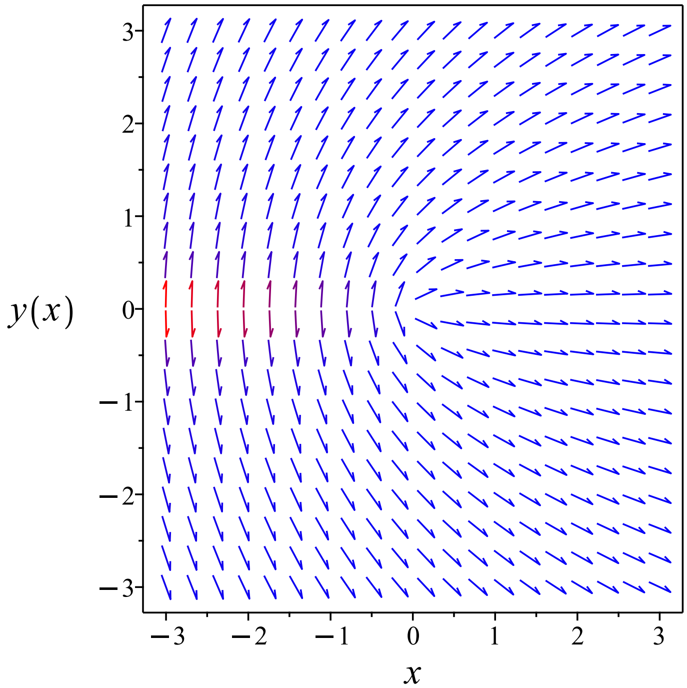


Figure 48: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln\left(2x+2\,\mathrm{e}^{c_1}\right)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 27

```
dsolve(y(x)*diff(y(x),x)=-x+sqrt(x^2+y(x)^2),y(x), singsol=all)
```

$$\frac{-c_1 y(x)^2 + \sqrt{x^2 + y(x)^2} + x}{y(x)^2} = 0$$

✓ Solution by Mathematica

Time used: 0.378 (sec). Leaf size: 57

```
DSolve[y[x]*y'[x]==-x+Sqrt[x^2+y[x]^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{\frac{c_1}{2}} \sqrt{2x + e^{c_1}}$$
$$y(x) \to e^{\frac{c_1}{2}} \sqrt{2x + e^{c_1}}$$
$$y(x) \to 0$$

4.9 problem 9

4.9.1 Solving as homogeneousTypeD2 ode 226
4.9.2 Solving as first order ode lie symmetry calculated ode 228
4.9.3 Solving as exact ode 233

Internal problem ID [4783]
Internal file name [`OUTPUT/4276_Sunday_June_05_2022_12_52_58_PM_73812867/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{xy + (y^2 - x^2)\, y' = 0}$$

4.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x^2u(x) + \left(u(x)^2 x^2 - x^2\right)\left(u'(x)\, x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3}{(u^2-1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^3}{u^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3}{u^2-1}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{u^3}{u^2-1}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln(u) + \frac{1}{2u^2} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)) + \frac{1}{2u(x)^2} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{2y^2} + \ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{2y^2} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) + \frac{x^2}{2y^2} + \ln(x) - c_2 = 0 \tag{1}$$

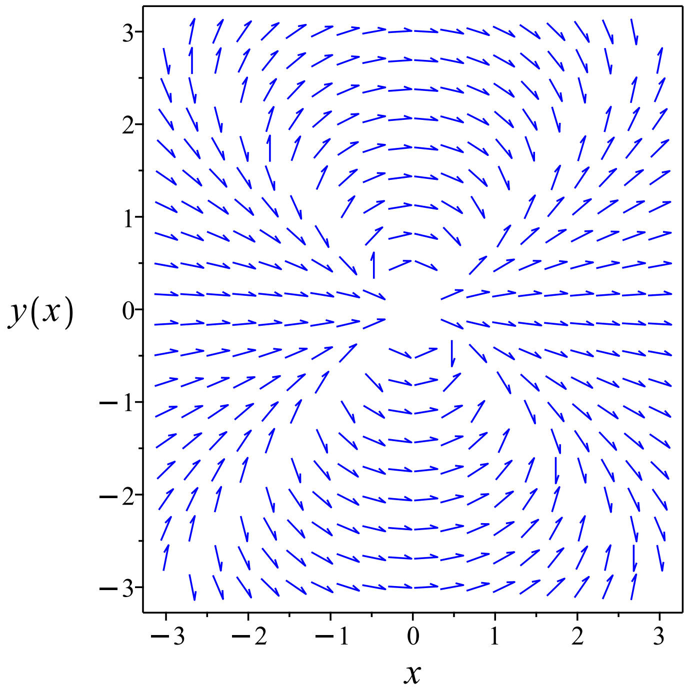


Figure 49: Slope field plot

<u>Verification of solutions</u>

$$\ln\left(\frac{y}{x}\right)+\frac{x^2}{2y^2}+\ln(x)-c_2=0$$

Verified OK.

4.9.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{xy}{-x^2+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{xy(b_3 - a_2)}{-x^2 + y^2} - \frac{x^2 y^2 a_3}{(-x^2 + y^2)^2} \\ & - \left(-\frac{y}{-x^2 + y^2} - \frac{2x^2 y}{(-x^2 + y^2)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{x}{-x^2 + y^2} + \frac{2x\,y^2}{(-x^2 + y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3x^2 y^2 b_2 - 2x\,y^3 a_2 + 2x\,y^3 b_3 - y^4 a_3 - y^4 b_2 + x^3 b_1 - x^2 y a_1 + x\,y^2 b_1 - y^3 a_1}{(x^2 - y^2)^2} = 0$$

Setting the numerator to zero gives

$$-3x^2 y^2 b_2 + 2x\,y^3 a_2 - 2x\,y^3 b_3 + y^4 a_3 + y^4 b_2 - x^3 b_1 + x^2 y a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2 v_1 v_2^3 + a_3 v_2^4 - 3b_2 v_1^2 v_2^2 + b_2 v_2^4 - 2b_3 v_1 v_2^3 + a_1 v_1^2 v_2 + a_1 v_2^3 - b_1 v_1^3 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
-b_1 v_1^3 - 3b_2 v_1^2 v_2^2 + a_1 v_1^2 v_2 + (2a_2 - 2b_3)\, v_1 v_2^3 - b_1 v_1 v_2^2 + (a_3 + b_2)\, v_2^4 + a_1 v_2^3 = 0 \quad \text{(8E)}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0\\
-b_1 &= 0\\
-3b_2 &= 0\\
2a_2 - 2b_3 &= 0\\
a_3 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi\\
&= y - \left(-\frac{xy}{-x^2 + y^2}\right)(x)\\
&= -\frac{y^3}{x^2 - y^2}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\frac{y^3}{x^2-y^2}} dy \end{aligned}$$

Which results in

$$S = \ln(y) + \frac{x^2}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy}{-x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x}{y^2} \\ S_y &= \frac{-x^2 + y^2}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\, y^2 + x^2}{2y^2} = c_1$$

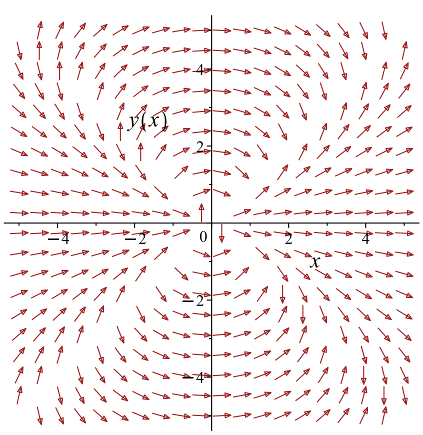
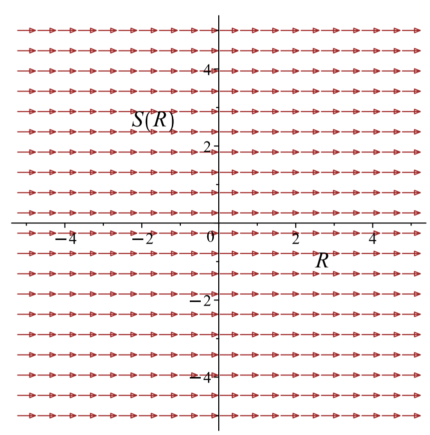
Which simplifies to

$$\frac{2\ln(y)\, y^2 + x^2}{2y^2} = c_1$$

Which gives

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-x^2 \mathrm{e}^{-2c_1}\right)}{2} + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy}{-x^2+y^2}$  	$R = x$ $S = \frac{2\ln(y)\, y^2 + x^2}{2y^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-x^{2} \mathrm{e}^{-2 c_{1}}\right)}{2}+c_{1}} \tag{1}$$

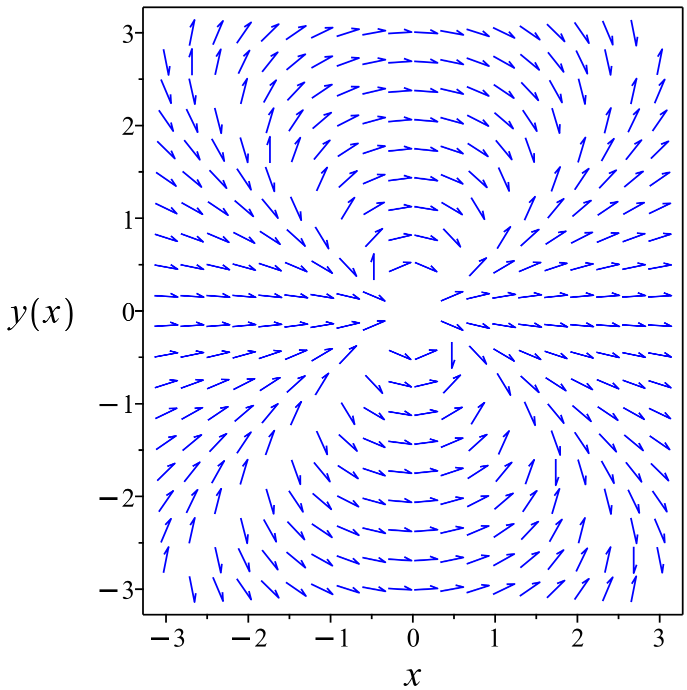


Figure 50: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-x^{2} \mathrm{e}^{-2 c_{1}}\right)}{2}+c_{1}}$$

Verified OK.

4.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x\partial y}=\frac{\partial^2 \phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x\partial y}=\frac{\partial^2 \phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2+y^2\right)\mathrm{d}y&=(-xy)\,\mathrm{d}x\\ (xy)\,\mathrm{d}x+\left(-x^2+y^2\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=xy\\ N(x,y)&=-x^2+y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(xy)\\ &=x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2 + y^2)\\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{-x^2 + y^2}((x) - (-2x))\\ &= -\frac{3x}{x^2 - y^2}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= \frac{1}{yx}((-2x) - (x))\\ &= -\frac{3}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{3}{y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(y)}\\ &= \frac{1}{y^3}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^3}(xy)\\ &= \frac{x}{y^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^3}(-x^2+y^2) \\
&= \frac{-x^2+y^2}{y^3}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x}{y^2}\right)+\left(\frac{-x^2+y^2}{y^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\overline{M}\,\mathrm{d}x \\
\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int\frac{x}{y^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2}{2y^2}+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = -\frac{x^2}{y^3}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\frac{-x^2+y^2}{y^3}$. Therefore equation (4) becomes

$$
\frac{-x^2+y^2}{y^3} = -\frac{x^2}{y^3}+f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) + \frac{x^2}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) + \frac{x^2}{2y^2}$$

The solution becomes

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-x^2\mathrm{e}^{-2c_1}\right)}{2} + c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\mathrm{LambertW}\left(-x^2\mathrm{e}^{-2c_1}\right)}{2} + c_1} \tag{1}$$

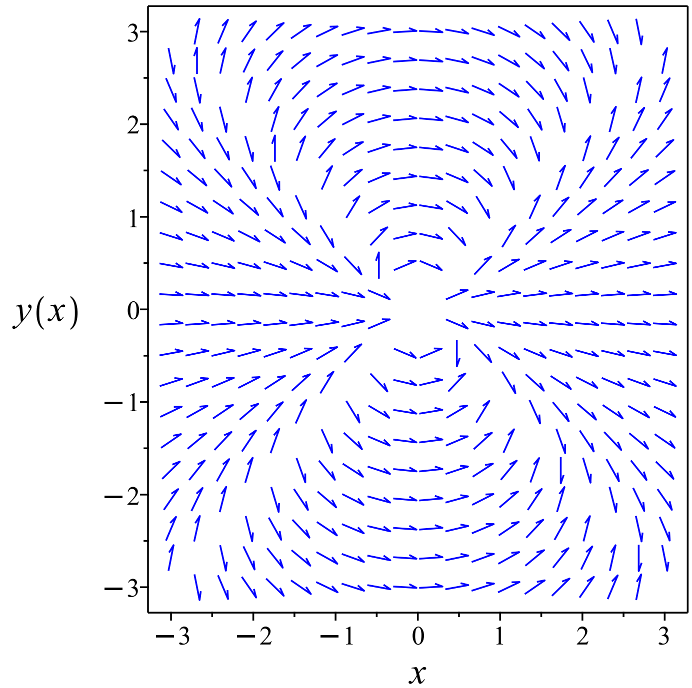


Figure 51: Slope field plot

Verification of solutions

$$y = e^{\frac{\text{LambertW}\left(-x^2 e^{-2c_1}\right)}{2} + c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x*y(x)+(y(x)^2-x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{-\frac{1}{\mathrm{LambertW}\left(-c_1 x^2\right)}}\, x$$

✓ Solution by Mathematica

Time used: 8.102 (sec). Leaf size: 56

```
DSolve[x*y[x]+(y[x]^2-x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ix}{\sqrt{W\left(-e^{-2c_1}x^2\right)}}$$
$$y(x) \to \frac{ix}{\sqrt{W\left(-e^{-2c_1}x^2\right)}}$$
$$y(x) \to 0$$

4.10 problem 10

4.10.1 Solving as homogeneousTypeD2 ode 240
4.10.2 Solving as first order ode lie symmetry calculated ode 242
4.10.3 Solving as exact ode 248

Internal problem ID [4784]
Internal file name [`OUTPUT/4277_Sunday_June_05_2022_12_53_07_PM_59688046/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 - xy + \left(xy + x^2\right) y' = 0}$$

4.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2 x^2 - x^2 u(x) + \left(x^2 u(x) + x^2\right) \left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2}{x\,(u+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u+1}}\, du = -\frac{2}{x}\, dx$$

$$\int \frac{1}{\frac{u^2}{u+1}}\, du = \int -\frac{2}{x}\, dx$$

$$\ln (u) - \frac{1}{u} = -2 \ln (x) + c_2$$

The solution is

$$\ln (u(x)) - \frac{1}{u (x)} + 2 \ln (x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln \left(\frac{y}{x}\right) - \frac{x}{y} + 2 \ln (x) - c_2 = 0$$

$$\ln \left(\frac{y}{x}\right) - \frac{x}{y} + 2 \ln (x) - c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\ln \left(\frac{y}{x}\right) - \frac{x}{y} + 2 \ln (x) - c_2 = 0 \tag{1}$$

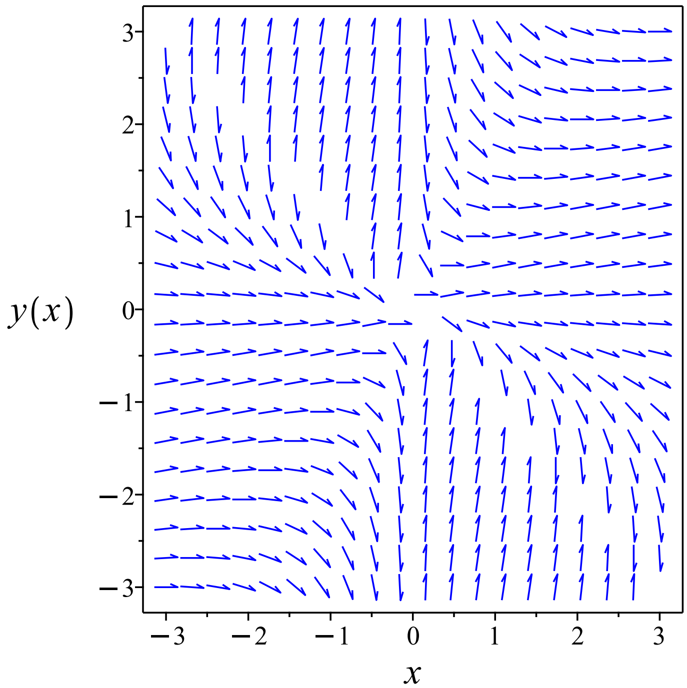


Figure 52: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln(x) - c_2 = 0$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{y(-x+y)}{x(x+y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{y(-x+y)\,(b_3-a_2)}{x\,(x+y)} - \frac{y^2(-x+y)^2\,a_3}{x^2\,(x+y)^2} \\ & - \left(\frac{y}{x\,(x+y)} + \frac{y(-x+y)}{x^2\,(x+y)} + \frac{y(-x+y)}{x\,(x+y)^2}\right)(xa_2+ya_3+a_1) \\ & - \left(-\frac{-x+y}{x\,(x+y)} - \frac{y}{x\,(x+y)} + \frac{y(-x+y)}{x\,(x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2b_2 + 2x^2y^2b_3 - 2y^4a_3 - x^3b_1 + x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1}{x^2\,(x+y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} & 4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2b_2 + 2x^2y^2b_3 - 2y^4a_3 - x^3b_1 \\ & + x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -2a_2v_1^2v_2^2 - 2a_3v_2^4 + 4b_2v_1^3v_2 + 2b_2v_1^2v_2^2 + 2b_3v_1^2v_2^2 + a_1v_1^2v_2 \\ & - 2a_1v_1v_2^2 - a_1v_2^3 - b_1v_1^3 + 2b_1v_1^2v_2 + b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & 4b_2v_1^3v_2 - b_1v_1^3 + (-2a_2 + 2b_2 + 2b_3)\,v_1^2v_2^2 \\ & + (a_1 + 2b_1)\,v_1^2v_2 + (-2a_1 + b_1)\,v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 &= 0 \\ -2a_3 &= 0 \\ -b_1 &= 0 \\ 4b_2 &= 0 \\ -2a_1 + b_1 &= 0 \\ a_1 + 2b_1 &= 0 \\ -2a_2 + 2b_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(-\frac{y(-x + y)}{x\,(x + y)}\right)(x) \\ &= \frac{2y^2}{x + y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y^2}{x+y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(y)}{2} - \frac{x}{2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x + y)}{x(x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2y} \\ S_y &= \frac{x + y}{2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y\ln(y) - x}{2y} = -\frac{\ln(x)}{2} + c_1$$

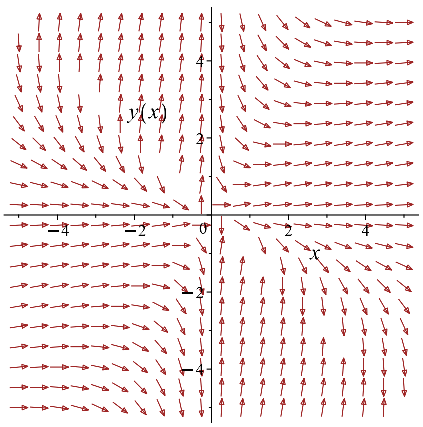
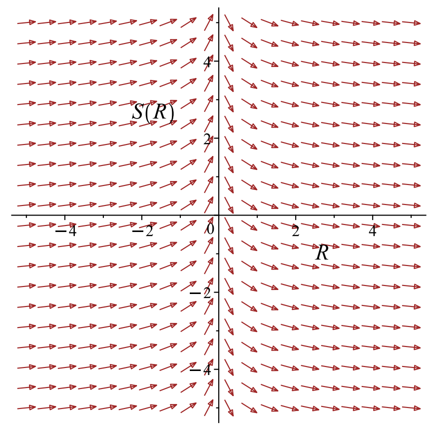
Which simplifies to

$$\frac{y\ln(y) - x}{2y} = -\frac{\ln(x)}{2} + c_1$$

Which gives

$$y = \frac{x}{\operatorname{LambertW}(x^2 e^{-2c_1})}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x(x+y)}$ 	$R = x$ $S = \frac{\ln(y)\, y - x}{2y}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{\mathrm{LambertW}\left(x^2 \mathrm{e}^{-2c_1}\right)} \tag{1}$$

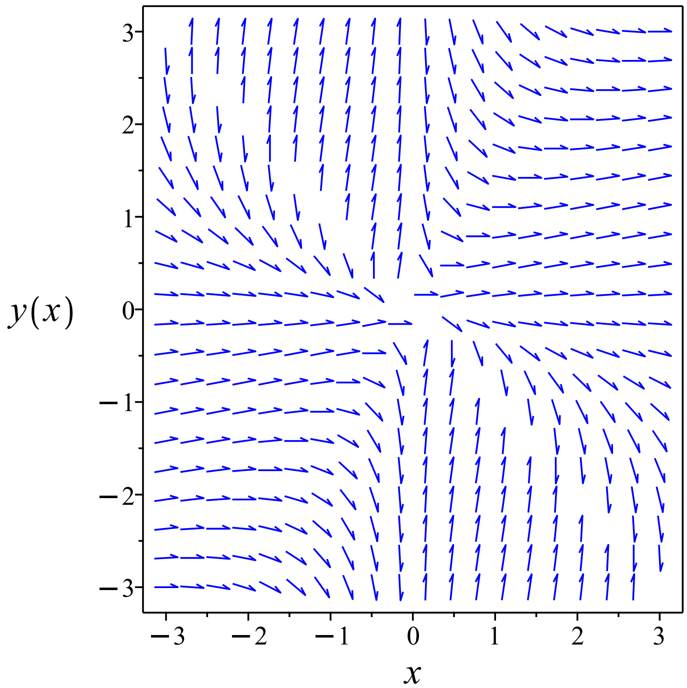


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{x}{\operatorname{LambertW}\left(x^{2} \mathrm{e}^{-2c_{1}}\right)}$$

Verified OK.

4.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x^2 + xy\right) \mathrm{d}y &= \left(xy - y^2\right) \mathrm{d}x \\
\left(-xy + y^2\right) \mathrm{d}x + \left(x^2 + xy\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -xy + y^2 \\
N(x, y) &= x^2 + xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-xy + y^2\right) \\
&= -x + 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 + xy\right) \\
&= 2x + y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = y^2 - xy$ and $N = xy + x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{y^2 - xy}{xy^2} \\
N &= \frac{xy + x^2}{xy^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{x^2+xy}{x\,y^2}\right)\mathrm{d}y = \left(-\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x$$

$$\left(\frac{-xy+y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{x^2+xy}{x\,y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \frac{-xy+y^2}{x\,y^2}\\ N(x,y) &= \frac{x^2+xy}{x\,y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-xy+y^2}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2+xy}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-xy+y^2}{x\,y^2}\,\mathrm{d}x$$

$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+xy}{x\,y^2}$. Therefore equation (4) becomes

$$\frac{x^2 + xy}{x\,y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y} + \ln(y)$$

The solution becomes

$$y = \frac{x}{\mathrm{LambertW}\left(\mathrm{e}^{-c_1} x^2\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{\mathrm{LambertW}\left(\mathrm{e}^{-c_1} x^2\right)} \tag{1}$$

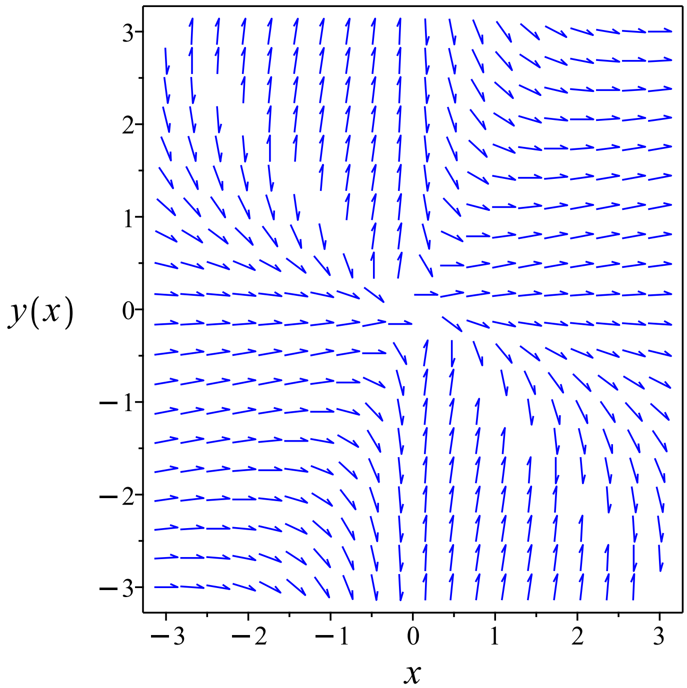


Figure 54: Slope field plot

Verification of solutions

$$y = \frac{x}{\mathrm{LambertW}\left(\mathrm{e}^{-c_1} x^2\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve((y(x)^2-x*y(x))+(x^2+x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{x}{\text{LambertW}\left(c_1 x^2\right)}$$

✓ Solution by Mathematica

Time used: 4.24 (sec). Leaf size: 25

```
DSolve[(y[x]^2-x*y[x])+(x^2+x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{W\left(e^{-c_1} x^2\right)}$$
$$y(x) \to 0$$

4.11 problem 11

4.11.1 Solving as first order ode lie symmetry calculated ode 255

Internal problem ID [4785]
Internal file name [`OUTPUT/4278_Sunday_June_05_2022_12_53_16_PM_69049410/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \cos(x+y) = 0}$$

4.11.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \cos(x+y)$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 + \cos(x+y)(b_3 - a_2) - \cos(x+y)^2 a_3 \\ &+ \sin(x+y)(xa_2 + ya_3 + a_1) + \sin(x+y)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}&\sin(x+y)\, xa_2 + \sin(x+y)\, xb_2 + \sin(x+y)\, ya_3 + \sin(x+y)\, yb_3 - \cos(x+y)^2 a_3 \\ &+ \sin(x+y)\, a_1 + \sin(x+y)\, b_1 - \cos(x+y)\, a_2 + \cos(x+y)\, b_3 + b_2 = 0\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}&\sin(x+y)\, xa_2 + \sin(x+y)\, xb_2 + \sin(x+y)\, ya_3 + \sin(x+y)\, yb_3 - \cos(x+y)^2 a_3 \\ &+ \sin(x+y)\, a_1 + \sin(x+y)\, b_1 - \cos(x+y)\, a_2 + \cos(x+y)\, b_3 + b_2 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}&b_2 - \frac{a_3}{2} + \sin(x+y)\, xa_2 + \sin(x+y)\, xb_2 + \sin(x+y)\, ya_3 \\ &+ \sin(x+y)\, yb_3 - \frac{a_3 \cos(2y+2x)}{2} + \sin(x+y)\, a_1 \\ &+ \sin(x+y)\, b_1 - \cos(x+y)\, a_2 + \cos(x+y)\, b_3 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(x+y), \cos(2y+2x), \sin(x+y)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(x+y) = v_3, \cos(2y+2x) = v_4, \sin(x+y) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 + v_5 v_1 a_2 + v_5 v_1 b_2 + v_5 v_2 a_3 + v_5 v_2 b_3 - \frac{1}{2}a_3 v_4 + v_5 a_1 + v_5 b_1 - v_3 a_2 + v_3 b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (b_3 - a_2)\, v_3 - \frac{a_3 v_4}{2} + (a_1 + b_1)\, v_5 + (a_2 + b_2)\, v_1 v_5 + (a_3 + b_3)\, v_2 v_5 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-\frac{a_3}{2} &= 0 \\
a_1 + b_1 &= 0 \\
a_2 + b_2 &= 0 \\
a_3 + b_3 &= 0 \\
b_2 - \frac{a_3}{2} &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 1 - (\cos(x + y))\, (-1) \\
&= 1 + \cos(x) \cos(y) - \sin(x) \sin(y) \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{1 + \cos(x)\cos(y) - \sin(x)\sin(y)} dy \end{aligned}$$

Which results in

$$S = \tan\left(\frac{x}{2} + \frac{y}{2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \cos(x + y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sec\left(\frac{x}{2} + \frac{y}{2}\right)^2}{2} \\ S_y &= \frac{\sec\left(\frac{x}{2} + \frac{y}{2}\right)^2}{2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos\left(\frac{x}{2}+\frac{y}{2}\right)^2 \sec\left(\frac{x}{2}+\frac{y}{2}\right)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\tan\left(\frac{x}{2}+\frac{y}{2}\right) = x + c_1$$

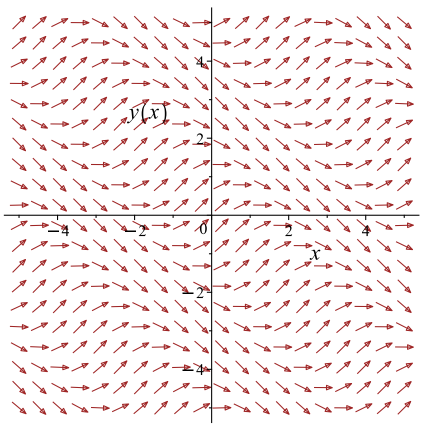
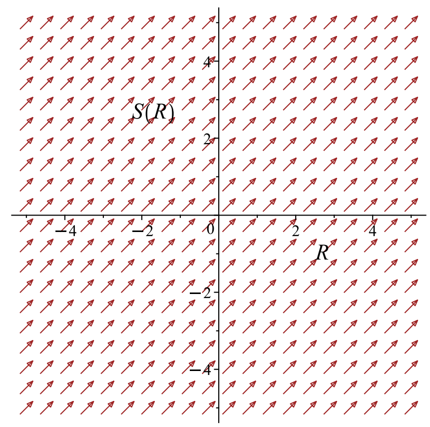
Which simplifies to

$$\tan\left(\frac{x}{2}+\frac{y}{2}\right) = x + c_1$$

Which gives

$$y = -x + 2\arctan(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \cos(x+y)$ 	$R = x$ $S = \tan\left(\frac{x}{2} + \frac{y}{2}\right)$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = -x + 2\arctan(x + c_1) \tag{1}$$

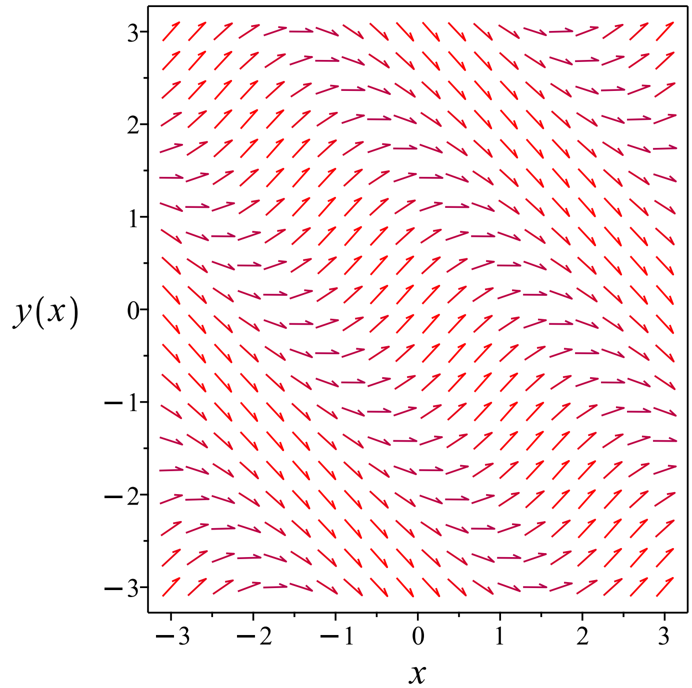


Figure 55: Slope field plot

Verification of solutions

$$y = -x + 2\arctan(x + c_1)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)=cos(x+y(x)),y(x), singsol=all)
```

$$y(x) = -x - 2\arctan(-x + c_1)$$

✓ Solution by Mathematica

Time used: 0.933 (sec). Leaf size: 59

```
DSolve[y'[x]==Cos[x+y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + 2\arctan\left(x + \frac{c_1}{2}\right)$$
$$y(x) \to -x + 2\arctan\left(x + \frac{c_1}{2}\right)$$
$$y(x) \to -x - \pi$$
$$y(x) \to \pi - x$$

4.12 problem 12

4.12.1 Solving as homogeneousTypeD ode 263
4.12.2 Solving as homogeneousTypeD2 ode 266
4.12.3 Solving as first order ode lie symmetry lookup ode 267

Internal problem ID [4786]
Internal file name [`OUTPUT/4279_Sunday_June_05_2022_12_53_47_PM_7335234/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y' - \frac{y}{x} + \tan\left(\frac{y}{x}\right) = 0}$$

4.12.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned}\frac{du}{dx}x+u &= u+g(x)\,f(bu)^{\frac{n}{m}}\\ u' &= \frac{1}{x}g(x)\,f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$\begin{aligned} g(x) &= -1\\ b &= 1\\ f\left(\frac{bx}{y}\right) &= \tan\left(\frac{y}{x}\right) \end{aligned}$$

Substituting the above in (2) results in the $u(x)$ ode as

$$u'(x) = -\frac{\tan(u(x))}{x}$$

Which is now solved as separable In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(u)}\,du &= -\frac{1}{x}\,dx\\ \int\frac{1}{\tan(u)}\,du &= \int -\frac{1}{x}\,dx\\ \ln(\sin(u)) &= -\ln(x)+c_1 \end{aligned}$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-\ln(x)+c_1}$$

Which simplifies to

$$\sin(u) = \frac{c_2}{x}$$

Therefore the solution is

$$
\begin{aligned}
y &= ux \\
&= x \arcsin\left(\frac{c_2 e^{c_1}}{x}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x \arcsin\left(\frac{c_2 e^{c_1}}{x}\right) \tag{1}
$$

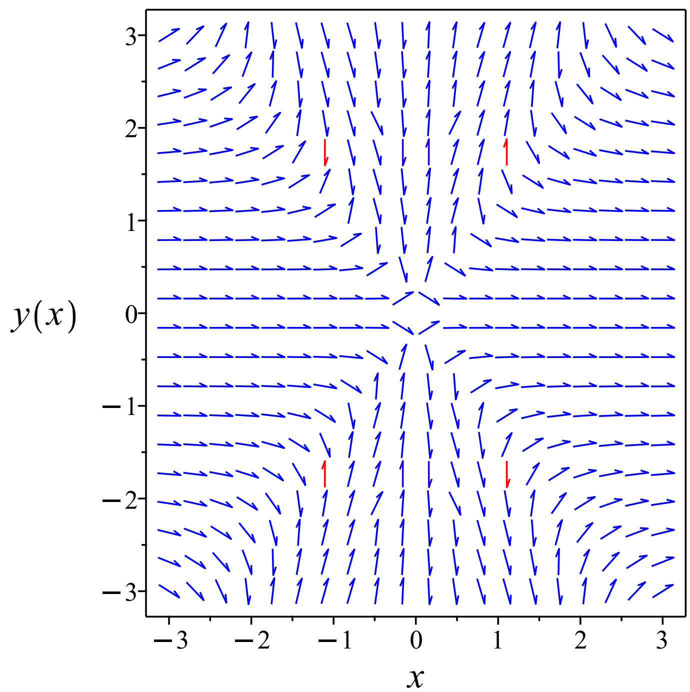


Figure 56: Slope field plot

Verification of solutions

$$
y = x \arcsin\left(\frac{c_2 e^{c_1}}{x}\right)
$$

Verified OK.

4.12.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + \tan(u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\tan(u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \tan(u)$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\tan(u)}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{\tan(u)}\,du &= \int -\frac{1}{x}\,dx \\ \ln(\sin(u)) &= -\ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sin(u) = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sin(u) = \frac{c_3}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\arcsin\left(\frac{c_3\,\mathrm{e}^{c_2}}{x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\arcsin\left(\frac{c_3\,\mathrm{e}^{c_2}}{x}\right) \tag{1}$$

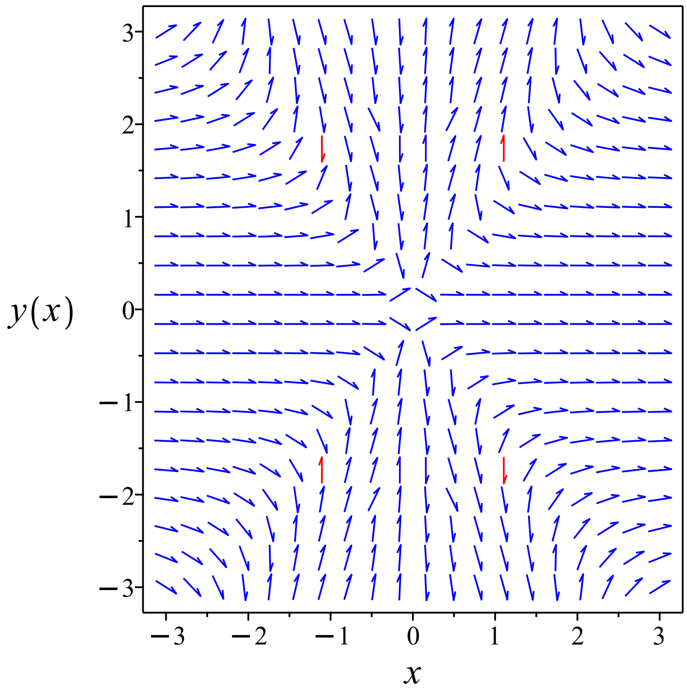


Figure 57: Slope field plot

Verification of solutions

$$
y = x \arcsin\left(\frac{c_3 e^{c_2}}{x}\right)
$$

Verified OK.

4.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{\tan\left(\frac{y}{x}\right) x - y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 40: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= x^2 \\ \eta(x,y) &= xy \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{xy}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{\tan\left(\frac{y}{x}\right) x - y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\cot\left(\frac{y}{x}\right)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = S(R)\cot(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \sin(R) \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 \sin\left(\frac{y}{x}\right)$$

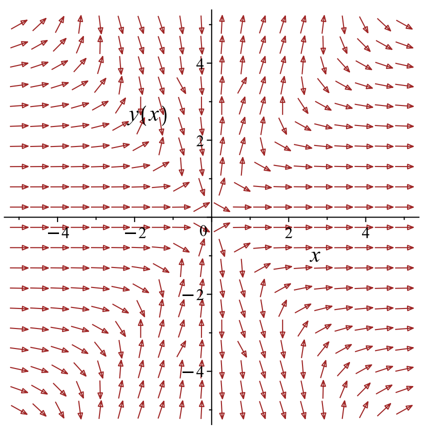
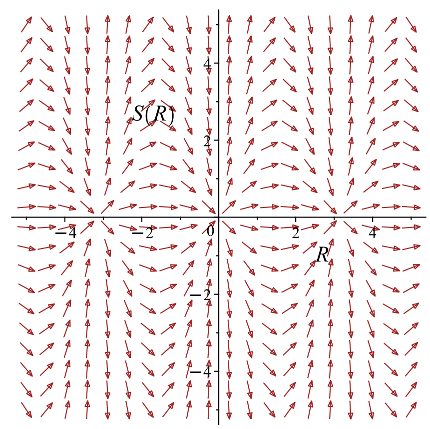
Which simplifies to

$$-\frac{1}{x} = c_1 \sin\left(\frac{y}{x}\right)$$

Which gives

$$y = -\arcsin\left(\frac{1}{c_1 x}\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{\tan\left(\frac{y}{x}\right)x - y}{x}$

</td><td>$R = \frac{y}{x}$
$S = -\frac{1}{x}$</td><td>$\frac{dS}{dR} = S(R)\cot(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = -\arcsin\left(\frac{1}{c_1 x}\right) x \tag{1}$$

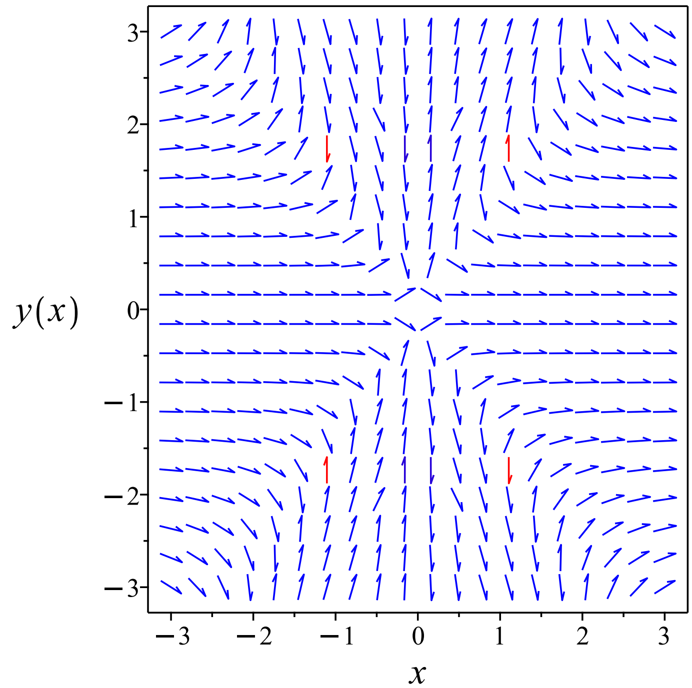


Figure 58: Slope field plot

Verification of solutions

$$y = -\arcsin\left(\frac{1}{c_1 x}\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)=y(x)/x- tan(y(x)/x),y(x), singsol=all)
```

$$y(x) = x \arcsin\left(\frac{1}{x c_1}\right)$$

✓ Solution by Mathematica

Time used: 12.97 (sec). Leaf size: 21

```
DSolve[y'[x]==y[x]/x- Tan[y[x]/x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x \arcsin\left(\frac{e^{c_1}}{x}\right)$$
$$y(x) \to 0$$

4.13 problem 13

4.13.1 Solving as linear ode 274
4.13.2 Solving as first order ode lie symmetry lookup ode 276
4.13.3 Solving as exact ode 280
4.13.4 Maple step by step solution 284

Internal problem ID [4787]
Internal file name [`OUTPUT/4280_Sunday_June_05_2022_12_53_57_PM_12700606/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first__order__ode__lie__symmetry__lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x-1)\,y'+y=\frac{1}{x^2}-\frac{2}{x^3}}$$

4.13.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{x-1}$$
$$q(x)=\frac{-2+x}{(x-1)\,x^3}$$

Hence the ode is

$$y'+\frac{y}{x-1}=\frac{-2+x}{(x-1)\,x^3}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{x-1}dx} \\
&= x-1
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{-2+x}{(x-1)\,x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(y(x-1)) &= (x-1)\left(\frac{-2+x}{(x-1)\,x^3}\right) \\
\mathrm{d}(y(x-1)) &= \left(\frac{-2+x}{x^3}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y(x-1) &= \int \frac{-2+x}{x^3}\,\mathrm{d}x \\
y(x-1) &= \frac{1}{x^2}-\frac{1}{x}+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x-1$ results in

$$
y = \frac{\frac{1}{x^2}-\frac{1}{x}}{x-1}+\frac{c_1}{x-1}
$$

which simplifies to

$$
y = \frac{c_1x^2-x+1}{x^2\,(x-1)}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{c_1x^2-x+1}{x^2\,(x-1)} \tag{1}
$$

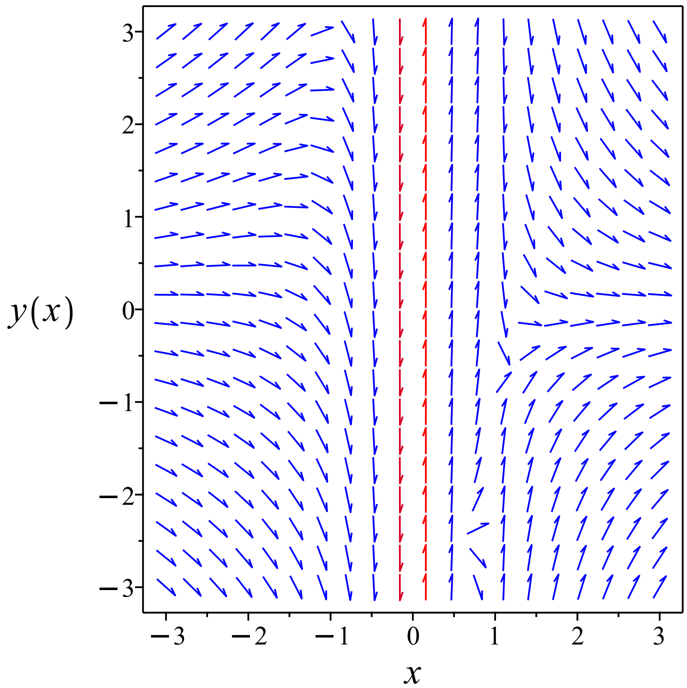


Figure 59: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)}$$

Verified OK.

4.13.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y\,x^3 - x + 2}{(x - 1)\,x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x-1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x-1}} dy\end{aligned}$$

Which results in

$$S = y(x - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y\, x^3 - x + 2}{(x - 1)\, x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x - 1\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{-2 + x}{x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{-2 + R}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{R^2} - \frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x-1)\, y = \frac{1}{x^2} - \frac{1}{x} + c_1$$

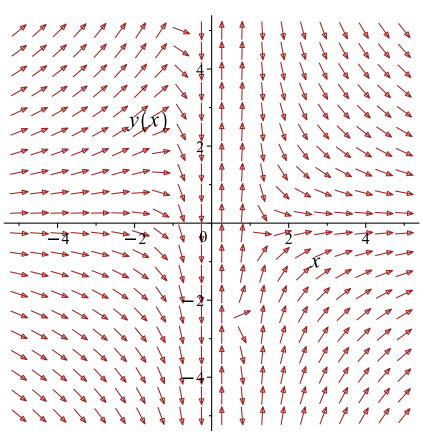
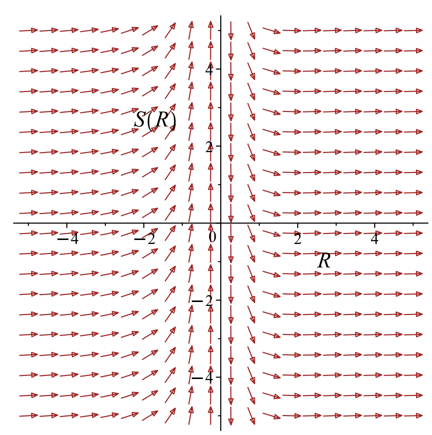
Which simplifies to

$$(x-1)\, y = \frac{1}{x^2} - \frac{1}{x} + c_1$$

Which gives

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x-1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y\,x^3 - x + 2}{(x-1)x^3}$ 	$R = x$ $S = y(x-1)$	$\frac{dS}{dR} = \frac{-2+R}{R^3}$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)} \tag{1}$$

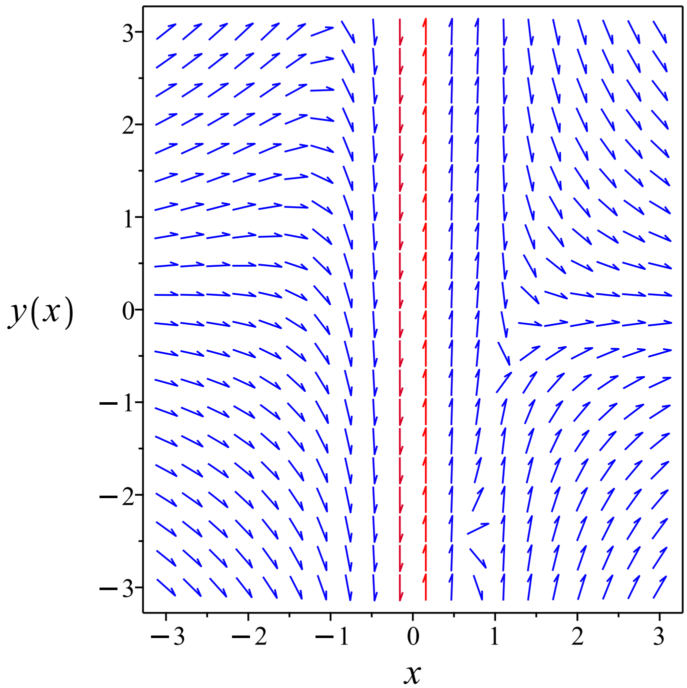


Figure 60: Slope field plot

Verification of solutions

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)}$$

Verified OK.

4.13.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x-1)\,\mathrm{d}y&=\left(-y+\frac{1}{x^2}-\frac{2}{x^3}\right)\mathrm{d}x\\ \left(y-\frac{1}{x^2}+\frac{2}{x^3}\right)\mathrm{d}x+(x-1)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y-\frac{1}{x^2}+\frac{2}{x^3}\\ N(x,y)&=x-1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - \frac{1}{x^2} + \frac{2}{x^3}\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-1)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int y - \frac{1}{x^2} + \frac{2}{x^3}\,\mathrm{d}x$$
$$\phi = \frac{y\,x^3 + x - 1}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x - 1$. Therefore equation (4) becomes

$$x - 1 = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y\,x^3 + x - 1}{x^2} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\,x^3 + x - 1}{x^2} - y$$

The solution becomes

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)} \tag{1}$$

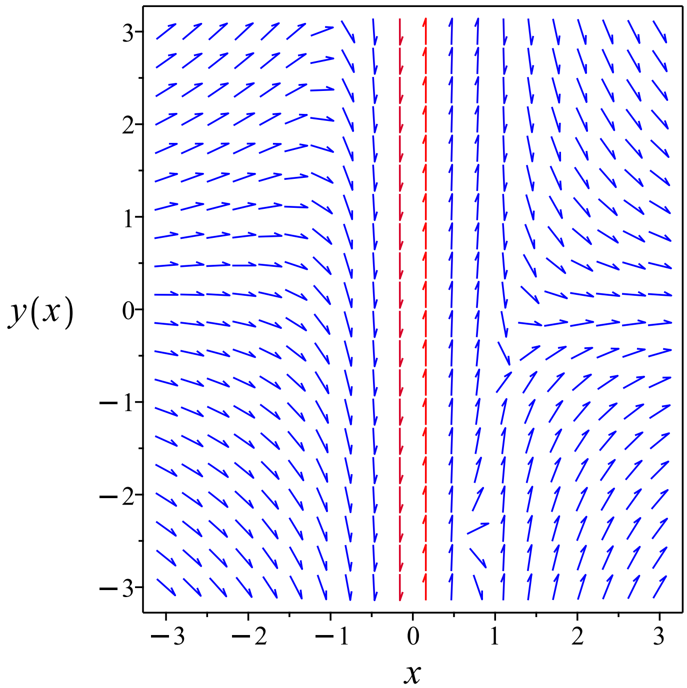


Figure 61: Slope field plot

Verification of solutions

$$y = \frac{c_1 x^2 - x + 1}{x^2 (x - 1)}$$

Verified OK.

4.13.4 Maple step by step solution

Let's solve

$(x-1)\,y' + y = \frac{1}{x^2} - \frac{2}{x^3}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x-1} + \frac{-2+x}{(x-1)x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x-1} = \frac{-2+x}{(x-1)x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y' + \frac{y}{x-1}\right) = \frac{\mu(x)(-2+x)}{(x-1)x^3}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x)\left(y' + \frac{y}{x-1}\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \frac{\mu(x)}{x-1}$$

- Solve to find the integrating factor

$$\mu(x) = x - 1$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(-2+x)}{(x-1)x^3} dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \frac{\mu(x)(-2+x)}{(x-1)x^3} dx + c_1$$

- Solve for y

$$y = \frac{\int \frac{\mu(x)(-2+x)}{(x-1)x^3} dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = x - 1$

$$y = \frac{\int \frac{-2+x}{x^3} dx + c_1}{x-1}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{1}{x^2} - \frac{1}{x} + c_1}{x-1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve((x-1)*diff(y(x),x)+y(x)-1/x^2+2/x^3=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1}{x-1} - \frac{1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 24

```
DSolve[(x-1)*y'[x]+y[x]-1/x^2+2/x^3==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1 x^2 + x - 1}{(x-1)x^2}$$

4.14 problem 25 part (a)

4.14.1 Solving as first order ode lie symmetry calculated ode 287
4.14.2 Solving as riccati ode 293

Internal problem ID [4788]
Internal file name [`OUTPUT/4281_Sunday_June_05_2022_12_54_08_PM_88776971/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 25 part (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Riccati]
```

$$\boxed{y' - xy^2 + \frac{2y}{x} = -\frac{1}{x^3}}$$

4.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2x^4 - 2y\,x^2 - 1}{x^3} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(y^2x^4 - 2y\,x^2 - 1)\,(b_3 - a_2)}{x^3} - \frac{(y^2x^4 - 2y\,x^2 - 1)^2\,a_3}{x^6} \\
&- \left(\frac{4y^2x^3 - 4xy}{x^3} - \frac{3(y^2x^4 - 2y\,x^2 - 1)}{x^4}\right)(xa_2 + ya_3 + a_1) \\
&- \frac{(2y\,x^4 - 2x^2)\,(xb_2 + yb_3 + b_1)}{x^3} = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^8y^4a_3 + 2x^8yb_2 + 2x^7y^2a_2 + x^7y^2b_3 - 3x^6y^3a_3 + 2x^7yb_1 + x^6y^2a_1 - 3b_2x^6 + 4x^4y^2a_3 - 2x^5b_1 + 2x^4ya_1}{x^6}
$$
$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^8y^4a_3 - 2x^8yb_2 - 2x^7y^2a_2 - x^7y^2b_3 + 3x^6y^3a_3 - 2x^7yb_1 - x^6y^2a_1 + 3b_2x^6 \\
&- 4x^4y^2a_3 + 2x^5b_1 - 2x^4ya_1 - 2x^3a_2 - x^3b_3 - 7x^2ya_3 - 3x^2a_1 - a_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_3v_1^8v_2^4 - 2a_2v_1^7v_2^2 + 3a_3v_1^6v_2^3 - 2b_2v_1^8v_2 - b_3v_1^7v_2^2 - a_1v_1^6v_2^2 - 2b_1v_1^7v_2 - 4a_3v_1^4v_2^2 \\
&+ 3b_2v_1^6 - 2a_1v_1^4v_2 + 2b_1v_1^5 - 2a_2v_1^3 - 7a_3v_1^2v_2 - b_3v_1^3 - 3a_1v_1^2 - a_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & -a_3 v_1^8 v_2^4 - 2b_2 v_1^8 v_2 + (-2a_2 - b_3) v_1^7 v_2^2 - 2b_1 v_1^7 v_2 + 3a_3 v_1^6 v_2^3 - a_1 v_1^6 v_2^2 + 3b_2 v_1^6 \\ & + 2b_1 v_1^5 - 4a_3 v_1^4 v_2^2 - 2a_1 v_1^4 v_2 + (-2a_2 - b_3) v_1^3 - 7a_3 v_1^2 v_2 - 3a_1 v_1^2 - a_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -3a_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ -7a_3 &= 0 \\ -4a_3 &= 0 \\ -a_3 &= 0 \\ 3a_3 &= 0 \\ -2b_1 &= 0 \\ 2b_1 &= 0 \\ -2b_2 &= 0 \\ 3b_2 &= 0 \\ -2a_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= -2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= -2y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= -2y - \left(\frac{y^2x^4 - 2y\,x^2 - 1}{x^3}\right)(x) \\
&= \frac{-y^2x^4 + 1}{x^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2x^4+1}{x^2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(y\,x^2 - 1)}{2} + \frac{\ln(y\,x^2 + 1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2x^4 - 2y\,x^2 - 1}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2xy}{y^2x^4-1} \\
S_y &= -\frac{x^2}{y^2x^4-1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\frac{\ln(yx^2-1)}{2} + \frac{\ln(yx^2+1)}{2} = -\ln(x) + c_1
$$

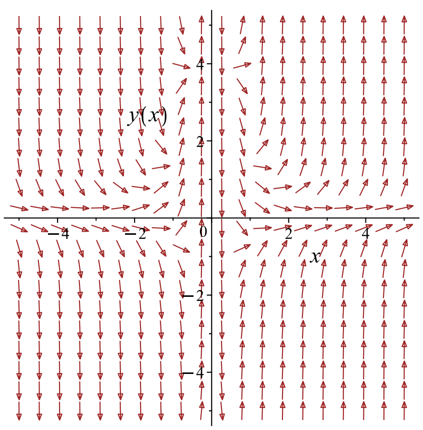
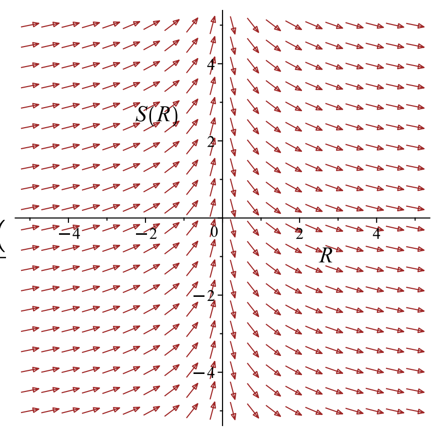
Which simplifies to

$$
-\frac{\ln(yx^2-1)}{2} + \frac{\ln(yx^2+1)}{2} = -\ln(x) + c_1
$$

Which gives

$$
y = \frac{e^{2c_1} + x^2}{(e^{2c_1} - x^2)\, x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2 x^4 - 2y\,x^2 - 1}{x^3}$ 	$R = x$ $S = -\frac{\ln\left(y\,x^2 - 1\right)}{2} + \frac{\ln\left(\right.}{}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$
y = \frac{e^{2c_1} + x^2}{\left(e^{2c_1} - x^2\right) x^2} \tag{1}
$$

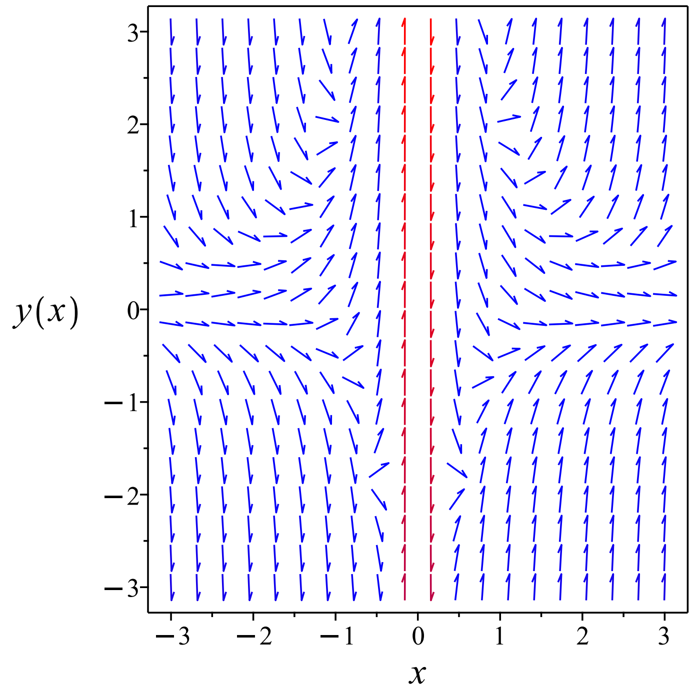


Figure 62: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{2c_1} + x^2}{(e^{2c_1} - x^2)\, x^2}$$

Verified OK.

4.14.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2 x^4 - 2y\, x^2 - 1}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2 x - \frac{2y}{x} - \frac{1}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{1}{x^3}$, $f_1(x) = -\frac{2}{x}$ and $f_2(x) = x$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{xu} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 1 \\
f_1 f_2 &= -2 \\
f_2^2 f_0 &= -\frac{1}{x}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
x u''(x) + u'(x) - \frac{u(x)}{x} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = \frac{c_2 x^2 + c_1}{x}
$$

The above shows that

$$
u'(x) = \frac{c_2 x^2 - c_1}{x^2}
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_2 x^2 - c_1}{x^2 (c_2 x^2 + c_1)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{-x^2 + c_3}{x^2 (x^2 + c_3)}
$$

Summary
The solution(s) found are the following

$$y = \frac{-x^2 + c_3}{x^2 (x^2 + c_3)} \tag{1}$$

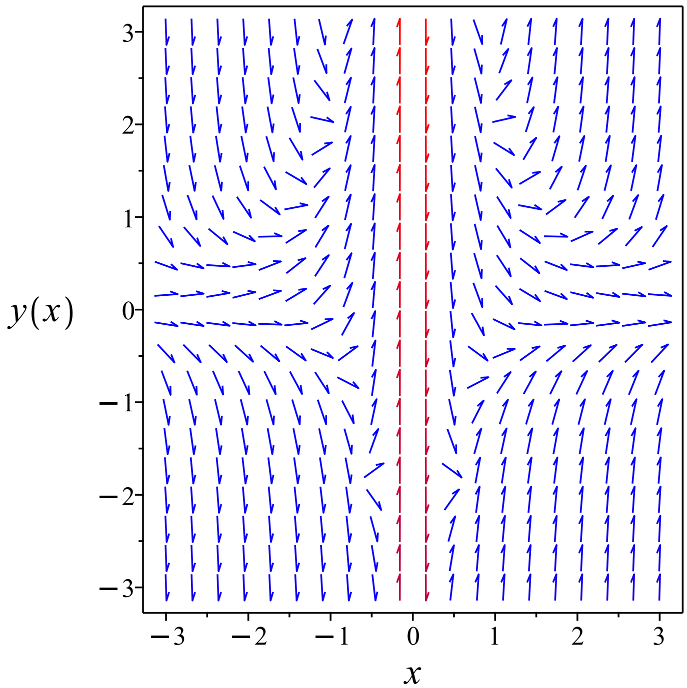


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + c_3}{x^2 (x^2 + c_3)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)= x*y(x)^2-2/x*y(x)-1/x^3,y(x), singsol=all)
```

$$y(x) = \frac{\tanh(-\ln(x) + c_1)}{x^2}$$

✓ Solution by Mathematica

Time used: 1.188 (sec). Leaf size: 63

```
DSolve[y'[x]== x*y[x]^2-2/x*y[x]-1/x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{i\tan(i\log(x) + c_1)}{x^2}$$
$$y(x) \to \frac{-x^2 + e^{2i\text{Interval}[\{0,\pi\}]}}{x^4 + x^2 e^{2i\text{Interval}[\{0,\pi\}]}}$$

4.15 problem 25 part (b)

4.15.1 Solving as homogeneousTypeD2 ode 297
4.15.2 Solving as exact ode 299
4.15.3 Solving as riccati ode 304

Internal problem ID [4789]
Internal file name [`OUTPUT/4282_Sunday_June_05_2022_12_54_18_PM_2594122/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 25 part (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "exactByInspection", "homogeneousTypeD2"**

Maple gives the following as the ode type

```
[[_homogeneous, `class D`], _rational, _Riccati]
```

$$\boxed{y' - \frac{2y^2}{x} - \frac{y}{x} = -2x}$$

4.15.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x - 2u(x)^2\,x = -2x$$

Integrating both sides gives

$$\int \frac{1}{2u^2 - 2}du = c_2 + x$$
$$-\frac{\operatorname{arctanh}(u)}{2} = c_2 + x$$

Solving for u gives these solutions

$$u_1 = -\tanh(2c_2 + 2x)$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= -x \tanh(2c_2 + 2x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -x \tanh(2c_2 + 2x) \tag{1}
$$

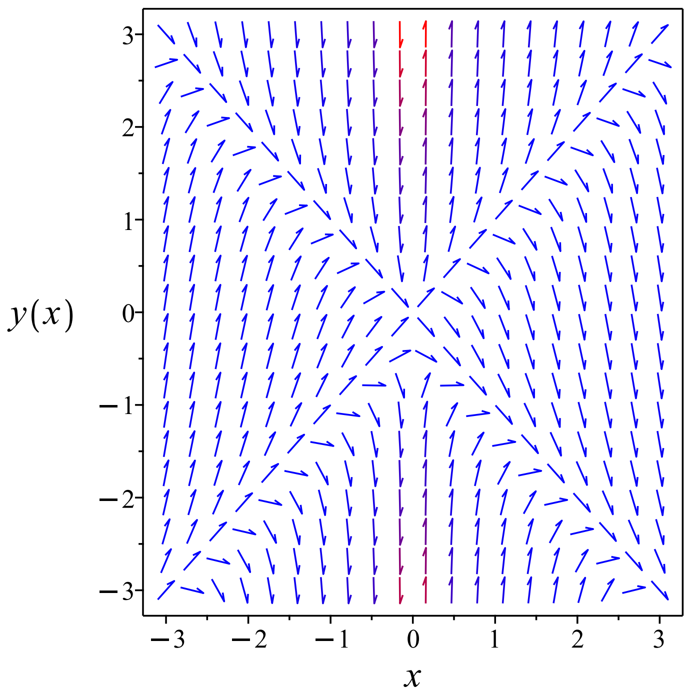


Figure 64: Slope field plot

Verification of solutions

$$
y = -x \tanh(2c_2 + 2x)
$$

Verified OK.

4.15.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(-2x^2+2y^2+y\right)\mathrm{d}x\\ \left(2x^2-2y^2-y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2x^2-2y^2-y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x^2 - 2y^2 - y\right)\\ &= -4y - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2-y^2}$ is an integrating factor. Therefore by multiplying $M = 2x^2 - 2y^2 - y$ and $N = x$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{2x^2 - 2y^2 - y}{x^2 - y^2}\\ N &= \frac{x}{x^2 - y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{gathered}
\left(\frac{x}{x^2 - y^2}\right)\mathrm{d}y = \left(-\frac{2x^2 - 2y^2 - y}{x^2 - y^2}\right)\mathrm{d}x \\
\left(\frac{2x^2 - 2y^2 - y}{x^2 - y^2}\right)\mathrm{d}x + \left(\frac{x}{x^2 - y^2}\right)\mathrm{d}y = 0 \qquad (2\mathrm{A})
\end{gathered}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \frac{2x^2 - 2y^2 - y}{x^2 - y^2} \\
N(x, y) &= \frac{x}{x^2 - y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2x^2 - 2y^2 - y}{x^2 - y^2}\right) \\
&= \frac{-x^2 - y^2}{(x^2 - y^2)^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{x^2 - y^2}\right) \\
&= \frac{-x^2 - y^2}{(x^2 - y^2)^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{2x^2 - 2y^2 - y}{x^2 - y^2} \mathrm{d}x$$
$$\phi = 2x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{2y+2x} + \frac{1}{-2y+2x} + f'(y) \tag{4}$$
$$= \frac{x}{x^2 - y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{x^2 - y^2}$. Therefore equation (4) becomes

$$\frac{x}{x^2 - y^2} = \frac{x}{x^2 - y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2x + \frac{\ln(x+y)}{2} - \frac{\ln(x-y)}{2}$$

The solution becomes

$$y = \frac{x(-1 + e^{-4x+2c_1})}{e^{-4x+2c_1} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x(-1 + e^{-4x+2c_1})}{e^{-4x+2c_1} + 1} \tag{1}$$

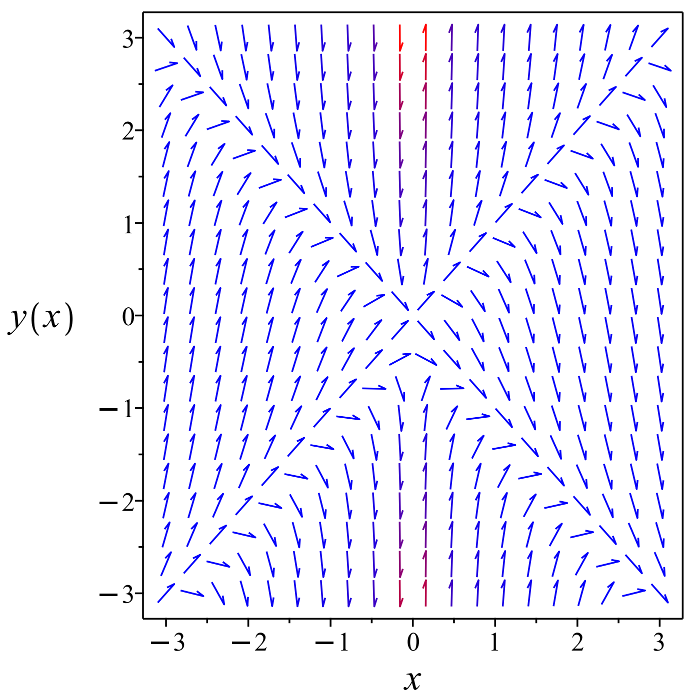


Figure 65: Slope field plot

Verification of solutions

$$y = \frac{x(-1 + e^{-4x+2c_1})}{e^{-4x+2c_1} + 1}$$

Verified OK.

4.15.3 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{-2x^2 + 2y^2 + y}{x}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = \frac{2y^2}{x} + \frac{y}{x} - 2x
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = -2x$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{2}{x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{2u}{x}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{x^2} \\
f_1 f_2 &= \frac{2}{x^2} \\
f_2^2 f_0 &= -\frac{8}{x}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{2u''(x)}{x} - \frac{8u(x)}{x} = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}
$$

The above shows that

$$u'(x) = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

Using the above in (1) gives the solution

$$y = -\frac{(2c_1 e^{2x} - 2c_2 e^{-2x})\, x}{2\,(c_1 e^{2x} + c_2 e^{-2x})}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x(e^{4x} c_3 - 1)}{e^{4x} c_3 + 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x(e^{4x} c_3 - 1)}{e^{4x} c_3 + 1} \tag{1}$$

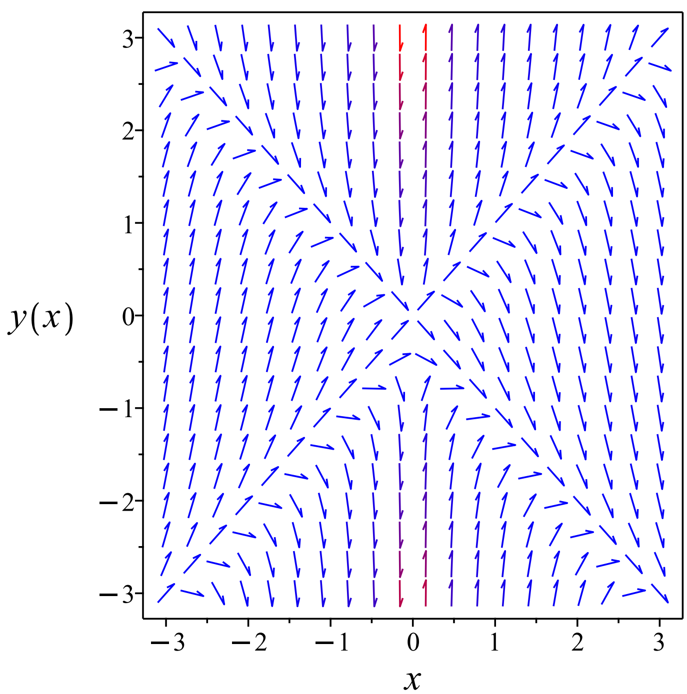


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{x(e^{4x}c_3 - 1)}{e^{4x}c_3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)= 2/x*y(x)^2+1/x*y(x)-2*x,y(x), singsol=all)
```

$$y(x) = -\tanh(2x + 2c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.716 (sec). Leaf size: 47

```
DSolve[y'[x]== 2/x*y[x]^2+1/x*y[x]-2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x - xe^{4x+2c_1}}{1 + e^{4x+2c_1}}$$
$$y(x) \to -x$$
$$y(x) \to x$$

4.16 problem 25 part (c)

4.16.1 Solving as first order ode lie symmetry calculated ode 307
4.16.2 Solving as riccati ode 313

Internal problem ID [4790]
Internal file name [`OUTPUT/4283_Sunday_June_05_2022_12_54_27_PM_46772709/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 4. OTHER METHODS FOR FIRST-ORDER EQUATIONS. page 406
Problem number: 25 part (c).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Riccati]
```

$$\boxed{y' - \mathrm{e}^{-x} y^2 - y = -\mathrm{e}^x}$$

4.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \mathrm{e}^{-x} y^2 + y - \mathrm{e}^x \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 + \left(\mathrm{e}^{-x}y^2 + y - \mathrm{e}^x\right)(b_3 - a_2) - \left(\mathrm{e}^{-x}y^2 + y - \mathrm{e}^x\right)^2 a_3 \\ - \left(-\mathrm{e}^{-x}y^2 - \mathrm{e}^x\right)(xa_2 + ya_3 + a_1) - \left(1 + 2\,\mathrm{e}^{-x}y\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} -\mathrm{e}^{-2x}y^4a_3 + 2\,\mathrm{e}^{-x}\mathrm{e}^x y^2 a_3 + \mathrm{e}^{-x}x\,y^2a_2 - \mathrm{e}^{-x}y^3a_3 - 2\,\mathrm{e}^{-x}xyb_2 \\ + \mathrm{e}^{-x}y^2a_1 - \mathrm{e}^{-x}y^2a_2 - \mathrm{e}^{-x}y^2b_3 - 2\,\mathrm{e}^{-x}yb_1 - \mathrm{e}^{2x}a_3 + \mathrm{e}^x xa_2 \\ + 3\,\mathrm{e}^x ya_3 - y^2a_3 + \mathrm{e}^x a_1 + \mathrm{e}^x a_2 - \mathrm{e}^x b_3 - xb_2 - ya_2 - b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} -\mathrm{e}^{-2x}y^4a_3 + 2\,\mathrm{e}^{-x}\mathrm{e}^x y^2 a_3 + \mathrm{e}^{-x}x\,y^2a_2 - \mathrm{e}^{-x}y^3a_3 - 2\,\mathrm{e}^{-x}xyb_2 \\ + \mathrm{e}^{-x}y^2a_1 - \mathrm{e}^{-x}y^2a_2 - \mathrm{e}^{-x}y^2b_3 - 2\,\mathrm{e}^{-x}yb_1 - \mathrm{e}^{2x}a_3 + \mathrm{e}^x xa_2 \\ + 3\,\mathrm{e}^x ya_3 - y^2a_3 + \mathrm{e}^x a_1 + \mathrm{e}^x a_2 - \mathrm{e}^x b_3 - xb_2 - ya_2 - b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} -\mathrm{e}^{-2x}y^4a_3 + y^2a_3 + \mathrm{e}^{-x}x\,y^2a_2 - \mathrm{e}^{-x}y^3a_3 - 2\,\mathrm{e}^{-x}xyb_2 \\ + \mathrm{e}^{-x}y^2a_1 - \mathrm{e}^{-x}y^2a_2 - \mathrm{e}^{-x}y^2b_3 - 2\,\mathrm{e}^{-x}yb_1 - \mathrm{e}^{2x}a_3 + \mathrm{e}^x xa_2 \\ + 3\,\mathrm{e}^x ya_3 + \mathrm{e}^x a_1 + \mathrm{e}^x a_2 - \mathrm{e}^x b_3 - xb_2 - ya_2 - b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^x, \mathrm{e}^{-2x}, \mathrm{e}^{-x}, \mathrm{e}^{2x}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^x = v_3, \mathrm{e}^{-2x} = v_4, \mathrm{e}^{-x} = v_5, \mathrm{e}^{2x} = v_6\}$$

The above PDE (6E) now becomes

$$\begin{aligned} -v_4v_2^4a_3 + v_5v_1v_2^2a_2 - v_5v_2^3a_3 + v_5v_2^2a_1 - v_5v_2^2a_2 - 2v_5v_1v_2b_2 - v_5v_2^2b_3 + v_3v_1a_2 \\ + v_2^2a_3 + 3v_3v_2a_3 - 2v_5v_2b_1 + v_3a_1 - v_2a_2 + v_3a_2 - v_6a_3 - v_1b_2 - v_3b_3 - b_1 + b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5, v_6\}$$

Equation (7E) now becomes

$$\begin{aligned} &v_5v_1v_2^2a_2 - 2v_5v_1v_2b_2 + v_3v_1a_2 - v_1b_2 - v_4v_2^4a_3 - v_5v_2^3a_3 + (a_1 - a_2 - b_3)\, v_2^2v_5 \\ &+ v_2^2a_3 + 3v_3v_2a_3 - 2v_5v_2b_1 - v_2a_2 + (a_1 + a_2 - b_3)\, v_3 - v_6a_3 - b_1 + b_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ -a_2 &= 0 \\ -a_3 &= 0 \\ 3a_3 &= 0 \\ -2b_1 &= 0 \\ -2b_2 &= 0 \\ -b_2 &= 0 \\ -b_1 + b_2 &= 0 \\ a_1 - a_2 - b_3 &= 0 \\ a_1 + a_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= b_3 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\mathrm{e}^{-x} y^2 + y - \mathrm{e}^{x}\right)(1) \\
&= -\mathrm{e}^{-x} y^2 + \mathrm{e}^{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\mathrm{e}^{-x} y^2 + \mathrm{e}^{x}} dy
\end{aligned}
$$

Which results in

$$
S = \operatorname{arctanh}\left(\mathrm{e}^{-x} y\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \mathrm{e}^{-x} y^2 + y - \mathrm{e}^{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{e^{-x} y}{e^{-2x} y^2 - 1} \\
S_y &= -\frac{e^{-x}}{e^{-2x} y^2 - 1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\operatorname{arctanh}\left(y\, e^{-x}\right) = -x + c_1
$$

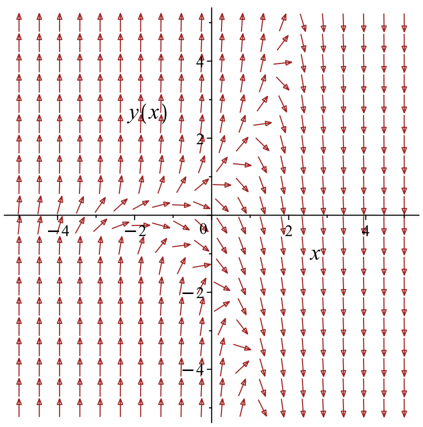
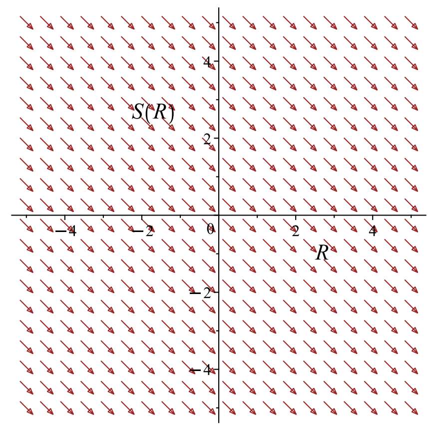
Which simplifies to

$$
\operatorname{arctanh}\left(y\, e^{-x}\right) = -x + c_1
$$

Which gives

$$
y = \tanh\left(-x + c_1\right) e^{x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \mathrm{e}^{-x} y^2 + y - \mathrm{e}^{x}$ 	$R = x$ $S = \operatorname{arctanh}\left(\mathrm{e}^{-x} y\right)$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \tanh\left(-x + c_1\right) \mathrm{e}^{x} \tag{1}$$

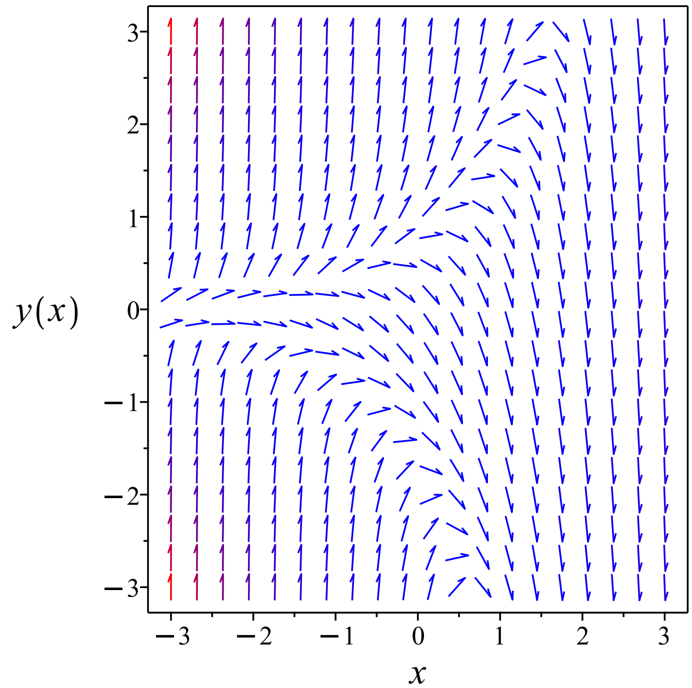


Figure 67: Slope field plot

Verification of solutions

$$y = \tanh(-x + c_1)\, e^{x}$$

Verified OK.

4.16.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= e^{-x}y^2 + y - e^{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = e^{-x}y^2 + y - e^{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\mathrm{e}^x$, $f_1(x) = 1$ and $f_2(x) = \mathrm{e}^{-x}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\mathrm{e}^{-x} u} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= -\mathrm{e}^{-x} \\
f_1 f_2 &= \mathrm{e}^{-x} \\
f_2^2 f_0 &= -\mathrm{e}^{-2x} \mathrm{e}^x
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\mathrm{e}^{-x} u''(x) - \mathrm{e}^{-2x} \mathrm{e}^x u(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^x
$$

The above shows that

$$
u'(x) = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^x
$$

Using the above in (1) gives the solution

$$
y = -\frac{(-c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^x)\, \mathrm{e}^x}{c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^x}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{\mathrm{e}^x (-\mathrm{e}^{2x} + c_3)}{\mathrm{e}^{2x} + c_3}
$$

Summary

The solution(s) found are the following

$$y = \frac{e^{x}\left(-e^{2x}+c_3\right)}{e^{2x}+c_3} \tag{1}$$

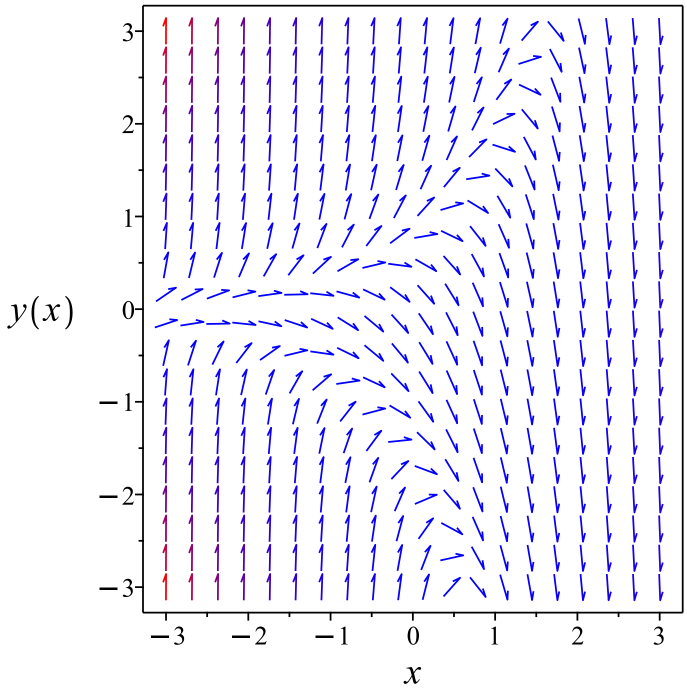


Figure 68: Slope field plot

Verification of solutions

$$y = \frac{e^{x}\left(-e^{2x}+c_3\right)}{e^{2x}+c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 16

```
dsolve(diff(y(x),x)= exp(-x)*y(x)^2+y(x)-exp(x),y(x), singsol=all)
```

$$y(x) = i \tan (ix + c_1) e^x$$

✓ Solution by Mathematica
Time used: 0.302 (sec). Leaf size: 19

```
DSolve[y'[x]== Exp[-x]*y[x]^2+y[x]-Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -e^x \tanh(x - ic_1)$$

5 Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414

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5.1 problem 1

5.1.1 Solving as second order linear constant coeff ode 318
5.1.2 Solving using Kovacic algorithm 320
5.1.3 Maple step by step solution 324

Internal problem ID [4791]
Internal file name [`OUTPUT/4284_Sunday_June_05_2022_12_54_37_PM_2236424/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 2y = 0}$$

5.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{3}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-2)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

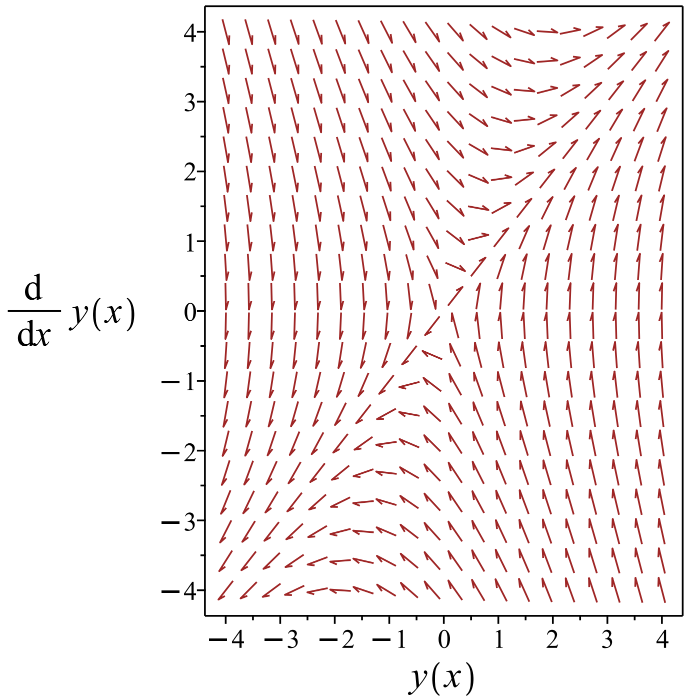

Figure 69: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Verified OK.

5.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 45: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{x}}{3} \tag{1}
$$

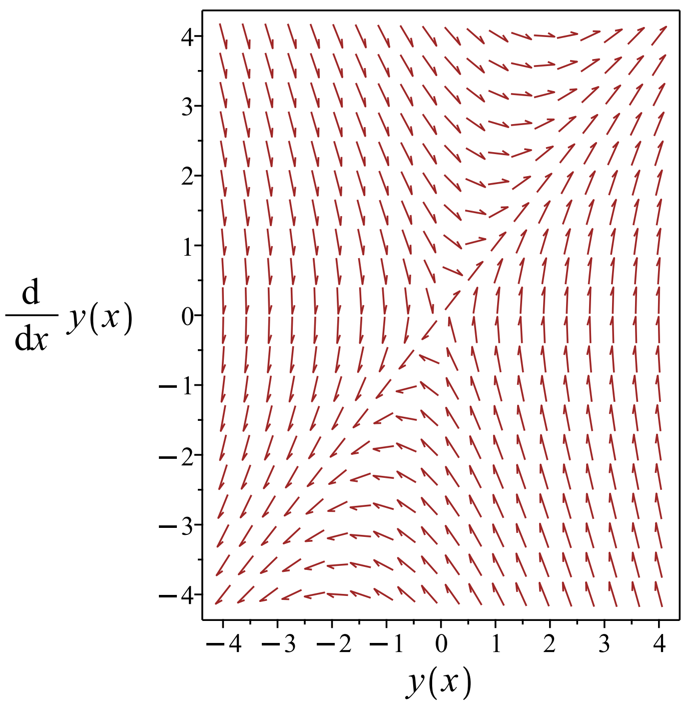

Figure 70: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{x}}{3}
$$

Verified OK.

5.1.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 1)$

- 1st solution of the ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-2x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{3x} + c_2\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[y''[x]+y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-2x} + c_2 e^x$$

5.2 problem 2

5.2.1 Solving as second order linear constant coeff ode 326
5.2.2 Solving as linear second order ode solved by an integrating factor ode 328
5.2.3 Solving using Kovacic algorithm 329
5.2.4 Maple step by step solution 333

Internal problem ID [4792]
Internal file name [`OUTPUT/4285_Sunday_June_05_2022_12_54_45_PM_36748892/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 4y = 0}$$

5.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)} \\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + x\, e^{2x} c_2 \tag{1}$$

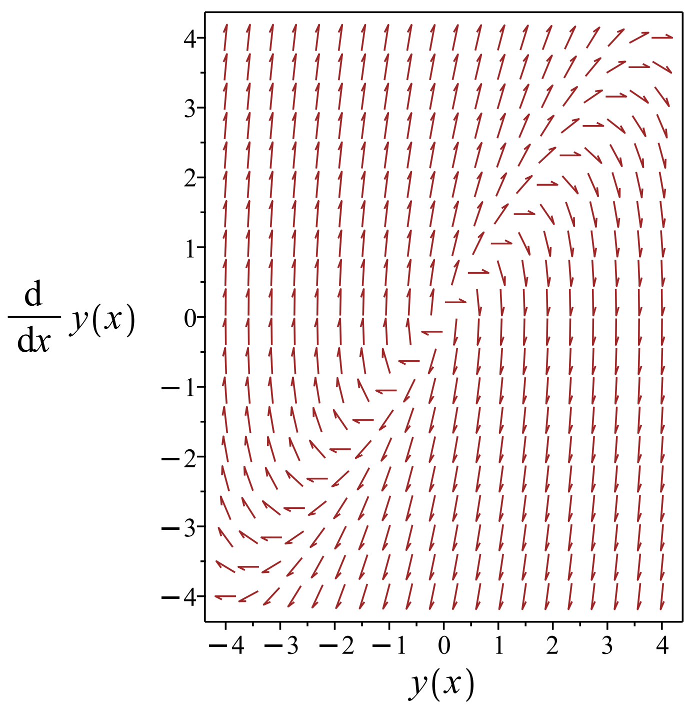

Figure 71: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + x\, e^{2x} c_2$$

Verified OK.

5.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(y\,\mathrm{e}^{-2x}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{-2x}\right)' = c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{-2x}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} \tag{1}$$

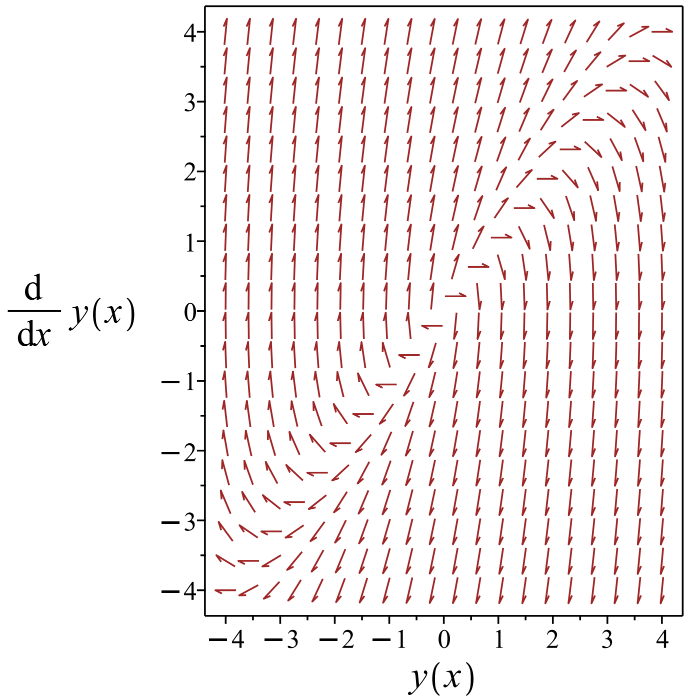

Figure 72: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x}$$

Verified OK.

5.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 47: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1 \left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2 \tag{1}
$$

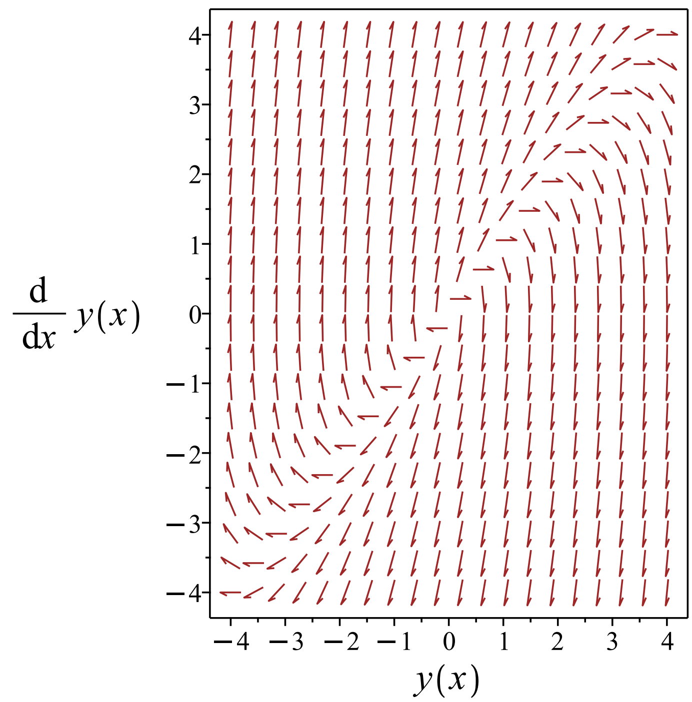

Figure 73: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2
$$

Verified OK.

5.2.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} + x\, e^{2x} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[y''[x]-4*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2x + c_1)$$

5.3 problem 3

5.3.1 Solving as second order linear constant coeff ode 336
5.3.2 Solving as second order integrable as is ode 337
5.3.3 Solving as second order ode missing y ode 339
5.3.4 Solving as type second_order_integrable_as_is (not using ABC version) 340
5.3.5 Solving using Kovacic algorithm 342
5.3.6 Solving as exact linear second order ode ode 346
5.3.7 Maple step by step solution 348

Internal problem ID [4793]
Internal file name [`OUTPUT/4286_Sunday_June_05_2022_12_54_52_PM_38244359/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 9y' = 0}$$

5.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 9, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\lambda \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 9, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-9}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{9^2 - (4)\,(1)\,(0)}\\ &= -\frac{9}{2} \pm \frac{9}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{9}{2} + \frac{9}{2}\\ \lambda_2 &= -\frac{9}{2} - \frac{9}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -9\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-9)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-9x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-9x} \tag{1}$$

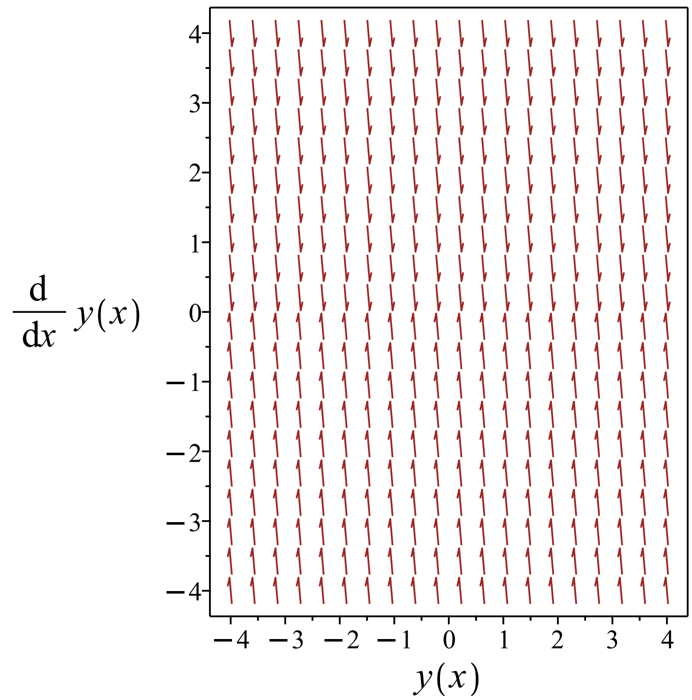

Figure 74: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-9x}$$

Verified OK.

5.3.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 9y')\, dx = 0$$
$$9y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$
\int \frac{1}{-9y + c_1} dy = \int dx
$$

$$
-\frac{\ln(-9y + c_1)}{9} = c_2 + x
$$

Raising both side to exponential gives

$$
\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = e^{c_2 + x}
$$

Which simplifies to

$$
\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = c_3 e^{x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{e^{-9x}}{9c_3^9} + \frac{c_1}{9} \tag{1}
$$

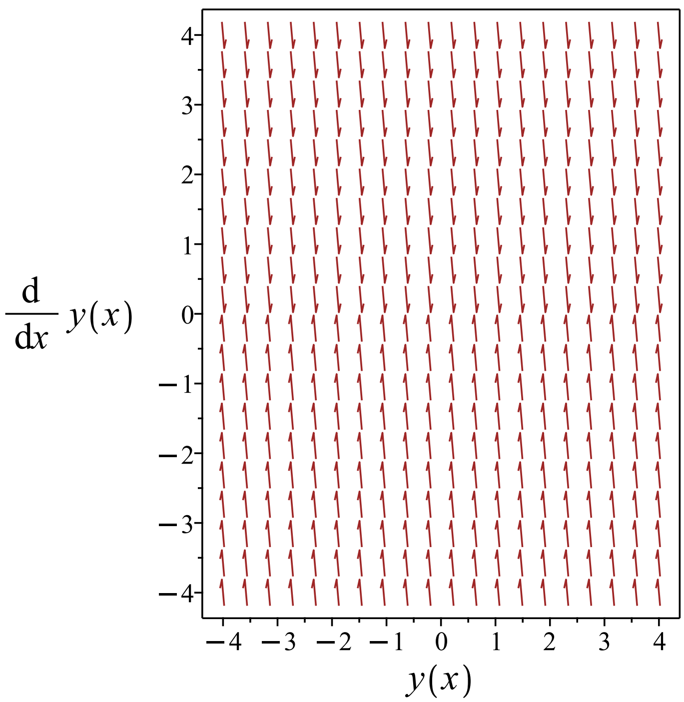

Figure 75: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-9x}}{9c_3^9} + \frac{c_1}{9}$$

Verified OK.

5.3.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 9p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int -\frac{1}{9p} dp &= \int dx \\ -\frac{\ln(p)}{9} &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{9}}} = \mathrm{e}^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{9}}} = c_2 \mathrm{e}^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{\mathrm{e}^{-9x}}{c_2^9}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{\mathrm{e}^{-9x}}{c_2^9} \,\mathrm{d}x \\ &= -\frac{\mathrm{e}^{-9x}}{9c_2^9} + c_3\end{aligned}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-9x}}{9c_2^9} + c_3 \tag{1}$$

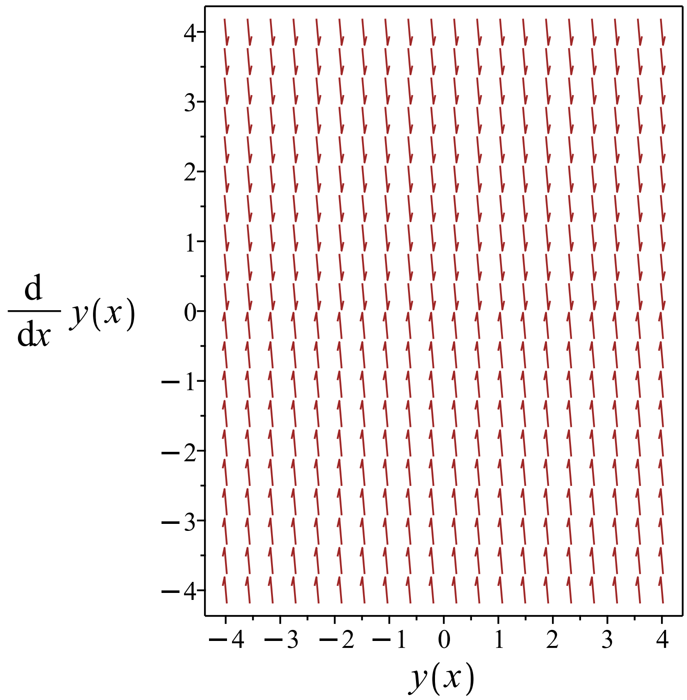

Figure 76: Slope field plot

Verification of solutions

$$y = -\frac{e^{-9x}}{9c_2^9} + c_3$$

Verified OK.

5.3.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 9y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 9y')\, dx = 0$$
$$9y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-9y + c_1} dy = \int dx$$
$$-\frac{\ln(-9y + c_1)}{9} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = e^{c_2 + x}$$

Which simplifies to

$$\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-9x}}{9c_3^9} + \frac{c_1}{9} \tag{1}$$

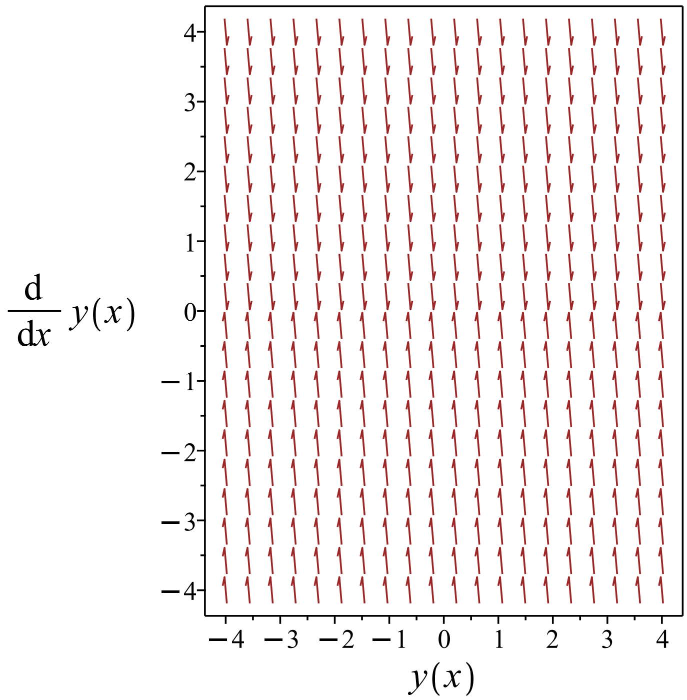

Figure 77: Slope field plot

Verification of solutions

$$y = -\frac{e^{-9x}}{9c_3^9} + \frac{c_1}{9}$$

Verified OK.

5.3.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 9 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{81z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 49: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{81}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{9x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{9}{1}\,dx} \\ &= z_1 e^{-\frac{9x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{9x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-9x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{9}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-9x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{9x}}{9}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-9x}\right) + c_2 \left(\mathrm{e}^{-9x} \left(\frac{\mathrm{e}^{9x}}{9}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-9x} + \frac{c_2}{9} \tag{1}
$$

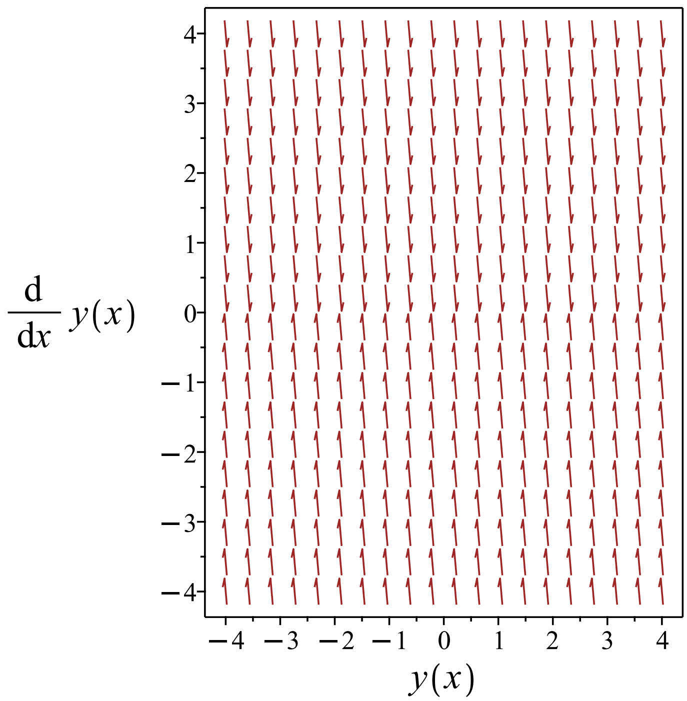

Figure 78: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-9x} + \frac{c_2}{9}
$$

Verified OK.

5.3.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 9 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(x)\,y' + (q(x) - p'(x))\,y\right)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$9y + y' = c_1$$

We now have a first order ode to solve which is

$$9y + y' = c_1$$

Integrating both sides gives

$$\int \frac{1}{-9y + c_1} dy = \int dx$$
$$-\frac{\ln(-9y + c_1)}{9} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = \mathrm{e}^{c_2 + x}$$

Which simplifies to

$$\frac{1}{(-9y + c_1)^{\frac{1}{9}}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-9x}}{9c_3^9} + \frac{c_1}{9} \tag{1}$$

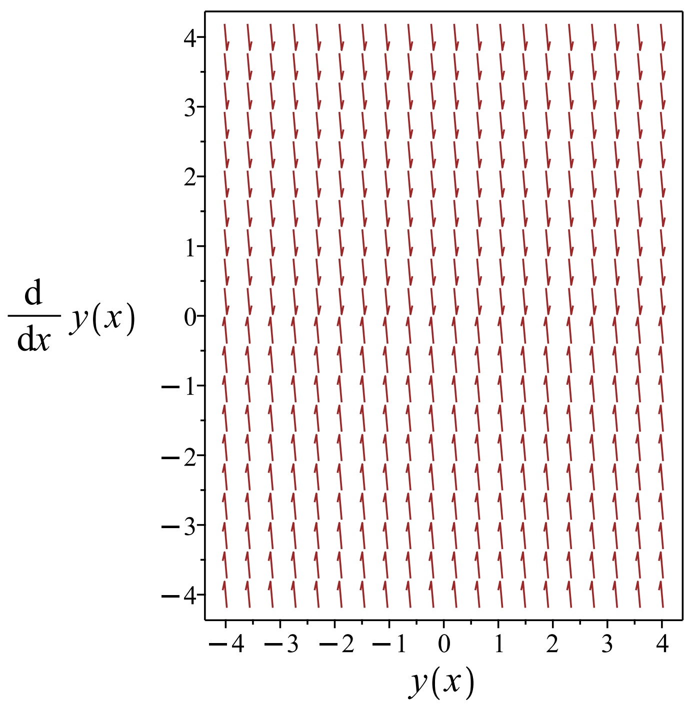

Figure 79: Slope field plot

Verification of solutions

$$
y = -\frac{e^{-9x}}{9c_3^9} + \frac{c_1}{9}
$$

Verified OK.

5.3.7 Maple step by step solution

Let's solve

$y'' + 9y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 9r = 0$

- Factor the characteristic polynomial

  $r(r + 9) = 0$

- Roots of the characteristic polynomial

  $r = (-9, 0)$

- 1st solution of the ODE

  $y_1(x) = e^{-9x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-9x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+9*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{-9x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 19

```
DSolve[y''[x]+9*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{9} c_1 e^{-9x}$$

5.4 problem 4

5.4.1 Solving as second order linear constant coeff ode 350
5.4.2 Solving using Kovacic algorithm 352
5.4.3 Maple step by step solution 356

Internal problem ID [4794]
Internal file name [`OUTPUT/4287_Sunday_June_05_2022_12_55_00_PM_28340165/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + 2y = 0}$$

5.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)} \\
&= -1 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -1 + i \\
\lambda_2 &= -1 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 + i \\
\lambda_2 &= -1 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(\cos(x) c_1 + c_2 \sin(x)) \tag{1}$$

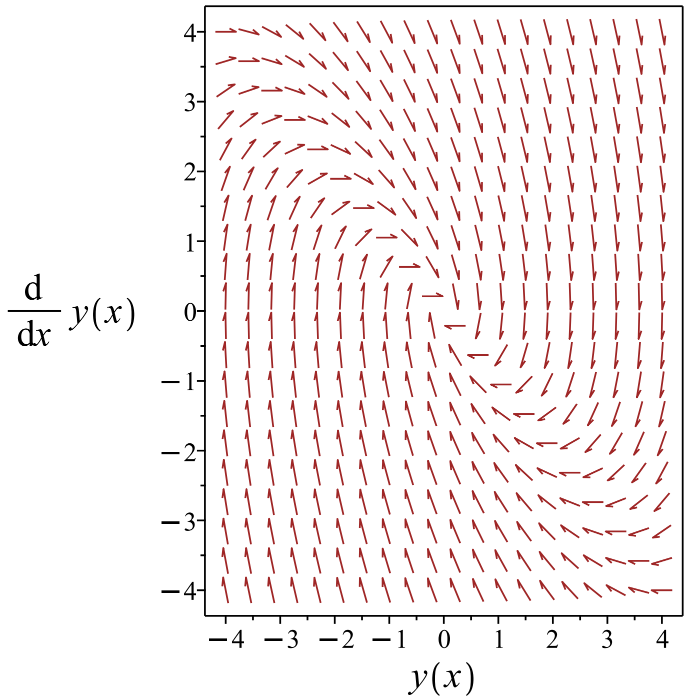

Figure 80: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x))$$

Verified OK.

5.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 51: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\cos(x)\right) + c_2\left(\mathrm{e}^{-x}\cos(x)\left(\tan(x)\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-x}\cos(x) + c_2\mathrm{e}^{-x}\sin(x) \tag{1}
$$

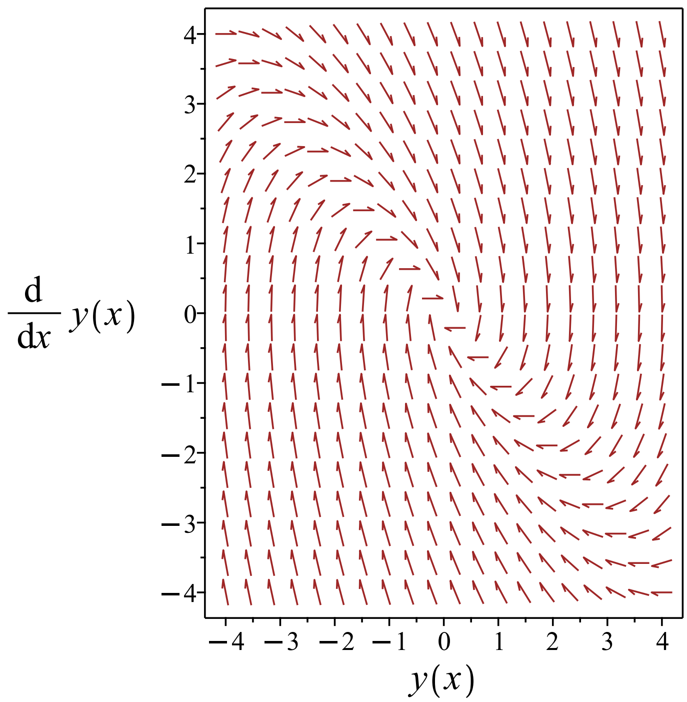

Figure 81: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{-x}\cos(x) + c_2\mathrm{e}^{-x}\sin(x)
$$

Verified OK.

5.4.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} \cos(x) + c_2 \mathrm{e}^{-x} \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-x}(c_1 \sin(x) + c_2 \cos(x))$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(c_2 \cos(x) + c_1 \sin(x))$$

5.5 problem 5

5.5.1 Solving as second order linear constant coeff ode 358
5.5.2 Solving using Kovacic algorithm 360
5.5.3 Maple step by step solution 364

Internal problem ID [4795]
Internal file name [`OUTPUT/4288_Sunday_June_05_2022_12_55_07_PM_98500309/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 6y = 0}$$

5.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(6)} \\ &= 1 \pm i\sqrt{5}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{5} \\ \lambda_2 &= 1 - i\sqrt{5}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{5} \\ \lambda_2 &= 1 - i\sqrt{5}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \sqrt{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x}\left(c_1 \cos\left(x\sqrt{5}\right) + c_2 \sin\left(x\sqrt{5}\right)\right) \tag{1}$$

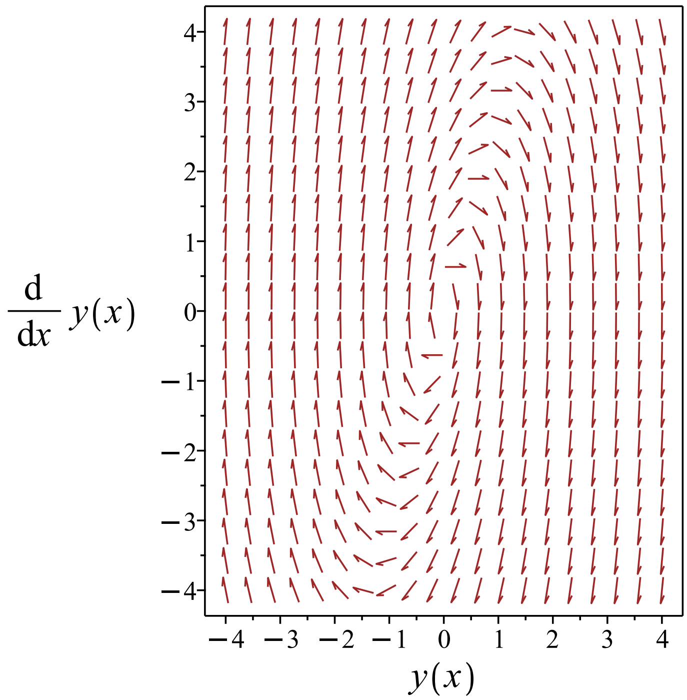

Figure 82: Slope field plot

Verification of solutions

$$y = e^x \left(c_1 \cos \left(x\sqrt{5} \right) + c_2 \sin \left(x\sqrt{5} \right) \right)$$

Verified OK.

5.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -5z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 53: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -5$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x} \cos\left(x\sqrt{5}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{5}\tan\left(x\sqrt{5}\right)}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x}\cos\left(x\sqrt{5}\right)\right) + c_2\left(\mathrm{e}^{x}\cos\left(x\sqrt{5}\right)\left(\frac{\sqrt{5}\,\tan\left(x\sqrt{5}\right)}{5}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x}\cos\left(x\sqrt{5}\right) + \frac{c_2 \mathrm{e}^{x}\sqrt{5}\,\sin\left(x\sqrt{5}\right)}{5} \tag{1}
$$

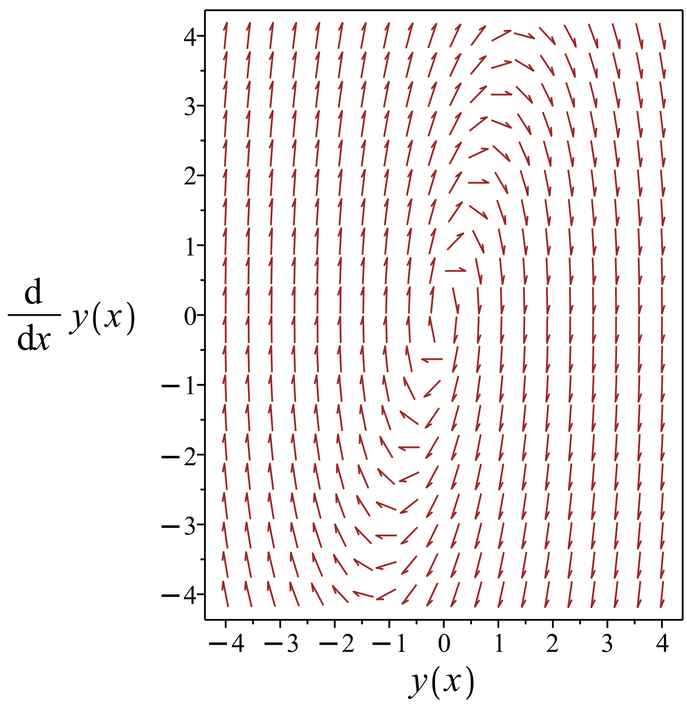

Figure 83: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{x}\cos\left(x\sqrt{5}\right) + \frac{c_2 \mathrm{e}^{x}\sqrt{5}\,\sin\left(x\sqrt{5}\right)}{5}
$$

Verified OK.

5.5.3 Maple step by step solution

Let's solve

$y'' - 2y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \mathrm{I}\sqrt{5}, 1 + \mathrm{I}\sqrt{5}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos\left(x\sqrt{5}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin\left(x\sqrt{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos\left(x\sqrt{5}\right) + c_2 \mathrm{e}^x \sin\left(x\sqrt{5}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x} \left(c_1 \sin\left(\sqrt{5}\, x\right) + c_2 \cos\left(\sqrt{5}\, x\right)\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x} \left(c_2 \cos\left(\sqrt{5} x\right) + c_1 \sin\left(\sqrt{5} x\right)\right)$$

5.6 problem 6

5.6.1 Solving as second order linear constant coeff ode 366
5.6.2 Solving as second order ode can be made integrable ode 368
5.6.3 Solving using Kovacic algorithm 370
5.6.4 Maple step by step solution 374

Internal problem ID [4796]
Internal file name [`OUTPUT/4289_Sunday_June_05_2022_12_55_15_PM_45865211/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 16y = 0}$$

5.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 16\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(16)} \\ &= \pm 4i\end{aligned}$$

Hence

$$\lambda_1 = +4i$$
$$\lambda_2 = -4i$$

Which simplifies to

$$\lambda_1 = 4i$$
$$\lambda_2 = -4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4x) + c_2 \sin(4x) \tag{1}$$

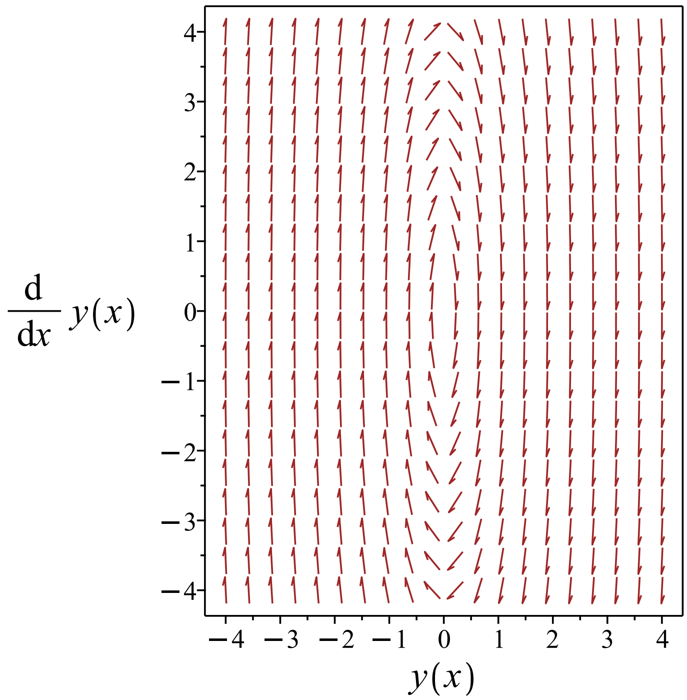

Figure 84: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Verified OK.

5.6.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 16y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 16y'y)\, dx = 0$$
$$\frac{y'^2}{2} + 8y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-16y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-16y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-16y^2+2c_1}}dy = \int dx$$

$$\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = c_2 + x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-16y^2+2c_1}}dy = \int dx$$

$$-\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = x + c_3$$

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = c_2 + x \tag{1}$$

$$-\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = x + c_3 \tag{2}$$

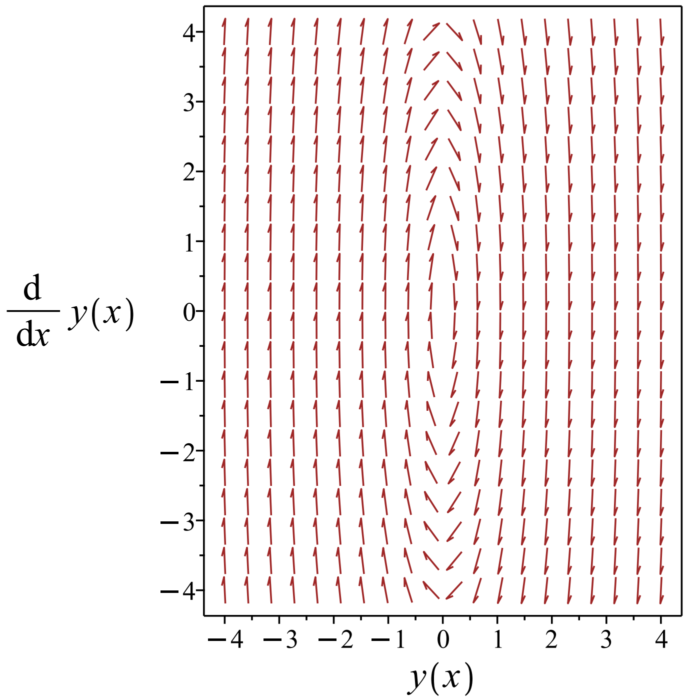

Figure 85: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = c_2 + x$$

Verified OK.

$$-\frac{\arctan\left(\frac{4y}{\sqrt{-16y^2+2c_1}}\right)}{4} = x + c_3$$

Verified OK.

5.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 55: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2}\, dx \\ &= \cos(4x) \left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2\left(\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} \tag{1}$$

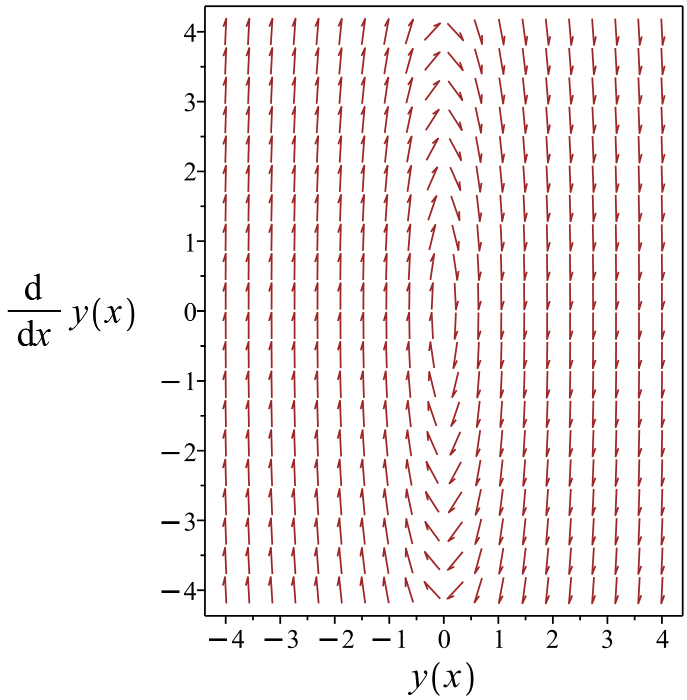

Figure 86: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

Verified OK.

5.6.4 Maple step by step solution

Let's solve

$y'' + 16y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$$r = (-4\,\mathrm{I}, 4\,\mathrm{I})$$

- 1st solution of the ODE

$$y_1(x) = \cos(4x)$$

- 2nd solution of the ODE

$$y_2(x) = \sin(4x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+16*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(4x) + c_2 \cos(4x)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 20

```
DSolve[y''[x]+16*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(4x) + c_2 \sin(4x)$$

5.7 problem 7

5.7.1 Solving as second order linear constant coeff ode 376
5.7.2 Solving using Kovacic algorithm 378
5.7.3 Maple step by step solution 382

Internal problem ID [4797]
Internal file name [`OUTPUT/4290_Sunday_June_05_2022_12_55_23_PM_78926956/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 5y' + 6y = 0}$$

5.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda \,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-5^2 - (4)(1)(6)}\\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(2)x}$$

Or

$$y = e^{3x} c_1 + c_2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{2x} \tag{1}$$

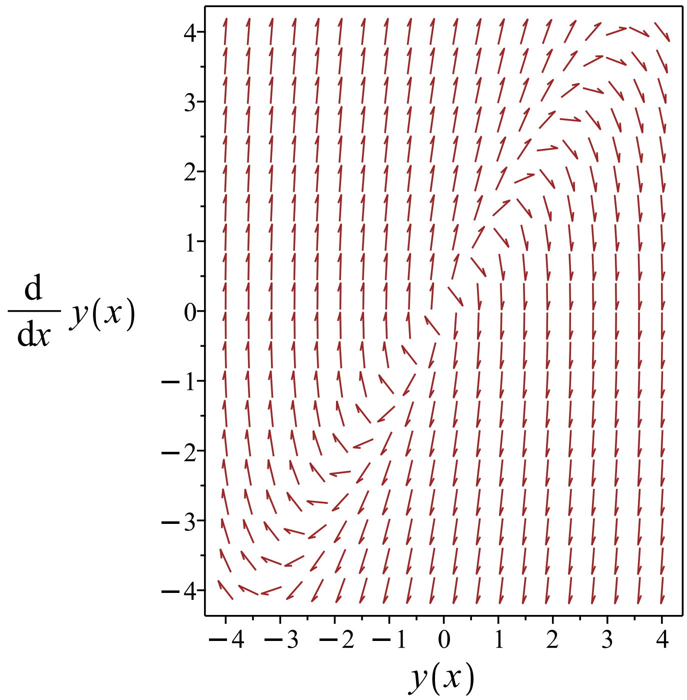

Figure 87: Slope field plot

<u>Verification of solutions</u>

$$y = e^{3x}c_1 + c_2 e^{2x}$$

Verified OK.

5.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx}\\ &= z_1 e^{\frac{5x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx\\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} (e^{x})\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 e^{2x} + c_2 e^{3x} \tag{1}
$$

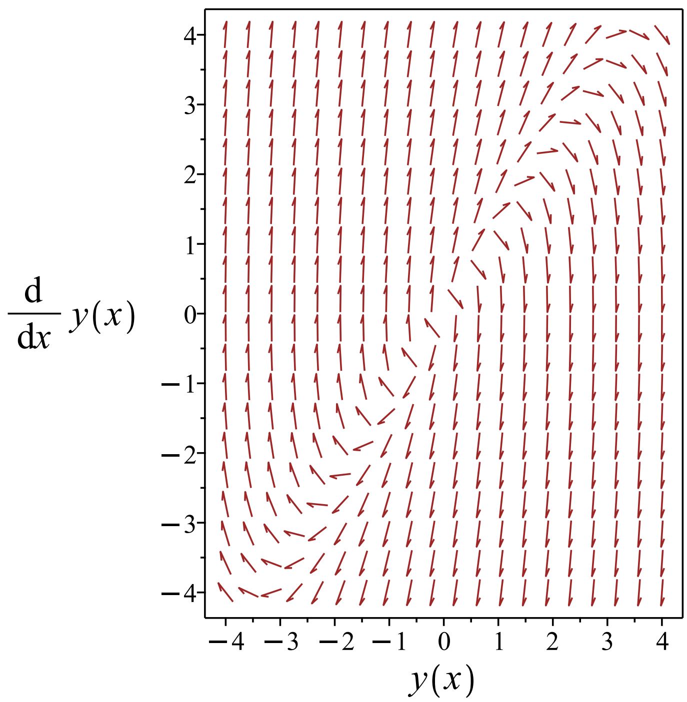

Figure 88: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 e^{2x} + c_2 e^{3x}
$$

Verified OK.

5.7.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{2x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{2x} c_1 + c_2 \mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[y''[x]-5*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 e^x + c_1)$$

5.8 problem 8

5.8.1 Solving as second order linear constant coeff ode 385
5.8.2 Solving as second order integrable as is ode 386
5.8.3 Solving as second order ode missing y ode 388
5.8.4 Solving as type second_order_integrable_as_is (not using ABC version) 389
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5.8.6 Solving as exact linear second order ode ode 395
5.8.7 Maple step by step solution 397

Internal problem ID [4798]
Internal file name [`OUTPUT/4291_Sunday_June_05_2022_12_55_30_PM_75179379/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' = 0}$$

5.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 5\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(0)}\\ &= -\frac{5}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{5}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-5x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-5x} \tag{1}$$

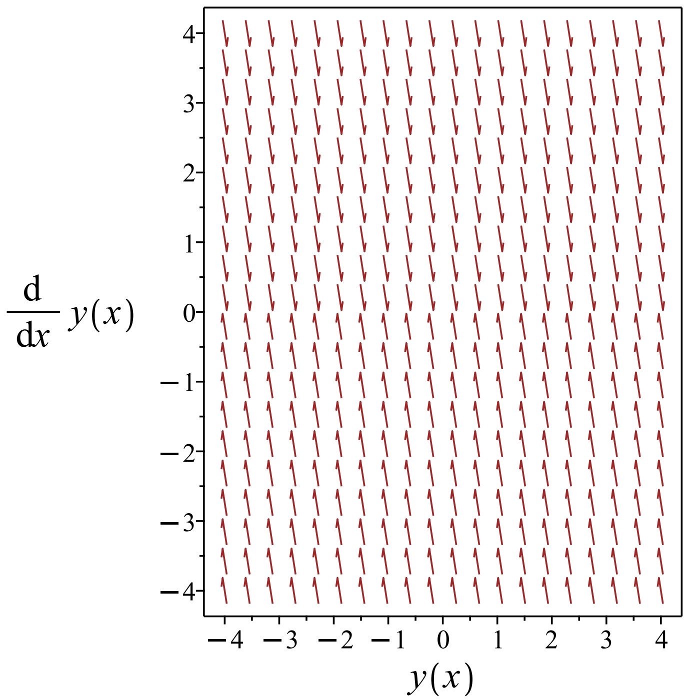

Figure 89: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-5x}$$

Verified OK.

5.8.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$
$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$
$$-\frac{\ln(-5y + c_1)}{5} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{c_2 + x}$$

Which simplifies to

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

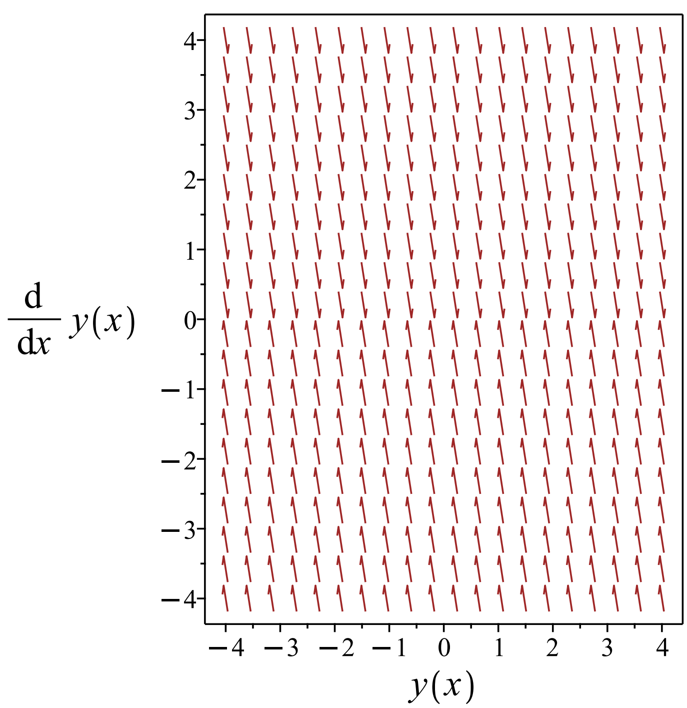

Figure 90: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

5.8.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 5p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{5p} dp = \int dx$$
$$-\frac{\ln(p)}{5} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{5}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{5}}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-5x}}{c_2^5}$$

Integrating both sides gives

$$y = \int \frac{e^{-5x}}{c_2^5} \, dx$$
$$= -\frac{e^{-5x}}{5c_2^5} + c_3$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3 \tag{1}$$

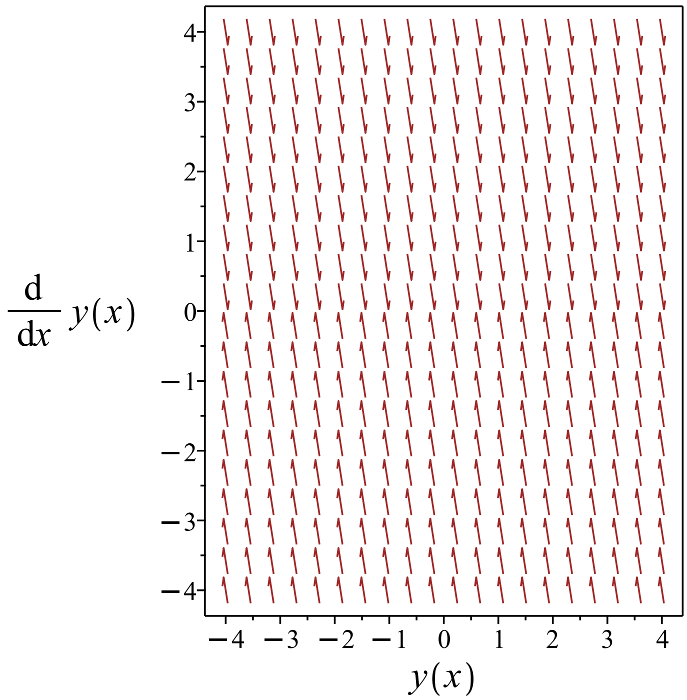

Figure 91: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3$$

Verified OK.

5.8.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 5y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$
$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$
$$-\frac{\ln(-5y + c_1)}{5} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{c_2 + x}$$

Which simplifies to

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

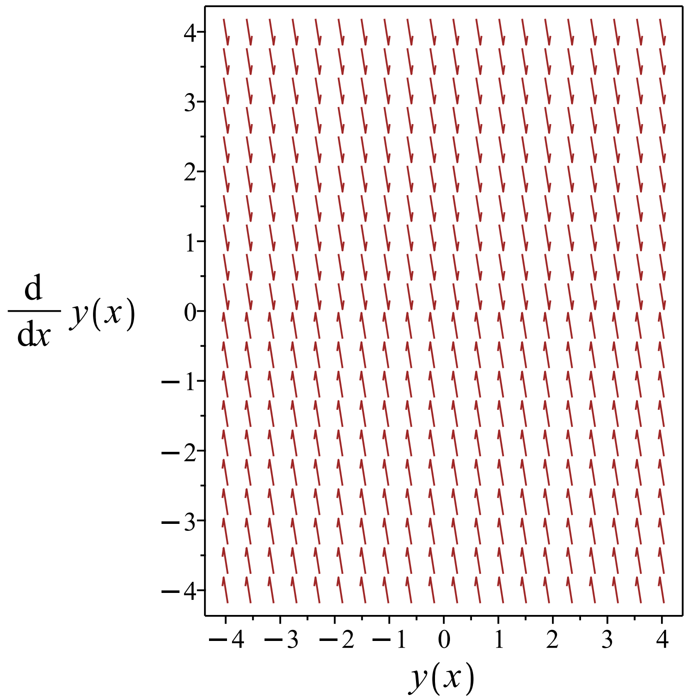

Figure 92: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

5.8.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 59: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-5x}\right) + c_2 \left(\mathrm{e}^{-5x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-5x} + \frac{c_2}{5} \tag{1}
$$

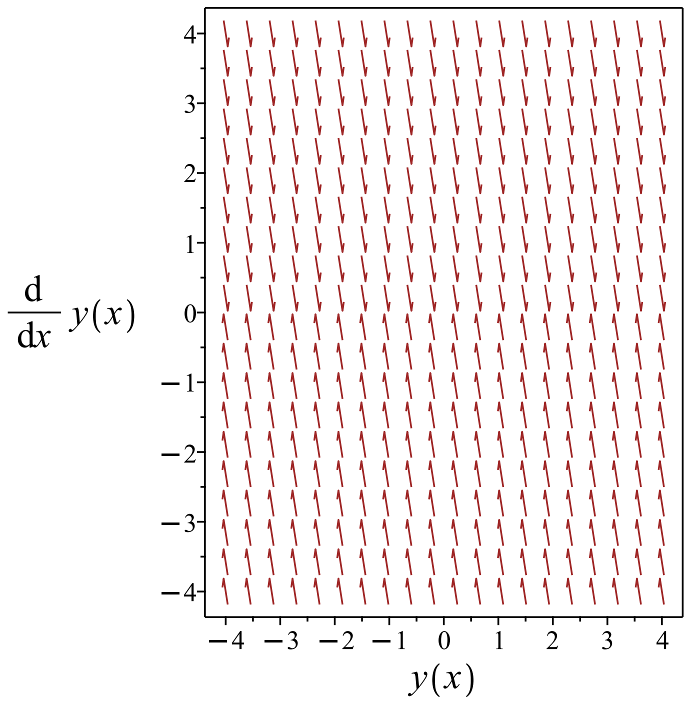

Figure 93: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-5x} + \frac{c_2}{5}
$$

Verified OK.

5.8.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= 5 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx$$

Substituting the above values for p, q, r, s gives

$$5y + y' = c_1$$

We now have a first order ode to solve which is

$$5y + y' = c_1$$

Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$
$$-\frac{\ln(-5y + c_1)}{5} = c_2 + x$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{c_2 + x}$$

Which simplifies to

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

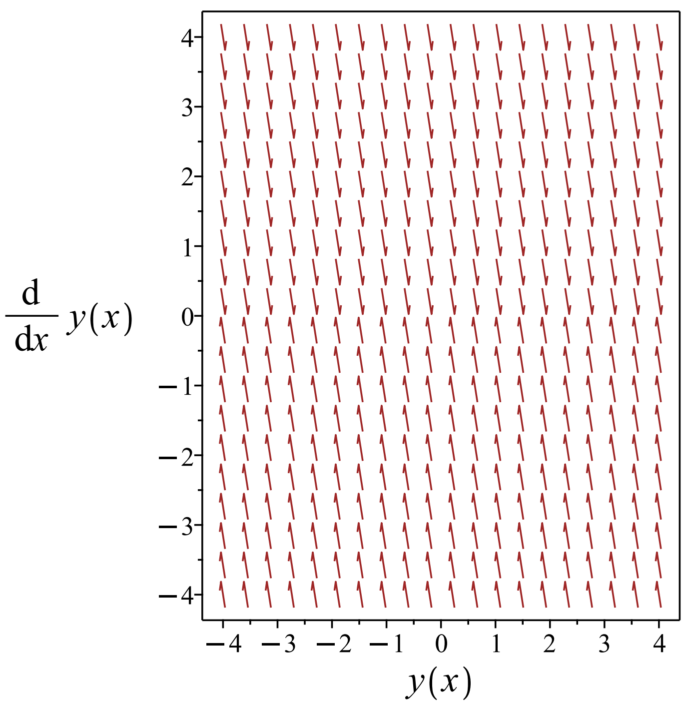

Figure 94: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

5.8.7 Maple step by step solution

Let's solve

$y'' + 5y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r = 0$

- Factor the characteristic polynomial

  $r(r+5) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 0)$

- 1st solution of the ODE

  $y_1(x) = e^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-5x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 19

```
DSolve[y''[x]+5*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{5} c_1 e^{-5x}$$

5.9 problem 9

5.9.1 Solving as second order linear constant coeff ode 399
5.9.2 Solving using Kovacic algorithm 401
5.9.3 Maple step by step solution 405

Internal problem ID [4799]
Internal file name [`OUTPUT/4292_Sunday_June_05_2022_12_55_38_PM_4563175/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 13y = 0}$$

5.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 13 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(13)} \\ &= 2 \pm 3i\end{aligned}$$

Hence

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

Which simplifies to

$$\lambda_1 = 2 + 3i$$
$$\lambda_2 = 2 - 3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 2$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{2x}(c_1 \cos (3x) + c_2 \sin (3x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{2x}(c_1 \cos (3x) + c_2 \sin (3x)) \tag{1}$$

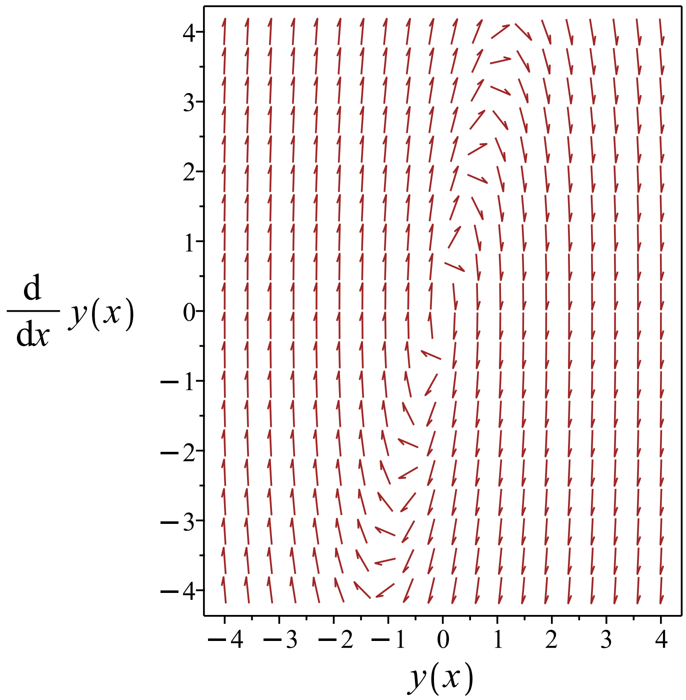

Figure 95: Slope field plot

Verification of solutions

$$
y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x))
$$

Verified OK.

5.9.2 Solving using Kovacic algorithm

Writing the ode as

$$
y'' - 4y' + 13y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -4 \\
C &= 13
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 61: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx}\\ &= z_1 e^{2x}\\ &= z_1\left(\mathrm{e}^{2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{2x}\cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\tan(3x)}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x} \cos(3x)\right) + c_2 \left(\mathrm{e}^{2x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{2x} \cos(3x) + \frac{c_2 \mathrm{e}^{2x} \sin(3x)}{3} \tag{1}
$$

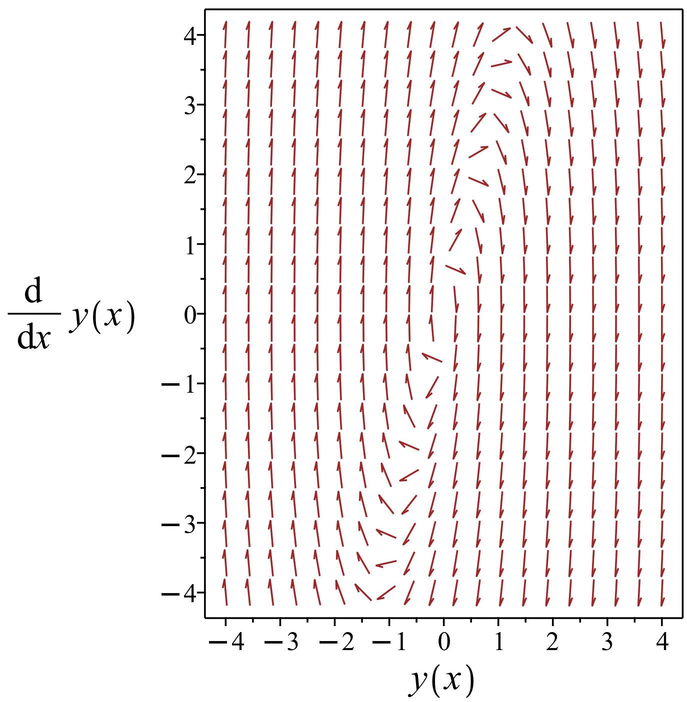

Figure 96: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{2x} \cos(3x) + \frac{c_2 \mathrm{e}^{2x} \sin(3x)}{3}
$$

Verified OK.

5.9.3 Maple step by step solution

Let's solve

$y'' - 4y' + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{4 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (2 - 3\,\mathrm{I}, 2 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{2x} \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{2x} \cos(3x) + c_2 \mathrm{e}^{2x} \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 26

```
DSolve[y''[x]-4*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(c_2 \cos(3x) + c_1 \sin(3x))$$

5.10 problem 12

5.10.1 Solving as second order linear constant coeff ode 407
5.10.2 Solving using Kovacic algorithm 409
5.10.3 Maple step by step solution 413

Internal problem ID [4800]
Internal file name [`OUTPUT/4293_Sunday_June_05_2022_12_55_46_PM_85544413/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + y' - y = 0}$$

5.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 1, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + \lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 1, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{1^2 - (4)\,(2)\,(-1)} \\ &= -\frac{1}{4} \pm \frac{3}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{4} + \frac{3}{4}$$
$$\lambda_2 = -\frac{1}{4} - \frac{3}{4}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2}$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{1}{2}\right)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{x}{2}} + c_2 \mathrm{e}^{-x} \tag{1}$$

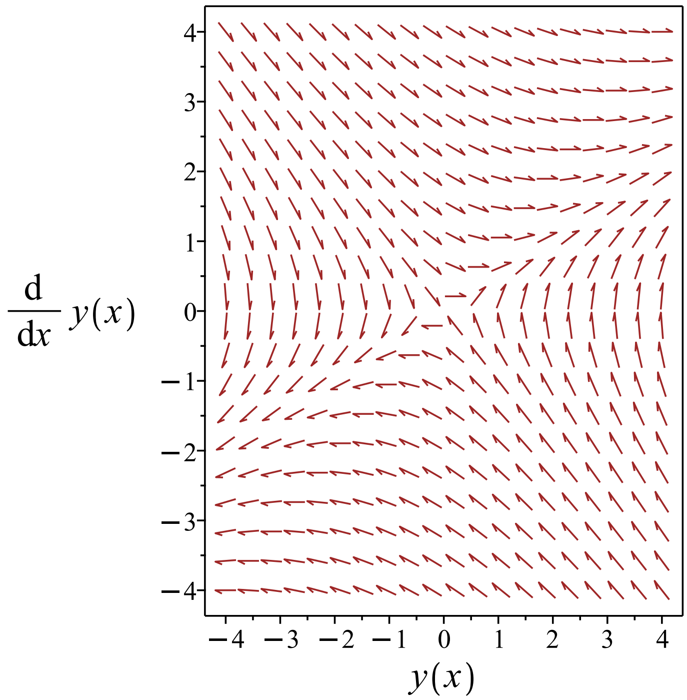

Figure 97: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{x}{2}} + c_2 e^{-x}$$

Verified OK.

5.10.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 1 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 63: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{2}\,dx}\\ &= z_1 e^{-\frac{x}{4}}\\ &= z_1\left(\mathrm{e}^{-\frac{x}{4}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\frac{x}{2}}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{2 c_2 \mathrm{e}^{\frac{x}{2}}}{3} \tag{1}
$$

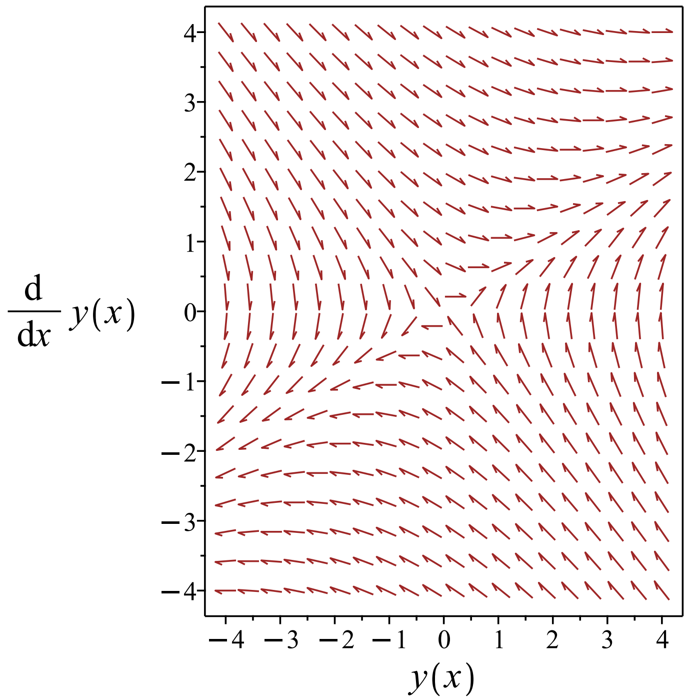

Figure 98: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-x} + \frac{2 c_2 \mathrm{e}^{\frac{x}{2}}}{3}
$$

Verified OK.

5.10.3 Maple step by step solution

Let's solve

$2y'' + y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2} + \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2} - \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{1}{2}r - \frac{1}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(r+1)(2r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(-1, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*diff(y(x),x$2)+diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 \mathrm{e}^{\frac{3x}{2}} + c_2\right) \mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 24

```
DSolve[2*y''[x]+y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x} \left(c_1 e^{3x/2} + c_2\right)$$

5.11 problem 19

5.11.1 Solving as second order linear constant coeff ode 415
5.11.2 Solving using Kovacic algorithm 417
5.11.3 Maple step by step solution 421

Internal problem ID [4801]
Internal file name [`OUTPUT/4294_Sunday_June_05_2022_12_55_53_PM_15240854/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' + (1+2i)\, y' + (-1+i)\, y = 0}$$

5.11.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1+2i, C = -1+i$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + (1+2i)\, \lambda\, \mathrm{e}^{\lambda x} + (-1+i)\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + (1+2i)\, \lambda - 1 + i = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1 + 2i, C = -1 + i$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1 - 2i}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1 + 2i^2 - (4)(1)(-1 + i)} \\ &= -\frac{1}{2} - i \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} - i + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - i - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -i \\ \lambda_2 &= -1 - i\end{aligned}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ &= c_1 e^{-ix} + c_2 e^{(-1-i)x}\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-ix} + c_2 \mathrm{e}^{(-1-i)x} \tag{1}$$

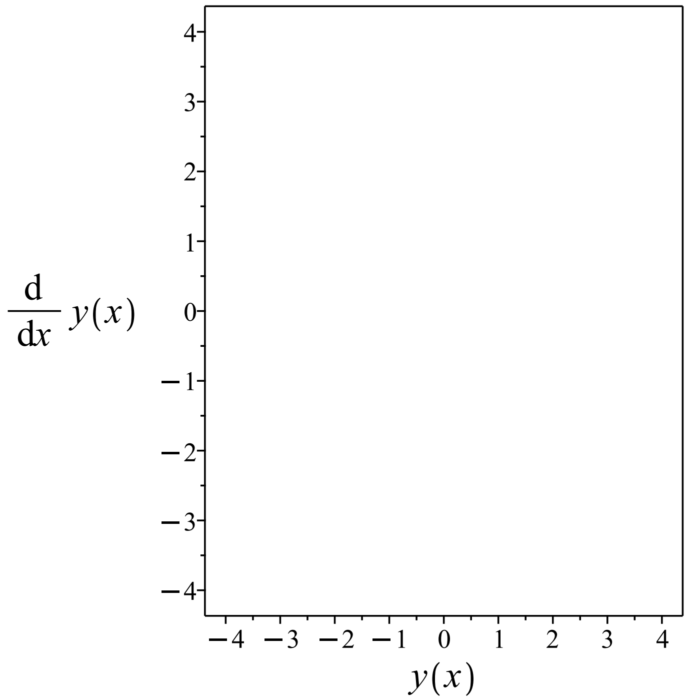

Figure 99: Slope field plot

Verification of solutions

$$y = c_1 e^{-ix} + c_2 e^{(-1-i)x}$$

Verified OK.

5.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (1+2i)\,y' + (-1+i)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1+2i \\ C &= -1+i \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 65: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+2i}{1}\,dx} \\
&= z_1 e^{\left(-\frac{1}{2}-i\right)x} \\
&= z_1 \left(\mathrm{e}^{\left(-\frac{1}{2}-i\right)x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{(-1-i)x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+2i}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{(-1-2i)x}}{(y_1)^2}\,dx \\
&= y_1 (\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{(-1-i)x}\right) + c_2\left(e^{(-1-i)x}\left(e^{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{(-1-i)x} + e^{-ix} c_2 \tag{1}
$$

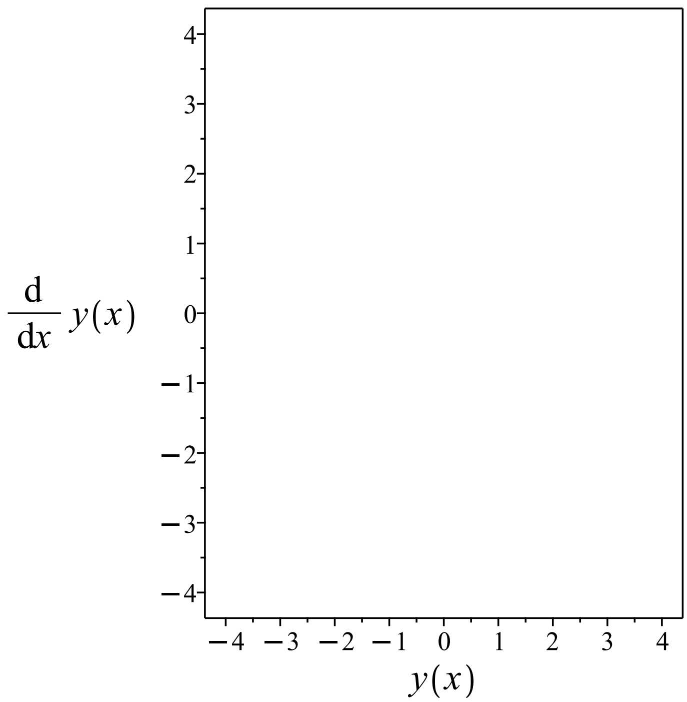

Figure 100: Slope field plot

Verification of solutions

$$
y = c_1 e^{(-1-i)x} + e^{-ix} c_2
$$

Verified OK.

5.11.3 Maple step by step solution

Let's solve

$y'' + (1+2\,\mathrm{I})\,y' + (-1+\mathrm{I})\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + (1+2\,\mathrm{I})\,r - 1 + \mathrm{I} = 0$

- Factor the characteristic polynomial

  $(r+\mathrm{I})\,(r+1+\mathrm{I}) = 0$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, -1-\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = \cos(x)\,c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+(1+2*I)*diff(y(x),x)+(I-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{(-1-i)x} + c_2 \mathrm{e}^{-ix}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+(1+2*I)*y'[x]+(I-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{(-1-i)x}(c_2 e^x + c_1)$$

5.12 problem 20

5.12.1 Solving as second order linear constant coeff ode 423
5.12.2 Solving using Kovacic algorithm 425
5.12.3 Maple step by step solution 429

Internal problem ID [4802]
Internal file name [`OUTPUT/4295_Sunday_June_05_2022_12_56_01_PM_40779441/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + (1+2i)\, y' + (-1+i)\, y = 0}$$

5.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1+2i, C = -1+i$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + (1+2i)\, \lambda\, \mathrm{e}^{\lambda x} + (-1+i)\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + (1+2i)\, \lambda - 1 + i = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1 + 2i, C = -1 + i$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1 - 2i}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1 + 2i^2 - (4)(1)(-1 + i)} \\ &= -\frac{1}{2} - i \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} - i + \frac{1}{2} \\ \lambda_2 &= -\frac{1}{2} - i - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -i \\ \lambda_2 &= -1 - i\end{aligned}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ &= c_1 e^{-ix} + c_2 e^{(-1-i)x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-ix} + c_2 \mathrm{e}^{(-1-i)x} \tag{1}$$

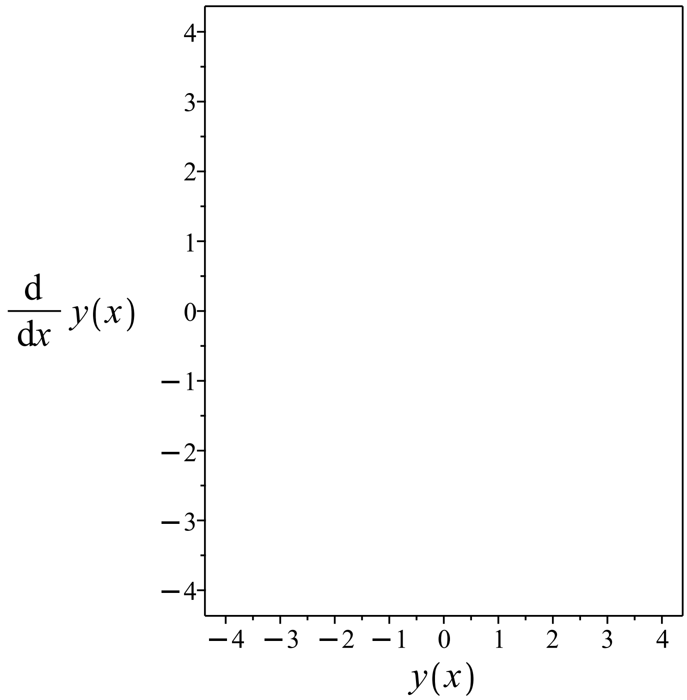

Figure 101: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-ix} + c_2 e^{(-1-i)x}$$

Verified OK.

5.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + (1+2i)\,y' + (-1+i)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1+2i \\ C &= -1+i \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 67: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1+2i}{1}\,dx} \\
&= z_1 e^{\left(-\frac{1}{2}-i\right)x} \\
&= z_1 \left(\mathrm{e}^{\left(-\frac{1}{2}-i\right)x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{(-1-i)x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1+2i}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{(-1-2i)x}}{(y_1)^2}\,dx \\
&= y_1 (\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{(-1-i)x}\right) + c_2\left(e^{(-1-i)x}\left(e^{x}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{(-1-i)x} + e^{-ix} c_2 \tag{1}
$$

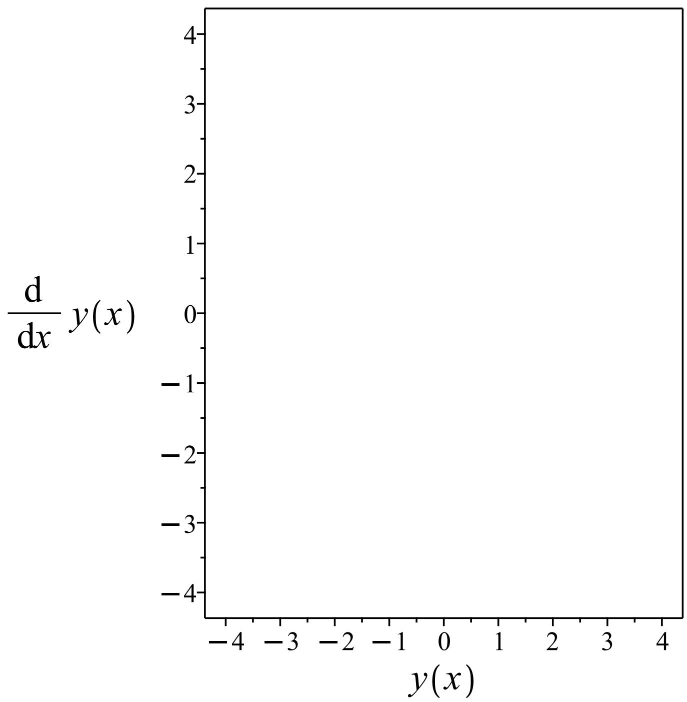

Figure 102: Slope field plot

Verification of solutions

$$
y = c_1 e^{(-1-i)x} + e^{-ix} c_2
$$

Verified OK.

5.12.3 Maple step by step solution

Let's solve

$y''+(1+2\,\mathrm{I})\,y'+(-1+\mathrm{I})\,y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+(1+2\,\mathrm{I})\,r-1+\mathrm{I}=0$

- Factor the characteristic polynomial

  $(r+\mathrm{I})\,(r+1+\mathrm{I})=0$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},-1-\mathrm{I})$

- 1st solution of the ODE

  $y_1(x)=\cos\,(x)$

- 2nd solution of the ODE

  $y_2(x)=\sin\,(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=\cos\,(x)\,c_1+c_2\sin\,(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+(1+2*I)*diff(y(x),x)+(I-1)*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{(-1-i)x} + c_2 \mathrm{e}^{-ix}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[y''[x]+(1+2*I)*y'[x]+(I-1)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{(-1-i)x}(c_2 e^x + c_1)$$

5.13 problem 24

5.13.1 Maple step by step solution 432

Internal problem ID [4803]
Internal file name [`OUTPUT/4296_Sunday_June_05_2022_12_56_08_PM_95641432/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 24.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y = 0}$$

The characteristic equation is

$$\lambda^3 + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -1$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$
$$\lambda_3 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{-x} + \mathrm{e}^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + \mathrm{e}^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{-x}$$
$$y_2 = e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x}$$
$$y_3 = e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 e^{-x} + e^{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)x} c_2 + e^{\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)x} c_3$$

Verified OK.

5.13.1 Maple step by step solution

Let's solve

$y''' + y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(x)$

    $y_1(x) = y$
  - ○ Define new variable $y_2(x)$

    $y_2(x) = y'$
  - ○ Define new variable $y_3(x)$

    $y_3(x) = y''$
  - ○ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_1(x)]$

- Define vector

$$\vec{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\vec{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right] \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{array}\right]$$

- Rewrite the system as

$\vec{y}'(x) = A \cdot \vec{y}(x)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right], \left[\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right], \left[\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)^2} \\ \frac{1}{\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair

$$\left[-1, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{\frac{x}{2}} \cdot \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{1}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\left(\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}\right)^{2}} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_{2}(x)=\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right], \vec{y}_{3}(x)=\mathrm{e}^{\frac{x}{2}} \cdot \left[\begin{array}{c} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}+\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2}-\frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y}=c_{1}\vec{y}_{1}+c_{2}\vec{y}_{2}(x)+c_{3}\vec{y}_{3}(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 e^{-x} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{\frac{x}{2}} \cdot \begin{bmatrix} -\frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \\ \cos\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix} + c_3 e^{\frac{x}{2}} \cdot \begin{bmatrix} \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ \frac{\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \\ -\sin\left(\frac{\sqrt{3}\,x}{2}\right) \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \left(-\frac{e^{\frac{3x}{2}}\left(-\sqrt{3}\,c_3 + c_2\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{\frac{3x}{2}}\left(c_2\sqrt{3} + c_3\right)\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + c_1\right) e^{-x}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$3)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{\frac{3x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + c_3 e^{\frac{3x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_1\right) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 56

```
DSolve[y'''[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}\left(c_3 e^{3x/2} \cos\left(\frac{\sqrt{3}x}{2}\right) + c_2 e^{3x/2} \sin\left(\frac{\sqrt{3}x}{2}\right) + c_1\right)$$

5.14 problem 25

5.14.1 Maple step by step solution 437

Internal problem ID [4804]
Internal file name [OUTPUT/4297_Sunday_June_05_2022_12_56_16_PM_48243206/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 25.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + y'' - 6y' = 0}$$

The characteristic equation is

$$\lambda^3 + \lambda^2 - 6\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = 2$$
$$\lambda_3 = -3$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{-3x} + \mathrm{e}^{2x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{-3x}$$
$$y_3 = \mathrm{e}^{2x}$$

Summary
The solution(s) found are the following

$$y = c_1 + c_2 e^{-3x} + e^{2x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{-3x} + e^{2x} c_3$$

Verified OK.

5.14.1 Maple step by step solution

Let's solve

$y''' + y'' - 6y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

- ☐ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -y_3(x) + 6y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -y_3(x) + 6y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\overrightarrow{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & -1 \end{bmatrix} \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & -1 \end{bmatrix}$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[-3, \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{y}_1 = \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3 = \mathrm{e}^{2x} \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-3x} \cdot \begin{bmatrix} \frac{1}{9} \\ -\frac{1}{3} \\ 1 \end{bmatrix} + \mathrm{e}^{2x} c_3 \cdot \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(9\,\mathrm{e}^{5x} c_3 + 36 c_2 \mathrm{e}^{3x} + 4 c_1)\,\mathrm{e}^{-3x}}{36}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$3)+diff(y(x),x$2)-6*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{5x} + c_1 e^{3x} + c_3\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 30

```
DSolve[y'''[x]+y''[x]-6*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3} c_1 e^{-3x} + \frac{1}{2} c_2 e^{2x} + c_3$$

5.15 problem 26

5.15.1 Maple step by step solution 443

Internal problem ID [4805]
Internal file name [`OUTPUT/4298_Sunday_June_05_2022_12_56_25_PM_92400865/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 26.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 3y'' - 9y' - 5y = 0}$$

The characteristic equation is

$$\lambda^3 + 3\lambda^2 - 9\lambda - 5 = 0$$

The roots of the above equation are

$$\lambda_1 = \left(-3 + i\sqrt{55}\right)^{\frac{1}{3}} + \frac{4}{\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}}} - 1$$

$$\lambda_2 = -\frac{\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}}} - 1 + \frac{i\sqrt{3}\left(\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}} - \frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}$$

$$\lambda_3 = -\frac{\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}}} - 1 - \frac{i\sqrt{3}\left(\left(-3 + i\sqrt{55}\right)^{\frac{1}{3}} - \frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x} c_1+\mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2+\mathrm{e}^{\left(-\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x}$$

$$y_2 = \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

$$y_3 = \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x}$$

<u>Summary</u>

The solution(s) found are the following

$$\begin{aligned} y = {} & \mathrm{e}^{\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x} c_1 \\ & + \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 \\ & + \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_3 \end{aligned} \tag{1}$$

Verification of solutions

$$
y = \mathrm{e}^{\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x} c_1 + \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_2 + \mathrm{e}^{\left(-\frac{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{i\sqrt{3}\left(\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+i\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} c_3
$$

Verified OK.

5.15.1 Maple step by step solution

Let's solve

$y''' + 3y'' - 9y' - 5y = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  - Define new variable $y_1(x)$

    $y_1(x) = y$

  - Define new variable $y_2(x)$

    $y_2(x) = y'$

  - Define new variable $y_3(x)$

    $y_3(x) = y''$

  - Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -3y_3(x) + 9y_2(x) + 5y_1(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -3y_3(x) + 9y_2(x) + 5y_1(x)]$

- Define vector

$$\overrightarrow{y}(x) = \left[\begin{array}{c} y_1(x) \\ y_2(x) \\ y_3(x) \end{array}\right]$$

- System to solve

$$\overrightarrow{y}'(x) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 9 & -3 \end{array}\right] \cdot \overrightarrow{y}(x)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 9 & -3 \end{array}\right]$$

- Rewrite the system as

$$\overrightarrow{y}'(x) = A \cdot \overrightarrow{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}} + \frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}} - 1, \left[\begin{array}{c} \frac{1}{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}} + \frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}} - 1\right)^{2}} \\ \frac{1}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}} + \frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}} - 1} \\ 1 \end{array}\right]\right], \left[-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}} \right.\right.$$

- Consider eigenpair

$$\left[\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1, \left[\begin{array}{c} \frac{1}{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1\right)^{2}} \\ \frac{1}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1\right)^{2}} \\ \frac{1}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{3}\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{\cdots}}{\cdots}\right.} \\ \frac{1}{-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{\cdots}}{\cdots}} \\ 1 \end{array}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_2 = \mathrm{e}^{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{3}\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{3}\cdots}{\cdots}\right.} \\ \frac{1}{-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1-\frac{\mathrm{I}\sqrt{3}\cdots}{\cdots}} \\ 1 \end{array}\right.$$

- Consider eigenpair

$$\left[-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{3}\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{}}{}\right.} \\ \frac{1}{-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{}}{}} \\ 1 \end{array}\right.\right.$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_3=\mathrm{e}^{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{3}\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}-\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}\right)}{2}\right)x}\cdot\left[\begin{array}{c} \frac{1}{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{3}}{}\right.} \\ \frac{1}{-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1+\frac{\mathrm{I}\sqrt{3}}{}} \\ 1 \end{array}\right.$$

- General solution to the system of ODEs

$\vec{y}=c_1\vec{y}_1+c_2\vec{y}_2+c_3\vec{y}_3$

- Substitute solutions into the general solution

$$\vec{y}=\mathrm{e}^{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1\right)x}c_1\cdot\left[\begin{array}{c} \frac{1}{\left(\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1\right)^2} \\ \frac{1}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}+\frac{4}{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}-1} \\ 1 \end{array}\right]+c_2\mathrm{e}^{\left(-\frac{\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{1}{3}}}{2}-\frac{2}{\left(-3+\mathrm{I}\sqrt{5}\right.}\right.}$$

- First component of the vector is the solution to the ODE

$$y = \frac{37 \left(c_2 \mathrm{e}^{\left(-1-2\sin\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)+2\sqrt{3}\,\cos\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)\right)x} \left(\frac{231\,\mathrm{I}}{37}+\frac{2\left(-3+\mathrm{I}\sqrt{55}\right)^{\frac{2}{3}}\left(-33\,\mathrm{I}+\sqrt{55}+33\sqrt{3}+\mathrm{I}\sqrt{165}\right)}{37}+\left(-3+\mathrm{I} \cdots \right.\right.\right.}{\cdots}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 105

```
dsolve(diff(y(x),x$3)+3*diff(y(x),x$2)-9*diff(y(x),x)-5*y(x)=0,y(x), singsol=all)
```

$$\begin{aligned} y(x) = {} & c_1 \mathrm{e}^{\left(-1-2\sin\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)+2\sqrt{3}\,\cos\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)\right)x} \\ & + c_2 \mathrm{e}^{-2\left(\sqrt{3}\,\cos\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)+\sin\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)+\frac{1}{2}\right)x} \\ & + c_3 \mathrm{e}^{\left(4\sin\left(\frac{\arctan\left(\frac{\sqrt{55}}{3}\right)}{3}+\frac{\pi}{6}\right)-1\right)x} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 87

```
DSolve[y'''[x]+3*y''[x]-9*y'[x]-5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) \to {} & c_2 \exp\left(x\mathrm{Root}\left[\#1^3+3\#1^2-9\#1-5\&,2\right]\right) \\ & + c_3 \exp\left(x\mathrm{Root}\left[\#1^3+3\#1^2-9\#1-5\&,3\right]\right) \\ & + c_1 \exp\left(x\mathrm{Root}\left[\#1^3+3\#1^2-9\#1-5\&,1\right]\right) \end{aligned}$$

5.16 problem 28

5.16.1 Maple step by step solution 449

Internal problem ID [4806]
Internal file name [`OUTPUT/4299_Sunday_June_05_2022_12_56_33_PM_41963450/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 5. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND ZERO RIGHT-HAND SIDE. page 414
Problem number: 28.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' + 4y = 0}$$

The characteristic equation is

$$\lambda^4 + 4 = 0$$

The roots of the above equation are

$$\begin{aligned}
\lambda_1 &= 1 - i \\
\lambda_2 &= 1 + i \\
\lambda_3 &= -1 - i \\
\lambda_4 &= -1 + i
\end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 \mathrm{e}^{(-1-i)x} + \mathrm{e}^{(-1+i)x} c_2 + \mathrm{e}^{(1+i)x} c_3 + \mathrm{e}^{(1-i)x} c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = e^{(-1-i)x}$$
$$y_2 = e^{(-1+i)x}$$
$$y_3 = e^{(1+i)x}$$
$$y_4 = e^{(1-i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{(-1-i)x} + e^{(-1+i)x} c_2 + e^{(1+i)x} c_3 + e^{(1-i)x} c_4 \tag{1}$$

Verification of solutions

$$y = c_1 e^{(-1-i)x} + e^{(-1+i)x} c_2 + e^{(1+i)x} c_3 + e^{(1-i)x} c_4$$

Verified OK.

5.16.1 Maple step by step solution

Let's solve

$y'''' + 4y = 0$

- Highest derivative means the order of the ODE is 4

  y''''

- □ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$

    $y_1(x) = y$
  - Define new variable $y_2(x)$

    $y_2(x) = y'$
  - Define new variable $y_3(x)$

    $y_3(x) = y''$
  - Define new variable $y_4(x)$

    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE

    $y_4'(x) = -4y_1(x)$

  Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = -4y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1-\mathrm{I}, \begin{bmatrix} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{1}{4}-\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1-\mathrm{I}, \begin{bmatrix} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[1+\mathrm{I}, \begin{bmatrix} -\frac{1}{4}-\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ \frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(-1-\mathrm{I})x} \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{-x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} \frac{1}{4}+\frac{\mathrm{I}}{4} \\ -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-x} \cdot \left[\begin{array}{c} \left(\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ -\frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(-\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_2(x) = \mathrm{e}^{-x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4}-\frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}, \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$\mathrm{e}^{(1-\mathrm{I})x} \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{x} \cdot (\cos(x) - \mathrm{I}\sin(x)) \cdot \left[\begin{array}{c} -\frac{1}{4}+\frac{\mathrm{I}}{4} \\ \frac{\mathrm{I}}{2} \\ \frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{x} \cdot \left[\begin{array}{c} \left(-\frac{1}{4}+\frac{\mathrm{I}}{4}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \frac{\mathrm{I}}{2}(\cos(x) - \mathrm{I}\sin(x)) \\ \left(\frac{1}{2}+\frac{\mathrm{I}}{2}\right)(\cos(x) - \mathrm{I}\sin(x)) \\ \cos(x) - \mathrm{I}\sin(x) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} -\frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ \cos(x) \end{array}\right], \vec{y}_4(x) = \mathrm{e}^{x} \cdot \left[\begin{array}{c} \frac{\cos(x)}{4}+\frac{\sin(x)}{4} \\ \frac{\cos(x)}{2} \\ \frac{\cos(x)}{2}-\frac{\sin(x)}{2} \\ -\sin(x) \end{array}\right]\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_1 \mathrm{e}^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} + \frac{\sin(x)}{4} \\ -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2} + \frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix} + c_2 \mathrm{e}^{-x} \cdot \begin{bmatrix} \frac{\cos(x)}{4} - \frac{\sin(x)}{4} \\ -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix} + c_3 \mathrm{e}^{x} \cdot \begin{bmatrix} -\frac{\cos(x)}{4} + \frac{\sin(x)}{4} \\ \frac{\sin(x)}{2} \\ \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \\ \cos(x) \end{bmatrix} + c_4 \mathrm{e}^{x} \cdot$$

- First component of the vector is the solution to the ODE

$$y = \frac{((c_1+c_2)\cos(x)+\sin(x)(c_1-c_2))\mathrm{e}^{-x}}{4} - \frac{((c_3-c_4)\cos(x)-\sin(x)(c_3+c_4))\mathrm{e}^{x}}{4}$$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$4)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(x) \mathrm{e}^{-x} + c_2 \cos(x) \mathrm{e}^{-x} + c_3 \mathrm{e}^{x} \sin(x) + c_4 \mathrm{e}^{x} \cos(x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 40

```
DSolve[y''''[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow e^{-x}\left(\left(c_4 e^{2x} + c_1\right)\cos(x) + \left(c_3 e^{2x} + c_2\right)\sin(x)\right)$$

6 Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422

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6.1 problem 1

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Internal problem ID [4807]
Internal file name [`OUTPUT/4300_Sunday_June_05_2022_12_56_41_PM_70665956/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' = 10}$$

6.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 0, f(x) = 10$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-4^2 - (4)(1)(0)} \\ &= 2 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 2 + 2 \\ \lambda_2 &= 2 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(4)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{4x}\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 10$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{4x} + c_2) + \left(-\frac{5x}{2}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 - \frac{5x}{2} \tag{1}$$

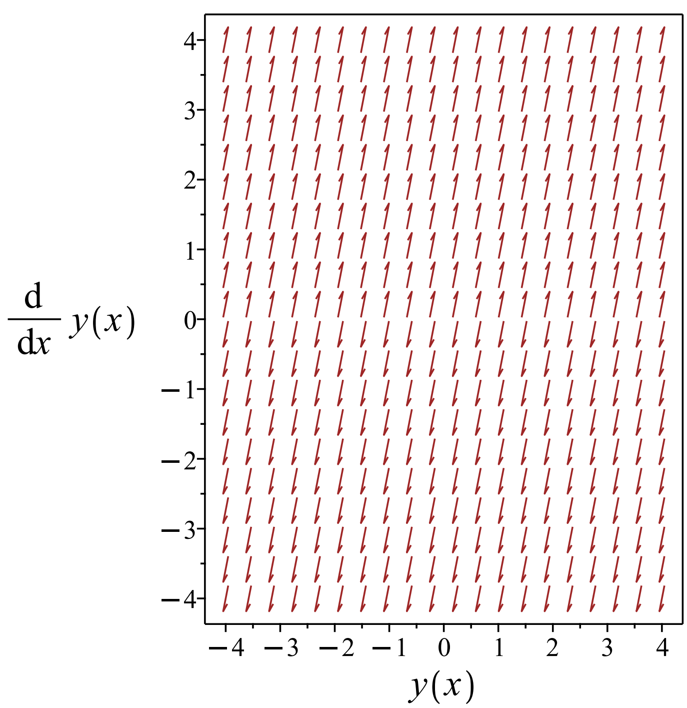

Figure 103: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{4x} + c_2 - \frac{5x}{2}
$$

Verified OK.

6.1.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (y'' - 4y')\, dx &= \int 10 dx \\
-4y + y' &= 10x + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= 10x + c_1
\end{aligned}
$$

Hence the ode is

$$
-4y + y' = 10x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-4) dx} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(10x + c_1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-4x} y\right) &= \left(\mathrm{e}^{-4x}\right)(10x + c_1) \\
\mathrm{d}\left(\mathrm{e}^{-4x} y\right) &= \left((10x + c_1)\,\mathrm{e}^{-4x}\right)\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-4x} y &= \int (10x + c_1)\,\mathrm{e}^{-4x}\, \mathrm{d}x \\
\mathrm{e}^{-4x} y &= -\frac{(20x + 2c_1 + 5)\,\mathrm{e}^{-4x}}{8} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$y = -\frac{e^{4x}(20x + 2c_1 + 5)\, e^{-4x}}{8} + c_2 e^{4x}$$

which simplifies to

$$y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x} \tag{1}$$

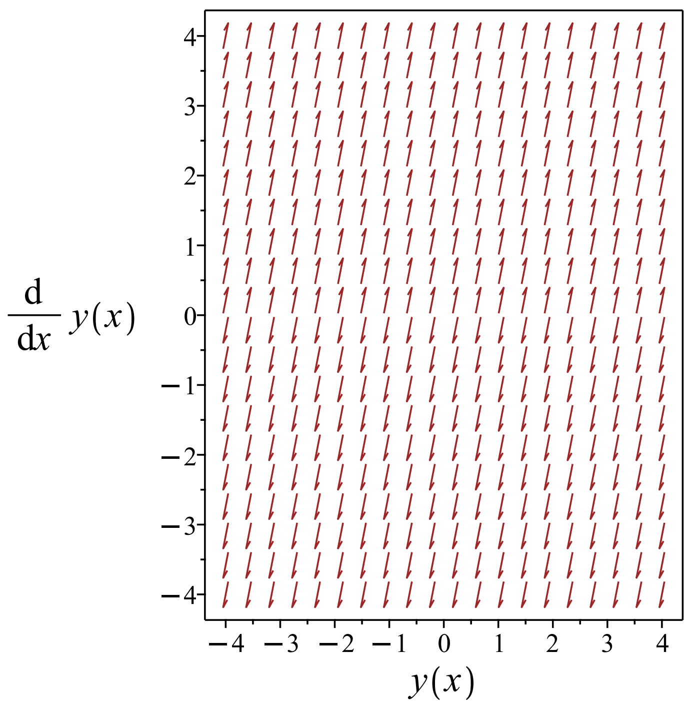

Figure 104: Slope field plot

Verification of solutions

$$y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x}$$

Verified OK.

6.1.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4p(x) - 10 = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int \frac{1}{4p+10}dp &= \int dx \\ \frac{\ln(2p+5)}{4} &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$(2p+5)^{\frac{1}{4}} = e^{x+c_1}$$

Which simplifies to

$$(2p+5)^{\frac{1}{4}} = c_2 e^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_2^4 e^{4x}}{2} - \frac{5}{2}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{c_2^4 e^{4x}}{2} - \frac{5}{2}\, \mathrm{d}x \\ &= -\frac{5x}{2} + \frac{c_2^4 e^{4x}}{8} + c_3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{5x}{2} + \frac{c_2^4 e^{4x}}{8} + c_3 \tag{1}$$

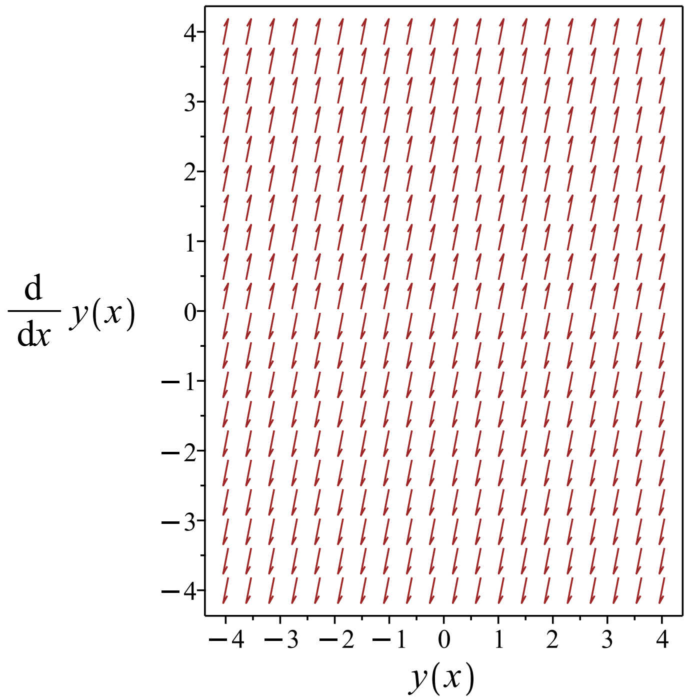

Figure 105: Slope field plot

Verification of solutions

$$y = -\frac{5x}{2} + \frac{c_2^4 e^{4x}}{8} + c_3$$

Verified OK.

6.1.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 4y' = 10$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 4y')\,dx = \int 10 dx$$

$$-4y + y' = 10x + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= 10x + c_1
\end{aligned}
$$

Hence the ode is

$$
-4y + y' = 10x + c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int (-4)dx} \\
&= e^{-4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(10x + c_1) \\
\frac{d}{dx}\left(e^{-4x} y\right) &= \left(e^{-4x}\right)(10x + c_1) \\
d\left(e^{-4x} y\right) &= \left((10x + c_1) e^{-4x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-4x} y &= \int (10x + c_1) e^{-4x} dx \\
e^{-4x} y &= -\frac{(20x + 2c_1 + 5) e^{-4x}}{8} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$
y = -\frac{e^{4x}(20x + 2c_1 + 5) e^{-4x}}{8} + c_2 e^{4x}
$$

which simplifies to

$$
y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x} \tag{1}
$$

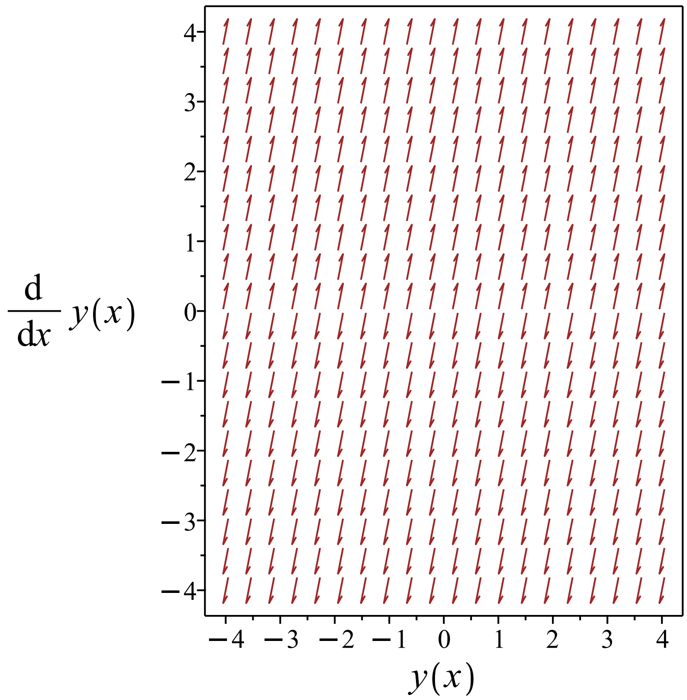

Figure 106: Slope field plot

<u>Verification of solutions</u>

$$y=-\frac{5x}{2}-\frac{c_1}{4}-\frac{5}{8}+c_2\mathrm{e}^{4x}$$

Verified OK.

6.1.5 Solving using Kovacic algorithm

Writing the ode as

$$y''-4y'=0 \tag{1}$$
$$Ay''+By'+Cy=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A&=1\\B&=-4\\C&=0\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x)=ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 73: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 e^{4x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{1, \frac{e^{4x}}{4}\right\}
$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 10$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{5x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 e^{4x}}{4}\right) + \left(-\frac{5x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{2} \tag{1}$$

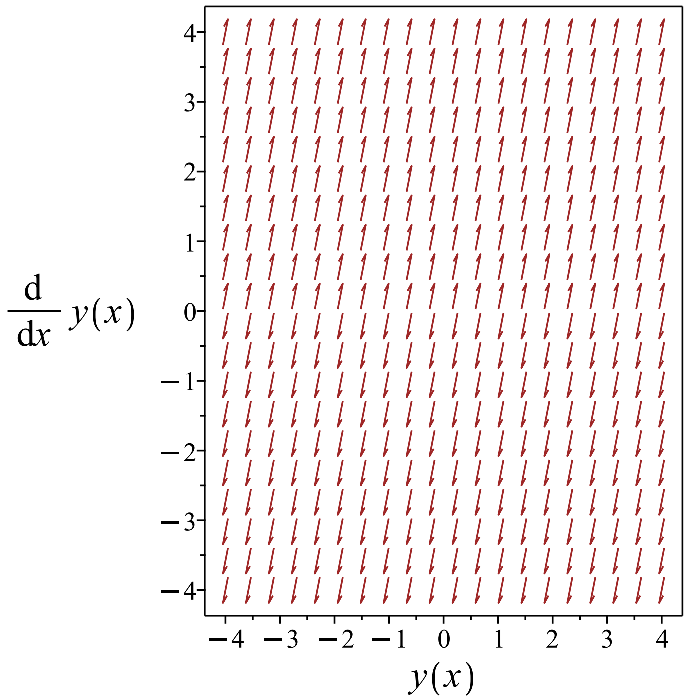

Figure 107: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 e^{4x}}{4} - \frac{5x}{2}$$

Verified OK.

6.1.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -4 \\ r(x) &= 0 \\ s(x) &= 10 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\;dx
$$

Substituting the above values for p,q,r,s gives

$$
-4y+y'=\int 10\,dx
$$

We now have a first order ode to solve which is

$$
-4y+y'=10x+c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -4 \\
q(x) &= 10x+c_1
\end{aligned}
$$

Hence the ode is

$$
-4y+y'=10x+c_1
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-4)dx} \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(10x + c_1) \\
\frac{d}{dx}\left(e^{-4x} y\right) &= \left(e^{-4x}\right)(10x + c_1) \\
d\left(e^{-4x} y\right) &= \left((10x + c_1) e^{-4x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-4x} y &= \int (10x + c_1) e^{-4x} dx \\
e^{-4x} y &= -\frac{(20x + 2c_1 + 5) e^{-4x}}{8} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-4x}$ results in

$$
y = -\frac{e^{4x}(20x + 2c_1 + 5) e^{-4x}}{8} + c_2 e^{4x}
$$

which simplifies to

$$
y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x} \tag{1}
$$

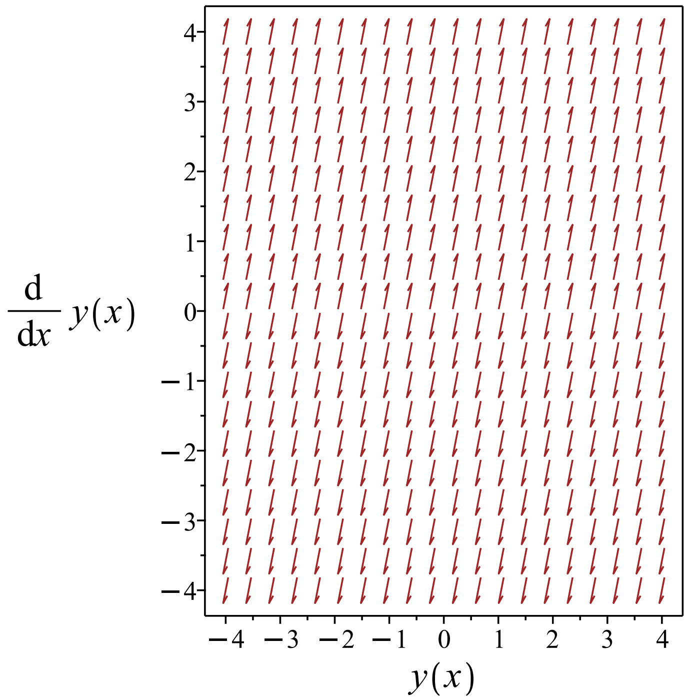

Figure 108: Slope field plot

Verification of solutions

$$y = -\frac{5x}{2} - \frac{c_1}{4} - \frac{5}{8} + c_2 e^{4x}$$

Verified OK.

6.1.7 Maple step by step solution

Let's solve

$y'' - 4y' = 10$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r = 0$

- Factor the characteristic polynomial

  $r(r-4) = 0$

- Roots of the characteristic polynomial

$r = (0, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 10\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{4x} \\ 0 & 4\,e^{4x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,e^{4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{5\left(\int 1 dx\right)}{2} + \frac{5\,e^{4x}\left(\int e^{-4x} dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = -\frac{5x}{2} - \frac{5}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{4x} - \frac{5x}{2} - \frac{5}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 4*_b(_a)+10, _b(_a)`   *** Sublevel 2 *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)=10,y(x), singsol=all)
```

$$y(x) = \frac{e^{4x} c_1}{4} - \frac{5x}{2} + c_2$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[y''[x]-4*y'[x]==10,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5x}{2} + \frac{1}{4} c_1 e^{4x} + c_2$$

6.2 problem 2

6.2.1 Solving as second order linear constant coeff ode 476
6.2.2 Solving as linear second order ode solved by an integrating factor ode 479
6.2.3 Solving using Kovacic algorithm 481
6.2.4 Maple step by step solution 486

Internal problem ID [4808]
Internal file name [`OUTPUT/4301_Sunday_June_05_2022_12_56_50_PM_41366954/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 4y = 16}$$

6.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 16$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\, \mathrm{e}^{\lambda x} + 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)} \\
&= 2
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + x\, \mathrm{e}^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}x, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 16$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + x\, e^{2x} c_2\right) + (4) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + 4$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + 4 \tag{1}$$

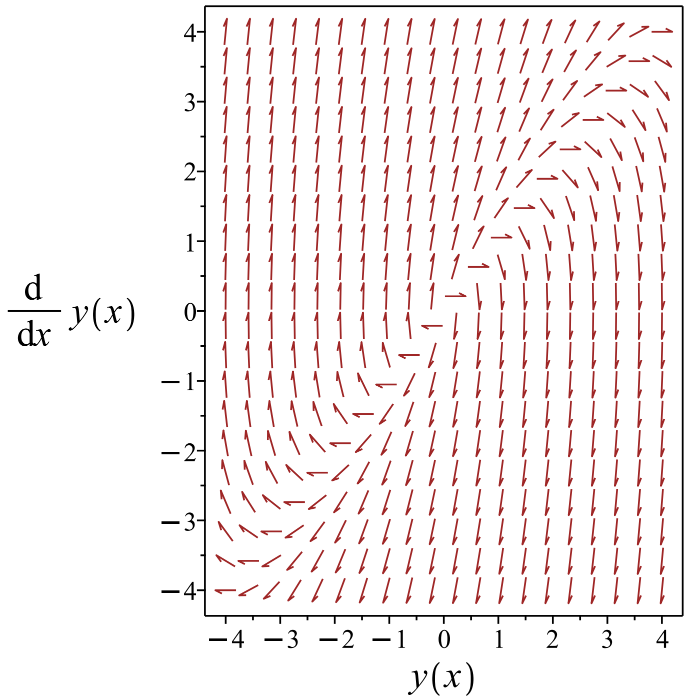

Figure 109: Slope field plot

<u>Verification of solutions</u>

$$y = e^{2x}(c_2 x + c_1) + 4$$

Verified OK.

6.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4\, dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 16\,\mathrm{e}^{-2x}$$
$$\left(y\,\mathrm{e}^{-2x}\right)'' = 16\,\mathrm{e}^{-2x}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{-2x}\right)' = -8\,\mathrm{e}^{-2x} + c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{-2x}\right) = c_1 x + 4\,\mathrm{e}^{-2x} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + 4\,\mathrm{e}^{-2x} + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} + 4$$

Summary
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{2x} + c_2 \mathrm{e}^{2x} + 4 \tag{1}$$

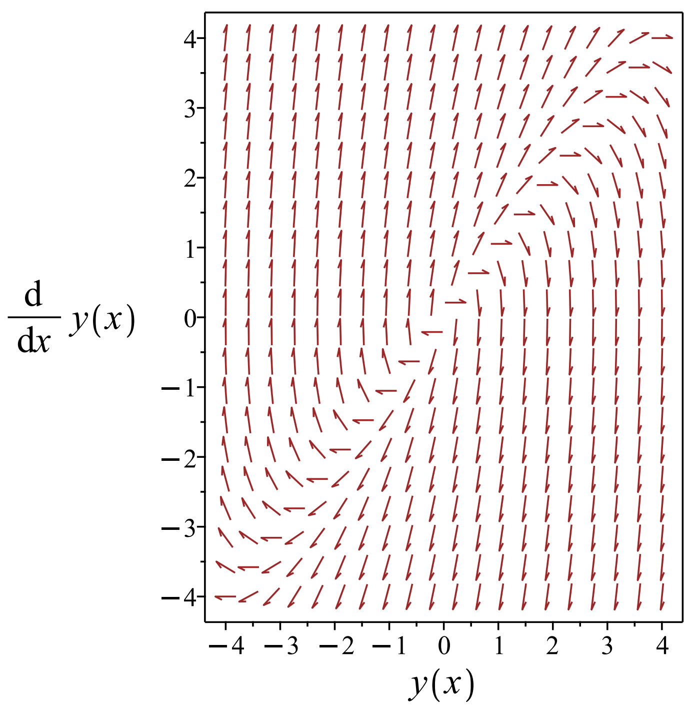

Figure 110: Slope field plot

Verification of solutions

$$y = c_1 x \, e^{2x} + c_2 e^{2x} + 4$$

Verified OK.

6.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 75: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{2x}\right) + c_2\left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 16$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2\right) + (4) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2x + c_1) + 4$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2x + c_1) + 4 \tag{1}$$

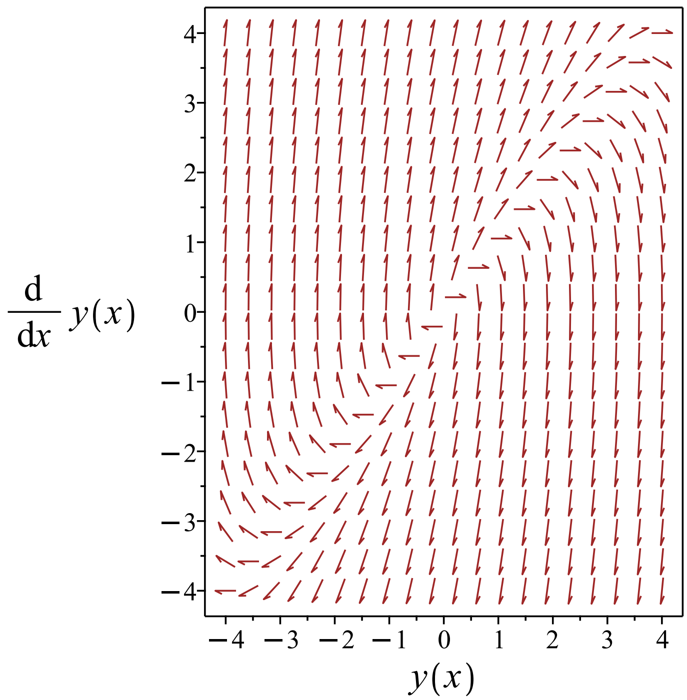

Figure 111: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + 4$$

Verified OK.

6.2.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 16$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{2x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + x\, e^{2x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 16\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & e^{2x} x \\ 2\, e^{2x} & 2\, e^{2x} x + e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -16\, e^{2x}\left(\int x\, e^{-2x} dx - \left(\int e^{-2x} dx\right) x\right)$

- Compute integrals

  $y_p(x) = 4$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{2x}c_2 + c_1e^{2x} + 4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=16,y(x), singsol=all)
```

$$y(x) = 4 + (c_1x + c_2)\,e^{2x}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[y''[x]-4*y'[x]+4*y[x]==16,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 4 + e^{2x}(c_2x + c_1)$$

6.3 problem 3

6.3.1 Solving as second order linear constant coeff ode 488
6.3.2 Solving using Kovacic algorithm 491
6.3.3 Maple step by step solution 496

Internal problem ID [4809]
Internal file name [`OUTPUT/4302_Sunday_June_05_2022_12_56_59_PM_53647118/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y' - 2y = e^{2x}}$$

6.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \, \mathrm{e}^{\lambda x} - 2 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)} \\
&= -\frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^x + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^x, e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + c_2 e^{-2x}\right) + \left(\frac{e^{2x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-2x} + \frac{e^{2x}}{4} \tag{1}$$

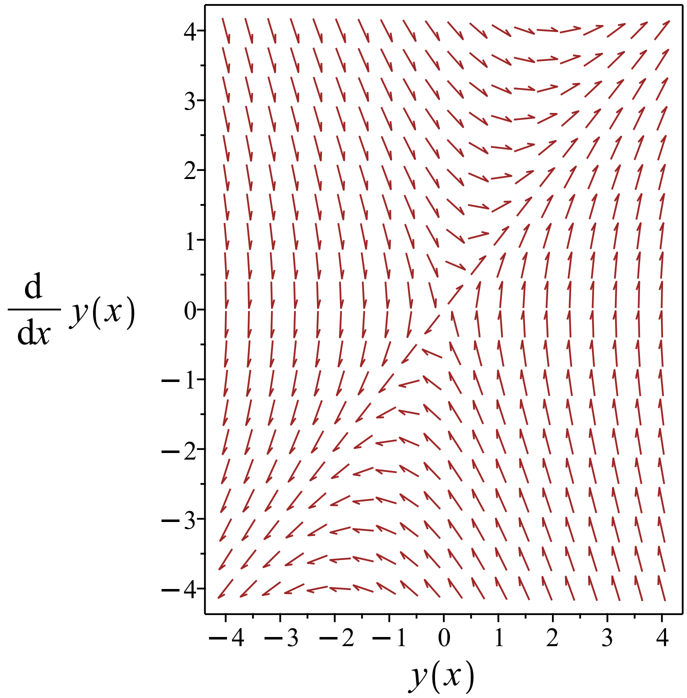

Figure 112: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-2x} + \frac{e^{2x}}{4}$$

Verified OK.

6.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= -2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 77: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{x}}{3}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{2x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{x}}{3}\right) + \left(\frac{e^{2x}}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} + \frac{e^{2x}}{4} \tag{1}$$

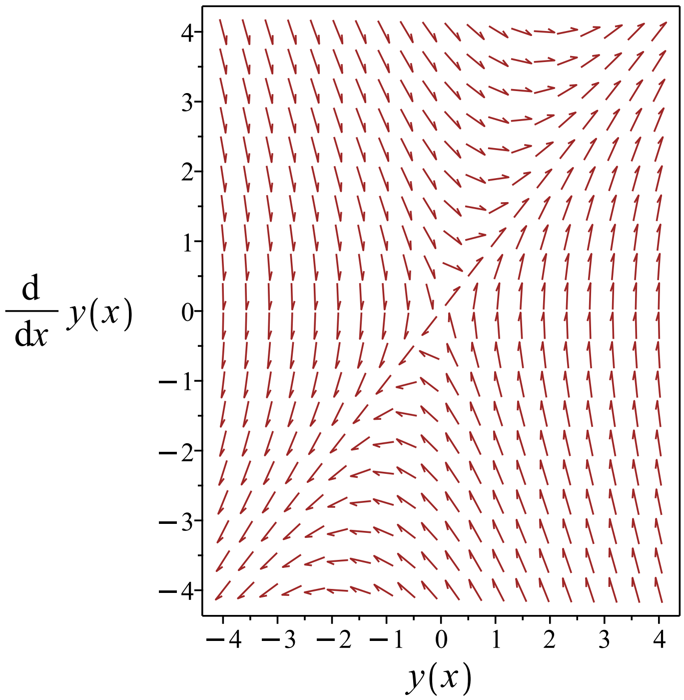

Figure 113: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^x}{3} + \frac{e^{2x}}{4}$$

Verified OK.

6.3.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 1) = 0$

- Roots of the characteristic polynomial

$r = (-2, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{(\mathrm{e}^{3x}(\int \mathrm{e}^{x} dx) - (\int \mathrm{e}^{4x} dx))\mathrm{e}^{-2x}}{3}$

- ○ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{2x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{x} + \frac{\mathrm{e}^{2x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(\mathrm{e}^{4x} + 4c_2\mathrm{e}^{3x} + 4c_1)\,\mathrm{e}^{-2x}}{4}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 29

```
DSolve[y''[x]+y'[x]-2*y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x}}{4} + c_1e^{-2x} + c_2e^x$$

6.4 problem 4

6.4.1 Solving as second order linear constant coeff ode 499
6.4.2 Solving using Kovacic algorithm 502
6.4.3 Maple step by step solution 507

Internal problem ID [4810]
Internal file name [`OUTPUT/4303_Sunday_June_05_2022_12_57_08_PM_35303653/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' - 3y = 24\,\mathrm{e}^{-3x}}$$

6.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -3, f(x) = 24\,\mathrm{e}^{-3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} - 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-3)}\\ &= 1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2\\ \lambda_2 &= 1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = e^{3x} c_1 + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24\,\mathrm{e}^{-3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}, \mathrm{e}^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12 A_1 \mathrm{e}^{-3x} = 24\,\mathrm{e}^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}\right) + \left(2\,\mathrm{e}^{-3x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x} + 2\,\mathrm{e}^{-3x} \tag{1}$$

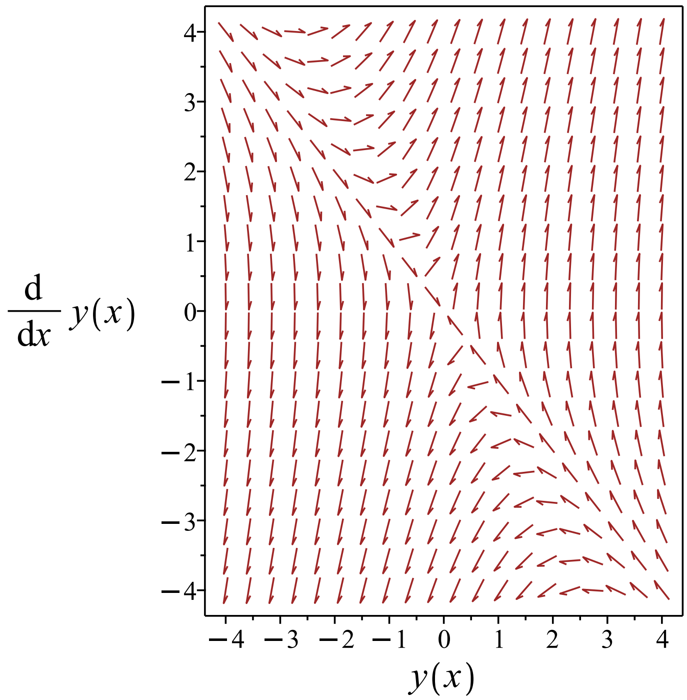

Figure 114: Slope field plot

Verification of solutions

$$y = e^{3x} c_1 + c_2 e^{-x} + 2\,e^{-3x}$$

Verified OK.

6.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 79: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' - 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
24\,\mathrm{e}^{-3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-3x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3x}}{4}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{-3x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^{-3x} = 24\,e^{-3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^{-3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{3x}}{4}\right) + \left(2\,e^{-3x}\right) \end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} + 2\,e^{-3x} \tag{1}$$

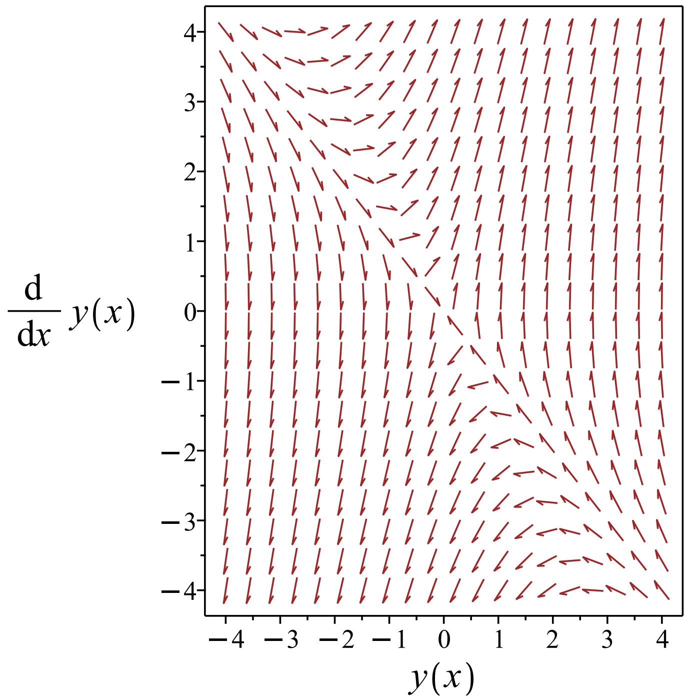

Figure 115: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} + 2\,e^{-3x}$$

Verified OK.

6.4.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 24\,e^{-3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

$r = (-1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 24 e^{-3x}\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3 e^{3x} \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4 e^{2x}$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -6 e^{-x}\left(\int e^{-2x} dx\right) + 6 e^{3x}\left(\int e^{-6x} dx\right)$
- ○ Compute integrals

  $y_p(x) = 2 e^{-3x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{3x} + 2 e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=24*exp(-3*x),y(x), singsol=all)
```

$$y(x) = \left(e^{6x} c_1 + c_2 e^{2x} + 2\right) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 29

```
DSolve[y''[x]-2*y'[x]-3*y[x]==24*Exp[-3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(c_1 e^{2x} + c_2 e^{6x} + 2\right)$$

6.5 problem 5

6.5.1 Solving as second order linear constant coeff ode 510
6.5.2 Solving using Kovacic algorithm 513
6.5.3 Maple step by step solution 518

Internal problem ID [4811]
Internal file name [`OUTPUT/4304_Sunday_June_05_2022_12_57_16_PM_65066005/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' + y = 2\,\mathrm{e}^{x}}$$

6.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{x} = 2\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (\mathrm{e}^{x}) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x \tag{1}$$

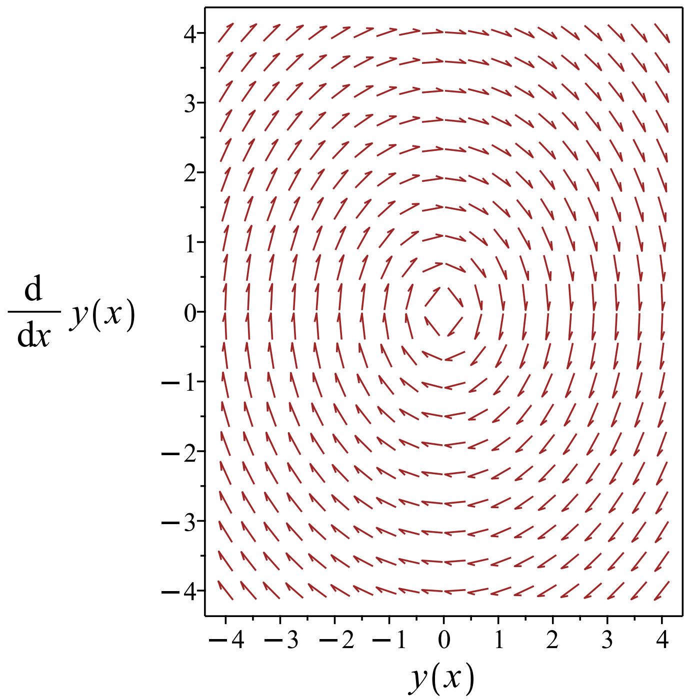

Figure 116: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x$$

Verified OK.

6.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 81: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\, c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x = 2\,e^x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2 \sin(x)) + (e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2 \sin(x) + e^x \tag{1}$$

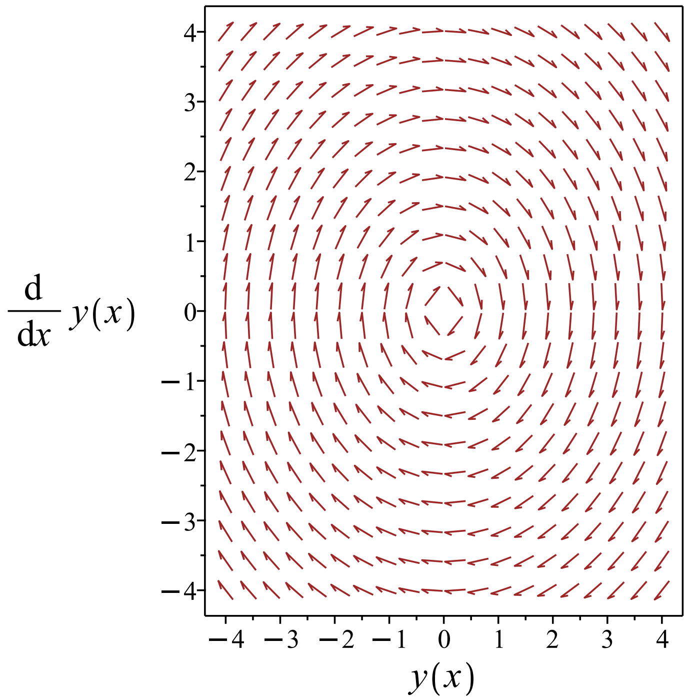

Figure 117: Slope field plot

<u>Verification of solutions</u>

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x$$

Verified OK.

6.5.3 Maple step by step solution

Let's solve

$y'' + y = 2\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\cos(x)\left(\int \sin(x)\,\mathrm{e}^x dx\right) + 2\sin(x)\left(\int \cos(x)\,\mathrm{e}^x dx\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^x$

- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + \mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)+y(x)=2*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + e^x$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 19

```
DSolve[y''[x]+y[x]==2*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x + c_1 \cos(x) + c_2 \sin(x)$$

6.6 problem 6

6.6.1 Solving as second order linear constant coeff ode 521
6.6.2 Solving as linear second order ode solved by an integrating factor ode 524
6.6.3 Solving using Kovacic algorithm 526
6.6.4 Maple step by step solution 531

Internal problem ID [4812]
Internal file name [`OUTPUT/4305_Sunday_June_05_2022_12_57_25_PM_73900820/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 9y = 12\,\mathrm{e}^{-x}}$$

6.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 9, f(x) = 12\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(6)^2 - (4)\,(1)\,(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x\, e^{-3x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-3x} + x\, e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\, e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-3x}, \mathrm{e}^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 \mathrm{e}^{-x} = 12\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2\right) + \left(3\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + 3\,\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(c_2 x + c_1) + 3\,\mathrm{e}^{-x} \tag{1}$$

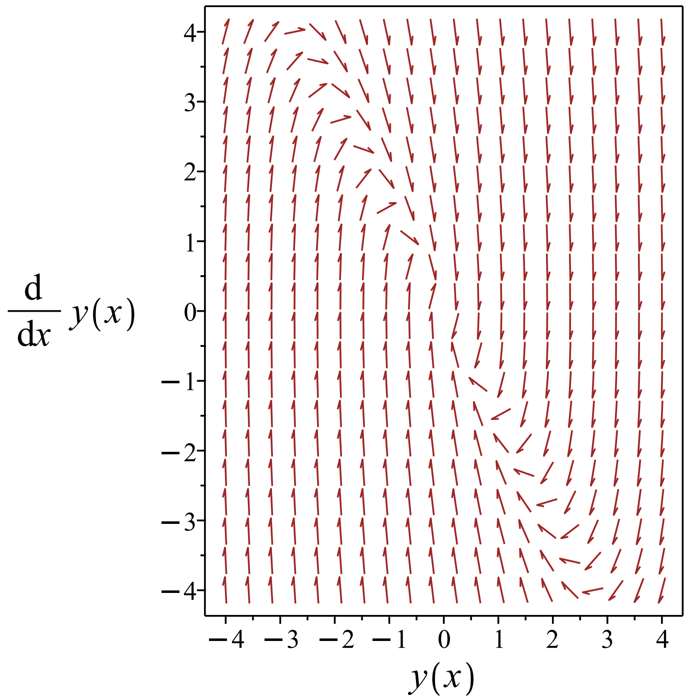

Figure 118: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-3x}(c_2x + c_1) + 3\,e^{-x}$$

Verified OK.

6.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= e^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 12\,\mathrm{e}^{-x}\mathrm{e}^{3x}$$
$$\left(\mathrm{e}^{3x}y\right)'' = 12\,\mathrm{e}^{-x}\mathrm{e}^{3x}$$

Integrating once gives

$$\left(\mathrm{e}^{3x}y\right)' = 6\,\mathrm{e}^{2x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{3x}y\right) = c_1x + 3\,\mathrm{e}^{2x} + c_2$$

Hence the solution is

$$y = \frac{c_1x + 3\,\mathrm{e}^{2x} + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1x\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{-3x} + 3\,\mathrm{e}^{-x} + c_2\mathrm{e}^{-3x} \tag{1}$$

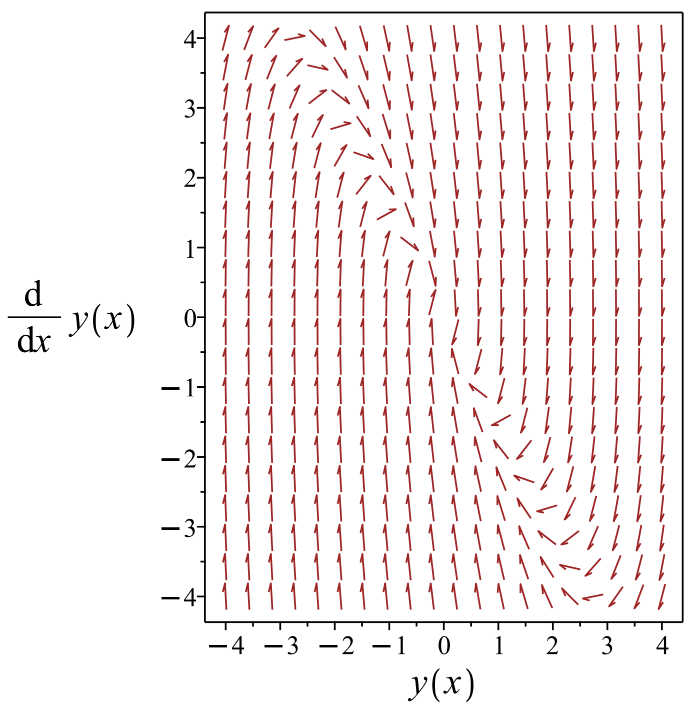

Figure 119: Slope field plot

Verification of solutions

$$
y = c_1 x \,\mathrm{e}^{-3x} + 3 \,\mathrm{e}^{-x} + c_2 \mathrm{e}^{-3x}
$$

Verified OK.

6.6.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + 6y' + 9y &= 0 \qquad (1)\\
Ay'' + By' + Cy &= 0 \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 6 \qquad (3)\\
C &= 9
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \qquad (4)
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \qquad (6)
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 0\\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \qquad (7)
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 83: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\
&= z_1 e^{-3x} \\
&= z_1\left(\mathrm{e}^{-3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-3x} + x\,e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,e^{-3x}, e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^{-x} = 12\,e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3\,e^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + x\,e^{-3x} c_2\right) + \left(3\,e^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-3x}(c_2x + c_1) + 3e^{-x}$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(c_2x + c_1) + 3e^{-x} \tag{1}$$

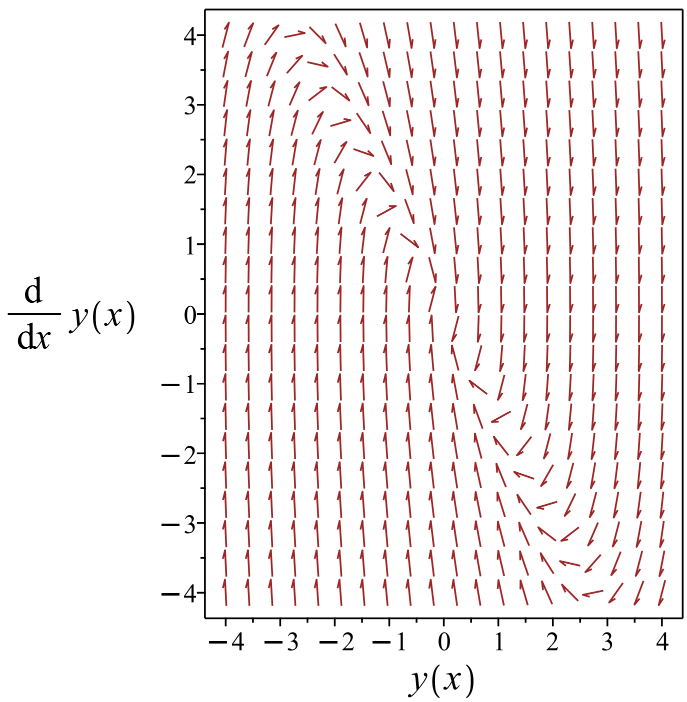

Figure 120: Slope field plot

Verification of solutions

$$y = e^{-3x}(c_2x + c_1) + 3e^{-x}$$

Verified OK.

6.6.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 12\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + x\,\mathrm{e}^{-3x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 12\,\mathrm{e}^{-x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & x\,\mathrm{e}^{-3x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{-3x} - 3x\,\mathrm{e}^{-3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -12\,\mathrm{e}^{-3x}\left(\int \mathrm{e}^{2x} x dx - \left(\int \mathrm{e}^{2x} dx\right) x\right)$

- Compute integrals

  $y_p(x) = 3\,e^{-x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{-3x}c_2 + c_1e^{-3x} + 3\,e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=12*exp(-x),y(x), singsol=all)
```

$$y(x) = (c_1x + c_2)\,e^{-3x} + 3\,e^{-x}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[y''[x]+6*y'[x]+9*y[x]==12*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(3e^{2x} + c_2x + c_1\right)$$

6.7 problem 7

6.7.1 Solving as second order linear constant coeff ode 533
6.7.2 Solving using Kovacic algorithm 537
6.7.3 Maple step by step solution 542

Internal problem ID [4813]
Internal file name [`OUTPUT/4306_Sunday_June_05_2022_12_57_34_PM_16528035/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 3\,\mathrm{e}^{2x}}$$

6.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 3\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda \,\mathrm{e}^{\lambda x} - 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-2)}\\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 e^{2x} = 3\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{2x} x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(e^{2x} x\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{2x} + c_2 e^{-x} + e^{2x} x \tag{1}
$$

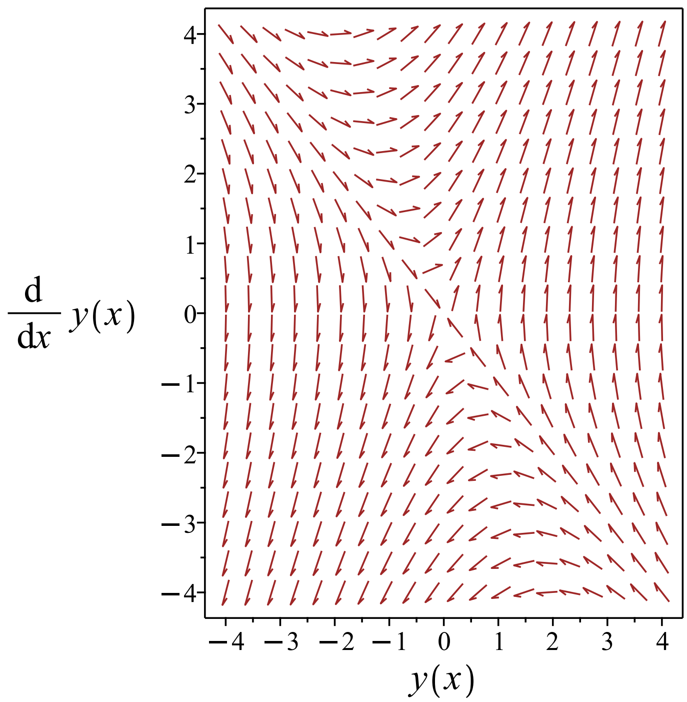

Figure 121: Slope field plot

Verification of solutions

$$
y = c_1 e^{2x} + c_2 e^{-x} + e^{2x} x
$$

Verified OK.

6.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 85: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-x} \\ y_2 &= \frac{\mathrm{e}^{2x}}{3} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ \frac{d}{dx}(\mathrm{e}^{-x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{2x}}{3} \\ -\mathrm{e}^{-x} & \frac{2\,\mathrm{e}^{2x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-x}\right)\left(\frac{2\,\mathrm{e}^{2x}}{3}\right) - \left(\frac{\mathrm{e}^{2x}}{3}\right)\left(-\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-x}\mathrm{e}^{2x}$$

Which simplifies to

$$W = e^{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{4x}}{e^{x}}\, dx$$

Which simplifies to

$$u_1 = -\int e^{3x} dx$$

Hence

$$u_1 = -\frac{e^{3x}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3\, e^{-x} e^{2x}}{e^{x}}\, dx$$

Which simplifies to

$$u_2 = \int 3 dx$$

Hence

$$u_2 = 3x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{e^{-x} e^{3x}}{3} + e^{2x} x$$

Which simplifies to

$$y_p(x) = e^{2x}\left(-\frac{1}{3} + x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{2x}}{3}\right) + \left(e^{2x}\left(-\frac{1}{3} + x\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + e^{2x}\left(-\frac{1}{3} + x\right) \tag{1}$$

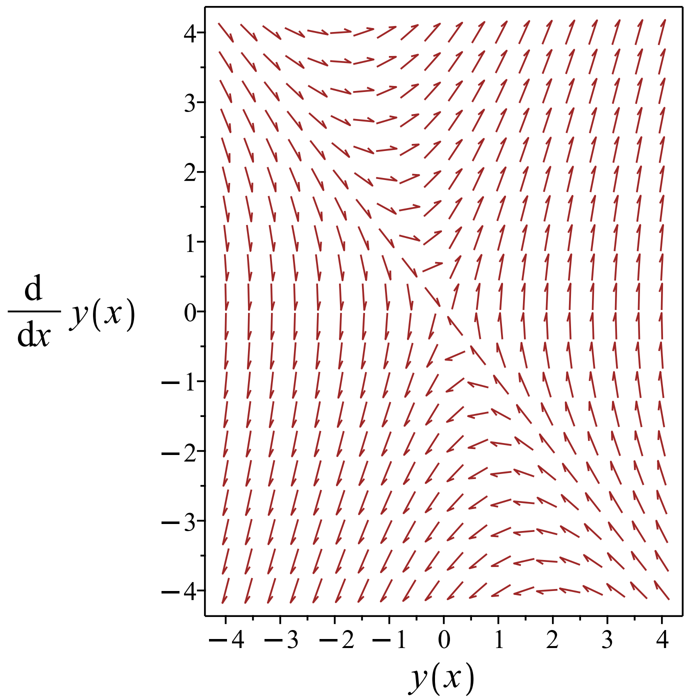

Figure 122: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} + e^{2x}\left(-\frac{1}{3} + x\right)$$

Verified OK.

6.7.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 3\,e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3\,\mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int \mathrm{e}^{3x}dx\right) + \mathrm{e}^{2x}\left(\int 1dx\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{2x}\left(-\frac{1}{3} + x\right)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + \mathrm{e}^{2x}\left(-\frac{1}{3} + x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=3*exp(2*x),y(x), singsol=all)
```

$$y(x) = (c_2 + x)\,\mathrm{e}^{2x} + \mathrm{e}^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 27

```
DSolve[y''[x]-y'[x]-2*y[x]==3*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + e^{2x}\left(x - \frac{1}{3} + c_2\right)$$

6.8 problem 8

6.8.1 Solving as second order linear constant coeff ode 545
6.8.2 Solving using Kovacic algorithm 548
6.8.3 Maple step by step solution 554

Internal problem ID [4814]
Internal file name [`OUTPUT/4307_Sunday_June_05_2022_12_57_43_PM_46914653/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 16y = 40\,\mathrm{e}^{4x}}$$

6.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -16, f(x) = 40\,\mathrm{e}^{4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 16\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-16)} \\ &= \pm 4\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4 \\ \lambda_2 &= -4\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(4)x} + c_2 e^{(-4)x}\end{aligned}$$

Or

$$y = c_1 e^{4x} + c_2 e^{-4x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{4x} + c_2 e^{-4x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$40\,\mathrm{e}^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-4x}, \mathrm{e}^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{4x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{4x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1\mathrm{e}^{4x} = 40\,\mathrm{e}^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 5x\,\mathrm{e}^{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{-4x}\right) + \left(5x\,\mathrm{e}^{4x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-4x} + 5x\,e^{4x} \tag{1}$$

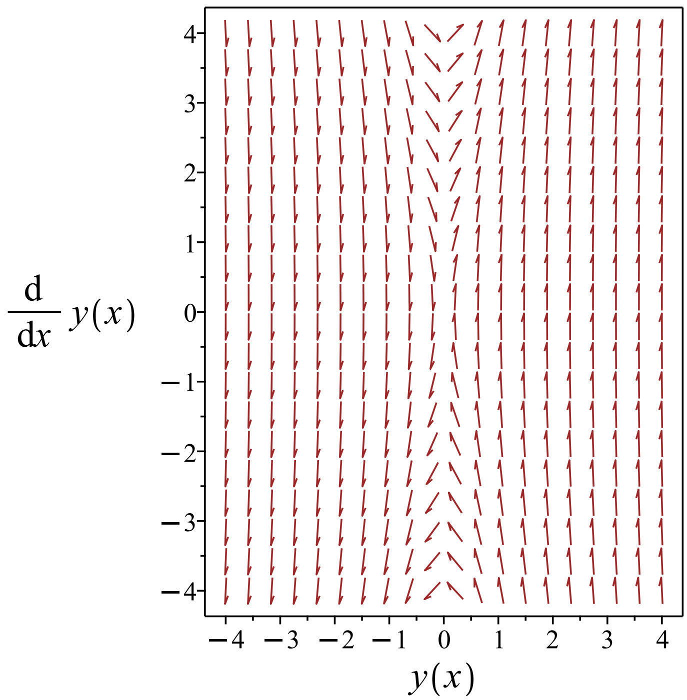

Figure 123: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-4x} + 5x\,e^{4x}$$

Verified OK.

6.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-4x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-4x}
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \mathrm{e}^{-4x} \int \frac{1}{\mathrm{e}^{-8x}} \, dx \\ &= \mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{8x}}{8} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-4x}\right) + c_2 \left(\mathrm{e}^{-4x} \left(\frac{\mathrm{e}^{8x}}{8} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-4x} + \frac{c_2 \mathrm{e}^{4x}}{8}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-4x}$$
$$y_2 = \frac{e^{4x}}{8}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-4x} & \frac{e^{4x}}{8} \\ \frac{d}{dx}\left(e^{-4x}\right) & \frac{d}{dx}\left(\frac{e^{4x}}{8}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-4x} & \frac{e^{4x}}{8} \\ -4\,e^{-4x} & \frac{e^{4x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-4x}\right)\left(\frac{e^{4x}}{2}\right) - \left(\frac{e^{4x}}{8}\right)\left(-4\,e^{-4x}\right)$$

Which simplifies to

$$W = e^{4x} e^{-4x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{5\,\mathrm{e}^{8x}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 5\,\mathrm{e}^{8x}dx$$

Hence

$$u_1 = -\frac{5\,\mathrm{e}^{8x}}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{40\,\mathrm{e}^{4x}\mathrm{e}^{-4x}}{1}\,dx$$

Which simplifies to

$$u_2 = \int 40dx$$

Hence

$$u_2 = 40x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{5\,\mathrm{e}^{-4x}\mathrm{e}^{8x}}{8} + 5x\,\mathrm{e}^{4x}$$

Which simplifies to

$$y_p(x) = \frac{5\,\mathrm{e}^{4x}(-1+8x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-4x} + \frac{c_2\mathrm{e}^{4x}}{8}\right) + \left(\frac{5\,\mathrm{e}^{4x}(-1+8x)}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-4x} + \frac{c_2 e^{4x}}{8} + \frac{5 e^{4x} (-1 + 8x)}{8} \tag{1}$$

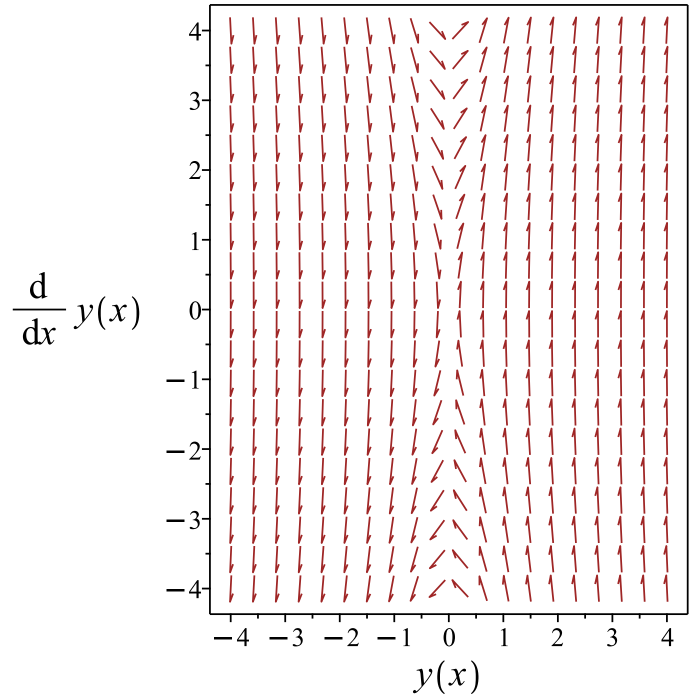

Figure 124: Slope field plot

Verification of solutions

$$y = c_1 e^{-4x} + \frac{c_2 e^{4x}}{8} + \frac{5 e^{4x} (-1 + 8x)}{8}$$

Verified OK.

6.8.3 Maple step by step solution

Let's solve

$y'' - 16y = 40 e^{4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 16 = 0$

- Factor the characteristic polynomial

  $(r-4)(r+4) = 0$

- Roots of the characteristic polynomial

  $r = (-4, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 40\,\mathrm{e}^{4x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-4x} & \mathrm{e}^{4x} \\ -4\,\mathrm{e}^{-4x} & 4\,\mathrm{e}^{4x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 8$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -5\,\mathrm{e}^{-4x}\left(\int \mathrm{e}^{8x} dx\right) + 5\,\mathrm{e}^{4x}\left(\int 1 dx\right)$

  - Compute integrals

    $y_p(x) = \frac{5\,\mathrm{e}^{4x}(-1+8x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-4x} + c_2 \mathrm{e}^{4x} + \frac{5\,\mathrm{e}^{4x}(-1+8x)}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-16*y(x)=40*exp(4*x),y(x), singsol=all)
```

$$y(x) = (5x + c_2)\,\mathrm{e}^{4x} + \mathrm{e}^{-4x}c_1$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 29

```
DSolve[y''[x]-16*y[x]==40*Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{4x}\left(5x - \frac{5}{8} + c_1\right) + c_2e^{-4x}$$

6.9 problem 9

6.9.1 Solving as second order linear constant coeff ode 557
6.9.2 Solving as linear second order ode solved by an integrating factor ode 560
6.9.3 Solving using Kovacic algorithm 562
6.9.4 Maple step by step solution 567

Internal problem ID [4815]
Internal file name [`OUTPUT/4308_Sunday_June_05_2022_12_57_52_PM_96118765/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = 2\,\mathrm{e}^{-x}}$$

6.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 2\,\mathrm{e}^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \,\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x \,\mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + x \,\mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 \,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{-x} = 2\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\mathrm{e}^{-x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2\right) + \left(x^2\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + x^2\mathrm{e}^{-x}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(c_2x + c_1) + x^2 e^{-x} \tag{1}$$

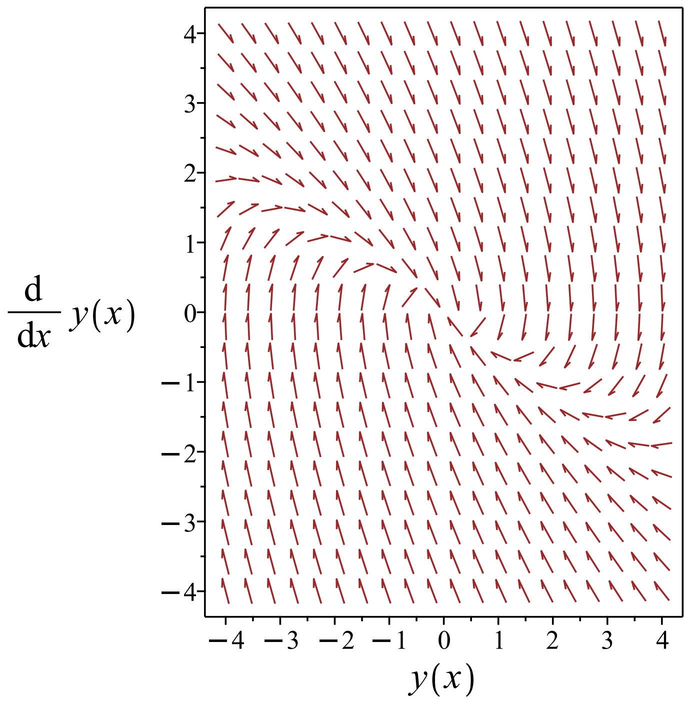

Figure 125: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + x^2 e^{-x}$$

Verified OK.

6.9.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 2\,\mathrm{e}^{x}\mathrm{e}^{-x} \\
(y\,\mathrm{e}^{x})'' &= 2\,\mathrm{e}^{x}\mathrm{e}^{-x}
\end{aligned}
$$

Integrating once gives

$$
(y\,\mathrm{e}^{x})' = 2x + c_1
$$

Integrating again gives

$$
(y\,\mathrm{e}^{x}) = x(x + c_1) + c_2
$$

Hence the solution is

$$
y = \frac{x(x + c_1) + c_2}{\mathrm{e}^{x}}
$$

Or

$$
y = c_1 x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x} + c_2\mathrm{e}^{-x}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x} + x^2\mathrm{e}^{-x} + c_2\mathrm{e}^{-x} \tag{1}
$$

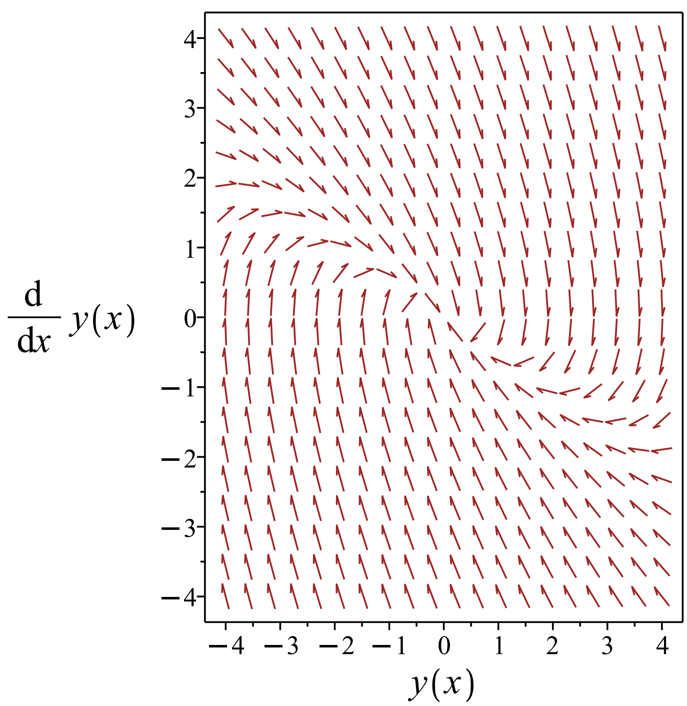

Figure 126: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-x} + x^2 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x}$$

Verified OK.

6.9.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 89: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}\}]$$

Since $x\,\mathrm{e}^{-x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{-x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{-x} = 2\,\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \mathrm{e}^{-x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2\right) + \left(x^2 \mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{-x}(c_2 x + c_1) + x^2 \mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x}(c_2 x + c_1) + x^2 \mathrm{e}^{-x} \tag{1}
$$

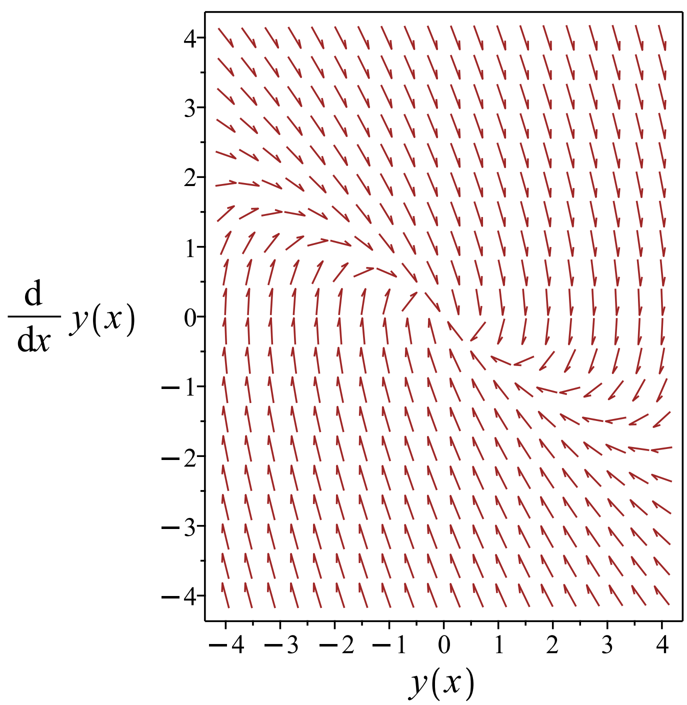

Figure 127: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-x}(c_2 x + c_1) + x^2 \mathrm{e}^{-x}
$$

Verified OK.

6.9.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 2\,\mathrm{e}^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^{-x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,\mathrm{e}^{-x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = x^2 e^{-x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{-x}c_2 + x^2 e^{-x} + c_1 e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=2*exp(-x),y(x), singsol=all)
```

$$y(x) = e^{-x}(c_1 x + x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 21

```
DSolve[y''[x]+2*y'[x]+y[x]==2*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(x^2 + c_2 x + c_1)$$

6.10 problem 10

6.10.1 Solving as second order linear constant coeff ode 569
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Internal problem ID [4816]
Internal file name [`OUTPUT/4309_Sunday_June_05_2022_12_58_01_PM_71311423/index.tex`]

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Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = 6\,\mathrm{e}^{3x}}$$

6.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 6\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda\, \mathrm{e}^{\lambda x} + 9\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-6)^2 - (4)\,(1)\,(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 x\, \mathrm{e}^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\, \mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x}x, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{3x}x\}]$$

Since $e^{3x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{3x} = 6\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x}c_1 + c_2x\,e^{3x}\right) + \left(3x^2e^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2x + c_1) + 3x^2e^{3x}$$

Summary
The solution(s) found are the following

$$y = e^{3x}(c_2x + c_1) + 3x^2e^{3x} \tag{1}$$

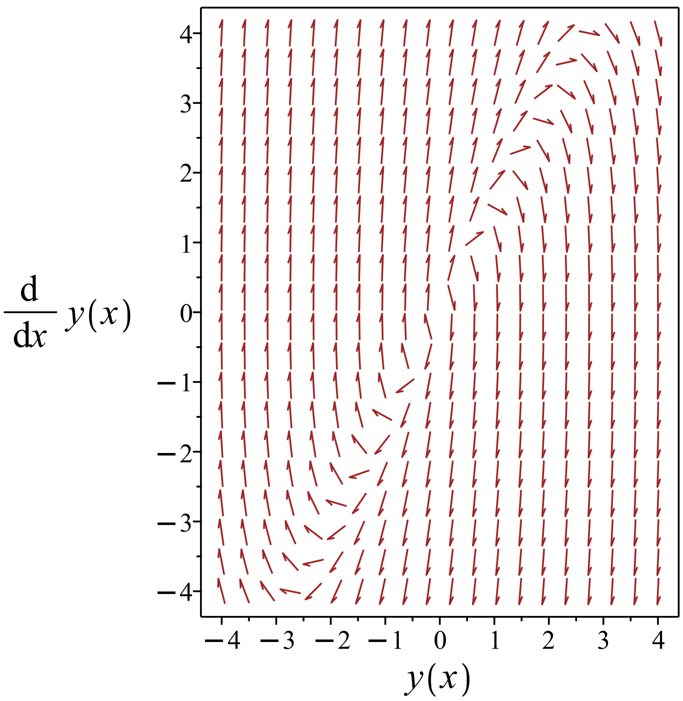

Figure 128: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2x + c_1) + 3x^2e^{3x}$$

Verified OK.

6.10.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -6\,dx} \\ &= e^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 6\,\mathrm{e}^{-3x}\mathrm{e}^{3x}$$
$$\left(\mathrm{e}^{-3x}y\right)'' = 6\,\mathrm{e}^{-3x}\mathrm{e}^{3x}$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)' = 6x + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right) = x(3x + c_1) + c_2$$

Hence the solution is

$$y = \frac{x(3x + c_1) + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{3x} + 3x^2\mathrm{e}^{3x} + c_2\mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{3x} + 3x^2\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} \tag{1}$$

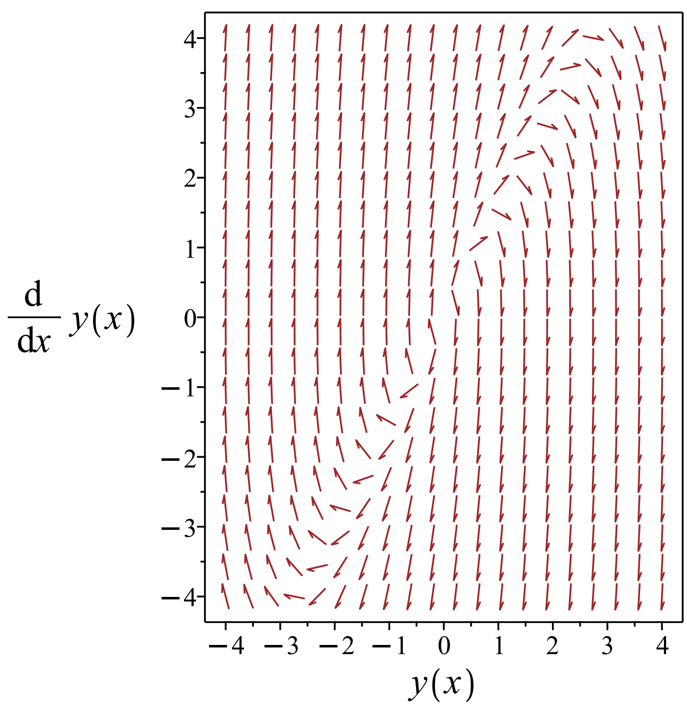

Figure 129: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{3x} + 3x^2 \mathrm{e}^{3x} + c_2 \mathrm{e}^{3x}$$

Verified OK.

6.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 91: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{3x} c_1 + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\, e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{3x} x\}]$$

Since $e^{3x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{3x} = 6\, e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 e^{3x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{3x} c_1 + c_2 x\,\mathrm{e}^{3x}\right) + \left(3x^2 \mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{3x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{3x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{3x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{3x} \tag{1}
$$

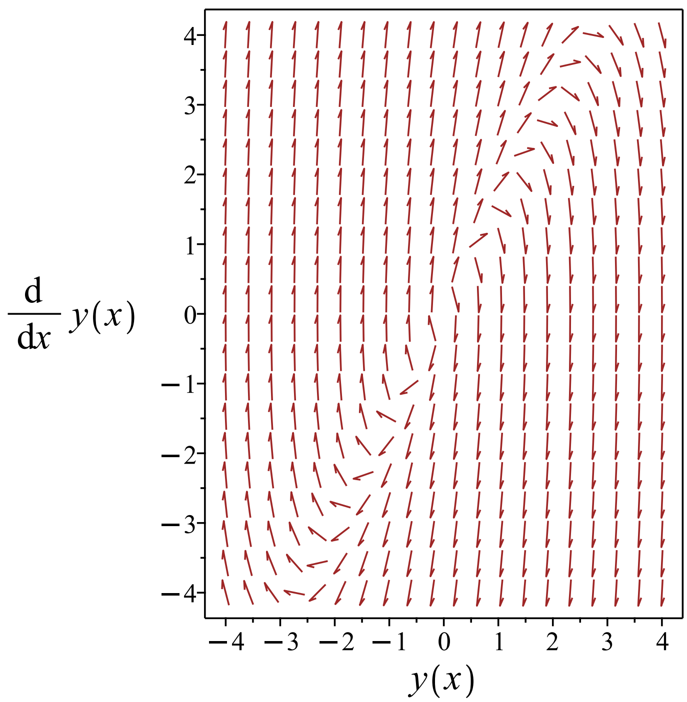

Figure 130: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{3x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{3x}
$$

Verified OK.

6.10.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 6\,\mathrm{e}^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{3x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{3x} c_1 + c_2 x\,\mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 6\,\mathrm{e}^{3x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{3x} x \\ 3\,\mathrm{e}^{3x} & 3\,\mathrm{e}^{3x} x + \mathrm{e}^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{6x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -6\,\mathrm{e}^{3x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 3x^2 e^{3x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x e^{3x} + 3x^2 e^{3x} + e^{3x} c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=6*exp(3*x),y(x), singsol=all)
```

$$y(x) = e^{3x}(c_1 x + 3x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[y''[x]-6*y'[x]+9*y[x]==6*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(3x^2 + c_2 x + c_1)$$

6.11 problem 11

6.11.1 Solving as second order linear constant coeff ode 581
6.11.2 Solving using Kovacic algorithm 584
6.11.3 Maple step by step solution 589

Internal problem ID [4817]
Internal file name [`OUTPUT/4310_Sunday_June_05_2022_12_58_09_PM_2727767/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 10y = 100\cos(4x)}$$

6.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 10, f(x) = 100\cos(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda \, \mathrm{e}^{\lambda x} + 10 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(10)} \\ &= -1 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 3i \\ \lambda_2 &= -1 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(3x), e^{-x} \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \cos(4x) - 6A_2 \sin(4x) - 8A_1 \sin(4x) + 8A_2 \cos(4x) = 100 \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -6, A_2 = 8]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -6 \cos(4x) + 8 \sin(4x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1 \cos(3x) + c_2 \sin(3x))\right) + (-6 \cos(4x) + 8 \sin(4x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x)) - 6\cos(4x) + 8\sin(4x) \tag{1}$$

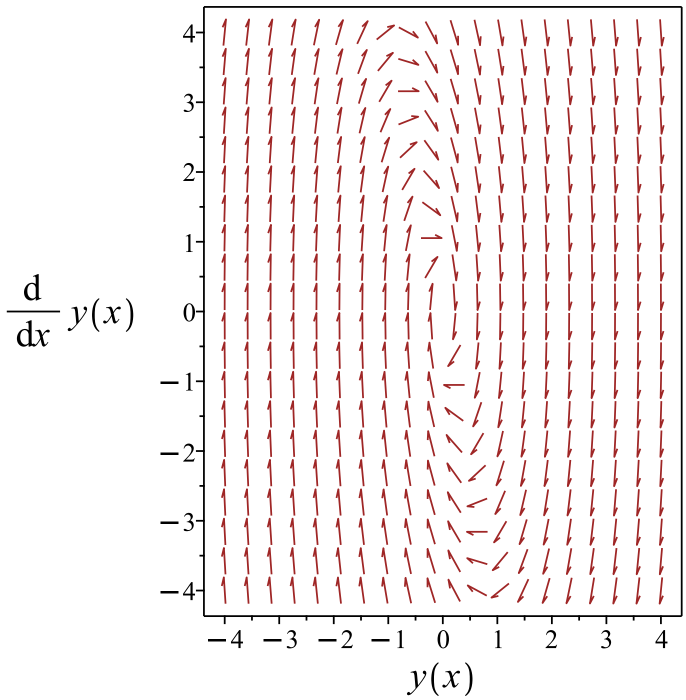

Figure 131: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(3x) + c_2 \sin(3x)) - 6\cos(4x) + 8\sin(4x)$$

Verified OK.

6.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 10y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x} \cos(3x)\right) + c_2 \left(\mathrm{e}^{-x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-x} \cos(3x)\, c_1 + \frac{\mathrm{e}^{-x} \sin(3x)\, c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(4x), \sin(4x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{ \mathrm{e}^{-x} \cos(3x), \frac{\mathrm{e}^{-x} \sin(3x)}{3} \right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(4x) + A_2 \sin(4x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-6A_1 \cos(4x) - 6A_2 \sin(4x) - 8A_1 \sin(4x) + 8A_2 \cos(4x) = 100 \cos(4x)
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = -6, A_2 = 8]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = -6 \cos(4x) + 8 \sin(4x)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-x} \cos(3x) c_1 + \frac{\mathrm{e}^{-x} \sin(3x) c_2}{3} \right) + (-6 \cos(4x) + 8 \sin(4x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x} \cos(3x) c_1 + \frac{\mathrm{e}^{-x} \sin(3x) c_2}{3} - 6 \cos(4x) + 8 \sin(4x) \tag{1}
$$

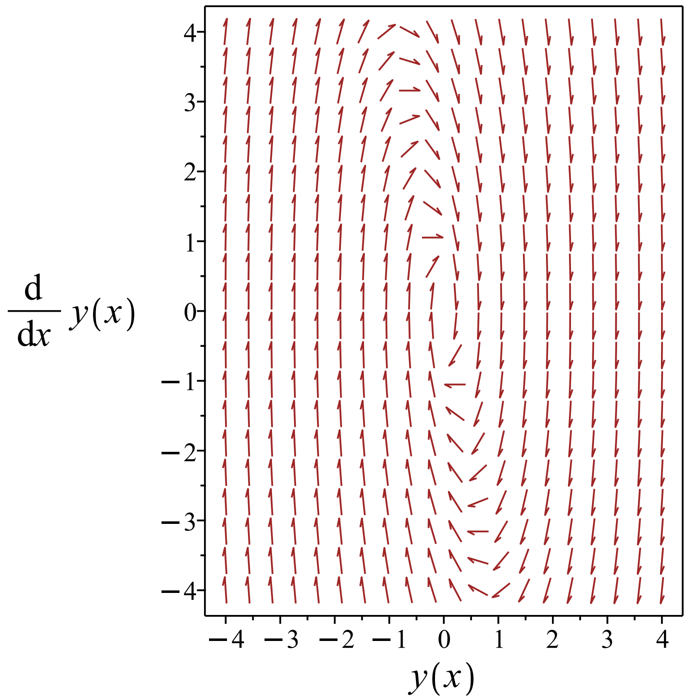

Figure 132: Slope field plot

Verification of solutions

$$y = e^{-x} \cos(3x) c_1 + \frac{e^{-x} \sin(3x) c_2}{3} - 6\cos(4x) + 8\sin(4x)$$

Verified OK.

6.11.3 Maple step by step solution

Let's solve

$y'' + 2y' + 10y = 100\cos(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 3\,\mathrm{I}, -1 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-x} \cos(3x)\, c_1 + \mathrm{e}^{-x} \sin(3x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 100 \cos(4x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(3x) & \mathrm{e}^{-x} \sin(3x) \\ -\mathrm{e}^{-x} \cos(3x) - 3\,\mathrm{e}^{-x} \sin(3x) & -\mathrm{e}^{-x} \sin(3x) + 3\,\mathrm{e}^{-x} \cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{50\,\mathrm{e}^{-x}\left(\cos(3x)\left(\int(-\sin(7x)+\sin(x))\mathrm{e}^{x} dx\right)+\sin(3x)\left(\int(\cos(x)+\cos(7x))\mathrm{e}^{x} dx\right)\right)}{3}$

  - Compute integrals

    $y_p(x) = -6\cos(4x) + 8\sin(4x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-x} \sin(3x)\, c_2 + \mathrm{e}^{-x} \cos(3x)\, c_1 - 6\cos(4x) + 8\sin(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+10*y(x)=100*cos(4*x),y(x), singsol=all)
```

$$y(x) = e^{-x} \sin(3x) c_2 + e^{-x} \cos(3x) c_1 + 8 \sin(4x) - 6 \cos(4x)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 42

```
DSolve[y''[x]+2*y'[x]+10*y[x]==100*Cos[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 8 \sin(4x) - 6 \cos(4x) + c_2 e^{-x} \cos(3x) + c_1 e^{-x} \sin(3x)$$

6.12 problem 12

6.12.1 Solving as second order linear constant coeff ode 592
6.12.2 Solving using Kovacic algorithm 595
6.12.3 Maple step by step solution 600

Internal problem ID [4818]
Internal file name [`OUTPUT/4311_Sunday_June_05_2022_12_58_19_PM_18790568/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 12y = 80 \sin(2x)}$$

6.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 12, f(x) = 80 \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 12y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 12\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(12)}\\ &= -2 \pm 2i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 2i\sqrt{2}\\ \lambda_2 &= -2 - 2i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 2i\sqrt{2}\\ \lambda_2 &= -2 - 2i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 2\sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}\left(c_1 \cos\left(2x\sqrt{2}\right) + c_2 \sin\left(2x\sqrt{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}\left(c_1 \cos\left(2x\sqrt{2}\right) + c_2 \sin\left(2x\sqrt{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$80 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-2x} \cos\left(2x\sqrt{2}\right), e^{-2x} \sin\left(2x\sqrt{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(2x) + 8A_2 \sin(2x) - 8A_1 \sin(2x) + 8A_2 \cos(2x) = 80 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -5, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5 \cos(2x) + 5 \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}\left(c_1 \cos\left(2x\sqrt{2}\right) + c_2 \sin\left(2x\sqrt{2}\right)\right)\right) + (-5 \cos(2x) + 5 \sin(2x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-2x}\left(c_1 \cos\left(2x\sqrt{2}\right) + c_2 \sin\left(2x\sqrt{2}\right)\right) - 5\cos(2x) + 5\sin(2x) \tag{1}$$

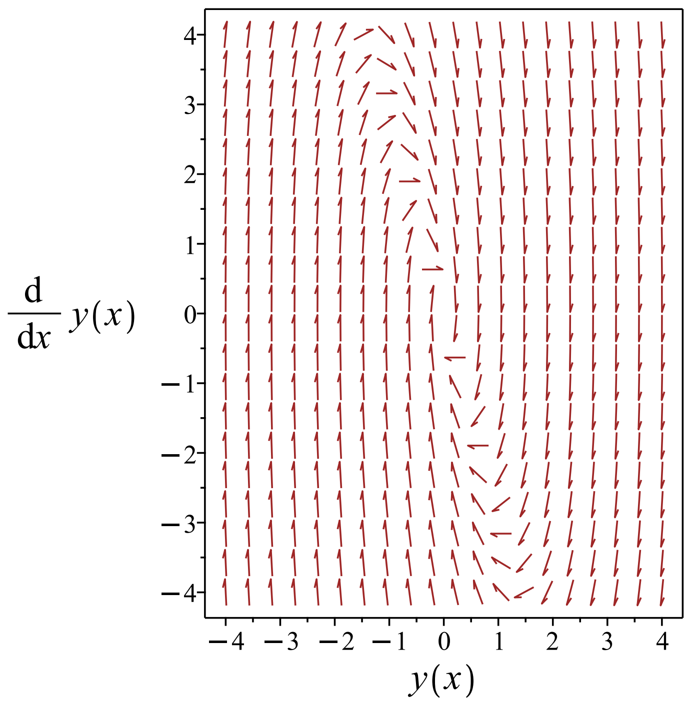

Figure 133: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2x}\left(c_1 \cos\left(2x\sqrt{2}\right) + c_2 \sin\left(2x\sqrt{2}\right)\right) - 5\cos(2x) + 5\sin(2x)$$

Verified OK.

6.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-8}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -8 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -8z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -8$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(2x\sqrt{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\
&= z_1 e^{-2x} \\
&= z_1\left(\mathrm{e}^{-2x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x} \cos\left(2x\sqrt{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{2}\, \tan\left(2x\sqrt{2}\right)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos\left(2x\sqrt{2}\right)\right) + c_2 \left(\mathrm{e}^{-2x} \cos\left(2x\sqrt{2}\right) \left(\frac{\sqrt{2}\, \tan\left(2x\sqrt{2}\right)}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 12y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{-2x} \cos\left(2x\sqrt{2}\right) c_1 + \frac{c_2 \mathrm{e}^{-2x} \sqrt{2}\, \sin\left(2x\sqrt{2}\right)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$80 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-2x} \cos\left(2x\sqrt{2}\right), \frac{e^{-2x}\sqrt{2}\, \sin\left(2x\sqrt{2}\right)}{4} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$8A_1 \cos(2x) + 8A_2 \sin(2x) - 8A_1 \sin(2x) + 8A_2 \cos(2x) = 80 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -5, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5 \cos(2x) + 5 \sin(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x} \cos\left(2x\sqrt{2}\right) c_1 + \frac{c_2 e^{-2x}\sqrt{2}\, \sin\left(2x\sqrt{2}\right)}{4} \right) + (-5 \cos(2x) + 5 \sin(2x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x} \cos\left(2x\sqrt{2}\right) c_1 + \frac{c_2 e^{-2x}\sqrt{2}\, \sin\left(2x\sqrt{2}\right)}{4} - 5\cos(2x) + 5\sin(2x) \tag{1}$$

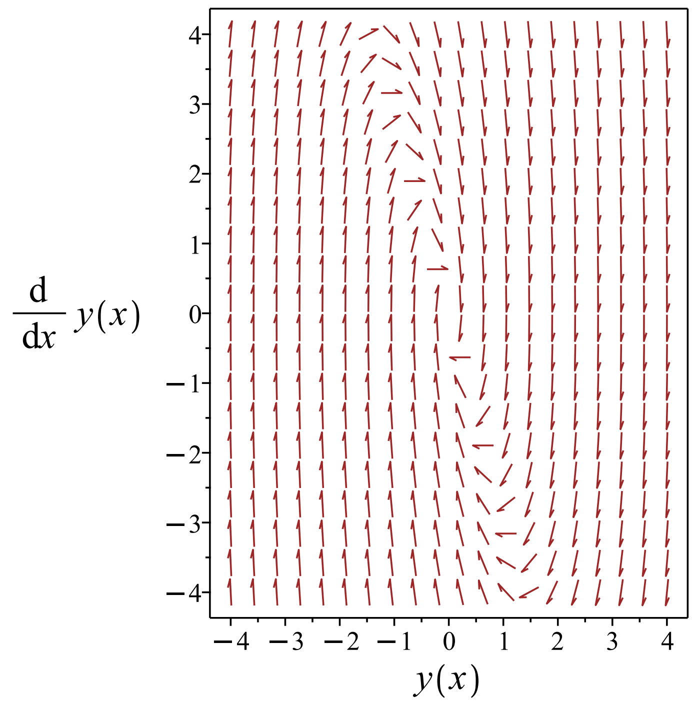

Figure 134: Slope field plot

Verification of solutions

$$y = e^{-2x} \cos\left(2x\sqrt{2}\right) c_1 + \frac{c_2 e^{-2x}\sqrt{2}\, \sin\left(2x\sqrt{2}\right)}{4} - 5\cos(2x) + 5\sin(2x)$$

Verified OK.

6.12.3 Maple step by step solution

Let's solve

$y'' + 4y' + 12y = 80\sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 12 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-32})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-2-2\,\mathrm{I}\sqrt{2}, -2+2\,\mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}\cos\left(2x\sqrt{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x}\sin\left(2x\sqrt{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2x}\cos\left(2x\sqrt{2}\right)c_1 + \mathrm{e}^{-2x}\sin\left(2x\sqrt{2}\right)c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 80\sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-2x}\cos\left(2x\sqrt{2}\right) & \mathrm{e}^{-2x}\sin\left(2x\sqrt{2}\right. \\ -2\,\mathrm{e}^{-2x}\cos\left(2x\sqrt{2}\right) - 2\,\mathrm{e}^{-2x}\sqrt{2}\sin\left(2x\sqrt{2}\right) & -2\,\mathrm{e}^{-2x}\sin\left(2x\sqrt{2}\right) + 2\,\mathrm{e}^{-2x} \end{array}\right.$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\sqrt{2}\,\mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -20\,\mathrm{e}^{-2x}\sqrt{2}\left(\cos\left(2x\sqrt{2}\right)\left(\int \mathrm{e}^{2x}\sin(2x)\sin\left(2x\sqrt{2}\right)dx\right) - \sin\left(2x\sqrt{2}\right)\left(\int \mathrm{e}^{2x}\sin(2x)\cos\right.\right.$

  - Compute integrals

    $y_p(x) = -5\cos(2x) + 5\sin(2x)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2x}\cos\left(2x\sqrt{2}\right)c_1 + \mathrm{e}^{-2x}\sin\left(2x\sqrt{2}\right)c_2 + 5\sin(2x) - 5\cos(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+12*y(x)=80*sin(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x} \sin \left(2x\sqrt{2}\right) c_2 + \mathrm{e}^{-2x} \cos \left(2x\sqrt{2}\right) c_1 + 5 \sin (2x) - 5 \cos (2x)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 52

```
DSolve[y''[x]+4*y'[x]+12*y[x]==80*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 5 \sin(2x) - 5 \cos(2x) + c_2 e^{-2x} \cos \left(2\sqrt{2}x\right) + c_1 e^{-2x} \sin \left(2\sqrt{2}x\right)$$

6.13 problem 13

6.13.1 Solving as second order linear constant coeff ode 603
6.13.2 Solving as linear second order ode solved by an integrating factor ode 606
6.13.3 Solving using Kovacic algorithm 608
6.13.4 Maple step by step solution 613

Internal problem ID [4819]
Internal file name [`OUTPUT/4312_Sunday_June_05_2022_12_58_28_PM_46124618/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 2\cos(x)}$$

6.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 2\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)} \\
&= 1
\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 x \, \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x} x, \mathrm{e}^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) - 2A_2 \cos(x) = 2\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + (-\sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) - \sin(x)$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) - \sin(x) \tag{1}$$

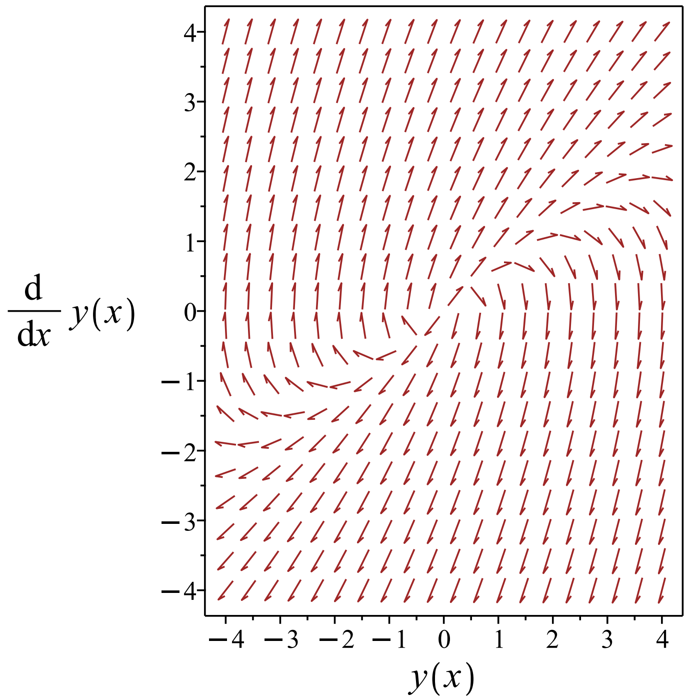

Figure 135: Slope field plot

Verification of solutions

$$y = e^{x}(c_2x + c_1) - \sin(x)$$

Verified OK.

6.13.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x) y' + \frac{(p(x)^2 + p'(x)) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -2\,dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}(M(x)y)'' &= 2\,\mathrm{e}^{-x}\cos(x)\\ \left(\mathrm{e}^{-x}y\right)'' &= 2\,\mathrm{e}^{-x}\cos(x)\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -\mathrm{e}^{-x}(-\sin(x)+\cos(x)) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1x - \mathrm{e}^{-x}\sin(x) + c_2$$

Hence the solution is

$$y = \frac{c_1x - \mathrm{e}^{-x}\sin(x) + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \sin(x)$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} - \sin(x) \tag{1}$$

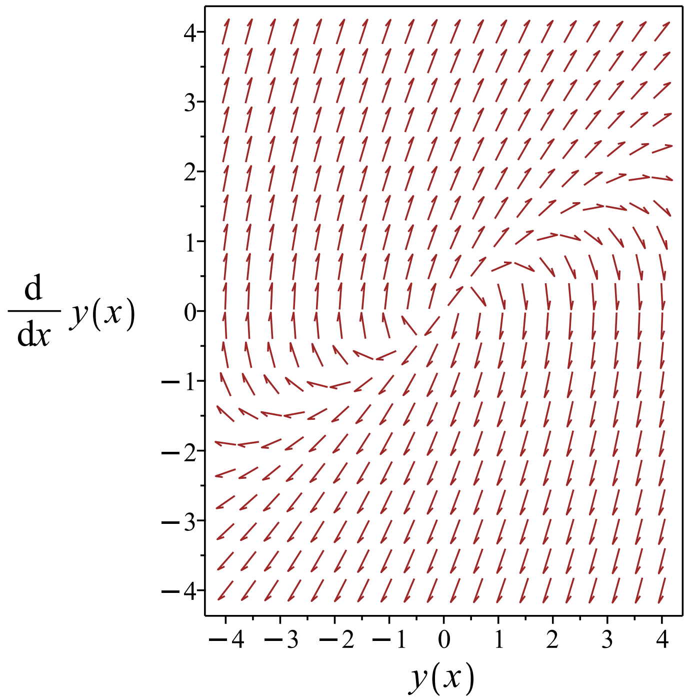

Figure 136: Slope field plot

Verification of solutions

$$y = c_1 x\, \mathrm{e}^{x} + c_2 \mathrm{e}^{x} - \sin(x)$$

Verified OK.

6.13.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2(\mathrm{e}^{x}(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{x} x, \mathrm{e}^{x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \sin(x) - 2A_2 \cos(x) = 2\cos(x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x e^x) + (-\sin(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) - \sin(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^x (c_2 x + c_1) - \sin(x) \tag{1}$$

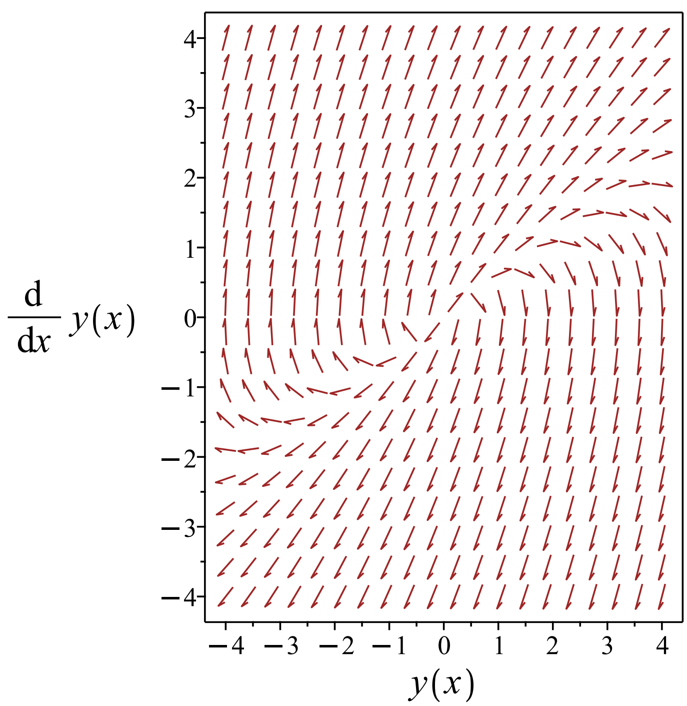

Figure 137: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) - \sin(x)$$

Verified OK.

6.13.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 2\cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{x} + c_2 x\, e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\cos(x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{x} & e^{x} x \\ e^{x} & e^{x} x + e^{x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 2\,e^{x}\left(-\left(\int \cos(x)\, x\, e^{-x} dx\right) + x\left(\int e^{-x} \cos(x)\, dx\right)\right)$

- Compute integrals

  $y_p(x) = -\sin(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{x} + c_1 e^{x} - \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=2*cos(x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\, e^{x} - \sin(x)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 21

```
DSolve[y''[x]-2*y'[x]+y[x]==2*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sin(x) + e^{x}(c_2 x + c_1)$$

6.14 problem 14

6.14.1 Solving as second order linear constant coeff ode 615
6.14.2 Solving using Kovacic algorithm 618
6.14.3 Maple step by step solution 623

Internal problem ID [4820]
Internal file name [`OUTPUT/4313_Sunday_June_05_2022_12_58_37_PM_80094949/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 8y' + 25y = 120\sin(5x)}$$

6.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 8, C = 25, f(x) = 120\sin(5x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 8y' + 25y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 8, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 8\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 8\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{8^2 - (4)\,(1)\,(25)}\\ &= -4 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 3i\\ \lambda_2 &= -4 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -4 + 3i\\ \lambda_2 &= -4 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -4$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$120 \sin(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4x} \cos(3x), e^{-4x} \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(5x) + A_2 \sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-40A_1 \sin(5x) + 40A_2 \cos(5x) = 120 \sin(5x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3 \cos(5x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x))\right) + (-3 \cos(5x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x)) - 3\cos(5x) \tag{1}$$

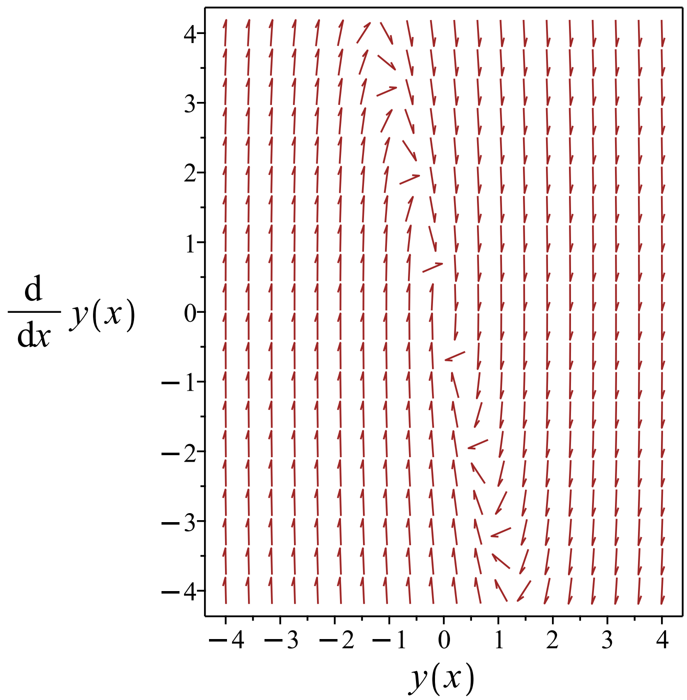

Figure 138: Slope field plot

Verification of solutions

$$y = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x)) - 3\cos(5x)$$

Verified OK.

6.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 8y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dx} \\
&= z_1 e^{-4x} \\
&= z_1\left(\mathrm{e}^{-4x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x} \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{8}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-8x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x} \cos(3x)\right) + c_2 \left(\mathrm{e}^{-4x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 8y' + 25y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(3x)\,\mathrm{e}^{-4x} c_1 + \frac{\sin(3x)\,\mathrm{e}^{-4x} c_2}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$120 \sin(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5x), \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-4x}\cos(3x), \frac{e^{-4x}\sin(3x)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(5x) + A_2\sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-40A_1\sin(5x) + 40A_2\cos(5x) = 120\sin(5x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3\cos(5x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(3x)e^{-4x}c_1 + \frac{\sin(3x)e^{-4x}c_2}{3}\right) + (-3\cos(5x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(3x)e^{-4x}c_1 + \frac{\sin(3x)e^{-4x}c_2}{3} - 3\cos(5x) \tag{1}$$

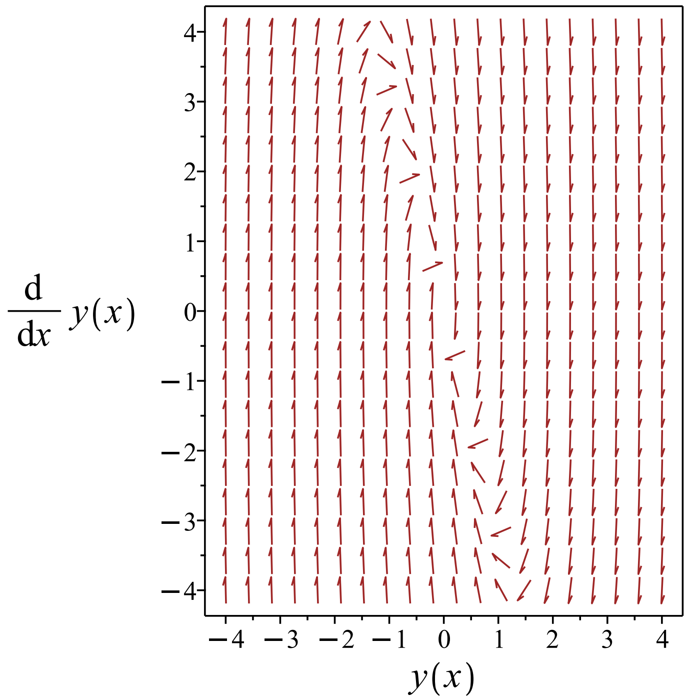

Figure 139: Slope field plot

Verification of solutions

$$y = \cos(3x)\,e^{-4x}c_1 + \frac{\sin(3x)\,e^{-4x}c_2}{3} - 3\cos(5x)$$

Verified OK.

6.14.3 Maple step by step solution

Let's solve

$y'' + 8y' + 25y = 120\sin(5x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 8r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-8)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

$r = (-4 - 3\,\mathrm{I}, -4 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-4x} \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-4x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(3x)\,\mathrm{e}^{-4x} c_1 + \sin(3x)\,\mathrm{e}^{-4x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 120 \sin(5x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-4x} \cos(3x) & \mathrm{e}^{-4x} \sin(3x) \\ -4\,\mathrm{e}^{-4x} \cos(3x) - 3\,\mathrm{e}^{-4x} \sin(3x) & -4\,\mathrm{e}^{-4x} \sin(3x) + 3\,\mathrm{e}^{-4x} \cos(3x) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{-8x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 40\,\mathrm{e}^{-4x}\left(-\cos(3x)\left(\int \sin(3x)\sin(5x)\,\mathrm{e}^{4x} dx\right) + \sin(3x)\left(\int \cos(3x)\sin(5x)\,\mathrm{e}^{4x} dx\right)\right)$

  - Compute integrals

    $y_p(x) = -3\cos(5x)$

- Substitute particular solution into general solution to ODE

  $y = \sin(3x)\,\mathrm{e}^{-4x} c_2 + \cos(3x)\,\mathrm{e}^{-4x} c_1 - 3\cos(5x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+8*diff(y(x),x)+25*y(x)=120*sin(5*x),y(x), singsol=all)
```

$$y(x) = e^{-4x} \sin(3x)\, c_2 + e^{-4x} \cos(3x)\, c_1 - 3\cos(5x)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 36

```
DSolve[y''[x]+8*y'[x]+25*y[x]==120*Sin[5*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3\cos(5x) + c_2 e^{-4x} \cos(3x) + c_1 e^{-4x} \sin(3x)$$

6.15 problem 15

6.15.1 Solving as second order linear constant coeff ode 626
6.15.2 Solving using Kovacic algorithm 630
6.15.3 Maple step by step solution 635

Internal problem ID [4821]
Internal file name [`OUTPUT/4314_Sunday_June_05_2022_12_58_46_PM_84261509/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{5y'' + 12y' + 20y = 120\sin(2x)}$$

6.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 5, B = 12, C = 20, f(x) = 120\sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$5y'' + 12y' + 20y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 5, B = 12, C = 20$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$5\lambda^2 \mathrm{e}^{\lambda x} + 12\lambda \, \mathrm{e}^{\lambda x} + 20 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$5\lambda^2 + 12\lambda + 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 5, B = 12, C = 20$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-12}{(2)(5)} \pm \frac{1}{(2)(5)}\sqrt{12^2 - (4)(5)(20)} \\
&= -\frac{6}{5} \pm \frac{8i}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{6}{5} + \frac{8i}{5} \\
\lambda_2 &= -\frac{6}{5} - \frac{8i}{5}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{6}{5} + \frac{8i}{5}$$

$$\lambda_2 = -\frac{6}{5} - \frac{8i}{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{6}{5}$ and $\beta = \frac{8}{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{6x}{5}}\left(c_1 \cos\left(\frac{8x}{5}\right) + c_2 \sin\left(\frac{8x}{5}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{6x}{5}}\left(c_1 \cos\left(\frac{8x}{5}\right) + c_2 \sin\left(\frac{8x}{5}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$120 \sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right), \mathrm{e}^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-24A_1 \sin(2x) + 24A_2 \cos(2x) = 120 \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -5, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5 \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{6x}{5}}\left(c_1\cos\left(\frac{8x}{5}\right)+c_2\sin\left(\frac{8x}{5}\right)\right)\right) + (-5\cos(2x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{-\frac{6x}{5}}\left(c_1\cos\left(\frac{8x}{5}\right)+c_2\sin\left(\frac{8x}{5}\right)\right) - 5\cos(2x) \tag{1}$$

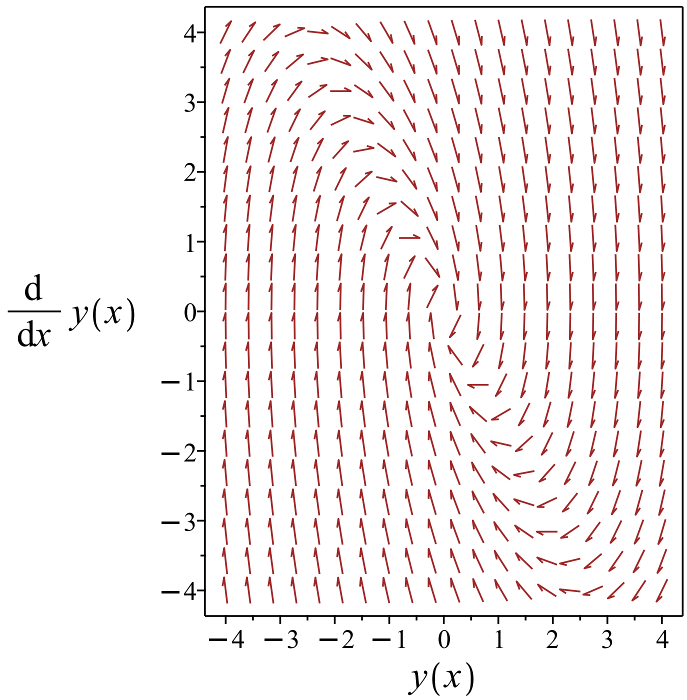

Figure 140: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\frac{6x}{5}}\left(c_1\cos\left(\frac{8x}{5}\right)+c_2\sin\left(\frac{8x}{5}\right)\right) - 5\cos(2x)$$

Verified OK.

6.15.2 Solving using Kovacic algorithm

Writing the ode as

$$5y'' + 12y' + 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= 12 \\ C &= 20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-64}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -64 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{64z(x)}{25} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{64}{25}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{8x}{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{12}{5}\, dx} \\
&= z_1 e^{-\frac{6x}{5}} \\
&= z_1 \left(\mathrm{e}^{-\frac{6x}{5}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{12}{5}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{12x}{5}}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{5 \tan\left(\frac{8x}{5}\right)}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right) \left(\frac{5 \tan\left(\frac{8x}{5}\right)}{8}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$5y''+12y'+20y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\cos\left(\frac{8x}{5}\right)\mathrm{e}^{-\frac{6x}{5}}c_1+\frac{5\sin\left(\frac{8x}{5}\right)\mathrm{e}^{-\frac{6x}{5}}c_2}{8}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$120\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x),\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{6x}{5}}\cos\left(\frac{8x}{5}\right),\frac{5\,\mathrm{e}^{-\frac{6x}{5}}\sin\left(\frac{8x}{5}\right)}{8}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\cos(2x)+A_2\sin(2x)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-24A_1\sin(2x)+24A_2\cos(2x)=120\sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=-5,A_2=0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5\cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_1 + \frac{5\sin\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_2}{8}\right) + (-5\cos(2x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_1 + \frac{5\sin\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_2}{8} - 5\cos(2x) \tag{1}$$

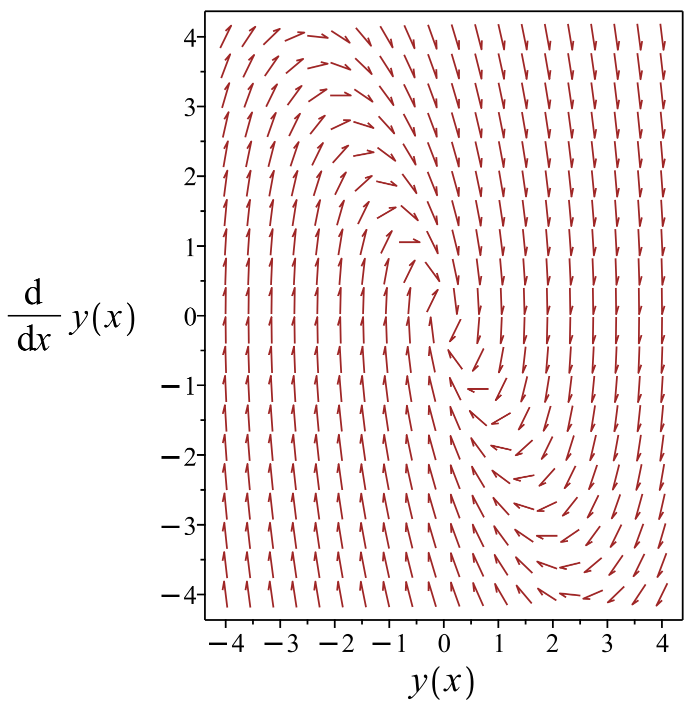

Figure 141: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_1 + \frac{5\sin\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_2}{8} - 5\cos(2x)$$

Verified OK.

6.15.3 Maple step by step solution

Let's solve

$5y'' + 12y' + 20y = 120 \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{12y'}{5} - 4y + 24 \sin(2x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{12y'}{5} + 4y = 24 \sin(2x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{12}{5}r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{12}{5}\right) \pm \left(\sqrt{-\frac{256}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{6}{5} - \frac{8\,\mathrm{I}}{5}, -\frac{6}{5} + \frac{8\,\mathrm{I}}{5}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{8x}{5}\right) \mathrm{e}^{-\frac{6x}{5}} c_1 + \sin\left(\frac{8x}{5}\right) \mathrm{e}^{-\frac{6x}{5}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 24 \sin(2x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right) & e^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right) \\ -\frac{6 e^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right)}{5} - \frac{8 e^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right)}{5} & -\frac{6 e^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right)}{5} + \frac{8 e^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right)}{5} \end{bmatrix}$$

- Compute Wronskian

$$W(y_1(x), y_2(x)) = \frac{8 e^{-\frac{12x}{5}}}{5}$$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -15 e^{-\frac{6x}{5}} \left(\cos\left(\frac{8x}{5}\right) \left(\int \sin\left(\frac{8x}{5}\right) \sin(2x) e^{\frac{6x}{5}} dx\right) - \sin\left(\frac{8x}{5}\right) \left(\int \cos\left(\frac{8x}{5}\right) \sin(2x) e^{\frac{6x}{5}} dx\right)\right)$$

- Compute integrals

$$y_p(x) = -80 \cos\left(\frac{2x}{5}\right)^5 + 100 \cos\left(\frac{2x}{5}\right)^3 - 25 \cos\left(\frac{2x}{5}\right)$$

- Substitute particular solution into general solution to ODE

$$y = \cos\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_1 + \sin\left(\frac{8x}{5}\right) e^{-\frac{6x}{5}} c_2 - 80 \cos\left(\frac{2x}{5}\right)^5 + 100 \cos\left(\frac{2x}{5}\right)^3 - 25 \cos\left(\frac{2x}{5}\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(5*diff(y(x),x$2)+12*diff(y(x),x)+20*y(x)=120*sin(2*x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{6x}{5}} \sin\left(\frac{8x}{5}\right) c_2 + e^{-\frac{6x}{5}} \cos\left(\frac{8x}{5}\right) c_1 - 5 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 44

```
DSolve[5*y''[x]+12*y'[x]+20*y[x]==120*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -5\cos(2x) + c_2 e^{-6x/5} \cos\left(\frac{8x}{5}\right) + c_1 e^{-6x/5} \sin\left(\frac{8x}{5}\right)$$

6.16 problem 16

6.16.1 Solving as second order linear constant coeff ode 638
6.16.2 Solving using Kovacic algorithm 642
6.16.3 Maple step by step solution 647

Internal problem ID [4822]
Internal file name [`OUTPUT/4315_Sunday_June_05_2022_12_58_59_PM_67796572/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 30 \sin (3x)}$$

6.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 30 \sin (3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$30 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(3x), x \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(3x) + A_2 x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) = 30 \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -5, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5x \cos(3x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + (-5x \cos(3x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - 5x \cos(3x) \tag{1}
$$

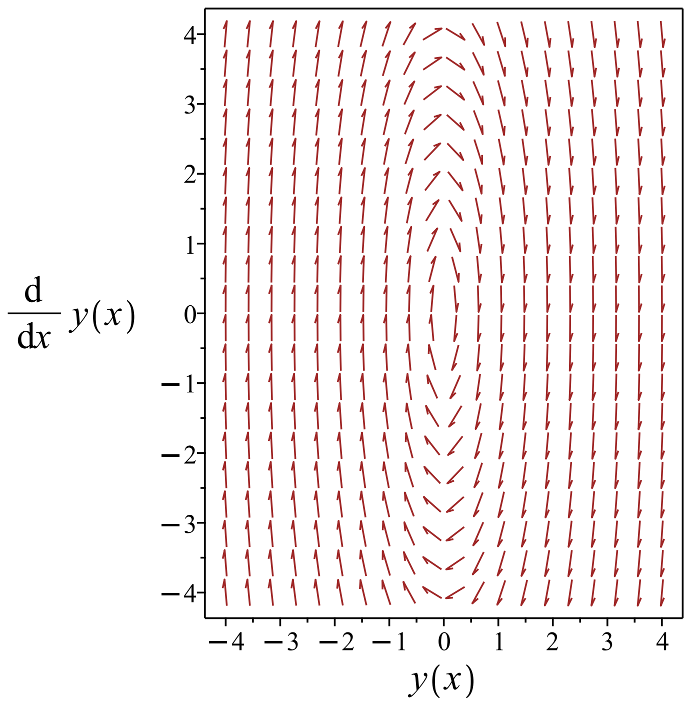

Figure 142: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - 5x \cos(3x)
$$

Verified OK.

6.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2} \, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2 \left(\cos(3x) \left(\frac{\tan(3x)}{3} \right) \right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$30 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(3x), x \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(3x) + A_2 x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) = 30 \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -5, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -5x \cos(3x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + (-5x \cos(3x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - 5x \cos(3x) \tag{1}
$$

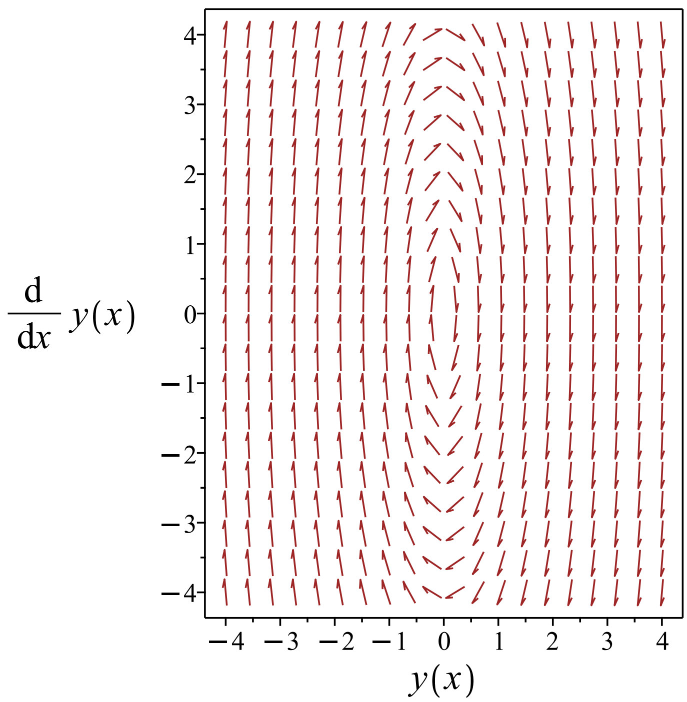

Figure 143: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - 5x \cos(3x)
$$

Verified OK.

6.16.3 Maple step by step solution

Let's solve

$y'' + 9y = 30\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(3x) + c_2\sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 30\sin(3x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -5\cos(3x)\left(\int (1-\cos(6x))\,dx\right) + 5\sin(3x)\left(\int \sin(6x)\,dx\right)$$

- Compute integrals

$$y_p(x) = \frac{5\sin(3x)}{6} - 5x\cos(3x)$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(3x) + c_2\sin(3x) + \frac{5\sin(3x)}{6} - 5x\cos(3x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)+9*y(x)=30*sin(3*x),y(x), singsol=all)
```

$$y(x) = (-5x + c_1)\cos(3x) + c_2\sin(3x)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 31

```
DSolve[y''[x]+9*y[x]==30*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-5x + c_1)\cos(3x) + \frac{1}{6}(5 + 6c_2)\sin(3x)$$

6.17 problem 17

6.17.1 Solving as second order linear constant coeff ode 649
6.17.2 Solving using Kovacic algorithm 653
6.17.3 Maple step by step solution 658

Internal problem ID [4823]
Internal file name [`OUTPUT/4316_Sunday_June_05_2022_12_59_08_PM_24672139/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 16y = 16\cos(4x)}$$

6.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = 16\cos(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 16\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(16)}\\ &= \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i\\ \lambda_2 &= -4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4i\\ \lambda_2 &= -4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos (4x) + c_2 \sin (4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4x), \sin(4x)\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(4x), \sin(4x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(4x) + A_2 \sin(4x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \sin(4x) + 8A_2 \cos(4x) = 16 \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 \sin(4x)\, x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(4x) + c_2 \sin(4x)) + (2 \sin(4x)\, x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(4x) + c_2 \sin(4x) + 2 \sin(4x)\, x \tag{1}
$$

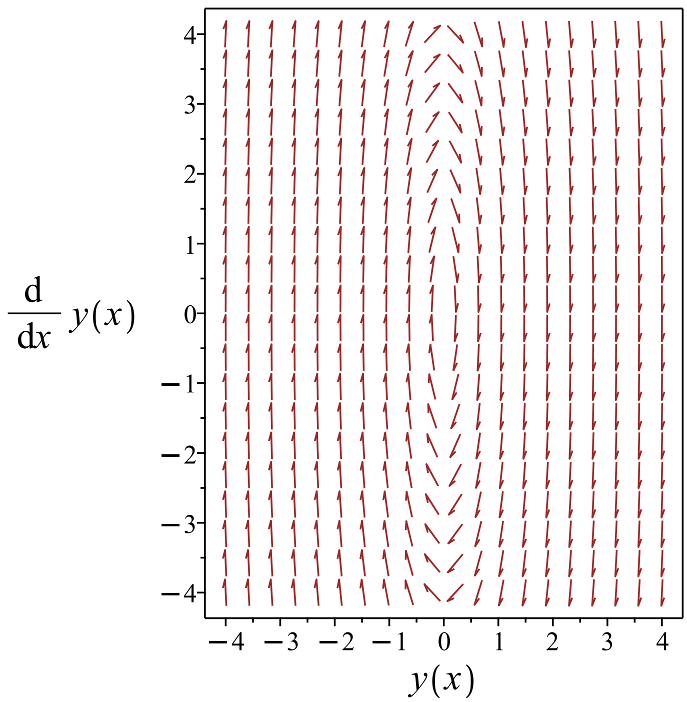

Figure 144: Slope field plot

Verification of solutions

$$
y = c_1 \cos(4x) + c_2 \sin(4x) + 2 \sin(4x)\, x
$$

Verified OK.

6.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2}\, dx \\ &= \cos(4x)\left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2\left(\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16 \cos(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(4x)}{4}, \cos(4x)\right\}$$

Since $\cos(4x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(4x), \sin(4x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(4x) + A_2 \sin(4x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_1 \sin(4x) + 8A_2 \cos(4x) = 16 \cos(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 \sin(4x)\, x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}\right) + (2\sin(4x)\, x)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + 2\sin(4x)\, x \tag{1}
$$

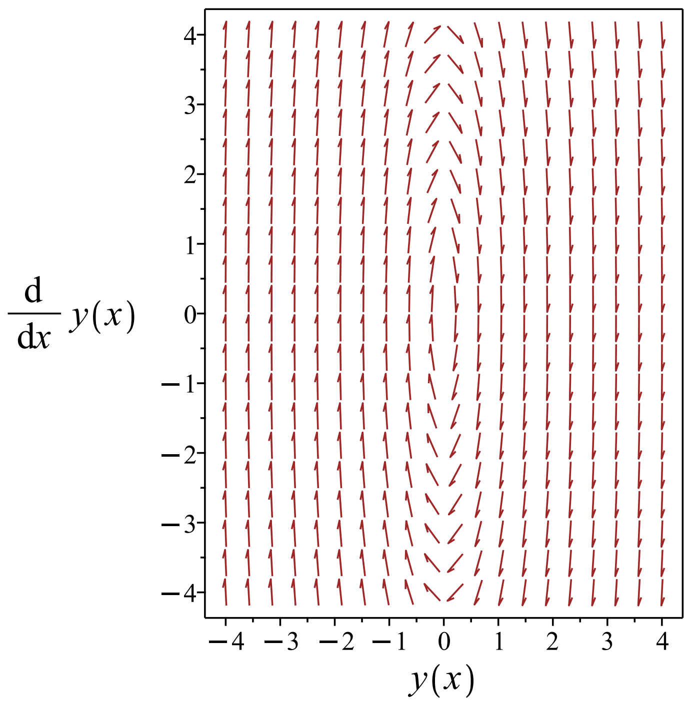

Figure 145: Slope field plot

Verification of solutions

$$
y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + 2\sin(4x)\, x
$$

Verified OK.

6.17.3 Maple step by step solution

Let's solve

$y'' + 16y = 16\cos(4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 16\cos(4x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -2\cos(4x)\left(\int \sin(8x)\,dx\right) + 2\sin(4x)\left(\int (1+\cos(8x))\,dx\right)$$

- Compute integrals

  $$y_p(x) = \frac{\cos(4x)}{4} + 2\sin(4x)\,x$$

- Substitute particular solution into general solution to ODE

  $$y = c_1\cos(4x) + c_2\sin(4x) + \frac{\cos(4x)}{4} + 2\sin(4x)\,x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+16*y(x)=16*cos(4*x),y(x), singsol=all)
```

$$y(x) = \frac{(4x+2c_2)\sin(4x)}{2} + \frac{(2c_1+1)\cos(4x)}{2}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 28

```
DSolve[y''[x]+16*y[x]==16*Cos[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(\frac{1}{4}+c_1\right)\cos(4x) + (2x+c_2)\sin(4x)$$

6.18 problem 18

6.18.1 Solving as second order linear constant coeff ode 660
6.18.2 Solving using Kovacic algorithm 663
6.18.3 Maple step by step solution 668

Internal problem ID [4824]
Internal file name [`OUTPUT/4317_Sunday_June_05_2022_12_59_17_PM_24834970/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 17y = 60\,\mathrm{e}^{-4x}\sin(5x)}$$

6.18.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 17, f(x) = 60\,\mathrm{e}^{-4x}\sin(5x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 17y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 17$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} + 17\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 17 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 17$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(17)}\\ &= -1 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 4i\\ \lambda_2 &= -1 - 4i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 4i\\ \lambda_2 &= -1 - 4i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$60\, e^{-4x} \sin(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4x} \cos(5x), e^{-4x} \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x} \cos(4x), e^{-x} \sin(4x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4x} \cos(5x) + A_2 e^{-4x} \sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$30A_1 e^{-4x} \sin(5x) - 30A_2 e^{-4x} \cos(5x) = 60\, e^{-4x} \sin(5x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\, e^{-4x} \cos(5x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-x}(c_1 \cos(4x) + c_2 \sin(4x))\right) + \left(2\, e^{-4x} \cos(5x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x)) + 2 e^{-4x} \cos(5x) \tag{1}$$

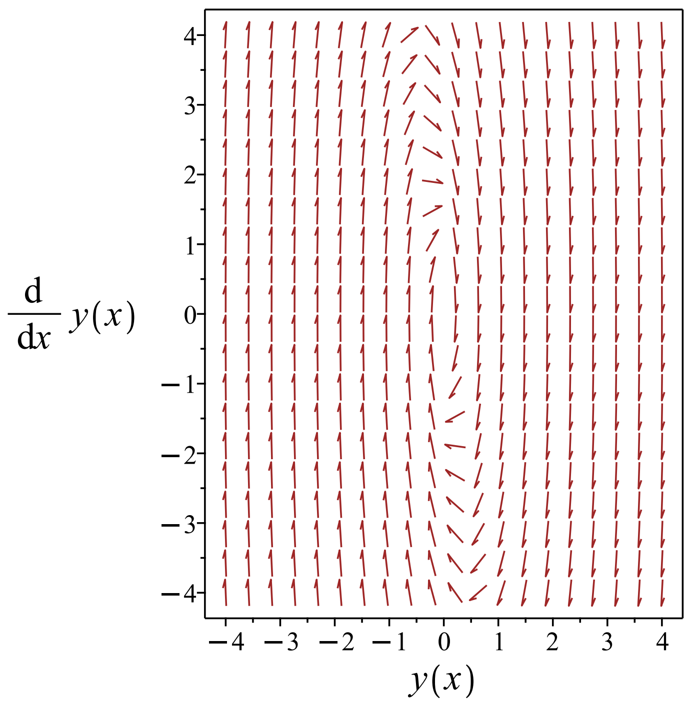

Figure 146: Slope field plot

Verification of solutions

$$y = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x)) + 2 e^{-4x} \cos(5x)$$

Verified OK.

6.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 17y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 17 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(4x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-x} \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(4x)}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x} \cos(4x)\right) + c_2 \left(e^{-x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 17y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(4x)\, e^{-x} c_1 + \frac{e^{-x} \sin(4x)\, c_2}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$60\, e^{-4x} \sin(5x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4x} \cos(5x), e^{-4x} \sin(5x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x} \cos(4x), \frac{e^{-x} \sin(4x)}{4}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-4x} \cos(5x) + A_2 e^{-4x} \sin(5x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$30A_1 e^{-4x} \sin(5x) - 30A_2 e^{-4x} \cos(5x) = 60 e^{-4x} \sin(5x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 e^{-4x} \cos(5x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(4x) e^{-x} c_1 + \frac{e^{-x} \sin(4x) c_2}{4}\right) + \left(2 e^{-4x} \cos(5x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(4x) e^{-x} c_1 + \frac{e^{-x} \sin(4x) c_2}{4} + 2 e^{-4x} \cos(5x) \tag{1}$$

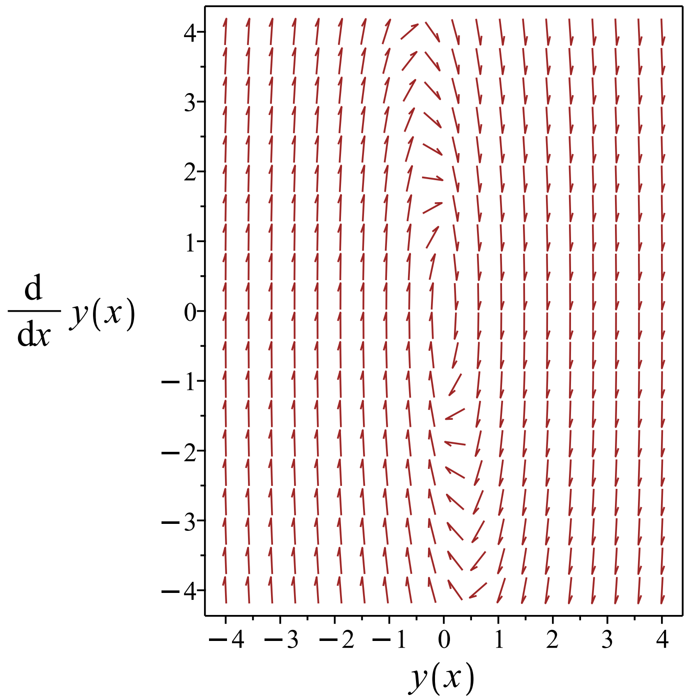

Figure 147: Slope field plot

Verification of solutions

$$y = \cos(4x)\,e^{-x}c_1 + \frac{e^{-x}\sin(4x)\,c_2}{4} + 2\,e^{-4x}\cos(5x)$$

Verified OK.

6.18.3 Maple step by step solution

Let's solve

$y'' + 2y' + 17y = 60\,e^{-4x}\sin(5x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 17 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-1 - 4\,\mathrm{I}, -1 + 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(4x)\,\mathrm{e}^{-x} c_1 + \mathrm{e}^{-x} \sin(4x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 60\,\mathrm{e}^{-4x} \sin(5x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x}\cos(4x) & \mathrm{e}^{-x}\sin(4x) \\ -\mathrm{e}^{-x}\cos(4x) - 4\,\mathrm{e}^{-x}\sin(4x) & -\mathrm{e}^{-x}\sin(4x) + 4\,\mathrm{e}^{-x}\cos(4x) \end{array}\right]$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -15\,\mathrm{e}^{-x}\left(\cos(4x)\left(\int \sin(4x)\sin(5x)\,\mathrm{e}^{-3x} dx\right) - \sin(4x)\left(\int \cos(4x)\sin(5x)\,\mathrm{e}^{-3x} dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{-4x} \cos(5x)$

- Substitute particular solution into general solution to ODE

  $y = \cos(4x)\,\mathrm{e}^{-x} c_1 + \mathrm{e}^{-x} \sin(4x)\, c_2 + 2\,\mathrm{e}^{-4x} \cos(5x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+17*y(x)=60*exp(-4*x)*sin(5*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} \sin (4x)\, c_2 + \mathrm{e}^{-x} \cos (4x)\, c_1 + 2\, \mathrm{e}^{-4x} \cos (5x)$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 42

```
DSolve[y''[x]+2*y'[x]+17*y[x]==60*Exp[-4*x]*Sin[5*x],y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to e^{-4x} \left(2 \cos(5x) + c_2 e^{3x} \cos(4x) + c_1 e^{3x} \sin(4x)\right)$$

6.19 problem 19

6.19.1 Solving as second order linear constant coeff ode 671
6.19.2 Solving using Kovacic algorithm 675
6.19.3 Maple step by step solution 680

Internal problem ID [4825]
Internal file name [`OUTPUT/4318_Sunday_June_05_2022_12_59_26_PM_34062207/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' + 4y' + 5y = 40\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)}$$

6.19.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4, B = 4, C = 5, f(x) = 40\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 5\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{4^2 - (4)(4)(5)} \\
&= -\frac{1}{2} \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + i \\
\lambda_2 &= -\frac{1}{2} - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + i \\
\lambda_2 &= -\frac{1}{2} - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}} (\cos (x) c_1 + c_2 \sin (x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} (\cos (x) c_1 + c_2 \sin (x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$40\,\mathrm{e}^{-\frac{3x}{2}} \sin (2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{-\frac{3x}{2}} \cos (2x)\,, \mathrm{e}^{-\frac{3x}{2}} \sin (2x)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos (x)\,, \mathrm{e}^{-\frac{x}{2}} \sin (x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-\frac{3x}{2}} \cos (2x) + A_2 \mathrm{e}^{-\frac{3x}{2}} \sin (2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &-8A_1 \mathrm{e}^{-\frac{3x}{2}} \cos (2x) + 16A_1 \mathrm{e}^{-\frac{3x}{2}} \sin (2x) - 8A_2 \mathrm{e}^{-\frac{3x}{2}} \sin (2x) - 16A_2 \mathrm{e}^{-\frac{3x}{2}} \cos (2x) \\ &= 40\,\mathrm{e}^{-\frac{3x}{2}} \sin (2x) \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-\frac{3x}{2}} \cos (2x) - \mathrm{e}^{-\frac{3x}{2}} \sin (2x)$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x))\right) + \left(2\,\mathrm{e}^{-\frac{3x}{2}} \cos(2x) - \mathrm{e}^{-\frac{3x}{2}} \sin(2x)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x)) + 2\,\mathrm{e}^{-\frac{3x}{2}} \cos(2x) - \mathrm{e}^{-\frac{3x}{2}} \sin(2x) \tag{1}
$$

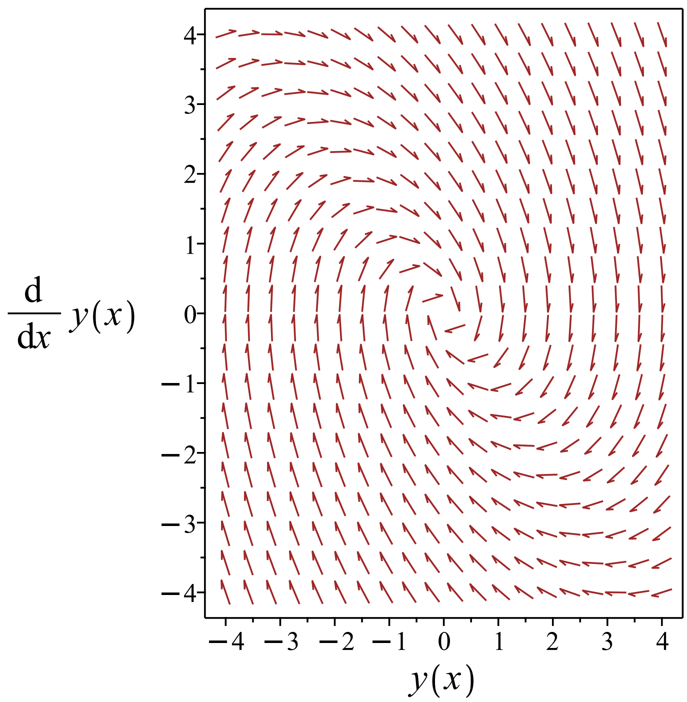

Figure 148: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x)) + 2\,\mathrm{e}^{-\frac{3x}{2}} \cos(2x) - \mathrm{e}^{-\frac{3x}{2}} \sin(2x)
$$

Verified OK.

6.19.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx}\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(x\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\
&= y_1 (\tan\left(x\right))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(x\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(x\right) \left(\tan\left(x\right)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
4y'' + 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos\left(x\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \sin\left(x\right) \mathrm{e}^{-\frac{x}{2}} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$40\,\mathrm{e}^{-\frac{3x}{2}}\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{\mathrm{e}^{-\frac{3x}{2}}\cos(2x), \mathrm{e}^{-\frac{3x}{2}}\sin(2x)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}\cos(x), \mathrm{e}^{-\frac{x}{2}}\sin(x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-\frac{3x}{2}}\cos(2x) + A_2\mathrm{e}^{-\frac{3x}{2}}\sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-8A_1\mathrm{e}^{-\frac{3x}{2}}\cos(2x) + 16A_1\mathrm{e}^{-\frac{3x}{2}}\sin(2x) - 8A_2\mathrm{e}^{-\frac{3x}{2}}\sin(2x) - 16A_2\mathrm{e}^{-\frac{3x}{2}}\cos(2x)\\ &= 40\,\mathrm{e}^{-\frac{3x}{2}}\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-\frac{3x}{2}}\cos(2x) - \mathrm{e}^{-\frac{3x}{2}}\sin(2x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\ &= \left(\cos(x)\,\mathrm{e}^{-\frac{x}{2}}c_1 + \sin(x)\,\mathrm{e}^{-\frac{x}{2}}c_2\right) + \left(2\,\mathrm{e}^{-\frac{3x}{2}}\cos(2x) - \mathrm{e}^{-\frac{3x}{2}}\sin(2x)\right)\end{aligned}$$

Which simplifies to

$$y = e^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{-\frac{3x}{2}} \cos(2x) - e^{-\frac{3x}{2}} \sin(2x)$$

Summary
The solution(s) found are the following

$$y = e^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{-\frac{3x}{2}} \cos(2x) - e^{-\frac{3x}{2}} \sin(2x) \tag{1}$$

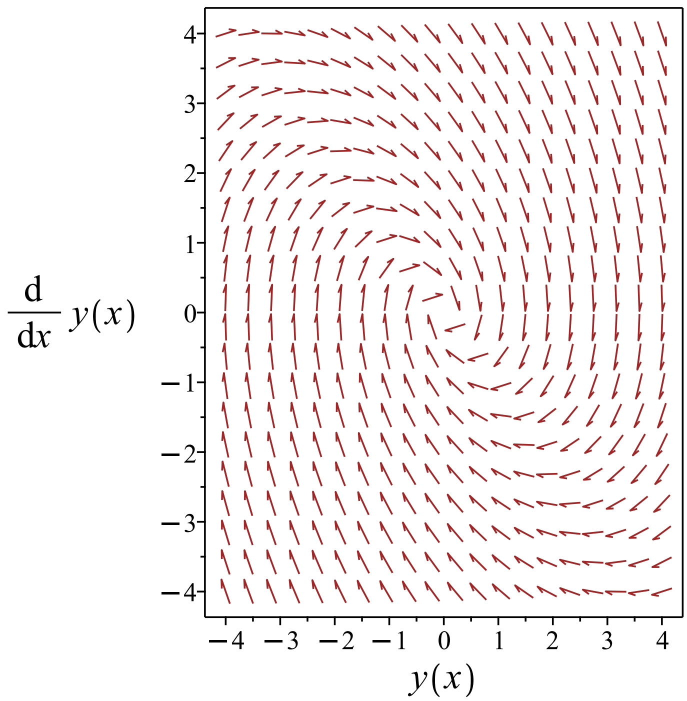

Figure 149: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{-\frac{3x}{2}} \cos(2x) - e^{-\frac{3x}{2}} \sin(2x)$$

Verified OK.

6.19.3 Maple step by step solution

Let's solve

$4y'' + 4y' + 5y = 40\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{5y}{4} + 10\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{5y}{4} = 10\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + \frac{5}{4} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \mathrm{I}, -\frac{1}{2} + \mathrm{I}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x)\,\mathrm{e}^{-\frac{x}{2}} c_1 + \sin(x)\,\mathrm{e}^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 10\,\mathrm{e}^{-\frac{3x}{2}} \sin(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos(x) & e^{-\frac{x}{2}} \sin(x) \\ -\frac{e^{-\frac{x}{2}} \cos(x)}{2} - e^{-\frac{x}{2}} \sin(x) & -\frac{e^{-\frac{x}{2}} \sin(x)}{2} + e^{-\frac{x}{2}} \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -20\, e^{-\frac{x}{2}} \left(\cos(x) \left(\int e^{-x} \cos(x) \sin(x)^2 dx\right) - \sin(x) \left(\int e^{-x} \cos(x)^2 \sin(x) dx\right)\right)$

- Compute integrals

  $y_p(x) = (2\cos(2x) - \sin(2x))\, e^{-\frac{3x}{2}}$

- Substitute particular solution into general solution to ODE

  $y = (2\cos(2x) - \sin(2x))\, e^{-\frac{3x}{2}} + \cos(x)\, e^{-\frac{x}{2}} c_1 + \sin(x)\, e^{-\frac{x}{2}} c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(4*diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=40*exp(-3*x/2)*sin(2*x),y(x), singsol=all)
```

$$y(x) = 4\cos(x)^2 e^{-\frac{3x}{2}} - 2\, e^{-\frac{3x}{2}} \cos(x) \sin(x) + e^{-\frac{x}{2}} \cos(x)\, c_1 + e^{-\frac{x}{2}} \sin(x)\, c_2 - 2\, e^{-\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 42

```
DSolve[4*y''[x]+4*y'[x]+5*y[x]==40*Exp[-3*x/2]*Sin[2*x],y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{-3x/2}(2\cos(2x) + c_1 e^x \sin(x) + \cos(x)(-2\sin(x) + c_2 e^x))$$

6.20 problem 20

6.20.1 Solving as second order linear constant coeff ode 683
6.20.2 Solving using Kovacic algorithm 687
6.20.3 Maple step by step solution 691

Internal problem ID [4826]
Internal file name [`OUTPUT/4319_Sunday_June_05_2022_12_59_35_PM_49159072/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 8y = 30\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)}$$

6.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 8, f(x) = 30\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 8\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(8)}\\ &= -2 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 2i\\ \lambda_2 &= -2 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + 2i\\ \lambda_2 &= -2 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$30\, e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$\left[\left\{e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)\right\}\right]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} \cos(2x), e^{-2x} \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right) + A_2 e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-\frac{15A_1 e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)}{2} + \frac{15A_2 e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)}{2} = 30\, e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4\, e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(4\,e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x)) + 4\,e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right) \tag{1}$$

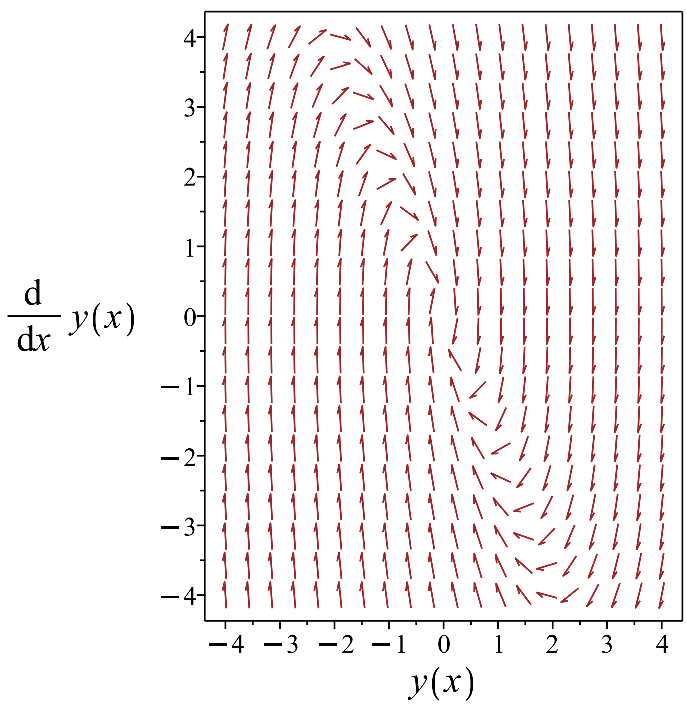

Figure 150: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_1 \cos(2x) + c_2 \sin(2x)) + 4\,e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$$

Verified OK.

6.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x} \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 8y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-2x} \cos(2x)\, c_1 + \frac{\mathrm{e}^{-2x} \sin(2x)\, c_2}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
30\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{5x}{2}\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
\left[\left\{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{5x}{2}\right), \mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{5x}{2}\right)\right\}\right]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-2x}\cos(2x), \frac{\mathrm{e}^{-2x}\sin(2x)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{5x}{2}\right) + A_2\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{5x}{2}\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-\frac{15A_1\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{5x}{2}\right)}{2} + \frac{15A_2\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{5x}{2}\right)}{2} = 30\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{5x}{2}\right)
$$

Solving for the unknowns by comparing coefficients results in

$$
[A_1 = 0, A_2 = 4]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = 4\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{5x}{2}\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-2x}\cos(2x)\,c_1 + \frac{\mathrm{e}^{-2x}\sin(2x)\,c_2}{2}\right) + \left(4\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{5x}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = e^{-2x} \cos(2x) c_1 + \frac{e^{-2x} \sin(2x) c_2}{2} + 4 e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right) \tag{1}$$

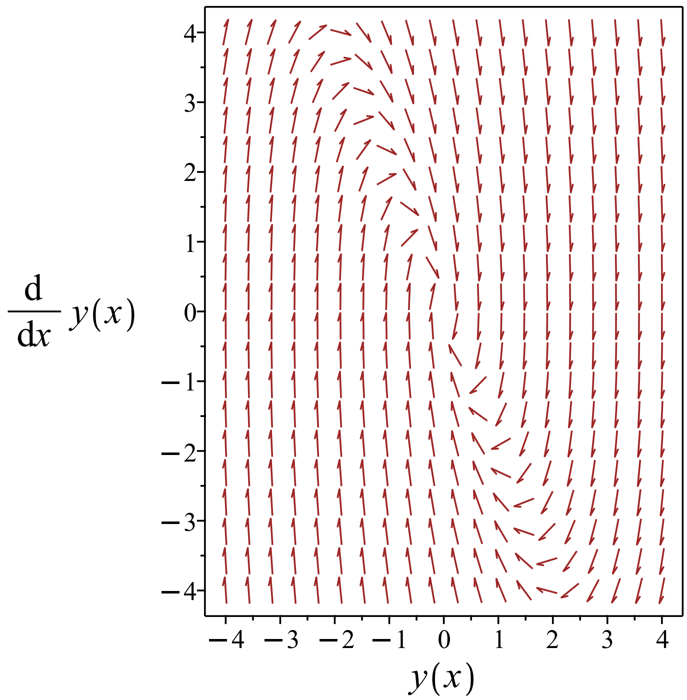

Figure 151: Slope field plot

Verification of solutions

$$y = e^{-2x} \cos(2x) c_1 + \frac{e^{-2x} \sin(2x) c_2}{2} + 4 e^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$$

Verified OK.

6.20.3 Maple step by step solution

Let's solve

$y'' + 4y' + 8y = 30 e^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4r + 8 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4) \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - 2\,\mathrm{I}, -2 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-2x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2x} \cos(2x)\, c_1 + \mathrm{e}^{-2x} \sin(2x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 30\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{5x}{2}\right)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} \cos(2x) & \mathrm{e}^{-2x} \sin(2x) \\ -2\,\mathrm{e}^{-2x} \cos(2x) - 2\,\mathrm{e}^{-2x} \sin(2x) & -2\,\mathrm{e}^{-2x} \sin(2x) + 2\,\mathrm{e}^{-2x} \cos(2x) \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{-4x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -15\,\mathrm{e}^{-2x} \left(\cos(2x) \left(\int \sin(2x) \cos\left(\frac{5x}{2}\right) \mathrm{e}^{\frac{3x}{2}} dx\right) - \sin(2x) \left(\int \cos(2x) \cos\left(\frac{5x}{2}\right) \mathrm{e}^{\frac{3x}{2}} dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = 4\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2x} \cos(2x)\, c_1 + \mathrm{e}^{-2x} \sin(2x)\, c_2 + 4\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 35

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+8*y(x)=30*exp(-x/2)*cos(5/2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x} \sin(2x) c_2 + \mathrm{e}^{-2x} \cos(2x) c_1 + 4\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{5x}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 41

```
DSolve[y''[x]+4*y'[x]+8*y[x]==30*Exp[-x/2]*Cos[5/2*x],y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-2x}\left(4e^{3x/2} \sin\left(\frac{5x}{2}\right) + c_2 \cos(2x) + c_1 \sin(2x)\right)$$

6.21 problem 21

6.21.1 Solving as second order linear constant coeff ode 694
6.21.2 Solving using Kovacic algorithm 698
6.21.3 Maple step by step solution 703

Internal problem ID [4827]
Internal file name [`OUTPUT/4320_Sunday_June_05_2022_12_59_45_PM_22082520/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{5y'' + 6y' + 2y = x^2 + 6x}$$

6.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 5, B = 6, C = 2, f(x) = x^2 + 6x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$5y'' + 6y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 5, B = 6, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$5\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda \, \mathrm{e}^{\lambda x} + 2 \, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$5\lambda^2 + 6\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 5, B = 6, C = 2$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-6}{(2)\,(5)} \pm \frac{1}{(2)\,(5)}\sqrt{6^2 - (4)\,(5)\,(2)} \\ &= -\frac{3}{5} \pm \frac{i}{5} \end{aligned}$$

Hence

$$\begin{aligned} \lambda_1 &= -\frac{3}{5} + \frac{i}{5} \\ \lambda_2 &= -\frac{3}{5} - \frac{i}{5} \end{aligned}$$

Which simplifies to

$$\begin{aligned} \lambda_1 &= -\frac{3}{5} + \frac{i}{5} \\ \lambda_2 &= -\frac{3}{5} - \frac{i}{5} \end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{3}{5}$ and $\beta = \frac{1}{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + c_2 \sin\left(\frac{x}{5}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + c_2 \sin\left(\frac{x}{5}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right), \mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 x^2 + 2A_2 x + 12x A_3 + 2A_1 + 6A_2 + 10A_3 = x^2 + 6x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{2}, A_2 = 0, A_3 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x^2}{2} - \frac{5}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + c_2 \sin\left(\frac{x}{5}\right)\right)\right) + \left(\frac{x^2}{2} - \frac{5}{2}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + c_2 \sin\left(\frac{x}{5}\right)\right) + \frac{x^2}{2} - \frac{5}{2} \tag{1}
$$

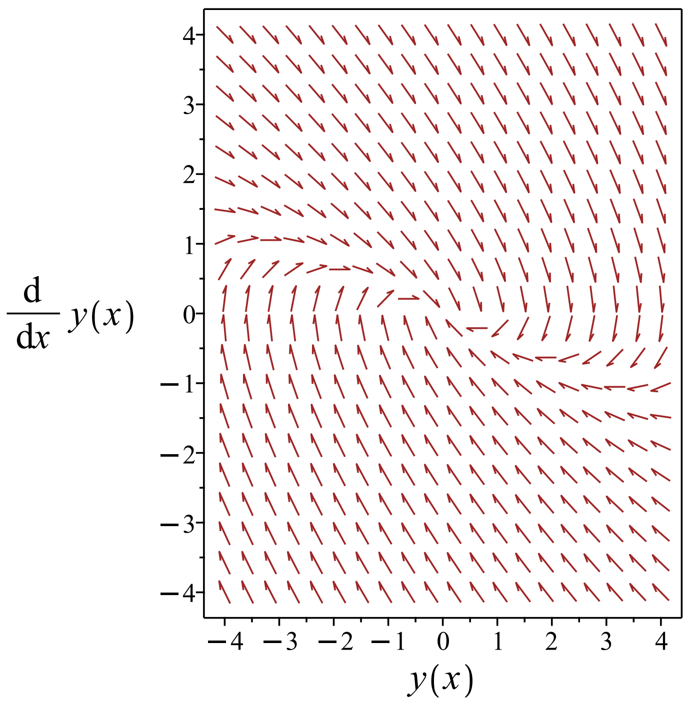

Figure 152: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + c_2 \sin\left(\frac{x}{5}\right)\right) + \frac{x^2}{2} - \frac{5}{2}
$$

Verified OK.

6.21.2 Solving using Kovacic algorithm

Writing the ode as

$$5y'' + 6y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 5 \\ B &= 6 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{25} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 25 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{25} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{1}{25}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{x}{5}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

$$
\begin{aligned}
&= z_1 e^{-\int \frac{1}{2}\frac{6}{5}\,dx} \\
&= z_1 e^{-\frac{3x}{5}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{5}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{6}{5}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{6x}{5}}}{(y_1)^2}\,dx \\
&= y_1 \left(5\tan\left(\frac{x}{5}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right) \left(5\tan\left(\frac{x}{5}\right)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
5y'' + 6y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right) c_1 + 5\,\mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right) c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right), 5\,\mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right)\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_3x^2 + 2A_2x + 12xA_3 + 2A_1 + 6A_2 + 10A_3 = x^2 + 6x
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = -\frac{5}{2}, A_2 = 0, A_3 = \frac{1}{2}\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = \frac{x^2}{2} - \frac{5}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right) c_1 + 5\,\mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right) c_2\right) + \left(\frac{x^2}{2} - \frac{5}{2}\right)
\end{aligned}
$$

Which simplifies to

$$y = e^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + 5c_2 \sin\left(\frac{x}{5}\right)\right) + \frac{x^2}{2} - \frac{5}{2}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + 5c_2 \sin\left(\frac{x}{5}\right)\right) + \frac{x^2}{2} - \frac{5}{2} \tag{1}$$

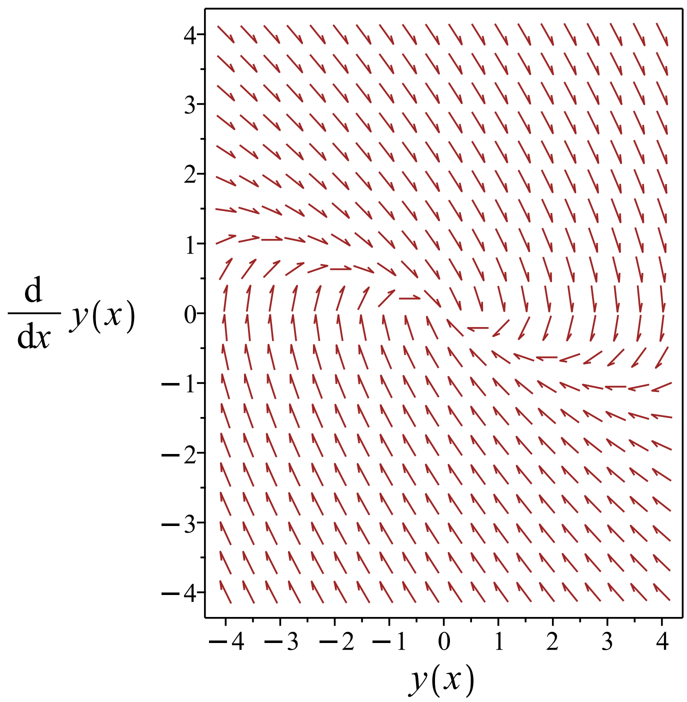

Figure 153: Slope field plot

Verification of solutions

$$y = e^{-\frac{3x}{5}} \left(c_1 \cos\left(\frac{x}{5}\right) + 5c_2 \sin\left(\frac{x}{5}\right)\right) + \frac{x^2}{2} - \frac{5}{2}$$

Verified OK.

6.21.3 Maple step by step solution

Let's solve

$5y'' + 6y' + 2y = x^2 + 6x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{6y'}{5} - \frac{2y}{5} + \frac{x^2}{5} + \frac{6x}{5}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{6y'}{5} + \frac{2y}{5} = \frac{1}{5}x^2 + \frac{6}{5}x$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{6}{5}r + \frac{2}{5} = 0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{6}{5}\right) \pm \left(\sqrt{-\frac{4}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{5} - \frac{I}{5}, -\frac{3}{5} + \frac{I}{5}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-\frac{3x}{5}} \cos\left(\frac{x}{5}\right) c_1 + \mathrm{e}^{-\frac{3x}{5}} \sin\left(\frac{x}{5}\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \frac{1}{5}x^2 + \frac{6}{5}x\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x)\,,y_2(x)) = \begin{bmatrix} e^{-\frac{3x}{5}}\cos\left(\frac{x}{5}\right) & e^{-\frac{3x}{5}}\sin\left(\frac{x}{5}\right) \\ -\frac{3\,e^{-\frac{3x}{5}}\cos\left(\frac{x}{5}\right)}{5} - \frac{e^{-\frac{3x}{5}}\sin\left(\frac{x}{5}\right)}{5} & -\frac{3\,e^{-\frac{3x}{5}}\sin\left(\frac{x}{5}\right)}{5} + \frac{e^{-\frac{3x}{5}}\cos\left(\frac{x}{5}\right)}{5} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x)\,,y_2(x)) = \frac{e^{-\frac{6x}{5}}}{5}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -e^{-\frac{3x}{5}}\left(\cos\left(\frac{x}{5}\right)\left(\int e^{\frac{3x}{5}}\sin\left(\frac{x}{5}\right)x(x+6)\,dx\right) - \sin\left(\frac{x}{5}\right)\left(\int \cos\left(\frac{x}{5}\right)e^{\frac{3x}{5}}x(x+6)\,dx\right)\right)$$

- Compute integrals

  $$y_p(x) = \frac{x^2}{2} - \frac{5}{2}$$

- Substitute particular solution into general solution to ODE

  $$y = e^{-\frac{3x}{5}}\cos\left(\frac{x}{5}\right)c_1 + e^{-\frac{3x}{5}}\sin\left(\frac{x}{5}\right)c_2 + \frac{x^2}{2} - \frac{5}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(5*diff(y(x),x$2)+6*diff(y(x),x)+2*y(x)=x^2+6*x,y(x), singsol=all)
```

$$y(x) = e^{-\frac{3x}{5}}\sin\left(\frac{x}{5}\right)c_2 + e^{-\frac{3x}{5}}\cos\left(\frac{x}{5}\right)c_1 + \frac{x^2}{2} - \frac{5}{2}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 47

```
DSolve[5*y''[x]+6*y'[x]+2*y[x]==x^2+6*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x^2-5\right)+c_2 e^{-3x/5}\cos\left(\frac{x}{5}\right)+c_1 e^{-3x/5}\sin\left(\frac{x}{5}\right)$$

6.22 problem 22

6.22.1 Solving as second order linear constant coeff ode 707
6.22.2 Solving as second order integrable as is ode 710
6.22.3 Solving as second order ode missing y ode 712
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6.22.7 Maple step by step solution 724

Internal problem ID [4828]
Internal file name [`OUTPUT/4321_Sunday_June_05_2022_12_59_54_PM_25321001/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{2y'' + y' = 2x}$$

6.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = 1, C = 0, f(x) = 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' + y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 1, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{1^2 - (4)\,(2)\,(0)}\\ &= -\frac{1}{4} \pm \frac{1}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{4} + \frac{1}{4}$$

$$\lambda_2 = -\frac{1}{4} - \frac{1}{4}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -\frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{\left(-\frac{1}{2}\right)x}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{1, \mathrm{e}^{-\frac{x}{2}}\right\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x, x^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_2 x^2 + A_1 x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2xA_2 + A_1 + 4A_2 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 - 4x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + c_2 e^{-\frac{x}{2}}\right) + \left(x^2 - 4x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{-\frac{x}{2}} + x^2 - 4x \tag{1}$$

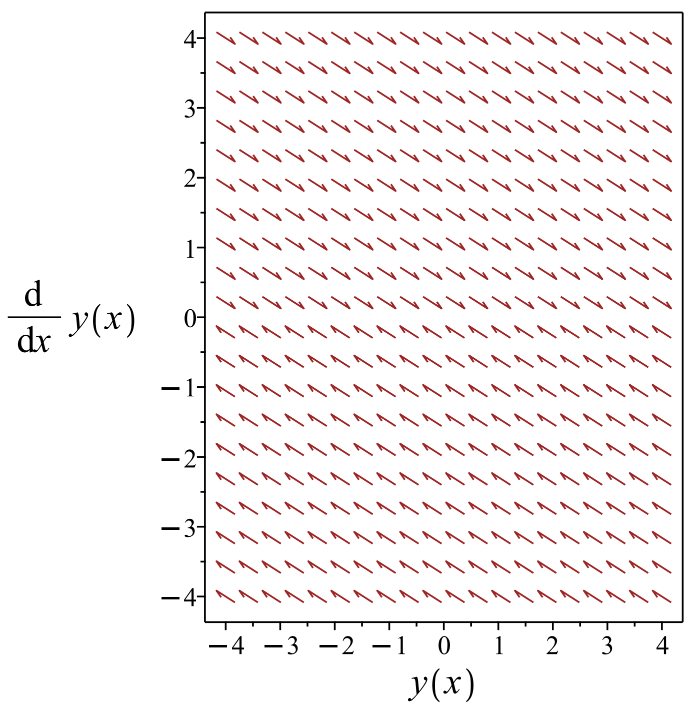

Figure 154: Slope field plot

Verification of solutions

$$
y = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}} + x^2 - 4x
$$

Verified OK.

6.22.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (2y'' + y')\,dx &= \int 2x dx \\
2y' + y &= x^2 + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2} \\
q(x) &= \frac{x^2}{2} + \frac{c_1}{2}
\end{aligned}
$$

Hence the ode is

$$
y' + \frac{y}{2} = \frac{x^2}{2} + \frac{c_1}{2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{2} dx} \\
&= \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2}{2} + \frac{c_1}{2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x}{2}} y\right) &= \left(\mathrm{e}^{\frac{x}{2}}\right)\left(\frac{x^2}{2} + \frac{c_1}{2}\right) \\
\mathrm{d}\left(\mathrm{e}^{\frac{x}{2}} y\right) &= \left(\frac{(x^2 + c_1)\,\mathrm{e}^{\frac{x}{2}}}{2}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$e^{\frac{x}{2}} y = \int \frac{(x^2 + c_1) e^{\frac{x}{2}}}{2} \, dx$$
$$e^{\frac{x}{2}} y = \left(x^2 + c_1 - 4x + 8\right) e^{\frac{x}{2}} + c_2$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$y = e^{-\frac{x}{2}} \left(x^2 + c_1 - 4x + 8\right) e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}$$

which simplifies to

$$y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}} \tag{1}$$

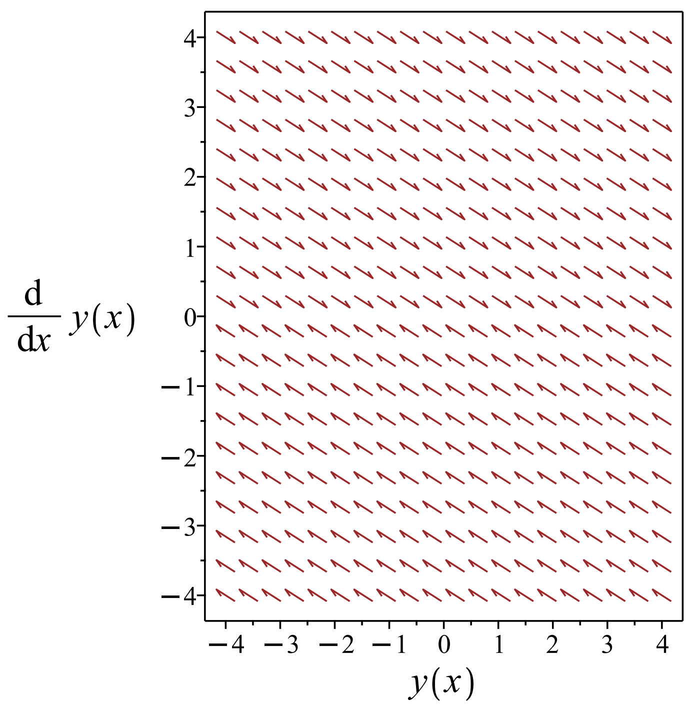

Figure 155: Slope field plot

Verification of solutions

$$y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}}$$

Verified OK.

6.22.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x) + p(x) - 2x = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$p(x) = \frac{1}{2}$$
$$q(x) = x$$

Hence the ode is

$$p'(x) + \frac{p(x)}{2} = x$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{2}dx}$$
$$= e^{\frac{x}{2}}$$

The ode becomes

$$\frac{d}{dx}(\mu p) = (\mu)(x)$$
$$\frac{d}{dx}\left(e^{\frac{x}{2}} p\right) = \left(e^{\frac{x}{2}}\right)(x)$$
$$d\left(e^{\frac{x}{2}} p\right) = \left(x\, e^{\frac{x}{2}}\right) dx$$

Integrating gives

$$e^{\frac{x}{2}} p = \int x\, e^{\frac{x}{2}}\, dx$$
$$e^{\frac{x}{2}} p = 2(-2 + x)\, e^{\frac{x}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$p(x) = 2\,e^{-\frac{x}{2}}(-2+x)\,e^{\frac{x}{2}} + c_1 e^{-\frac{x}{2}}$$

which simplifies to

$$p(x) = 2x - 4 + c_1 e^{-\frac{x}{2}}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 2x - 4 + c_1 e^{-\frac{x}{2}}$$

Integrating both sides gives

$$\begin{aligned} y &= \int 2x - 4 + c_1 e^{-\frac{x}{2}}\, dx \\ &= -4x - 2c_1 e^{-\frac{x}{2}} + x^2 + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -4x - 2c_1 e^{-\frac{x}{2}} + x^2 + c_2 \tag{1}$$

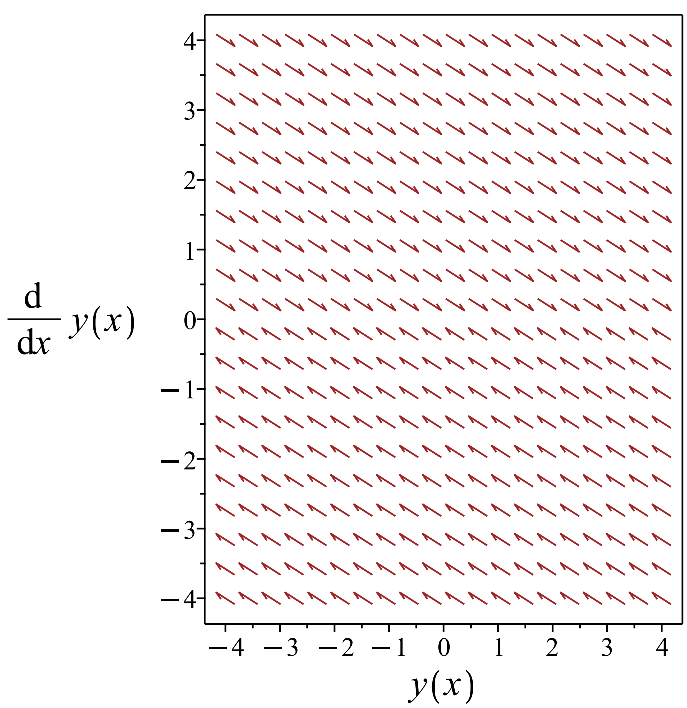

Figure 156: Slope field plot

Verification of solutions

$$
y=-4x-2c_1\mathrm{e}^{-\frac{x}{2}}+x^2+c_2
$$

Verified OK.

6.22.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
2y''+y'=2x
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (2y''+y')\,dx &= \int 2x dx\\
2y'+y &= x^2+c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{2}\\
q(x) &= \frac{x^2}{2}+\frac{c_1}{2}
\end{aligned}
$$

Hence the ode is

$$
y'+\frac{y}{2}=\frac{x^2}{2}+\frac{c_1}{2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{1}{2}dx}\\
&= \mathrm{e}^{\frac{x}{2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2}{2}+\frac{c_1}{2}\right)\\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x}{2}}y\right) &= \left(\mathrm{e}^{\frac{x}{2}}\right)\left(\frac{x^2}{2}+\frac{c_1}{2}\right)\\
\mathrm{d}\left(\mathrm{e}^{\frac{x}{2}}y\right) &= \left(\frac{(x^2+c_1)\,\mathrm{e}^{\frac{x}{2}}}{2}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{\frac{x}{2}} y &= \int \frac{(x^2 + c_1)\, e^{\frac{x}{2}}}{2} \, dx \\
e^{\frac{x}{2}} y &= \left(x^2 + c_1 - 4x + 8\right) e^{\frac{x}{2}} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{\frac{x}{2}}$ results in

$$
y = e^{-\frac{x}{2}} \left(x^2 + c_1 - 4x + 8\right) e^{\frac{x}{2}} + c_2 e^{-\frac{x}{2}}
$$

which simplifies to

$$
y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}}
$$

Summary

The solution(s) found are the following

$$
y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}} \tag{1}
$$

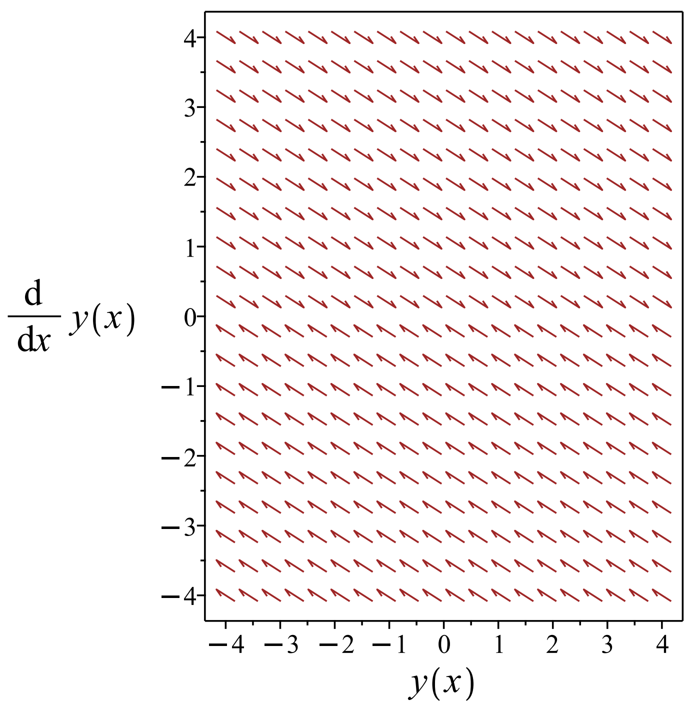

Figure 157: Slope field plot

Verification of solutions

$$
y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}}
$$

Verified OK.

6.22.5 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{1}{2} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-\frac{x}{2}}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(2\,\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \left(2\,\mathrm{e}^{\frac{x}{2}}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
2y'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + 2c_2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{-\frac{x}{2}} \\ y_2 &= 2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x}{2}} & 2 \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x}{2}}\right) & \frac{d}{dx}(2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x}{2}} & 2 \\ -\frac{\mathrm{e}^{-\frac{x}{2}}}{2} & 0 \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-\frac{x}{2}}\right)(0) - (2)\left(-\frac{\mathrm{e}^{-\frac{x}{2}}}{2}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x}{2}}$$

Which simplifies to

$$W = \mathrm{e}^{-\frac{x}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x}{2\,\mathrm{e}^{-\frac{x}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int 2x\,\mathrm{e}^{\frac{x}{2}}dx$$

Hence

$$u_1 = -4(-2+x)\,\mathrm{e}^{\frac{x}{2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{-\frac{x}{2}}x}{2\,\mathrm{e}^{-\frac{x}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -4\,\mathrm{e}^{-\frac{x}{2}}(-2+x)\,\mathrm{e}^{\frac{x}{2}} + x^2$$

Which simplifies to

$$y_p(x) = x^2 - 4x + 8$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-\frac{x}{2}} + 2c_2\right) + \left(x^2 - 4x + 8\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + 2c_2 + x^2 - 4x + 8 \tag{1}$$

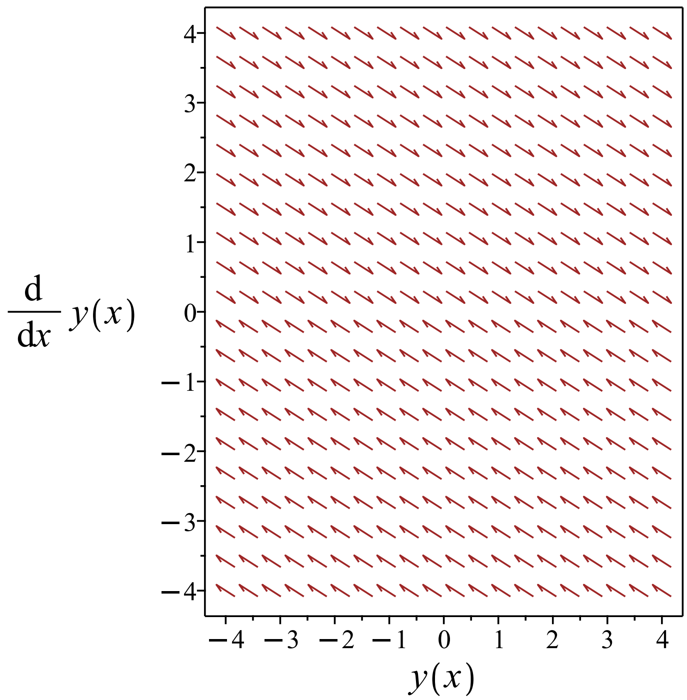

Figure 158: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + 2c_2 + x^2 - 4x + 8$$

Verified OK.

6.22.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 2\\
q(x) &= 1\\
r(x) &= 0\\
s(x) &= 2x
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$2y'+y=\int 2x\,dx$$

We now have a first order ode to solve which is

$$2y'+y=x^2+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}
p(x) &= \frac{1}{2}\\
q(x) &= \frac{x^2}{2}+\frac{c_1}{2}
\end{aligned}$$

Hence the ode is

$$y' + \frac{y}{2} = \frac{x^2}{2} + \frac{c_1}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{2}dx}\\ &= \mathrm{e}^{\frac{x}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x^2}{2} + \frac{c_1}{2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\frac{x}{2}} y\right) &= \left(\mathrm{e}^{\frac{x}{2}}\right)\left(\frac{x^2}{2} + \frac{c_1}{2}\right)\\ \mathrm{d}\left(\mathrm{e}^{\frac{x}{2}} y\right) &= \left(\frac{(x^2 + c_1)\,\mathrm{e}^{\frac{x}{2}}}{2}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{\frac{x}{2}} y &= \int \frac{(x^2 + c_1)\,\mathrm{e}^{\frac{x}{2}}}{2}\,\mathrm{d}x\\ \mathrm{e}^{\frac{x}{2}} y &= \left(x^2 + c_1 - 4x + 8\right)\mathrm{e}^{\frac{x}{2}} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{\frac{x}{2}}$ results in

$$y = \mathrm{e}^{-\frac{x}{2}}\left(x^2 + c_1 - 4x + 8\right)\mathrm{e}^{\frac{x}{2}} + c_2\mathrm{e}^{-\frac{x}{2}}$$

which simplifies to

$$y = x^2 + c_1 - 4x + 8 + c_2\mathrm{e}^{-\frac{x}{2}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = x^2 + c_1 - 4x + 8 + c_2\mathrm{e}^{-\frac{x}{2}} \tag{1}$$

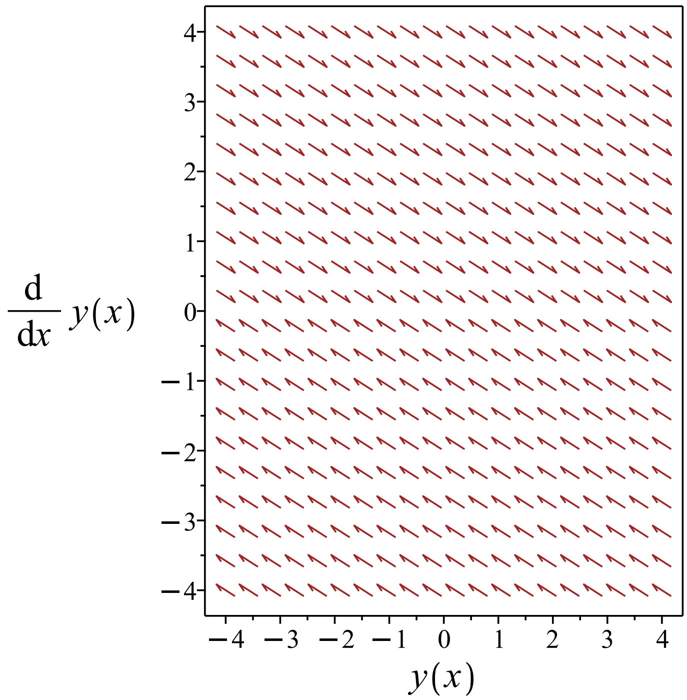

Figure 159: Slope field plot

Verification of solutions

$$y = x^2 + c_1 - 4x + 8 + c_2 e^{-\frac{x}{2}}$$

Verified OK.

6.22.7 Maple step by step solution

Let's solve

$2y'' + y' = 2x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{2} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{2} = x$

- Characteristic polynomial of homogeneous ODE

$r^2 + \frac{1}{2}r = 0$

- Factor the characteristic polynomial

  $\frac{r(2r+1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(0, -\frac{1}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 \mathrm{e}^{-\frac{x}{2}} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & \mathrm{e}^{-\frac{x}{2}} \\ 0 & -\frac{\mathrm{e}^{-\frac{x}{2}}}{2} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = -\frac{\mathrm{e}^{-\frac{x}{2}}}{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 2\left(\int x dx\right) - 2\,\mathrm{e}^{-\frac{x}{2}}\left(\int x\,\mathrm{e}^{\frac{x}{2}} dx\right)$

  - Compute integrals

    $y_p(x) = x^2 - 4x + 8$

- Substitute particular solution into general solution to ODE

  $y = x^2 + c_1 - 4x + 8 + c_2 \mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(1/2)*_b(_a)+_a, _b(_a)`   *** Subleve
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(2*diff(y(x),x$2)+diff(y(x),x)=2*x,y(x), singsol=all)
```

$$y(x) = -2\,\mathrm{e}^{-\frac{x}{2}}c_1 + x^2 - 4x + c_2$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 23

```
DSolve[y''[x]+y'[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 - 2x - c_1 e^{-x} + c_2$$

6.23 problem 23

6.23.1 Solving as second order linear constant coeff ode 727
6.23.2 Solving using Kovacic algorithm 730
6.23.3 Maple step by step solution 735

Internal problem ID [4829]
Internal file name [`OUTPUT/4322_Sunday_June_05_2022_01_00_02_PM_70035153/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 2\,\mathrm{e}^{x} x}$$

6.23.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\,\mathrm{e}^{x} x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 e^x x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x x + 2A_1 e^x + 2A_2 e^x = 2 e^x x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x x - e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (e^x x - e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x x - e^x \tag{1}$$

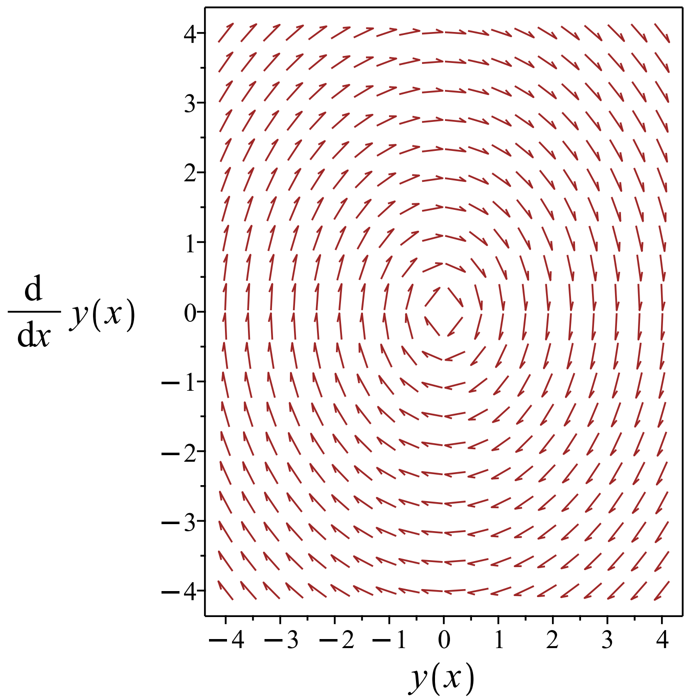

Figure 160: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x x - e^x$$

Verified OK.

6.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos(x)\,c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x x + 2A_1 e^x + 2A_2 e^x = 2\,e^x x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x x - e^x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2 \sin(x)) + (e^x x - e^x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2 \sin(x) + e^x x - e^x \tag{1}$$

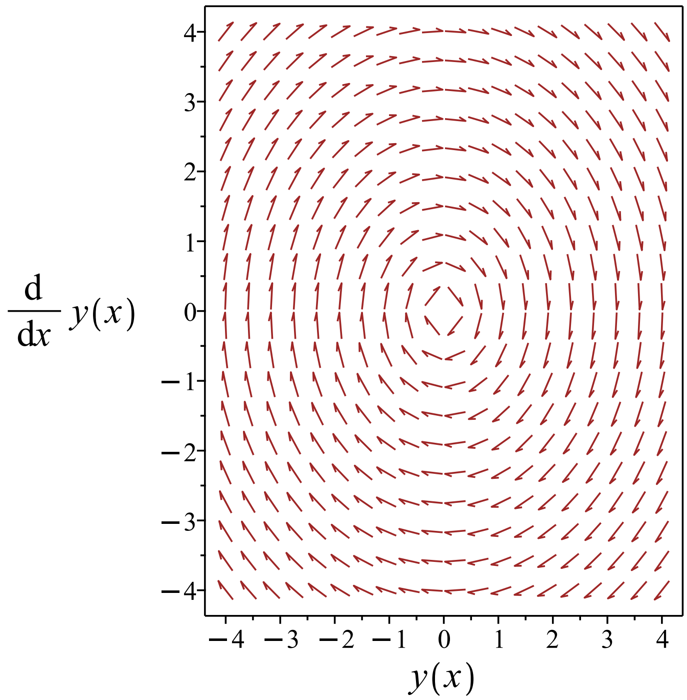

Figure 161: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x x - e^x$$

Verified OK.

6.23.3 Maple step by step solution

Let's solve

$y'' + y = 2 e^x x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^x x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\cos(x)\left(\int \sin(x)\,\mathrm{e}^x x dx\right) + 2\sin(x)\left(\int \cos(x)\,\mathrm{e}^x x dx\right)$

  ◦ Compute integrals

  $y_p(x) = (x-1)\,\mathrm{e}^x$

- Substitute particular solution into general solution to ODE

  $y = (x-1)\,\mathrm{e}^x + \cos(x) c_1 + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+y(x)=2*x*exp(x),y(x), singsol=all)
```

$$y(x) = c_2 \sin(x) + \cos(x) c_1 + (x - 1) e^x$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==2*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x (x - 1) + c_1 \cos(x) + c_2 \sin(x)$$

6.24 problem 24

6.24.1 Solving as second order linear constant coeff ode 738
6.24.2 Solving as linear second order ode solved by an integrating factor ode 741
6.24.3 Solving using Kovacic algorithm 743
6.24.4 Maple step by step solution 748

Internal problem ID [4830]
Internal file name [`OUTPUT/4323_Sunday_June_05_2022_01_00_11_PM_66479638/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 9y = 12\,e^{3x}x}$$

6.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 12\,e^{3x}x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 e^{3x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\, e^{3x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x} x, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x}x, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{3x}, e^{3x}x\}]$$

Since $e^{3x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{3x}, x^3 e^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{3x} + A_2 x^3 e^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{3x} + 6A_2 x\, e^{3x} = 12\, e^{3x} x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^3 e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 x\, e^{3x}\right) + \left(2x^3 e^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x} \tag{1}$$

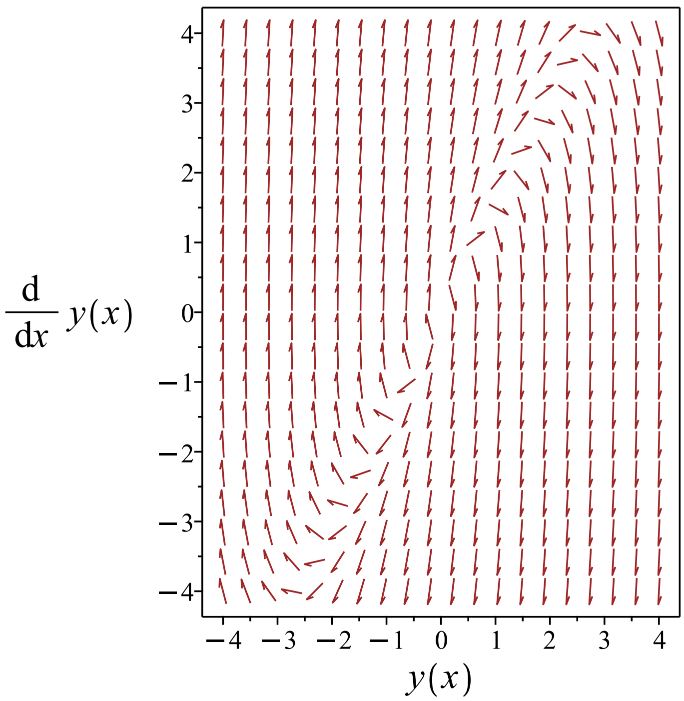

Figure 162: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x}$$

Verified OK.

6.24.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -6\,dx} \\ &= e^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 12\,\mathrm{e}^{-3x}\mathrm{e}^{3x}x$$
$$\left(\mathrm{e}^{-3x}y\right)'' = 12\,\mathrm{e}^{-3x}\mathrm{e}^{3x}x$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)' = 6x^2 + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right) = 2x^3 + c_1x + c_2$$

Hence the solution is

$$y = \frac{2x^3 + c_1x + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = 2x^3\mathrm{e}^{3x} + c_1x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x}$$

Summary

The solution(s) found are the following

$$y = 2x^3\mathrm{e}^{3x} + c_1x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} \tag{1}$$

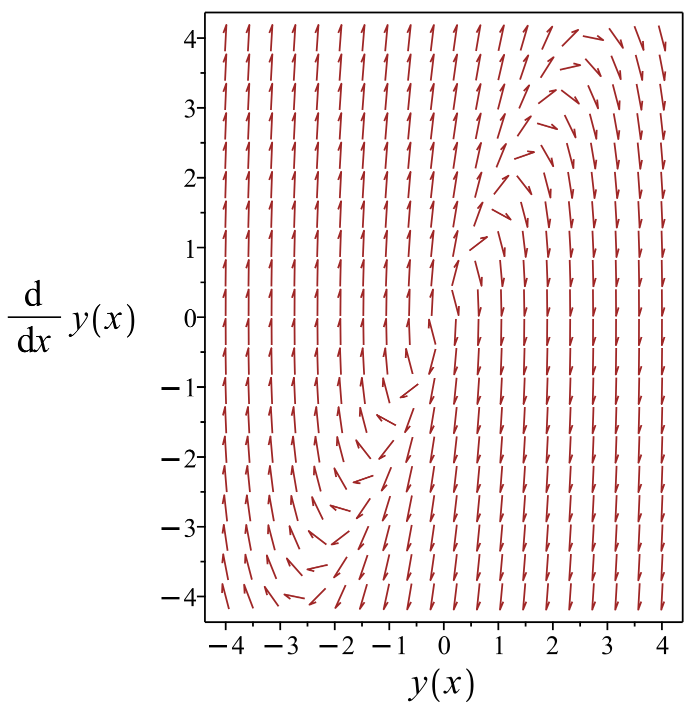

Figure 163: Slope field plot

Verification of solutions

$$y = 2x^3 e^{3x} + c_1 x\, e^{3x} + c_2 e^{3x}$$

Verified OK.

6.24.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{3x}\right) + c_2 \left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{3x} c_1 + c_2 x\,\mathrm{e}^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$12\,\mathrm{e}^{3x} x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x} x, \mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{3x} x, \mathrm{e}^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{3x}, \mathrm{e}^{3x} x\}]$$

Since $\mathrm{e}^{3x} x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \mathrm{e}^{3x}, x^3 \mathrm{e}^{3x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \mathrm{e}^{3x} + A_2 x^3 \mathrm{e}^{3x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \mathrm{e}^{3x} + 6A_2 x\,\mathrm{e}^{3x} = 12\,\mathrm{e}^{3x} x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2x^3 \mathrm{e}^{3x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{3x} c_1 + c_2 x\, e^{3x}\right) + \left(2x^3 e^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x} \tag{1}
$$

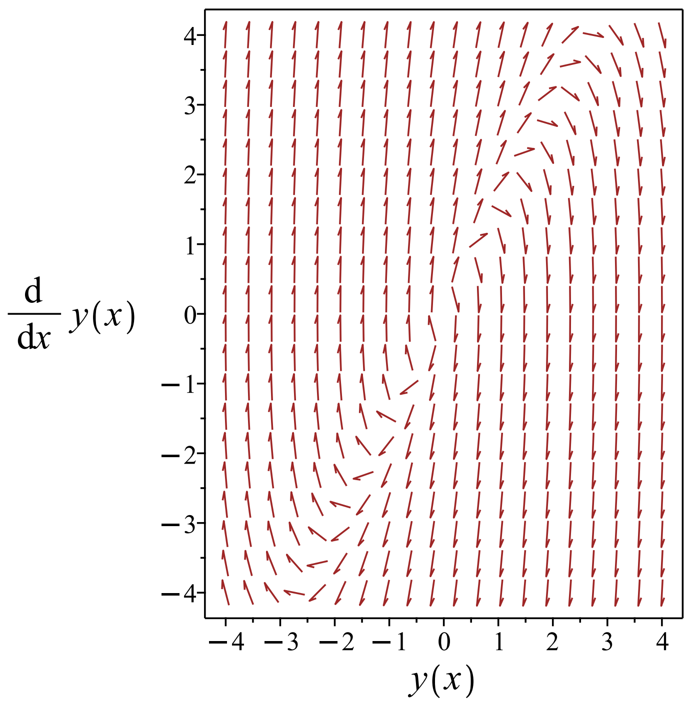

Figure 164: Slope field plot

<u>Verification of solutions</u>

$$
y = e^{3x}(c_2 x + c_1) + 2x^3 e^{3x}
$$

Verified OK.

6.24.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 12\,\mathrm{e}^{3x} x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{3x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{3x} c_1 + c_2 x\,\mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 12\,\mathrm{e}^{3x} x\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} & \mathrm{e}^{3x} x \\ 3\,\mathrm{e}^{3x} & 3\,\mathrm{e}^{3x} x + \mathrm{e}^{3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{6x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -12\,\mathrm{e}^{3x}\left(\int x^2 dx - \left(\int x dx\right) x\right)$

- Compute integrals

  $y_p(x) = 2x^3 e^{3x}$

- Substitute particular solution into general solution to ODE

  $y = 2x^3 e^{3x} + c_2 x\, e^{3x} + e^{3x} c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=12*x*exp(3*x),y(x), singsol=all)
```

$$y(x) = e^{3x}(2x^3 + c_1 x + c_2)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[y''[x]-6y'[x]+9*y[x]==12*x*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(2x^3 + c_2 x + c_1)$$

6.25 problem 25

6.25.1 Solving as second order linear constant coeff ode 750
6.25.2 Solving using Kovacic algorithm 753
6.25.3 Maple step by step solution 758

Internal problem ID [4831]
Internal file name [`OUTPUT/4324_Sunday_June_05_2022_01_00_20_PM_62826217/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' - 3y = 16x^2 e^{-x}}$$

6.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -3, f(x) = 16x^2 e^{-x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - 3\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-3)}\\ &= 1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2\\ \lambda_2 &= 1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16x^2 e^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{-x}, x^2 e^{-x}, e^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{3x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-x}, x^2 e^{-x}, x^3 e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-x} + A_2 x^2 e^{-x} + A_3 x^3 e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 e^{-x} + 2A_2 e^{-x} - 8A_2 x\,e^{-x} + 6A_3 x\,e^{-x} - 12A_3 x^2 e^{-x} = 16x^2 e^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -1, A_3 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{-x}\right) + \left(-\frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{3x}c_1 + c_2 e^{-x} - \frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3} \tag{1}$$

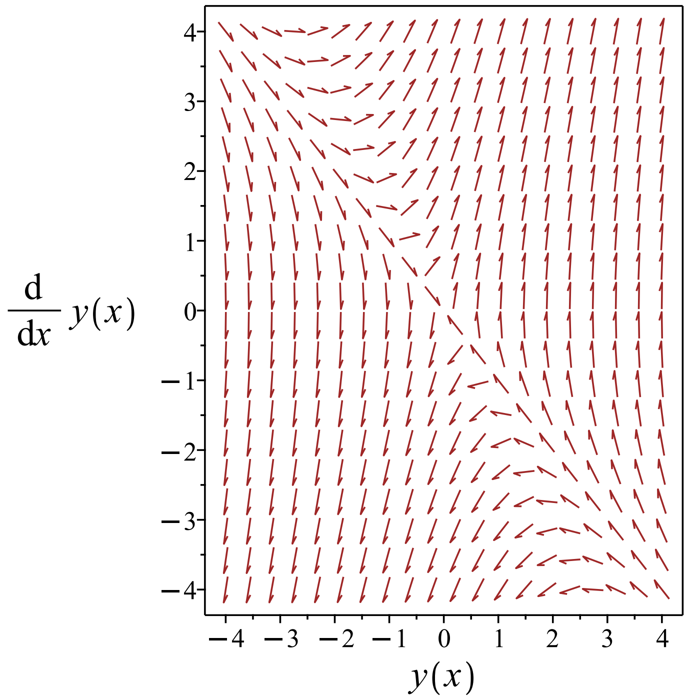

Figure 165: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2 e^{-x} - \frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3}$$

Verified OK.

6.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= -3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx} \\
&= z_1 e^{x} \\
&= z_1 (\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-x}\right) + c_2 \left(e^{-x} \left(\frac{e^{4x}}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-x} + \frac{c_2 e^{3x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16x^2\mathrm{e}^{-x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, \mathrm{e}^{-x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{3x}}{4}, \mathrm{e}^{-x}\right\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}, x^3\mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 x^2\mathrm{e}^{-x} + A_3 x^3\mathrm{e}^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1\mathrm{e}^{-x} + 2A_2\mathrm{e}^{-x} - 8A_2 x\,\mathrm{e}^{-x} + 6A_3 x\,\mathrm{e}^{-x} - 12A_3 x^2\mathrm{e}^{-x} = 16x^2\mathrm{e}^{-x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = -1, A_3 = -\frac{4}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,\mathrm{e}^{-x}}{2} - x^2\mathrm{e}^{-x} - \frac{4x^3\mathrm{e}^{-x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{3x}}{4}\right) + \left(-\frac{x\,\mathrm{e}^{-x}}{2} - x^2\mathrm{e}^{-x} - \frac{4x^3\mathrm{e}^{-x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} - \frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3} \tag{1}$$

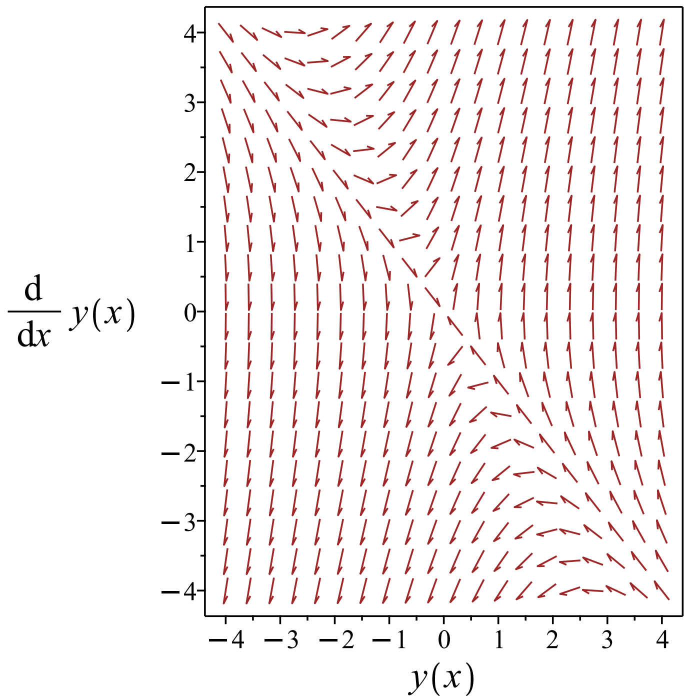

Figure 166: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{3x}}{4} - \frac{x\,e^{-x}}{2} - x^2 e^{-x} - \frac{4x^3 e^{-x}}{3}$$

Verified OK.

6.25.3 Maple step by step solution

Let's solve

$y'' - 2y' - 3y = 16x^2 e^{-x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 16x^2\mathrm{e}^{-x}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & \mathrm{e}^{3x} \\ -\mathrm{e}^{-x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,\mathrm{e}^{-x}\left(\int x^2 dx\right) + 4\,\mathrm{e}^{3x}\left(\int \mathrm{e}^{-4x} x^2 dx\right)$

  ◦ Compute integrals

  $y_p(x) = -\frac{\mathrm{e}^{-x}(32x^3+24x^2+12x+3)}{24}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{3x} - \frac{\mathrm{e}^{-x}(32x^3+24x^2+12x+3)}{24}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=16*x^2*exp(-x),y(x), singsol=all)
```

$$y(x) = \frac{(-8x^3 - 6x^2 + 6c_2 - 3x)\,\mathrm{e}^{-x}}{6} + c_1 \mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 37

```
DSolve[y''[x]-2*y'[x]-3*y[x]==16*x*Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}e^{-x}\left(-8x^2 - 4x + 4c_2 e^{4x} - 1 + 4c_1\right)$$

6.26 problem 26

6.26.1 Solving as second order linear constant coeff ode 761
6.26.2 Solving using Kovacic algorithm 765
6.26.3 Maple step by step solution 769

Internal problem ID [4832]
Internal file name [`OUTPUT/4325_Sunday_June_05_2022_01_00_29_PM_21740273/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = 8 \sin(x) x}$$

6.26.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 8 \sin(x) x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x)\, c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ &= 8 \sin(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -2, A_4 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\cos(x)\,x^2 + 2\sin(x)\,x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2\sin(x)) + \left(-2\cos(x)\,x^2 + 2\sin(x)\,x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2\sin(x) - 2\cos(x)\,x^2 + 2\sin(x)\,x \tag{1}$$

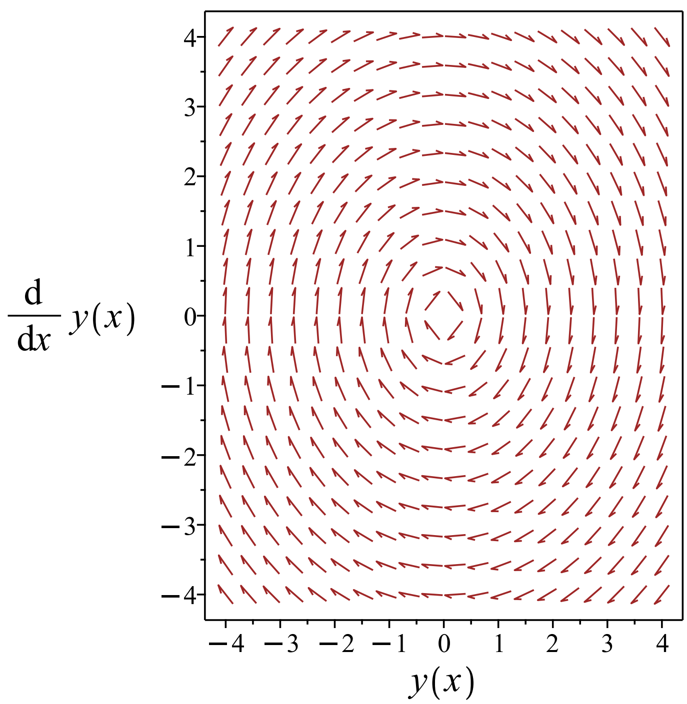

Figure 167: Slope field plot

Verification of solutions

$$y = \cos(x)\,c_1 + c_2\sin(x) - 2\cos(x)\,x^2 + 2\sin(x)\,x$$

Verified OK.

6.26.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(x) c_1 + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8 \sin(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ &= 8 \sin(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -2, A_4 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2 \cos(x) x^2 + 2 \sin(x) x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + \left(-2 \cos(x) x^2 + 2 \sin(x) x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\cos(x) x^2 + 2\sin(x) x \tag{1}$$

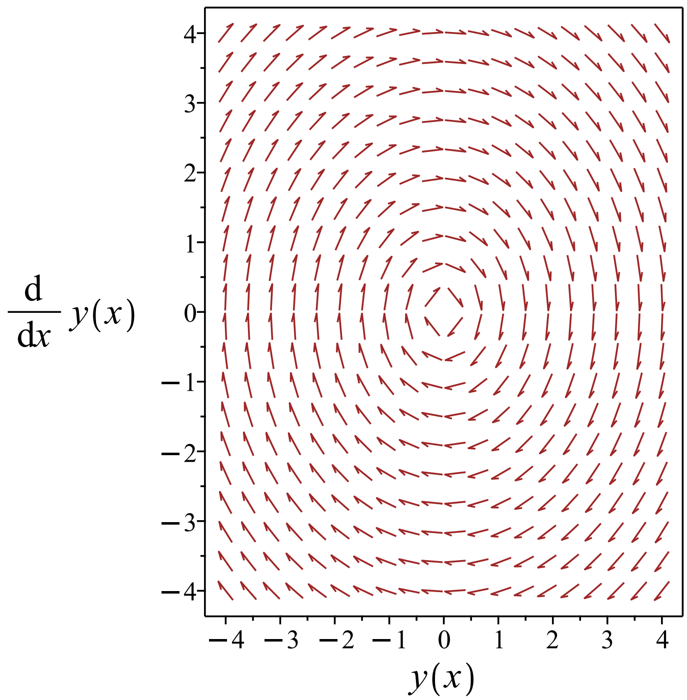

Figure 168: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\cos(x) x^2 + 2\sin(x) x$$

Verified OK.

6.26.3 Maple step by step solution

Let's solve

$y'' + y = 8\sin(x) x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\sin(x)x\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -8\cos(x)\left(\int \sin(x)^2 x dx\right) + 4\sin(x)\left(\int x\sin(2x) dx\right)$
  - Compute integrals

    $y_p(x) = 2x(-\cos(x)x + \sin(x))$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + 2x(-\cos(x)x + \sin(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.016 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+y(x)=8*x*sin(x),y(x), singsol=all)
```

$$y(x) = (-2x^2 + c_1)\cos(x) + 2\sin(x)\left(x + \frac{c_2}{2}\right)$$

✓ Solution by Mathematica
Time used: 0.031 (sec). Leaf size: 27

```
DSolve[y''[x]+y[x]==8*x*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (-2x^2 + 1 + c_1)\cos(x) + (2x + c_2)\sin(x)$$

6.27 problem 33

6.27.1 Solving as second order linear constant coeff ode 772
6.27.2 Solving using Kovacic algorithm 776
6.27.3 Maple step by step solution 780

Internal problem ID [4833]
Internal file name [`OUTPUT/4326_Sunday_June_05_2022_01_00_38_PM_93148565/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = x^3 - 1 + 2\cos(x) + (-4x + 2)\,\mathrm{e}^{x}}$$

6.27.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x^3 - 1 + 2\cos(x) + (-4x + 2)\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 - 1 + 2\cos(x) + (-4x + 2) e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x x, e^x\}, \{\cos(x), \sin(x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x x, e^x\}, \{\cos(x) x, \sin(x) x\}, \{1, x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x x + A_2 e^x + A_3 \cos(x) x + A_4 \sin(x) x + A_5 + A_6 x + A_7 x^2 + A_8 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}2A_1 e^x x + 2A_1 e^x + 2A_2 e^x - 2A_3 \sin(x) + 2A_4 \cos(x) + 2A_7 + 6A_8 x \\ + A_5 + A_6 x + A_7 x^2 + A_8 x^3 = x^3 - 1 + 2\cos(x) + (-4x + 2) e^x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 3, A_3 = 0, A_4 = 1, A_5 = -1, A_6 = -6, A_7 = 0, A_8 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,e^x x + 3\,e^x - 1 + \sin(x)\,x - 6x + x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2 \sin(x)) + \left(-2\,e^x x + 3\,e^x - 1 + \sin(x)\,x - 6x + x^3\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x)\,c_1 + c_2 \sin(x) - 2\,e^x x + 3\,e^x - 1 + \sin(x)\,x - 6x + x^3 \tag{1}$$

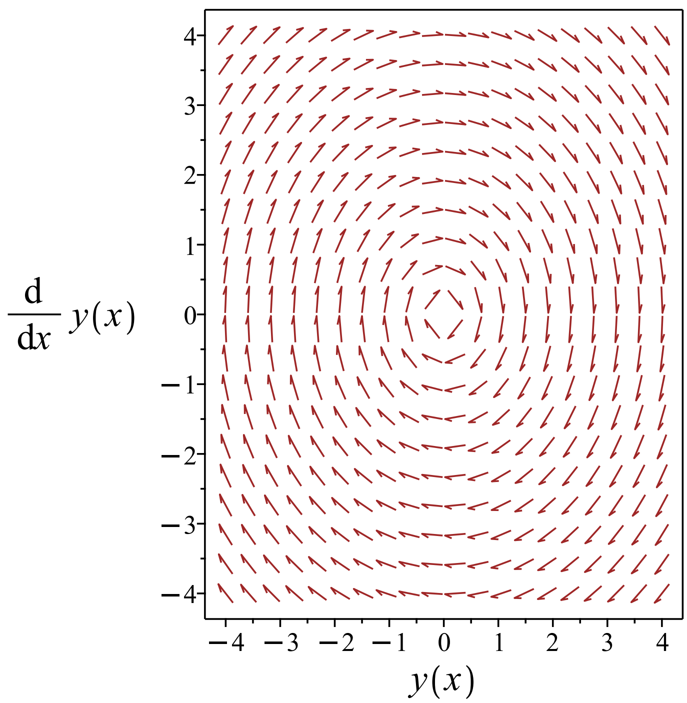

Figure 169: Slope field plot

Verification of solutions

$$y = \cos(x)\,c_1 + c_2 \sin(x) - 2\,e^x x + 3\,e^x - 1 + \sin(x)\,x - 6x + x^3$$

Verified OK.

6.27.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)\, (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)\, (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(x)\, c_1 + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 - 1 + 2\cos(x) + (-4x + 2)\,\mathrm{e}^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^x x, \mathrm{e}^x\}, \{\cos(x), \sin(x)\}, \{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^x x, \mathrm{e}^x\}, \{\cos(x)\,x, \sin(x)\,x\}, \{1, x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^x x + A_2 \mathrm{e}^x + A_3 \cos(x)\,x + A_4 \sin(x)\,x + A_5 + A_6 x + A_7 x^2 + A_8 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1 \mathrm{e}^x x + 2A_1 \mathrm{e}^x + 2A_2 \mathrm{e}^x - 2A_3 \sin(x) + 2A_4 \cos(x) + 2A_7 + 6A_8 x \\ &+ A_5 + A_6 x + A_7 x^2 + A_8 x^3 = x^3 - 1 + 2\cos(x) + (-4x + 2)\,\mathrm{e}^x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 3, A_3 = 0, A_4 = 1, A_5 = -1, A_6 = -6, A_7 = 0, A_8 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2\,\mathrm{e}^x x + 3\,\mathrm{e}^x - 1 + \sin(x)\,x - 6x + x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x)\,c_1 + c_2 \sin(x)) + \left(-2\,\mathrm{e}^x x + 3\,\mathrm{e}^x - 1 + \sin(x)\,x - 6x + x^3\right)\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\,\mathrm{e}^{x} x + 3\,\mathrm{e}^{x} - 1 + \sin(x) x - 6x + x^3 \tag{1}$$

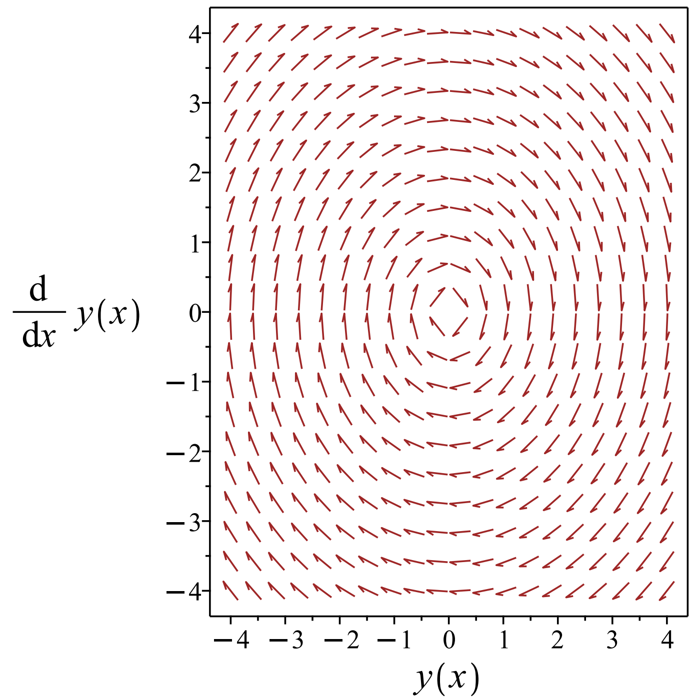

Figure 170: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) - 2\,\mathrm{e}^{x} x + 3\,\mathrm{e}^{x} - 1 + \sin(x) x - 6x + x^3$$

Verified OK.

6.27.3 Maple step by step solution

Let's solve

$y'' + y = x^3 - 1 + 2\cos(x) + (-4x + 2)\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = x^3 - 4\,\mathrm{e}^{x} x - y + 2\cos(x) + 2\,\mathrm{e}^{x} - 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+y=-1+x^3-4\,\mathrm{e}^{x}x+2\cos(x)+2\,\mathrm{e}^{x}$
- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(x)$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=\cos(x)\,c_1+c_2\sin(x)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=-1+x^3-4\,\mathrm{e}^{x}x+2\,\mathrm{co}\right.$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x),y_2(x))=\left[\begin{array}{cc}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{array}\right]$
  - Compute Wronskian

    $W(y_1(x),y_2(x))=1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\cos(x)\left(\int-(x^3-1+2\cos(x)+(-4x+2)\,\mathrm{e}^{x})\sin(x)\,dx\right)-\sin(x)\left(\int-(x^3-1+2\cos\right.$
  - Compute integrals

    $y_p(x)=\mathrm{e}^{x}(-2x+3)+x^3+\sin(x)\,x-6x+\cos(x)-1$
- Substitute particular solution into general solution to ODE

$$y = \cos(x) c_1 + c_2 \sin(x) + e^x(-2x+3) + x^3 + \sin(x) x - 6x + \cos(x) - 1$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+y(x)=x^3-1+2*cos(x)+(2-4*x)*exp(x),y(x), singsol=all)
```

$$y(x) = (1 + c_1) \cos(x) + (-2x+3) e^x + \sin(x) (c_2 + x) + x^3 - 6x - 1$$

✓ Solution by Mathematica

Time used: 0.572 (sec). Leaf size: 40

```
DSolve[y''[x]+y[x]==x^3-1+2*Cos[x]+(2-4*x)*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 - 2e^x x - 6x + 3e^x + \left(\frac{1}{2} + c_1\right) \cos(x) + (x + c_2) \sin(x) - 1$$

6.28 problem 34

6.28.1 Solving as second order linear constant coeff ode 783
6.28.2 Solving using Kovacic algorithm 786
6.28.3 Maple step by step solution 791

Internal problem ID [4834]
Internal file name [`OUTPUT/4327_Sunday_June_05_2022_01_00_48_PM_6401573/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 6y = 2\,e^{x} + 6x - 5}$$

6.28.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = 2\,e^{x} + 6x - 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda\, \mathrm{e}^{\lambda x} + 6\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)}\\ &= \frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{5}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{5}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 2$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(2)x}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 e^{x} + 6x - 5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} - 5A_3 + 6A_2 + 6A_3 x = 2 e^{x} + 6x - 5$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{x} + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{2x}\right) + \left(e^{x} + x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{2x} + e^x + x \tag{1}$$

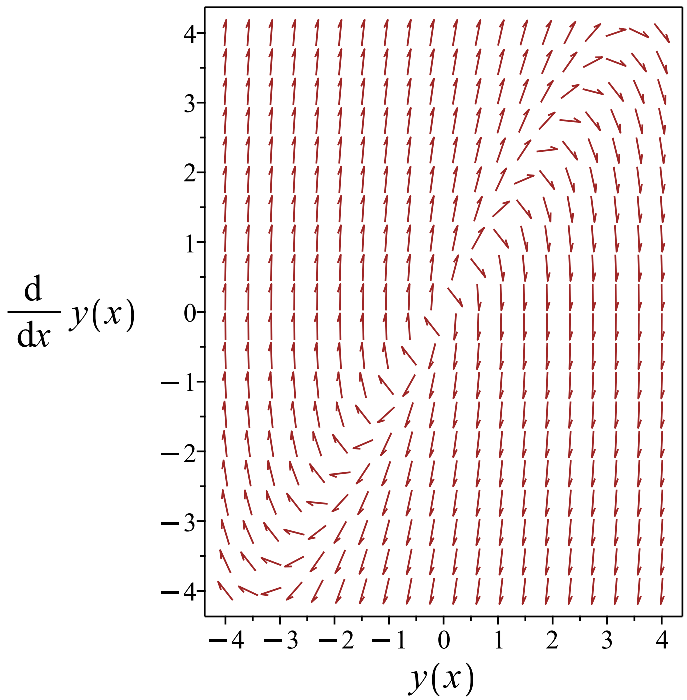

Figure 171: Slope field plot

Verification of solutions

$$y = e^{3x} c_1 + c_2 e^{2x} + e^x + x$$

Verified OK.

6.28.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\ &= z_1 e^{\frac{5x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{5x}}{(y_1)^2}\, dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{2x}\right) + c_2\left(e^{2x}(e^x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + c_2 e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^x + 6x - 5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x - 5A_3 + 6A_2 + 6A_3 x = 2\,e^x + 6x - 5$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x + x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{3x}\right) + (e^x + x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{3x} + e^x + x \tag{1}$$

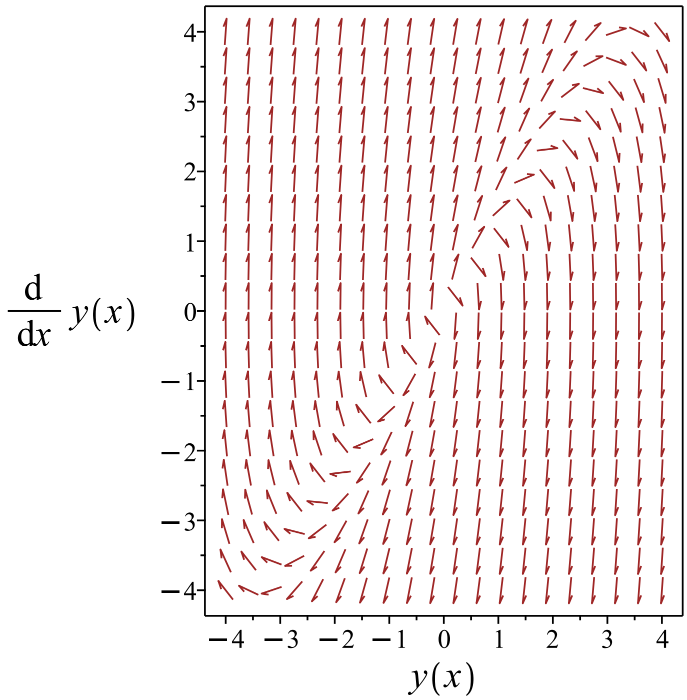

Figure 172: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{3x} + e^x + x$$

Verified OK.

6.28.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 2\,e^x + 6x - 5$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r - 2)\,(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^x + 6x - 5\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{2x} & \mathrm{e}^{3x} \\ 2\,\mathrm{e}^{2x} & 3\,\mathrm{e}^{3x} \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{5x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{2x}\left(\int (2\,\mathrm{e}^x + 6x - 5)\,\mathrm{e}^{-2x}dx\right) + \mathrm{e}^{3x}\left(\int (2\,\mathrm{e}^x + 6x - 5)\,\mathrm{e}^{-3x}dx\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^x + x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x} + \mathrm{e}^x + x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=2*exp(x)+6*x-5,y(x), singsol=all)
```

$$y(x) = c_2 e^{2x} + c_1 e^{3x} + x + e^x$$

✓ Solution by Mathematica

Time used: 0.191 (sec). Leaf size: 26

```
DSolve[y''[x]-5*y'[x]+6*y[x]==2*Exp[x]+6*x-5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + e^x + c_1 e^{2x} + c_2 e^{3x}$$

6.29 problem 35

6.29.1 Solving as second order linear constant coeff ode 794
6.29.2 Solving using Kovacic algorithm 799
6.29.3 Maple step by step solution 805

Internal problem ID [4835]
Internal file name [`OUTPUT/4328_Sunday_June_05_2022_01_00_57_PM_788493/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' - y = \sinh(x)}$$

6.29.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -1, f(x) = \sinh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= \mathrm{e}^{-x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(\mathrm{e}^{-x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-x} \\ \mathrm{e}^{x} & -\mathrm{e}^{-x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x}) \left(-\mathrm{e}^{-x}\right) - \left(\mathrm{e}^{-x}\right) (\mathrm{e}^{x})$$

Which simplifies to

$$W = -2\,\mathrm{e}^{x} \mathrm{e}^{-x}$$

Which simplifies to

$$W = -2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-x} \sinh(x)}{-2} dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-x} \sinh(x)}{2} dx$$

Hence

$$u_1 = \frac{x}{4} - \frac{\sinh(2x)}{8} + \frac{\cosh(2x)}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{x} \sinh(x)}{-2} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{x} \sinh(x)}{2} dx$$

Hence

$$u_2 = -\frac{\cosh(x) \sinh(x)}{4} + \frac{x}{4} - \frac{\cosh(x)^2}{4}$$

Which simplifies to

$$u_1 = \frac{x}{4} - \frac{\sinh(2x)}{8} + \frac{\cosh(2x)}{8}$$
$$u_2 = \frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{4} - \frac{\sinh(2x)}{8} + \frac{\cosh(2x)}{8}\right) e^{x} + \left(\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}\right) e^{-x}$$

Which simplifies to

$$y_p(x) = \frac{(2x - 1 - \sinh(2x) - \cosh(2x))\,\mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{(2x - 1 - \sinh(2x) - \cosh(2x))\,\mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{(2x - 1 - \sinh(2x) - \cosh(2x))\,\mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right)\mathrm{e}^{x}}{4} \tag{1}$$

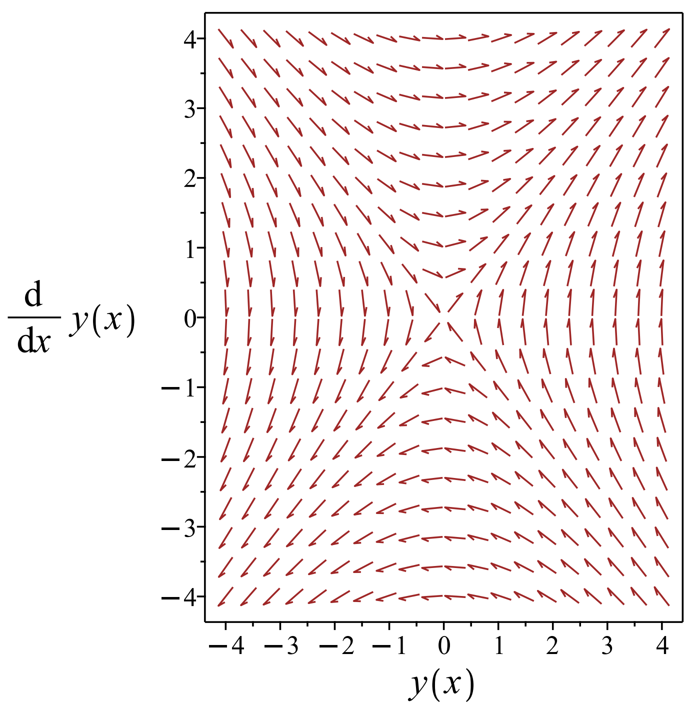

Figure 173: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} + \frac{(2x - 1 - \sinh(2x) - \cosh(2x))\, \mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right) \mathrm{e}^{x}}{4}$$

Verified OK.

6.29.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-x} & \frac{\mathrm{e}^{x}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & \frac{e^{x}}{2} \\ -e^{-x} & \frac{e^{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(\frac{e^{x}}{2}\right) - \left(\frac{e^{x}}{2}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{x} e^{-x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{x} \sinh(x)}{2}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{x} \sinh(x)}{2} dx$$

Hence

$$u_1 = -\frac{\cosh(x)\sinh(x)}{4} + \frac{x}{4} - \frac{\cosh(x)^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-x} \sinh(x)}{1} dx$$

Which simplifies to

$$u_2 = \int e^{-x} \sinh(x)\, dx$$

Hence

$$u_2 = \frac{x}{2} - \frac{\sinh(2x)}{4} + \frac{\cosh(2x)}{4}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= \frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8} \\
u_2 &= \frac{x}{2} - \frac{\sinh(2x)}{4} + \frac{\cosh(2x)}{4}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
y_p(x) = \left(\frac{x}{4} - \frac{\sinh(2x)}{8} - \frac{\cosh(2x)}{8} - \frac{1}{8}\right) \mathrm{e}^{-x} + \frac{\left(\frac{x}{2} - \frac{\sinh(2x)}{4} + \frac{\cosh(2x)}{4}\right) \mathrm{e}^{x}}{2}
$$

Which simplifies to

$$
y_p(x) = \frac{(2x - 1 - \sinh(2x) - \cosh(2x))\, \mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right) \mathrm{e}^{x}}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2}\right) + \left(\frac{(2x - 1 - \sinh(2x) - \cosh(2x))\, \mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right) \mathrm{e}^{x}}{4}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} + \frac{(2x - 1 - \sinh(2x) - \cosh(2x))\, \mathrm{e}^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right) \mathrm{e}^{x}}{4} \tag{1}
$$

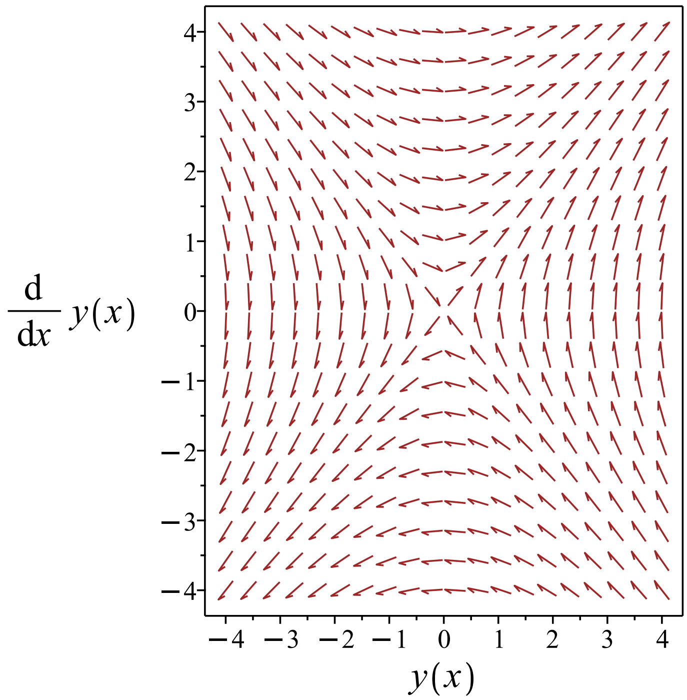

Figure 174: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^{x}}{2} + \frac{(2x - 1 - \sinh(2x) - \cosh(2x)) e^{-x}}{8} + \frac{\left(x - \frac{\sinh(2x)}{2} + \frac{\cosh(2x)}{2}\right) e^{x}}{4}$$

Verified OK.

6.29.3 Maple step by step solution

Let's solve

$y'' - y = \sinh(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sinh(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x}\left(\int e^{x}\sinh(x)dx\right)}{2} + \frac{e^{x}\left(\int e^{-x}\sinh(x)dx\right)}{2}$

  ◦ Compute integrals

  $y_p(x) = \frac{(2x-1-\sinh(2x)-\cosh(2x))e^{-x}}{8} + \frac{\left(x-\frac{\sinh(2x)}{2}+\frac{\cosh(2x)}{2}\right)e^{x}}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{x} + \frac{(2x-1-\sinh(2x)-\cosh(2x))e^{-x}}{8} + \frac{\left(x-\frac{\sinh(2x)}{2}+\frac{\cosh(2x)}{2}\right)e^{x}}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 28

```
dsolve(diff(y(x),x$2)-y(x)=sinh(x),y(x), singsol=all)
```

$$y(x) = \frac{(2x + 8c_1)\,\mathrm{e}^{-x}}{8} + \frac{\left(x + 4c_2 - \frac{1}{2}\right)\mathrm{e}^{x}}{4}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 38

```
DSolve[y''[x]-y[x]==Sinh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}e^{-x}\left(2x + e^{2x}(2x - 1 + 8c_1) + 1 + 8c_2\right)$$

6.30 problem 36

6.30.1 Solving as second order linear constant coeff ode 808
6.30.2 Solving using Kovacic algorithm 812
6.30.3 Maple step by step solution 816

Internal problem ID [4836]
Internal file name [`OUTPUT/4329_Sunday_June_05_2022_01_01_05_PM_29190235/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y = 2\sin(x) + 4\cos(x)\, x}$$

6.30.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 2\sin(x) + 4\cos(x)\, x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(\cos(x) c_1 + c_2 \sin(x))$$

Or

$$y = \cos(x) c_1 + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = \cos(x) c_1 + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(x) + 4\cos(x) x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x) x, \sin(x) x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 \sin(x), \cos(x) x, \cos(x) x^2, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x) x + A_3 \cos(x) x^2 + A_4 \sin(x) x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &2A_1 \sin(x) + 4A_1 x \cos(x) - 2A_2 \sin(x) - 4A_3 \sin(x) x + 2A_3 \cos(x) + 2A_4 \cos(x) \\ &= 2\sin(x) + 4\cos(x) x \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = 0, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\cos(x) c_1 + c_2 \sin(x)) + (x^2 \sin(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) \tag{1}$$

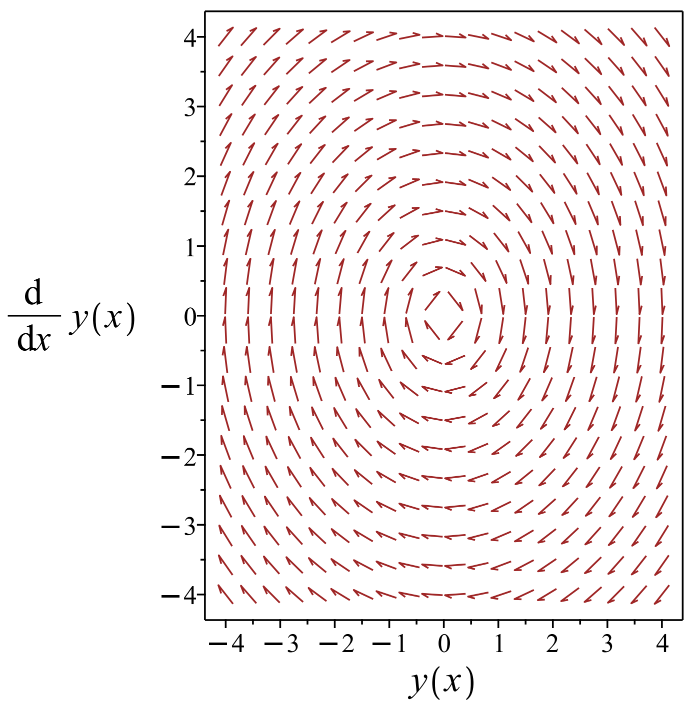

Figure 175: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x)$$

Verified OK.

6.30.2 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + y &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 0 \tag{3} \\
C &= 1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x) (\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x) (\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(x)\, c_1 + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(x) + 4\cos(x)\, x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\, x, \sin(x)\, x, \cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\sin(x), \cos(x)\, x, \cos(x)\, x^2, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 \sin(x) + A_2 \cos(x)\, x + A_3 \cos(x)\, x^2 + A_4 \sin(x)\, x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_1\sin(x) + 4A_1 x\cos(x) - 2A_2\sin(x) - 4A_3\sin(x)\, x + 2A_3\cos(x) + 2A_4\cos(x)\\&= 2\sin(x) + 4\cos(x)\, x\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 0, A_3 = 0, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2\sin(x)$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= (\cos(x)\, c_1 + c_2\sin(x)) + (x^2\sin(x))\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x) \tag{1}$$

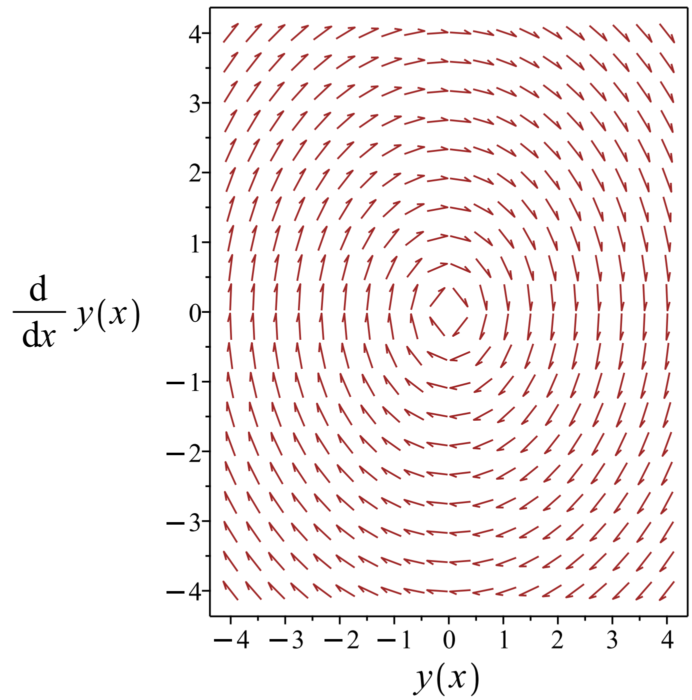

Figure 176: Slope field plot

Verification of solutions

$$y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x)$$

Verified OK.

6.30.3 Maple step by step solution

Let's solve

$y'' + y = 2\sin(x) + 4\cos(x) x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\sin(x) + 4\cos(x) x\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int (2x\sin(2x) + 1 - \cos(2x)) dx\right) + \sin(x)\left(\int (2x\cos(2x) + 2x + \sin(2x)) dx\right.$
  - Compute integrals

    $y_p(x) = x^2 \sin(x)$
- Substitute particular solution into general solution to ODE

  $y = \cos(x) c_1 + c_2 \sin(x) + x^2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)+y(x)=2*sin(x)+4*x*cos(x),y(x), singsol=all)
```

$$y(x) = \left(x^2 + c_2 - 1\right) \sin\left(x\right) + \cos\left(x\right) c_1$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 28

```
DSolve[y''[x]+y[x]==2*Sin[x]+4*x*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(2x^2 - 1 + 2c_2\right) \sin(x) + c_1 \cos(x)$$

6.31 problem 37

6.31.1 Solving as second order linear constant coeff ode 819
6.31.2 Solving as linear second order ode solved by an integrating factor ode 822
6.31.3 Solving using Kovacic algorithm 824
6.31.4 Maple step by step solution 829

Internal problem ID [4837]
Internal file name [`OUTPUT/4330_Sunday_June_05_2022_01_01_14_PM_18273352/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = 4\,e^{x} + (1 - x)\left(e^{2x} - 1\right)}$$

6.31.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = -1 - e^{2x}x + 4\,e^{x} + e^{2x} + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$.

y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x\, e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-1 - e^{2x}x + 4\, e^{x} + e^{2x} + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{x}\}, \{1, x\}, \{e^{2x}x, e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{x} + A_2 + A_3 x + A_4\mathrm{e}^{2x}x + A_5\mathrm{e}^{2x}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{x} + 9A_4\mathrm{e}^{2x}x + 6A_4\mathrm{e}^{2x} + 9A_5\mathrm{e}^{2x} + 2A_3 + A_2 + A_3 x = -1 - \mathrm{e}^{2x}x + 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -3, A_3 = 1, A_4 = -\frac{1}{9}, A_5 = \frac{5}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x}x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2\right) + \left(\mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x}x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x}x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x}x}{9} + \frac{5\,\mathrm{e}^{2x}}{27} \tag{1}$$

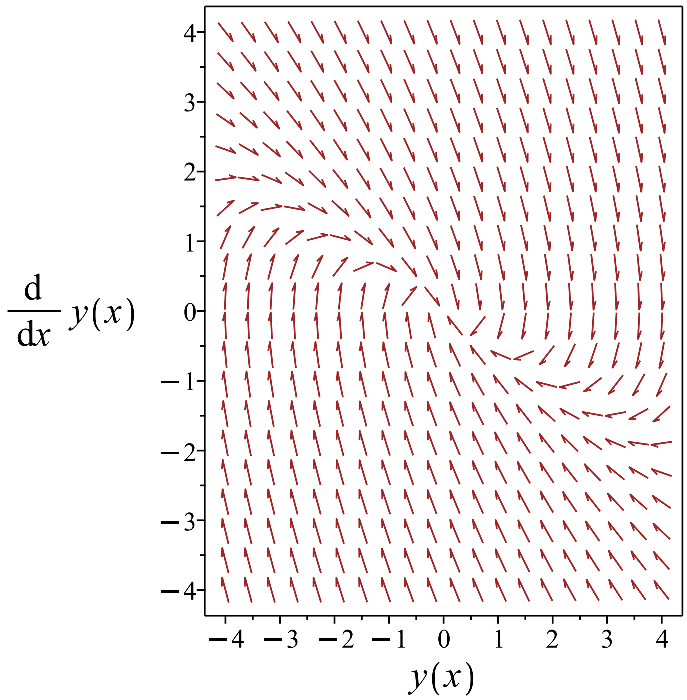

Figure 177: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^x - 3 + x - \frac{e^{2x}x}{9} + \frac{5\,e^{2x}}{27}$$

Verified OK.

6.31.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^x \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= e^x\left(-1 - e^{2x}x + 4\,e^x + e^{2x} + x\right) \\
(y\,e^x)'' &= e^x\left(-1 - e^{2x}x + 4\,e^x + e^{2x} + x\right)
\end{aligned}
$$

Integrating once gives

$$
(y\,e^x)' = \frac{(-3x+4)\,e^{3x}}{9} + 2\,e^{2x} + (-2+x)\,e^x + c_1
$$

Integrating again gives

$$
(y\,e^x) = \frac{(-3x+5)\,e^{3x}}{27} + e^{2x} + (x-3)\,e^x + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{(-3x+5)e^{3x}}{27} + e^{2x} + (x-3)\,e^x + c_1 x + c_2}{e^x}
$$

Or

$$
y = -\frac{e^{2x}x}{9} + \frac{5\,e^{2x}}{27} + c_1 x\,e^{-x} + x + e^x + c_2 e^{-x} - 3
$$

Summary

The solution(s) found are the following

$$
y = -\frac{e^{2x}x}{9} + \frac{5\,e^{2x}}{27} + c_1 x\,e^{-x} + x + e^x + c_2 e^{-x} - 3 \tag{1}
$$

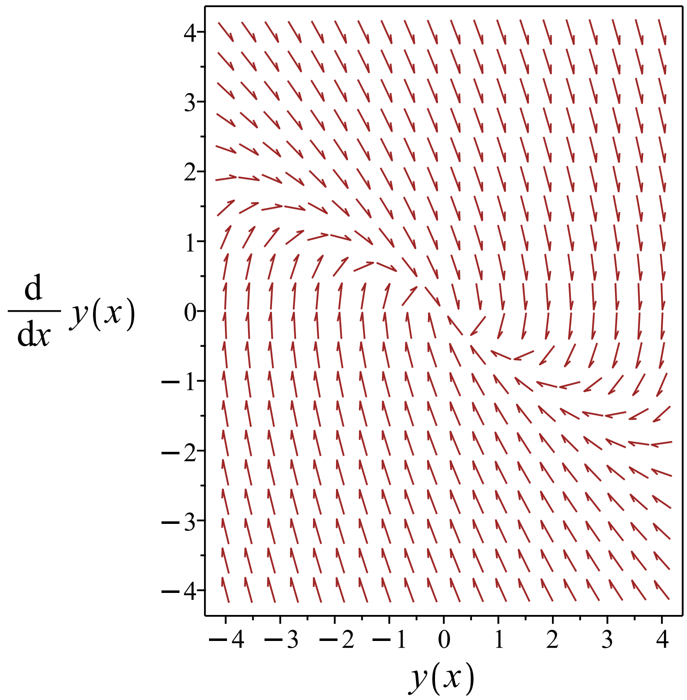

Figure 178: Slope field plot

Verification of solutions

$$y = -\frac{e^{2x}x}{9} + \frac{5\,e^{2x}}{27} + c_1 x\,e^{-x} + x + e^{x} + c_2 e^{-x} - 3$$

Verified OK.

6.31.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
-1 - \mathrm{e}^{2x} x + 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} + x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}, \{1, x\}, \{\mathrm{e}^{2x} x, \mathrm{e}^{2x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x} + A_2 + A_3 x + A_4 \mathrm{e}^{2x} x + A_5 \mathrm{e}^{2x}
$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_1 \mathrm{e}^{x} + 9A_4 \mathrm{e}^{2x} x + 6A_4 \mathrm{e}^{2x} + 9A_5 \mathrm{e}^{2x} + 2A_3 + A_2 + A_3 x = -1 - \mathrm{e}^{2x} x + 4\,\mathrm{e}^{x} + \mathrm{e}^{2x} + x
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = -3, A_3 = 1, A_4 = -\frac{1}{9}, A_5 = \frac{5}{27}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x} x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2\right) + \left(\mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x} x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x} x}{9} + \frac{5\,\mathrm{e}^{2x}}{27}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{x} - 3 + x - \frac{\mathrm{e}^{2x} x}{9} + \frac{5\,\mathrm{e}^{2x}}{27} \tag{1}$$

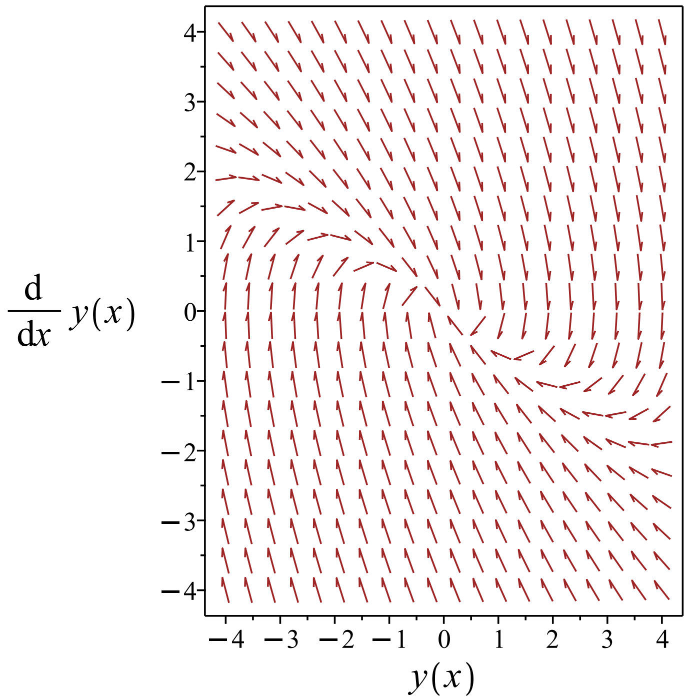

Figure 179: Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x}(c_2x + c_1) + e^x - 3 + x - \frac{e^{2x}x}{9} + \frac{5\,e^{2x}}{27}$$

Verified OK.

6.31.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = -1 - e^{2x}x + 4\,e^x + e^{2x} + x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -1 - \mathrm{e}^{2x}x + 4\,\mathrm{e}^{x} + \mathrm{e}^{2x}\right.$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \mathrm{e}^{-x}\left(\int \left((x-1)\,\mathrm{e}^{2x} - x - 4\,\mathrm{e}^{x} + 1\right)\mathrm{e}^{x} x dx - \left(\int \left((x-1)\,\mathrm{e}^{2x} - x - 4\,\mathrm{e}^{x} + 1\right)\mathrm{e}^{x} dx\right) x\right)$

  - Compute integrals

    $y_p(x) = \frac{(-3x+5)\mathrm{e}^{2x}}{27} + x + \mathrm{e}^{x} - 3$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{(-3x+5)\mathrm{e}^{2x}}{27} + x + \mathrm{e}^{x} - 3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=4*exp(x)+(1-x)*(exp(2*x)-1),y(x), singsol=all)
```

$$y(x) = -3 + (c_1 x + c_2)\,\mathrm{e}^{-x} + \frac{(-3x+5)\,\mathrm{e}^{2x}}{27} + x + \mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.539 (sec). Leaf size: 38

```
DSolve[y''[x]+2*y'[x]+y[x]==4*Exp[x]+(1-x)*(Exp[2*x]-1),y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \frac{1}{27} e^{2x}(5-3x) + e^{x} + x + e^{-x}(c_2 x + c_1) - 3$$

6.32 problem 38

6.32.1 Solving as second order linear constant coeff ode 833
6.32.2 Solving as second order integrable as is ode 836
6.32.3 Solving as second order ode missing y ode 838
6.32.4 Solving as type second_order_integrable_as_is (not using ABC version) 841
6.32.5 Solving using Kovacic algorithm 843
6.32.6 Solving as exact linear second order ode ode 849
6.32.7 Maple step by step solution 851

Internal problem ID [4838]
Internal file name [`OUTPUT/4331_Sunday_June_05_2022_01_01_24_PM_43404379/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 6. SECOND-ORDER LINEAR EQUATIONSWITH CONSTANT COEFFICIENTS AND RIGHT-HAND SIDE NOT ZERO. page 422
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' - 2y' = 9x\,e^{-x} - 6x^2 + 4\,e^{2x}}$$

6.32.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 0, f(x) = 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(0)}\\ &= 1 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 1\\ \lambda_2 &= 1 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{2x}\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, \mathrm{e}^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{2x} x\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{1, x, x^2\}]$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{2x} x\}, \{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{x, x^2, x^3\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \mathrm{e}^{2x} x + A_2 x\,\mathrm{e}^{-x} + A_3 \mathrm{e}^{-x} + A_4 x + A_5 x^2 + A_6 x^3$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&2A_1\mathrm{e}^{2x} - 4A_2\mathrm{e}^{-x} + 3A_2x\,\mathrm{e}^{-x} + 3A_3\mathrm{e}^{-x} + 2A_5 + 6A_6x - 2A_4 - 4A_5x - 6A_6x^2 \\
&\quad = 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$
\left[A_1 = 2, A_2 = 3, A_3 = 4, A_4 = \frac{3}{2}, A_5 = \frac{3}{2}, A_6 = 1\right]
$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$
y_p = 2\,\mathrm{e}^{2x}x + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{3x}{2} + \frac{3x^2}{2} + x^3
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\mathrm{e}^{2x} + c_2\right) + \left(2\,\mathrm{e}^{2x}x + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{3x}{2} + \frac{3x^2}{2} + x^3\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{2x} + c_2 + 2\,\mathrm{e}^{2x}x + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{3x}{2} + \frac{3x^2}{2} + x^3 \tag{1}
$$

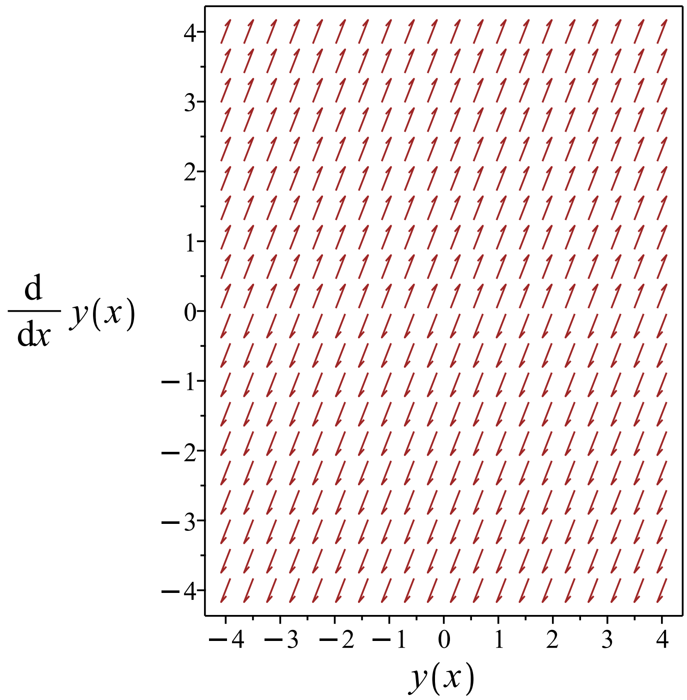

Figure 180: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 + 2\,e^{2x}x + 3x\,e^{-x} + 4\,e^{-x} + \frac{3x}{2} + \frac{3x^2}{2} + x^3$$

Verified OK.

6.32.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2y')\,dx = \int \left(9x\,e^{-x} - 6x^2 + 4\,e^{2x}\right) dx$$

$$-2y + y' = -2x^3 - 9x\,e^{-x} - 9\,e^{-x} + 2\,e^{2x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= (-9x - 9)\,e^{-x} - 2x^3 + c_1 + 2\,e^{2x} \end{aligned}$$

Hence the ode is

$$-2y + y' = (-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-2)dx} \\ &= \mathrm{e}^{-2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left((-9x-9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\mathrm{e}^{-2x}\right)\left((-9x-9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,\mathrm{e}^{-2x} &= \int \left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\,\mathrm{d}x \\ y\,\mathrm{e}^{-2x} &= 2x - \frac{c_1\mathrm{e}^{-2x}}{2} + 4\,\mathrm{e}^{-3x} + 3x\,\mathrm{e}^{-3x} + x^3\mathrm{e}^{-2x} + \frac{3x^2\mathrm{e}^{-2x}}{2} + \frac{3x\,\mathrm{e}^{-2x}}{2} + \frac{3\,\mathrm{e}^{-2x}}{4} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$y = \mathrm{e}^{2x}\left(2x - \frac{c_1\mathrm{e}^{-2x}}{2} + 4\,\mathrm{e}^{-3x} + 3x\,\mathrm{e}^{-3x} + x^3\mathrm{e}^{-2x} + \frac{3x^2\mathrm{e}^{-2x}}{2} + \frac{3x\,\mathrm{e}^{-2x}}{2} + \frac{3\,\mathrm{e}^{-2x}}{4}\right) + c_2\mathrm{e}^{2x}$$

which simplifies to

$$y = \frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x + c_2)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x + c_2)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2} \tag{1}$$

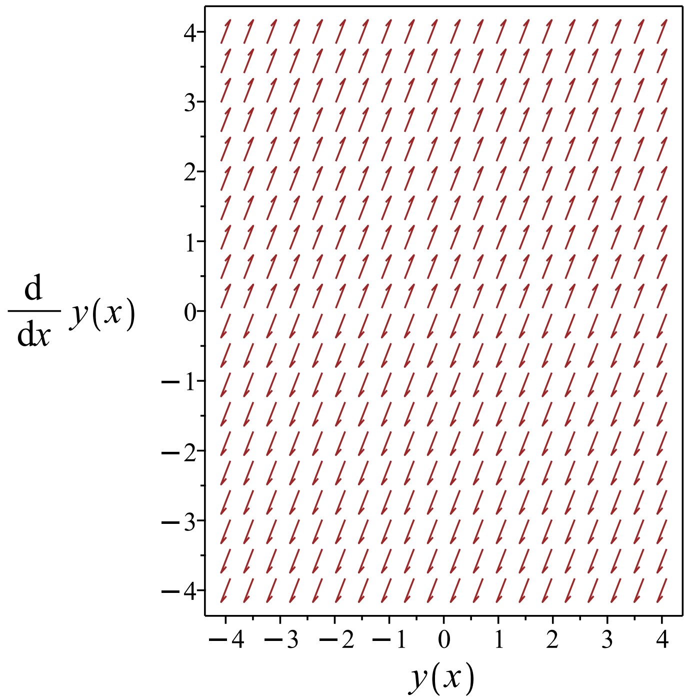

Figure 181: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{3}{4} + (3x + 4)\,e^{-x} + (2x + c_2)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

Verified OK.

6.32.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 2p(x) - 9x\,e^{-x} + 6x^2 - 4\,e^{2x} = 0$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$p'(x) + p(x)p(x) = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -2 \\
q(x) &= 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}
\end{aligned}$$

Hence the ode is

$$p'(x) - 2p(x) = 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int (-2)dx} \\
&= \mathrm{e}^{-2x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x} p\right) &= \left(\mathrm{e}^{-2x}\right)\left(9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}\right) \\
\mathrm{d}\left(\mathrm{e}^{-2x} p\right) &= \left(\left(4\,\mathrm{e}^{3x} - 6x^2\mathrm{e}^{x} + 9x\right)\mathrm{e}^{-3x}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathrm{e}^{-2x} p &= \int \left(4\,\mathrm{e}^{3x} - 6x^2\mathrm{e}^{x} + 9x\right)\mathrm{e}^{-3x}\,\mathrm{d}x \\
\mathrm{e}^{-2x} p &= 4x + 3x^2\mathrm{e}^{-2x} + 3x\,\mathrm{e}^{-2x} + \frac{3\,\mathrm{e}^{-2x}}{2} - 3x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$p(x) = \mathrm{e}^{2x}\left(4x + 3x^2\mathrm{e}^{-2x} + 3x\,\mathrm{e}^{-2x} + \frac{3\,\mathrm{e}^{-2x}}{2} - 3x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x}\right) + c_1\mathrm{e}^{2x}$$

which simplifies to

$$p(x) = \frac{3}{2} + (-3x - 1)\,\mathrm{e}^{-x} + (4x + c_1)\,\mathrm{e}^{2x} + 3x^2 + 3x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{3}{2} + (-3x - 1)\,\mathrm{e}^{-x} + (4x + c_1)\,\mathrm{e}^{2x} + 3x^2 + 3x$$

Integrating both sides gives

$$
\begin{aligned}
y &= \int 4\,\mathrm{e}^{2x} x + 3x^2 + 3x + \frac{3}{2} - 3x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} + c_1 \mathrm{e}^{2x}\,\mathrm{d}x \\
&= x^3 + \frac{3x^2}{2} + \frac{3x}{2} + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{c_1 \mathrm{e}^{2x}}{2} + 2\,\mathrm{e}^{2x} x - \mathrm{e}^{2x} + c_2
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x^3 + \frac{3x^2}{2} + \frac{3x}{2} + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{c_1 \mathrm{e}^{2x}}{2} + 2\,\mathrm{e}^{2x} x - \mathrm{e}^{2x} + c_2 \tag{1}
$$

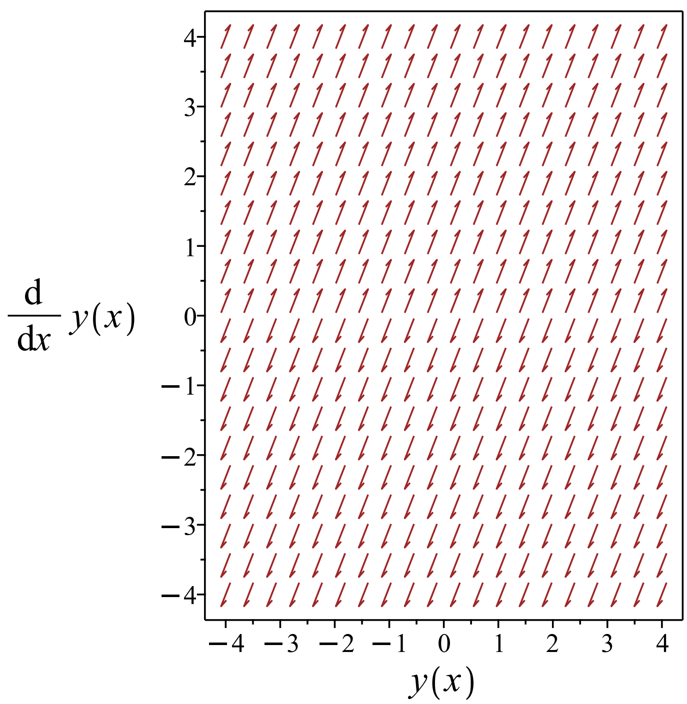

Figure 182: Slope field plot

Verification of solutions

$$
y = x^3 + \frac{3x^2}{2} + \frac{3x}{2} + 3x\,\mathrm{e}^{-x} + 4\,\mathrm{e}^{-x} + \frac{c_1 \mathrm{e}^{2x}}{2} + 2\,\mathrm{e}^{2x} x - \mathrm{e}^{2x} + c_2
$$

Verified OK.

6.32.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 2y' = 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 2y')\, dx = \int \left(9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}\right) dx$$
$$-2y + y' = -2x^3 - 9x\,\mathrm{e}^{-x} - 9\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{2x} + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= (-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x} \end{aligned}$$

Hence the ode is

$$-2y + y' = (-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-2)dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left((-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\mathrm{e}^{-2x}\right)\left((-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-2x} &= \int \left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\,\mathrm{d}x \\ y\,\mathrm{e}^{-2x} &= 2x - \frac{c_1\mathrm{e}^{-2x}}{2} + 4\,\mathrm{e}^{-3x} + 3x\,\mathrm{e}^{-3x} + x^3\mathrm{e}^{-2x} + \frac{3x^2\mathrm{e}^{-2x}}{2} + \frac{3x\,\mathrm{e}^{-2x}}{2} + \frac{3\,\mathrm{e}^{-2x}}{4} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-2x}$ results in

$$y = e^{2x}\left(2x - \frac{c_1 e^{-2x}}{2} + 4\,e^{-3x} + 3x\,e^{-3x} + x^3 e^{-2x} + \frac{3x^2 e^{-2x}}{2} + \frac{3x\,e^{-2x}}{2} + \frac{3\,e^{-2x}}{4}\right) + c_2 e^{2x}$$

which simplifies to

$$y = \frac{3}{4} + (3x+4)\,e^{-x} + (2x+c_2)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{3}{4} + (3x+4)\,e^{-x} + (2x+c_2)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2} \tag{1}$$

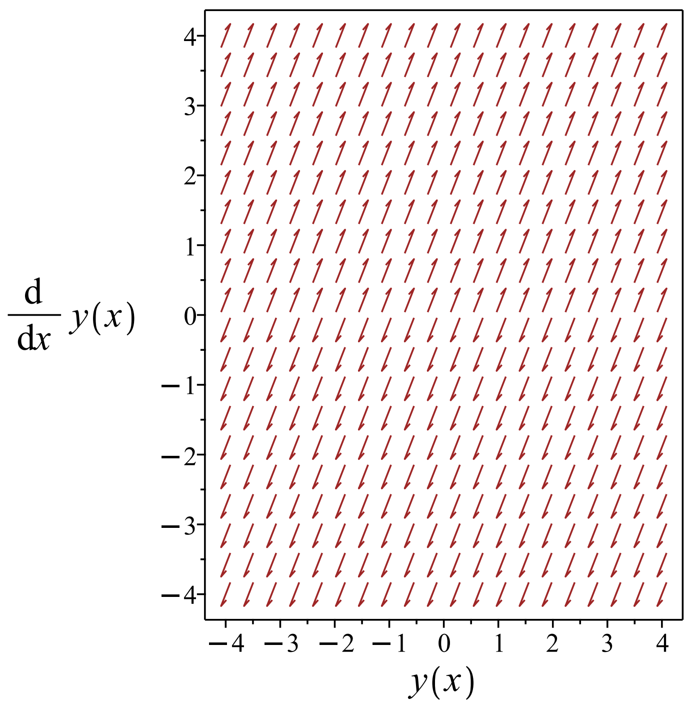

Figure 183: Slope field plot

Verification of solutions

$$y = \frac{3}{4} + (3x+4)\,e^{-x} + (2x+c_2)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

Verified OK.

6.32.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2 \left(1 \left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 + \frac{c_2 \mathrm{e}^{2x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = \frac{e^{2x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \frac{e^{2x}}{2} \\ \frac{d}{dx}(1) & \frac{d}{dx}\left(\frac{e^{2x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \frac{e^{2x}}{2} \\ 0 & e^{2x} \end{vmatrix}$$

Therefore

$$W = (1)\left(e^{2x}\right) - \left(\frac{e^{2x}}{2}\right)(0)$$

Which simplifies to

$$W = e^{2x}$$

Which simplifies to

$$W = \mathrm{e}^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^{2x}\left(9x\,\mathrm{e}^{-x}-6x^2+4\,\mathrm{e}^{2x}\right)}{2}}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \left(-3x^2 + \frac{9x\,\mathrm{e}^{-x}}{2} + 2\,\mathrm{e}^{2x}\right) dx$$

Hence

$$u_1 = x^3 + \frac{9x\,\mathrm{e}^{-x}}{2} + \frac{9\,\mathrm{e}^{-x}}{2} - \mathrm{e}^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}}{\mathrm{e}^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(4\,\mathrm{e}^{3x} - 6x^2\mathrm{e}^{x} + 9x\right)\mathrm{e}^{-3x}dx$$

Hence

$$u_2 = 4x + 3x^2\mathrm{e}^{-2x} + 3x\,\mathrm{e}^{-2x} + \frac{3\,\mathrm{e}^{-2x}}{2} - 3x\,\mathrm{e}^{-3x} - \mathrm{e}^{-3x}$$

Which simplifies to

$$u_1 = \frac{(9x+9)\,\mathrm{e}^{-x}}{2} + x^3 - \mathrm{e}^{2x}$$
$$u_2 = \frac{3(2x^2+2x+1)\,\mathrm{e}^{-2x}}{2} + (-3x-1)\,\mathrm{e}^{-3x} + 4x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{(9x+9)\,\mathrm{e}^{-x}}{2} + x^3 - \mathrm{e}^{2x} + \frac{\left(\frac{3(2x^2+2x+1)\mathrm{e}^{-2x}}{2} + (-3x-1)\,\mathrm{e}^{-3x} + 4x\right)\mathrm{e}^{2x}}{2}$$

Which simplifies to

$$y_p(x) = \frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x-1)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 + \frac{c_2 \mathrm{e}^{2x}}{2}\right) + \left(\frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x-1)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{c_2 \mathrm{e}^{2x}}{2} + \frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x-1)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} \tag{1}$$

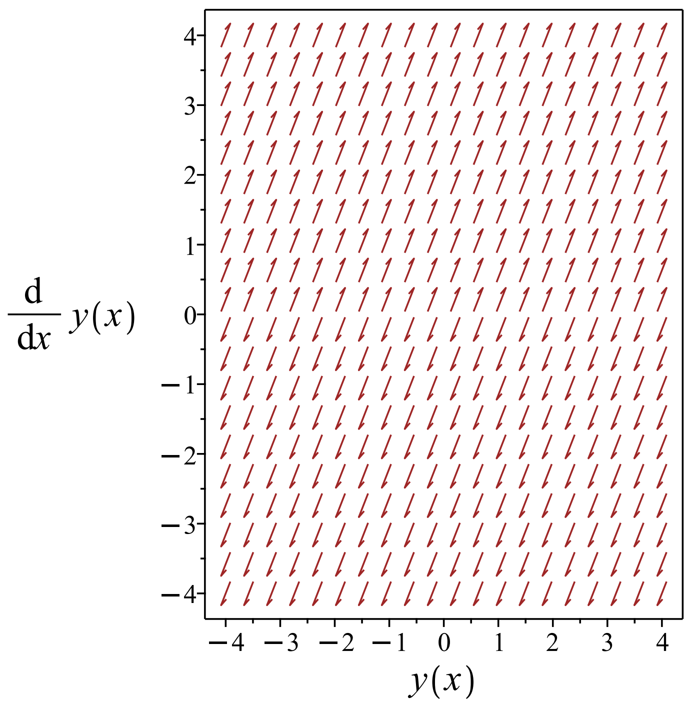

Figure 184: Slope field plot

Verification of solutions

$$y = c_1 + \frac{c_2 \mathrm{e}^{2x}}{2} + \frac{3}{4} + (3x+4)\,\mathrm{e}^{-x} + (2x-1)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2}$$

Verified OK.

6.32.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1 \\
q(x) &= -2 \\
r(x) &= 0 \\
s(x) &= 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$-2y + y' = \int 9x\,\mathrm{e}^{-x} - 6x^2 + 4\,\mathrm{e}^{2x}\,dx$$

We now have a first order ode to solve which is

$$-2y + y' = -2x^3 - 9x\,\mathrm{e}^{-x} - 9\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{2x} + c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= (-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x} \end{aligned}$$

Hence the ode is

$$-2y + y' = (-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int (-2)dx} \\ &= \mathrm{e}^{-2x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left((-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\mathrm{e}^{-2x}\right)\left((-9x - 9)\,\mathrm{e}^{-x} - 2x^3 + c_1 + 2\,\mathrm{e}^{2x}\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{-2x}\right) &= \left(\left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-2x} &= \int \left(2\,\mathrm{e}^{3x} + \left(-2x^3 + c_1\right)\mathrm{e}^{x} - 9x - 9\right)\mathrm{e}^{-3x}\,\mathrm{d}x \\ y\,\mathrm{e}^{-2x} &= 2x - \frac{c_1\mathrm{e}^{-2x}}{2} + 4\,\mathrm{e}^{-3x} + 3x\,\mathrm{e}^{-3x} + x^3\mathrm{e}^{-2x} + \frac{3x^2\mathrm{e}^{-2x}}{2} + \frac{3x\,\mathrm{e}^{-2x}}{2} + \frac{3\,\mathrm{e}^{-2x}}{4} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$y = \mathrm{e}^{2x}\left(2x - \frac{c_1\mathrm{e}^{-2x}}{2} + 4\,\mathrm{e}^{-3x} + 3x\,\mathrm{e}^{-3x} + x^3\mathrm{e}^{-2x} + \frac{3x^2\mathrm{e}^{-2x}}{2} + \frac{3x\,\mathrm{e}^{-2x}}{2} + \frac{3\,\mathrm{e}^{-2x}}{4}\right) + c_2\mathrm{e}^{2x}$$

which simplifies to

$$y = \frac{3}{4} + (3x + 4)\,\mathrm{e}^{-x} + (2x + c_2)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{3}{4} + (3x + 4)\,\mathrm{e}^{-x} + (2x + c_2)\,\mathrm{e}^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2} \tag{1}$$

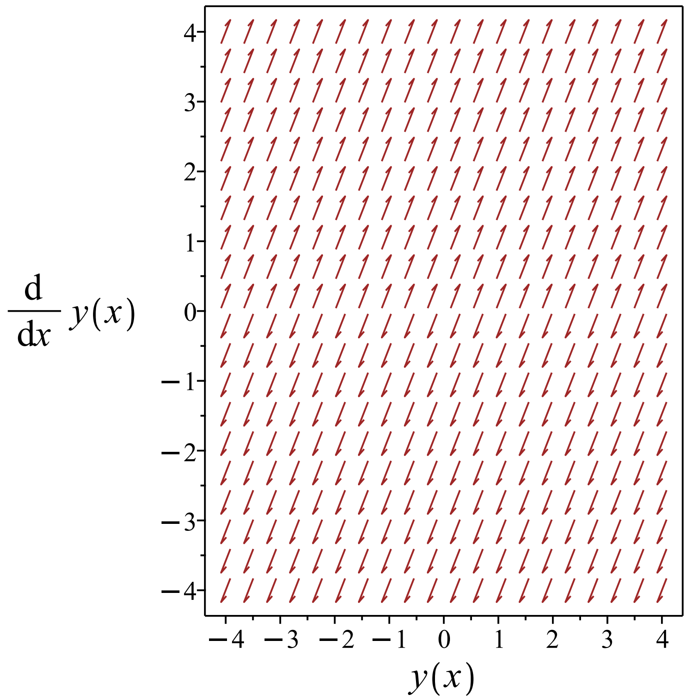

Figure 185: Slope field plot

Verification of solutions

$$y = \frac{3}{4} + (3x+4)\,e^{-x} + (2x+c_2)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2} - \frac{c_1}{2}$$

Verified OK.

6.32.7 Maple step by step solution

Let's solve

$y'' - 2y' = 9x\,e^{-x} - 6x^2 + 4\,e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r = 0$

- Factor the characteristic polynomial

  $r(r-2) = 0$

- Roots of the characteristic polynomial

$r = (0, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 + c_2 e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 9x\,e^{-x} - 6x^2 + 4\,e^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & e^{2x} \\ 0 & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\left(\int (9x\,e^{-x} - 6x^2 + 4\,e^{2x})dx\right)}{2} - \frac{e^{2x}\left(\int (-4\,e^{3x} + 6x^2 e^{x} - 9x)e^{-3x}dx\right)}{2}$

  - Compute integrals

    $y_p(x) = \frac{3}{4} + (3x+4)\,e^{-x} + (2x-1)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 + c_2 e^{2x} + \frac{3}{4} + (3x+4)\,e^{-x} + (2x-1)\,e^{2x} + x^3 + \frac{3x^2}{2} + \frac{3x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = 9*exp(-_a)*_a-6*_a^2+2*_b(_a)+4*exp(2*_
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)=9*x*exp(-x)-6*x^2+4*exp(2*x),y(x), singsol=all)
```

$$y(x)=\frac{(4x+c_1-2)\,\mathrm{e}^{2x}}{2}+(3x+4)\,\mathrm{e}^{-x}+x^3+\frac{3x^2}{2}+\frac{3x}{2}+c_2$$

✓ Solution by Mathematica

Time used: 0.492 (sec). Leaf size: 49

```
DSolve[y''[x]-2*y'[x]==9*x*Exp[-x]-6*x^2+4*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{1}{2}\left(x\left(2x^2+3x+3\right)+e^{-x}(6x+8)+e^{2x}(4x-2+c_1)\right)+c_2$$

7 Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435

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7.1 problem 1 (a)

7.1.1 Solving as second order integrable as is ode 856
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Internal problem ID [4839]
Internal file name [`OUTPUT/4332_Sunday_June_05_2022_01_01_33_PM_90742386/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 1 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' + y'y = 0}$$

With initial conditions

$$[y(0) = 5, y'(0) = 0]$$

7.1.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$5 = \frac{\left(e^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{4\, e^{c_2\sqrt{2}\,\sqrt{c_1}} c_1}{e^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\, e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

7.1.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)\, y = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int -y \,\mathrm{d}y \\ &= -\frac{y^2}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 5$ and $p = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{25}{2} + c_1$$

$$c_1 = \frac{25}{2}$$

Substituting c_1 found above in the general solution gives

$$p(y) = -\frac{y^2}{2} + \frac{25}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^2}{2} + \frac{25}{2}$$

Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2}+\frac{25}{2}}dy = \int dx$$
$$\frac{\ln(y+5)}{5}-\frac{\ln(y-5)}{5} = c_2 + x$$

The above can be written as

$$\left(\frac{1}{5}\right)(\ln(y+5)-\ln(y-5)) = c_2 + x$$
$$\ln(y+5)-\ln(y-5) = (5)(c_2+x)$$
$$= 5c_2 + 5x$$

Raising both side to exponential gives

$$e^{\ln(y+5)-\ln(y-5)} = 5c_2 e^{5x}$$

Which simplifies to

$$\frac{y+5}{y-5} = e^{5x}c_3$$

Initial conditions are used to solve for c_3. Substituting $x=0$ and $y=5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{5c_3+5}{-1+c_3}$$

Unable to solve for constant of integration. Since $\lim_{c_1\to\infty}$ gives $y = \frac{5\,e^{5x}c_3+5}{-1+e^{5x}c_3} = y = 5$ and this result satisfies the given initial condition. Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = 5 \tag{1}$$

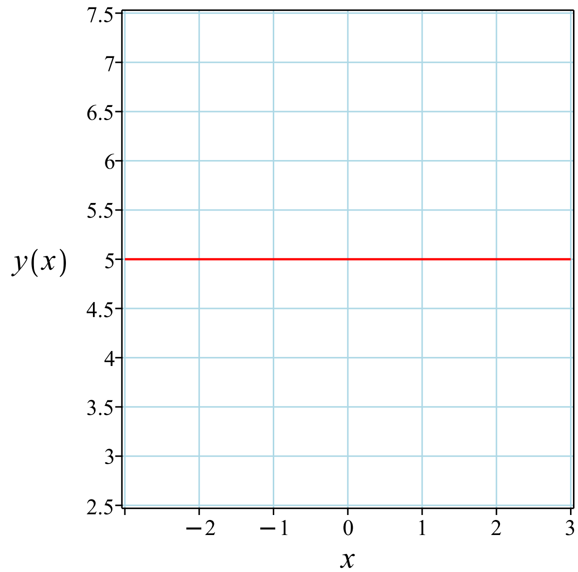

Figure 186: Solution plot

Verification of solutions

$$y = 5$$

Verified OK.

7.1.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y'y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=5$ and $x=0$ in the above gives

$$5 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}-1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}+1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1-\tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y'=0$ and $x=0$ in the above gives

$$0 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}}+2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}+1} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1,c_2\}$. There is no solution for the constants of integrations. This solution is removed.

7.1.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x,y,y')\,y'' + a_1(x,y,y')\,y' + a_0(x,y,y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$
\begin{aligned}
a_2 &= 1 \\
a_1 &= y \\
a_0 &= 0
\end{aligned}
$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$
\begin{aligned}
\int a_2 \, dy' + \int a_1 \, dy + \int a_0 \, dx &= c_1 \\
\int 1 \, dy' + \int y \, dy + \int 0 \, dx &= c_1
\end{aligned}
$$

Which results in

$$
\frac{y^2}{2} + y' = c_1
$$

Which is now solved Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-\frac{y^2}{2} + c_1} dy &= c_2 + x \\
\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} &= c_2 + x
\end{aligned}
$$

Solving for y gives these solutions

$$
y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 5$ and $x = 0$ in the above gives

$$
5 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}
$$

Taking derivative of the solution gives

$$y' = c_1 \left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

7.1.5 Maple step by step solution

Let's solve

$$\left[y'' + y'y = 0, y(0) = 5, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) + u(y)\,y = 0$

- Separate variables

  $\frac{d}{dy}u(y) = -y$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int -ydy + c_1$
- Evaluate integral

  $u(y) = -\frac{y^2}{2} + c_1$
- Solve for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$
- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$
- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y^2}{2} + c_1$
- Separate variables

  $\frac{y'}{-\frac{y^2}{2}+c_1} = 1$
- Integrate both sides with respect to x

  $\int \frac{y'}{-\frac{y^2}{2}+c_1} dx = \int 1dx + c_2$
- Evaluate integral

  $\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$
- Solve for y

  $y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

□ Check validity of solution $y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

  - Use initial condition $y(0) = 5$

    $5 = \tanh\left(\frac{c_2\sqrt{2}\,\sqrt{c_1}}{2}\right)\sqrt{c_1}\,\sqrt{2}$
  - Compute derivative of the solution

    $y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)^2\right)$
  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = c_1\left(1 - \tanh\left(\frac{c_2\sqrt{2}\,\sqrt{c_1}}{2}\right)^2\right)$

- Solve for c_1 and c_2
- The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_a*_b(_a) = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 5

```
dsolve([diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(0) = 5, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = 5$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]*y'[x]==0,{y[0]==5,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

{}

7.2 problem 1 (b)

7.2.1 Solving as second order integrable as is ode 866
7.2.2 Solving as second order ode missing x ode 867
7.2.3 Solving as type second_order_integrable_as_is (not using ABC version) 869
7.2.4 Solving as exact nonlinear second order ode ode 870
7.2.5 Maple step by step solution 872

Internal problem ID [4840]
Internal file name [`OUTPUT/4333_Sunday_June_05_2022_01_01_47_PM_18861716/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 1 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' + y'y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2]$$

7.2.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = \frac{\left(e^{c_2\sqrt{2}\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = \frac{4\, e^{c_2\sqrt{2}\sqrt{c_1}} c_1}{e^{2c_2\sqrt{2}\sqrt{c_1}} + 2\, e^{c_2\sqrt{2}\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

7.2.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)\, y = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int -y \, \mathrm{d}y \\ &= -\frac{y^2}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 2$ and $p = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -2 + c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$p(y) = -\frac{y^2}{2}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^2}{2}$$

Integrating both sides gives

$$\int -\frac{2}{y^2}dy = c_2 + x$$
$$\frac{2}{y} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \frac{2}{c_2 + x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{2}{c_2}$$

$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = \frac{2}{x + 1}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{2}{x + 1} \tag{1}$$

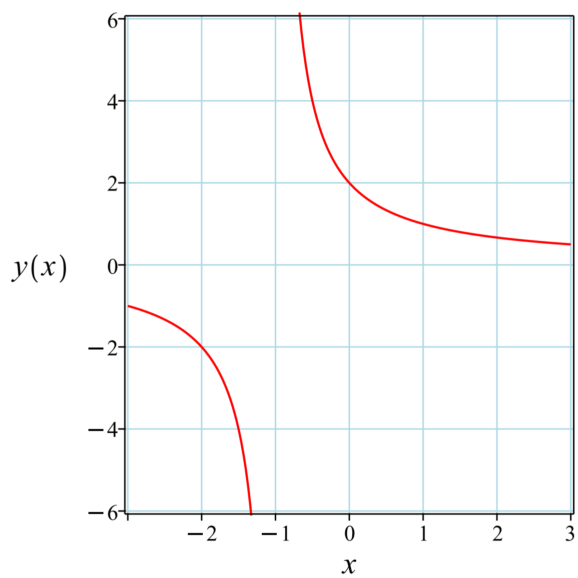

Figure 187: Solution plot

Verification of solutions

$$y = \frac{2}{x+1}$$

Verified OK.

7.2.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y'y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations $\{1\mathrm{A},2\mathrm{A}\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

7.2.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\,y'' + a_1(x, y, y')\,y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'}\\ \frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'}\\ \frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}\end{aligned}$$

Looking at the the ode given we see that

$$\begin{aligned}a_2 &= 1\\ a_1 &= y\\ a_0 &= 0\end{aligned}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\begin{aligned}\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx &= c_1\\ \int 1\,dy' + \int y\,dy + \int 0\,dx &= c_1\end{aligned}$$

Which results in

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2}+c_1}dy = c_2 + x$$
$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}-1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}+1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1-\tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}}+2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}+1} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

7.2.5 Maple step by step solution

Let's solve

$\left[y'' + y'y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y), y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) + u(y)\,y = 0$

- Separate variables

  $\frac{d}{dy}u(y) = -y$

- Integrate both sides with respect to y

  $\int \left(\frac{d}{dy}u(y)\right) dy = \int -y dy + c_1$

- Evaluate integral

  $u(y) = -\frac{y^2}{2} + c_1$

- Solve for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$

- Solve 1st ODE for $u(y)$

  $u(y) = -\frac{y^2}{2} + c_1$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{y^2}{2} + c_1$

- Separate variables

  $\frac{y'}{-\frac{y^2}{2}+c_1} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-\frac{y^2}{2}+c_1} dx = \int 1 dx + c_2$

- Evaluate integral

  $\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$

- Solve for y

  $y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

- [ ] Check validity of solution $y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

  - Use initial condition $y(0) = 2$

    $2 = \tanh\left(\frac{c_2\sqrt{2}\,\sqrt{c_1}}{2}\right)\sqrt{c_1}\,\sqrt{2}$

  - Compute derivative of the solution

    $y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)^2\right)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -2$

    $-2 = c_1\left(1 - \tanh\left(\frac{c_2\sqrt{2}\,\sqrt{c_1}}{2}\right)^2\right)$

  - Solve for c_1 and c_2
  - The solution does not satisfy the initial conditions

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(0) = 2, D(y)(0) = -2],y(x), singsol=all)
```

$$y(x) = \frac{2}{1+x}$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]*y'[x]==0,{y[0]==2,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

{}

7.3 problem 1 (c)

7.3.1 Solving as second order integrable as is ode 876
7.3.2 Solving as second order ode missing x ode 877
7.3.3 Solving as type second_order_integrable_as_is (not using ABC version) 879
7.3.4 Solving as exact nonlinear second order ode ode 881

Internal problem ID [4841]
Internal file name [`OUTPUT/4334_Sunday_June_05_2022_01_01_58_PM_41809141/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 1 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
   _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
    _reducible, _mu_xy]]
```

$$\boxed{y'' + y'y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

7.3.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\,dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{2} \\
c_2 &= -\frac{\pi}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{-\mathrm{e}^{ix} + i}{i\mathrm{e}^{ix} - 1}
$$

Summary
The solution(s) found are the following

$$
y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i} \tag{1}
$$

Verification of solutions

$$
y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i}
$$

Verified OK.

7.3.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y) \left(\frac{d}{dy}p(y)\right) + p(y)\, y = 0
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
p(y) &= \int -y \,\mathrm{d}y \\
&= -\frac{y^2}{2} + c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $y = 1$ and $p = -1$ in the above solution gives an equation to solve for the constant of integration.

$$
-1 = -\frac{1}{2} + c_1
$$

$$
c_1 = -\frac{1}{2}
$$

Substituting c_1 found above in the general solution gives

$$
p(y) = -\frac{y^2}{2} - \frac{1}{2}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = -\frac{y^2}{2} - \frac{1}{2}
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-\frac{y^2}{2} - \frac{1}{2}} dy &= c_2 + x \\
-2\arctan(y) &= c_2 + x
\end{aligned}
$$

Solving for y gives these solutions

$$
y_1 = -\tan\left(\frac{c_2}{2} + \frac{x}{2}\right)
$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
1 = -\tan\left(\frac{c_2}{2}\right)
$$

$$
c_2 = -\frac{\pi}{2}
$$

Substituting c_2 found above in the general solution gives

$$y = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right) \tag{1}$$

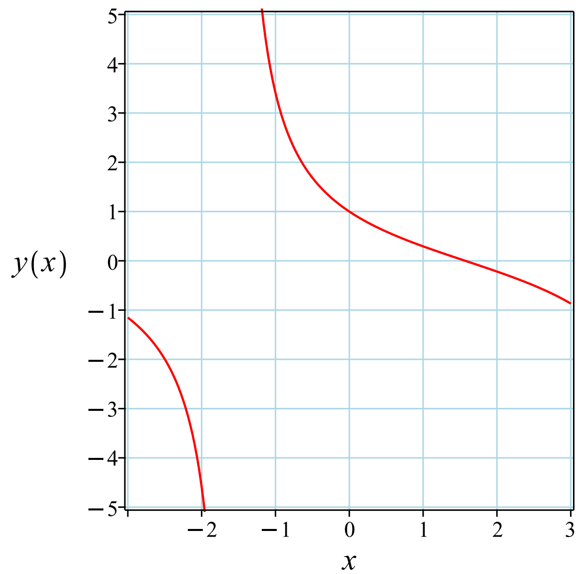

Figure 188: Solution plot

Verification of solutions

$$y = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

Verified OK.

7.3.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y'y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(e^{c_2\sqrt{2}\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = \frac{4\, e^{c_2\sqrt{2}\sqrt{c_1}} c_1}{e^{2c_2\sqrt{2}\sqrt{c_1}} + 2\, e^{c_2\sqrt{2}\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -\frac{1}{2} \\
c_2 &= -\frac{\pi}{2}
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{-\mathrm{e}^{ix} + i}{i\mathrm{e}^{ix} - 1}
$$

Summary
The solution(s) found are the following

$$
y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i} \tag{1}
$$

Verification of solutions

$$
y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i}
$$

Verified OK.

7.3.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$
a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0
$$

Where the following conditions are satisfied

$$
\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}
$$

Looking at the the ode given we see that

$$
\begin{aligned}
a_2 &= 1 \\
a_1 &= y \\
a_0 &= 0
\end{aligned}
$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2 \, dy' + \int a_1 \, dy + \int a_0 \, dx = c_1$$
$$\int 1 \, dy' + \int y \, dy + \int 0 \, dx = c_1$$

Which results in

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$
$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = \frac{\left(\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{2}$$
$$c_2 = -\frac{\pi}{2}$$

Substituting these values back in above solution results in

$$y = \frac{-\mathrm{e}^{ix} + i}{i\mathrm{e}^{ix} - 1}$$

Summary

The solution(s) found are the following

$$y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i} \tag{1}$$

Verification of solutions

$$y = \frac{i\mathrm{e}^{ix} + 1}{\mathrm{e}^{ix} + i}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

✓ Solution by Maple

Time used: 0.062 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(0) = 1, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = \cot\left(\frac{x}{2} + \frac{\pi}{4}\right)$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[{y''[x]+y[x]*y'[x]==0,{y[0]==1,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

{}

7.4 problem 1 (d)

7.4.1 Solving as second order integrable as is ode 886
7.4.2 Solving as second order ode missing x ode 888
7.4.3 Solving as type second_order_integrable_as_is (not using ABC version) 890
7.4.4 Solving as exact nonlinear second order ode ode 892

Internal problem ID [4842]
Internal file name [`OUTPUT/4335_Sunday_June_05_2022_01_02_16_PM_16787110/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 1 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear],
  _Lagerstrom, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order,
   _reducible, _mu_xy]]
```

$$\boxed{y'' + y'y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 2]$$

7.4.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\, \operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{\left(e^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{4\, e^{c_2\sqrt{2}\,\sqrt{c_1}} c_1}{e^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\, e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1} \tag{1}
$$

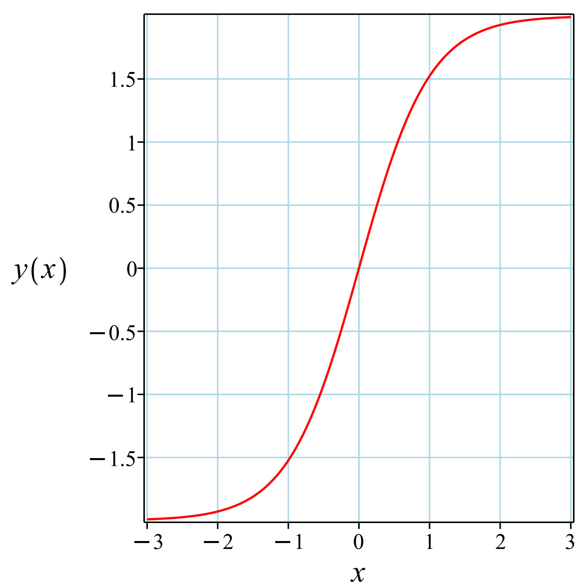

Figure 189: Solution plot

Verification of solutions

$$
y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1}
$$

Verified OK.

7.4.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + p(y)\,y = 0$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$\begin{aligned} p(y) &= \int -y \,\mathrm{d}y \\ &= -\frac{y^2}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $y = 0$ and $p = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$p(y) = -\frac{y^2}{2} + 2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{y^2}{2} + 2$$

Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2}+2}dy = \int dx$$
$$\frac{\ln(y+2)}{2} - \frac{\ln(y-2)}{2} = c_2 + x$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(y+2) - \ln(y-2)) = c_2 + x$$
$$\ln(y+2) - \ln(y-2) = (2)(c_2 + x)$$
$$= 2c_2 + 2x$$

Raising both side to exponential gives

$$e^{\ln(y+2)-\ln(y-2)} = 2c_2 e^{2x}$$

Which simplifies to

$$\frac{y+2}{y-2} = e^{2x}c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2c_3 + 2}{-1 + c_3}$$

$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

$$y = \frac{2\,e^{2x} - 2}{e^{2x} + 1}$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$y = \frac{2\,e^{2x} - 2}{e^{2x} + 1} \tag{1}$$

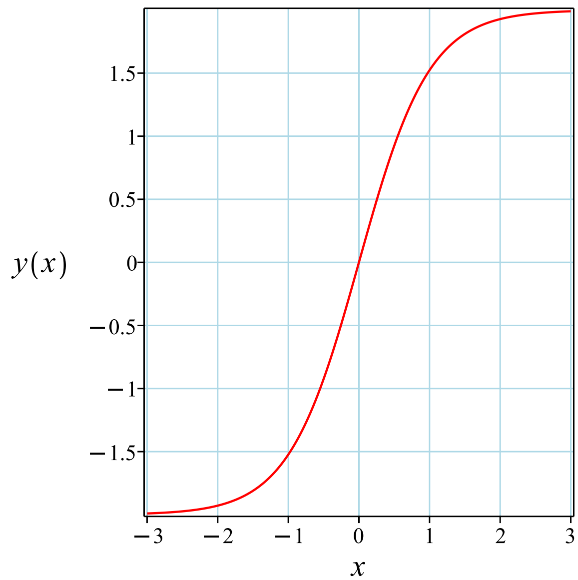

Figure 190: Solution plot

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{2x}-2}{\mathrm{e}^{2x}+1}$$

Verified OK.

7.4.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y'y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y'y)\, dx = 0$$

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2}+c_1} dy = c_2 + x$$

$$\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{\left(e^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2+x)\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{4\,e^{c_2\sqrt{2}\,\sqrt{c_1}}\,c_1}{e^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2\,e^{2x} - 2}{e^{2x} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,e^{2x} - 2}{e^{2x} + 1} \tag{1}$$

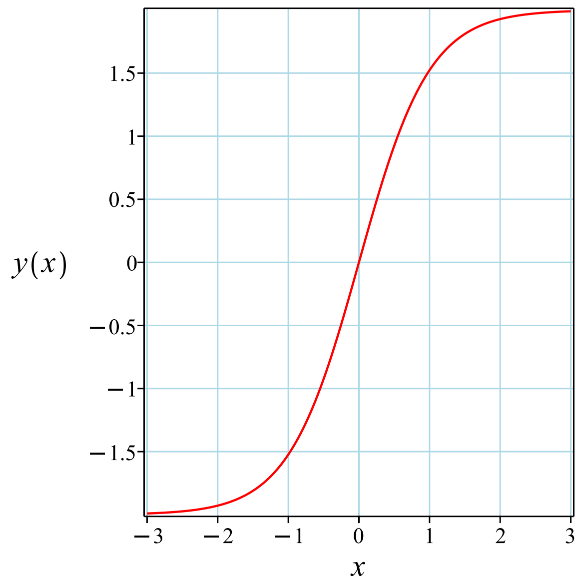

Figure 191: Solution plot

<u>Verification of solutions</u>

$$y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1}$$

Verified OK.

7.4.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$$
$$\frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}$$
$$\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$$

Looking at the the ode given we see that

$$a_2 = 1$$
$$a_1 = y$$
$$a_0 = 0$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$\int a_2\,dy' + \int a_1\,dy + \int a_0\,dx = c_1$$
$$\int 1\,dy' + \int y\,dy + \int 0\,dx = c_1$$

Which results in

$$\frac{y^2}{2} + y' = c_1$$

Which is now solved Integrating both sides gives

$$\int \frac{1}{-\frac{y^2}{2} + c_1}dy = c_2 + x$$
$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = c_2 + x$$

Solving for y gives these solutions

$$y_1 = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)\sqrt{c_1}\,\sqrt{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{\left(e^{c_2\sqrt{2}\,\sqrt{c_1}} - 1\right)\sqrt{2}\,\sqrt{c_1}}{e^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1\left(1 - \tanh\left(\frac{\sqrt{c_1}\,(c_2 + x)\,\sqrt{2}}{2}\right)^2\right)$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{4\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}}\,c_1}{\mathrm{e}^{2c_2\sqrt{2}\,\sqrt{c_1}} + 2\,\mathrm{e}^{c_2\sqrt{2}\,\sqrt{c_1}} + 1} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1} \tag{1}$$

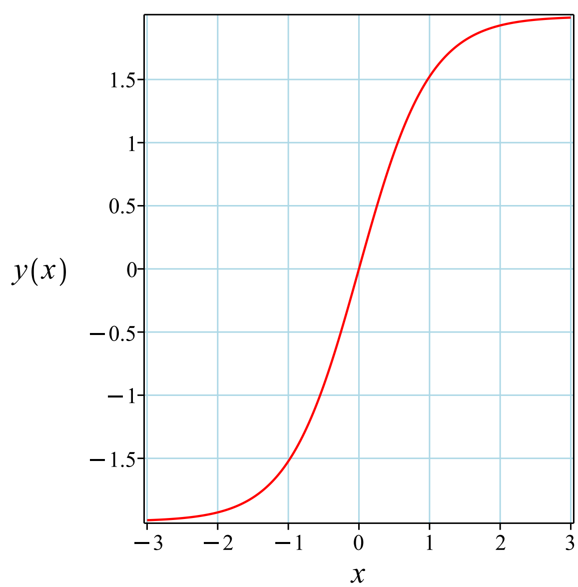

Figure 192: Solution plot

Verification of solutions

$$y = \frac{2\,\mathrm{e}^{2x} - 2}{\mathrm{e}^{2x} + 1}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+_a*_b(_a) = 0, _b(_a), HINT = [[
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 2*_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 8

```
dsolve([diff(y(x),x$2)+y(x)*diff(y(x),x)=0,y(0) = 0, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = 2\tanh(x)$$

✓ Solution by Mathematica

Time used: 10.835 (sec). Leaf size: 9

```
DSolve[{y''[x]+y[x]*y'[x]==0,{y[0]==0,y'[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\tanh(x)$$

7.5 problem 2

7.5.1 Solving as second order ode missing y ode 896
7.5.2 Solving using Kovacic algorithm 897
7.5.3 Maple step by step solution 903

Internal problem ID [4843]
Internal file name [`OUTPUT/4336_Sunday_June_05_2022_01_02_30_PM_65238584/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{y'' + 2xy' = 0}$$

7.5.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 2xp(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -2xp \end{aligned}$$

Where $f(x) = -2x$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= -2x\,dx \\
\int \frac{1}{p}\,dp &= \int -2x\,dx \\
\ln(p) &= -x^2 + c_1 \\
p &= \mathrm{e}^{-x^2+c_1} \\
&= c_1 \mathrm{e}^{-x^2}
\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_1 \mathrm{e}^{-x^2}$$

Integrating both sides gives

$$\begin{aligned}
y &= \int c_1 \mathrm{e}^{-x^2}\,\mathrm{d}x \\
&= \frac{c_1 \sqrt{\pi}\,\operatorname{erf}(x)}{2} + c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \sqrt{\pi}\,\operatorname{erf}(x)}{2} + c_2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \sqrt{\pi}\,\operatorname{erf}(x)}{2} + c_2$$

Verified OK.

7.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2xy' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= 1 \\
B &= 2x \\
C &= 0
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 + 1\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{1}{2x} - \frac{1}{8x^3} + \frac{1}{16x^5} - \frac{5}{128x^7} + \frac{7}{256x^9} - \frac{21}{1024x^{11}} + \frac{33}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x
\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 1}{1} \\
&= Q + \frac{R}{1} \\
&= \left(x^2 + 1\right) + (0) \\
&= x^2 + 1
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 1. Now b can be found.

$$\begin{aligned}
b &= (1) - (0) \\
&= 1
\end{aligned}$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{1}{1} - 1\right) &&= 0 \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{1}{1} - 1\right) &&= -1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = x^2 + 1
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	0	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{+} = 0$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^{+} \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + (x) \\
&= x \\
&= x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2(x)(0) + \left((1) + (x)^2 - \left(x^2 + 1\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega \, dx}\\ &= \mathrm{e}^{\int x dx}\\ &= \mathrm{e}^{\frac{x^2}{2}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx}\\ &= z_1 e^{-\frac{x^2}{2}}\\ &= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\sqrt{\pi}\, \operatorname{erf}(x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\sqrt{\pi}\, \operatorname{erf}(x)}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 + \frac{c_2\sqrt{\pi}\, \operatorname{erf}(x)}{2} \tag{1}
$$

Verification of solutions

$$
y = c_1 + \frac{c_2\sqrt{\pi}\, \operatorname{erf}(x)}{2}
$$

Verified OK.

7.5.3 Maple step by step solution

Let's solve

$y'' + 2xy' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

- □ Rewrite DE with series expansions
  - Convert $x \cdot y'$ to series expansion

$$x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- > k+2$

$$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite DE with series expansions

$$\sum_{k=0}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + 2a_k k\right) x^k = 0$$

- Each term in the series must be 0, giving the recursion relation

$$\left(k^2+3k+2\right) a_{k+2} + 2a_k k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{2a_k k}{k^2+3k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve(diff(y(x),x$2)+2*x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + \operatorname{erf}(x)\, c_2$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 21

```
DSolve[y''[x]+2*x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt{\pi}c_1\mathrm{erf}(x) + c_2$$

7.6 problem 3

7.6.1 Solving as second order ode missing x ode 906
7.6.2 Maple step by step solution 908

Internal problem ID [4844]
Internal file name [`OUTPUT/4337_Sunday_June_05_2022_01_02_41_PM_45749230/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible,
    _mu_x_y1], [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{2yy'' - {y'}^2 = 0}$$

7.6.1 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$2yp(y)\left(\frac{d}{dy}p(y)\right)-p(y)^2=0$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}
p' &= F(y,p)\\
&= f(y)g(p)\\
&= \frac{p}{2y}
\end{aligned}$$

Where $f(y)=\frac{1}{2y}$ and $g(p)=p$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{p}\,dp &= \frac{1}{2y}\,dy\\
\int\frac{1}{p}\,dp &= \int\frac{1}{2y}\,dy\\
\ln(p) &= \frac{\ln(y)}{2}+c_1\\
p &= e^{\frac{\ln(y)}{2}+c_1}\\
&= c_1\sqrt{y}
\end{aligned}$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=c_1\sqrt{y}$$

Integrating both sides gives

$$\begin{aligned}
\int\frac{1}{c_1\sqrt{y}}dy &= \int dx\\
\frac{2\sqrt{y}}{c_1} &= c_2+x
\end{aligned}$$

Summary
The solution(s) found are the following

$$y=\frac{1}{4}c_2^2c_1^2+\frac{1}{2}c_2c_1^2x+\frac{1}{4}x^2c_1^2 \tag{1}$$

Verification of solutions

$$y=\frac{1}{4}c_2^2c_1^2+\frac{1}{2}c_2c_1^2x+\frac{1}{4}x^2c_1^2$$

Verified OK.

7.6.2 Maple step by step solution

Let's solve

$2yy'' - y'^2 = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x) = y'$

- Compute y''

  $u'(x) = y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right) = y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = y''$

- Make substitutions $y' = u(y)\,, y'' = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $2yu(y)\left(\frac{d}{dy}u(y)\right) - u(y)^2 = 0$

- Separate variables

  $\frac{\frac{d}{dy}u(y)}{u(y)} = \frac{1}{2y}$

- Integrate both sides with respect to y

  $\int \frac{\frac{d}{dy}u(y)}{u(y)}dy = \int \frac{1}{2y}dy + c_1$

- Evaluate integral

  $\ln(u(y)) = \frac{\ln(y)}{2} + c_1$

- Solve for $u(y)$

  $\left\{u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}, u(y) = -\frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y) = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = \frac{\sqrt{e^{-2c_1}y}}{e^{-2c_1}}$

- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{-2c_1} y}} = \frac{1}{\mathrm{e}^{-2c_1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{-2c_1} y}} dx = \int \frac{1}{\mathrm{e}^{-2c_1}} dx + c_2$

- Evaluate integral

  $\frac{2\sqrt{\mathrm{e}^{-2c_1} y}}{\mathrm{e}^{-2c_1}} = \frac{x}{\mathrm{e}^{-2c_1}} + c_2$

- Solve for y

  $y = \frac{c_2^2 (\mathrm{e}^{-2c_1})^2 + 2c_2 \mathrm{e}^{-2c_1} x + x^2}{4\,\mathrm{e}^{-2c_1}}$

- Solve 2nd ODE for $u(y)$

  $u(y) = -\frac{\sqrt{\mathrm{e}^{-2c_1} y}}{\mathrm{e}^{-2c_1}}$

- Revert to original variables with substitution $u(y) = y', y = y$

  $y' = -\frac{\sqrt{\mathrm{e}^{-2c_1} y}}{\mathrm{e}^{-2c_1}}$

- Separate variables

  $\frac{y'}{\sqrt{\mathrm{e}^{-2c_1} y}} = -\frac{1}{\mathrm{e}^{-2c_1}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{\mathrm{e}^{-2c_1} y}} dx = \int -\frac{1}{\mathrm{e}^{-2c_1}} dx + c_2$

- Evaluate integral

  $\frac{2\sqrt{\mathrm{e}^{-2c_1} y}}{\mathrm{e}^{-2c_1}} = -\frac{x}{\mathrm{e}^{-2c_1}} + c_2$

- Solve for y

  $y = \frac{c_2^2 (\mathrm{e}^{-2c_1})^2 - 2c_2 \mathrm{e}^{-2c_1} x + x^2}{4\,\mathrm{e}^{-2c_1}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(2*y(x)*diff(y(x),x$2)=(diff(y(x),x))^2,y(x), singsol=all)
```

$$y(x) = 0$$
$$y(x) = \frac{(c_1 x + c_2)^2}{4}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 29

```
DSolve[2*y[x]*y''[x]==(y'[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(c_1 x + 2c_2)^2}{4c_2}$$
$$y(x) \to \text{Indeterminate}$$

7.7 problem 4

7.7.1 Solving as second order ode missing y ode 911
7.7.2 Maple step by step solution 913

Internal problem ID [4845]
Internal file name [`OUTPUT/4338_Sunday_June_05_2022_01_02_58_PM_31761600/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]
```

$$\boxed{xy'' - y' - {y'}^3 = 0}$$

7.7.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x)\, x + \left(-p(x)^2 - 1\right) p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= \frac{p(p^2+1)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(p) = p(p^2 + 1)$. Integrating both sides gives

$$\frac{1}{p\,(p^2+1)}\,dp = \frac{1}{x}\,dx$$
$$\int \frac{1}{p\,(p^2+1)}\,dp = \int \frac{1}{x}\,dx$$
$$-\frac{\ln(p^2+1)}{2} + \ln(p) = \ln(x) + c_1$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(p^2+1\right)}{2}+\ln(p)} = \mathrm{e}^{\ln(x)+c_1}$$

Which simplifies to

$$\frac{p}{\sqrt{p^2+1}} = c_2 x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = c_2 x\sqrt{-\frac{1}{c_2^2x^2-1}}$$

Integrating both sides gives

$$\begin{aligned} y &= \int c_2 x\sqrt{-\frac{1}{c_2^2x^2-1}}\,\mathrm{d}x \\ &= \frac{(c_2x-1)(c_2x+1)\sqrt{-\frac{1}{c_2^2x^2-1}}}{c_2} + c_3 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_2x-1)(c_2x+1)\sqrt{-\frac{1}{c_2^2x^2-1}}}{c_2} + c_3 \tag{1}$$

Verification of solutions

$$y = \frac{(c_2x-1)(c_2x+1)\sqrt{-\frac{1}{c_2^2x^2-1}}}{c_2} + c_3$$

Verified OK.

7.7.2 Maple step by step solution

Let's solve

$y''x+\left(-y'^2-1\right)y'=0$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u=y'$ to reduce order of ODE

  $u'(x)\,x+\left(-u(x)^2-1\right)u(x)=0$

- Separate variables

  $\frac{u'(x)}{\left(-u(x)^2-1\right)u(x)}=-\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\left(-u(x)^2-1\right)u(x)}dx=\int -\frac{1}{x}dx+c_1$

- Evaluate integral

  $\frac{\ln\left(u(x)^2+1\right)}{2}-\ln\left(u(x)\right)=-\ln\left(x\right)+c_1$

- Solve for $u(x)$

  $\left\{u(x)=\frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}, u(x)=-\frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}\right\}$

- Solve 1st ODE for $u(x)$

  $u(x)=\frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}$

- Make substitution $u=y'$

  $y'=\frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}$

- Integrate both sides to solve for y

  $\int y'dx=\int \frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}dx+c_2$

- Compute integrals

  $y=-\sqrt{\left(e^{c_1}\right)^2-x^2}+c_2$

- Solve 2nd ODE for $u(x)$

  $u(x)=-\frac{x}{\sqrt{\left(e^{c_1}\right)^2-x^2}}$

- Make substitution $u = y'$

$$y' = -\frac{x}{\sqrt{(e^{c_1})^2 - x^2}}$$

- Integrate both sides to solve for y

$$\int y' dx = \int -\frac{x}{\sqrt{(e^{c_1})^2 - x^2}} dx + c_2$$

- Compute integrals

$$y = \sqrt{(e^{c_1})^2 - x^2} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = _b(_a)*(_b(_a)^2+1)/_a, _b(_a), HINT =
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(x*diff(y(x),x$2)=diff(y(x),x)+(diff(y(x),x))^3,y(x), singsol=all)
```

$$y(x) = -\sqrt{-x^2 + c_1} + c_2$$
$$y(x) = \sqrt{-x^2 + c_1} + c_2$$

✓ Solution by Mathematica

Time used: 1.486 (sec). Leaf size: 103

```
DSolve[x*y''[x]==y'[x]+(y'[x])^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - ie^{-c_1}\sqrt{-1 + e^{2c_1}x^2}$$
$$y(x) \to ie^{-c_1}\sqrt{-1 + e^{2c_1}x^2} + c_2$$
$$y(x) \to c_2 - i\sqrt{x^2}$$
$$y(x) \to i\sqrt{x^2} + c_2$$

7.8 problem 5

7.8.1 Solving as second order ode missing y ode 916
7.8.2 Solving as second order ode missing x ode 918
7.8.3 Maple step by step solution 923

Internal problem ID [4846]
Internal file name [`OUTPUT/4339_Sunday_June_05_2022_01_03_09_PM_8720060/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 5.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y''^2 - k^2\left(1 + y'^2\right) = 0}$$

7.8.1 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$-p(x)^2 k^2 + p'(x)^2 - k^2 = 0$$

Which is now solve for $p(x)$ as first order ode. Solving the given ode for $p'(x)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$p'(x) = \sqrt{p(x)^2+1}\,k \tag{1}$$
$$p'(x) = -\sqrt{p(x)^2+1}\,k \tag{2}$$

Now each one of the above ODE is solved.

<u>Solving equation (1)</u>

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{\sqrt{p^2+1}\,k}dp &= x+c_1\\ \frac{\operatorname{arcsinh}(p)}{k} &= x+c_1\end{aligned}$$

Solving for p gives these solutions

$$p_1 = \sinh(c_1k+kx)$$

<u>Solving equation (2)</u>

Integrating both sides gives

$$\begin{aligned}\int -\frac{1}{\sqrt{p^2+1}\,k}dp &= c_2+x\\ -\frac{\operatorname{arcsinh}(p)}{k} &= c_2+x\end{aligned}$$

Solving for p gives these solutions

$$p_1 = -\sinh(c_2k+kx)$$

For solution (1) found earlier, since $p=y'$ then the new first order ode to solve is

$$y' = \sinh(c_1k+kx)$$

Integrating both sides gives

$$\begin{aligned}y &= \int \sinh(c_1k+kx)\,\mathrm{d}x\\ &= \frac{\cosh(c_1k+kx)}{k}+c_3\end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -\sinh(c_2k + kx)$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\sinh(c_2k + kx) \,\mathrm{d}x \\ &= -\frac{\cosh(c_2k + kx)}{k} + c_4 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\cosh(c_1k + kx)}{k} + c_3 \tag{1}$$
$$y = -\frac{\cosh(c_2k + kx)}{k} + c_4 \tag{2}$$

<u>Verification of solutions</u>

$$y = \frac{\cosh(c_1k + kx)}{k} + c_3$$

Verified OK.

$$y = -\frac{\cosh(c_2k + kx)}{k} + c_4$$

Verified OK.

7.8.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)^2 \left(\frac{d}{dy}p(y)\right)^2 - p(y)^2 k^2 = k^2$$

Which is now solved as first order ode for $p(y)$. Solving the given ode for $\frac{d}{dy}p(y)$ results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$\frac{d}{dy}p(y) = \frac{\sqrt{p(y)^2 + 1}\, k}{p(y)} \tag{1}$$

$$\frac{d}{dy}p(y) = -\frac{\sqrt{p(y)^2 + 1}\, k}{p(y)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{p}{\sqrt{p^2 + 1}\, k} dp = \int dy$$

$$\frac{\sqrt{p(y)^2 + 1}}{k} = y + c_1$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{p}{\sqrt{p^2 + 1}\, k} dp = \int dy$$

$$-\frac{\sqrt{p(y)^2 + 1}}{k} = y + c_2$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$\frac{\sqrt{1 + y'^2}}{k} = y + c_1$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-1 + c_1^2k^2 + 2yc_1k^2 + y^2k^2} \tag{1}$$

$$y' = -\sqrt{-1 + c_1^2k^2 + 2yc_1k^2 + y^2k^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_1^2k^2 + 2c_1k^2y + y^2k^2 - 1}}dy = \int dx$$

$$\frac{\ln\left(\frac{c_1k^2+y\,k^2}{\sqrt{k^2}} + \sqrt{c_1^2k^2 + 2c_1k^2y + y^2k^2 - 1}\right)}{\sqrt{k^2}} = x + c_3$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\frac{c_1k^2+y\,k^2}{\sqrt{k^2}} + \sqrt{c_1^2k^2+2c_1k^2y+y^2k^2-1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{x+c_3}$$

Which simplifies to

$$\left(\sqrt{-1 + (y + c_1)^2\, k^2} + k(y + c_1)\operatorname{csgn}(k)\right)^{\frac{1}{\sqrt{k^2}}} = c_4\mathrm{e}^{x}$$

Simplifying the solution $y = \frac{\operatorname{csgn}(k)\left(-2k\operatorname{csgn}(k)c_1 + (c_4\mathrm{e}^{x})^{\operatorname{csgn}(k)k} + (c_4\mathrm{e}^{x})^{-\operatorname{csgn}(k)k}\right)}{2k}$ to $y = \frac{-2c_1k + (c_4\mathrm{e}^{x})^{k} + (c_4\mathrm{e}^{x})^{-k}}{2k}$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_1^2k^2 + 2c_1k^2y + y^2k^2 - 1}}dy = \int dx$$

$$-\frac{\ln\left(\frac{c_1k^2+y\,k^2}{\sqrt{k^2}} + \sqrt{c_1^2k^2 + 2c_1k^2y + y^2k^2 - 1}\right)}{\sqrt{k^2}} = x + c_5$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(\frac{c_1k^2+y\,k^2}{\sqrt{k^2}} + \sqrt{c_1^2k^2+2c_1k^2y+y^2k^2-1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{x+c_5}$$

Which simplifies to

$$\left(\sqrt{-1+(y+c_1)^2 k^2}+k(y+c_1)\operatorname{csgn}(k)\right)^{-\frac{\operatorname{csgn}(k)}{k}} = c_6 \mathrm{e}^{x}$$

Simplifying the solution $y = \frac{\operatorname{csgn}(k)\left(-2k \operatorname{csgn}(k)c_1+(c_6\mathrm{e}^{x})^{-\operatorname{csgn}(k)k}+(c_6\mathrm{e}^{x})^{\operatorname{csgn}(k)k}\right)}{2k}$ to $y = \frac{-2c_1k+(c_6\mathrm{e}^{x})^{-k}+(c_6\mathrm{e}^{x})^{k}}{2k}$
For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{\sqrt{1+y'^2}}{k} = y + c_2$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-1+c_2^2k^2+2yc_2k^2+y^2k^2} \tag{1}$$
$$y' = -\sqrt{-1+c_2^2k^2+2yc_2k^2+y^2k^2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}}dy = \int dx$$
$$\frac{\ln\left(\frac{c_2k^2+y\,k^2}{\sqrt{k^2}}+\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}\right)}{\sqrt{k^2}} = x + c_7$$

Raising both side to exponential gives

$$\mathrm{e}^{\frac{\ln\left(\frac{c_2k^2+y\,k^2}{\sqrt{k^2}}+\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{x+c_7}$$

Which simplifies to

$$\left(\sqrt{-1+(y+c_2)^2 k^2}+k(y+c_2)\operatorname{csgn}(k)\right)^{\frac{1}{\sqrt{k^2}}} = c_8 \mathrm{e}^{x}$$

Simplifying the solution $y = \frac{\operatorname{csgn}(k)\left(-2k \operatorname{csgn}(k)c_2+(c_8\mathrm{e}^{x})^{\operatorname{csgn}(k)k}+(c_8\mathrm{e}^{x})^{-\operatorname{csgn}(k)k}\right)}{2k}$ to $y = \frac{-2c_2k+(c_8\mathrm{e}^{x})^{k}+(c_8\mathrm{e}^{x})^{-k}}{2k}$
Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}}dy = \int dx$$

$$-\frac{\ln\left(\frac{c_2k^2+y\,k^2}{\sqrt{k^2}}+\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}\right)}{\sqrt{k^2}} = x+c_9$$

Raising both side to exponential gives

$$\mathrm{e}^{-\frac{\ln\left(\frac{c_2k^2+y\,k^2}{\sqrt{k^2}}+\sqrt{c_2^2k^2+2c_2k^2y+y^2k^2-1}\right)}{\sqrt{k^2}}} = \mathrm{e}^{x+c_9}$$

Which simplifies to

$$\left(\sqrt{-1+(y+c_2)^2\,k^2}+k(y+c_2)\,\mathrm{csgn}\,(k)\right)^{-\frac{\mathrm{csgn}(k)}{k}} = \_C10\,\mathrm{e}^{x}$$

Simplifying the solution $y = \frac{\mathrm{csgn}(k)\left(-2k\ \mathrm{csgn}(k)c_2+\left(\_C10\,\mathrm{e}^{x}\right)^{-\,\mathrm{csgn}(k)k}+\left(\_C10\,\mathrm{e}^{x}\right)^{\mathrm{csgn}(k)k}\right)}{2k}$ to

$$y = \frac{-2c_2k+\left(\_C10\,\mathrm{e}^{x}\right)^{-k}+\left(\_C10\,\mathrm{e}^{x}\right)^{k}}{2k}$$

Summary

The solution(s) found are the following

$$y = \frac{-2c_1k+\left(c_4\mathrm{e}^{x}\right)^{k}+\left(c_4\mathrm{e}^{x}\right)^{-k}}{2k}$$

$$y = \frac{-2c_1k+\left(c_6\mathrm{e}^{x}\right)^{-k}+\left(c_6\mathrm{e}^{x}\right)^{k}}{2k}$$

$$y = \frac{-2c_2k+\left(c_8\mathrm{e}^{x}\right)^{k}+\left(c_8\mathrm{e}^{x}\right)^{-k}}{2k}$$

$$y = \frac{-2c_2k+\left(\_C10\,\mathrm{e}^{x}\right)^{-k}+\left(\_C10\,\mathrm{e}^{x}\right)^{k}}{2k}$$

Verification of solutions

$$y = \frac{-2c_1k + (c_4\mathrm{e}^x)^k + (c_4\mathrm{e}^x)^{-k}}{2k}$$

Verified OK.

$$y = \frac{-2c_1k + (c_6\mathrm{e}^x)^{-k} + (c_6\mathrm{e}^x)^k}{2k}$$

Verified OK.

$$y = \frac{-2c_2k + (c_8\mathrm{e}^x)^k + (c_8\mathrm{e}^x)^{-k}}{2k}$$

Verified OK.

$$y = \frac{-2c_2k + (\_C10\,\mathrm{e}^x)^{-k} + (\_C10\,\mathrm{e}^x)^k}{2k}$$

Verified OK.

7.8.3 Maple step by step solution

Let's solve

$y''^2 - y'^2k^2 = k^2$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)^2 - u(x)^2 k^2 = k^2$

- Separate variables

  $\frac{u'(x)}{\sqrt{u(x)^2+1}} = k$

- Integrate both sides with respect to x

  $\int \frac{u'(x)}{\sqrt{u(x)^2+1}}dx = \int kdx + c_1$

- Evaluate integral

  $\operatorname{arcsinh}(u(x)) = kx + c_1$

- Solve for $u(x)$

  $u(x) = \sinh(kx + c_1)$

- Solve 1st ODE for $u(x)$

  $u(x) = \sinh(kx + c_1)$
- Make substitution $u = y'$

  $y' = \sinh(kx + c_1)$
- Integrate both sides to solve for y

  $\int y'dx = \int \sinh(kx + c_1)\, dx + c_2$
- Compute integrals

  $y = \frac{\cosh(kx+c_1)}{k} + c_2$

Maple trace

```
`Methods for second order ODEs:
   *** Sublevel 2 ***
   Methods for second order ODEs:
   Successful isolation of d^2y/dx^2: 2 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      -> Calling odsolve with the ODE`, diff(diff(diff(y(x), x), x), x)-k^2*(diff(y(x), x)),
         Methods for third order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         <- constant coefficients successful
      <- 2nd order ODE linearizable_by_differentiation successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying 2nd order Liouville
      trying 2nd order WeierstrassP
      trying 2nd order JacobiSN
      differential order: 2; trying a linearization to 3rd order
      trying 2nd order ODE linearizable_by_differentiation
      <- 2nd order ODE linearizable_by_differentiation successful`
```

✓ Solution by Maple

Time used: 0.5 (sec). Leaf size: 55

```
dsolve((diff(y(x),x$2))^2=k^2*(1+ (diff(y(x),x))^2),y(x), singsol=all)
```

$$y(x) = -ix + c_1$$
$$y(x) = ix + c_1$$
$$y(x) = \frac{4c_2^2 e^{kx} k^2 + 4c_1 c_2 k^2 + e^{-kx}}{4c_2 k^2}$$

✓ Solution by Mathematica

Time used: 0.451 (sec). Leaf size: 71

```
DSolve[(y''[x])^2==k^2*(1+ (y'[x])^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{kx-c_1} + e^{-kx+c_1} - 2c_2 k}{2k}$$
$$y(x) \to \frac{e^{kx+c_1}\left(1 + e^{-2(kx+c_1)}\right)}{2k} + c_2$$

7.9 problem 6

7.9.1 Solving as second order integrable as is ode 927
7.9.2 Solving as second order ode missing y ode 928
7.9.3 Solving as second order ode missing x ode 929
7.9.4 Maple step by step solution 931

Internal problem ID [4847]
Internal file name [`OUTPUT/4340_Sunday_June_05_2022_01_03_31_PM_49431190/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 6.
ODE order: 2.
ODE degree: 2.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear]]
```

$$\boxed{-\frac{y''}{(1+y')^{\frac{3}{2}}} = -k}$$

7.9.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int -\frac{y''}{(1+y')^{\frac{3}{2}}} dx = \int -k dx$$

$$\frac{2}{\sqrt{1+y'}} = -kx + c_1$$

Which is now solved for y. Integrating both sides gives

$$y = \int -\frac{k^2x^2 - 2c_1kx + c_1^2 - 4}{(-kx+c_1)^2} \,\mathrm{d}x$$

$$= -x - \frac{4}{k(kx-c_1)} + c_2$$

Summary
The solution(s) found are the following

$$y = -x - \frac{4}{k\left(kx - c_1\right)} + c_2 \tag{1}$$

Verification of solutions

$$y = -x - \frac{4}{k\left(kx - c_1\right)} + c_2$$

Verified OK.

7.9.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$k - \frac{p'(x)}{\left(p\left(x\right)+1\right)^{\frac{3}{2}}} = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{k\left(p+1\right)^{\frac{3}{2}}} dp = \int dx$$

$$-\frac{2}{\sqrt{p\left(x\right)+1}\, k} = x + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$-\frac{2}{\sqrt{1+y'}\, k} = x + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int -\frac{c_1^2 k^2 + 2c_1 k^2 x + k^2 x^2 - 4}{k^2 \left(x+c_1\right)^2}\, \mathrm{d}x \\ &= -\frac{k^2 x + \frac{4}{x+c_1}}{k^2} + c_2 \end{aligned}$$

Summary
The solution(s) found are the following

$$
y=-\frac{k^2x+\frac{4}{x+c_1}}{k^2}+c_2 \tag{1}
$$

Verification of solutions

$$
y=-\frac{k^2x+\frac{4}{x+c_1}}{k^2}+c_2
$$

Verified OK.

7.9.3 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y'=p(y)
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dy}{dx}\frac{dp}{dy} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
-\frac{p(y)\left(\frac{d}{dy}p(y)\right)}{(1+p(y))^{\frac{3}{2}}}=-k
$$

Which is now solved as first order ode for $p(y)$. Integrating both sides gives

$$
\begin{aligned}
\int \frac{p}{k(1+p)^{\frac{3}{2}}}dp &= \int dy \\
\frac{2p(y)+4}{\sqrt{1+p(y)}\,k} &= y+c_1
\end{aligned}
$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
\frac{2y'+4}{k\sqrt{1+y'}}=y+c_1
$$

Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -2+\frac{\left(\frac{c_1 k}{4}+\frac{y k}{4}+\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{4}\right) c_1 k}{2}+\frac{\left(\frac{c_1 k}{4}+\frac{y k}{4}+\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{4}\right) y k}{2} \tag{1}$$

$$y' = -2+\frac{\left(\frac{c_1 k}{4}+\frac{y k}{4}-\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{4}\right) c_1 k}{2}+\frac{\left(\frac{c_1 k}{4}+\frac{y k}{4}-\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{4}\right) y k}{2} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{-2+\frac{c_1^2 k^2}{8}+\frac{y c_1 k^2}{4}+\frac{c_1 k \sqrt{c_1^2 k^2+2 y c_1 k^2+y^2 k^2-16}}{8}+\frac{y^2 k^2}{8}+\frac{y k \sqrt{c_1^2 k^2+2 y c_1 k^2+y^2 k^2-16}}{8}} d y=\int d x$$

$$\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{2 k}-\frac{y}{2}=c_2+x$$

Solving equation (2)

Integrating both sides gives

$$\int \frac{1}{-2+\frac{c_1^2 k^2}{8}+\frac{y c_1 k^2}{4}-\frac{c_1 k \sqrt{c_1^2 k^2+2 y c_1 k^2+y^2 k^2-16}}{8}+\frac{y^2 k^2}{8}-\frac{y k \sqrt{c_1^2 k^2+2 y c_1 k^2+y^2 k^2-16}}{8}} d y=\int d x$$

$$-\frac{y}{2}-\frac{\sqrt{y^2 k^2+2 y c_1 k^2+c_1^2 k^2-16}}{2 k}=x+c_3$$

Summary

The solution(s) found are the following

$$y=-\frac{c_1^2 k^2-4 c_2^2 k^2-8 c_2 k^2 x-4 k^2 x^2-16}{2 k^2\left(c_1-2 c_2-2 x\right)} \tag{1}$$

$$y=-\frac{c_1^2 k^2-4 c_3^2 k^2-8 c_3 k^2 x-4 k^2 x^2-16}{2 k^2\left(c_1-2 c_3-2 x\right)} \tag{2}$$

Verification of solutions

$$y = -\frac{c_1^2 k^2 - 4c_2^2 k^2 - 8c_2 k^2 x - 4k^2 x^2 - 16}{2k^2 (c_1 - 2c_2 - 2x)}$$

Verified OK.

$$y = -\frac{c_1^2 k^2 - 4c_3^2 k^2 - 8c_3 k^2 x - 4k^2 x^2 - 16}{2k^2 (c_1 - 2c_3 - 2x)}$$

Verified OK.

7.9.4 Maple step by step solution

Let's solve

$-\frac{y''}{(1+y')^{\frac{3}{2}}} = -k$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $-\frac{u'(x)}{(1+u(x))^{\frac{3}{2}}} = -k$

- Integrate both sides with respect to x

  $\int -\frac{u'(x)}{(1+u(x))^{\frac{3}{2}}} dx = \int -k dx + c_1$

- Evaluate integral

  $\frac{2}{\sqrt{1+u(x)}} = -kx + c_1$

- Solve for $u(x)$

  $u(x) = -\frac{k^2 x^2 - 2c_1 kx + c_1^2 - 4}{(-kx+c_1)^2}$

- Solve 1st ODE for $u(x)$

  $u(x) = -\frac{k^2 x^2 - 2c_1 kx + c_1^2 - 4}{(-kx+c_1)^2}$

- Make substitution $u = y'$

  $y' = -\frac{k^2 x^2 - 2c_1 kx + c_1^2 - 4}{(-kx+c_1)^2}$

- Integrate both sides to solve for y

  $\int y' dx = \int -\frac{k^2 x^2 - 2c_1 kx + c_1^2 - 4}{(-kx+c_1)^2} dx + c_2$

- Compute integrals

$$y = -x - \frac{4}{k(kx - c_1)} + c_2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = k*(_b(_a)+1)^(3/2), _b(_a), HINT = [[1,
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 0], [_a, -2-2*_b]
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 19

```
dsolve(k=diff(y(x),x$2)*(1+ (diff(y(x),x))^(-3/2),y(x), singsol=all)
```

$$y(x) = -x - \frac{4}{k^2 (x + c_1)} + c_2$$

✓ Solution by Mathematica

Time used: 0.515 (sec). Leaf size: 75

```
DSolve[k==y''[x]*(1+ (y'[x])^2)^(-3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{i\sqrt{k^2x^2 + 2c_1kx - 1 + c_1{}^2}}{k}$$
$$y(x) \to \frac{i\sqrt{k^2x^2 + 2c_1kx - 1 + c_1{}^2}}{k} + c_2$$

7.10 problem 16 (a)

7.10.1 Solving as second order euler ode ode 933
7.10.2 Solving as second order change of variable on x method 2 ode . 934
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7.10.4 Solving as second order ode non constant coeff transformation on B ode 939
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Internal problem ID [4848]
Internal file name [`OUTPUT/4341_Sunday_June_05_2022_01_03_54_PM_90851396/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 16 (a).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 3xy' - 3y = 0}$$

7.10.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r - 3 = 0$$

Or

$$r^2 + 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + c_2 x$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + c_2 x$$

Verified OK.

7.10.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 3xy' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = -\frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int \frac{3}{x}dx\right)} dx \\
&= \int e^{-3\ln(x)}\, dx \\
&= \int \frac{1}{x^3} dx \\
&= -\frac{1}{2x^2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{x^2}}{\frac{1}{x^6}} \\
&= -3x^4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 3x^4 y(\tau) = 0$$

But in terms of τ

$$-3x^4 = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(4c_1x^4 + c_2)\sqrt{2}}{4x^4\sqrt{-\frac{1}{x^2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{(4c_1x^4 + c_2)\sqrt{2}}{4x^4\sqrt{-\frac{1}{x^2}}} \tag{1}$$

Verification of solutions

$$y = \frac{(4c_1x^4 + c_2)\sqrt{2}}{4x^4\sqrt{-\frac{1}{x^2}}}$$

Verified OK.

7.10.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 3xy' - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{3}{x} \\ q(x) &= -\frac{3}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right)v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right)v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{3n}{x^2} - \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u}{x} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{5}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{5}{x}\,dx \\ \ln(u) &= -5\ln(x) + c_1 \\ u &= \mathrm{e}^{-5\ln(x)+c_1} \\ &= \frac{c_1}{x^5} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x
$$

Verified OK.

7.10.4 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0
\end{aligned} \tag{1}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= 3x \\
C &= -3 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= (x^2)(0) + (3x)(3) + (-3)(3x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$3x^3 v'' + (15x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$3x^2 (u'(x)\, x + 5u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{5u}{x}
\end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{5}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{5}{x}\, dx \\
\ln(u) &= -5\ln(x) + c_1 \\
u &= \mathrm{e}^{-5\ln(x) + c_1} \\
&= \frac{c_1}{x^5}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^5} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^5}\, \mathrm{d}x \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (3x)\left(-\frac{c_1}{4x^4} + c_2\right) \\ &= \frac{3c_2x^4 - \frac{3c_1}{4}}{x^3} \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{3c_2x^4 - \frac{3c_1}{4}}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{3c_2x^4 - \frac{3c_1}{4}}{x^3}$$

Verified OK.

7.10.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 3xy' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 3x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)\,(0) \\
&= -\frac{3}{2x} \\
&= -\frac{3}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3}{2x} dx} \\
&= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3x}{x^2}\,dx} \\
&= z_1 e^{-\frac{3\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{3}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^3} + \frac{c_2 x}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^3} + \frac{c_2 x}{4}
$$

Verified OK.

7.10.6 Maple step by step solution

Let's solve

$x^2y'' + 3xy' - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{x} + \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{x} - \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + 3xy' - 3y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 3\frac{d}{dt}y(t) - 3y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) - 3y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-1)=0$
- Roots of the characteristic polynomial

  $r=(-3,1)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^{-3t}$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{t}$
- Change variables back using $t=\ln(x)$

  $y=\frac{c_1}{x^3}+c_2x$
- Simplify

  $y=\frac{c_1}{x^3}+c_2x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+3*x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x)=\frac{c_1x^4+c_2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+3*x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x^3} + c_2 x$$

7.11 problem 16 (b)

7.11.1 Solving as second order euler ode ode 950
7.11.2 Solving as second order change of variable on x method 2 ode . 951
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Internal problem ID [4849]
Internal file name [`OUTPUT/4342_Sunday_June_05_2022_01_04_01_PM_87728878/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 16 (b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + xy' - 4y = 0}$$

7.11.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 4 = 0$$

Or

$$r^2 - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2 x^2$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^2} + c_2 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^2} + c_2 x^2$$

Verified OK.

7.11.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + x y' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\
&= -4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda \tau} - 4\, e^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\
&= \pm 2
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2 \\
\lambda_2 &= -2
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}
\end{aligned}$$

Or

$$y(\tau) = c_1 e^{2\tau} + c_2 e^{-2\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^4 + c_2}{x^2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^4 + c_2}{x^2}$$

Verified OK.

7.11.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + x y' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cosh(2\ln(x)) + i c_2 \sinh(2\ln(x))
$$

Summary
The solution(s) found are the following

$$y = c_1 \cosh(2\ln(x)) + ic_2 \sinh(2\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cosh(2\ln(x)) + ic_2 \sinh(2\ln(x))$$

Verified OK.

7.11.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x)+\frac{5v'(x)}{x}&=0\\
v''(x)+\frac{5v'(x)}{x}&=0 \qquad (7)
\end{aligned}
$$

Using the substitution

$$
u(x)=v'(x)
$$

Then (7) becomes

$$
u'(x)+\frac{5u(x)}{x}=0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=-\frac{5u}{x}
\end{aligned}
$$

Where $f(x)=-\frac{5}{x}$ and $g(u)=u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du&=-\frac{5}{x}\,dx\\
\int\frac{1}{u}\,du&=\int-\frac{5}{x}\,dx\\
\ln(u)&=-5\ln(x)+c_1\\
u&=\mathrm{e}^{-5\ln(x)+c_1}\\
&=\frac{c_1}{x^5}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x)&=u(x)\\
v(x)&=\int u(x)\,dx+c_2\\
&=-\frac{c_1}{4x^4}+c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \\
&= \frac{4c_2x^4 - c_1}{4x^2}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2
$$

Verified OK.

7.11.5 Solving using Kovacic algorithm

Writing the ode as

$$
x^2y'' + xy' - 4y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 147: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)\,(0) \\
&= -\frac{3}{2x} \\
&= -\frac{3}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{3}{2x} dx} \\
&= \frac{1}{x^{\frac{3}{2}}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^4}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^2} + \frac{c_2 x^2}{4}
$$

Verified OK.

7.11.6 Maple step by step solution

Let's solve

$x^2 y'' + x y' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + x y' - 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the ODE

  $y_1(t) = e^{-2t}$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$

- Substitute in solutions

  $y(t) = c_1e^{-2t} + c_2e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^2} + c_2x^2$

- Simplify

  $y = \frac{c_1}{x^2} + c_2x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^4 + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+x*y'[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^4 + c_1}{x^2}$$

7.12 problem 16 (c)

7.12.1 Solving as second order euler ode ode 966
7.12.2 Solving as second order change of variable on x method 2 ode . 967
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Internal problem ID [4850]
Internal file name [`OUTPUT/4343_Sunday_June_05_2022_01_04_09_PM_26643473/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 16 (c).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 7xy' + 9y = 0}$$

7.12.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 7xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 7r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1)+7r+9=0$$

Or

$$r^2+6r+9=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = -3$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3}$$

Verified OK.

7.12.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y''+7xy'+9y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x) = \frac{7}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{7}{x}dx\right)}dx \\
&= \int e^{-7\ln(x)}\,dx \\
&= \int \frac{1}{x^7}dx \\
&= -\frac{1}{6x^6}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{9}{x^2}}{\frac{1}{x^{14}}} \\
&= 9x^{12}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 9x^{12}y(\tau) = 0$$

But in terms of τ

$$9x^{12} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{6}\,\sqrt{-\frac{1}{x^6}}\left(c_1 - c_2\ln(2) - c_2\ln(3) + c_2\ln\left(-\frac{1}{x^6}\right)\right)}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{6}\,\sqrt{-\frac{1}{x^6}}\,\left(c_1 - c_2 \ln(2) - c_2 \ln(3) + c_2 \ln\left(-\frac{1}{x^6}\right)\right)}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\,\sqrt{-\frac{1}{x^6}}\,\left(c_1 - c_2 \ln(2) - c_2 \ln(3) + c_2 \ln\left(-\frac{1}{x^6}\right)\right)}{6}$$

Verified OK.

7.12.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + 7xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = \frac{7}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{7}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{-c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1}{x^3} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1}{x^3}
$$

Verified OK.

7.12.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 7xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{7}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{7n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= e^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \frac{c_1 \ln(x) + c_2}{x^3} \\
&= \frac{c_1 \ln(x) + c_2}{x^3}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \ln(x) + c_2}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \ln(x) + c_2}{x^3}$$

Verified OK.

7.12.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 7xy' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 7x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 149: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{7x}{x^2} \, dx} \\
&= z_1 e^{-\frac{7 \ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{7}{2}}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-7 \ln(x)}}{(y_1)^2} \, dx \\
&= y_1 (\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^3}\right) + c_2 \left(\frac{1}{x^3} (\ln(x))\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3}$$

Verified OK.

7.12.6 Maple step by step solution

Let's solve

$x^2 y'' + 7xy' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{7y'}{x} - \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7y'}{x} + \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 7xy' + 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  ◦ Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 7\frac{d}{dt}y(t) + 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r+3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t}$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^{-3t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-3t} + c_2 t\,\mathrm{e}^{-3t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3}$

- Simplify

  $y = \frac{c_1}{x^3} + \frac{c_2 \ln(x)}{x^3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)+7*x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 \ln(x) + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+7*x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3c_2 \log(x) + c_1}{x^3}$$

7.13 problem 16 (d)

7.13.1 Solving as second order euler ode ode 982
7.13.2 Solving as second order change of variable on x method 2 ode . 984
7.13.3 Solving as second order change of variable on x method 1 ode . 987
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Internal problem ID [4851]
Internal file name [`OUTPUT/4344_Sunday_June_05_2022_01_04_20_PM_41427607/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 16 (d).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - xy' + 6y = 0}$$

7.13.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + 6x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + 6x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 6 = 0$$

Or

$$r^2 - 2r + 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1 - i\sqrt{5}$$
$$r_2 = 1 + i\sqrt{5}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = 1$ and $\beta = -\sqrt{5}$. Hence the solution becomes

$$\begin{aligned}
y &= c_1 x^{r_1} + c_2 x^{r_2} \\
&= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\
&= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\
&= x^\alpha \left(c_1 e^{\ln(x^{i\beta})} + c_2 e^{\ln(x^{-i\beta})}\right) \\
&= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right)
\end{aligned}$$

Using the values for $\alpha = 1, \beta = -\sqrt{5}$, the above becomes

$$y = x^1 \left(c_1 e^{-i\sqrt{5}\, \ln(x)} + c_2 e^{i\sqrt{5}\, \ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = x\left(c_1 \cos\left(\sqrt{5}\, \ln(x)\right) + c_2 \sin\left(\sqrt{5}\, \ln(x)\right)\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = x\left(c_1 \cos\left(\sqrt{5}\, \ln(x)\right) + c_2 \sin\left(\sqrt{5}\, \ln(x)\right)\right) \tag{1}$$

<u>Verification of solutions</u>

$$y = x\left(c_1 \cos\left(\sqrt{5}\, \ln(x)\right) + c_2 \sin\left(\sqrt{5}\, \ln(x)\right)\right)$$

Verified OK.

7.13.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{6}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^2} \\
&= \frac{6}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{6}{x^4} = \frac{3}{2\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{2\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$2\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$2\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$2r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$2r(r-1) + 0 + 3 = 0$$

Or

$$2r^2 - 2r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= \frac{1}{2} - \frac{i\sqrt{5}}{2} \\
r_2 &= \frac{1}{2} + \frac{i\sqrt{5}}{2}
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{5}}{2}$. Hence the solution becomes

$$
\begin{aligned}
y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\
&= c_1 \tau^{\alpha + i\beta} + c_2 \tau^{\alpha - i\beta} \\
&= \tau^{\alpha} \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\
&= \tau^{\alpha} \left(c_1 e^{\ln\left(\tau^{i\beta}\right)} + c_2 e^{\ln\left(\tau^{-i\beta}\right)}\right) \\
&= \tau^{\alpha} \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right)
\end{aligned}
$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{5}}{2}$, the above becomes

$$
y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{i\sqrt{5}\,\ln(\tau)}{2}} + c_2 e^{\frac{i\sqrt{5}\,\ln(\tau)}{2}}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y(\tau) = \sqrt{\tau} \left(c_1 \cos\left(\frac{\sqrt{5}\,\ln(\tau)}{2}\right) + c_2 \sin\left(\frac{\sqrt{5}\,\ln(\tau)}{2}\right)\right)
$$

The above solution is now transformed back to y using (6) which results in

$$
y = -\frac{\sqrt{2}\,x \left(c_2 \sin\left(\frac{\sqrt{5}\,(\ln(2) - 2\ln(x))}{2}\right) - c_1 \cos\left(\frac{\sqrt{5}\,(\ln(2) - 2\ln(x))}{2}\right)\right)}{2}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\sqrt{2}\,x \left(c_2 \sin\left(\frac{\sqrt{5}\,(\ln(2) - 2\ln(x))}{2}\right) - c_1 \cos\left(\frac{\sqrt{5}\,(\ln(2) - 2\ln(x))}{2}\right)\right)}{2} \tag{1}
$$

Verification of solutions

$$y = -\frac{\sqrt{2}\, x \left(c_2 \sin\left(\frac{\sqrt{5}\, (\ln(2)-2\ln(x))}{2}\right) - c_1 \cos\left(\frac{\sqrt{5}\, (\ln(2)-2\ln(x))}{2}\right)\right)}{2}$$

Verified OK.

7.13.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - x y' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\, \sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c \sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{6}\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -\frac{c\sqrt{6}}{3}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{\frac{\sqrt{6}\,c\tau}{6}}\left(c_1 \cos\left(\frac{c\sqrt{30}\,\tau}{6}\right) + c_2 \sin\left(\frac{c\sqrt{30}\,\tau}{6}\right)\right)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x\left(c_1 \cos\left(\sqrt{5}\,\ln(x)\right) + c_2 \sin\left(\sqrt{5}\,\ln(x)\right)\right)
$$

Summary

The solution(s) found are the following

$$
y = x\left(c_1 \cos\left(\sqrt{5}\,\ln(x)\right) + c_2 \sin\left(\sqrt{5}\,\ln(x)\right)\right) \qquad (1)
$$

Verification of solutions

$$
y = x\left(c_1 \cos\left(\sqrt{5}\,\ln(x)\right) + c_2 \sin\left(\sqrt{5}\,\ln(x)\right)\right)
$$

Verified OK.

7.13.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - xy' + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 + i\sqrt{5} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2+2i\sqrt{5}}{x} - \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(2i\sqrt{5}+1)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{\left(2i\sqrt{5}+1\right)u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{\left(-2i\sqrt{5}-1\right)u}{x}
\end{aligned}$$

Where $f(x)=\frac{-2i\sqrt{5}-1}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{-2i\sqrt{5}-1}{x}\,dx\\
\int \frac{1}{u}\,du &= \int \frac{-2i\sqrt{5}-1}{x}\,dx\\
\ln(u) &= \left(-2i\sqrt{5}-1\right)\ln(x)+c_1\\
u &= \mathrm{e}^{\left(-2i\sqrt{5}-1\right)\ln(x)+c_1}\\
&= c_1\mathrm{e}^{\left(-2i\sqrt{5}-1\right)\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1x^{-2i\sqrt{5}}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x)\\
v(x) &= \int u(x)\,dx+c_2\\
&= \frac{i\sqrt{5}\,c_1x^{-2i\sqrt{5}}}{10}+c_2
\end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i\sqrt{5}\, c_1 x^{-2i\sqrt{5}}}{10} + c_2\right) x^{1+i\sqrt{5}} \\
&= x\left(x^{i\sqrt{5}} c_2 + \frac{i x^{-i\sqrt{5}} \sqrt{5}\, c_1}{10}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{i\sqrt{5}\, c_1 x^{-2i\sqrt{5}}}{10} + c_2\right) x^{1+i\sqrt{5}} \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{i\sqrt{5}\, c_1 x^{-2i\sqrt{5}}}{10} + c_2\right) x^{1+i\sqrt{5}}
$$

Verified OK.

7.13.5 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' - x y' + 6y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 6
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-21}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -21 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{21}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 151: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{21}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{21}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{5}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{5}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{21}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{21}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i\sqrt{5}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i\sqrt{5}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{21}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i\sqrt{5}$	$\frac{1}{2} - i\sqrt{5}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i\sqrt{5}$	$\frac{1}{2} - i\sqrt{5}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i\sqrt{5}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - i\sqrt{5} - \left(\frac{1}{2} - i\sqrt{5}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - i\sqrt{5}}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - i\sqrt{5}}{x} \\ &= \frac{-2i\sqrt{5} + 1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i\sqrt{5}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - i\sqrt{5}}{x^2}\right) + \left(\frac{\frac{1}{2} - i\sqrt{5}}{x}\right)^2 - \left(-\frac{21}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{\frac{1}{2} - i\sqrt{5}}{x} dx} \\ &= x^{\frac{1}{2} - i\sqrt{5}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{1 - i\sqrt{5}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{i x^{2i\sqrt{5}} \sqrt{5}}{10}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{1-i\sqrt{5}}\right) + c_2 \left(x^{1-i\sqrt{5}} \left(-\frac{i x^{2i\sqrt{5}} \sqrt{5}}{10}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{1-i\sqrt{5}} - \frac{i c_2 \sqrt{5}\, x^{1+i\sqrt{5}}}{10} \tag{1}$$

Verification of solutions

$$y = c_1 x^{1-i\sqrt{5}} - \frac{i c_2 \sqrt{5}\, x^{1+i\sqrt{5}}}{10}$$

Verified OK.

7.13.6 Maple step by step solution

Let's solve

$x^2 y'' - x y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{x} - \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{y'}{x} + \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - x y' + 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) - \frac{d}{dt} y(t) + 6y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2} y(t) - 2\frac{d}{dt} y(t) + 6y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 6 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-20})}{2}$

- Roots of the characteristic polynomial

  $r = \left(1 - \mathrm{I}\sqrt{5}, 1 + \mathrm{I}\sqrt{5}\right)$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^t \cos\left(\sqrt{5}\, t\right)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^t \sin\left(\sqrt{5}\, t\right)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^t \cos\left(\sqrt{5}\, t\right) + c_2 e^t \sin\left(\sqrt{5}\, t\right)$

- Change variables back using $t = \ln(x)$

  $y = c_1 x \cos\left(\sqrt{5}\, \ln(x)\right) + c_2 x \sin\left(\sqrt{5}\, \ln(x)\right)$

- Simplify

  $y = x\left(c_1 \cos\left(\sqrt{5}\, \ln(x)\right) + c_2 \sin\left(\sqrt{5}\, \ln(x)\right)\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = x\left(c_1 \sin\left(\sqrt{5}\, \ln(x)\right) + c_2 \cos\left(\sqrt{5}\, \ln(x)\right)\right)$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 32

```
DSolve[x^2*y''[x]-x*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(c_2 \cos\left(\sqrt{5}\log(x)\right) + c_1 \sin\left(\sqrt{5}\log(x)\right)\right)$$

7.14 problem 17

7.14.1 Solving as second order euler ode ode 999
7.14.2 Solving as second order change of variable on x method 2 ode . 1003
7.14.3 Solving as second order change of variable on x method 1 ode . 1008
7.14.4 Solving as second order change of variable on y method 2 ode . 1013
7.14.5 Solving using Kovacic algorithm 1018

Internal problem ID [4852]
Internal file name [`OUTPUT/4345_Sunday_June_05_2022_01_04_28_PM_19216924/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' - 16y = 8x^4}$$

7.14.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -16, f(x) = 8x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' + xy' - 16y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 16x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 16x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 16 = 0$$

Or

$$r^2 - 16 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -4$$
$$r_2 = 4$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^4} + c_2x^4$$

Next, we find the particular solution to the ODE

$$x^2y'' + xy' - 16y = 8x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^4}$$

$$y_2 = x^4$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ \frac{d}{dx}\left(\frac{1}{x^4}\right) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ -\frac{4}{x^5} & 4x^3 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^4}\right)(4x^3) - (x^4)\left(-\frac{4}{x^5}\right)$$

Which simplifies to

$$W = \frac{8}{x}$$

Which simplifies to

$$W = \frac{8}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^8}{8x}\,dx$$

Which simplifies to

$$u_1 = -\int x^7 dx$$

Hence

$$u_1 = -\frac{x^8}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8}{8x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4}{8} + \ln(x)\,x^4$$

Which simplifies to

$$y_p(x) = x^4\left(-\frac{1}{8} + \ln(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^4\left(-\frac{1}{8} + \ln(x)\right) + \frac{c_1}{x^4} + c_2 x^4 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^4\left(-\frac{1}{8}+\ln(x)\right)+\frac{c_1}{x^4}+c_2x^4 \tag{1}$$

Verification of solutions

$$y = x^4\left(-\frac{1}{8}+\ln(x)\right)+\frac{c_1}{x^4}+c_2x^4$$

Verified OK.

7.14.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+xy'-16y=0$$

In normal form the ode

$$x^2y''+xy'-16y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{1}{x}$$
$$q(x)=-\frac{16}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau)+p_1\left(\frac{d}{d\tau}y(\tau)\right)+q_1y(\tau)=0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{16}{x^2}}{\frac{1}{x^2}} \\
&= -16
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2} y(\tau) - 16y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -16$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda \tau} - 16\, \mathrm{e}^{\lambda \tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 - 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-16)} \\ &= \pm 4\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4 \\ \lambda_2 &= -4\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= -4\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\ y(\tau) &= c_1 e^{(4)\tau} + c_2 e^{(-4)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{4\tau} + c_2 \mathrm{e}^{-4\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^8 + c_2}{x^4}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^8 + c_2}{x^4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^4}$$
$$y_2 = x^4$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ \frac{d}{dx}\left(\frac{1}{x^4}\right) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ -\frac{4}{x^5} & 4x^3 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^4}\right)\left(4x^3\right) - \left(x^4\right)\left(-\frac{4}{x^5}\right)$$

Which simplifies to

$$W = \frac{8}{x}$$

Which simplifies to

$$W = \frac{8}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^8}{8x}\,dx$$

Which simplifies to

$$u_1 = -\int x^7 dx$$

Hence

$$u_1 = -\frac{x^8}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8}{8x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4}{8} + \ln(x)\,x^4$$

Which simplifies to

$$y_p(x) = x^4 \left(-\frac{1}{8} + \ln(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^8 + c_2}{x^4}\right) + \left(x^4 \left(-\frac{1}{8} + \ln(x)\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^8 + c_2}{x^4} + x^4 \left(-\frac{1}{8} + \ln(x)\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^8 + c_2}{x^4} + x^4 \left(-\frac{1}{8} + \ln(x)\right)$$

Verified OK.

7.14.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -16, f(x) = 8x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 16y = 0$$

In normal form the ode

$$x^2y'' + xy' - 16y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{16}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{4\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{4}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{4\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{4\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int 4\sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{4\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cosh(4\ln(x)) + ic_2 \sinh(4\ln(x))$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - 16y = 8x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \frac{1}{x^4} \\
y_2 &= x^4
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ \frac{d}{dx}\left(\frac{1}{x^4}\right) & \frac{d}{dx}(x^4) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ -\frac{4}{x^5} & 4x^3 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^4}\right)(4x^3) - (x^4)\left(-\frac{4}{x^5}\right)$$

Which simplifies to

$$W = \frac{8}{x}$$

Which simplifies to

$$W = \frac{8}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^8}{8x}\,dx$$

Which simplifies to

$$u_1 = -\int x^7 dx$$

Hence

$$u_1 = -\frac{x^8}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8}{8x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4}{8} + \ln(x)\, x^4$$

Which simplifies to

$$y_p(x) = x^4 \left(-\frac{1}{8} + \ln(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cosh(4\ln(x)) + ic_2 \sinh(4\ln(x))) + \left(x^4\left(-\frac{1}{8} + \ln(x)\right)\right) \\ &= x^4\left(-\frac{1}{8} + \ln(x)\right) + c_1 \cosh(4\ln(x)) + ic_2 \sinh(4\ln(x)) \end{aligned}$$

Which simplifies to

$$y = \ln(x)\, x^4 - \frac{x^4}{8} + ic_2 \sinh(4\ln(x)) + c_1 \cosh(4\ln(x))$$

Summary

The solution(s) found are the following

$$y = \ln(x)\, x^4 - \frac{x^4}{8} + ic_2 \sinh(4\ln(x)) + c_1 \cosh(4\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x)\, x^4 - \frac{x^4}{8} + ic_2 \sinh(4\ln(x)) + c_1 \cosh(4\ln(x))$$

Verified OK.

7.14.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -16, f(x) = 8x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 16y = 0$$

In normal form the ode

$$x^2y'' + xy' - 16y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{16}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{16}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 4 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{9v'(x)}{x} &= 0 \\ v''(x) + \frac{9v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{9u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{9u}{x} \end{aligned}$$

Where $f(x) = -\frac{9}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{9}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{9}{x}\,dx \\ \ln(u) &= -9\ln(x) + c_1 \\ u &= \mathrm{e}^{-9\ln(x)+c_1} \\ &= \frac{c_1}{x^9} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{8x^8} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{8x^8} + c_2\right) x^4 \\
&= \frac{8c_2x^8 - c_1}{8x^4}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2y'' + xy' - 16y = 8x^4
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^4} \\
y_2 &= x^4
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ \frac{d}{dx}\left(\frac{1}{x^4}\right) & \frac{d}{dx}\left(x^4\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^4} & x^4 \\ -\frac{4}{x^5} & 4x^3 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^4}\right)\left(4x^3\right) - \left(x^4\right)\left(-\frac{4}{x^5}\right)$$

Which simplifies to

$$W = \frac{8}{x}$$

Which simplifies to

$$W = \frac{8}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^8}{8x}\,dx$$

Which simplifies to

$$u_1 = -\int x^7 dx$$

Hence

$$u_1 = -\frac{x^8}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8}{8x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4}{8} + \ln(x)\, x^4$$

Which simplifies to

$$y_p(x) = x^4\left(-\frac{1}{8} + \ln(x)\right)$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{8x^8} + c_2\right) x^4\right) + \left(x^4\left(-\frac{1}{8} + \ln(x)\right)\right) \\
&= x^4\left(-\frac{1}{8} + \ln(x)\right) + \left(-\frac{c_1}{8x^8} + c_2\right) x^4
\end{aligned}$$

Which simplifies to

$$y = \frac{8x^8 \ln(x) + 8c_2 x^8 - x^8 - c_1}{8x^4}$$

Summary

The solution(s) found are the following

$$y = \frac{8x^8 \ln(x) + 8c_2 x^8 - x^8 - c_1}{8x^4} \tag{1}$$

Verification of solutions

$$y = \frac{8x^8 \ln(x) + 8c_2 x^8 - x^8 - c_1}{8x^4}$$

Verified OK.

7.14.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x^2y'' + xy' - 16y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x^2\\
B &= x \tag{3}\\
C &= -16
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{63}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 63\\
t &= 4x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{63}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{63}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{63}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{63}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{63}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{9}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{7}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{63}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{9}{2}$	$-\frac{7}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{9}{2}$	$-\frac{7}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{7}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{7}{2} - \left(-\frac{7}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{7}{2x} + (-)\,(0) \\
&= -\frac{7}{2x} \\
&= -\frac{7}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{7}{2x}\right)(0) + \left(\left(\frac{7}{2x^2}\right) + \left(-\frac{7}{2x}\right)^2 - \left(\frac{63}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{7}{2x} dx} \\
&= \frac{1}{x^{\frac{7}{2}}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^4}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^8}{8}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4}\left(\frac{x^8}{8}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^4} + \frac{c_2x^4}{8}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^4}$$

$$y_2 = \frac{x^4}{8}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^4} & \frac{x^4}{8} \\ \frac{d}{dx}\left(\frac{1}{x^4}\right) & \frac{d}{dx}\left(\frac{x^4}{8}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^4} & \frac{x^4}{8} \\ -\frac{4}{x^5} & \frac{x^3}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^4}\right)\left(\frac{x^3}{2}\right) - \left(\frac{x^4}{8}\right)\left(-\frac{4}{x^5}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^8}{x}\,dx$$

Which simplifies to

$$u_1 = -\int x^7 dx$$

Hence

$$u_1 = -\frac{x^8}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{8}{x} dx$$

Hence

$$u_2 = 8\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4}{8} + \ln(x)\, x^4$$

Which simplifies to

$$y_p(x) = x^4\left(-\frac{1}{8} + \ln(x)\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^4} + \frac{c_2 x^4}{8}\right) + \left(x^4\left(-\frac{1}{8} + \ln(x)\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2 x^4}{8} + x^4\left(-\frac{1}{8} + \ln(x)\right) \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2 x^4}{8} + x^4\left(-\frac{1}{8} + \ln(x)\right)$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-16*y(x)=8*x^4,y(x), singsol=all)
```

$$y(x) = \frac{8x^8 \ln (x) + (8c_2 - 1) x^8 + 8c_1}{8x^4}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 28

```
DSolve[x^2*y''[x]+x*y'[x]-16*y[x]==8*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^4 \log(x) + \left(-\frac{1}{8} + c_2\right) x^4 + \frac{c_1}{x^4}$$

7.15 problem 18

7.15.1 Solving as second order euler ode ode 1028
7.15.2 Solving as second order change of variable on x method 2 ode . 1031
7.15.3 Solving as second order change of variable on x method 1 ode . 1036
7.15.4 Solving as second order change of variable on y method 2 ode . 1041
7.15.5 Solving as second order integrable as is ode 1046
7.15.6 Solving as second order ode non constant coeff transformation on B ode 1048
7.15.7 Solving as type second_order_integrable_as_is (not using ABC version) 1052
7.15.8 Solving using Kovacic algorithm 1054
7.15.9 Solving as exact linear second order ode ode 1061

Internal problem ID [4853]
Internal file name [`OUTPUT/4346_Sunday_June_05_2022_01_04_35_PM_66108900/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + xy' - y = x - \frac{1}{x}}$$

7.15.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x - \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + xy' - y = x - \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x\left(x - \frac{1}{x}\right)}{2x} \, dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{x}{2} - \frac{1}{2x}\right) dx$$

Hence

$$u_1 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x - \frac{1}{x}}{x}}{2x} \, dx$$

Which simplifies to

$$u_2 = \int \frac{x^2 - 1}{2x^3} dx$$

Hence

$$u_2 = \frac{\ln(x)}{2} + \frac{1}{4x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \left(\frac{\ln(x)}{2} + \frac{1}{4x^2}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{2\ln(x)\, x^2 + 4c_2 x^2 - x^2 + 2\ln(x) + 4c_1 + 1}{4x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\ln(x)\, x^2 + 4c_2 x^2 - x^2 + 2\ln(x) + 4c_1 + 1}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{2\ln(x)\, x^2 + 4c_2 x^2 - x^2 + 2\ln(x) + 4c_1 + 1}{4x}
$$

Verified OK.

7.15.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' + xy' - y = 0
$$

In normal form the ode

$$
x^2 y'' + xy' - y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)}dx \\
&= \int e^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda\tau} - e^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1\tau} + c_2 e^{\lambda_2\tau} \\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 e^{\tau} + c_2 e^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x-\frac{1}{x}}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+1}{2x^3}dx$$

Hence

$$u_1 = \frac{\ln(x)}{2} + \frac{1}{4x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(x-\frac{1}{x}\right)}{-2x}\,dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{x}{2} + \frac{1}{2x}\right)dx$$

Hence

$$u_2 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \left(\frac{\ln(x)}{2} + \frac{1}{4x^2}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2 + c_2}{x}\right) + \left(\frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x} + \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Verified OK.

7.15.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x - \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + xy' - y = x - \frac{1}{x}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x-\frac{1}{x}}{x}}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+1}{2x^3}dx$$

Hence

$$u_1 = \frac{\ln(x)}{2} + \frac{1}{4x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(x - \frac{1}{x}\right)}{-2x} \, dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{x}{2} + \frac{1}{2x}\right) dx$$

Hence

$$u_2 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \left(\frac{\ln(x)}{2} + \frac{1}{4x^2}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}\right) + \left(\frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}\right) \\ &= \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x} + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \end{aligned}$$

Which simplifies to

$$y = \frac{(2x^2 + 2)\ln(x) + (2ic_2 + 2c_1 - 1)\, x^2 - 2ic_2 + 2c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x^2+2)\ln(x) + (2ic_2+2c_1-1)x^2 - 2ic_2 + 2c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{(2x^2+2)\ln(x) + (2ic_2+2c_1-1)x^2 - 2ic_2 + 2c_1 + 1}{4x}$$

Verified OK.

7.15.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = x - \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - y = 0$$

In normal form the ode

$$x^2y'' + xy' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{3}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right)x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right)x
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + xy' - y = x - \frac{1}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \\
y_2 &= \frac{1}{x}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x - \frac{1}{x}}{x}}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2 + 1}{2x^3} dx$$

Hence

$$u_1 = \frac{\ln(x)}{2} + \frac{1}{4x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(x - \frac{1}{x}\right)}{-2x} \, dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{x}{2} + \frac{1}{2x}\right) dx$$

Hence

$$u_2 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \left(\frac{\ln(x)}{2} + \frac{1}{4x^2}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}\right) \\ &= \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x} + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = \frac{2\ln(x)\, x^2 + 4c_2 x^2 - x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Summary
The solution(s) found are the following

$$y = \frac{2\ln(x)\, x^2 + 4c_2x^2 - x^2 + 2\ln(x) - 2c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{2\ln(x)\, x^2 + 4c_2x^2 - x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Verified OK.

7.15.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2y'' + xy' - y)\, dx = \int \left(x - \frac{1}{x}\right) dx$$
$$x^2y' - xy = \frac{x^2}{2} - \ln(x) + c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\frac{x^2}{2}-\ln(x)+c_1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{\frac{x^2}{2}-\ln(x)+c_1}{x^2}\right) \\
d\left(\frac{y}{x}\right) &= \left(\frac{\frac{x^2}{2}-\ln(x)+c_1}{x^3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{\frac{x^2}{2}-\ln(x)+c_1}{x^3}\, dx \\
\frac{y}{x} &= \frac{\ln(x)}{2}+\frac{\ln(x)}{2x^2}+\frac{1}{4x^2}-\frac{c_1}{2x^2}+c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x\left(\frac{\ln(x)}{2}+\frac{\ln(x)}{2x^2}+\frac{1}{4x^2}-\frac{c_1}{2x^2}\right)+c_2 x
$$

which simplifies to

$$
y = \frac{2\ln(x)\,x^2+4c_2x^2+2\ln(x)-2c_1+1}{4x}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2\ln(x)\,x^2+4c_2x^2+2\ln(x)-2c_1+1}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{2\ln(x)\,x^2+4c_2x^2+2\ln(x)-2c_1+1}{4x}
$$

Verified OK.

7.15.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= x - \frac{1}{x} \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^2)\,(0) + (x)\,(1) + (-1)\,(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + \left(3x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\,x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right)x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x-\frac{1}{x}}{x}}{-2x} dx$$

Which simplifies to

$$u_1 = -\int \frac{-x^2+1}{2x^3} dx$$

Hence

$$u_1 = \frac{\ln(x)}{2} + \frac{1}{4x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x\left(x-\frac{1}{x}\right)}{-2x} dx$$

Which simplifies to

$$u_2 = \int \left(-\frac{x}{2} + \frac{1}{2x}\right) dx$$

Hence

$$u_2 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \left(\frac{\ln(x)}{2} + \frac{1}{4x^2}\right) x$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(\frac{2\ln(x)\,x^2 - x^2 + 2\ln(x) + 1}{4x}\right) \\
&= \frac{(2x^2+2)\ln(x) + (4c_2-1)\,x^2 - 2c_1 + 1}{4x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{(2x^2+2)\ln(x) + (4c_2-1)\,x^2 - 2c_1 + 1}{4x} \tag{1}
$$

Verification of solutions

$$
y = \frac{(2x^2+2)\ln(x) + (4c_2-1)\,x^2 - 2c_1 + 1}{4x}
$$

Verified OK.

7.15.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
x^2y'' + xy' - y = x - \frac{1}{x}
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (x^2y'' + xy' - y)\, dx &= \int \left(x - \frac{1}{x}\right) dx \\
x^2y' - xy &= \frac{x^2}{2} - \ln(x) + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^3}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^3} dx \\ \frac{y}{x} &= \frac{\ln(x)}{2} + \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} - \frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{\ln(x)}{2} + \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} - \frac{c_1}{2x^2}\right) + c_2 x$$

which simplifies to

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Verified OK.

7.15.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + xy' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 154: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+xy'-y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x}+\frac{c_2x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1=\frac{1}{x}$$
$$y_2=\frac{x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x\left(x-\frac{1}{x}\right)}{2}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \left(\frac{x}{2} - \frac{1}{2x}\right) dx$$

Hence

$$u_1 = -\frac{x^2}{4} + \frac{\ln(x)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{x-\frac{1}{x}}{x}}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^2-1}{x^3} dx$$

Hence

$$u_2 = \ln(x) + \frac{1}{2x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{-\frac{x^2}{4} + \frac{\ln(x)}{2}}{x} + \frac{\left(\ln(x) + \frac{1}{2x^2}\right) x}{2}$$

Which simplifies to

$$y_p(x) = \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + \left(\frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + \frac{2\ln(x)\, x^2 - x^2 + 2\ln(x) + 1}{4x}$$

Verified OK.

7.15.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= x - \frac{1}{x} \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2-(1)+(-1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\,y'+\left(q(x)-p'(x)\right)y\right)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+\left(q(x)-p'(x)\right)y=\int s(x)\,dx
$$

Substituting the above values for p, q, r, s gives

$$
x^2y'-xy=\int x-\frac{1}{x}\,dx
$$

We now have a first order ode to solve which is

$$
x^2y'-xy=\frac{x^2}{2}-\ln\left(x\right)+c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{\frac{x^2}{2}-\ln\left(x\right)+c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{y}{x}=\frac{\frac{x^2}{2}-\ln\left(x\right)+c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^2}\right)$$

$$\mathrm{d}\left(\frac{y}{x}\right) = \left(\frac{\frac{x^2}{2} - \ln(x) + c_1}{x^3}\right)\mathrm{d}x$$

Integrating gives

$$\frac{y}{x} = \int \frac{\frac{x^2}{2} - \ln(x) + c_1}{x^3}\,\mathrm{d}x$$

$$\frac{y}{x} = \frac{\ln(x)}{2} + \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(\frac{\ln(x)}{2} + \frac{\ln(x)}{2x^2} + \frac{1}{4x^2} - \frac{c_1}{2x^2}\right) + c_2 x$$

which simplifies to

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x} \tag{1}$$

Verification of solutions

$$y = \frac{2\ln(x)\,x^2 + 4c_2x^2 + 2\ln(x) - 2c_1 + 1}{4x}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=x-1/x,y(x), singsol=all)
```

$$y(x) = \frac{2\ln(x)\, x^2 + 4c_2x^2 + 2\ln(x) + 4c_1 + 1}{4x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 37

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==x-1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2(x^2+1)\log(x) + (-1+4c_2)x^2 + 1 + 4c_1}{4x}$$

7.16 problem 19

7.16.1 Solving as second order euler ode ode 1065
7.16.2 Solving as second order change of variable on x method 2 ode . 1069
7.16.3 Solving as second order change of variable on x method 1 ode . 1074
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7.16.5 Solving using Kovacic algorithm 1083

Internal problem ID [4854]
Internal file name [`OUTPUT/4347_Sunday_June_05_2022_01_04_43_PM_36534505/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 5xy' + 9y = 2x^3}$$

7.16.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 9, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 5xy' + 9y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 5xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 5r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 5r + 9 = 0$$

Or

$$r^2 - 6r + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 3$$
$$r_2 = 3$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^3 + c_2x^3 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 5xy' + 9y = 2x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^3 \\
y_2 &= x^3 \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(x^3 \ln(x)) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ 3x^2 & 3\ln(x)\, x^2 + x^2 \end{vmatrix}
$$

Therefore

$$
W = (x^3)\left(3\ln(x)\, x^2 + x^2\right) - \left(x^3 \ln(x)\right)(3x^2)
$$

Which simplifies to

$$
W = x^5
$$

Which simplifies to

$$
W = x^5
$$

Therefore Eq. (2) becomes

$$
u_1 = -\int \frac{2x^6 \ln(x)}{x^7}\, dx
$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x)}{x} dx$$

Hence

$$u_1 = -\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^6}{x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x} dx$$

Hence

$$u_2 = 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2\, x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^3\left(\ln(x)^2 + c_1 + c_2\ln(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^3\left(\ln(x)^2 + c_1 + c_2\ln(x)\right) \tag{1}$$

Verification of solutions

$$y = x^3\left(\ln(x)^2 + c_1 + c_2\ln(x)\right)$$

Verified OK.

7.16.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y'' - 5xy' + 9y = 0$$

In normal form the ode

$$x^2y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{5}{x}\\ q(x) &= \frac{9}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{5}{x}dx\right)} dx \\
&= \int e^{5\ln(x)}\, dx \\
&= \int x^5 dx \\
&= \frac{x^6}{6} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{9}{x^2}}{x^{10}} \\
&= \frac{9}{x^{12}} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{9y(\tau)}{x^{12}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{9}{x^{12}} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^{r}+0\,\tau^{r}+\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$4r(r-1)+0+1=0$$

Or

$$4r^{2}-4r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=\frac{1}{2}$$
$$r_2=\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau)=c_1y_1+c_2y_2$$

Where $y_1=\tau^{r}$ and $y_2=\tau^{r}\ln(\tau)$. Hence

$$y(\tau)=c_1\sqrt{\tau}+c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{\sqrt{6}\,\sqrt{x^{6}}\,(c_1-c_2\ln(2)-c_2\ln(3)+c_2\ln(x^{6}))}{6}$$

Therefore the homogeneous solution y_h is

$$y_h=\frac{\sqrt{6}\,\sqrt{x^{6}}\,(c_1-c_2\ln(2)-c_2\ln(3)+c_2\ln(x^{6}))}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^6} \\
y_2 &= -\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \sqrt{x^6} & -\frac{\sqrt{6}\sqrt{x^6}\ln(2)}{6} - \frac{\sqrt{6}\sqrt{x^6}\ln(3)}{6} + \frac{\sqrt{6}\sqrt{x^6}\ln(x^6)}{6} \\ \frac{d}{dx}\left(\sqrt{x^6}\right) & \frac{d}{dx}\left(-\frac{\sqrt{6}\sqrt{x^6}\ln(2)}{6} - \frac{\sqrt{6}\sqrt{x^6}\ln(3)}{6} + \frac{\sqrt{6}\sqrt{x^6}\ln(x^6)}{6}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} \sqrt{x^6} & -\frac{\sqrt{6}\sqrt{x^6}\ln(2)}{6} - \frac{\sqrt{6}\sqrt{x^6}\ln(3)}{6} + \frac{\sqrt{6}\sqrt{x^6}\ln(x^6)}{6} \\ \frac{3x^5}{\sqrt{x^6}} & -\frac{\sqrt{6}\ln(2)x^5}{2\sqrt{x^6}} - \frac{\sqrt{6}\ln(3)x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\ln(x^6)x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\sqrt{x^6}}{x} \end{vmatrix}
$$

Therefore

$$
\begin{aligned}
W = {} & \left(\sqrt{x^6}\right)\left(-\frac{\sqrt{6}\,\ln(2)\,x^5}{2\sqrt{x^6}} - \frac{\sqrt{6}\,\ln(3)\,x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\ln(x^6)\,x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\sqrt{x^6}}{x}\right) \\
& - \left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right)\left(\frac{3x^5}{\sqrt{x^6}}\right)
\end{aligned}
$$

Which simplifies to

$$
W = \sqrt{6}\,x^5
$$

Which simplifies to

$$W = \sqrt{6}\, x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right) x^3}{x^7 \sqrt{6}}\, dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \frac{-\ln(2) - \ln(3) + 6\ln(x)}{3x} dx$$

Hence

$$u_1 = \frac{\ln(x)\left(\ln(2) + \ln(3) - 3\ln(x)\right)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\sqrt{x^6}\, x^3}{x^7 \sqrt{6}}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\sqrt{6}}{3x} dx$$

Hence

$$u_2 = \frac{\sqrt{6}\,\ln(x)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)\left(\ln(2) + \ln(3) - 3\ln(x)\right)\sqrt{x^6}}{3} + \frac{\sqrt{6}\,\ln(x)\left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right)}{3}$$

Which simplifies to

$$y_p(x) = \ln(x)^2\, x^3$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2\ln(2) - c_2\ln(3) + c_2\ln(x^6))}{6}\right) + \left(\ln(x)^2 x^3\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2\ln(2) - c_2\ln(3) + c_2\ln(x^6))}{6} + \ln(x)^2 x^3 \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{6}\,\sqrt{x^6}\,(c_1 - c_2\ln(2) - c_2\ln(3) + c_2\ln(x^6))}{6} + \ln(x)^2 x^3$$

Verified OK. `{0 < x}`

7.16.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 9, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5xy' + 9y = 0$$

In normal form the ode

$$x^2y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= -\frac{5}{x} \\
q(x) &= \frac{9}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{3\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{5}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{c\tau} c_1$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^3
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 5xy' + 9y = 2x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x^6} \\
y_2 &= -\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^6} & -\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6} \\ \frac{d}{dx}\left(\sqrt{x^6}\right) & \frac{d}{dx}\left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^6} & -\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6} \\ \frac{3x^5}{\sqrt{x^6}} & -\frac{\sqrt{6}\,\ln(2)x^5}{2\sqrt{x^6}} - \frac{\sqrt{6}\,\ln(3)x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\ln(x^6)x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\sqrt{x^6}}{x} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x^6}\right)\left(-\frac{\sqrt{6}\,\ln(2)\,x^5}{2\sqrt{x^6}} - \frac{\sqrt{6}\,\ln(3)\,x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\ln(x^6)\,x^5}{2\sqrt{x^6}} + \frac{\sqrt{6}\,\sqrt{x^6}}{x}\right) \\ & - \left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right)\left(\frac{3x^5}{\sqrt{x^6}}\right) \end{aligned}$$

Which simplifies to

$$W = \sqrt{6}\,x^5$$

Which simplifies to

$$W = \sqrt{6}\,x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\left(-\frac{\sqrt{6}\,\sqrt{x^6}\,\ln(2)}{6} - \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(3)}{6} + \frac{\sqrt{6}\,\sqrt{x^6}\,\ln(x^6)}{6}\right)x^3}{x^7\sqrt{6}}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int \frac{-\ln(2) - \ln(3) + 6\ln(x)}{3x}\,dx$$

Hence

$$u_1 = \frac{\ln(x)\left(\ln(2)+\ln(3)-3\ln(x)\right)}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\sqrt{x^6}\, x^3}{x^7\sqrt{6}}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{\sqrt{6}}{3x} dx$$

Hence

$$u_2 = \frac{\sqrt{6}\, \ln(x)}{3}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \frac{\ln(x)\left(\ln(2)+\ln(3)-3\ln(x)\right)\sqrt{x^6}}{3} \\ & + \frac{\sqrt{6}\, \ln(x)\left(-\frac{\sqrt{6}\sqrt{x^6}\, \ln(2)}{6} - \frac{\sqrt{6}\sqrt{x^6}\, \ln(3)}{6} + \frac{\sqrt{6}\sqrt{x^6}\, \ln(x^6)}{6}\right)}{3} \end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)^2 x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^3\right) + \left(\ln(x)^2 x^3\right) \\ &= \ln(x)^2 x^3 + c_1 x^3 \end{aligned}$$

Which simplifies to

$$y = x^3\left(\ln(x)^2 + c_1\right)$$

Summary
The solution(s) found are the following

$$y = x^3 \left(\ln(x)^2 + c_1\right) \tag{1}$$

Verification of solutions

$$y = x^3 \left(\ln(x)^2 + c_1\right)$$

Verified OK. `{0 < x}`

7.16.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -5x, C = 9, f(x) = 2x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 5xy' + 9y = 0$$

In normal form the ode

$$x^2y'' - 5xy' + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{5}{x}$$
$$q(x) = \frac{9}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{5n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^3 \\
&= (c_1 \ln(x) + c_2)\, x^3
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 5xy' + 9y = 2x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^3 \\
y_2 &= x^3 \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE.

The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(x^3 \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ 3x^2 & 3\ln(x)\, x^2 + x^2 \end{vmatrix}$$

Therefore

$$W = (x^3)\left(3\ln(x)\, x^2 + x^2\right) - \left(x^3 \ln(x)\right)(3x^2)$$

Which simplifies to

$$W = x^5$$

Which simplifies to

$$W = x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^6 \ln(x)}{x^7}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x)}{x} dx$$

Hence

$$u_1 = -\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^6}{x^7}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x} dx$$

Hence

$$u_2 = 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left((c_1 \ln(x) + c_2) x^3\right) + \left(\ln(x)^2 x^3\right) \\ &= \ln(x)^2 x^3 + (c_1 \ln(x) + c_2) x^3 \end{aligned}$$

Which simplifies to

$$y = x^3 \left(\ln(x)^2 + c_1 \ln(x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x^3 \left(\ln(x)^2 + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^3 \left(\ln(x)^2 + c_1 \ln(x) + c_2\right)$$

Verified OK. `{0 < x}`

7.16.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 5xy' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -5x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 155: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5x}{x^2}\,dx} \\
&= z_1 e^{\frac{5\ln(x)}{2}} \\
&= z_1\left(x^{\frac{5}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^3
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^3\right) + c_2\left(x^3(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 5xy' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^3 + c_2 x^3 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^3 \\ y_2 &= x^3 \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ \frac{d}{dx}(x^3) & \frac{d}{dx}(x^3 \ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^3 & x^3 \ln(x) \\ 3x^2 & 3\ln(x)\, x^2 + x^2 \end{vmatrix}$$

Therefore

$$W = (x^3)\left(3\ln(x)\, x^2 + x^2\right) - \left(x^3 \ln(x)\right)\left(3x^2\right)$$

Which simplifies to

$$W = x^5$$

Which simplifies to

$$W = x^5$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^6 \ln(x)}{x^7} dx$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x)}{x} dx$$

Hence

$$u_1 = -\ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^6}{x^7} dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x} dx$$

Hence

$$u_2 = 2\ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^2 x^3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^3 + c_2 x^3 \ln(x)\right) + \left(\ln(x)^2 x^3\right) \end{aligned}$$

Which simplifies to

$$y = x^3(c_1 + c_2 \ln(x)) + \ln(x)^2 x^3$$

Summary

The solution(s) found are the following

$$y = x^3(c_1 + c_2 \ln(x)) + \ln(x)^2 x^3 \tag{1}$$

Verification of solutions

$$y = x^3(c_1 + c_2 \ln(x)) + \ln(x)^2 x^3$$

Verified OK. {0 < x}

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-5*x*diff(y(x),x)+9*y(x)=2*x^3,y(x), singsol=all)
```

$$y(x) = x^3\left(c_2 + c_1 \ln(x) + \ln(x)^2\right)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-5*x*y'[x]+9*y[x]==2*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3\left(\log^2(x) + 3c_2 \log(x) + c_1\right)$$

7.17 problem 20

7.17.1 Solving as second order euler ode ode 1092
7.17.2 Solving as second order change of variable on x method 2 ode . 1096
7.17.3 Solving as second order change of variable on x method 1 ode . 1101
7.17.4 Solving as second order change of variable on y method 2 ode . 1106
7.17.5 Solving using Kovacic algorithm 1110

Internal problem ID [4855]
Internal file name [`OUTPUT/4348_Sunday_June_05_2022_01_04_51_PM_40921221/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' - 3xy' + 4y = 6\ln(x)\, x^2}$$

7.17.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 6\ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3xy' + 4y = 6\ln(x)\,x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2\ln(x)\, x) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\ln(x)^2\, x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{6 \ln(x)^2}{x} dx$$

Hence

$$u_1 = -2 \ln(x)^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{6 \ln(x) x^4}{x^5} dx$$

Which simplifies to

$$u_2 = \int \frac{6 \ln(x)}{x} dx$$

Hence

$$u_2 = 3 \ln(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^3 x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^2 \left(\ln(x)^3 + c_1 + c_2 \ln(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2 \left(\ln(x)^3 + c_1 + c_2 \ln(x)\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\ln(x)^3 + c_1 + c_2 \ln(x)\right)$$

Verified OK.

7.17.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{4}{x^8} = \frac{1}{4\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0
$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(-2c_2\ln(2) + c_2\ln(x^4) + c_1\right)\sqrt{x^4}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{\left(-2c_2\ln(2) + c_2\ln(x^4) + c_1\right)\sqrt{x^4}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)\,x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\left(-\ln \left(2\right) \sqrt{x^4}+\frac{\ln \left(x^4\right) \sqrt{x^4}}{2}\right) \ln \left(x\right) x^2}{2 x^5} d x$$

Which simplifies for $0<x$ to

$$u_1 = -\int -\frac{3(\ln \left(2\right)-2 \ln \left(x\right)) \ln \left(x\right)}{x} d x$$

Hence

$$u_1 = -2 \ln \left(x\right)^3+\frac{3 \ln \left(2\right) \ln \left(x\right)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6 \sqrt{x^4}\, \ln \left(x\right) x^2}{2 x^5} d x$$

Which simplifies for $0<x$ to

$$u_2 = \int \frac{3 \ln \left(x\right)}{x} d x$$

Hence

$$u_2 = \frac{3 \ln \left(x\right)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-2 \ln \left(x\right)^3+\frac{3 \ln \left(2\right) \ln \left(x\right)^2}{2}\right) \sqrt{x^4}+\frac{3 \ln \left(x\right)^2\left(-\ln \left(2\right) \sqrt{x^4}+\frac{\ln \left(x^4\right) \sqrt{x^4}}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \ln \left(x\right)^3 x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(\frac{\left(-2 c_2 \ln \left(2\right)+c_2 \ln \left(x^4\right)+c_1\right) \sqrt{x^4}}{2}\right)+\left(\ln \left(x\right)^3 x^2\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1\right)\sqrt{x^4}}{2} + \ln(x)^3 x^2 \tag{1}$$

Verification of solutions

$$y = \frac{\left(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1\right)\sqrt{x^4}}{2} + \ln(x)^3 x^2$$

Verified OK. `{0 < x}`

7.17.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 6\ln(x)\,x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3xy' + 4y = 6\ln(x)\, x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \sqrt{x^4} \\ y_2 &= -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)\,x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\ln(x)\,x^2}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int -\frac{3(\ln(2) - 2\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = -2\ln(x)^3 + \frac{3\ln(2)\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6\sqrt{x^4}\,\ln(x)\,x^2}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{3\ln(x)}{x}dx$$

Hence

$$u_2 = \frac{3 \ln(x)^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-2 \ln(x)^3 + \frac{3 \ln(2) \ln(x)^2}{2}\right) \sqrt{x^4} + \frac{3 \ln(x)^2 \left(-\ln(2) \sqrt{x^4} + \frac{\ln(x^4) \sqrt{x^4}}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \ln(x)^3 x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + (\ln(x)^3 x^2) \\ &= \ln(x)^3 x^2 + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 (\ln(x)^3 + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2 (\ln(x)^3 + c_1) \tag{1}$$

Verification of solutions

$$y = x^2 (\ln(x)^3 + c_1)$$

Verified OK. `{0 < x}`

7.17.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = 6\ln(x)\, x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3xy' + 4y = 0$$

In normal form the ode

$$x^2y'' - 3xy' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^2 \\
&= (c_1 \ln(x) + c_2)\, x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3xy' + 4y = 6 \ln(x)\, x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\,x^2 \\ 2x & x + 2\ln(x)\,x \end{vmatrix}$$

Therefore

$$W = \left(x^2\right)\left(x + 2\ln(x)\,x\right) - \left(\ln(x)\,x^2\right)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\ln(x)^2\,x^4}{x^5}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{6\ln(x)^2}{x}dx$$

Hence

$$u_1 = -2\ln(x)^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{6\ln(x)\,x^4}{x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{6\ln(x)}{x}dx$$

Hence

$$u_2 = 3\ln(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^3 x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 \ln(x) + c_2\right) x^2\right) + \left(\ln(x)^3 x^2\right) \\ &= \ln(x)^3 x^2 + \left(c_1 \ln(x) + c_2\right) x^2 \end{aligned}$$

Which simplifies to

$$y = x^2 \left(\ln(x)^3 + c_1 \ln(x) + c_2\right)$$

Summary
The solution(s) found are the following

$$y = x^2 \left(\ln(x)^3 + c_1 \ln(x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x^2 \left(\ln(x)^3 + c_1 \ln(x) + c_2\right)$$

Verified OK. `{0 < x}`

7.17.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' - 3xy' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 156: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0 \\ 0 = 0\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-3x}{x^2} \, dx} \\ &= z_1 e^{\frac{3 \ln(x)}{2}} \\ &= z_1 \left(x^{\frac{3}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
x^2y'' - 3xy' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1x^2 + c_2x^2\ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\,x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2\ln(x)\, x \end{vmatrix}$$

Therefore

$$W = (x^2)\,(x + 2\ln(x)\, x) - (\ln(x)\, x^2)\,(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{6\ln(x)^2\, x^4}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{6\ln(x)^2}{x} dx$$

Hence

$$u_1 = -2\ln(x)^3$$

And Eq. (3) becomes

$$u_2 = \int \frac{6\ln(x)\,x^4}{x^5}\,dx$$

Which simplifies to

$$u_2 = \int \frac{6\ln(x)}{x}dx$$

Hence

$$u_2 = 3\ln(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(x)^3\,x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1x^2 + c_2x^2\ln(x)\right) + \left(\ln(x)^3\,x^2\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_1 + c_2\ln(x)) + \ln(x)^3\,x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^2(c_1 + c_2\ln(x)) + \ln(x)^3\,x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = x^2(c_1 + c_2\ln(x)) + \ln(x)^3\,x^2$$

Verified OK. `{0 < x}`

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=6*x^2*ln(x),y(x), singsol=all)
```

$$y(x) = x^2\left(c_2 + c_1 \ln(x) + \ln(x)^3\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==6*x^2*Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2\left(\log^3(x) + 2c_2 \log(x) + c_1\right)$$

7.18 problem 21

7.18.1 Solving as second order euler ode ode 1119
7.18.2 Solving using Kovacic algorithm 1124

Internal problem ID [4856]
Internal file name [OUTPUT/4349_Sunday_June_05_2022_01_04_59_PM_43903136/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y = 3x^2}$$

7.18.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = 0, C = 1, f(x) = 3x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 0rx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 0\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 0 + 1 = 0$$

Or

$$r^2 - r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{2} - \frac{i\sqrt{3}}{2} \\ r_2 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{\sqrt{3}}{2}$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{\sqrt{3}}{2}$, the above becomes

$$y = x^{\frac{1}{2}} \left(c_1 e^{-\frac{i\sqrt{3}\,\ln(x)}{2}} + c_2 e^{\frac{i\sqrt{3}\,\ln(x)}{2}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \sqrt{x}\left(c_1 \cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)$$

Next, we find the particular solution to the ODE

$$x^2y'' + y = 3x^2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)$$

$$y_2 = -\sqrt{x}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) & -\sqrt{x}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right) & \frac{d}{dx}\left(-\sqrt{x}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) & -\sqrt{x}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right) \\ \frac{\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{2\sqrt{x}} - \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{2\sqrt{x}} & -\frac{\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{2\sqrt{x}} - \frac{\sqrt{3}\,\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{2\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\, \cos\left(\frac{\sqrt{3}\, \ln\left(x\right)}{2}\right)\right) \left(-\frac{\sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{2\sqrt{x}} - \frac{\sqrt{3}\, \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{2\sqrt{x}}\right)$$
$$- \left(-\sqrt{x}\, \sin\left(\frac{\sqrt{3}\, \ln\left(x\right)}{2}\right)\right) \left(\frac{\cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{2\sqrt{x}} - \frac{\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = -\frac{\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)^{2} + \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)^{2}\right)}{2}$$

Which simplifies to

$$W = -\frac{\sqrt{3}}{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-3x^{\frac{5}{2}} \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{-\frac{\sqrt{3}\, x^2}{2}}\, dx$$

Which simplifies to

$$u_1 = -\int 2\sqrt{x}\, \sin\left(\frac{\sqrt{3}\, \ln\left(x\right)}{2}\right) \sqrt{3} dx$$

Hence

$$u_1 = \frac{\sqrt{3}\left(\sqrt{3}\, \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) - 3\sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)\right) x^{\frac{3}{2}}}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^{\frac{5}{2}} \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right)}{-\frac{\sqrt{3}\, x^2}{2}}\, dx$$

Which simplifies to

$$u_2 = \int -2\sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\sqrt{3}dx$$

Hence

$$u_2 = -\frac{\sqrt{3}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)+3\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right)x^{\frac{3}{2}}}{3}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = {} & \frac{\sqrt{3}\left(\sqrt{3}\,\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)-3\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right)x^2\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{3} \\ & + \frac{\sqrt{3}\left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)+3\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right)\right)x^2\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)}{3}\end{aligned}$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_1 + \sqrt{x}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_2 + x^2\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_1 + \sqrt{x}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_2 + x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \sqrt{x}\,\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_1 + \sqrt{x}\,\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)c_2 + x^2$$

Verified OK.

7.18.2 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 157: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -1$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + \frac{i\sqrt{3}}{2}$	$\frac{1}{2} - \frac{i\sqrt{3}}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \frac{i\sqrt{3}}{2} - \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \\
&= \frac{1 - i\sqrt{3}}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x}\right)(0) + \left(\left(-\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x^2} \right) + \left(\frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} \right)^2 - \left(-\frac{1}{x^2} \right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - \frac{i\sqrt{3}}{2}}{x} dx} \\
&= x^{\frac{1}{2} - \frac{i\sqrt{3}}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \end{aligned}$$

Which simplifies to

$$y_1 = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \int \frac{1}{x^{1-i\sqrt{3}}}\, dx \\ &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}} \sqrt{3}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) + c_2 \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(-\frac{i x^{i\sqrt{3}} \sqrt{3}}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2 \sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$
$$y_2 = -\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} & -\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} \\ \frac{d}{dx}\left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right) & \frac{d}{dx}\left(-\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} & -\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} \\ \frac{x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}{x} & -\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{3x} \end{vmatrix}$$

Therefore

$$W = \left(x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\right)\left(-\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}{3x}\right) - \left(-\frac{i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}\right)\left(\frac{x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)}{x}\right)$$

Which simplifies to

$$W = \frac{x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} x^2}{x^2}\, dx$$

Which simplifies to

$$u_1 = -\int -i\sqrt{3}\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\, dx$$

Hence

$$u_1 = \frac{\sqrt{3}\, x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}}\left(\sqrt{3}+3i\right)}{6}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} x^2}{x^2}\, dx$$

Which simplifies to

$$u_2 = \int 3x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\, dx$$

Hence

$$u_2 = \frac{x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}}\left(i\sqrt{3}+3\right)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\sqrt{3}\,x^{\frac{3}{2}+\frac{i\sqrt{3}}{2}}\left(\sqrt{3}+3i\right)x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}}{6} - \frac{i x^{\frac{3}{2}-\frac{i\sqrt{3}}{2}}\left(i\sqrt{3}+3\right)\sqrt{3}\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{6}$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2\sqrt{3}\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3}\right) + \left(x^2\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2\sqrt{3}\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} + x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_1 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} - \frac{i c_2\sqrt{3}\,x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}}{3} + x^2$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(x^2*diff(y(x),x$2)+y(x)=3*x^2,y(x), singsol=all)
```

$$y(x) = \sqrt{x}\, \sin\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) c_2 + \sqrt{x}\, \cos\left(\frac{\sqrt{3}\, \ln(x)}{2}\right) c_1 + x^2$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 47

```
DSolve[x^2*y''[x]+y[x]==3*x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{x}\left(x^{3/2} + c_1 \cos\left(\frac{1}{2}\sqrt{3}\log(x)\right) + c_2 \sin\left(\frac{1}{2}\sqrt{3}\log(x)\right)\right)$$

7.19 problem 22

7.19.1 Solving as second order euler ode ode 1133
7.19.2 Solving as second order change of variable on x method 2 ode . 1137
7.19.3 Solving as second order change of variable on x method 1 ode . 1143
7.19.4 Solving as second order change of variable on y method 2 ode . 1147
7.19.5 Solving using Kovacic algorithm 1152

Internal problem ID [4857]
Internal file name [`OUTPUT/4350_Sunday_June_05_2022_01_05_10_PM_31427534/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + xy' + y = 2x}$$

7.19.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$.

Solving for y_h from

$$x^2y'' + xy' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 1 = 0$$

Or

$$r^2 + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -i \\ r_2 &= i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \\ &= c_1x^{\alpha+i\beta} + c_2x^{\alpha-i\beta} \\ &= x^\alpha\left(c_1x^{i\beta} + c_2x^{-i\beta}\right) \\ &= x^\alpha\left(c_1e^{\ln\left(x^{i\beta}\right)} + c_2e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha\left(c_1e^{i(\beta\ln x)} + c_2e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -1$, the above becomes

$$y = x^0\left(c_1e^{-i\ln(x)} + c_2e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Next, we find the particular solution to the ODE

$$x^2 y'' + xy' + y = 2x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(\ln(x)) \\ y_2 &= -\sin(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(-\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & -\frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(-\frac{\cos(\ln(x))}{x}\right) - (-\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-2\sin(\ln(x))\,x}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int 2\sin(\ln(x))\,dx$$

Hence

$$u_1 = \cos(\ln(x))\,x - \sin(\ln(x))\,x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(\ln(x))\,x}{-x}\,dx$$

Which simplifies to

$$u_2 = \int -2\cos(\ln(x))\,dx$$

Hence

$$u_2 = -\cos(\ln(x))\,x - \sin(\ln(x))\,x$$

Which simplifies to

$$u_1 = x(\cos(\ln(x)) - \sin(\ln(x)))$$
$$u_2 = x(-\cos(\ln(x)) - \sin(\ln(x)))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x(\cos(\ln(x)) - \sin(\ln(x)))\cos(\ln(x)) - x(-\cos(\ln(x)) - \sin(\ln(x)))\sin(\ln(x))$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = x + c_1\cos(\ln(x)) + c_2\sin(\ln(x))$$

Verified OK.

7.19.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' + y = 0$$

In normal form the ode

$$x^2y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $p_1 = 0$. Eq (4) simplifies to

$$
\tau''(x) + p(x)\,\tau'(x) = 0
$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\
&= \int e^{-\ln(x)}\,dx \\
&= \int \frac{1}{x}dx \\
&= \ln(x)
\end{aligned} \tag{6}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= 1
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos(\ln(x)) \\ y_2 &= \sin(\ln(x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(\ln(x))\,x}{x}\,dx$$

Which simplifies to

$$u_1 = -\int 2\sin(\ln(x))\,dx$$

Hence

$$u_1 = \cos(\ln(x))\,x - \sin(\ln(x))\,x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(\ln(x))\,x}{x}\,dx$$

Which simplifies to

$$u_2 = \int 2\cos(\ln(x))\, dx$$

Hence

$$u_2 = \cos(\ln(x))\, x + \sin(\ln(x))\, x$$

Which simplifies to

$$\begin{aligned} u_1 &= x(\cos(\ln(x)) - \sin(\ln(x))) \\ u_2 &= x(\cos(\ln(x)) + \sin(\ln(x))) \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x(\cos(\ln(x)) - \sin(\ln(x)))\cos(\ln(x)) + x(\cos(\ln(x)) + \sin(\ln(x)))\sin(\ln(x))$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) + (x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Verified OK.

7.19.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' + y = 0$$

In normal form the ode

$$x^2y'' + xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + x y' + y = 2x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(\ln(x))$$
$$y_2 = \sin(\ln(x))$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2\sin(\ln(x))\,x}{x}\,dx$$

Which simplifies to

$$u_1 = -\int 2\sin(\ln(x))\,dx$$

Hence

$$u_1 = \cos(\ln(x))\,x - \sin(\ln(x))\,x$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(\ln(x))\,x}{x}\,dx$$

Which simplifies to

$$u_2 = \int 2\cos(\ln(x))\,dx$$

Hence

$$u_2 = \cos(\ln(x))\,x + \sin(\ln(x))\,x$$

Which simplifies to

$$u_1 = x(\cos(\ln(x)) - \sin(\ln(x)))$$
$$u_2 = x(\cos(\ln(x)) + \sin(\ln(x)))$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x(\cos(\ln(x)) - \sin(\ln(x)))\cos(\ln(x)) + x(\cos(\ln(x)) + \sin(\ln(x)))\sin(\ln(x))$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) + (x) \\
&= x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))
\end{aligned}
$$

Which simplifies to

$$
y = x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))
$$

Summary

The solution(s) found are the following

$$
y = x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}
$$

Verification of solutions

$$
y = x + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))
$$

Verified OK.

7.19.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$
Ay''(x) + By'(x) + Cy(x) = f(x)
$$

Where $A = x^2, B = x, C = 1, f(x) = 2x$. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$
x^2y'' + xy' + y = 0
$$

In normal form the ode

$$
x^2y'' + xy' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{1}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$
\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}
$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$
\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}
$$

Solving (5) for n gives

$$
n = i \tag{6}
$$

Substituting this value in (3) gives

$$
\begin{aligned}
v''(x) + \left(\frac{2i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\
v''(x) + \frac{(1+2i)\, v'(x)}{x} &= 0
\end{aligned} \tag{7}
$$

Using the substitution

$$
u(x) = v'(x)
$$

Then (7) becomes

$$
u'(x) + \frac{(1+2i)\, u(x)}{x} = 0 \tag{8}
$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{(-1-2i)\,u}{x}
\end{aligned}
$$

Where $f(x) = \frac{-1-2i}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{-1-2i}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{-1-2i}{x}\,dx \\
\ln(u) &= (-1-2i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-2i)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-2i}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= \frac{i c_1 x^{-2i}}{2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i \\
&= x^i c_2 + \frac{i x^{-i} c_1}{2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + x y' + y = 2x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^i$$
$$y_2 = x^{-i}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{d}{dx}(x^i) & \frac{d}{dx}(x^{-i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{i x^i}{x} & -\frac{i x^{-i}}{x} \end{vmatrix}$$

Therefore

$$W = (x^i)\left(-\frac{i x^{-i}}{x}\right) - (x^{-i})\left(\frac{i x^i}{x}\right)$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^{-i}x}{-2ix}\, dx$$

Which simplifies to

$$u_1 = -\int i x^{-i} dx$$

Hence

$$u_1 = \left(\frac{1}{2} - \frac{i}{2}\right) x^{1-i}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x\, x^{i}}{-2ix}\, dx$$

Which simplifies to

$$u_2 = \int i x^{i} dx$$

Hence

$$u_2 = \left(\frac{1}{2} + \frac{i}{2}\right) x^{1+i}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{1}{2} - \frac{i}{2}\right) x^{1-i} x^{i} + \left(\frac{1}{2} + \frac{i}{2}\right) x^{1+i} x^{-i}$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i\right) + (x) \\
&= x + \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i
\end{aligned}
$$

Which simplifies to

$$
y = x^i c_2 + \frac{i x^{-i} c_1}{2} + x
$$

Summary
The solution(s) found are the following

$$
y = x^i c_2 + \frac{i x^{-i} c_1}{2} + x \tag{1}
$$

Verification of solutions

$$
y = x^i c_2 + \frac{i x^{-i} c_1}{2} + x
$$

Verified OK.

7.19.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align}
x^2 y'' + x y' + y &= 0 \tag{1} \\
A y'' + B y' + C y &= 0 \tag{2}
\end{align}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 158: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4x^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} + i \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+i$	$\frac{1}{2}-i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2}-i}{x} + (-)(0) \\
&= \frac{\frac{1}{2}-i}{x} \\
&= \frac{\frac{1}{2}-i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{\frac{1}{2} - i}{x} dx} \\ &= x^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x^{-i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(-\frac{i x^{2i}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^{-i}\right) + c_2\left(x^{-i}\left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' + x y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x^{-i} c_1 - \frac{i c_2 x^{i}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-i}$$

$$y_2 = -\frac{i x^i}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-i} & -\frac{i x^i}{2} \\ \frac{d}{dx}\left(x^{-i}\right) & \frac{d}{dx}\left(-\frac{i x^i}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-i} & -\frac{i x^i}{2} \\ -\frac{i x^{-i}}{x} & \frac{x^i}{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-i}\right)\left(\frac{x^i}{2x}\right) - \left(-\frac{i x^i}{2}\right)\left(-\frac{i x^{-i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^i x^{-i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-i x^i x}{x}\, dx$$

Which simplifies to

$$u_1 = -\int -i x^i dx$$

Hence

$$u_1 = \left(\frac{1}{2} + \frac{i}{2}\right) x^{1+i}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2x^{-i} x}{x}\, dx$$

Which simplifies to

$$u_2 = \int 2x^{-i} dx$$

Hence

$$u_2 = (1+i)\, x^{1-i}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{1}{2} - \frac{i}{2}\right) x^{1-i} x^i + \left(\frac{1}{2} + \frac{i}{2}\right) x^{1+i} x^{-i}$$

Which simplifies to

$$y_p(x) = x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-i} c_1 - \frac{i c_2 x^i}{2}\right) + (x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^{-i}c_1 - \frac{i c_2 x^i}{2} + x \tag{1}$$

Verification of solutions

$$y = x^{-i}c_1 - \frac{i c_2 x^i}{2} + x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 16

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=2*x,y(x), singsol=all)
```

$$y(x) = \sin(\ln(x))\, c_2 + \cos(\ln(x))\, c_1 + x$$

✓ Solution by Mathematica
Time used: 0.027 (sec). Leaf size: 19

```
DSolve[x^2*y''[x]+x*y'[x]+y[x]==2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 \cos(\log(x)) + c_2 \sin(\log(x))$$

7.20 problem 25

7.20.1 Maple step by step solution 1162

Internal problem ID [4858]
Internal file name [OUTPUT/4351_Sunday_June_05_2022_01_05_18_PM_50334032/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2(2-x)\,y''+2xy'-2y=0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2}dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{2x}{-x^3+2x^2}$$

Therefore

$$y_2(x) = x\left(\int \frac{\mathrm{e}^{-\left(\int \frac{2x}{-x^3+2x^2}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{\mathrm{e}^{-\ln(x)+\ln(-2+x)}}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{-2+x}{x^3}dx\right)$$

$$y_2(x) = x\left(\frac{1}{x^2} - \frac{1}{x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x + c_2x\left(\frac{1}{x^2} - \frac{1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1x + c_2x\left(\frac{1}{x^2} - \frac{1}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1x + c_2x\left(\frac{1}{x^2} - \frac{1}{x}\right)$$

Verified OK.

7.20.1 Maple step by step solution

Let's solve

$(-x^3 + 2x^2)\,y'' + 2xy' - 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x^2(-2+x)} + \frac{2y'}{x(-2+x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x(-2+x)} + \frac{2y}{x^2(-2+x)} = 0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x) = -\frac{2}{x(-2+x)}, P_3(x) = \frac{2}{x^2(-2+x)}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = 1$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -1$

  - $x = 0$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $y''x^2(-2+x) - 2xy' + 2y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x \cdot y'$ to series expansion

    $x \cdot y' = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

  - Convert $x^m \cdot y''$ to series expansion for $m = 2..3$

    $x^m \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$

  - Shift index using $k- >k+2-m$

    $x^m \cdot y'' = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$

    Rewrite ODE with series expansions

    $-2a_0(1+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(-2a_k(k+r+1)\,(k+r-1) + a_{k-1}(k+r-1)\,(k-2+r)\right) x^{k+}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2(1+r)(-1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 1\}$
- Each term in the series must be 0, giving the recursion relation

  $-2\left(\left(-\frac{k}{2}-\frac{r}{2}+1\right)a_{k-1}+a_k(k+r+1)\right)(k+r-1) = 0$
- Shift index using $k->k+1$

  $-2\left(\left(-\frac{k}{2}+\frac{1}{2}-\frac{r}{2}\right)a_k+a_{k+1}(k+2+r)\right)(k+r) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{(k+r-1)a_k}{2(k+2+r)}$
- Recursion relation for $r = -1$; series terminates at $k = 2$

  $a_{k+1} = \frac{(k-2)a_k}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = -1$. Use reduction of order to find the second

  $y = a_0 \cdot \left(1 - x + \frac{1}{4}x^2\right)$
- Recursion relation for $r = 1$

  $a_{k+1} = \frac{k a_k}{2(k+3)}$
- Solution for $r = 1$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{k a_k}{2(k+3)}\right]$
- Combine solutions and rename parameters

  $\left[y = a_0 \cdot \left(1 - x + \frac{1}{4}x^2\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), b_{k+1} = \frac{k b_k}{2(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve([x^2*(2-x)*diff(y(x),x$2)+2*x*diff(y(x),x)-2*y(x)=0,x],singsol=all)
```

$$y(x) = \frac{c_1 x^2 + c_2(x-1)}{x}$$

✓ Solution by Mathematica
Time used: 0.044 (sec). Leaf size: 24

```
DSolve[x^2*(2-x)*y''[x]+2*x*y'[x]-2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1(x-2)^2 + c_2(x-1)}{x}$$

7.21 problem 26

Internal problem ID [4859]
Internal file name [`OUTPUT/4352_Sunday_June_05_2022_01_05_26_PM_3763595/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

Given that one solution of the ode is

$$y_1 = x$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = -\frac{2x}{x^2+1}$$

Therefore

$$y_2(x) = x\left(\int \frac{e^{-\left(\int -\frac{2x}{x^2+1}dx\right)}}{x^2}dx\right)$$

$$y_2(x) = x\int \frac{x^2+1}{x^2}, dx$$

$$y_2(x) = x\left(\int \frac{x^2+1}{x^2}dx\right)$$

$$y_2(x) = x\left(x - \frac{1}{x}\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x + c_2 x\left(x - \frac{1}{x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x + c_2 x\left(x - \frac{1}{x}\right) \tag{1}$$

Verification of solutions

$$y = c_1 x + c_2 x\left(x - \frac{1}{x}\right)$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([(x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,x],singsol=all)
```

$$y(x) = c_2x^2 + c_1x - c_2$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x - c_1(x - i)^2$$

7.22 problem 27

7.22.1 Maple step by step solution 1170

Internal problem ID [4860]
Internal file name [`OUTPUT/4353_Sunday_June_05_2022_01_05_33_PM_60988632/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{xy'' - 2(x+1)\,y' + (x+2)\,y = 0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-2-2x}{x}$$

Therefore

$$
\begin{aligned}
y_2(x) &= \mathrm{e}^{x}\left(\int \mathrm{e}^{-\left(\int \frac{-2-2x}{x}dx\right)}\mathrm{e}^{-2x}dx\right)\\
y_2(x) &= \mathrm{e}^{x}\int \frac{\mathrm{e}^{2x+2\ln(x)}}{\mathrm{e}^{2x}}, dx\\
y_2(x) &= \mathrm{e}^{x}\left(\int x^{2}dx\right)\\
y_2(x) &= \frac{x^{3}\mathrm{e}^{x}}{3}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= c_1y_1(x) + c_2y_2(x)\\
&= c_1\mathrm{e}^{x} + \frac{c_2x^{3}\mathrm{e}^{x}}{3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1\mathrm{e}^{x} + \frac{c_2x^{3}\mathrm{e}^{x}}{3} \tag{1}
$$

Verification of solutions

$$
y = c_1\mathrm{e}^{x} + \frac{c_2x^{3}\mathrm{e}^{x}}{3}
$$

Verified OK.

7.22.1 Maple step by step solution

Let's solve

$y''x + (-2-2x)\,y' + (x+2)\,y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(x+2)y}{x} + \frac{2(x+1)y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(x+1)y'}{x} + \frac{(x+2)y}{x} = 0$

- ☐ Check to see if $x_0 = 0$ is a regular singular point
  - ◦ Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x}, P_3(x) = \frac{x+2}{x}\right]$
  - ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

    $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -2$$
  - ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$
  - ◦ $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$
- • Multiply by denominators

  $y''x + (-2-2x)\,y' + (x+2)\,y = 0$
- • Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$
- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..1$

    $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$
  - ◦ Shift index using $k- >k-m$

    $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$
  - ◦ Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

    $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$$
  - ◦ Shift index using $k- >k+1-m$

    $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k+1-m+r)\, x^{k+r}$$
  - ◦ Convert $x \cdot y''$ to series expansion

$$x \cdot y'' = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k+1$

  $$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0r(-3+r)\, x^{-1+r} + (a_1(1+r)(-2+r) - 2a_0(-1+r))\, x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(k-2+r)\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $$r(-3+r) = 0$$

- Values of r that satisfy the indicial equation

  $$r \in \{0, 3\}$$

- Each term must be 0

  $$a_1(1+r)(-2+r) - 2a_0(-1+r) = 0$$

- Each term in the series must be 0, giving the recursion relation

  $$a_{k+1}(k+1+r)(k-2+r) - 2a_kk - 2a_kr + 2a_k + a_{k-1} = 0$$

- Shift index using $k- >k+1$

  $$a_{k+2}(k+2+r)(k+r-1) - 2a_{k+1}(k+1) - 2ra_{k+1} + 2a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

  $$a_{k+2} = \frac{2ka_{k+1}+2ra_{k+1}-a_k}{(k+2+r)(k+r-1)}$$

- Recursion relation for $r = 0$

  $$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$$

- Series not valid for $r = 0$, division by 0 in the recursion relation at $k = 1$

  $$a_{k+2} = \frac{2ka_{k+1}-a_k}{(k+2)(k-1)}$$

- Recursion relation for $r = 3$

  $$a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}$$

- Solution for $r = 3$

  $$\left[y = \sum_{k=0}^{\infty} a_kx^{k+3}, a_{k+2} = \frac{2ka_{k+1}-a_k+6a_{k+1}}{(k+5)(k+2)}, 4a_1 - 4a_0 = 0\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+(x+2)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = \mathrm{e}^{x}\left(c_2x^3 + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 23

```
DSolve[x*y''[x]-2*(x+1)*y'[x]+(x+2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}e^{x}\left(c_2x^3 + 3c_1\right)$$

7.23 problem 28

7.23.1 Maple step by step solution 1175

Internal problem ID [4861]
Internal file name [`OUTPUT/4354_Sunday_June_05_2022_01_05_41_PM_95419094/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{3xy'' - 2(3x-1)\,y' + (3x-2)\,y = 0}$$

Given that one solution of the ode is

$$y_1 = \mathrm{e}^{x}$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-(\int p dx)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x)\,y' + q(x)\,y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{-6x+2}{3x}$$

Therefore

$$y_2(x) = e^x \left(\int e^{-\left(\int \frac{-6x+2}{3x} dx\right)} e^{-2x} dx \right)$$

$$y_2(x) = e^x \int \frac{e^{2x - \frac{2\ln(x)}{3}}}{e^{2x}}, dx$$

$$y_2(x) = e^x \left(\int \frac{1}{x^{\frac{2}{3}}} dx \right)$$

$$y_2(x) = 3\, e^x x^{\frac{1}{3}}$$

Hence the solution is

$$\begin{aligned} y &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 e^x + 3c_2 e^x x^{\frac{1}{3}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + 3c_2 e^x x^{\frac{1}{3}} \tag{1}$$

Verification of solutions

$$y = c_1 e^x + 3c_2 e^x x^{\frac{1}{3}}$$

Verified OK.

7.23.1 Maple step by step solution

Let's solve

$3y''x + (-6x+2)\, y' + (3x-2)\, y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{(3x-2)y}{3x} + \frac{2(3x-1)y'}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2(3x-1)y'}{3x} + \frac{(3x-2)y}{3x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$$\left[P_2(x) = -\frac{2(3x-1)}{3x}, P_3(x) = \frac{3x-2}{3x}\right]$$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = \frac{2}{3}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $$x_0 = 0$$

- Multiply by denominators

  $$3y''x + (-6x + 2)\, y' + (3x - 2)\, y = 0$$

- Assume series solution for y

  $$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y$ to series expansion for $m = 0..1$

  $$x^m \cdot y = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- >k - m$

  $$x^m \cdot y = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot y'$ to series expansion for $m = 0..1$

  $$x^m \cdot y' = \sum_{k=0}^{\infty} a_k (k + r)\, x^{k+r-1+m}$$

- Shift index using $k- >k + 1 - m$

  $$x^m \cdot y' = \sum_{k=-1+m}^{\infty} a_{k+1-m} (k + 1 - m + r)\, x^{k+r}$$

- Convert $x \cdot y''$ to series expansion

  $$x \cdot y'' = \sum_{k=0}^{\infty} a_k (k + r)\,(k + r - 1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

$$x \cdot y'' = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(-1+3r)\, x^{-1+r} + \left(a_1(1+r)(2+3r) - 2a_0(1+3r)\right) x^r + \left(\sum_{k=1}^{\infty} (a_{k+1}(k+1+r)(3k+2+3r$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-1+3r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{3}\right\}$

- Each term must be 0

  $a_1(1+r)(2+3r) - 2a_0(1+3r) = 0$

- Each term in the series must be 0, giving the recursion relation

  $3\left(k+\frac{2}{3}+r\right)(k+1+r)\, a_{k+1} - 6a_k k - 6a_k r - 2a_k + 3a_{k-1} = 0$

- Shift index using $k- >k+1$

  $3\left(k+\frac{5}{3}+r\right)(k+2+r)\, a_{k+2} - 6a_{k+1}(k+1) - 6ra_{k+1} - 2a_{k+1} + 3a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{6ka_{k+1}+6ra_{k+1}-3a_k+8a_{k+1}}{(3k+5+3r)(k+2+r)}$

- Recursion relation for $r = 0$

  $a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}$

- Solution for $r = 0$

  $\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1 - 2a_0 = 0\right]$

- Recursion relation for $r = \frac{1}{3}$

  $a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}$

- Solution for $r = \frac{1}{3}$

  $\left[y = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{3}}, a_{k+2} = \frac{6ka_{k+1}-3a_k+10a_{k+1}}{(3k+6)\left(k+\frac{7}{3}\right)}, 4a_1 - 4a_0 = 0\right]$

- Combine solutions and rename parameters

  $\left[y = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{3}}\right), a_{k+2} = \frac{6ka_{k+1}-3a_k+8a_{k+1}}{(3k+5)(k+2)}, 2a_1 - 2a_0 = 0, b_{k+2} = \frac{6kb_{k+1}-3b_k+10b_k}{(3k+6)\left(k+\frac{7}{3}\right)}\right.$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([3*x*diff(y(x),x$2)-2*(3*x-1)*diff(y(x),x)+(3*x-2)*y(x)=0,exp(x)],singsol=all)
```

$$y(x) = e^x \left(c_1 + x^{\frac{1}{3}} c_2\right)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 21

```
DSolve[3*x*y''[x]-2*(3*x-1)*y'[x]+(3*x-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x \left(3c_2 \sqrt[3]{x} + c_1\right)$$

7.24 problem 29

Internal problem ID [4862]
Internal file name [OUTPUT/4355_Sunday_June_05_2022_01_05_49_PM_32176453/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 29.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2 y'' + (x+1) y' - y = 0}$$

Given that one solution of the ode is

$$y_1 = x + 1$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1 \left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx \right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y'' + p(x) y' + q(x) y = f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{x+1}{x^2}$$

Therefore

$$y_2(x) = (x+1)\left(\int \frac{e^{-\left(\int \frac{x+1}{x^2}dx\right)}}{(x+1)^2}dx\right)$$

$$y_2(x) = x+1\int \frac{e^{-\ln(x)+\frac{1}{x}}}{(x+1)^2}, dx$$

$$y_2(x) = (x+1)\left(\int \frac{e^{\frac{1}{x}}}{x(x+1)^2}dx\right)$$

$$y_2(x) = -x\,e^{\frac{1}{x}}$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x) + c_2y_2(x) \\ &= (x+1)\,c_1 - c_2x\,e^{\frac{1}{x}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = (x+1)\,c_1 - c_2x\,e^{\frac{1}{x}} \tag{1}$$

Verification of solutions

$$y = (x+1)\,c_1 - c_2x\,e^{\frac{1}{x}}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([x^2*diff(y(x),x$2)+(x+1)*diff(y(x),x)-y(x)=0,x+1],singsol=all)
```

$$y(x) = c_2 e^{\frac{1}{x}} x + c_1 x + c_1$$

✓ Solution by Mathematica

Time used: 0.077 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+(x+1)*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1}{x}} x + c_2(x+1)$$

7.25 problem 30

7.25.1 Maple step by step solution 1183

Internal problem ID [4863]
Internal file name [`OUTPUT/4356_Sunday_June_05_2022_01_05_57_PM_69438561/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 7. Other second-Order equations. page 435
Problem number: 30.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"reduction_of_order", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x(x+1)\,y''-(x-1)\,y'+y=0}$$

Given that one solution of the ode is

$$y_1 = x-1$$

Given one basis solution $y_1(x)$, then the second basis solution is given by

$$y_2(x) = y_1\left(\int \frac{\mathrm{e}^{-\left(\int p dx\right)}}{y_1^2} dx\right)$$

Where $p(x)$ is the coefficient of y' when the ode is written in the normal form

$$y''+p(x)\,y'+q(x)\,y=f(x)$$

Looking at the ode to solve shows that

$$p(x) = \frac{1-x}{x^2+x}$$

Therefore

$$y_2(x) = (x-1)\left(\int \frac{e^{-\left(\int \frac{1-x}{x^2+x}dx\right)}}{(x-1)^2}dx\right)$$

$$y_2(x) = x-1\int \frac{e^{2\ln(x+1)-\ln(x)}}{(x-1)^2}, dx$$

$$y_2(x) = (x-1)\left(\int \frac{(x+1)^2}{x(x-1)^2}dx\right)$$

$$y_2(x) = (x-1)\left(-\frac{4}{x-1}+\ln(x)\right)$$

Hence the solution is

$$\begin{aligned} y &= c_1y_1(x)+c_2y_2(x) \\ &= (x-1)\,c_1 + c_2(x-1)\left(-\frac{4}{x-1}+\ln(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = (x-1)\,c_1 + c_2(x-1)\left(-\frac{4}{x-1}+\ln(x)\right) \tag{1}$$

Verification of solutions

$$y = (x-1)\,c_1 + c_2(x-1)\left(-\frac{4}{x-1}+\ln(x)\right)$$

Verified OK.

7.25.1 Maple step by step solution

Let's solve

$(x^2+x)\,y''+(1-x)\,y'+y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = -\frac{y}{x(x+1)} + \frac{(x-1)y'}{x(x+1)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{(x-1)y'}{x(x+1)} + \frac{y}{x(x+1)} = 0$

□ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{x-1}{x(x+1)}, P_3(x) = \frac{1}{x(x+1)}\right]$

- ◦ $(x+1) \cdot P_2(x)$ is analytic at $x = -1$

  $\left((x+1) \cdot P_2(x)\right)\Big|_{x=-1} = -2$

- ◦ $(x+1)^2 \cdot P_3(x)$ is analytic at $x = -1$

  $\left((x+1)^2 \cdot P_3(x)\right)\Big|_{x=-1} = 0$

- ◦ $x = -1$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = -1$

- Multiply by denominators

  $x(x+1)\, y'' + (1-x)\, y' + y = 0$

- Change variables using $x = u - 1$ so that the regular singular point is at $u = 0$

  $(u^2 - u)\left(\frac{d^2}{du^2}y(u)\right) + (2-u)\left(\frac{d}{du}y(u)\right) + y(u) = 0$

- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- ◦ Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right) = \sum_{k=0}^{\infty} a_{k} (k+r) (k+r-1) u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^{m} \cdot \left(\frac{d^{2}}{d u^{2}}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m} (k+2-m+r) (k+1-m+r) u^{k+r}$$

Rewrite ODE with series expansions

$$-a_{0} r (-3+r) u^{-1+r}+\left(\sum_{k=0}^{\infty} \left(-a_{k+1} (k+1+r) (k-2+r)+a_{k} (k+r-1)^{2}\right) u^{k+r}\right) = 0$$

- a_{0}cannot be 0 by assumption, giving the indicial equation

$$-r (-3+r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \{0, 3\}$$

- Each term in the series must be 0, giving the recursion relation

$$-a_{k+1} (k+1+r) (k-2+r)+a_{k} (k+r-1)^{2} = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+1} = \frac{a_{k} (k+r-1)^{2}}{(k+1+r) (k-2+r)}$$

- Recursion relation for $r = 0$; series terminates at $k = 1$

$$a_{k+1} = \frac{a_{k} (k-1)^{2}}{(k+1) (k-2)}$$

- Apply recursion relation for $k = 0$

$$a_{1} = -\frac{a_{0}}{2}$$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

$$y(u) = a_{0} \cdot \left(1-\frac{u}{2}\right)$$

- Revert the change of variables $u = x+1$

$$\left[y = a_{0} \left(-\frac{x}{2}+\frac{1}{2}\right)\right]$$

- Recursion relation for $r = 3$

$$a_{k+1} = \frac{a_{k} (k+2)^{2}}{(k+4) (k+1)}$$

- Solution for $r = 3$

$$\left[y(u) = \sum_{k=0}^{\infty} a_{k} u^{k+3}, a_{k+1} = \frac{a_{k} (k+2)^{2}}{(k+4) (k+1)}\right]$$

- Revert the change of variables $u = x+1$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+1)^{k+3}, a_{k+1} = \frac{a_k (k+2)^2}{(k+4)(k+1)}\right]$$

- Combine solutions and rename parameters

$$\left[y = a_0\left(-\frac{x}{2}+\frac{1}{2}\right) + \left(\sum_{k=0}^{\infty} b_k (x+1)^{k+3}\right), b_{k+1} = \frac{b_k (k+2)^2}{(k+4)(k+1)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([x*(x+1)*diff(y(x),x$2)-(x-1)*diff(y(x),x)+y(x)=0,x-1],singsol=all)
```

$$y(x) = (x-1)\, c_2 \ln(x) - 4c_2 + c_1(x-1)$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 23

```
DSolve[x*(x+1)*y''[x]-(x-1)*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x-1) + c_2((x-1)\log(x) - 4)$$

8 Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466

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8.16 problem 16 1381
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8.1 problem 1

8.1.1 Solving as linear ode 1188
8.1.2 Solving as homogeneousTypeD2 ode 1190
8.1.3 Solving as first order ode lie symmetry lookup ode 1191
8.1.4 Solving as exact ode 1195
8.1.5 Maple step by step solution 1200

Internal problem ID [4864]
Internal file name [`OUTPUT/4357_Sunday_June_05_2022_01_06_04_PM_42320168/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{x^2y' - xy = \frac{1}{x}}$$

8.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^3}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{1}{x^3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x^3}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \frac{1}{x^4}\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{1}{x^4}\,\mathrm{d}x \\ \frac{y}{x} &= -\frac{1}{3x^3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{1}{3x^2} + c_1 x$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{3x^2} + c_1 x \tag{1}$$

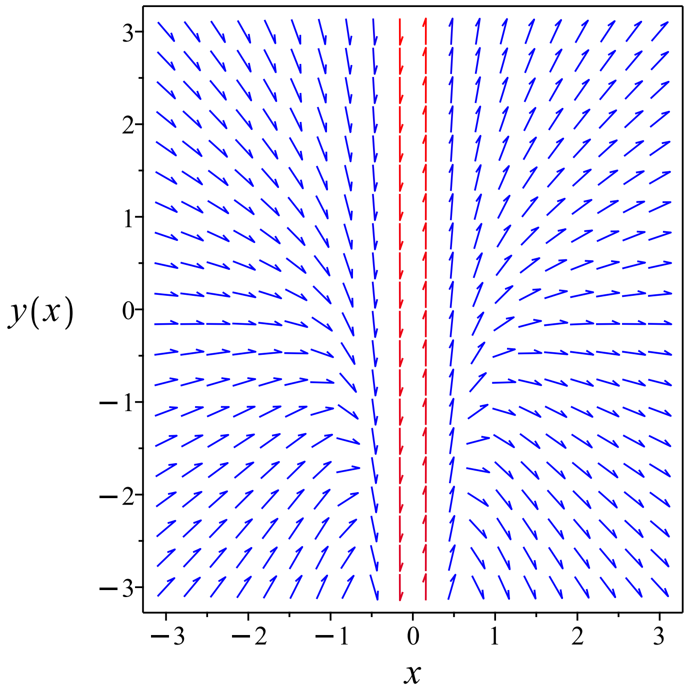


Figure 193: Slope field plot

Verification of solutions

$$y = -\frac{1}{3x^2} + c_1 x$$

Verified OK.

8.1.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) - x^2 u(x) = \frac{1}{x}$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x^4}\,\mathrm{d}x \\ &= -\frac{1}{3x^3} + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= x\left(-\frac{1}{3x^3} + c_2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(-\frac{1}{3x^3} + c_2\right) \tag{1}$$

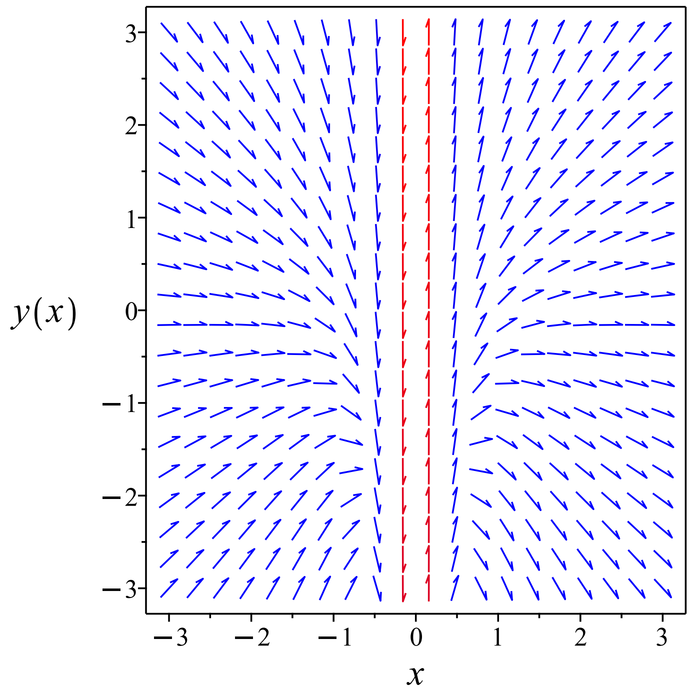


Figure 194: Slope field plot

Verification of solutions

$$y = x\left(-\frac{1}{3x^3} + c_2\right)$$

Verified OK.

8.1.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y\,x^2 + 1}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 163: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\,x^2 + 1}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{3R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = -\frac{1}{3x^3} + c_1$$

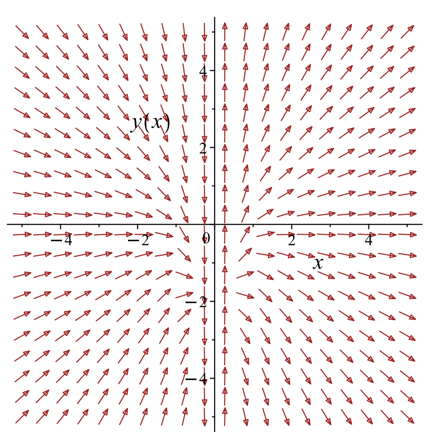
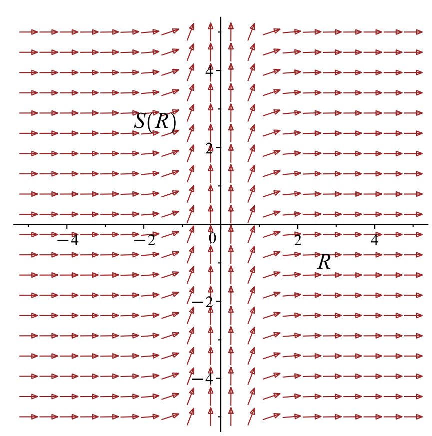
Which simplifies to

$$\frac{y}{x} = -\frac{1}{3x^3} + c_1$$

Which gives

$$y = \frac{3c_1x^3 - 1}{3x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y\,x^2+1}{x^3}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R^4}$ 

Summary
The solution(s) found are the following

$$y = \frac{3c_1x^3 - 1}{3x^2} \tag{1}$$

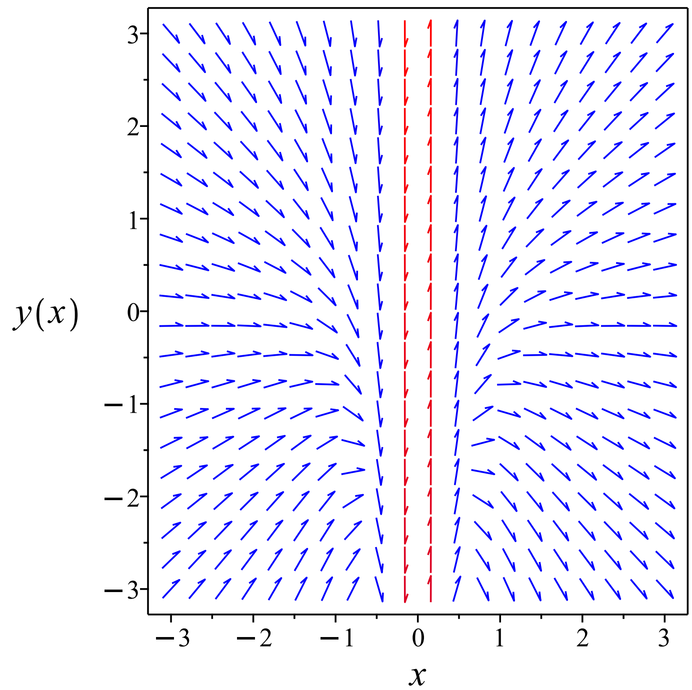


Figure 195: Slope field plot

Verification of solutions

$$y = \frac{3c_1x^3 - 1}{3x^2}$$

Verified OK.

8.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2\right)\mathrm{d}y&=\left(xy+\frac{1}{x}\right)\mathrm{d}x\\ \left(-xy-\frac{1}{x}\right)\mathrm{d}x+\left(x^2\right)\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xy-\frac{1}{x}\\ N(x,y)&=x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-xy-\frac{1}{x}\right)\\ &=-x\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^{2}\right) \\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^{2}}((-x) - (2x)) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^{3}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^{3}}\left(-xy - \frac{1}{x}\right) \\
&= \frac{-y\,x^{2} - 1}{x^{4}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{3}}\left(x^{2}\right) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(\frac{-y\,x^2-1}{x^4}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-y\,x^2-1}{x^4}\,\mathrm{d}x$$
$$\phi = \frac{1}{3x^3} + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3x^3} + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3x^3} + \frac{y}{x}$$

The solution becomes

$$y = \frac{3c_1 x^3 - 1}{3x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{3c_1 x^3 - 1}{3x^2} \tag{1}$$

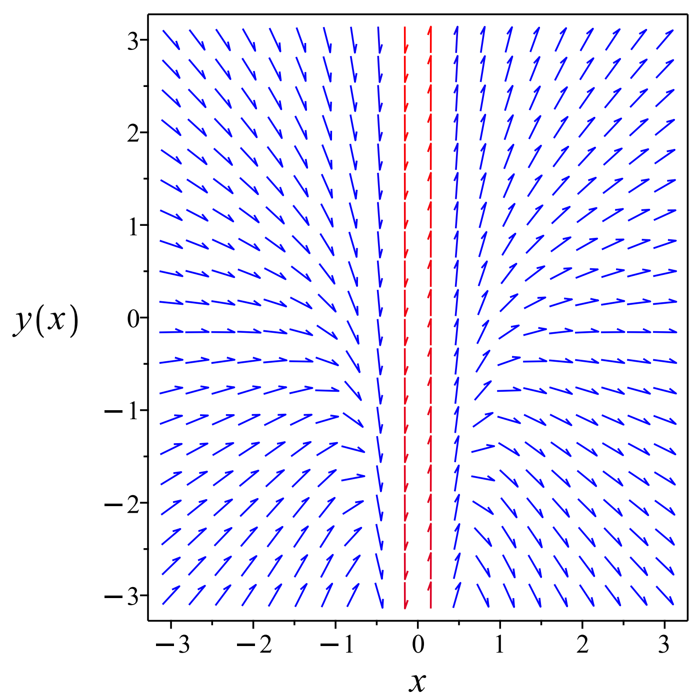


Figure 196: Slope field plot

Verification of solutions

$$y = \frac{3c_1x^3 - 1}{3x^2}$$

Verified OK.

8.1.5 Maple step by step solution

Let's solve

$x^2y' - xy = \frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + \frac{1}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \frac{1}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \frac{\mu(x)}{x^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

$y = x\left(\int \frac{1}{x^4}dx + c_1\right)$

- Evaluate the integrals on the rhs

$y = x\left(-\frac{1}{3x^3} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x)-x*y(x)=1/x,y(x), singsol=all)
```

$$y(x) = \left(-\frac{1}{3x^3} + c_1\right)x$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[x^2*y'[x]-x*y[x]==1/x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{3x^2} + c_1x$$

8.2 problem 2

8.2.1 Solving as separable ode 1202
8.2.2 Solving as first order ode lie symmetry lookup ode 1204
8.2.3 Solving as exact ode 1208
8.2.4 Maple step by step solution 1212

Internal problem ID [4865]
Internal file name [`OUTPUT/4358_Sunday_June_05_2022_01_06_15_PM_51042620/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{x \ln(y) y' - \ln(x) y = 0}$$

8.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\ln(x) y}{x \ln(y)} \end{aligned}$$

Where $f(x) = \frac{\ln(x)}{x}$ and $g(y) = \frac{y}{\ln(y)}$. Integrating both sides gives

$$\frac{1}{\frac{y}{\ln(y)}} \, dy = \frac{\ln(x)}{x} \, dx$$

$$\int \frac{1}{\frac{y}{\ln(y)}} dy = \int \frac{\ln(x)}{x} dx$$

$$\frac{\ln(y)^2}{2} = \frac{\ln(x)^2}{2} + c_1$$

Raising both side to exponential gives

$$e^{\frac{\ln(y)^2}{2}} = e^{\frac{\ln(x)^2}{2}+c_1}$$

Which simplifies to

$$e^{\frac{\ln(y)^2}{2}} = c_2 e^{\frac{\ln(x)^2}{2}}$$

The solution is

$$e^{\frac{\ln(y)^2}{2}} = c_2 e^{\frac{\ln(x)^2}{2}+c_1}$$

Summary

The solution(s) found are the following

$$e^{\frac{\ln(y)^2}{2}} = c_2 e^{\frac{\ln(x)^2}{2}+c_1} \tag{1}$$

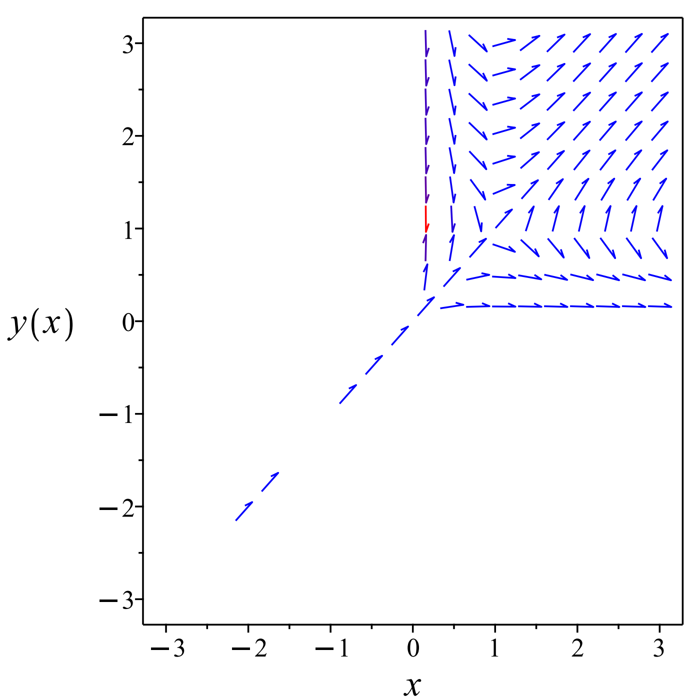


Figure 197: Slope field plot

Verification of solutions

$$
e^{\frac{\ln(y)^2}{2}} = c_2 e^{\frac{\ln(x)^2}{2}+c_1}
$$

Verified OK.

8.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\ln(x)\, y}{x \ln(y)} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 166: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{x}{\ln (x)} \\
\eta(x, y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x}{\ln(x)}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x)^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{\ln(x)\, y}{x \ln(y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{\ln(x)}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\ln(y)}{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\ln(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R)^2}{2} + c_1 \tag{4}$$

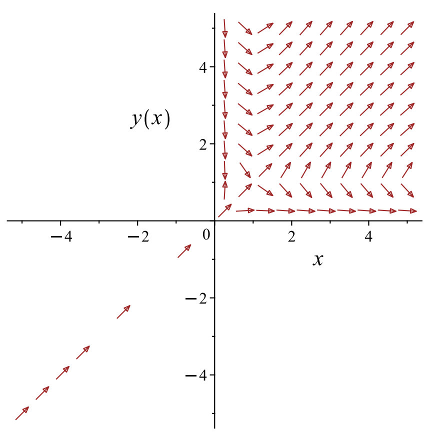
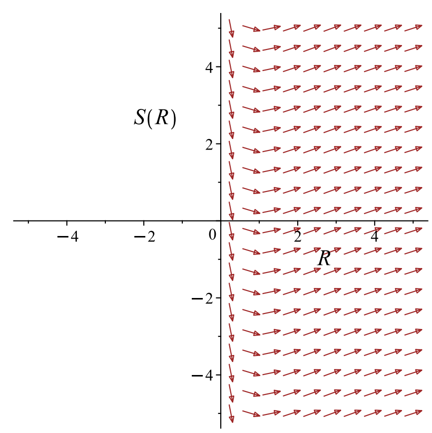
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x)^2}{2} = \frac{\ln(y)^2}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x)^2}{2} = \frac{\ln(y)^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\ln(x)y}{x\ln(y)}$ 	$R = y$ $S = \frac{\ln(x)^2}{2}$	$\frac{dS}{dR} = \frac{\ln(R)}{R}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x)^2}{2} = \frac{\ln(y)^2}{2} + c_1 \tag{1}$$

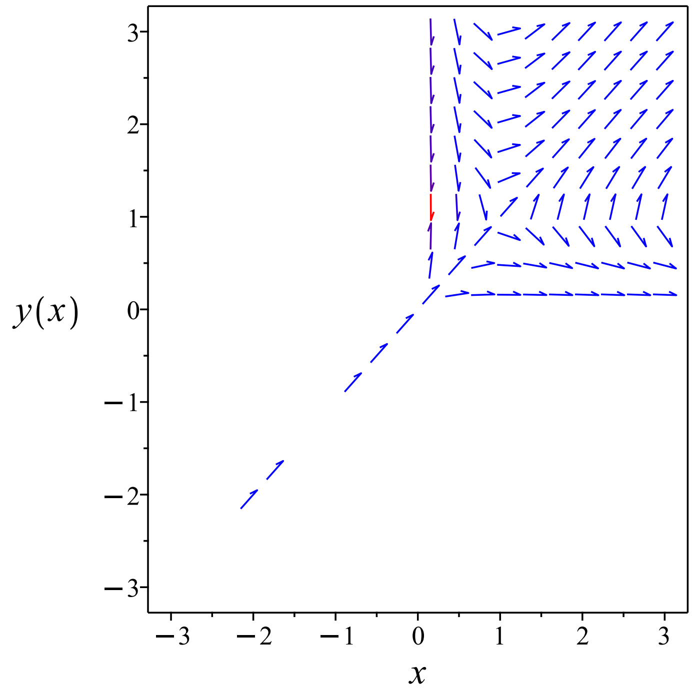


Figure 198: Slope field plot

Verification of solutions

$$
\frac{\ln(x)^2}{2} = \frac{\ln(y)^2}{2} + c_1
$$

Verified OK.

8.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{\ln(y)}{y}\right) \mathrm{d}y &= \left(\frac{\ln(x)}{x}\right) \mathrm{d}x \\ \left(-\frac{\ln(x)}{x}\right) \mathrm{d}x + \left(\frac{\ln(y)}{y}\right) \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{\ln(x)}{x} \\ N(x, y) &= \frac{\ln(y)}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\ln(x)}{x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\ln (y)}{y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{\ln (x)}{x} \mathrm{d}x\\
\phi &= -\frac{\ln (x)^2}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\ln(y)}{y}$. Therefore equation (4) becomes

$$
\frac{\ln (y)}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{\ln (y)}{y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\ln(y)}{y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y)^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x)^2}{2} + \frac{\ln(y)^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x)^2}{2} + \frac{\ln(y)^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x)^2}{2} + \frac{\ln(y)^2}{2} = c_1 \tag{1}$$

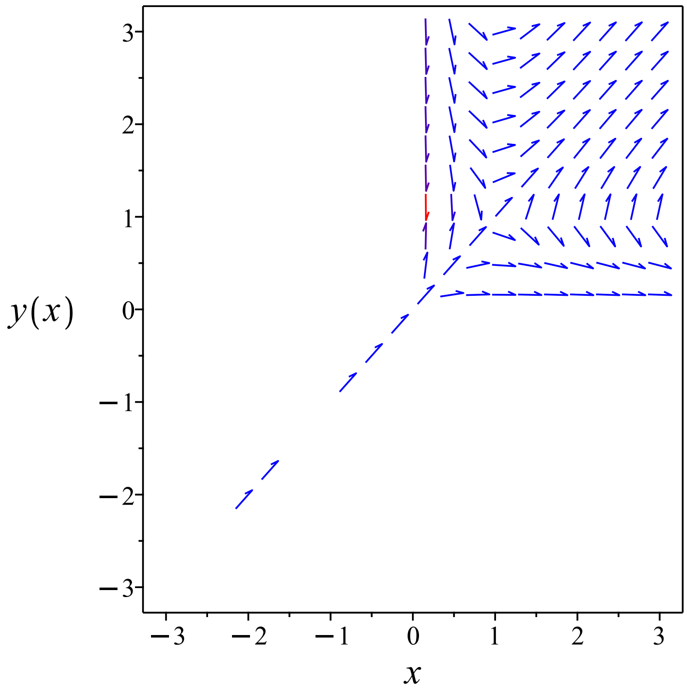


Figure 199: Slope field plot

Verification of solutions

$$-\frac{\ln \left(x \right)^{2}}{2}+\frac{\ln \left(y \right)^{2}}{2} = c_{1}$$

Verified OK.

8.2.4 Maple step by step solution

Let's solve

$x \ln \left(y \right) y^{\prime}-\ln \left(x \right) y = 0$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

- Separate variables

  $\frac{\ln(y) y^{\prime}}{y} = \frac{\ln(x)}{x}$

- Integrate both sides with respect to x

  $\int \frac{\ln(y) y^{\prime}}{y}dx = \int \frac{\ln(x)}{x}dx +c_{1}$

- Evaluate integral

$$\frac{\ln(y)^2}{2} = \frac{\ln(x)^2}{2} + c_1$$

- Solve for y

$$\left\{y = e^{\sqrt{\ln(x)^2+2c_1}}, y = e^{-\sqrt{\ln(x)^2+2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 31

```
dsolve(x*ln(y(x))*diff(y(x),x)-y(x)*ln(x)=0,y(x), singsol=all)
```

$$y(x) = e^{\sqrt{\ln(x)^2+2c_1}}$$
$$y(x) = e^{-\sqrt{\ln(x)^2+2c_1}}$$

✓ Solution by Mathematica

Time used: 0.815 (sec). Leaf size: 60

```
DSolve[x*Log[y[x]]*y'[x]-y[x]*Log[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sqrt{\log^2(x)+2c_1}}$$
$$y(x) \to e^{\sqrt{\log^2(x)+2c_1}}$$
$$y(x) \to 0$$
$$y(x) \to e^{2i\text{Interval}[\{0,\pi\}]}$$

8.3 problem 3

8.3.1 Maple step by step solution 1215

Internal problem ID [4866]
Internal file name [`OUTPUT/4359_Sunday_June_05_2022_01_06_29_PM_27966175/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 3.
ODE order: 3.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_3rd_order, _missing_x]]
```

$$\boxed{y''' + 2y'' + 2y' = 0}$$

The characteristic equation is

$$\lambda^3 + 2\lambda^2 + 2\lambda = 0$$

The roots of the above equation are

$$\lambda_1 = 0$$
$$\lambda_2 = -1 + i$$
$$\lambda_3 = -1 - i$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 + c_2 \mathrm{e}^{(-1-i)x} + \mathrm{e}^{(-1+i)x} c_3$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = 1$$
$$y_2 = \mathrm{e}^{(-1-i)x}$$
$$y_3 = \mathrm{e}^{(-1+i)x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 e^{(-1-i)x} + e^{(-1+i)x} c_3 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 e^{(-1-i)x} + e^{(-1+i)x} c_3$$

Verified OK.

8.3.1 Maple step by step solution

Let's solve

$y''' + 2y'' + 2y' = 0$

- Highest derivative means the order of the ODE is 3

  y'''

□ Convert linear ODE into a system of first order ODEs

  ◦ Define new variable $y_1(x)$

    $y_1(x) = y$

  ◦ Define new variable $y_2(x)$

    $y_2(x) = y'$

  ◦ Define new variable $y_3(x)$

    $y_3(x) = y''$

  ◦ Isolate for $y_3'(x)$ using original ODE

    $y_3'(x) = -2y_3(x) - 2y_2(x)$

  Convert linear ODE into a system of first order ODEs

  $[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_3'(x) = -2y_3(x) - 2y_2(x)]$

- Define vector

  $$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right], \left[-1-\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right], \left[-1+\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{2} \\ -\frac{1}{2}-\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1-\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{2} \\ -\frac{1}{2}+\frac{\mathrm{I}}{2} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{(-1-I)x} \cdot \begin{bmatrix} -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{-x} \cdot (\cos(x) - I\sin(x)) \cdot \begin{bmatrix} -\frac{I}{2} \\ -\frac{1}{2}+\frac{I}{2} \\ 1 \end{bmatrix}$$

- Simplify expression

$$e^{-x} \cdot \begin{bmatrix} -\frac{I}{2}(\cos(x) - I\sin(x)) \\ \left(-\frac{1}{2}+\frac{I}{2}\right)(\cos(x) - I\sin(x)) \\ \cos(x) - I\sin(x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_2(x) = e^{-x} \cdot \begin{bmatrix} -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix}, \vec{y}_3(x) = e^{-x} \cdot \begin{bmatrix} -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2(x) + c_3\vec{y}_3(x)$$

- Substitute solutions into the general solution

$$\vec{y} = c_2 e^{-x} \cdot \begin{bmatrix} -\frac{\sin(x)}{2} \\ -\frac{\cos(x)}{2}+\frac{\sin(x)}{2} \\ \cos(x) \end{bmatrix} + c_3 e^{-x} \cdot \begin{bmatrix} -\frac{\cos(x)}{2} \\ \frac{\sin(x)}{2}+\frac{\cos(x)}{2} \\ -\sin(x) \end{bmatrix} + \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

$$y = \frac{(-c_2\sin(x) - \cos(x)c_3)e^{-x}}{2} + c_1$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$3)+2*diff(y(x),x$2)+2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + e^{-x} \sin(x) c_2 + c_3 \cos(x) e^{-x}$$

✓ Solution by Mathematica

Time used: 0.113 (sec). Leaf size: 37

```
DSolve[y'''[x]+2*y''[x]+2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-x} ((c_2 - c_1) \sin(x) - (c_1 + c_2) \cos(x)) + c_3$$

8.4 problem 4

8.4.1 Solving as second order linear constant coeff ode 1219
8.4.2 Solving as linear second order ode solved by an integrating factor ode 1221
8.4.3 Solving using Kovacic algorithm 1222
8.4.4 Maple step by step solution 1226

Internal problem ID [4867]
Internal file name [`OUTPUT/4360_Sunday_June_05_2022_01_06_37_PM_86233007/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{r'' - 6r' + 9r = 0}$$

8.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ar''(t) + Br'(t) + Cr(t) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $r = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 6\lambda \,\mathrm{e}^{\lambda t} + 9\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-6)^2 - (4)(1)(9)} \\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$r = c_1 e^{3t} + c_2 t\, e^{3t} \tag{1}$$

Summary

The solution(s) found are the following

$$r = c_1 e^{3t} + c_2 t\, e^{3t} \tag{1}$$

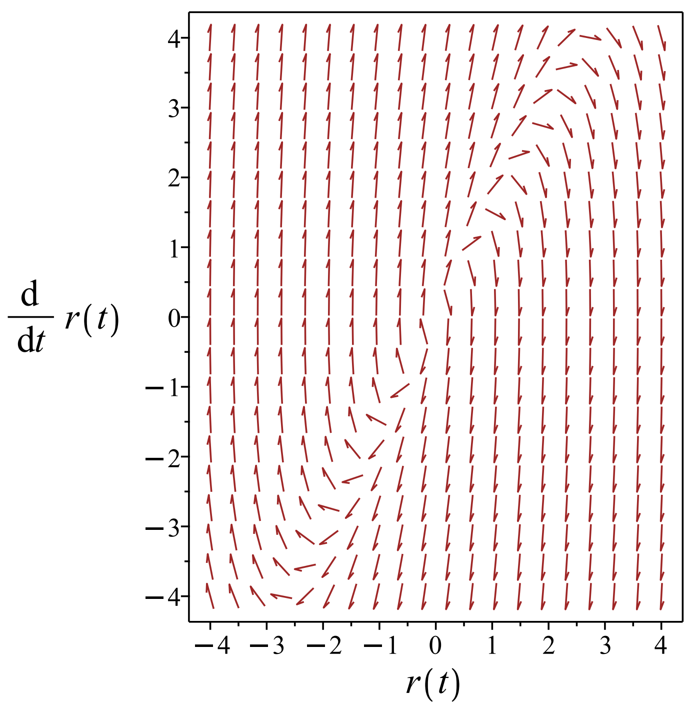


Figure 200: Slope field plot

Verification of solutions

$$r = c_1 e^{3t} + c_2 t\, e^{3t}$$

Verified OK.

8.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$r'' + p(t)\, r' + \frac{\left(p(t)^2 + p'(t)\right) r}{2} = f(t)$$

Where $p(t) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -6\, dx} \\ &= \mathrm{e}^{-3t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x) r)'' &= 0 \\ \left(\mathrm{e}^{-3t} r\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-3t} r\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3t} r\right) = c_1 t + c_2$$

Hence the solution is

$$r = \frac{c_1 t + c_2}{\mathrm{e}^{-3t}}$$

Or

$$r = c_1 t\, \mathrm{e}^{3t} + \mathrm{e}^{3t} c_2$$

Summary

The solution(s) found are the following

$$r = c_1 t\, \mathrm{e}^{3t} + \mathrm{e}^{3t} c_2 \tag{1}$$

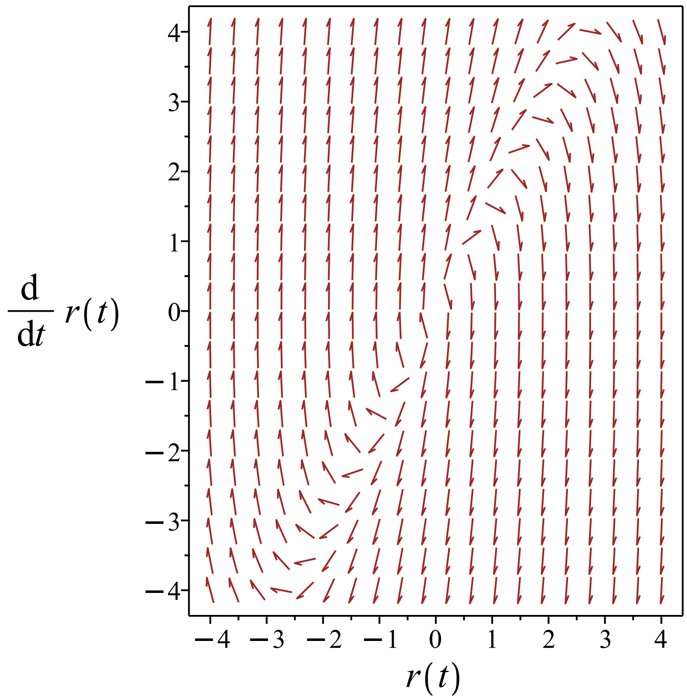


Figure 201: Slope field plot

<u>Verification of solutions</u>

$$r = c_1 t\, e^{3t} + e^{3t} c_2$$

Verified OK.

8.4.3 Solving using Kovacic algorithm

Writing the ode as

$$r'' - 6r' + 9r = 0 \tag{1}$$
$$Ar'' + Br' + Cr = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = r e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then r is found using the inverse transformation

$$r = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 170: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in r is found from

$$
\begin{aligned}
r_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dt} \\
&= z_1 e^{3t} \\
&= z_1\left(\mathrm{e}^{3t}\right)
\end{aligned}
$$

Which simplifies to

$$
r_1 = \mathrm{e}^{3t}
$$

The second solution r_2 to the original ode is found using reduction of order

$$
r_2 = r_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{r_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
r_2 &= r_1 \int \frac{e^{\int -\frac{-6}{1}\,dt}}{(r_1)^2}\,dt \\
&= r_1 \int \frac{e^{6t}}{(r_1)^2}\,dt \\
&= r_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
r &= c_1 r_1 + c_2 r_2 \\
&= c_1\left(e^{3t}\right) + c_2\left(e^{3t}(t)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
r = c_1 e^{3t} + c_2 t\, e^{3t} \tag{1}
$$

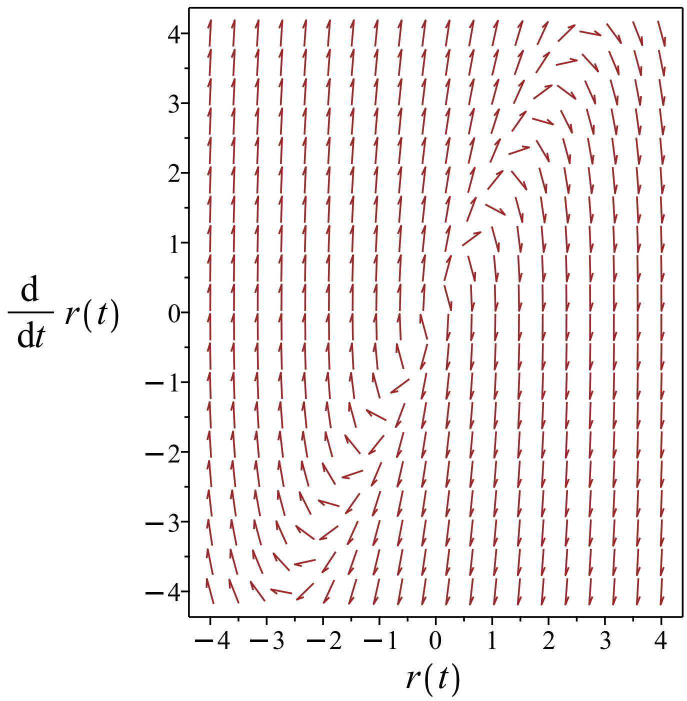


Figure 202: Slope field plot

Verification of solutions

$$
r = c_1 e^{3t} + c_2 t\, e^{3t}
$$

Verified OK.

8.4.4 Maple step by step solution

Let's solve

$r'' - 6r' + 9r = 0$

- Highest derivative means the order of the ODE is 2

  r''

- Characteristic polynomial of ODE

  $s^2 - 6s + 9 = 0$

- Factor the characteristic polynomial

  $(s-3)^2 = 0$

- Root of the characteristic polynomial

  $s = 3$

- 1st solution of the ODE

  $r_1(t) = e^{3t}$

- Repeated root, multiply $r_1(t)$ by t to ensure linear independence

  $r_2(t) = t\,e^{3t}$

- General solution of the ODE

  $r = c_1 r_1(t) + c_2 r_2(t)$

- Substitute in solutions

  $r = c_1 e^{3t} + c_2 t\,e^{3t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(r(t),t$2)-6*diff(r(t),t)+9*r(t)=0,r(t), singsol=all)
```

$$r(t) = \mathrm{e}^{3t}(c_2t + c_1)$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 18

```
DSolve[r''[t]-6*r'[t]+9*r[t]==0,r[t],t,IncludeSingularSolutions -> True]
```

$$r(t) \to e^{3t}(c_2t + c_1)$$

8.5 problem 5

8.5.1 Solving as exact ode 1228
8.5.2 Maple step by step solution 1232

Internal problem ID [4868]
Internal file name [`OUTPUT/4361_Sunday_June_05_2022_01_06_45_PM_71712485/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, [_1st_order, `_with_symmetry_[F(x),G(x)]`], [_Abel, `2
   nd type`, `class A`]]
```

$$\boxed{-y\sin(2x) - \left(\sin(x)^2 - 2y\right) y' = -2x}$$

8.5.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\sin(x)^2 + 2y\right)\mathrm{d}y &= (-2x + \sin(2x)\,y)\,\mathrm{d}x \\
(-\sin(2x)\,y + 2x)\,\mathrm{d}x + \left(-\sin(x)^2 + 2y\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\sin(2x)\,y + 2x \\
N(x, y) &= -\sin(x)^2 + 2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(2x)\,y + 2x) \\
&= -\sin(2x)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\sin(x)^2 + 2y\right) \\
&= -\sin(2x)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\sin(2x)\, y + 2x \,\mathrm{d}x$$
$$\phi = x^2 + \frac{\cos(2x)\, y}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\cos(2x)}{2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\sin(x)^2 + 2y$. Therefore equation (4) becomes

$$-\sin(x)^2 + 2y = \frac{\cos(2x)}{2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{\cos(2x)}{2} - \sin(x)^2 + 2y$$
$$= 2y - \frac{1}{2}$$

Integrating the above w.r.t y results in

$$\int f'(y) \,\mathrm{d}y = \int \left(2y - \frac{1}{2}\right) \mathrm{d}y$$
$$f(y) = y^2 - \frac{1}{2}y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2 + \frac{\cos(2x)\, y}{2} + y^2 - \frac{y}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2 + \frac{\cos(2x)\, y}{2} + y^2 - \frac{y}{2}$$

Summary

The solution(s) found are the following

$$x^2 + \frac{y\cos(2x)}{2} + y^2 - \frac{y}{2} = c_1 \tag{1}$$

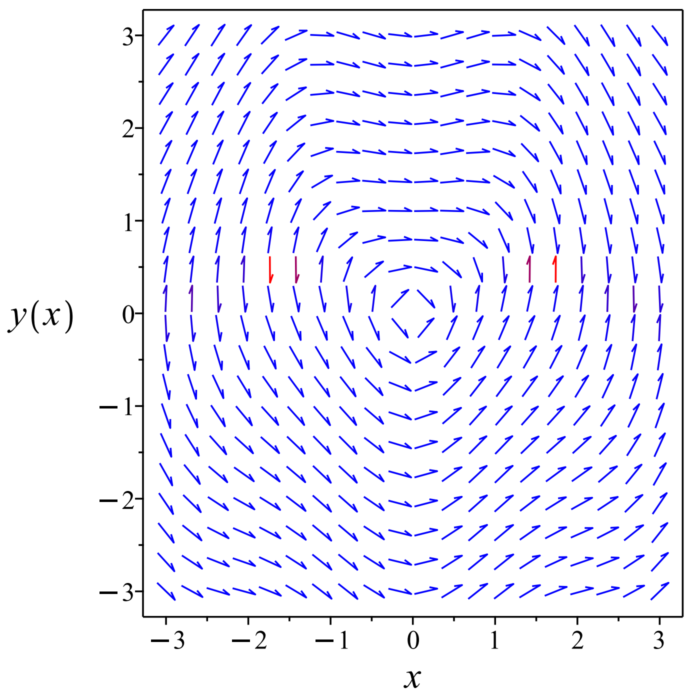


Figure 203: Slope field plot

Verification of solutions

$$x^2 + \frac{y\cos(2x)}{2} + y^2 - \frac{y}{2} = c_1$$

Verified OK.

8.5.2 Maple step by step solution

Let's solve

$-y\sin(2x) - \left(\sin(x)^2 - 2y\right)y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  ◦ Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$

  ◦ Evaluate derivatives

    $-\sin(2x) = -2\cos(x)\sin(x)$

  ◦ Simplify

    $-\sin(2x) = -\sin(2x)$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int(-\sin(2x)y + 2x)\,dx + f_1(y)$

- Evaluate integral

  $F(x,y) = x^2 + \frac{\cos(2x)y}{2} + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-\sin(x)^2 + 2y = \frac{\cos(2x)}{2} + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -\frac{\cos(2x)}{2} - \sin(x)^2 + 2y$

- Solve for $f_1(y)$

$f_1(y) = -\frac{\cos(2x)y}{2} - \sin(x)^2 y + y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = x^2 - \sin(x)^2 y + y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $x^2 - \sin(x)^2 y + y^2 = c_1$

- Solve for y

  $\left\{y = \frac{\sin(x)^2}{2} - \frac{\sqrt{\sin(x)^4 - 4x^2 + 4c_1}}{2}, y = \frac{\sin(x)^2}{2} + \frac{\sqrt{\sin(x)^4 - 4x^2 + 4c_1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 75

```
dsolve(2*x-y(x)*sin(2*x)=(sin(x)^2-2*y(x))*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = \frac{1}{4} - \frac{\cos(2x)}{4} - \frac{\sqrt{\cos(2x)^2 - 16x^2 - 2\cos(2x) - 16c_1 + 1}}{4}$$
$$y(x) = \frac{1}{4} - \frac{\cos(2x)}{4} + \frac{\sqrt{\cos(2x)^2 - 16x^2 - 2\cos(2x) - 16c_1 + 1}}{4}$$

✓ Solution by Mathematica

Time used: 0.259 (sec). Leaf size: 89

```
DSolve[2*x-y[x]*Sin[2*x]==(Sin[x]^2-2*y[x])*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(-\sqrt{-16x^2 + \cos^2(2x) - 2\cos(2x) + 1 + 16c_1} - \cos(2x) + 1\right)$$
$$y(x) \to \frac{1}{4}\left(\sqrt{-16x^2 + \cos^2(2x) - 2\cos(2x) + 1 + 16c_1} - \cos(2x) + 1\right)$$

8.6 problem 6

8.6.1 Solving as second order linear constant coeff ode 1235
8.6.2 Solving using Kovacic algorithm 1238
8.6.3 Maple step by step solution 1243

Internal problem ID [4869]
Internal file name [`OUTPUT/4362_Sunday_June_05_2022_01_06_57_PM_74292215/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + 2y = 10\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x}\cos(x)}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 2, f(x) = 10\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x}\cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}(\cos(x)\, c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x}(\cos(x)\, c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\,\mathrm{e}^{x}+6\,\mathrm{e}^{-x}\cos\left(x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\},\{\mathrm{e}^{-x}\cos\left(x\right),\mathrm{e}^{-x}\sin\left(x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-x}\cos\left(x\right),\mathrm{e}^{-x}\sin\left(x\right)\}$$

Since $\mathrm{e}^{-x}\cos\left(x\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{x}\},\{x\,\mathrm{e}^{-x}\sin\left(x\right),\cos\left(x\right)x\,\mathrm{e}^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p=A_1\mathrm{e}^{x}+A_2x\,\mathrm{e}^{-x}\sin\left(x\right)+A_3\cos\left(x\right)x\,\mathrm{e}^{-x}$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1\mathrm{e}^{x}+2A_2\mathrm{e}^{-x}\cos\left(x\right)-2A_3\sin\left(x\right)\mathrm{e}^{-x}=10\,\mathrm{e}^{x}+6\,\mathrm{e}^{-x}\cos\left(x\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1=2,A_2=3,A_3=0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=2\,\mathrm{e}^{x}+3x\,\mathrm{e}^{-x}\sin\left(x\right)$$

Therefore the general solution is

$$\begin{aligned}y&=y_h+y_p\\&=\left(\mathrm{e}^{-x}(\cos\left(x\right)c_1+c_2\sin\left(x\right))\right)+\left(2\,\mathrm{e}^{x}+3x\,\mathrm{e}^{-x}\sin\left(x\right)\right)\end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{x} + 3x e^{-x} \sin(x) \tag{1}$$

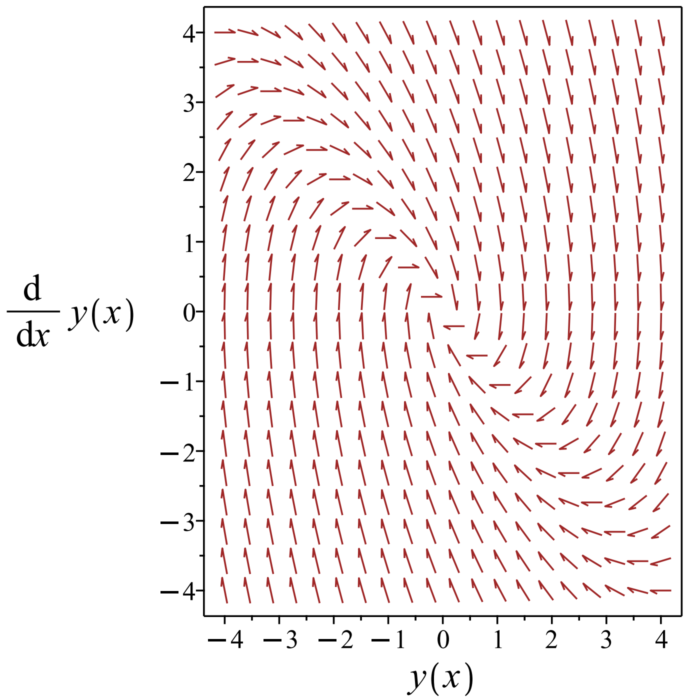


Figure 204: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{x} + 3x e^{-x} \sin(x)$$

Verified OK.

8.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 173: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x} \cos(x)\right) + c_2\left(\mathrm{e}^{-x} \cos(x) (\tan(x))\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\mathrm{e}^{-x} \cos(x) + c_2\mathrm{e}^{-x} \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{e^{-x}\cos(x), e^{-x}\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}\cos(x), e^{-x}\sin(x)\}$$

Since $e^{-x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{x\,e^{-x}\sin(x), \cos(x)\,x\,e^{-x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 x\,e^{-x}\sin(x) + A_3\cos(x)\,x\,e^{-x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^x + 2A_2 e^{-x}\cos(x) - 2A_3\sin(x)\,e^{-x} = 10\,e^x + 6\,e^{-x}\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,e^x + 3x\,e^{-x}\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x}\cos(x) + c_2 e^{-x}\sin(x)\right) + \left(2\,e^x + 3x\,e^{-x}\sin(x)\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(\cos(x)\,c_1 + c_2\sin(x)) + 2\,e^x + 3x\,e^{-x}\sin(x)$$

Summary
The solution(s) found are the following

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{x} + 3x e^{-x} \sin(x) \tag{1}$$

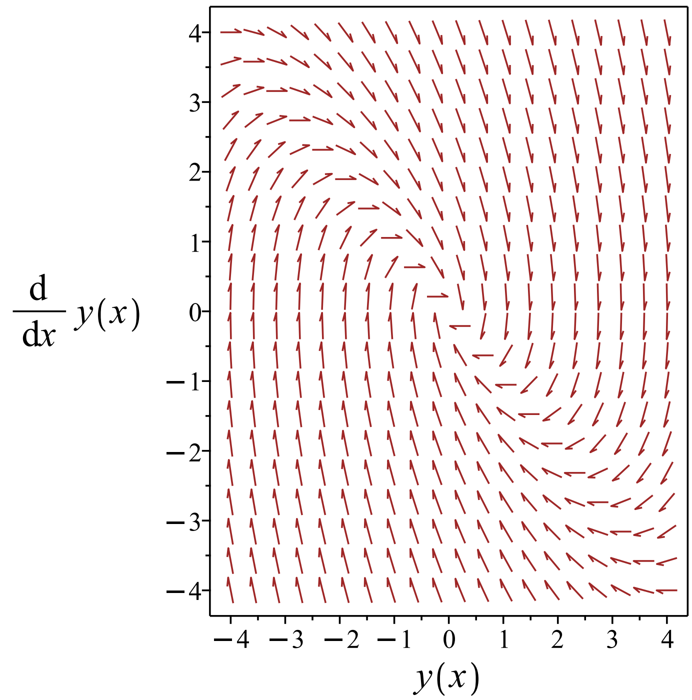


Figure 205: Slope field plot

Verification of solutions

$$y = e^{-x}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{x} + 3x e^{-x} \sin(x)$$

Verified OK.

8.6.3 Maple step by step solution

Let's solve

$y'' + 2y' + 2y = 10 e^{x} + 6 e^{-x} \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} \cos(x) + c_2 \mathrm{e}^{-x} \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 10\,\mathrm{e}^{x} + 6\,\mathrm{e}^{-x} \cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} \cos(x) & \mathrm{e}^{-x} \sin(x) \\ -\mathrm{e}^{-x} \cos(x) - \mathrm{e}^{-x} \sin(x) & -\mathrm{e}^{-x} \sin(x) + \mathrm{e}^{-x} \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,\mathrm{e}^{-x}\left(\cos(x)\left(\int \sin(x)(5\,\mathrm{e}^{2x} + 3\cos(x))\,dx\right) - \sin(x)\left(\int \cos(x)(5\,\mathrm{e}^{2x} + 3\cos(x))\,dx\right)\right)$

  ◦ Compute integrals

    $y_p(x) = (3\sin(x)\,x + 3\cos(x))\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} \cos(x) + c_2 \mathrm{e}^{-x} \sin(x) + (3\sin(x)\,x + 3\cos(x))\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+2*y(x)=10*exp(x)+6*exp(-x)*cos(x),y(x), singsol=all)
```

$$y(x) = \left((c_1 + 3)\cos(x) + 3\left(x + \frac{c_2}{3}\right)\sin(x)\right)\mathrm{e}^{-x} + 2\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.212 (sec). Leaf size: 41

```
DSolve[y''[x]+2*y'[x]+2*y[x]==10*Exp[x]+6*Exp[-x]*Cos[x],y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{2}e^{-x}\left(4e^{2x} + (3 + 2c_2)\cos(x) + 2(3x + c_1)\sin(x)\right)$$

8.7 problem 7

8.7.1 Solving as first order ode lie symmetry lookup ode 1246
8.7.2 Solving as bernoulli ode 1250
8.7.3 Solving as exact ode 1254

Internal problem ID [4870]
Internal file name [`OUTPUT/4363_Sunday_June_05_2022_01_07_07_PM_40950087/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3x^3y^2y' - y^3x^2 = 1}$$

8.7.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^3x^2 + 1}{3x^3y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 175: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x}{y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{y^3}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^3 x^2 + 1}{3x^3 y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^3}{3x^2} \\
S_y &= \frac{y^2}{x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{9R^3} + c_1 \tag{4}$$

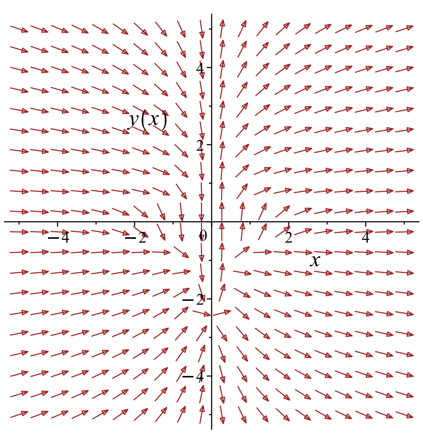
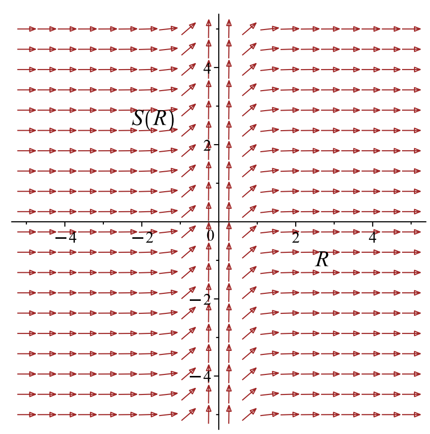
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x} = -\frac{1}{9x^3} + c_1$$

Which simplifies to

$$\frac{y^3}{3x} = -\frac{1}{9x^3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^3x^2+1}{3x^3y^2}$ 	$R = x$ $S = \frac{y^3}{3x}$	$\frac{dS}{dR} = \frac{1}{3R^4}$ 

Summary

The solution(s) found are the following

$$\frac{y^3}{3x} = -\frac{1}{9x^3} + c_1 \tag{1}$$

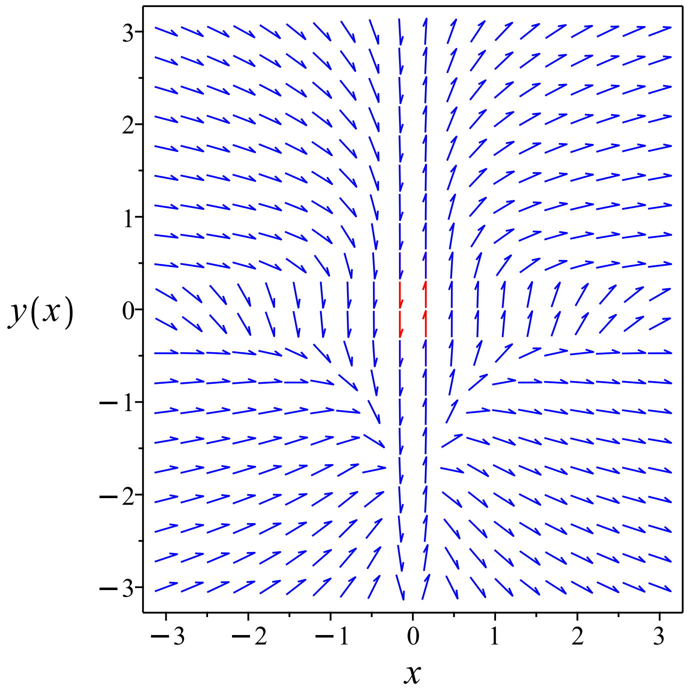


Figure 206: Slope field plot

Verification of solutions

$$\frac{y^3}{3x} = -\frac{1}{9x^3} + c_1$$

Verified OK.

8.7.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= \frac{y^3x^2 + 1}{3x^3y^2}
\end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{3x}y + \frac{1}{3x^3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{3x} \\ f_1(x) &= \frac{1}{3x^3} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{y^3}{3x} + \frac{1}{3x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{w(x)}{3x} + \frac{1}{3x^3} \\ w' &= \frac{w}{x} + \frac{1}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{1}{x^3}\right) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{1}{x^3}\right) \\ d\left(\frac{w}{x}\right) &= \frac{1}{x^4}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int \frac{1}{x^4}\, dx \\ \frac{w}{x} &= -\frac{1}{3x^3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = -\frac{1}{3x^2} + c_1 x$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = -\frac{1}{3x^2} + c_1 x$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{3^{\frac{2}{3}}(3c_1x^4 - x)^{\frac{1}{3}}}{3x} \\ y(x) &= \frac{\left(3i3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right)(3c_1x^4 - x)^{\frac{1}{3}}}{6x} \\ y(x) &= -\frac{3^{\frac{2}{3}}(3c_1x^4 - x)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{6x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{3^{\frac{2}{3}} (3c_1 x^4 - x)^{\frac{1}{3}}}{3x} \tag{1}$$

$$y = \frac{\left(3i 3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right) (3c_1 x^4 - x)^{\frac{1}{3}}}{6x} \tag{2}$$

$$y = -\frac{3^{\frac{2}{3}} (3c_1 x^4 - x)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{6x} \tag{3}$$

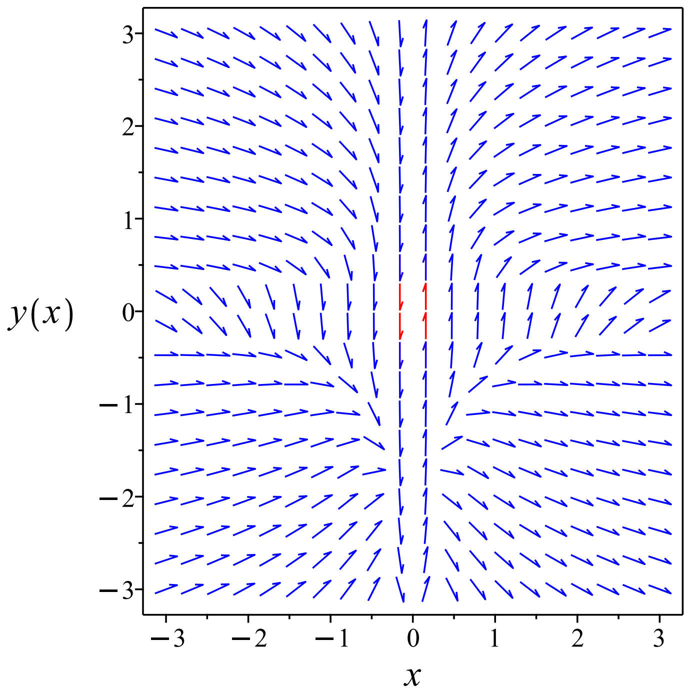


Figure 207: Slope field plot

Verification of solutions

$$y = \frac{3^{\frac{2}{3}} (3c_1x^4 - x)^{\frac{1}{3}}}{3x}$$

Verified OK.

$$y = \frac{\left(3i3^{\frac{1}{6}} - 3^{\frac{2}{3}}\right) (3c_1x^4 - x)^{\frac{1}{3}}}{6x}$$

Verified OK.

$$y = -\frac{3^{\frac{2}{3}} (3c_1x^4 - x)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{6x}$$

Verified OK.

8.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
(3y^2x^3)\,\mathrm{d}y &= (y^3x^2+1)\,\mathrm{d}x \\
(-y^3x^2-1)\,\mathrm{d}x + (3y^2x^3)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -y^3x^2-1 \\
N(x,y) &= 3y^2x^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^3x^2-1) \\
&= -3y^2x^2
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3y^2x^3) \\
&= 9y^2x^2
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{3x^3y^2}\left((-3y^2x^2) - (9y^2x^2)\right) \\
&= -\frac{4}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}(-y^3x^2 - 1)\\ &= \frac{-y^3x^2 - 1}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}(3y^2x^3)\\ &= \frac{3y^2}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-y^3x^2 - 1}{x^4}\right) + \left(\frac{3y^2}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-y^3x^2 - 1}{x^4} \mathrm{d}x$$

$$\phi = \frac{1}{3x^3} + \frac{y^3}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3y^2}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x}$. Therefore equation (4) becomes

$$\frac{3y^2}{x} = \frac{3y^2}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{3x^3} + \frac{y^3}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{3x^3} + \frac{y^3}{x}$$

Summary

The solution(s) found are the following

$$\frac{1}{3x^3} + \frac{y^3}{x} = c_1 \tag{1}$$

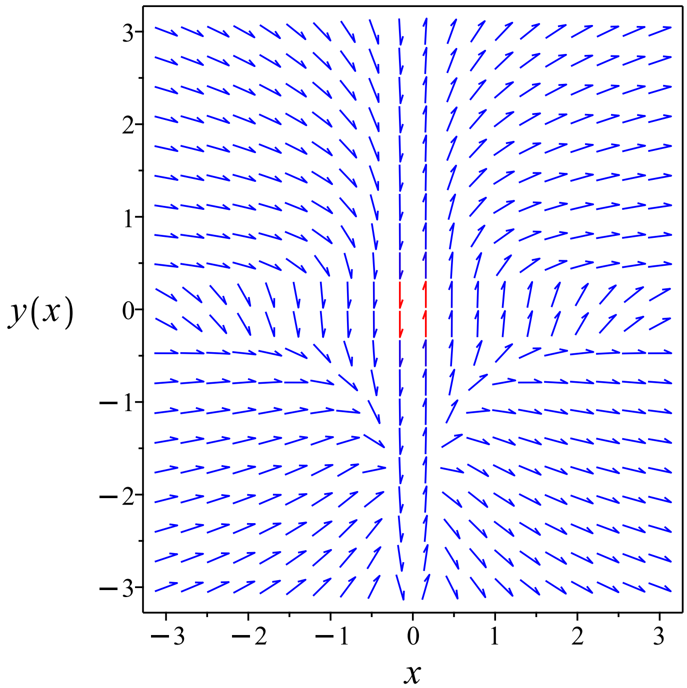


Figure 208: Slope field plot

Verification of solutions

$$\frac{1}{3x^3} + \frac{y^3}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 85

```
dsolve(3*x^3*y(x)^2*diff(y(x),x)-x^2*y(x)^3=1,y(x), singsol=all)
```

$$y(x) = \frac{3^{\frac{2}{3}}(3c_1x^4 - x)^{\frac{1}{3}}}{3x}$$
$$y(x) = -\frac{3^{\frac{2}{3}}(3c_1x^4 - x)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{6x}$$
$$y(x) = -\frac{\left(3^{\frac{2}{3}} - 3i3^{\frac{1}{6}}\right)(3c_1x^4 - x)^{\frac{1}{3}}}{6x}$$

✓ Solution by Mathematica

Time used: 0.518 (sec). Leaf size: 85

```
DSolve[3*x^3*y[x]^2*y'[x]-x^2*y[x]^3==1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-\frac{1}{3}}\sqrt[3]{-1 + 3c_1x^3}}{x^{2/3}}$$
$$y(x) \to \frac{\sqrt[3]{-\frac{1}{3} + c_1x^3}}{x^{2/3}}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{-\frac{1}{3} + c_1x^3}}{x^{2/3}}$$

8.8 problem 8

8.8.1 Solving as second order euler ode ode 1261
8.8.2 Solving as second order change of variable on x method 2 ode . 1264
8.8.3 Solving as second order change of variable on x method 1 ode . 1270
8.8.4 Solving as second order change of variable on y method 2 ode . 1274
8.8.5 Solving as second order ode non constant coeff transformation on B ode 1279
8.8.6 Solving using Kovacic algorithm 1283

Internal problem ID [4871]
Internal file name [`OUTPUT/4364_Sunday_June_05_2022_01_07_26_PM_95227566/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2 y'' - x y' + y = x}$$

8.8.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x + c_2x\ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - xy' + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x\left(\frac{\ln(x)^2}{2} + c_1 + c_2 \ln(x)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2\ln(x)\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1 + c_2\ln(x)\right)$$

Verified OK.

8.8.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + y = 0$$

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \frac{1}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\,(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\,(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-(\int p(x)dx)}dx \\
&= \int \mathrm{e}^{-(\int -\frac{1}{x}dx)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'\,(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{x^4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2} \\ 1 & \sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2}\right) - \left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)(1)$$

Which simplifies to

$$W = x\sqrt{2}$$

Which simplifies to

$$W = x\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)x}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{-\ln(2) + 2\ln(x)}{2x}dx$$

Hence

$$u_1 = \frac{\ln(2)\ln(x)}{2} - \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3\sqrt{2}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)\left(-\ln(2)+\ln(x)\right)}{2}$$
$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)\left(-\ln(2)+\ln(x)\right)x}{2} + \frac{\sqrt{2}\,\ln(x)\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2}\right) + \left(\frac{\ln(x)^2 x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2} + \frac{\ln(x)^2 x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2} + \frac{\ln(x)^2 x}{2}$$

Verified OK.

8.8.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - xy' + y = 0$$

In normal form the ode

$$x^2y'' - xy' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - xy' + y = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$

$$y_2 = x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\ln(2)}{2} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\ln(2)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\ln(2)}{2} \\ 1 & \sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\ln(2)}{2} \end{vmatrix}$$

Therefore

$$W = (x)\left(\sqrt{2}\,\ln(x) + \sqrt{2} - \frac{\sqrt{2}\,\ln(2)}{2}\right) - \left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)(1)$$

Which simplifies to

$$W = x\sqrt{2}$$

Which simplifies to

$$W = x\sqrt{2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)x}{x^3\sqrt{2}}dx$$

Which simplifies to

$$u_1 = -\int \frac{-\ln(2) + 2\ln(x)}{2x}dx$$

Hence

$$u_1 = \frac{\ln(2)\ln(x)}{2} - \frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3\sqrt{2}}dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{2}}{2x}dx$$

Hence

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Which simplifies to

$$u_1 = -\frac{\ln(x)(-\ln(2) + \ln(x))}{2}$$

$$u_2 = \frac{\sqrt{2}\,\ln(x)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)(-\ln(2) + \ln(x))\,x}{2} + \frac{\sqrt{2}\,\ln(x)\left(x\sqrt{2}\,\ln(x) - \frac{x\sqrt{2}\,\ln(2)}{2}\right)}{2}$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_1 x) + \left(\frac{\ln(x)^2 x}{2}\right) \\
&= \frac{\ln(x)^2 x}{2} + c_1 x
\end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln(x)^2}{2} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln(x)^2}{2} + c_1\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln(x)^2}{2} + c_1\right)$$

Verified OK.

8.8.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -x, C = 1, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-xy'+y=0$$

In normal form the ode

$$x^2y''-xy'+y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$\begin{aligned}p(x)&=-\frac{1}{x}\\q(x)&=\frac{1}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y=v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right)v'(x)+\left(\frac{n(n-1)}{x^2}+\frac{np}{x}+q\right)v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2}-\frac{n}{x^2}+\frac{1}{x^2}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned}v''(x)+\frac{v'(x)}{x}&=0\\v''(x)+\frac{v'(x)}{x}&=0\end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= (c_1 \ln(x) + c_2)\,x \\ &= (c_1 \ln(x) + c_2)\,x \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - xy' + y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)(1 + \ln(x)) - (\ln(x)\, x)(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x) x^2}{x^3} dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= ((c_1 \ln(x) + c_2) x) + \left(\frac{\ln(x)^2 x}{2}\right) \\ &= \frac{\ln(x)^2 x}{2} + (c_1 \ln(x) + c_2) x \end{aligned}$$

Which simplifies to

$$y = x\left(\frac{\ln (x)^2}{2} + c_1 \ln (x) + c_2\right)$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{\ln (x)^2}{2} + c_1 \ln (x) + c_2\right) \tag{1}$$

Verification of solutions

$$y = x\left(\frac{\ln (x)^2}{2} + c_1 \ln (x) + c_2\right)$$

Verified OK.

8.8.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1 \\
F &= x
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3 v'' + \left(-x^2\right) v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2 (u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{u}{x}
\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= e^{-\ln(x) + c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (-x)(c_1 \ln(x) + c_2) \\ &= -(c_1 \ln(x) + c_2)\, x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\, x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\, x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\, x \\ 1 & 1 + \ln(x) \end{vmatrix}$$

Therefore

$$W = (x)\,(1 + \ln(x)) - (\ln(x)\, x)\,(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^2}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x} dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2\, x}{2}$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= (-(c_1 \ln(x) + c_2)\, x) + \left(\frac{\ln(x)^2 x}{2}\right) \\
&= -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x \tag{1}
$$

Verification of solutions

$$
y = -\left(c_1 \ln(x) + c_2 - \frac{\ln(x)^2}{2}\right) x
$$

Verified OK.

8.8.6 Solving using Kovacic algorithm

Writing the ode as

$$
x^2 y'' - x y' + y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= -x \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 177: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\,(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2} \, dx} \\ &= z_1 e^{\frac{\ln(x)}{2}} \\ &= z_1\left(\sqrt{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\, dx \\ &= y_1(\ln(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(x) + c_2(x(\ln(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - xy' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1x + c_2x\ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \ln(x)\, x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \ln(x)\,x \\ \frac{d}{dx}(x) & \frac{d}{dx}(\ln(x)\,x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \ln(x)\,x \\ 1 & 1+\ln(x) \end{vmatrix}$$

Therefore

$$W = (x)\,(1+\ln(x)) - (\ln(x)\,x)\,(1)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\,x^2}{x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\ln(x)}{x}dx$$

Hence

$$u_1 = -\frac{\ln(x)^2}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^2}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{\ln(x)^2 x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x + c_2 x \ln(x)) + \left(\frac{\ln(x)^2 x}{2}\right) \end{aligned}$$

Which simplifies to

$$y = x(c_1 + c_2 \ln(x)) + \frac{\ln(x)^2 x}{2}$$

Summary

The solution(s) found are the following

$$y = x(c_1 + c_2 \ln(x)) + \frac{\ln(x)^2 x}{2} \tag{1}$$

Verification of solutions

$$y = x(c_1 + c_2 \ln(x)) + \frac{\ln(x)^2 x}{2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 18

```
dsolve(x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=x,y(x), singsol=all)
```

$$y(x) = x\left(c_2 + c_1 \ln(x) + \frac{\ln(x)^2}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 25

```
DSolve[x^2*y''[x]-x*y'[x]+y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}x\left(\log^2(x) + 2c_2\log(x) + 2c_1\right)$$

8.9 problem 9

8.9.1 Solving as first order ode lie symmetry lookup ode 1292
8.9.2 Solving as bernoulli ode 1296
8.9.3 Solving as riccati ode 1299

Internal problem ID [4872]
Internal file name [`OUTPUT/4365_Sunday_June_05_2022_01_07_34_PM_7427823/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{y' - 2y - y^2 e^{3x} = 0}$$

8.9.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= 2y + y^2 e^{3x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 178: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x} y^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x} y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{2x}}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2y + y^2 e^{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2 e^{2x}}{y} \\ S_y &= \frac{e^{2x}}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{5x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{5R}}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{2x}}{y} = \frac{e^{5x}}{5} + c_1$$

Which simplifies to

$$-\frac{e^{2x}}{y} = \frac{e^{5x}}{5} + c_1$$

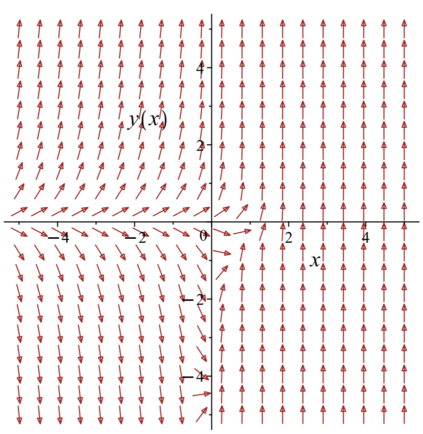
Which gives

$$y = -\frac{5\,e^{2x}}{e^{5x} + 5c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = 2y + y^2 e^{3x}$



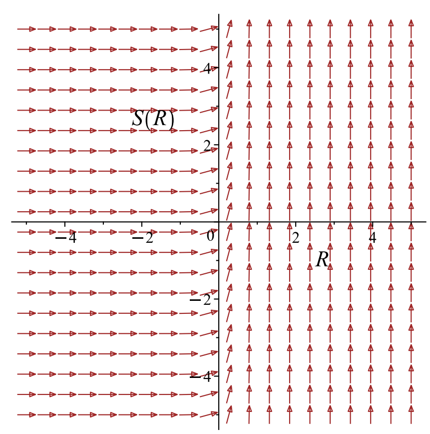
</td><td>

$R = x$

$S = -\frac{e^{2x}}{y}$

</td><td>

$\frac{dS}{dR} = e^{5R}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = -\frac{5\,\mathrm{e}^{2x}}{\mathrm{e}^{5x} + 5c_1} \tag{1}$$

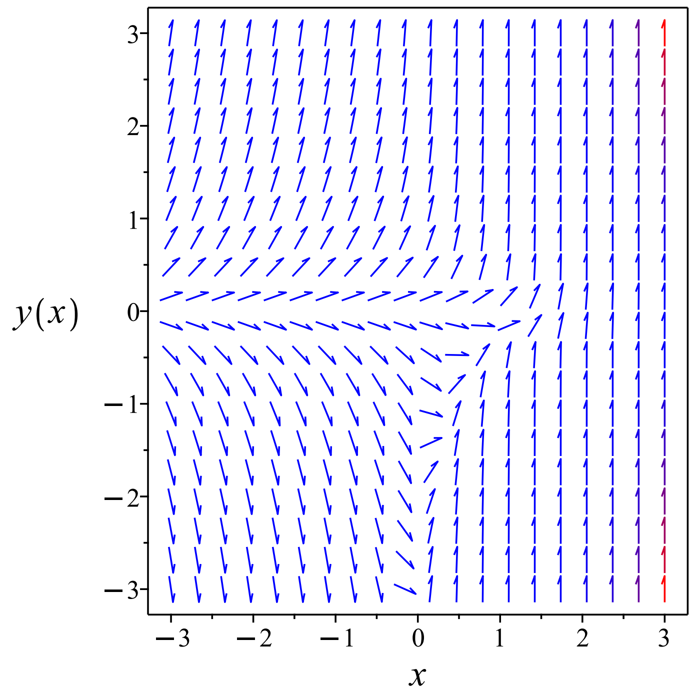


Figure 209: Slope field plot

Verification of solutions

$$y = -\frac{5\,\mathrm{e}^{2x}}{\mathrm{e}^{5x} + 5c_1}$$

Verified OK.

8.9.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2y + y^2 \mathrm{e}^{3x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 2y + \mathrm{e}^{3x} y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= 2 \\
f_1(x) &= \mathrm{e}^{3x} \\
n &= 2
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y' \frac{1}{y^2} = \frac{2}{y} + \mathrm{e}^{3x} \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-w'(x) &= 2w(x) + \mathrm{e}^{3x} \\
w' &= -2w - \mathrm{e}^{3x}
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= -\mathrm{e}^{3x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 2w(x) = -\mathrm{e}^{3x}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2dx} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\mathrm{e}^{3x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x} w\right) &= \left(\mathrm{e}^{2x}\right)\left(-\mathrm{e}^{3x}\right) \\
\mathrm{d}\left(\mathrm{e}^{2x} w\right) &= \left(-\mathrm{e}^{5x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{2x} w &= \int -\mathrm{e}^{5x}\,\mathrm{d}x \\
\mathrm{e}^{2x} w &= -\frac{\mathrm{e}^{5x}}{5} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{2x}$ results in

$$
w(x) = -\frac{\mathrm{e}^{-2x}\mathrm{e}^{5x}}{5} + c_1 \mathrm{e}^{-2x}
$$

which simplifies to

$$
w(x) = -\frac{\left(\mathrm{e}^{5x} - 5c_1\right)\mathrm{e}^{-2x}}{5}
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = -\frac{\left(\mathrm{e}^{5x} - 5c_1\right)\mathrm{e}^{-2x}}{5}
$$

Or

$$
y = -\frac{5\,\mathrm{e}^{2x}}{\mathrm{e}^{5x} - 5c_1}
$$

Summary
The solution(s) found are the following

$$y = -\frac{5\,e^{2x}}{e^{5x} - 5c_1} \tag{1}$$

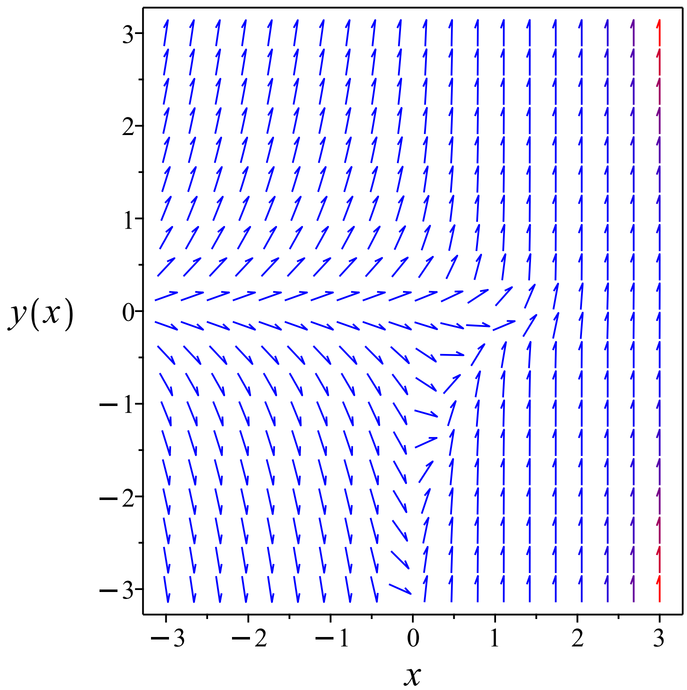


Figure 210: Slope field plot

Verification of solutions

$$y = -\frac{5\,e^{2x}}{e^{5x} - 5c_1}$$

Verified OK.

8.9.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2y + y^2 e^{3x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2y + y^2 e^{3x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 2$ and $f_2(x) = e^{3x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{e^{3x} u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 3\,e^{3x} \\ f_1 f_2 &= 2\,e^{3x} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$e^{3x} u''(x) - 5\,e^{3x} u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + c_2 e^{5x}$$

The above shows that

$$u'(x) = 5c_2 e^{5x}$$

Using the above in (1) gives the solution

$$y = -\frac{5c_2 e^{5x} e^{-3x}}{c_1 + c_2 e^{5x}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{5\,e^{2x}}{c_3 + e^{5x}}$$

Summary

The solution(s) found are the following

$$y = -\frac{5\,e^{2x}}{c_3 + e^{5x}} \tag{1}$$

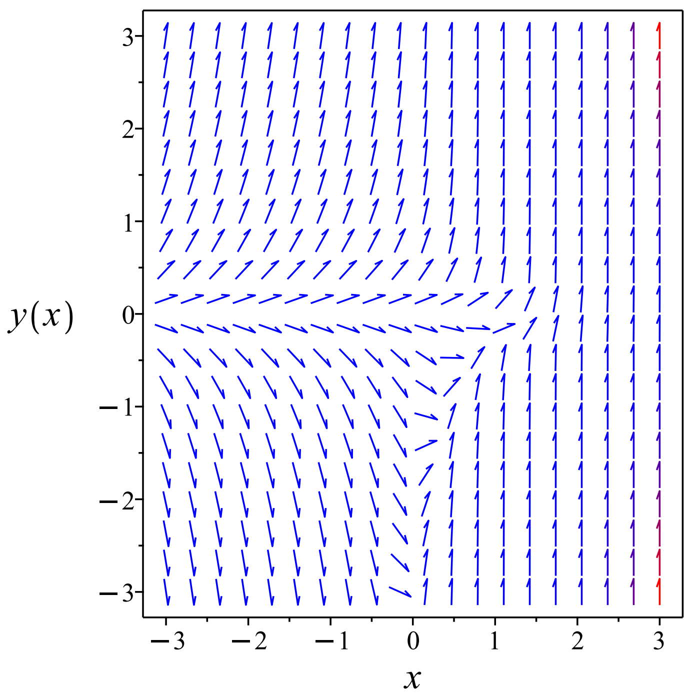


Figure 211: Slope field plot

Verification of solutions

$$y = -\frac{5\,e^{2x}}{c_3 + e^{5x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)-(2*y(x)+y(x)^2*exp(3*x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{5\,\mathrm{e}^{2x}}{\mathrm{e}^{5x} - 5c_1}$$

✓ Solution by Mathematica

Time used: 0.223 (sec). Leaf size: 29

```
DSolve[y'[x]-(2*y[x]+y[x]^2*Exp[3*x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{5e^{2x}}{e^{5x} - 5c_1}$$
$$y(x) \to 0$$

8.10 problem 10

8.10.1 Solving as separable ode 1303
8.10.2 Solving as first order ode lie symmetry lookup ode 1305
8.10.3 Solving as exact ode 1309
8.10.4 Maple step by step solution 1313

Internal problem ID [4873]
Internal file name [`OUTPUT/4366_Sunday_June_05_2022_01_07_44_PM_28761801/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{u(1-v)+v^2(1-u)\,u'=0}$$

8.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} u' &= F(v,u) \\ &= f(v)g(u) \\ &= -\frac{u(v-1)}{v^2\,(-1+u)} \end{aligned}$$

Where $f(v)=-\frac{v-1}{v^2}$ and $g(u)=\frac{u}{-1+u}$. Integrating both sides gives

$$\frac{1}{\frac{u}{-1+u}}\,du=-\frac{v-1}{v^2}\,dv$$

$$\int \frac{1}{\frac{u}{-1+u}} du = \int -\frac{v-1}{v^2} dv$$

$$u - \ln(u) = -\frac{1}{v} - \ln(v) + c_1$$

Which results in

$$u = e^{\frac{\ln(v)v - \mathrm{LambertW}\left(-e^{\frac{\ln(v)v - c_1 v + 1}{v}}\right)v - c_1 v + 1}{v}}$$

Which simplifies to

$$u = v\, e^{-\mathrm{LambertW}\left(-v\, e^{-c_1 + \frac{1}{v}}\right)} e^{-c_1} e^{\frac{1}{v}}$$

<u>Summary</u>

The solution(s) found are the following

$$u = v\, e^{-\mathrm{LambertW}\left(-v\, e^{-c_1 + \frac{1}{v}}\right)} e^{-c_1} e^{\frac{1}{v}} \tag{1}$$

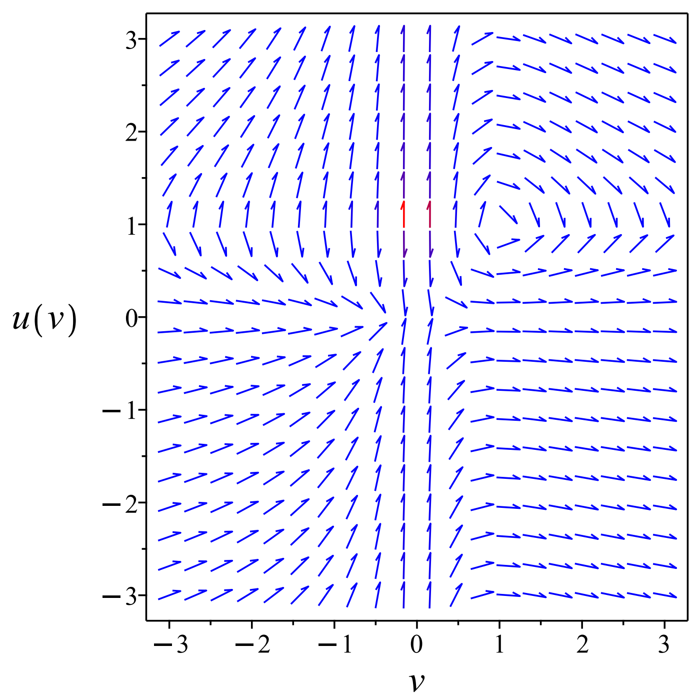


Figure 212: Slope field plot

<u>Verification of solutions</u>

$$u = v\, e^{-\mathrm{LambertW}\left(-v\, e^{-c_1 + \frac{1}{v}}\right)} e^{-c_1} e^{\frac{1}{v}}$$

Verified OK.

8.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
u' &= -\frac{u(v-1)}{v^2(-1+u)} \\
u' &= \omega(v,u)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_v + \omega(\eta_u - \xi_v) - \omega^2 \xi_u - \omega_v \xi - \omega_u \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 180: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(v,u) = -\frac{v^2}{v-1}$$
$$\eta(v,u) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(v,u) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dv}{\xi} = \frac{du}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial v} + \eta\frac{\partial}{\partial u}\right) S(v,u) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = u$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dv \\ &= \int \frac{1}{-\frac{v^2}{v-1}} dv \end{aligned}$$

Which results in

$$S = -\frac{1}{v} - \ln(v)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_v + \omega(v,u) S_u}{R_v + \omega(v,u) R_u} \tag{2}$$

Where in the above R_v, R_u, S_v, S_u are all partial derivatives and $\omega(v,u)$ is the right hand side of the original ode given by

$$\omega(v,u) = -\frac{u(v-1)}{v^2(-1+u)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_v &= 0 \\
R_u &= 1 \\
S_v &= \frac{1-v}{v^2} \\
S_u &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{-1+u}{u} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for v, u in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{-1+R}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R - \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to v, u coordinates. This results in

$$
\frac{-\ln(v)\,v - 1}{v} = u - \ln(u) + c_1
$$

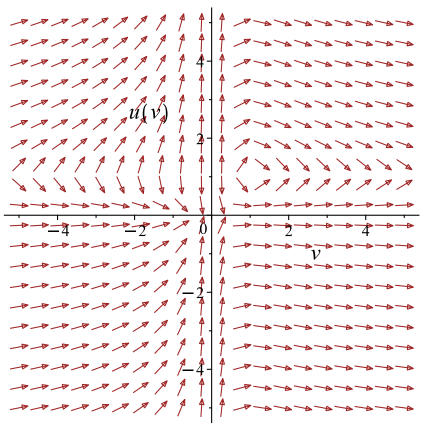
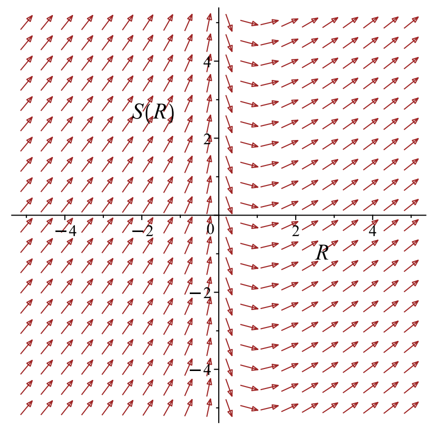
Which simplifies to

$$
\frac{-\ln(v)\,v - 1}{v} = u - \ln(u) + c_1
$$

Which gives

$$
u = -\operatorname{LambertW}\left(-\mathrm{e}^{\frac{\ln(v)v+c_1v+1}{v}}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in v, u coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{du}{dv} = -\frac{u(v-1)}{v^2(-1+u)}$
</td><td>$R = u$
$S = \frac{-\ln(v)\, v - 1}{v}$</td><td>$\frac{dS}{dR} = \frac{-1+R}{R}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$u = -\operatorname{LambertW}\left(-e^{\frac{\ln(v)v+c_1v+1}{v}}\right) \tag{1}$$

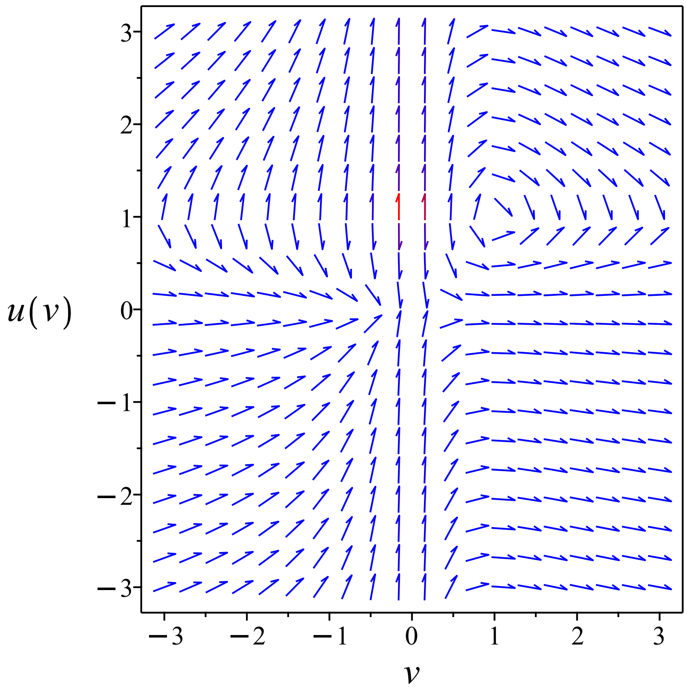


Figure 213: Slope field plot

Verification of solutions

$$u = -\operatorname{LambertW}\left(-\mathrm{e}^{\frac{\ln(v)v+c_1v+1}{v}}\right)$$

Verified OK.

8.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(v,u)\,\mathrm{d}v + N(v,u)\,\mathrm{d}u = 0 \tag{1A}$$

Therefore

$$\left(-\frac{-1+u}{u}\right)\mathrm{d}u = \left(\frac{v-1}{v^2}\right)\mathrm{d}v$$
$$\left(-\frac{v-1}{v^2}\right)\mathrm{d}v + \left(-\frac{-1+u}{u}\right)\mathrm{d}u = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(v,u) = -\frac{v-1}{v^2}$$
$$N(v,u) = -\frac{-1+u}{u}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial u} = \frac{\partial N}{\partial v}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial}{\partial u}\left(-\frac{v-1}{v^2}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{-1+u}{u}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial u} = \frac{\partial N}{\partial v}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(v,u)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial v} &= M \qquad (1) \\
\frac{\partial \phi}{\partial u} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. v gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v &= \int M\,\mathrm{d}v \\
\int \frac{\partial \phi}{\partial v}\,\mathrm{d}v &= \int -\frac{v-1}{v^2}\,\mathrm{d}v \\
\phi &= -\frac{1}{v} - \ln(v) + f(u) \qquad (3)
\end{aligned}
$$

Where $f(u)$ is used for the constant of integration since ϕ is a function of both v and u. Taking derivative of equation (3) w.r.t u gives

$$
\frac{\partial \phi}{\partial u} = 0 + f'(u) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial u} = -\frac{-1+u}{u}$. Therefore equation (4) becomes

$$
-\frac{-1+u}{u} = 0 + f'(u) \qquad (5)
$$

Solving equation (5) for $f'(u)$ gives

$$
f'(u) = -\frac{-1+u}{u}
$$

Integrating the above w.r.t u gives

$$
\begin{aligned}
\int f'(u)\,\mathrm{d}u &= \int \left(\frac{1-u}{u}\right)\mathrm{d}u \\
f(u) &= -u + \ln(u) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(u)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{v} - \ln(v) - u + \ln(u) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{v} - \ln(v) - u + \ln(u)$$

The solution becomes

$$u = -\operatorname{LambertW}\left(-e^{\frac{\ln(v)v + c_1 v + 1}{v}}\right)$$

Summary

The solution(s) found are the following

$$u = -\operatorname{LambertW}\left(-e^{\frac{\ln(v)v + c_1 v + 1}{v}}\right) \tag{1}$$

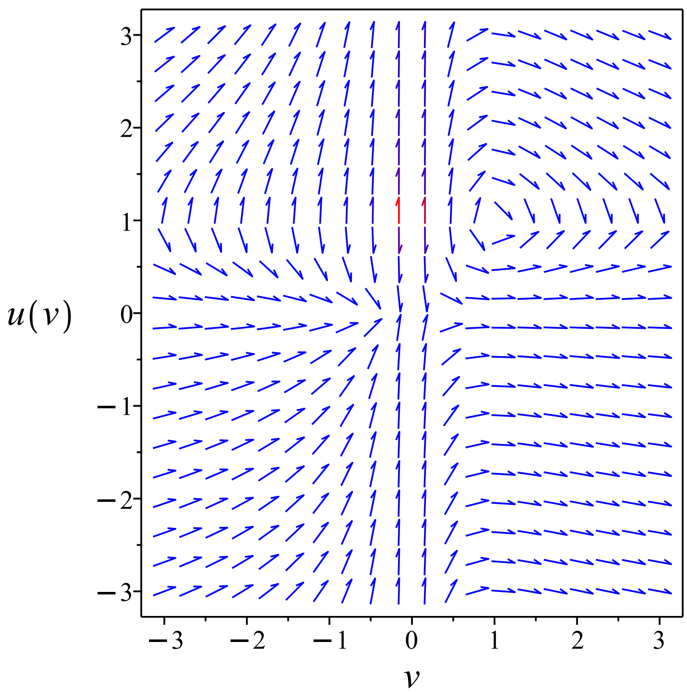


Figure 214: Slope field plot

Verification of solutions

$$u = -\operatorname{LambertW}\left(-\mathrm{e}^{\frac{\ln(v)v+c_1v+1}{v}}\right)$$

Verified OK.

8.10.4 Maple step by step solution

Let's solve

$u(1-v)+v^2(1-u)\,u'=0$

- Highest derivative means the order of the ODE is 1

  u'

- Separate variables

  $\frac{u'(1-u)}{u}=-\frac{1-v}{v^2}$

- Integrate both sides with respect to v

  $\int \frac{u'(1-u)}{u}dv=\int -\frac{1-v}{v^2}dv+c_1$

- Evaluate integral

  $-u+\ln(u)=c_1+\ln(v)+\frac{1}{v}$

- Solve for u

  $u=-\mathit{LambertW}\left(-\mathrm{e}^{\frac{\ln(v)v+c_1v+1}{v}}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 33

```
dsolve(u(v)*(1-v)+v^2*(1-u(v))*diff(u(v),v)=0,u(v), singsol=all)
```

$$u(v) = v\,e^{\frac{-\operatorname{LambertW}\left(-v\,e^{\frac{c_1 v+1}{v}}\right)v+c_1 v+1}{v}}$$

✓ Solution by Mathematica

Time used: 2.999 (sec). Leaf size: 26

```
DSolve[u[v]*(1-v)+v^2*(1-u[v])*u'[v]==0,u[v],v,IncludeSingularSolutions -> True]
```

$$u(v) \to -W\left(v\left(-e^{\frac{1}{v}-c_1}\right)\right)$$
$$u(v) \to 0$$

8.11 problem 11

8.11.1 Solving as linear ode 1315
8.11.2 Solving as homogeneousTypeD2 ode 1317
8.11.3 Solving as first order ode lie symmetry lookup ode 1318
8.11.4 Solving as exact ode 1322
8.11.5 Maple step by step solution 1327

Internal problem ID [4874]
Internal file name [`OUTPUT/4367_Sunday_June_05_2022_01_07_53_PM_28414310/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-xy' + y = -2x}$$

8.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 2$$

Hence the ode is

$$y' - \frac{y}{x} = 2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(2) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(2) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{2}{x}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{2}{x} \mathrm{d}x \\
\frac{y}{x} &= 2\ln(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = 2\ln(x)\, x + c_1 x
$$

which simplifies to

$$
y = x(2\ln(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x(2\ln(x) + c_1) \tag{1}
$$

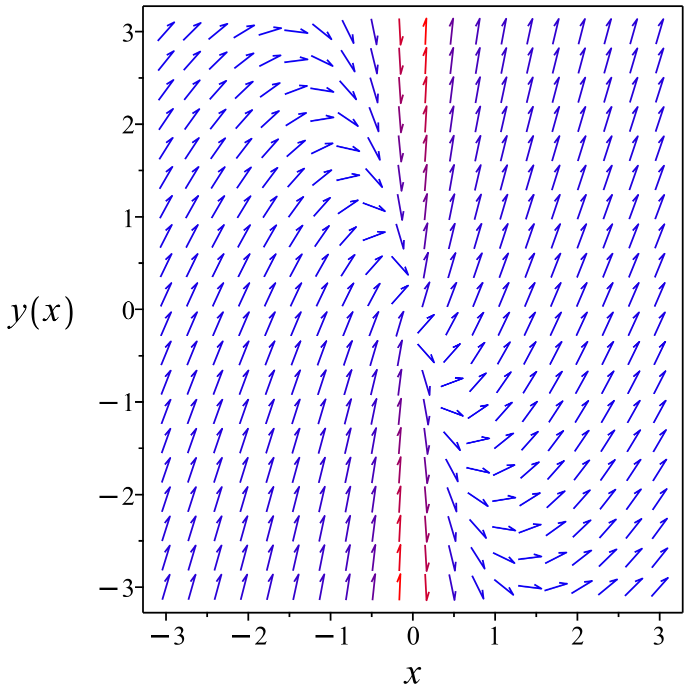


Figure 215: Slope field plot

Verification of solutions

$$y = x(2\ln(x) + c_1)$$

Verified OK.

8.11.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-x(u'(x)\,x + u(x)) + u(x)\,x = -2x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{2}{x}\,\mathrm{d}x \\ &= 2\ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(2\ln(x) + c_2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x(2\ln(x) + c_2) \tag{1}$$

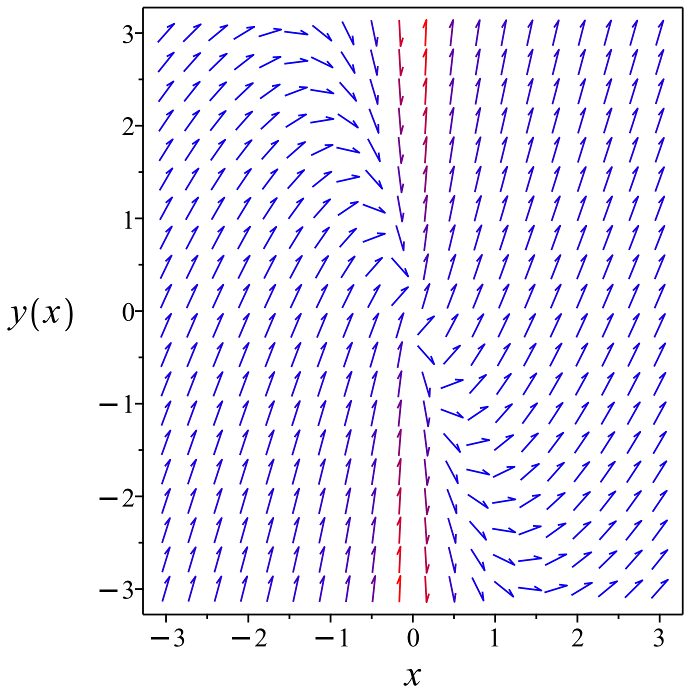


Figure 216: Slope field plot

Verification of solutions

$$y = x(2\ln(x) + c_2)$$

Verified OK.

8.11.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y + 2x}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 183: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned}S&=\int \frac{1}{\eta}dy\\&=\int \frac{1}{x}dy\end{aligned}$$

Which results in

$$S=\frac{y}{x}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y+2x}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x&=1\\R_y&=0\\S_x&=-\frac{y}{x^2}\\S_y&=\frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR}=\frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR}=\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = 2\ln(x) + c_1$$

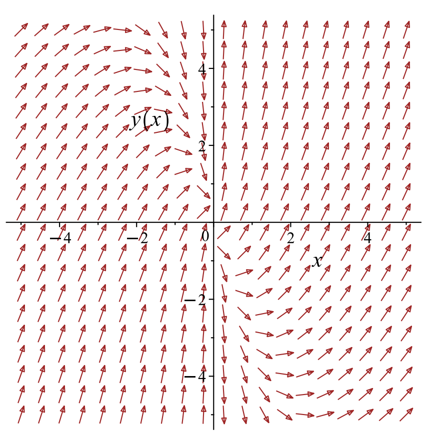
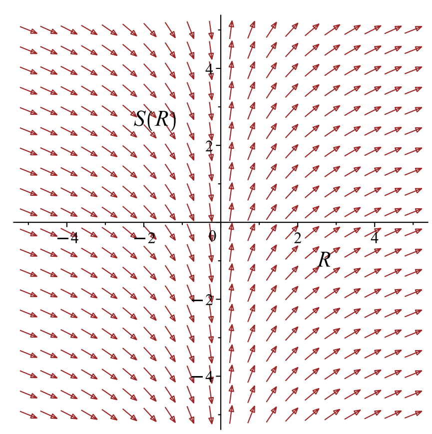
Which simplifies to

$$\frac{y}{x} = 2\ln(x) + c_1$$

Which gives

$$y = x(2\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y+2x}{x}$
</td><td>$R = x$
$S = \frac{y}{x}$</td><td>$\frac{dS}{dR} = \frac{2}{R}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = x(2\ln(x) + c_1) \tag{1}$$

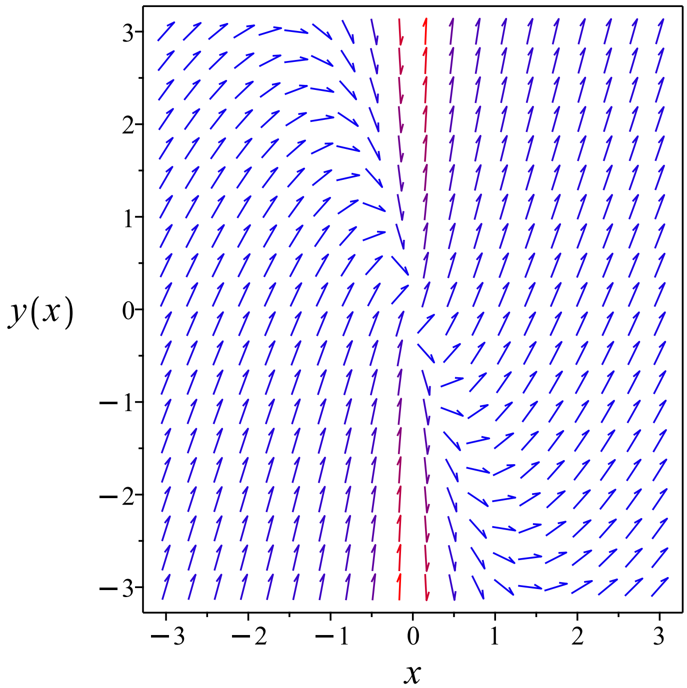


Figure 217: Slope field plot

Verification of solutions

$$y = x(2\ln(x) + c_1)$$

Verified OK.

8.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x)\,\mathrm{d}y &= (-y - 2x)\,\mathrm{d}x \\ (y + 2x)\,\mathrm{d}x + (-x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y + 2x$$
$$N(x, y) = -x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y + 2x) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x}((1) - (-1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(y + 2x) \\
&= \frac{y + 2x}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(-x) \\
&= -\frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y + 2x}{x^2}\right) + \left(-\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y + 2x}{x^2} \, \mathrm{d}x$$
$$\phi = 2\ln(x) - \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x}$. Therefore equation (4) becomes

$$-\frac{1}{x} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 2\ln(x) - \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 2\ln(x) - \frac{y}{x}$$

The solution becomes

$$y = (2\ln(x) - c_1)\, x$$

Summary

The solution(s) found are the following

$$y = (2\ln(x) - c_1)\, x \tag{1}$$

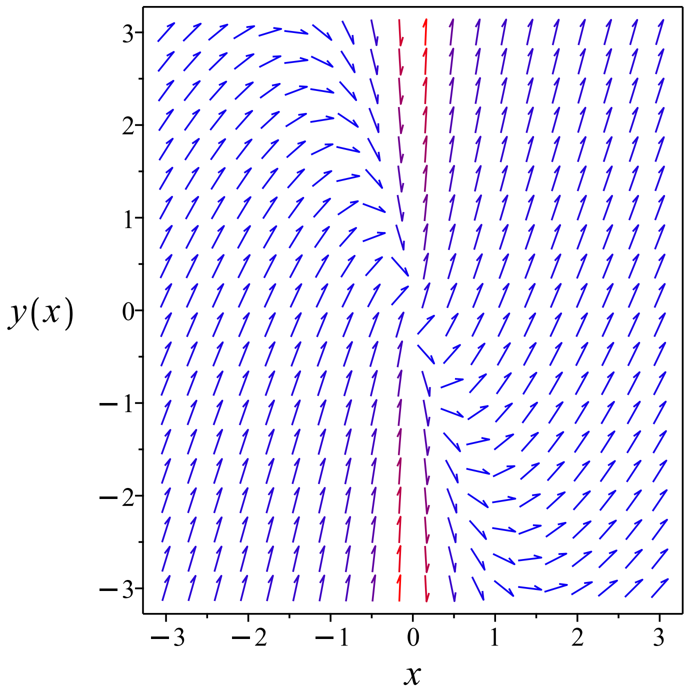


Figure 218: Slope field plot

Verification of solutions

$$y = (2\ln(x) - c_1)\, x$$

Verified OK.

8.11.5 Maple step by step solution

Let's solve

$-xy' + y = -2x$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = 2 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - \frac{y}{x} = 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y' - \frac{y}{x}\right) = 2\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

$\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\,dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int 2\mu(x)\,dx + c_1$

- Solve for y

$y = \frac{\int 2\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

$y = x\left(\int \frac{2}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

$y = x(2\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve((y(x)+2*x)-x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = (2\ln(x) + c_1)\, x$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 14

```
DSolve[(y[x]+2*x)-x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(2\log(x) + c_1)$$

8.12 problem 12

8.12.1 Solving as second order integrable as is ode 1330
8.12.2 Solving as second order ode missing y ode 1330
8.12.3 Solving as second order ode non constant coeff transformation on B ode 1332
8.12.4 Solving as type second_order_integrable_as_is (not using ABC version) 1336
8.12.5 Solving using Kovacic algorithm 1337
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8.12.7 Maple step by step solution 1345

Internal problem ID [4875]
Internal file name [`OUTPUT/4368_Sunday_June_05_2022_01_08_03_PM_26789376/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{xy'' + y' = 4x}$$

8.12.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (xy'' + y')\, dx &= \int 4x dx \\
xy' &= 2x^2 + c_1
\end{aligned}
$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{2x^2 + c_1}{x} \,\mathrm{d}x \\
&= x^2 + c_1 \ln(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x^2 + c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = x^2 + c_1 \ln(x) + c_2
$$

Verified OK.

8.12.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$
p(x) = y'
$$

Then

$$
p'(x) = y''
$$

Hence the ode becomes

$$
p'(x)\, x + p(x) - 4x = 0
$$

Which is now solve for $p(x)$ as first order ode.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
p'(x) + p(x)p(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= 4
\end{aligned}
$$

Hence the ode is

$$p'(x) + \frac{p(x)}{x} = 4$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu p) &= (\mu)(4) \\ \frac{d}{dx}(xp) &= (x)(4) \\ d(xp) &= (4x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xp &= \int 4x\, dx \\ xp &= 2x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$p(x) = 2x + \frac{c_1}{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 2x + \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned}y &= \int \frac{2x^2 + c_1}{x}\, dx \\ &= x^2 + c_1 \ln(x) + c_2\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = x^2 + c_1 \ln(x) + c_2 \tag{1}$$

<u>Verification of solutions</u>

$$y = x^2 + c_1 \ln(x) + c_2$$

Verified OK.

8.12.3 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x \\ B &= 1 \\ C &= 0 \\ F &= 4x \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x)(0) + (1)(0) + (0)(1) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$xv'' + (1)\,v' = 0$$

Now by applying $v' = u$ the above becomes

$$xu'(x) + u(x) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\,\mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (1)\,(c_1 \ln(x) + c_2) \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\ln(x)\,x}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\,x dx$$

Hence

$$u_1 = -2\ln(x)\,x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x}{1}\,dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2(1 - 2\ln(x)) + 2\ln(x)\,x^2$$

Which simplifies to

$$y_p(x) = x^2$$

Hence the complete solution is

$$
\begin{aligned}
y(x) &= y_h + y_p \\
&= (c_1 \ln(x) + c_2) + (x^2) \\
&= x^2 + c_1 \ln(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x^2 + c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = x^2 + c_1 \ln(x) + c_2
$$

Verified OK.

8.12.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$
xy'' + y' = 4x
$$

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int (xy'' + y')\, dx &= \int 4x dx \\
xy' &= 2x^2 + c_1
\end{aligned}
$$

Which is now solved for y. Integrating both sides gives

$$
\begin{aligned}
y &= \int \frac{2x^2 + c_1}{x}\, \mathrm{d}x \\
&= x^2 + c_1 \ln(x) + c_2
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x^2 + c_1 \ln(x) + c_2 \tag{1}
$$

Verification of solutions

$$
y = x^2 + c_1 \ln(x) + c_2
$$

Verified OK.

8.12.5 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
xy'' + y' &= 0 \tag{1} \\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x \\
B &= 1 \tag{3} \\
C &= 0
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1 \\
t &= 4x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 186: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{x}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
xy'' + y' = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 + c_2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \ln(x) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & \ln(x) \\ \frac{d}{dx}(1) & \frac{d}{dx}(\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & \ln(x) \\ 0 & \frac{1}{x} \end{vmatrix}$$

Therefore

$$W = (1)\left(\frac{1}{x}\right) - (\ln(x))(0)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\ln(x)\,x}{1}\,dx$$

Which simplifies to

$$u_1 = -\int 4\ln(x)\,x dx$$

Hence

$$u_1 = -2\ln(x)\,x^2 + x^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{4x}{1}\,dx$$

Which simplifies to

$$u_2 = \int 4x dx$$

Hence

$$u_2 = 2x^2$$

Which simplifies to

$$u_1 = x^2(1 - 2\ln(x))$$
$$u_2 = 2x^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = x^2(1 - 2\ln(x)) + 2\ln(x)\,x^2$$

Which simplifies to

$$y_p(x) = x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 + c_2 \ln(x)) + (x^2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \ln(x) + x^2 \tag{1}$$

Verification of solutions

$$y = c_1 + c_2 \ln(x) + x^2$$

Verified OK.

8.12.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x \\ q(x) &= 1 \\ r(x) &= 0 \\ s(x) &= 4x \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$xy' = \int 4x\,dx$$

We now have a first order ode to solve which is

$$xy' = 2x^2 + c_1$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{2x^2 + c_1}{x}\,\mathrm{d}x \\ &= x^2 + c_1 \ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^2 + c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = x^2 + c_1 \ln(x) + c_2$$

Verified OK.

8.12.7 Maple step by step solution

Let's solve

$y''x + y' = 4x$

- Highest derivative means the order of the ODE is 2

  y''

- Make substitution $u = y'$ to reduce order of ODE

  $u'(x)\,x + u(x) = 4x$

- Isolate the derivative

  $u'(x) = 4 - \frac{u(x)}{x}$

- Group terms with $u(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $u'(x) + \frac{u(x)}{x} = 4$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(u'(x) + \frac{u(x)}{x}\right) = 4\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, u(x))$

  $\mu(x) \left(u'(x) + \frac{u(x)}{x}\right) = \mu'(x)\, u(x) + \mu(x)\, u'(x)$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, u(x))\right) dx = \int 4\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, u(x) = \int 4\mu(x)\, dx + c_1$

- Solve for $u(x)$

  $u(x) = \frac{\int 4\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $u(x) = \frac{\int 4x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $u(x) = \frac{2x^2 + c_1}{x}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{2x^2 + c_1}{x}$

- Make substitution $u = y'$

  $y' = \frac{2x^2 + c_1}{x}$

- Integrate both sides to solve for y

  $\int y' dx = \int \frac{2x^2 + c_1}{x} dx + c_2$

- Compute integrals

  $y = x^2 + c_1 \ln(x) + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)-4*_a)/_a, _b(_a)`   *** Sublev
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x*diff(y(x),x$2)+diff(y(x),x)=4*x,y(x), singsol=all)
```

$$y(x) = x^2 + c_1 \ln(x) + c_2$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 16

```
DSolve[x*y''[x]+y'[x]==4*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + c_1 \log(x) + c_2$$

8.13 problem 13

8.13.1 Solving as second order linear constant coeff ode 1348
8.13.2 Solving using Kovacic algorithm 1351
8.13.3 Maple step by step solution 1356

Internal problem ID [4876]
Internal file name [`OUTPUT/4369_Sunday_June_05_2022_01_08_14_PM_27673658/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 5y = 26\,\mathrm{e}^{3x}}$$

8.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = 26\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\, \mathrm{e}^{\lambda x} + 5\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)} \\
&= -2 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$26\,e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), \sin(x)\,e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$26A_1 e^{3x} = 26\,e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(\cos(x)\,c_1 + c_2\sin(x))\right) + \left(e^{3x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{-2x}(\cos(x)\,c_1 + c_2\sin(x)) + e^{3x} \tag{1}$$

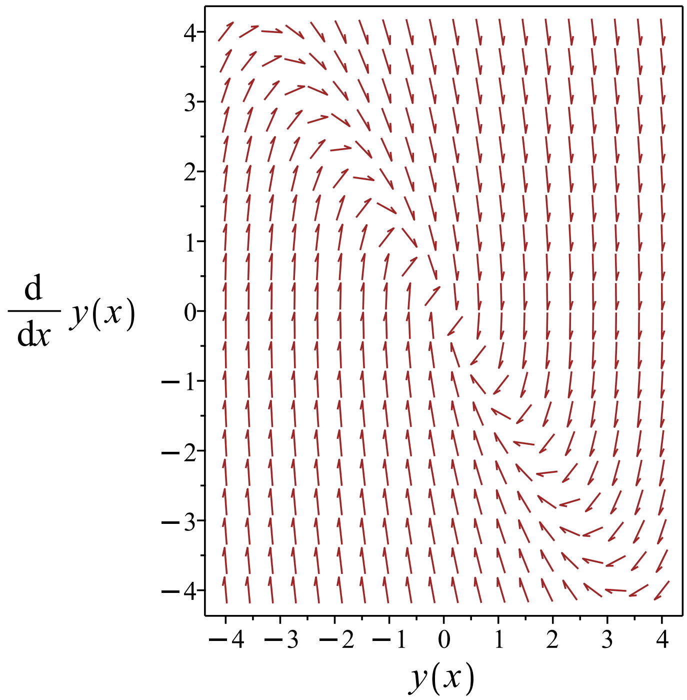


Figure 219: Slope field plot

Verification of solutions

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x)) + e^{3x}$$

Verified OK.

8.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 188: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}\cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos (x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos (x) (\tan (x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-2x} \cos (x) c_1 + \mathrm{e}^{-2x} \sin (x) c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
26\,\mathrm{e}^{3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{3x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x} \cos (x) , \sin (x) \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{3x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
26A_1 \mathrm{e}^{3x} = 26\,\mathrm{e}^{3x}
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}\cos(x)\, c_1 + e^{-2x}\sin(x)\, c_2\right) + \left(e^{3x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x)) + e^{3x}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x)) + e^{3x} \tag{1}$$

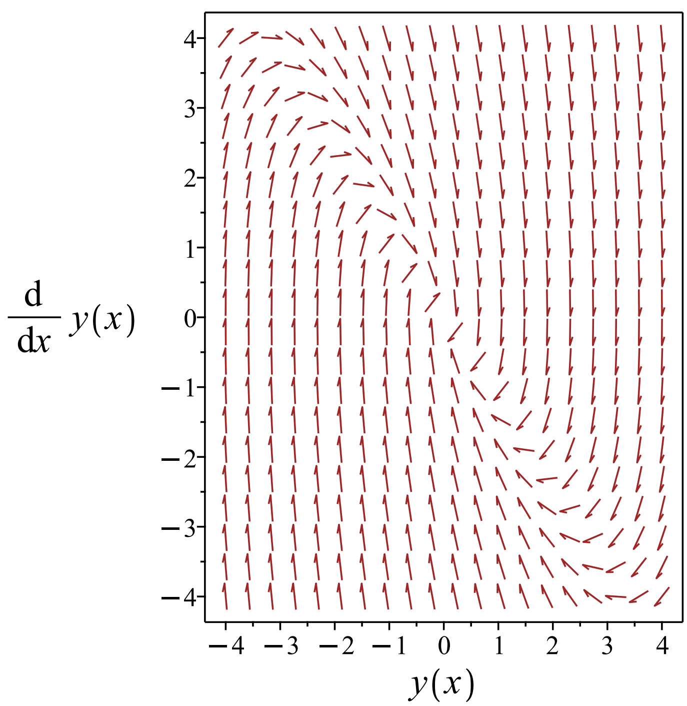


Figure 220: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-2x}(\cos(x)\, c_1 + c_2 \sin(x)) + \mathrm{e}^{3x}$$

Verified OK.

8.13.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = 26\,\mathrm{e}^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2-\mathrm{I}, -2+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2x}\cos(x)\, c_1 + \mathrm{e}^{-2x}\sin(x)\, c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 26\,\mathrm{e}^{3x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x}\cos(x) & \sin(x)\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x}\cos(x) - \sin(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\cos(x) - 2\sin(x)\,\mathrm{e}^{-2x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -26\,e^{-2x}\left(\cos(x)\left(\int \sin(x)\,e^{5x}dx\right) - \sin(x)\left(\int \cos(x)\,e^{5x}dx\right)\right)$
- Compute integrals

  $y_p(x) = e^{3x}$

- Substitute particular solution into general solution to ODE

  $y = e^{-2x}\cos(x)\,c_1 + e^{-2x}\sin(x)\,c_2 + e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=26*exp(3*x),y(x), singsol=all)
```

$$y(x) = \left(e^{5x} + c_2\sin(x) + \cos(x)\,c_1\right)e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 27

```
DSolve[y''[x]+4*y'[x]+5*y[x]==26*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(e^{5x} + c_2\cos(x) + c_1\sin(x)\right)$$

8.14 problem 14

8.14.1 Solving as second order linear constant coeff ode 1358
8.14.2 Solving using Kovacic algorithm 1361
8.14.3 Maple step by step solution 1366

Internal problem ID [4877]
Internal file name [`OUTPUT/4370_Sunday_June_05_2022_01_08_23_PM_94890598/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = 2\,\mathrm{e}^{-2x} \cos(x)}$$

8.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = 2\,\mathrm{e}^{-2x} \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda \,\mathrm{e}^{\lambda x} + 5 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2x}\cos\left(x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\cos\left(x\right), \sin\left(x\right)\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}\cos\left(x\right), \sin\left(x\right)\mathrm{e}^{-2x}\}$$

Since $\mathrm{e}^{-2x}\cos\left(x\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-2x}\cos\left(x\right), x\sin\left(x\right)\mathrm{e}^{-2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x\,\mathrm{e}^{-2x}\cos\left(x\right) + A_2x\sin\left(x\right)\mathrm{e}^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\mathrm{e}^{-2x}\sin\left(x\right) + 2A_2\cos\left(x\right)\mathrm{e}^{-2x} = 2\,\mathrm{e}^{-2x}\cos\left(x\right)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\sin\left(x\right)\mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2x}(\cos\left(x\right)c_1 + c_2\sin\left(x\right))\right) + \left(x\sin\left(x\right)\mathrm{e}^{-2x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + x \sin(x) e^{-2x} \tag{1}$$

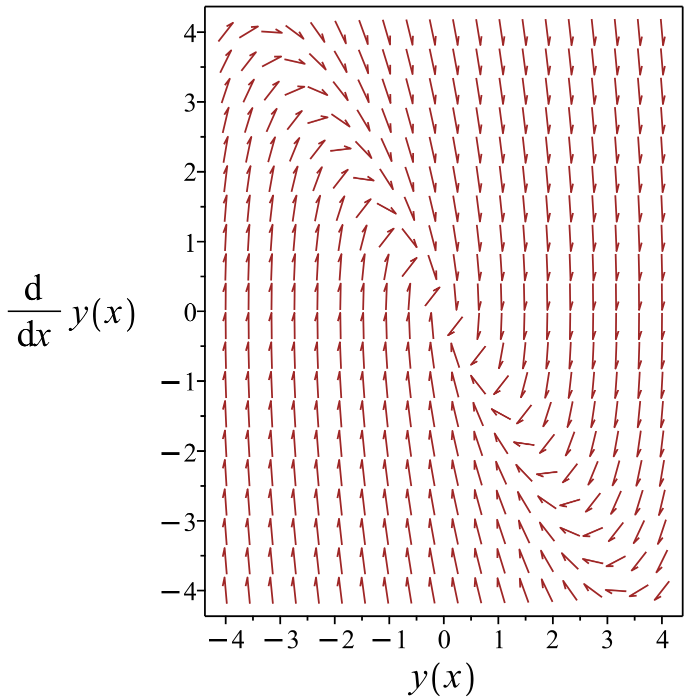


Figure 221: Slope field plot

Verification of solutions

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + x \sin(x) e^{-2x}$$

Verified OK.

8.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 190: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{4}{1} \, dx} \\
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-2x} \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2} \, dx \\
&= y_1(\tan(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x} \cos(x)\right) + c_2 \left(e^{-2x} \cos(x) (\tan(x))\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{-2x} \cos(x) c_1 + e^{-2x} \sin(x) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 e^{-2x} \cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\cos(x), \sin(x)\,e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}\cos(x), \sin(x)\,e^{-2x}\}$$

Since $e^{-2x}\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{-2x}\cos(x), x\sin(x)\,e^{-2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,e^{-2x}\cos(x) + A_2 x\sin(x)\,e^{-2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-2x}\sin(x) + 2A_2\cos(x)\,e^{-2x} = 2\,e^{-2x}\cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x\sin(x)\,e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}\cos(x)\,c_1 + e^{-2x}\sin(x)\,c_2\right) + \left(x\sin(x)\,e^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-2x}(\cos(x)\,c_1 + c_2\sin(x)) + x\sin(x)\,e^{-2x}$$

Summary

The solution(s) found are the following

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + x \sin(x) e^{-2x} \tag{1}$$

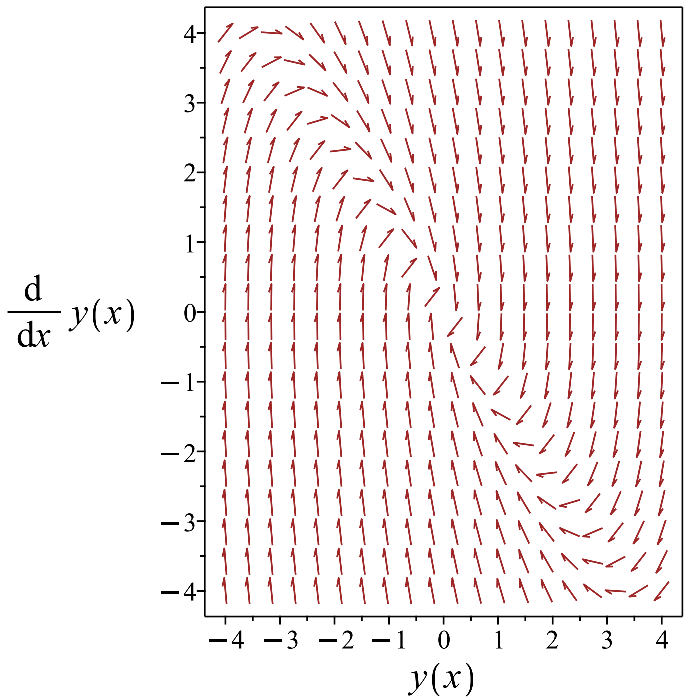


Figure 222: Slope field plot

Verification of solutions

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + x \sin(x) e^{-2x}$$

Verified OK.

8.14.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = 2 e^{-2x} \cos(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2-\mathrm{I}, -2+\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2x}\cos(x)\,c_1 + \mathrm{e}^{-2x}\sin(x)\,c_2 + y_p(x)$

- ☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,\mathrm{e}^{-2x}\cos(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x}\cos(x) & \sin(x)\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x}\cos(x) - \sin(x)\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x}\cos(x) - 2\sin(x)\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-2x}\left(\cos(x)\left(\int \sin(2x)\,dx\right) - 2\sin(x)\left(\int \cos(x)^2\,dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{(2\sin(x)x+\cos(x))\mathrm{e}^{-2x}}{2}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2x}\cos(x)\,c_1 + \mathrm{e}^{-2x}\sin(x)\,c_2 + \frac{(2\sin(x)x+\cos(x))\mathrm{e}^{-2x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=2*exp(-2*x)*cos(x),y(x), singsol=all)
```

$$y(x) = (\sin(x)(c_2 + x) + \cos(x) c_1) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 26

```
DSolve[y''[x]+4*y'[x]+5*y[x]==2*Exp[-2*x]*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}((1 + c_2)\cos(x) + (x + c_1)\sin(x))$$

8.15 problem 15

8.15.1 Solving as second order linear constant coeff ode 1369
8.15.2 Solving as linear second order ode solved by an integrating factor ode 1372
8.15.3 Solving using Kovacic algorithm 1374
8.15.4 Maple step by step solution 1379

Internal problem ID [4878]
Internal file name [`OUTPUT/4371_Sunday_June_05_2022_01_08_33_PM_56527564/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = 6\,\mathrm{e}^{2x}}$$

8.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 6\,\mathrm{e}^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-4)^2 - (4)\,(1)\,(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 e^{2x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + x\, e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x} x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x^2e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1e^{2x} = 6\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1e^{2x} + x\,e^{2x}c_2\right) + \left(3x^2e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2x + c_1) + 3x^2e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2x + c_1) + 3x^2e^{2x} \tag{1}$$

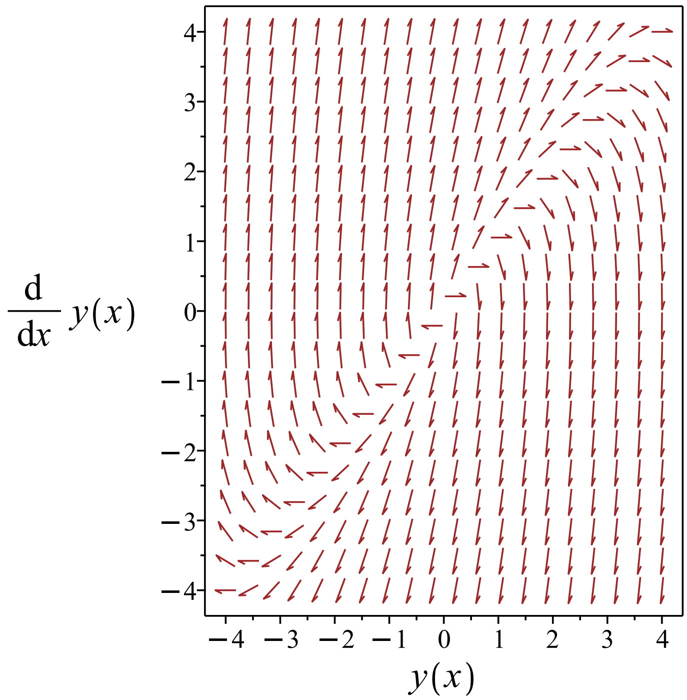


Figure 223: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + 3x^2e^{2x}$$

Verified OK.

8.15.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{(p(x)^2 + p'(x))\,y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 6\,\mathrm{e}^{-2x}\mathrm{e}^{2x}$$
$$\left(y\,\mathrm{e}^{-2x}\right)'' = 6\,\mathrm{e}^{-2x}\mathrm{e}^{2x}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{-2x}\right)' = 6x + c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{-2x}\right) = x(3x + c_1) + c_2$$

Hence the solution is

$$y = \frac{x(3x + c_1) + c_2}{\mathrm{e}^{-2x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{2x} + 3x^2\mathrm{e}^{2x} + c_2\mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{2x} + 3x^2\mathrm{e}^{2x} + c_2\mathrm{e}^{2x} \tag{1}$$

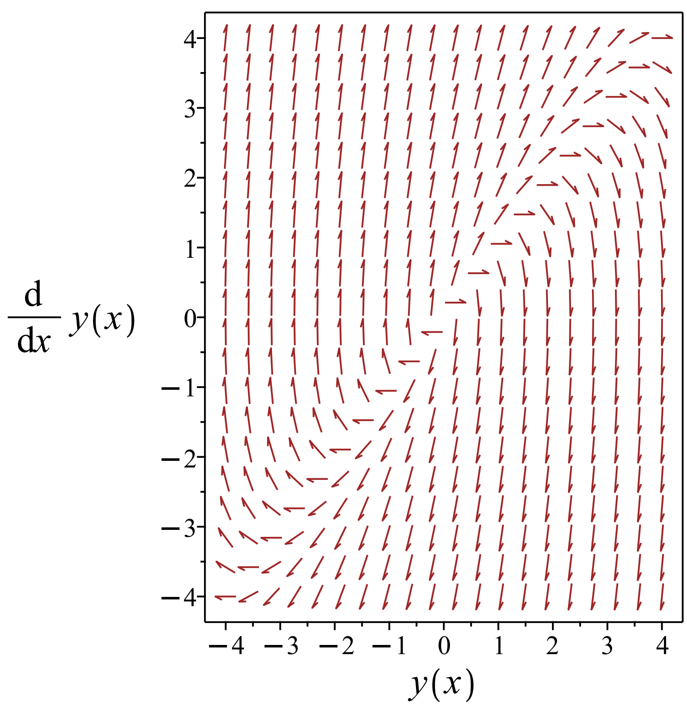


Figure 224: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{2x} + 3x^2 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x}$$

Verified OK.

8.15.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 192: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + x\,e^{2x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\,e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}x, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since $e^{2x}x$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} = 6\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x^2 e^{2x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{2x} + x\,\mathrm{e}^{2x} c_2\right) + \left(3x^2 \mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y = \mathrm{e}^{2x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{2x}
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{2x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{2x} \tag{1}
$$

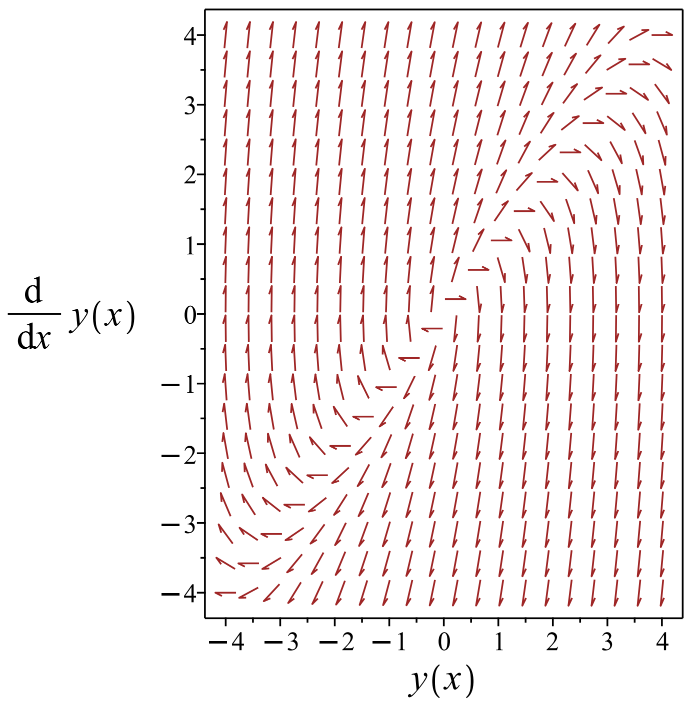


Figure 225: Slope field plot

Verification of solutions

$$
y = \mathrm{e}^{2x}(c_2 x + c_1) + 3x^2 \mathrm{e}^{2x}
$$

Verified OK.

8.15.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 6\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^{2x}x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\mathrm{e}^{2x} + x\,\mathrm{e}^{2x}c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 6\,\mathrm{e}^{2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{2x}x \\ 2\,\mathrm{e}^{2x} & 2\,\mathrm{e}^{2x}x + \mathrm{e}^{2x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -6\,\mathrm{e}^{2x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = 3x^2 e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{2x}c_2 + 3x^2 e^{2x} + c_1 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=6*exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 x + 3x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 23

```
DSolve[y''[x]-4*y'[x]+4*y[x]==6*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(3x^2 + c_2 x + c_1)$$

8.16 problem 16

8.16.1 Solving as second order linear constant coeff ode 1381
8.16.2 Solving using Kovacic algorithm 1384
8.16.3 Maple step by step solution 1389

Internal problem ID [4879]
Internal file name [`OUTPUT/4372_Sunday_June_05_2022_01_08_42_PM_75002421/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 5y' + 6y = e^{2x}}$$

8.16.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda\, \mathrm{e}^{\lambda x} + 6\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)} \\
&= \frac{5}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{5}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(2)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{2x}\right) + \left(-e^{2x} x\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{2x} - e^{2x} x \tag{1}$$

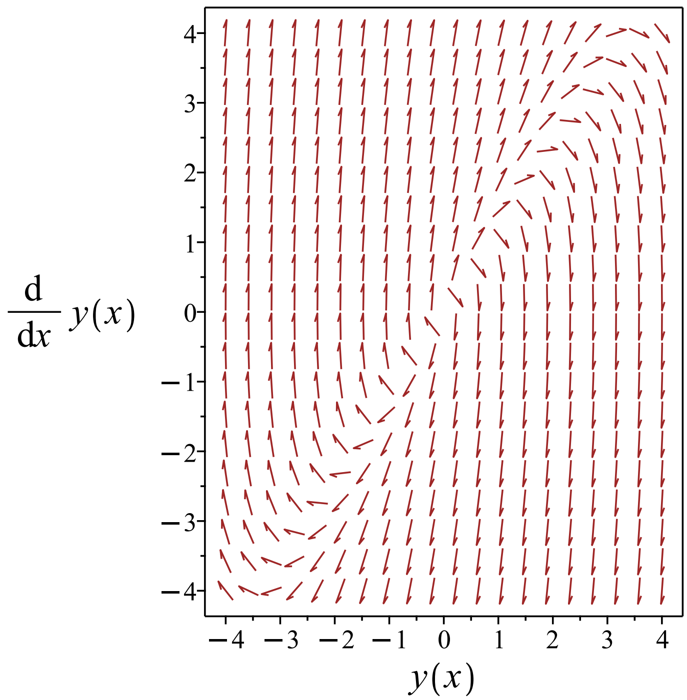


Figure 226: Slope field plot

Verification of solutions

$$y = e^{3x} c_1 + c_2 e^{2x} - e^{2x} x$$

Verified OK.

8.16.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 194: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx}\\ &= z_1 e^{\frac{5x}{2}}\\ &= z_1\left(\mathrm{e}^{\frac{5x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{5x}}{(y_1)^2} \, dx \\ &= y_1(e^x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x}(e^x)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + c_2 e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} = e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{2x} x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^{2x} + c_2 e^{3x}) + (-e^{2x} x) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{3x} - e^{2x} x \tag{1}$$

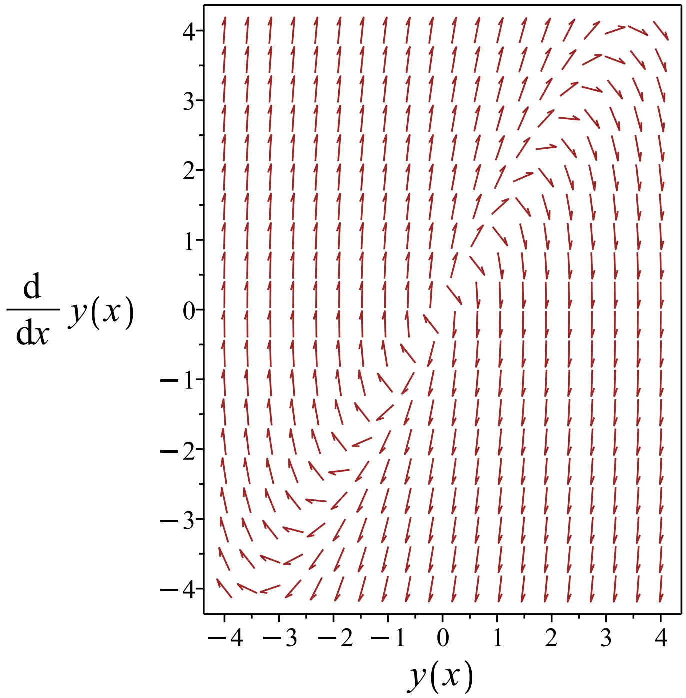


Figure 227: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{3x} - e^{2x} x$$

Verified OK.

8.16.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{3x} \\ 2\,\mathrm{e}^{2x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{5x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{2x} \left(\int 1 dx\right) + \mathrm{e}^{3x} \left(\int \mathrm{e}^{-x} dx\right)$

  - Compute integrals

    $y_p(x) = \mathrm{e}^{2x}(-1 - x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x} + \mathrm{e}^{2x}(-1 - x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=exp(2*x),y(x), singsol=all)
```

$$y(x) = (-x + c_1)\,\mathrm{e}^{2x} + c_2\mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 24

```
DSolve[y''[x]-5*y'[x]+6*y[x]==Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(-x + c_2e^x - 1 + c_1)$$

8.17 problem 17

8.17.1 Solving as homogeneousTypeD2 ode 1392
8.17.2 Solving as differentialType ode 1394
8.17.3 Solving as first order ode lie symmetry calculated ode 1396
8.17.4 Solving as exact ode 1401
8.17.5 Maple step by step solution 1405

Internal problem ID [4880]
Internal file name [`OUTPUT/4373_Sunday_June_05_2022_01_08_51_PM_47939461/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(2x+y)\,y'+2y=x}$$

8.17.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(2x+u(x)\,x)\,(u'(x)\,x+u(x))+2u(x)\,x=x$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2+4u-1}{x\,(u+2)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+4u-1}{u+2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^2+4u-1}{u+2}}\, du &= -\frac{1}{x}\, dx \\
\int \frac{1}{\frac{u^2+4u-1}{u+2}}\, du &= \int -\frac{1}{x}\, dx \\
\frac{\ln(u^2+4u-1)}{2} &= -\ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{u^2+4u-1} = \mathrm{e}^{-\ln(x)+c_2}
$$

Which simplifies to

$$
\sqrt{u^2+4u-1} = \frac{c_3}{x}
$$

Which simplifies to

$$
\sqrt{u(x)^2+4u(x)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

The solution is

$$
\sqrt{u(x)^2+4u(x)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\sqrt{\frac{y^2}{x^2}+\frac{4y}{x}-1} &= \frac{c_3 \mathrm{e}^{c_2}}{x} \\
\sqrt{\frac{y^2+4xy-x^2}{x^2}} &= \frac{c_3 \mathrm{e}^{c_2}}{x}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\sqrt{\frac{y^2+4xy-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}
$$

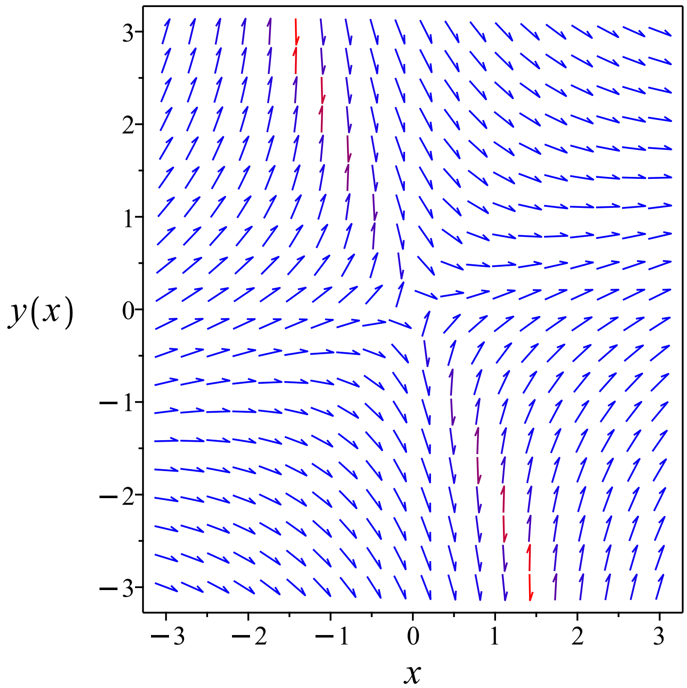


Figure 228: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2+4xy-x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

8.17.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x-2y}{2x+y} \tag{1}$$

Which becomes

$$(y)\,dy = (-2x)\,dy + (x-2y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\,dy + (x-2y)\,dx = d\left(\frac{1}{2}x^2 - 2xy\right)$$

Hence (2) becomes

$$(y)\,dy = d\left(\frac{1}{2}x^2 - 2xy\right)$$

Integrating both sides gives gives these solutions

$$y = -2x + \sqrt{5x^2 + 2c_1} + c_1$$
$$y = -2x - \sqrt{5x^2 + 2c_1} + c_1$$

Summary
The solution(s) found are the following

$$y = -2x + \sqrt{5x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -2x - \sqrt{5x^2 + 2c_1} + c_1 \tag{2}$$

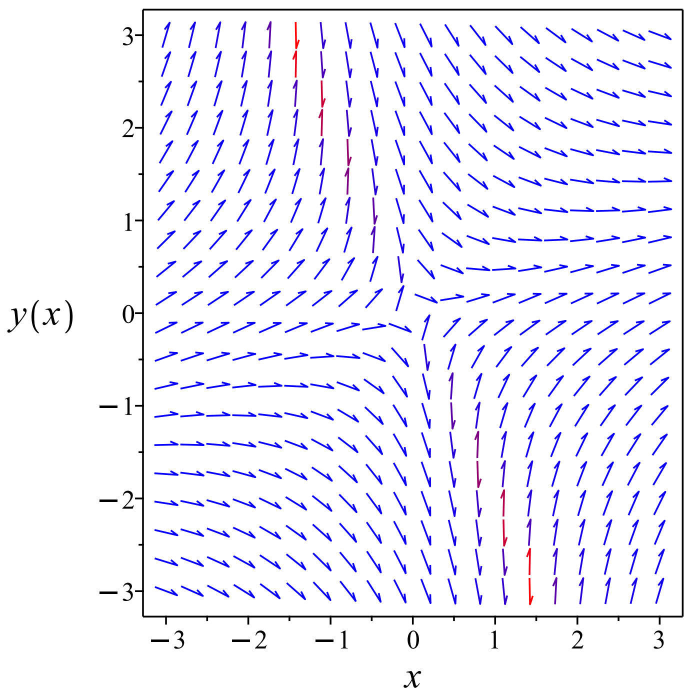


Figure 229: Slope field plot

Verification of solutions

$$y = -2x + \sqrt{5x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -2x - \sqrt{5x^2 + 2c_1} + c_1$$

Verified OK.

8.17.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{-x+2y}{y+2x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(-x+2y)(b_3 - a_2)}{y+2x} - \frac{(-x+2y)^2 a_3}{(y+2x)^2} \\
&- \left(\frac{1}{y+2x} + \frac{-2x+4y}{(y+2x)^2}\right)(x a_2 + y a_3 + a_1) \\
&- \left(-\frac{2}{y+2x} + \frac{-x+2y}{(y+2x)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^2 a_2 + x^2 a_3 - 9x^2 b_2 - 2x^2 b_3 + 2xy a_2 - 4xy a_3 - 4xy b_2 - 2xy b_3 - 2y^2 a_2 + 9y^2 a_3 - y^2 b_2 + 2y^2 b_3 - 5xb}{(y+2x)^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-2x^2 a_2 - x^2 a_3 + 9x^2 b_2 + 2x^2 b_3 - 2xy a_2 + 4xy a_3 + 4xy b_2 \\
&+ 2xy b_3 + 2y^2 a_2 - 9y^2 a_3 + y^2 b_2 - 2y^2 b_3 + 5x b_1 - 5y a_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_2v_1^2 - 2a_2v_1v_2 + 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 9a_3v_2^2 + 9b_2v_1^2 \\ &+ 4b_2v_1v_2 + b_2v_2^2 + 2b_3v_1^2 + 2b_3v_1v_2 - 2b_3v_2^2 - 5a_1v_2 + 5b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 - a_3 + 9b_2 + 2b_3)\, v_1^2 + (-2a_2 + 4a_3 + 4b_2 + 2b_3)\, v_1v_2 \\ &+ 5b_1v_1 + (2a_2 - 9a_3 + b_2 - 2b_3)\, v_2^2 - 5a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -5a_1 &= 0 \\ 5b_1 &= 0 \\ -2a_2 - a_3 + 9b_2 + 2b_3 &= 0 \\ -2a_2 + 4a_3 + 4b_2 + 2b_3 &= 0 \\ 2a_2 - 9a_3 + b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 4b_2 + b_3 \\ a_3 &= b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(-\frac{-x + 2y}{y + 2x}\right)(x) \\
&= \frac{-x^2 + 4xy + y^2}{y + 2x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2 + 4xy + y^2}{y + 2x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 + 4xy + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x+2y}{y+2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x-2y}{x^2-4xy-y^2} \\
S_y &= \frac{-y-2x}{x^2-4xy-y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

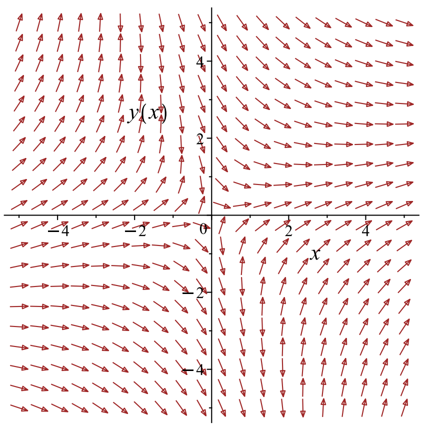
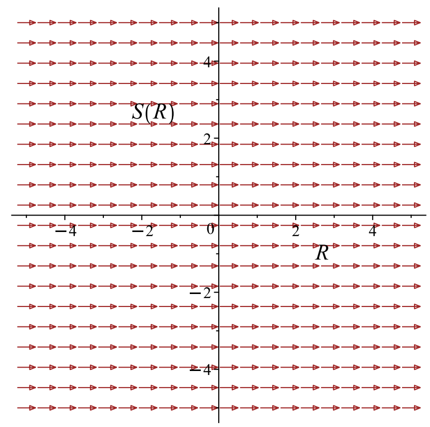
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2+4xy-x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y^2+4xy-x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x+2y}{y+2x}$ 	$R = x$ $S = \frac{\ln(-x^2+4xy+y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(y^2+4xy-x^2)}{2} = c_1 \tag{1}$$

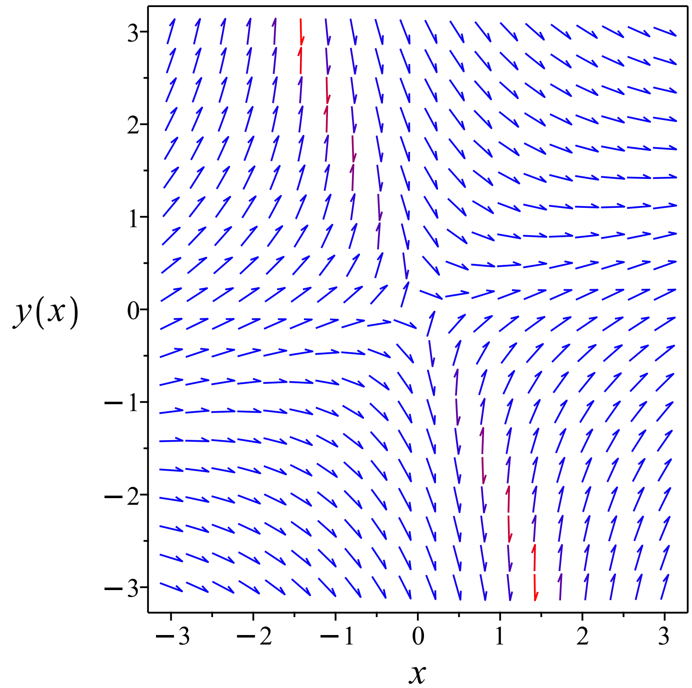


Figure 230: Slope field plot

Verification of solutions

$$\frac{\ln(y^2+4xy-x^2)}{2} = c_1$$

Verified OK.

8.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M \\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(y+2x)\,\mathrm{d}y &= (x-2y)\,\mathrm{d}x \\ (-x+2y)\,\mathrm{d}x + (y+2x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x+2y \\ N(x,y) &= y+2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x+2y) \\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y+2x) \\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M & (1) \\
\frac{\partial \phi}{\partial y} &= N & (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x + 2y \, \mathrm{d}x \\
\phi &= -\frac{x(x - 4y)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y + 2x$. Therefore equation (4) becomes

$$
y + 2x = 2x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (y) \, \mathrm{d}y \\
f(y) &= \frac{y^2}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x-4y)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x-4y)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(x-4y)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

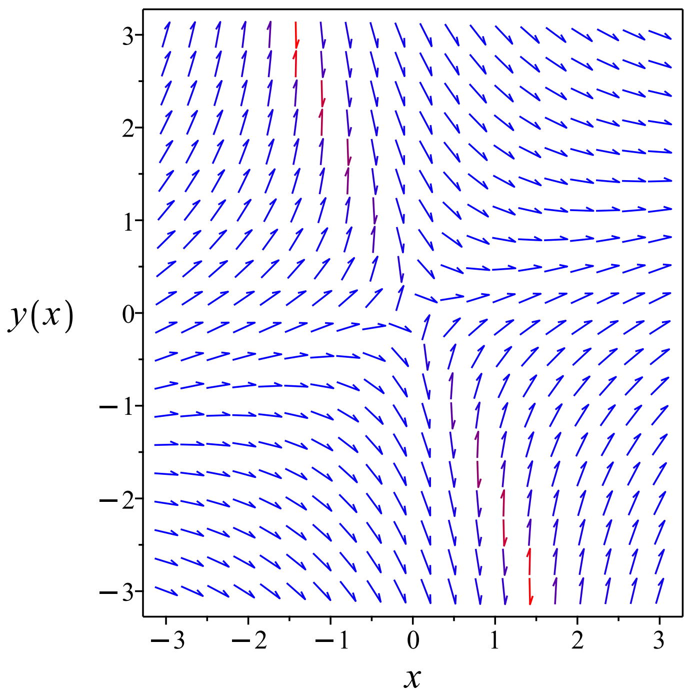


Figure 231: Slope field plot

Verification of solutions

$$-\frac{x(x-4y)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

8.17.5 Maple step by step solution

Let's solve

$(2x+y)\,y'+2y=x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $2=2$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y)\,, N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(-x+2y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-\frac{x^2}{2}+2xy+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $y+2x=2x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$F(x,y) = -\frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $-\frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2 = c_1$

- Solve for y

  $\left\{y = -2x - \sqrt{5x^2 + 2c_1}, y = -2x + \sqrt{5x^2 + 2c_1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 51

```
dsolve((2*x+y(x))*diff(y(x),x)-(x-2*y(x))=0,y(x), singsol=all)
```

$$y(x) = \frac{-2c_1x - \sqrt{5c_1^2x^2 + 1}}{c_1}$$
$$y(x) = \frac{-2c_1x + \sqrt{5c_1^2x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.458 (sec). Leaf size: 94

```
DSolve[(2*x+y[x])*y'[x]-(x-2*y[x])==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x - \sqrt{5x^2 + e^{2c_1}}$$
$$y(x) \to -2x + \sqrt{5x^2 + e^{2c_1}}$$
$$y(x) \to -\sqrt{5}\sqrt{x^2} - 2x$$
$$y(x) \to \sqrt{5}\sqrt{x^2} - 2x$$

8.18 problem 18

8.18.1 Solving as exact ode 1408

Internal problem ID [4881]
Internal file name [`OUTPUT/4374_Sunday_June_05_2022_01_09_00_PM_14549309/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x)*G(y),0]`]]
```

$$\boxed{\left(\cos\left(y\right)x-\mathrm{e}^{-\sin\left(y\right)}\right)y'=-1}$$

8.18.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\cos(y)\,x - \mathrm{e}^{-\sin(y)}\right)\mathrm{d}y &= (-1)\,\mathrm{d}x \\
(1)\,\mathrm{d}x + \left(\cos(y)\,x - \mathrm{e}^{-\sin(y)}\right)\mathrm{d}y &= 0 \qquad (2\mathrm{A})
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 1 \\
N(x, y) &= \cos(y)\,x - \mathrm{e}^{-\sin(y)}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1) \\
&= 0
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\cos(y)\,x - \mathrm{e}^{-\sin(y)}\right) \\
&= \cos(y)
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{\cos(y)\,x - \mathrm{e}^{-\sin(y)}}((0) - (\cos(y))) \\
&= \frac{\cos(y)}{-\cos(y)\,x + \mathrm{e}^{-\sin(y)}}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= 1((\cos(y)) - (0)) \\
&= \cos(y)
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int \cos(y) \,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\sin(y)} \\
&= \mathrm{e}^{\sin(y)}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{\sin(y)}(1) \\
&= \mathrm{e}^{\sin(y)}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{\sin(y)}\left(\cos(y)\,x - \mathrm{e}^{-\sin(y)}\right) \\
&= \cos(y)\,x\,\mathrm{e}^{\sin(y)} - 1
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\mathrm{e}^{\sin(y)}\right) + \left(\cos\left(y\right) x\,\mathrm{e}^{\sin(y)} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \mathrm{e}^{\sin(y)}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{\sin(y)}x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = \cos\left(y\right) x\,\mathrm{e}^{\sin(y)} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \cos\left(y\right) x\,\mathrm{e}^{\sin(y)} - 1$. Therefore equation (4) becomes

$$
\cos\left(y\right) x\,\mathrm{e}^{\sin(y)} - 1 = \cos\left(y\right) x\,\mathrm{e}^{\sin(y)} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (-1)\,\mathrm{d}y \\
f(y) &= -y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = e^{\sin(y)} x - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = e^{\sin(y)} x - y$$

Summary

The solution(s) found are the following

$$e^{\sin(y)} x - y = c_1 \tag{1}$$

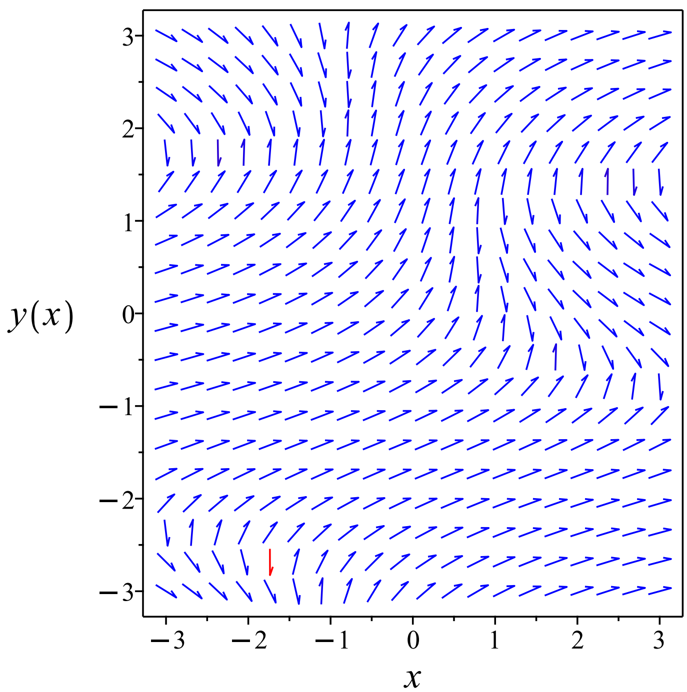


Figure 232: Slope field plot

Verification of solutions

$$e^{\sin(y)} x - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 20

```
dsolve((x*cos(y(x)) - exp(-sin(y(x))))*diff(y(x),x)+1=0,y(x), singsol=all)
```

$$(-y(x) - c_1)\, e^{-\sin(y(x))} + x = 0$$

✓ Solution by Mathematica

Time used: 0.734 (sec). Leaf size: 26

```
DSolve[(x*Cos[y[x]] - Exp[-Sin[y[x]]])*y'[x]+1==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x = y(x)e^{-\sin(y(x))} + c_1 e^{-\sin(y(x))}, y(x)\right]$$

8.19 problem 19

8.19.1 Solving as linear ode 1414
8.19.2 Solving as homogeneousTypeD2 ode 1416
8.19.3 Solving as first order ode lie symmetry lookup ode 1418
8.19.4 Solving as exact ode 1422
8.19.5 Maple step by step solution 1426

Internal problem ID [4882]
Internal file name [`OUTPUT/4375_Sunday_June_05_2022_01_09_13_PM_2439457/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' \sin(x)^2 + (x+y)\sin(2x) = -\sin(x)^2}$$

8.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2\cot(x)$$
$$q(x) = -1 - 2x\cot(x)$$

Hence the ode is

$$y' + 2y\cot(x) = -1 - 2x\cot(x)$$

The integrating factor μ is

$$\begin{aligned}
\mu &= \mathrm{e}^{\int 2\cot(x)dx} \\
&= \sin\left(x\right)^{2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-1-2x\cot\left(x\right)\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(x\right)^{2} y\right) &= \left(\sin\left(x\right)^{2}\right)\left(-1-2x\cot\left(x\right)\right) \\
\mathrm{d}\left(\sin\left(x\right)^{2} y\right) &= \left(\left(-1-2x\cot\left(x\right)\right)\sin\left(x\right)^{2}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sin\left(x\right)^{2} y &= \int \left(-1-2x\cot\left(x\right)\right)\sin\left(x\right)^{2}\,\mathrm{d}x \\
\sin\left(x\right)^{2} y &= -x+\cos\left(x\right)^{2} x+c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin\left(x\right)^{2}$ results in

$$y = \csc\left(x\right)^{2}\left(-x+\cos\left(x\right)^{2} x\right)+c_1\csc\left(x\right)^{2}$$

which simplifies to

$$y = -x+c_1\csc\left(x\right)^{2}$$

Summary

The solution(s) found are the following

$$y = -x+c_1\csc\left(x\right)^{2} \tag{1}$$

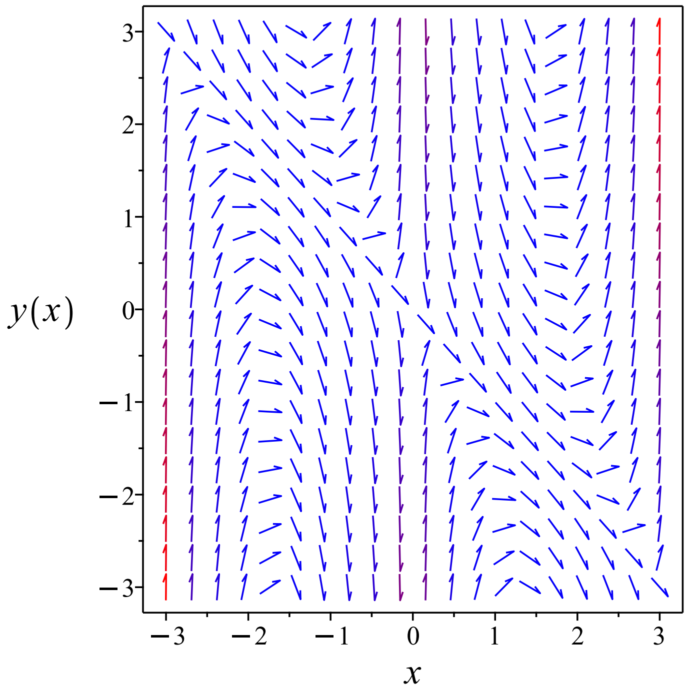


Figure 233: Slope field plot

Verification of solutions

$$y = -x + c_1 \csc(x)^2$$

Verified OK.

8.19.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\sin(x)^2 + (x + u(x)\,x)\sin(2x) = -\sin(x)^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{\left(1 + x\left(-\tan\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right)\right)(1+u)}{x} \end{aligned}$$

Where $f(x) = -\frac{1+x\left(-\tan\left(\frac{x}{2}\right)+\cot\left(\frac{x}{2}\right)\right)}{x}$ and $g(u) = 1 + u$. Integrating both sides gives

$$\frac{1}{1+u}\,du = -\frac{1 + x\left(-\tan\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)\right)}{x}\,dx$$

$$\int \frac{1}{1+u}\,du = \int -\frac{1+x\left(-\tan\left(\frac{x}{2}\right)+\cot\left(\frac{x}{2}\right)\right)}{x}\,dx$$

$$\ln(1+u) = -2\ln\left(\cos\left(\frac{x}{2}\right)\right) - 2\ln\left(\sin\left(\frac{x}{2}\right)\right) - \ln\left(\frac{x}{2}\right) + c_2$$

Raising both side to exponential gives

$$1+u = \mathrm{e}^{-2\ln\left(\cos\left(\frac{x}{2}\right)\right)-2\ln\left(\sin\left(\frac{x}{2}\right)\right)-\ln\left(\frac{x}{2}\right)+c_2}$$

Which simplifies to

$$1+u = c_3\mathrm{e}^{-2\ln\left(\cos\left(\frac{x}{2}\right)\right)-2\ln\left(\sin\left(\frac{x}{2}\right)\right)-\ln\left(\frac{x}{2}\right)}$$

Which simplifies to

$$u(x) = \frac{2c_3\mathrm{e}^{c_2}}{\cos\left(\frac{x}{2}\right)^2\sin\left(\frac{x}{2}\right)^2 x} - 1$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x\left(\frac{2c_3\mathrm{e}^{c_2}}{\cos\left(\frac{x}{2}\right)^2\sin\left(\frac{x}{2}\right)^2 x} - 1\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x\left(\frac{2c_3\mathrm{e}^{c_2}}{\cos\left(\frac{x}{2}\right)^2\sin\left(\frac{x}{2}\right)^2 x} - 1\right) \tag{1}$$

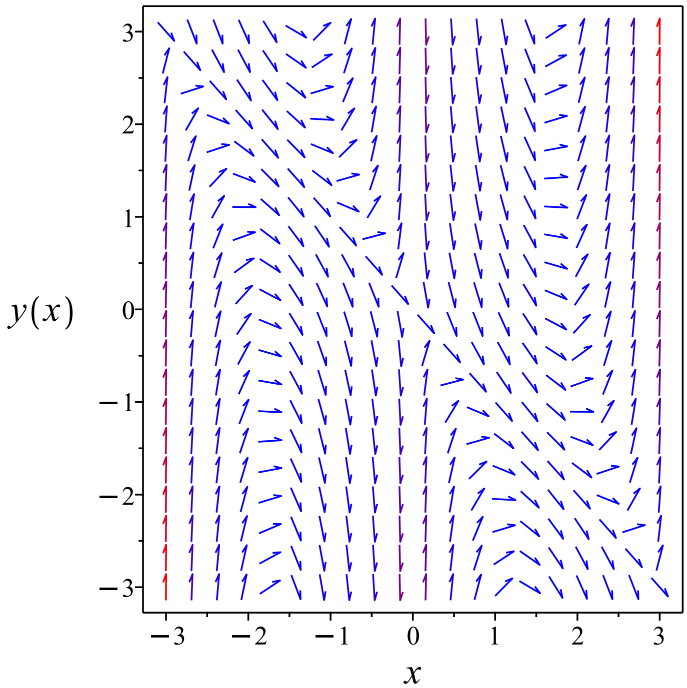


Figure 234: Slope field plot

Verification of solutions

$$y = x\left(\frac{2c_3 e^{c_2}}{\cos\left(\frac{x}{2}\right)^2 \sin\left(\frac{x}{2}\right)^2 x} - 1\right)$$

Verified OK.

8.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x\sin(2x) + \sin(2x)\, y + \sin(x)^2}{\sin(x)^2}$$

$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 197: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0\\
\eta(x,y) &= \frac{1}{\sin\left(x\right)^{2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)^2}} dy \end{aligned}$$

Which results in

$$S = \sin(x)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x \sin(2x) + \sin(2x) y + \sin(x)^2}{\sin(x)^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sin(2x) y \\ S_y &= \sin(x)^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(x)^2 - x \sin(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(R)^2 - R \sin(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 + \frac{R\cos(2R)}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)^2 y = \frac{x\cos(2x)}{2} + c_1 - \frac{x}{2}$$

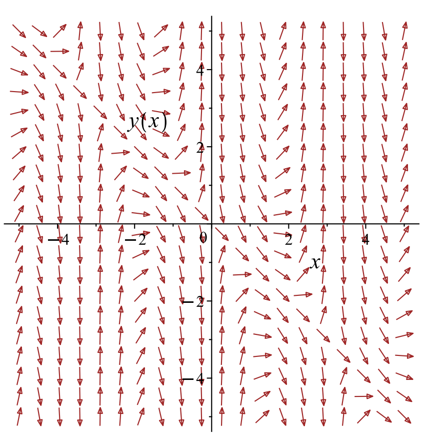
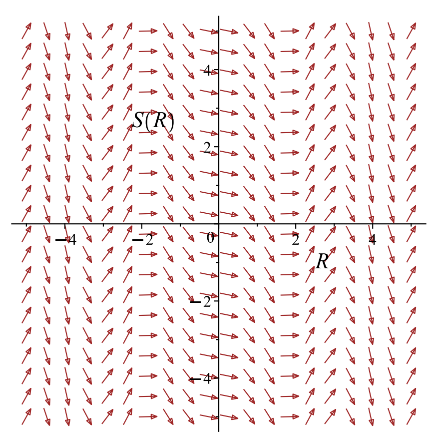
Which simplifies to

$$\sin(x)^2 (x + y) - c_1 = 0$$

Which gives

$$y = -\frac{\sin(x)^2 x - c_1}{\sin(x)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x\sin(2x)+\sin(2x)y+\sin(x)^2}{\sin(x)^2}$ 	$R = x$ $S = \sin(x)^2 y$	$\frac{dS}{dR} = -\sin(R)^2 - R\sin(2R)$ 

Summary
The solution(s) found are the following

$$y = -\frac{\sin(x)^2 x - c_1}{\sin(x)^2} \tag{1}$$

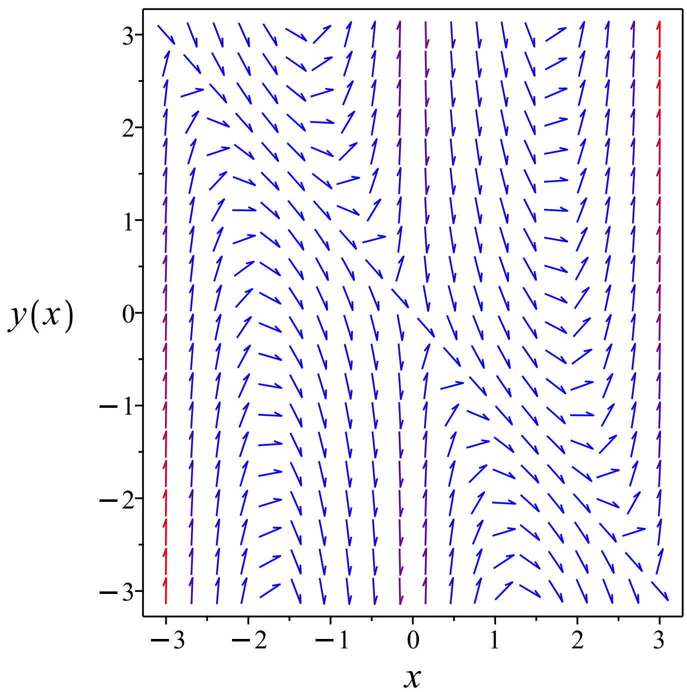


Figure 235: Slope field plot

Verification of solutions

$$y = -\frac{\sin(x)^2 x - c_1}{\sin(x)^2}$$

Verified OK.

8.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sin(x)^2\right)\mathrm{d}y&=\left(-\sin(x)^2-(x+y)\sin(2x)\right)\mathrm{d}x\\ \left(\sin(x)^2+(x+y)\sin(2x)\right)\mathrm{d}x+\left(\sin(x)^2\right)\mathrm{d}y&=0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\sin(x)^2+(x+y)\sin(2x)\\ N(x,y)&=\sin(x)^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sin\left(x\right)^{2}+\left(x+y\right)\sin\left(2x\right)\right) \\
&= \sin\left(2x\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin\left(x\right)^{2}\right) \\
&= \sin\left(2x\right)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \sin\left(x\right)^{2}+\left(x+y\right)\sin\left(2x\right)\mathrm{d}x \\
\phi &= \left(-y-x\right)\cos\left(x\right)^{2}+\frac{y}{2}+x+f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\cos\left(x\right)^{2}+\frac{1}{2}+f'(y) \qquad (4) \\
&= -\frac{\cos\left(2x\right)}{2}+f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\sin\left(x\right)^{2}$. Therefore equation (4) becomes

$$
\sin\left(x\right)^{2} = -\frac{\cos\left(2x\right)}{2}+f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
\begin{aligned}
f'(y) &= \frac{\cos(2x)}{2} + \sin(x)^2 \\
&= \frac{1}{2}
\end{aligned}
$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{2}\right) \mathrm{d}y \\
f(y) &= \frac{y}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = (-y - x)\cos(x)^2 + y + x + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = (-y - x)\cos(x)^2 + y + x
$$

The solution becomes

$$
y = -\frac{\cos(x)^2 x + c_1 - x}{-1 + \cos(x)^2}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\cos(x)^2 x + c_1 - x}{-1 + \cos(x)^2} \tag{1}
$$

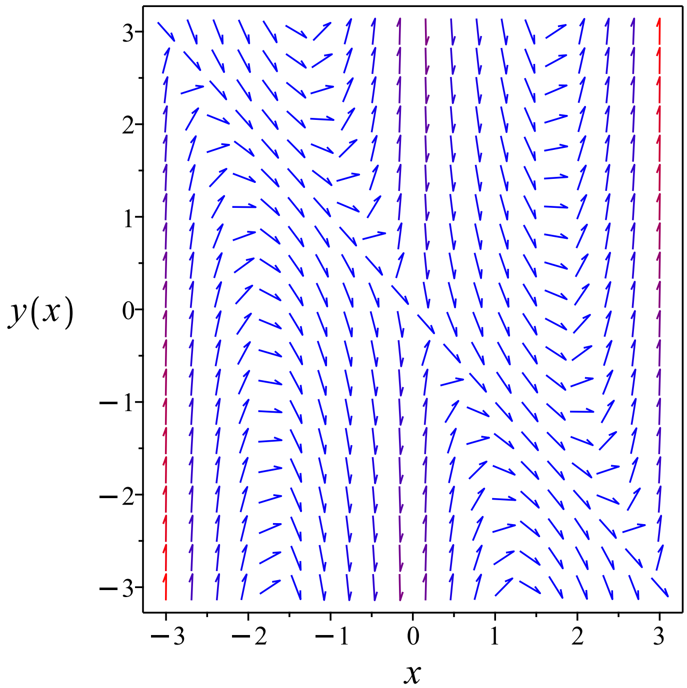


Figure 236: Slope field plot

Verification of solutions

$$y = -\frac{\cos(x)^2 x + c_1 - x}{-1 + \cos(x)^2}$$

Verified OK.

8.19.5 Maple step by step solution

Let's solve

$y' \sin(x)^2 + (x + y)\sin(2x) = -\sin(x)^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{\sin(2x)y}{\sin(x)^2} - \frac{\sin(x)^2 + x\sin(2x)}{\sin(x)^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{\sin(2x)y}{\sin(x)^2} = -\frac{\sin(x)^2 + x\sin(2x)}{\sin(x)^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{\sin(2x)y}{\sin(x)^2}\right)=-\frac{\mu(x)\left(\sin(x)^2+x\sin(2x)\right)}{\sin(x)^2}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{\sin(2x)y}{\sin(x)^2}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)\sin(2x)}{\sin(x)^2}$
- Solve to find the integrating factor

  $\mu(x)=\sin(x)^2$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int-\frac{\mu(x)\left(\sin(x)^2+x\sin(2x)\right)}{\sin(x)^2}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int-\frac{\mu(x)\left(\sin(x)^2+x\sin(2x)\right)}{\sin(x)^2}dx+c_1$
- Solve for y

  $y=\frac{\int-\frac{\mu(x)\left(\sin(x)^2+x\sin(2x)\right)}{\sin(x)^2}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\sin(x)^2$

  $y=\frac{\int\left(-\sin(x)^2-x\sin(2x)\right)dx+c_1}{\sin(x)^2}$
- Evaluate the integrals on the rhs

  $y=\frac{\frac{\cos(x)\sin(x)}{2}-\frac{x}{2}-\frac{\sin(2x)}{4}+\frac{x\cos(2x)}{2}+c_1}{\sin(x)^2}$
- Simplify

  $y=-x+c_1\csc(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(sin(x)^2*diff(y(x),x)+(sin(x)^2+(x+y(x))*sin(2*x))=0,y(x), singsol=all)
```

$$y(x) = -\frac{2c_1}{-1 + \cos(2x)} - x$$

✓ Solution by Mathematica

Time used: 0.056 (sec). Leaf size: 27

```
DSolve[Sin[x]^2*y'[x]+(Sin[x]^2+(x+y[x])*Sin[2*x])==0,y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{2}\csc^2(x)(-x + x\cos(2x) + 2c_1)$$

8.20 problem 20

8.20.1 Solving as second order linear constant coeff ode 1429
8.20.2 Solving using Kovacic algorithm 1432
8.20.3 Maple step by step solution 1437

Internal problem ID [4883]
Internal file name [`OUTPUT/4376_Sunday_June_05_2022_01_09_28_PM_80328307/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 5y = 5x + 4\,\mathrm{e}^{x}(1 + \sin(2x))}$$

8.20.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 5, f(x) = 4\,\mathrm{e}^{x}\sin(2x) + 4\,\mathrm{e}^{x} + 5x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} + 5\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(5)}\\ &= 1 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + 2i\\ \lambda_2 &= 1 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{x} \sin\left(2x\right)+4\,\mathrm{e}^{x}+5x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}, \{1, x\}, \{\mathrm{e}^{x} \cos\left(2x\right), \mathrm{e}^{x} \sin\left(2x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x} \cos\left(2x\right), \mathrm{e}^{x} \sin\left(2x\right)\}$$

Since $\mathrm{e}^{x} \cos\left(2x\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{x}\}, \{1, x\}, \{x\,\mathrm{e}^{x} \cos\left(2x\right), x\,\mathrm{e}^{x} \sin\left(2x\right)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\mathrm{e}^{x}+A_2+A_3x+A_4x\,\mathrm{e}^{x} \cos\left(2x\right)+A_5x\,\mathrm{e}^{x} \sin\left(2x\right)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\mathrm{e}^{x}-4A_4\mathrm{e}^{x} \sin\left(2x\right)+4A_5\mathrm{e}^{x} \cos\left(2x\right)-2A_3+5A_2+5A_3x = 4\,\mathrm{e}^{x} \sin\left(2x\right)+4\,\mathrm{e}^{x}+5x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{2}{5}, A_3 = 1, A_4 = -1, A_5 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{x}+\frac{2}{5}+x-x\,\mathrm{e}^{x} \cos\left(2x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{x} \left(c_1 \cos\left(2x\right)+c_2 \sin\left(2x\right)\right)\right) + \left(\mathrm{e}^{x}+\frac{2}{5}+x-x\,\mathrm{e}^{x} \cos\left(2x\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{x} + \frac{2}{5} + x - x\,e^{x} \cos(2x) \tag{1}$$

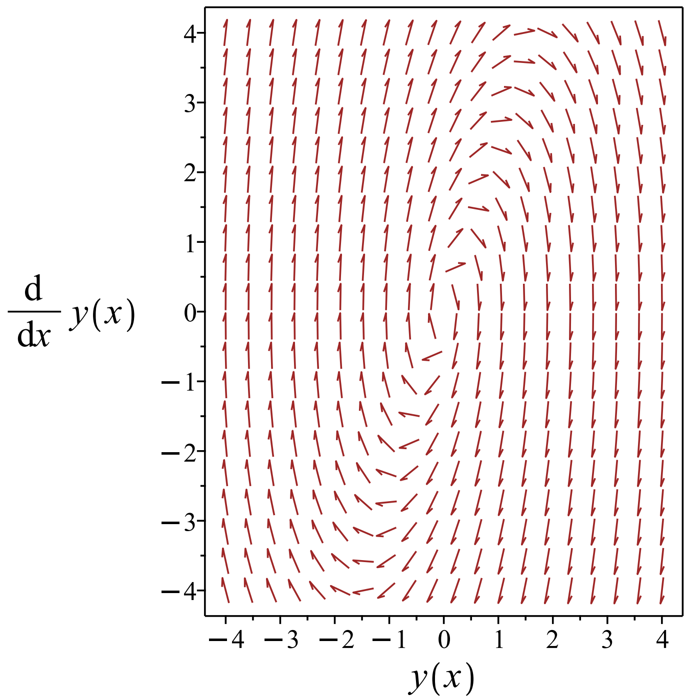


Figure 237: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(2x) + c_2 \sin(2x)) + e^{x} + \frac{2}{5} + x - x\,e^{x} \cos(2x)$$

Verified OK.

8.20.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= -2 \\
C &= 5
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 200: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = e^x \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 (e^x \cos(2x)) + c_2 \left(e^x \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 5y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^x \cos(2x)\, c_1 + \frac{e^x \sin(2x)\, c_2}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\, e^x \sin(2x) + 4\, e^x + 5x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}, \{1, x\}, \{e^x \cos(2x), e^x \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x \cos(2x), \frac{e^x \sin(2x)}{2}\right\}$$

Since $e^x \cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^x\}, \{1, x\}, \{x\, e^x \cos(2x), x\, e^x \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^x + A_2 + A_3 x + A_4 x\, e^x \cos(2x) + A_5 x\, e^x \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 e^x - 4A_4 e^x \sin(2x) + 4A_5 e^x \cos(2x) - 2A_3 + 5A_2 + 5A_3 x = 4\, e^x \sin(2x) + 4\, e^x + 5x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{2}{5}, A_3 = 1, A_4 = -1, A_5 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^x + \frac{2}{5} + x - x\, e^x \cos(2x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x \cos(2x)\, c_1 + \frac{e^x \sin(2x)\, c_2}{2}\right) + \left(e^x + \frac{2}{5} + x - x\, e^x \cos(2x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{x} \cos(2x) c_1 + \frac{e^{x} \sin(2x) c_2}{2} + e^{x} + \frac{2}{5} + x - x e^{x} \cos(2x) \tag{1}$$

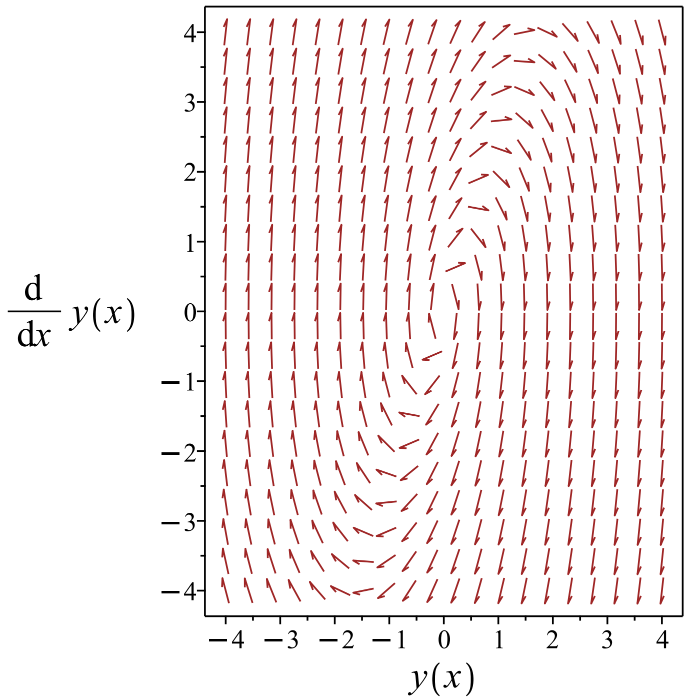


Figure 238: Slope field plot

Verification of solutions

$$y = e^{x} \cos(2x) c_1 + \frac{e^{x} \sin(2x) c_2}{2} + e^{x} + \frac{2}{5} + x - x e^{x} \cos(2x)$$

Verified OK.

8.20.3 Maple step by step solution

Let's solve

$y'' - 2y' + 5y = 4 e^{x} \sin(2x) + 4 e^{x} + 5x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r = (1 - 2\,\mathrm{I}, 1 + 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x \cos(2x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^x \sin(2x)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^x \cos(2x)\, c_1 + \mathrm{e}^x \sin(2x)\, c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,\mathrm{e}^x \sin(2x) + 4\,\mathrm{e}^x + 5\right.$
- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x \cos(2x) & \mathrm{e}^x \sin(2x) \\ \mathrm{e}^x \cos(2x) - 2\,\mathrm{e}^x \sin(2x) & \mathrm{e}^x \sin(2x) + 2\,\mathrm{e}^x \cos(2x) \end{bmatrix}$
- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{2x}$
- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^x\left(-\cos(2x)\left(\int \sin(2x)(4\sin(2x)+5x\,\mathrm{e}^{-x}+4)dx\right)+\sin(2x)\left(\int \cos(2x)(4\sin(2x)+5x\,\mathrm{e}^{-x}+4)dx\right)\right)}{2}$
- ○ Compute integrals

  $y_p(x) = -\mathrm{e}^x(x-1)\cos(2x) + \frac{\mathrm{e}^x \sin(2x)}{2} + x + \mathrm{e}^x + \frac{2}{5}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x \cos(2x)\, c_1 + \mathrm{e}^x \sin(2x)\, c_2 - \mathrm{e}^x(x-1)\cos(2x) + \frac{\mathrm{e}^x \sin(2x)}{2} + x + \mathrm{e}^x + \frac{2}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+5*y(x)=5*x+4*exp(x)*(1+sin(2*x)),y(x), singsol=all)
```

$$y(x) = \frac{2}{5} - e^{x}(x - c_1 - 1)\cos(2x) + \frac{(2c_2 + 1)\, e^{x} \sin(2x)}{2} + x + e^{x}$$

✓ Solution by Mathematica

Time used: 1.313 (sec). Leaf size: 45

```
DSolve[y''[x]-2*y'[x]+5*y[x]==5*x+4*Exp[x]*(1+Sin[2*x]),y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to x + e^x - e^x(x - c_2)\cos(2x) + \frac{1}{4}(1 + 4c_1)e^x \sin(2x) + \frac{2}{5}$$

8.21 problem 21

8.21.1 Solving as separable ode 1440
8.21.2 Solving as first order ode lie symmetry lookup ode 1442
8.21.3 Solving as bernoulli ode 1446
8.21.4 Solving as exact ode 1449
8.21.5 Maple step by step solution 1453

Internal problem ID [4884]
Internal file name [`OUTPUT/4377_Sunday_June_05_2022_01_09_38_PM_11157354/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + xy - \frac{x}{y} = 0}$$

8.21.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x(y^2-1)}{y} \end{aligned}$$

Where $f(x) = -x$ and $g(y) = \frac{y^2-1}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{y^2-1}{y}}\, dy &= -x\, dx \\
\int \frac{1}{\frac{y^2-1}{y}}\, dy &= \int -x\, dx \\
\frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} &= -\frac{x^2}{2} + c_1
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\left(\frac{1}{2}\right)(\ln(y-1) + \ln(1+y)) &= -\frac{x^2}{2} + 2c_1 \\
\ln(y-1) + \ln(1+y) &= (2)\left(-\frac{x^2}{2} + 2c_1\right) \\
&= -x^2 + 4c_1
\end{aligned}$$

Raising both side to exponential gives

$$e^{\ln(y-1)+\ln(1+y)} = e^{-x^2+2c_1}$$

Which simplifies to

$$\begin{aligned}
y^2 - 1 &= 2c_1 e^{-x^2} \\
&= c_2 e^{-x^2}
\end{aligned}$$

The solution is

$$y^2 - 1 = c_2 e^{-x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y^2 - 1 = c_2 e^{-x^2} \tag{1}$$

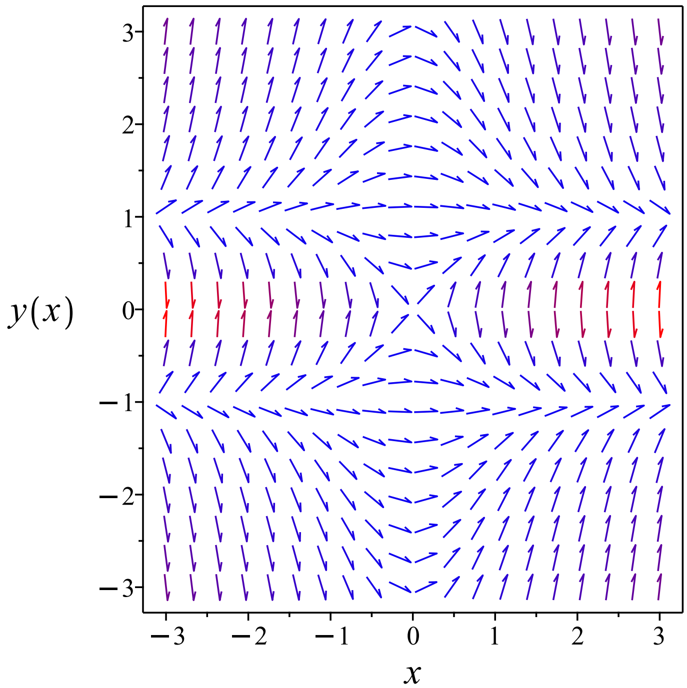


Figure 239: Slope field plot

Verification of solutions

$$y^2 - 1 = c_2 \mathrm{e}^{-x^2}$$

Verified OK.

8.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(y^2-1)}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 202: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -\frac{1}{x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(y^2-1)}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= -x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R-1)}{2} + \frac{\ln(R+1)}{2} + c_1 \tag{4}$$

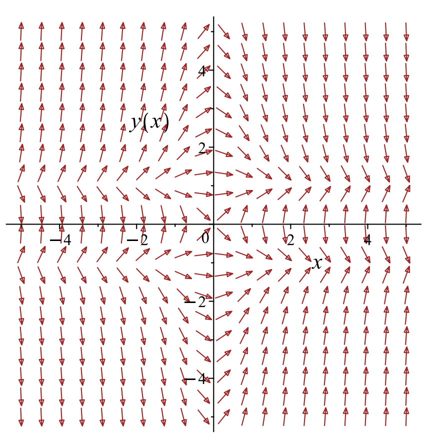
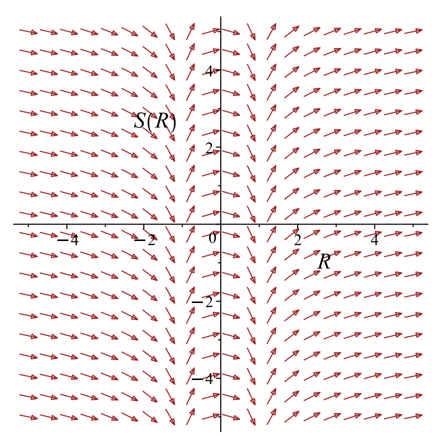
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

Which simplifies to

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(y^2-1)}{y}$ 	$R = y$ $S = -\frac{x^2}{2}$	$\frac{dS}{dR} = \frac{R}{R^2-1}$ 

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1 \tag{1}$$

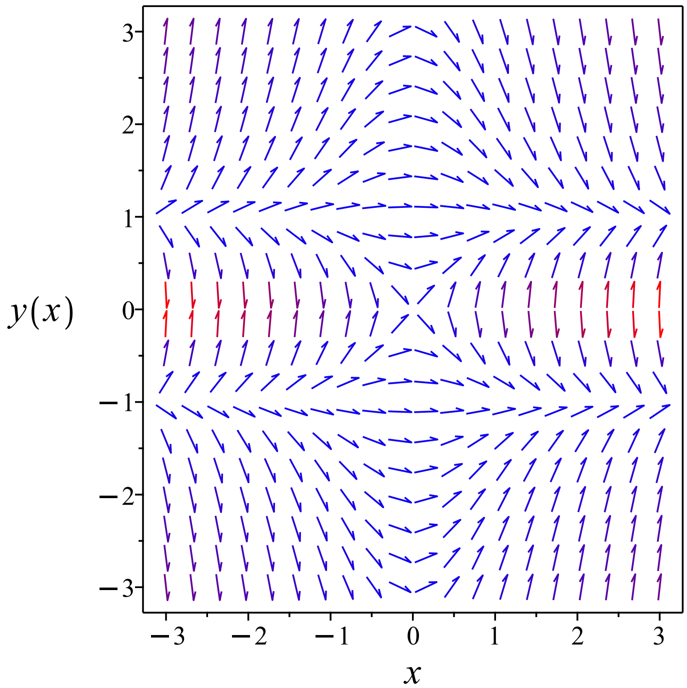


Figure 240: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} = \frac{\ln(y-1)}{2} + \frac{\ln(1+y)}{2} + c_1$$

Verified OK.

8.21.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{x(y^2-1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -xy + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -x \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -y^2x + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -w(x)\,x + x \\
w' &= -2xw + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 2w(x)\,x = 2x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2x dx} \\
&= \mathrm{e}^{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\,(2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} w\right) &= \left(\mathrm{e}^{x^2}\right)(2x) \\
\mathrm{d}\left(\mathrm{e}^{x^2} w\right) &= \left(2x\,\mathrm{e}^{x^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{x^2} w &= \int 2x\,\mathrm{e}^{x^2}\,\mathrm{d}x \\
\mathrm{e}^{x^2} w &= \mathrm{e}^{x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$
w(x) = \mathrm{e}^{-x^2}\mathrm{e}^{x^2} + c_1 \mathrm{e}^{-x^2}
$$

which simplifies to

$$
w(x) = 1 + c_1 \mathrm{e}^{-x^2}
$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$
y^2 = 1 + c_1 \mathrm{e}^{-x^2}
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \sqrt{1 + c_1 \mathrm{e}^{-x^2}} \\
y(x) &= -\sqrt{1 + c_1 \mathrm{e}^{-x^2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \sqrt{1 + c_1 \mathrm{e}^{-x^2}} \tag{1}
$$

$$
y = -\sqrt{1 + c_1 \mathrm{e}^{-x^2}} \tag{2}
$$

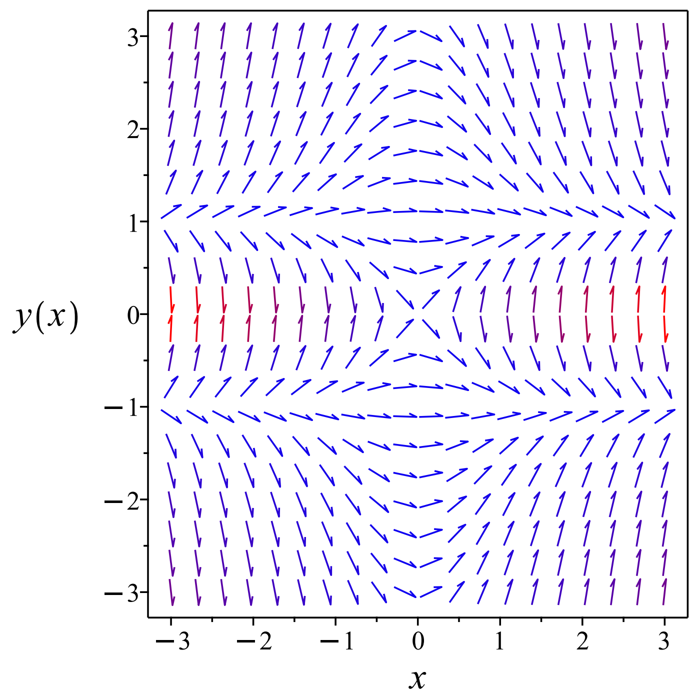


Figure 241: Slope field plot

Verification of solutions

$$y = \sqrt{1 + c_1 \mathrm{e}^{-x^2}}$$

Verified OK.

$$y = -\sqrt{1 + c_1 \mathrm{e}^{-x^2}}$$

Verified OK.

8.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y}{y^2-1}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(-\frac{y}{y^2-1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= -\frac{y}{y^2-1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y}{y^2-1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{align*}
\frac{\partial \phi}{\partial x} &= M \tag{1} \\
\frac{\partial \phi}{\partial y} &= N \tag{2}
\end{align*}
$$

Integrating (1) w.r.t. x gives

$$
\begin{align*}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y) \tag{3}
\end{align*}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y}{y^2-1}$. Therefore equation (4) becomes

$$
-\frac{y}{y^2-1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y}{y^2-1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{y}{y^2-1}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2} = c_1 \tag{1}$$

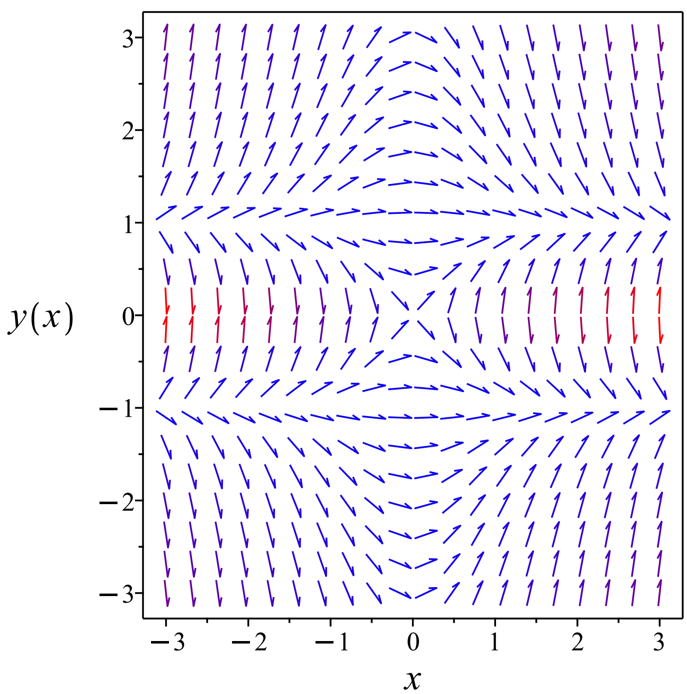


Figure 242: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2} = c_1$$

Verified OK.

8.21.5 Maple step by step solution

Let's solve

$y' + xy - \frac{x}{y} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{(y-1)(1+y)} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'y}{(y-1)(1+y)}dx = \int -xdx + c_1$

- Evaluate integral

  $\frac{\ln((y-1)(1+y))}{2} = -\frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{1 + \mathrm{e}^{-x^2+2c_1}}, y = -\sqrt{1 + \mathrm{e}^{-x^2+2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x)+x*y(x)=x/y(x),y(x), singsol=all)
```

$$y(x) = \sqrt{\mathrm{e}^{-x^2}c_1 + 1}$$
$$y(x) = -\sqrt{\mathrm{e}^{-x^2}c_1 + 1}$$

✓ Solution by Mathematica

Time used: 1.922 (sec). Leaf size: 57

```
DSolve[y'[x]+x*y[x]==x/y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\sqrt{1+e^{-x^2+2c_1}} \\
y(x) &\to \sqrt{1+e^{-x^2+2c_1}} \\
y(x) &\to -1 \\
y(x) &\to 1
\end{aligned}
$$

8.22 problem 22

8.22.1 Maple step by step solution 1456

Internal problem ID [4885]
Internal file name [`OUTPUT/4378_Sunday_June_05_2022_01_11_22_PM_13716379/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 22.
ODE order: 4.
ODE degree: 1.

The type(s) of ODE detected by this program : **"higher_order_linear_constant_coefficients_ODE"**

Maple gives the following as the ode type

```
[[_high_order, _missing_x]]
```

$$\boxed{y'''' - 2y''' + 13y'' - 18y' + 36y = 0}$$

The characteristic equation is

$$\lambda^4 - 2\lambda^3 + 13\lambda^2 - 18\lambda + 36 = 0$$

The roots of the above equation are

$$\lambda_1 = 1 - i\sqrt{3}$$
$$\lambda_2 = 1 + i\sqrt{3}$$
$$\lambda_3 = 3i$$
$$\lambda_4 = -3i$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{-3ix}c_1 + \mathrm{e}^{\left(1-i\sqrt{3}\right)x}c_2 + \mathrm{e}^{3ix}c_3 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x}c_4$$

The fundamental set of solutions for the homogeneous solution are the following

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-3ix} \\
y_2 &= \mathrm{e}^{\left(1-i\sqrt{3}\right)x} \\
y_3 &= \mathrm{e}^{3ix} \\
y_4 &= \mathrm{e}^{\left(1+i\sqrt{3}\right)x}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \mathrm{e}^{-3ix}c_1 + \mathrm{e}^{\left(1-i\sqrt{3}\right)x}c_2 + \mathrm{e}^{3ix}c_3 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x}c_4 \tag{1}
$$

Verification of solutions

$$
y = \mathrm{e}^{-3ix}c_1 + \mathrm{e}^{\left(1-i\sqrt{3}\right)x}c_2 + \mathrm{e}^{3ix}c_3 + \mathrm{e}^{\left(1+i\sqrt{3}\right)x}c_4
$$

Verified OK.

8.22.1 Maple step by step solution

Let's solve
$y'''' - 2y''' + 13y'' - 18y' + 36y = 0$

- Highest derivative means the order of the ODE is 4
  y''''

□ Convert linear ODE into a system of first order ODEs
  - Define new variable $y_1(x)$
    $y_1(x) = y$
  - Define new variable $y_2(x)$
    $y_2(x) = y'$
  - Define new variable $y_3(x)$
    $y_3(x) = y''$
  - Define new variable $y_4(x)$
    $y_4(x) = y'''$
  - Isolate for $y_4'(x)$ using original ODE
    $y_4'(x) = 2y_4(x) - 13y_3(x) + 18y_2(x) - 36y_1(x)$

Convert linear ODE into a system of first order ODEs

$$[y_2(x) = y_1'(x), y_3(x) = y_2'(x), y_4(x) = y_3'(x), y_4'(x) = 2y_4(x) - 13y_3(x) + 18y_2(x) - 36y_1(x)]$$

- Define vector

$$\vec{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \end{bmatrix}$$

- System to solve

$$\vec{y}'(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 18 & -13 & 2 \end{bmatrix} \cdot \vec{y}(x)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -36 & 18 & -13 & 2 \end{bmatrix}$$

- Rewrite the system as

$$\vec{y}'(x) = A \cdot \vec{y}(x)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} \frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ -\frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right], \left[1 - \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^3} \\ \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right], \left[1 + \mathrm{I}\sqrt{3}, \begin{bmatrix} \frac{1}{\left(1+\mathrm{I}\sqrt{3}\right)^3} \\ \frac{1}{\left(1+\mathrm{I}\sqrt{3}\right)^2} \\ \frac{1}{1+\mathrm{I}\sqrt{3}} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{-3\,\mathrm{I}x} \cdot \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(3x) - \mathrm{I}\sin(3x)) \cdot \begin{bmatrix} -\frac{\mathrm{I}}{27} \\ -\frac{1}{9} \\ \frac{\mathrm{I}}{3} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\frac{\mathrm{I}}{27}(\cos(3x) - \mathrm{I}\sin(3x)) \\ -\frac{\cos(3x)}{9} + \frac{\mathrm{I}\sin(3x)}{9} \\ \frac{\mathrm{I}}{3}(\cos(3x) - \mathrm{I}\sin(3x)) \\ \cos(3x) - \mathrm{I}\sin(3x) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_1(x) = \begin{bmatrix} -\frac{\sin(3x)}{27} \\ -\frac{\cos(3x)}{9} \\ \frac{\sin(3x)}{3} \\ \cos(3x) \end{bmatrix}, \vec{y}_2(x) = \begin{bmatrix} -\frac{\cos(3x)}{27} \\ \frac{\sin(3x)}{9} \\ \frac{\cos(3x)}{3} \\ -\sin(3x) \end{bmatrix}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[1-\mathrm{I}\sqrt{3}, \left[\begin{array}{c} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{3}} \\ \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{2}} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{\left(1-\mathrm{I}\sqrt{3}\right)x} \cdot \left[\begin{array}{c} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{3}} \\ \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{2}} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$e^{x} \cdot \left(\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)\right) \cdot \left[\begin{array}{c} \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{3}} \\ \frac{1}{\left(1-\mathrm{I}\sqrt{3}\right)^{2}} \\ \frac{1}{1-\mathrm{I}\sqrt{3}} \\ 1 \end{array}\right]$$

- Simplify expression

$$e^{x} \cdot \left[\begin{array}{c} \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{\left(1-\mathrm{I}\sqrt{3}\right)^{3}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{\left(1-\mathrm{I}\sqrt{3}\right)^{2}} \\ \frac{\cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right)}{1-\mathrm{I}\sqrt{3}} \\ \cos\left(\sqrt{3}\,x\right)-\mathrm{I}\sin\left(\sqrt{3}\,x\right) \end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{y}_3(x) = e^x \cdot \begin{bmatrix} -\frac{\cos(\sqrt{3}\,x)}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{8} + \frac{\sqrt{3}\,\sin(\sqrt{3}\,x)}{8} \\ \frac{\cos(\sqrt{3}\,x)}{4} + \frac{\sqrt{3}\,\sin(\sqrt{3}\,x)}{4} \\ \cos(\sqrt{3}\,x) \end{bmatrix}, \vec{y}_4(x) = e^x \cdot \begin{bmatrix} \frac{\sin(\sqrt{3}\,x)}{8} \\ \frac{\sqrt{3}\,\cos(\sqrt{3}\,x)}{8} + \frac{\sin(\sqrt{3}\,x)}{8} \\ \frac{\sqrt{3}\,\cos(\sqrt{3}\,x)}{4} - \frac{\sin(\sqrt{3}\,x)}{4} \\ -\sin(\sqrt{3}\,x) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{y} = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) + c_3\vec{y}_3(x) + c_4\vec{y}_4(x)$

- Substitute solutions into the general solution

$$\vec{y} = c_3 e^x \cdot \begin{bmatrix} -\frac{\cos(\sqrt{3}\,x)}{8} \\ -\frac{\cos(\sqrt{3}\,x)}{8} + \frac{\sqrt{3}\,\sin(\sqrt{3}\,x)}{8} \\ \frac{\cos(\sqrt{3}\,x)}{4} + \frac{\sqrt{3}\,\sin(\sqrt{3}\,x)}{4} \\ \cos(\sqrt{3}\,x) \end{bmatrix} + c_4 e^x \cdot \begin{bmatrix} \frac{\sin(\sqrt{3}\,x)}{8} \\ \frac{\sqrt{3}\,\cos(\sqrt{3}\,x)}{8} + \frac{\sin(\sqrt{3}\,x)}{8} \\ \frac{\sqrt{3}\,\cos(\sqrt{3}\,x)}{4} - \frac{\sin(\sqrt{3}\,x)}{4} \\ -\sin(\sqrt{3}\,x) \end{bmatrix} + \begin{bmatrix} -\frac{c_1\sin(3x)}{27} - c \\ \frac{c_2\sin(3x)}{9} - c_1 \\ \frac{c_2\cos(3x)}{3} + c_1 \\ c_1\cos(3x) - c_2 \end{bmatrix}$$

- First component of the vector is the solution to the ODE

  $y = -\frac{c_3 e^x \cos(\sqrt{3}\,x)}{8} + \frac{c_4 e^x \sin(\sqrt{3}\,x)}{8} - \frac{c_2\cos(3x)}{27} - \frac{c_1\sin(3x)}{27}$

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve(diff(y(x),x$4)-2*diff(y(x),x$3)+13*diff(y(x),x$2)-18*diff(y(x),x)+36*y(x)=0,y(x), sin
```

$$y(x) = c_1 e^x \sin\left(\sqrt{3}\,x\right) + c_2 e^x \cos\left(\sqrt{3}\,x\right) + c_3 \sin(3x) + c_4 \cos(3x)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 48

```
DSolve[y''''[x]-2*y'''[x]+13*y''[x]-18*y'[x]+36*y[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to c_3 \cos(3x) + c_2 e^x \cos\left(\sqrt{3}x\right) + c_4 \sin(3x) + c_1 e^x \sin\left(\sqrt{3}x\right)$$

8.23 problem 23

8.23.1 Solving as linear ode 1462
8.23.2 Solving as first order ode lie symmetry lookup ode 1464
8.23.3 Solving as exact ode 1468
8.23.4 Maple step by step solution 1473

Internal problem ID [4886]
Internal file name [`OUTPUT/4379_Sunday_June_05_2022_01_11_31_PM_50757174/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{\sin(\theta)\cos(\theta)\,r' - r\cos(\theta)^2 = \sin(\theta)^2}$$

8.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$r' + p(\theta)r = q(\theta)$$

Where here

$$\begin{aligned} p(\theta) &= -\cot(\theta) \\ q(\theta) &= \tan(\theta) \end{aligned}$$

Hence the ode is

$$r' - r\cot(\theta) = \tan(\theta)$$

The integrating factor μ is

$$\mu = e^{\int -\cot(\theta)d\theta}$$
$$= \frac{1}{\sin(\theta)}$$

Which simplifies to

$$\mu = \csc(\theta)$$

The ode becomes

$$\frac{d}{d\theta}(\mu r) = (\mu)(\tan(\theta))$$
$$\frac{d}{d\theta}(\csc(\theta) r) = (\csc(\theta))(\tan(\theta))$$
$$d(\csc(\theta) r) = \sec(\theta)\, d\theta$$

Integrating gives

$$\csc(\theta) r = \int \sec(\theta)\, d\theta$$
$$\csc(\theta) r = \ln(\sec(\theta) + \tan(\theta)) + c_1$$

Dividing both sides by the integrating factor $\mu = \csc(\theta)$ results in

$$r = \sin(\theta)\ln(\sec(\theta) + \tan(\theta)) + c_1 \sin(\theta)$$

which simplifies to

$$r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1)$$

Summary
The solution(s) found are the following

$$r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1) \tag{1}$$

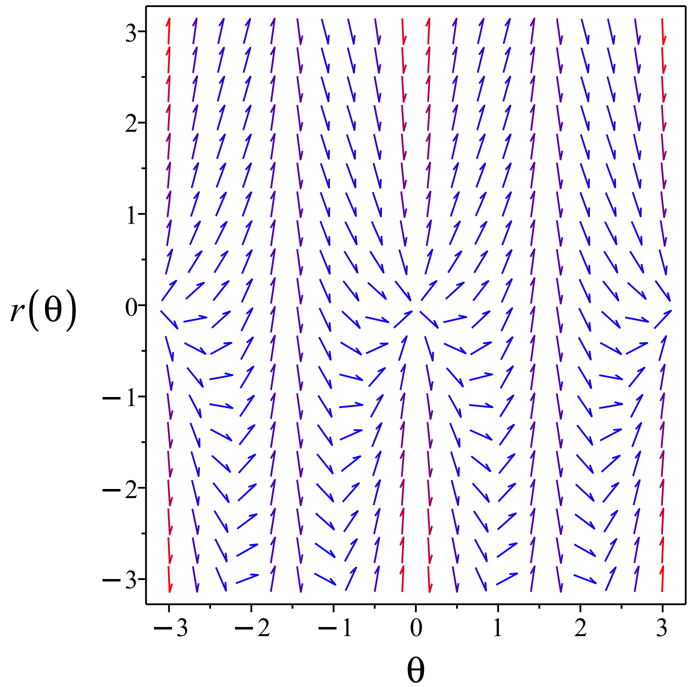


Figure 243: Slope field plot

Verification of solutions

$$r = \sin(\theta)\left(\ln\left(\sec(\theta) + \tan(\theta)\right) + c_1\right)$$

Verified OK.

8.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$r' = \frac{r\cos(\theta)^2 + \sin(\theta)^2}{\sin(\theta)\cos(\theta)}$$
$$r' = \omega(\theta, r)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2\xi_r - \omega_\theta\xi - \omega_r\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= 0 \\
\eta(\theta, r) &= \sin(\theta)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = \theta$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\sin(\theta)} dy
\end{aligned}$$

Which results in

$$S = \frac{r}{\sin(\theta)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$\omega(\theta, r) = \frac{r \cos(\theta)^2 + \sin(\theta)^2}{\sin(\theta) \cos(\theta)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_\theta &= 1 \\
R_r &= 0 \\
S_\theta &= -\csc(\theta) \cot(\theta) r \\
S_r &= \csc(\theta)
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(\theta) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln\left(\sec\left(R\right) + \tan\left(R\right)\right) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$\csc\left(\theta\right) r = \ln\left(\sec\left(\theta\right) + \tan\left(\theta\right)\right) + c_1$$

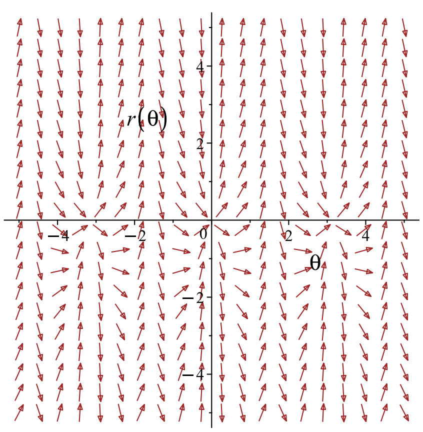
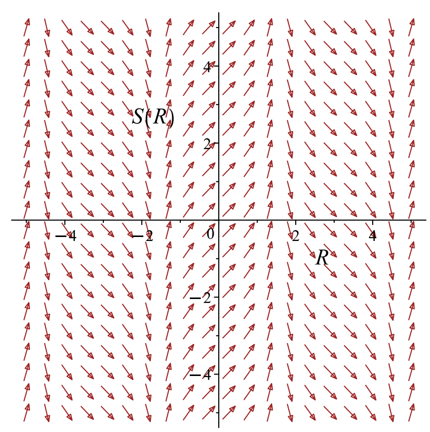
Which simplifies to

$$\csc\left(\theta\right) r = \ln\left(\sec\left(\theta\right) + \tan\left(\theta\right)\right) + c_1$$

Which gives

$$r = \frac{\ln\left(\sec\left(\theta\right) + \tan\left(\theta\right)\right) + c_1}{\csc\left(\theta\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in θ, r coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dr}{d\theta} = \frac{r\cos(\theta)^2+\sin(\theta)^2}{\sin(\theta)\cos(\theta)}$ 	$R = \theta$ $S = \csc\left(\theta\right) r$	$\frac{dS}{dR} = \sec\left(R\right)$ 

Summary

The solution(s) found are the following

$$r = \frac{\ln\left(\sec\left(\theta\right) + \tan\left(\theta\right)\right) + c_1}{\csc\left(\theta\right)} \tag{1}$$

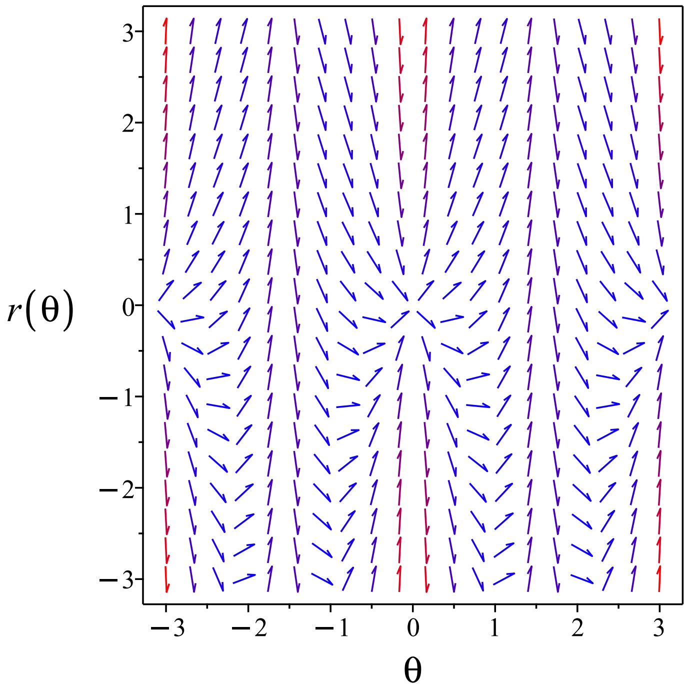


Figure 244: Slope field plot

Verification of solutions

$$r = \frac{\ln(\sec(\theta) + \tan(\theta)) + c_1}{\csc(\theta)}$$

Verified OK.

8.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\, \mathrm{d}\theta + N(\theta, r)\, \mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\cos(\theta) \sin(\theta))\, \mathrm{d}r &= \left(r \cos(\theta)^2 + \sin(\theta)^2\right) \mathrm{d}\theta \\ \left(-r \cos(\theta)^2 - \sin(\theta)^2\right) \mathrm{d}\theta + (\cos(\theta) \sin(\theta))\, \mathrm{d}r &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(\theta, r) &= -r \cos(\theta)^2 - \sin(\theta)^2 \\ N(\theta, r) &= \cos(\theta) \sin(\theta) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-r \cos(\theta)^2 - \sin(\theta)^2\right) \\ &= -\cos(\theta)^2 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}(\cos(\theta) \sin(\theta)) \\ &= \cos(2\theta) \end{aligned}$$

Since $\frac{\partial M}{\partial r} \neq \frac{\partial N}{\partial \theta}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial r} - \frac{\partial N}{\partial \theta}\right) \\
&= \sec(\theta)\csc(\theta)\left(\left(-\cos(\theta)^2\right) - \left(-\sin(\theta)^2 + \cos(\theta)^2\right)\right) \\
&= -2\cot(\theta) + \tan(\theta)
\end{aligned}
$$

Since A does not depend on r, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}\theta} \\
&= e^{\int -2\cot(\theta)+\tan(\theta) \, \mathrm{d}\theta}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(\sin(\theta))-\ln(\cos(\theta))} \\
&= \frac{1}{\sin(\theta)^2 \cos(\theta)}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sin(\theta)^2 \cos(\theta)}\left(-r\cos(\theta)^2 - \sin(\theta)^2\right) \\
&= -\csc(\theta)\cot(\theta) r - \sec(\theta)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sin(\theta)^2 \cos(\theta)}(\cos(\theta)\sin(\theta)) \\
&= \csc(\theta)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0 \\
(-\csc(\theta)\cot(\theta) r - \sec(\theta)) + (\csc(\theta))\frac{\mathrm{d}r}{\mathrm{d}\theta} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial\phi}{\partial\theta} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial r} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial\phi}{\partial\theta} \,\mathrm{d}\theta = \int \overline{M} \,\mathrm{d}\theta$$

$$\int \frac{\partial\phi}{\partial\theta} \,\mathrm{d}\theta = \int -\csc(\theta)\cot(\theta)\, r - \sec(\theta) \,\mathrm{d}\theta$$

$$\phi = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta)) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial\phi}{\partial r} = \csc(\theta) + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial r} = \csc(\theta)$. Therefore equation (4) becomes

$$\csc(\theta) = \csc(\theta) + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = 0$$

Therefore

$$f(r) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(r)$ into equation (3) gives ϕ

$$\phi = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \csc(\theta)\, r - \ln(\sec(\theta) + \tan(\theta))$$

The solution becomes

$$r = \frac{\ln\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)+c_1}{\csc\left(\theta\right)}$$

Summary

The solution(s) found are the following

$$r = \frac{\ln\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)+c_1}{\csc\left(\theta\right)} \tag{1}$$

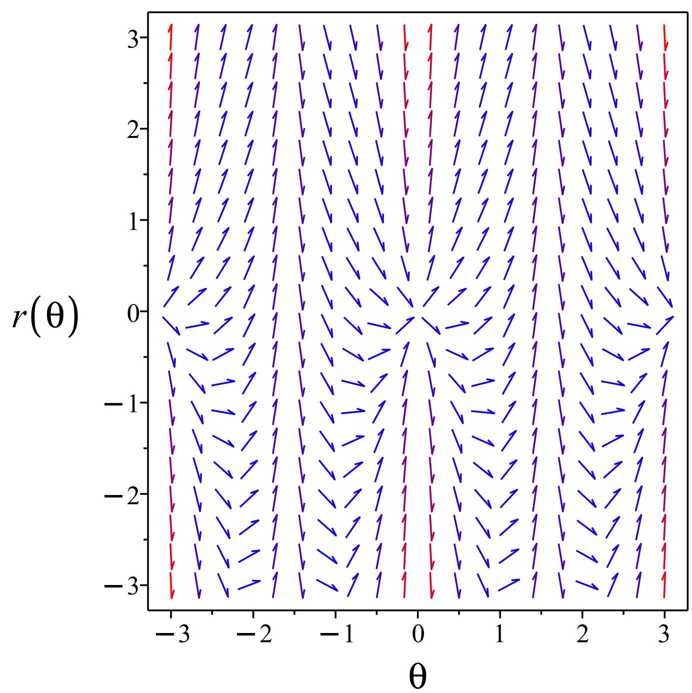


Figure 245: Slope field plot

Verification of solutions

$$r = \frac{\ln\left(\sec\left(\theta\right)+\tan\left(\theta\right)\right)+c_1}{\csc\left(\theta\right)}$$

Verified OK.

8.23.4 Maple step by step solution

Let's solve

$\sin(\theta)\cos(\theta) r' - r\cos(\theta)^2 = \sin(\theta)^2$

- Highest derivative means the order of the ODE is 1

  r'

- Isolate the derivative

  $r' = \frac{\cos(\theta) r}{\sin(\theta)} + \frac{\sin(\theta)}{\cos(\theta)}$

- Group terms with r on the lhs of the ODE and the rest on the rhs of the ODE

  $r' - \frac{\cos(\theta) r}{\sin(\theta)} = \frac{\sin(\theta)}{\cos(\theta)}$

- The ODE is linear; multiply by an integrating factor $\mu(\theta)$

  $\mu(\theta)\left(r' - \frac{\cos(\theta) r}{\sin(\theta)}\right) = \frac{\mu(\theta)\sin(\theta)}{\cos(\theta)}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{d\theta}(\mu(\theta) r)$

  $\mu(\theta)\left(r' - \frac{\cos(\theta) r}{\sin(\theta)}\right) = \mu'(\theta) r + \mu(\theta) r'$

- Isolate $\mu'(\theta)$

  $\mu'(\theta) = -\frac{\mu(\theta)\cos(\theta)}{\sin(\theta)}$

- Solve to find the integrating factor

  $\mu(\theta) = \frac{1}{\sin(\theta)}$

- Integrate both sides with respect to θ

  $\int \left(\frac{d}{d\theta}(\mu(\theta) r)\right) d\theta = \int \frac{\mu(\theta)\sin(\theta)}{\cos(\theta)} d\theta + c_1$

- Evaluate the integral on the lhs

  $\mu(\theta) r = \int \frac{\mu(\theta)\sin(\theta)}{\cos(\theta)} d\theta + c_1$

- Solve for r

  $r = \frac{\int \frac{\mu(\theta)\sin(\theta)}{\cos(\theta)} d\theta + c_1}{\mu(\theta)}$

- Substitute $\mu(\theta) = \frac{1}{\sin(\theta)}$

  $r = \sin(\theta)\left(\int \frac{1}{\cos(\theta)} d\theta + c_1\right)$

- Evaluate the integrals on the rhs

  $r = \sin(\theta)(\ln(\sec(\theta) + \tan(\theta)) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(sin(theta)*cos(theta)*diff(r(theta),theta)-sin(theta)^2=r(theta)*cos(theta)^2,r(theta
```

$$r(\theta) = (\ln(\sec(\theta) + \tan(\theta)) + c_1)\sin(\theta)$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 14

```
DSolve[Sin[\[Theta]]*Cos[\[Theta]]*r'[\[Theta]]-Sin[\[Theta]]^2==r[\[Theta]]*Cos[\[Theta]]^2
```

$$r(\theta) \to \sin(\theta)\left(\coth^{-1}(\sin(\theta)) + c_1\right)$$

8.24 problem 24

8.24.1 Solving as second order integrable as is ode 1475
8.24.2 Solving as second order nonlinear solved by mainardi lioville method ode 1477
8.24.3 Solving as type second_order_integrable_as_is (not using ABC version) 1479

Internal problem ID [4887]
Internal file name [`OUTPUT/4380_Sunday_June_05_2022_01_11_42_PM_4620951/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_nonlinear_solved_by_mainardi_lioville_method"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _nonlinear], _Liouville, [_2nd_order,
   _with_linear_symmetries], [_2nd_order, _reducible, _mu_x_y1],
    [_2nd_order, _reducible, _mu_xy]]
```

$$\boxed{x\left(yy''+{y'}^{2}\right)-y'y=0}$$

8.24.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (xyy''+(-y+xy')\,y')\,dx=0$$
$$xyy'-y^2=c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+c_1}{xy} \end{aligned}$$

Where $f(x)=\frac{1}{x}$ and $g(y)=\frac{y^2+c_1}{y}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y^2+c_1}{y}}\,dy &= \frac{1}{x}\,dx \\ \int \frac{1}{\frac{y^2+c_1}{y}}\,dy &= \int \frac{1}{x}\,dx \\ \frac{\ln(y^2+c_1)}{2} &= \ln(x)+c_2 \end{aligned}$$

Raising both side to exponential gives

$$\sqrt{y^2+c_1} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{y^2+c_1} = c_3 x$$

Which simplifies to

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2}$$

The solution is

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2} \tag{1}$$

Verification of solutions

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2}$$

Verified OK.

8.24.2 Solving as second order nonlinear solved by mainardi lioville method ode

The ode has the Liouville form given by

$$y'' + f(x)y' + g(y)y'^2 = 0 \tag{1A}$$

Where in this problem

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(y) &= \frac{1}{y} \end{aligned}$$

Dividing through by y' then Eq (1A) becomes

$$\frac{y''}{y'} + f + gy' = 0 \tag{2A}$$

But the first term in Eq (2A) can be written as

$$\frac{y''}{y'} = \frac{d}{dx}\ln(y') \tag{3A}$$

And the last term in Eq (2A) can be written as

$$\begin{aligned} g\frac{dy}{dx} &= \left(\frac{d}{dy}\int g dy\right)\frac{dy}{dx} \\ &= \frac{d}{dx}\int g dy \end{aligned} \tag{4A}$$

Substituting (3A,4A) back into (2A) gives

$$\frac{d}{dx}\ln(y') + \frac{d}{dx}\int g dy = -f \tag{5A}$$

Integrating the above w.r.t. x gives

$$\ln(y') + \int g dy = -\int f dx + c_1$$

Where c_1 is arbitrary constant. Taking the exponential of the above gives

$$y' = c_2 e^{\int -g dy}\, e^{\int -f dx} \tag{6A}$$

Where c_2 is a new arbitrary constant. But since $g = \frac{1}{y}$ and $f = -\frac{1}{x}$, then

$$
\begin{aligned}
\int -g dy &= \int -\frac{1}{y} dy \\
&= -\ln(y) \\
\int -f dx &= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned}
$$

Substituting the above into Eq(6A) gives

$$
y' = \frac{c_2 x}{y}
$$

Which is now solved as first order separable ode. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x) g(y) \\
&= \frac{c_2 x}{y}
\end{aligned}
$$

Where $f(x) = c_2 x$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}} dy &= c_2 x \, dx \\
\int \frac{1}{\frac{1}{y}} dy &= \int c_2 x \, dx \\
\frac{y^2}{2} &= \frac{c_2 x^2}{2} + c_3
\end{aligned}
$$

The solution is

$$
\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0
$$

Summary

The solution(s) found are the following

$$
\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0 \tag{1}
$$

Verification of solutions

$$
\frac{y^2}{2} - \frac{c_2 x^2}{2} - c_3 = 0
$$

Verified OK.

8.24.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$xyy'' + (-y + xy')\, y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (xyy'' + (-y + xy')\, y')\, dx = 0$$
$$xyy' - y^2 = c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{y^2 + c_1}{xy} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \frac{y^2+c_1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+c_1}{y}}\, dy = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{y^2+c_1}{y}}\, dy = \int \frac{1}{x}\, dx$$
$$\frac{\ln (y^2 + c_1)}{2} = \ln (x) + c_2$$

Raising both side to exponential gives

$$\sqrt{y^2 + c_1} = e^{\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{y^2 + c_1} = c_3 x$$

Which simplifies to

$$\sqrt{y^2 + c_1} = c_3 x\, e^{c_2}$$

The solution is

$$\sqrt{y^2 + c_1} = c_3 x\, e^{c_2}$$

Summary

The solution(s) found are the following

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2} \tag{1}$$

Verification of solutions

$$\sqrt{y^2+c_1} = c_3 x\,\mathrm{e}^{c_2}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 35

```
dsolve(x*(y(x)*diff(y(x),x$2) + diff(y(x),x)^2)= y(x)*diff(y(x),x),y(x), singsol=all)
```

$$\begin{aligned} y(x) &= 0 \\ y(x) &= \sqrt{c_1 x^2+2c_2} \\ y(x) &= -\sqrt{c_1 x^2+2c_2} \end{aligned}$$

✓ Solution by Mathematica

Time used: 0.234 (sec). Leaf size: 18

```
DSolve[x*(y[x]*y''[x]+(y'[x])^2)==y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2\sqrt{x^2+c_1}$$

8.25 problem 25

8.25.1 Existence and uniqueness analysis 1482
8.25.2 Solving as separable ode 1482
8.25.3 Solving as linear ode 1483
8.25.4 Solving as homogeneousTypeD2 ode 1485
8.25.5 Solving as first order ode lie symmetry lookup ode 1486
8.25.6 Solving as exact ode 1490
8.25.7 Maple step by step solution 1494

Internal problem ID [4888]
Internal file name [`OUTPUT/4381_Sunday_June_05_2022_01_11_51_PM_68055552/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{3yx^2 + y'x^3 = 0}$$

With initial conditions

$$[y(1) = 2]$$

8.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= \frac{3}{x}\\ q(x) &= 0\end{aligned}$$

Hence the ode is

$$y' + \frac{3y}{x} = 0$$

The domain of $p(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

8.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= -\frac{3y}{x}\end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y}\,dy &= -\frac{3}{x}\,dx\\ \int \frac{1}{y}\,dy &= \int -\frac{3}{x}\,dx\\ \ln(y) &= -3\ln(x) + c_1\\ y &= \mathrm{e}^{-3\ln(x)+c_1}\\ &= \frac{c_1}{x^3}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

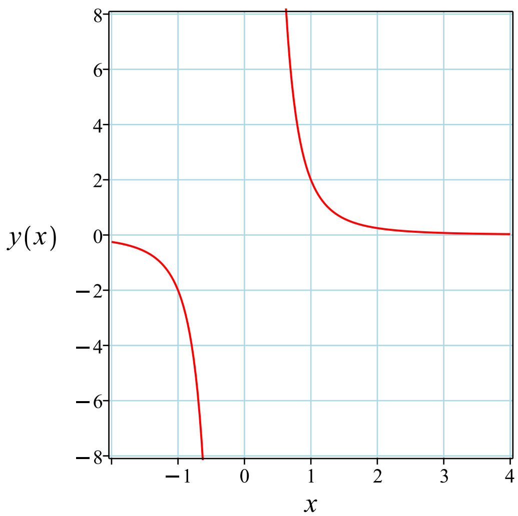
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{x^3}$$

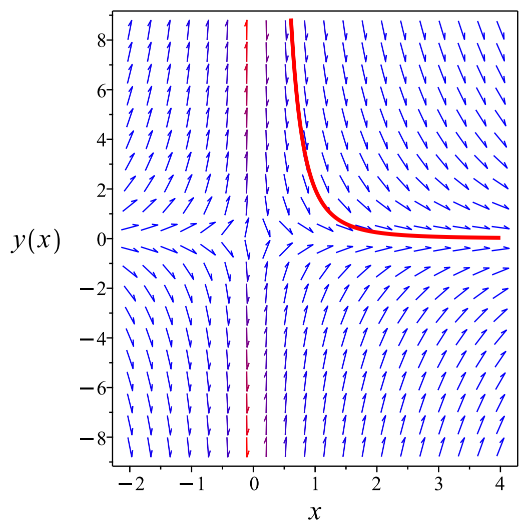
Summary

The solution(s) found are the following

$$y = \frac{2}{x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{x^3}$$

Verified OK.

8.25.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int \frac{3}{x} dx}$$
$$= x^3$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(y\,x^3\right) = 0$$

Integrating gives

$$y\,x^3 = c_1$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{c_1}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

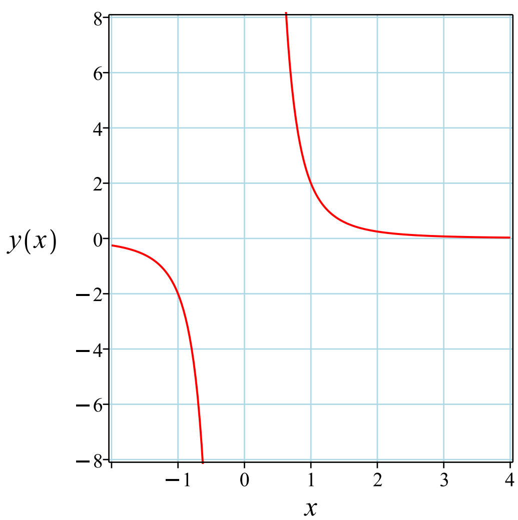
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{x^3}$$

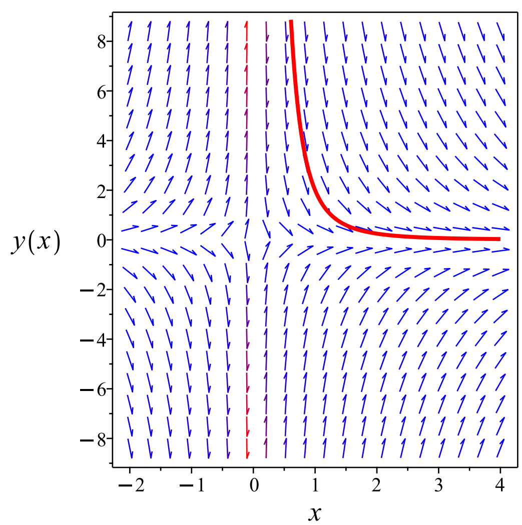
Summary

The solution(s) found are the following

$$y = \frac{2}{x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{x^3}$$

Verified OK.

8.25.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)\,x^3 + (u'(x)\,x + u(x))\,x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{4u}{x} \end{aligned}$$

Where $f(x) = -\frac{4}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{4}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{4}{x}\,dx \\ \ln(u) &= -4\ln(x) + c_2 \\ u &= \mathrm{e}^{-4\ln(x)+c_2} \\ &= \frac{c_2}{x^4} \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \frac{c_2}{x^3} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_2$$

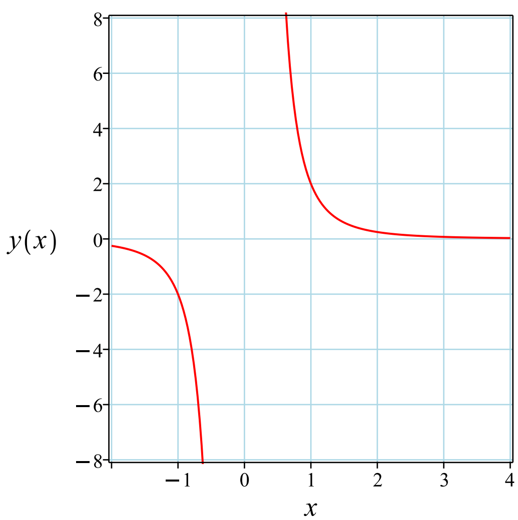
$$c_2 = 2$$

Substituting c_2 found above in the general solution gives

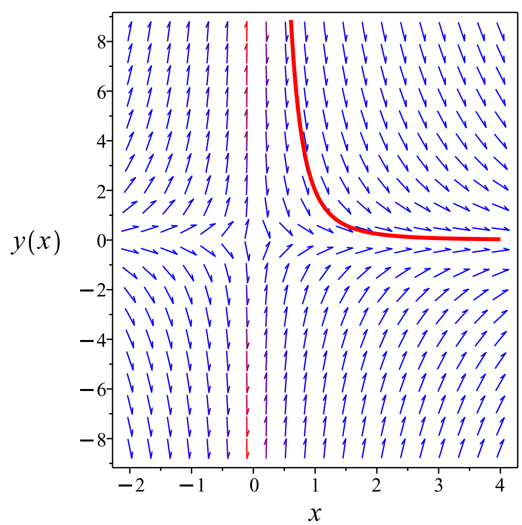
$$y = \frac{2}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{2}{x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{x^3}$$

Verified OK.

8.25.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 209: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\,x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\,x^2 \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^3 = c_1$$

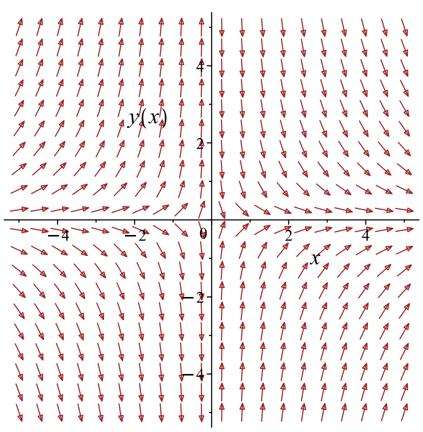
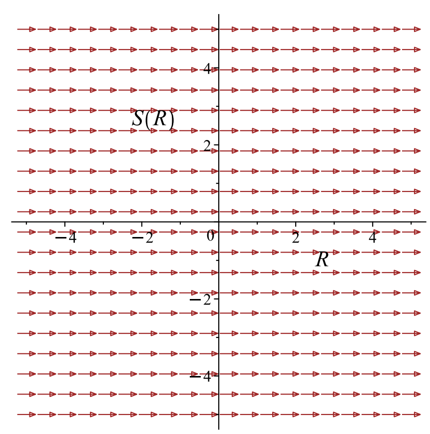
Which simplifies to

$$y x^3 = c_1$$

Which gives

$$y = \frac{c_1}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = -\frac{3y}{x}$
</td><td>$R = x$
$S = y\,x^3$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = c_1$$

$$c_1 = 2$$

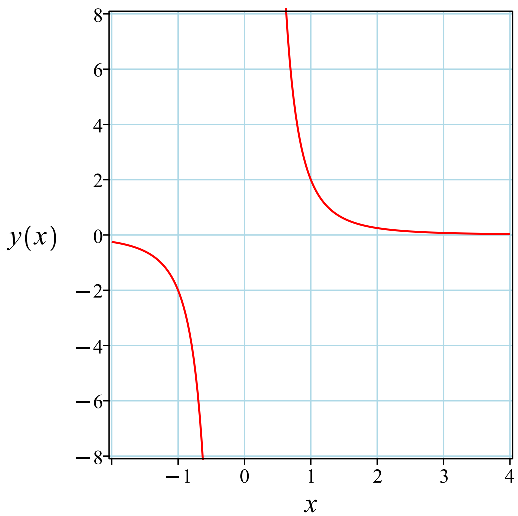
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{x^3}$$

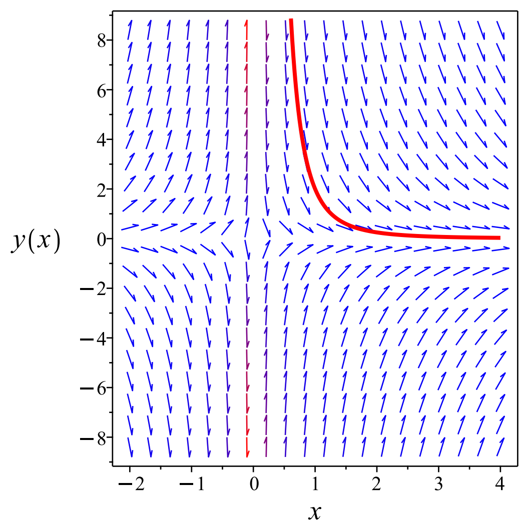
Summary

The solution(s) found are the following

$$y = \frac{2}{x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{x^3}$$

Verified OK.

8.25.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{3y}\right)\mathrm{d}y&=\left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x+\left(-\frac{1}{3y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x}\\ N(x,y)&=-\frac{1}{3y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{3y}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\frac{1}{3y}$. Therefore equation (4) becomes

$$
-\frac{1}{3y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{3y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{3y}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{\ln(y)}{3}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-3c_1}}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \mathrm{e}^{-3c_1}$$

$$c_1 = -\frac{\ln(2)}{3}$$

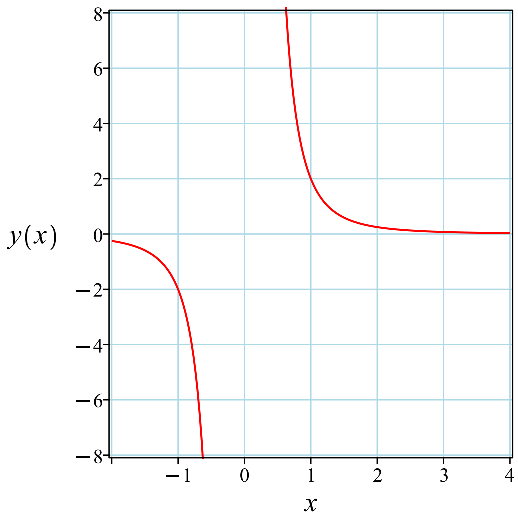
Substituting c_1 found above in the general solution gives

$$y = \frac{2}{x^3}$$

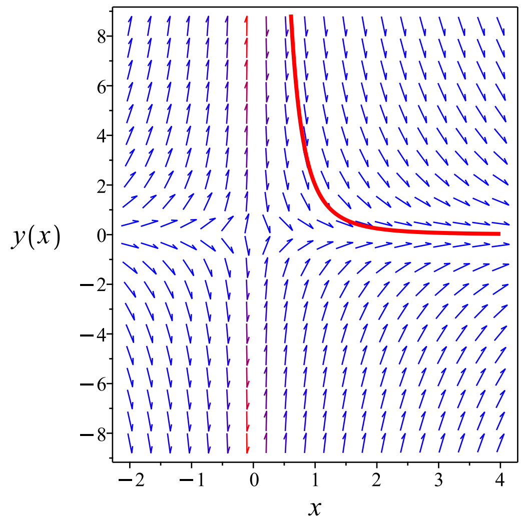
Summary

The solution(s) found are the following

$$y = \frac{2}{x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2}{x^3}$$

Verified OK.

8.25.7 Maple step by step solution

Let's solve

$[3yx^2 + y'x^3 = 0, y(1) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (3yx^2 + y'x^3)\, dx = \int 0 dx + c_1$

- Evaluate integral

  $yx^3 = c_1$

- Solve for y

  $y = \frac{c_1}{x^3}$

- Use initial condition $y(1) = 2$

  $2 = c_1$

- Solve for c_1

  $c_1 = 2$
- Substitute $c_1 = 2$ into general solution and simplify

  $y = \frac{2}{x^3}$
- Solution to the IVP

  $y = \frac{2}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([3*x^2*y(x)+x^3*diff(y(x),x)=0,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{2}{x^3}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 10

```
DSolve[{3*x^2*y[x]+x^3*y'[x]==0,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{x^3}$$

8.26 problem 26

8.26.1 Existence and uniqueness analysis 1497
8.26.2 Solving as linear ode 1497
8.26.3 Solving as homogeneousTypeD2 ode 1499
8.26.4 Solving as first order ode lie symmetry lookup ode 1500
8.26.5 Solving as exact ode 1504
8.26.6 Maple step by step solution 1509

Internal problem ID [4889]
Internal file name [`OUTPUT/4382_Sunday_June_05_2022_01_12_02_PM_16551554/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + xy' = x^2}$$

With initial conditions

$$[y(2) = 6]$$

8.26.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= x\end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = x$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

8.26.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx}\\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(x)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(x)\\ \mathrm{d}\left(\frac{y}{x}\right) &= \mathrm{d}x\end{aligned}$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \mathrm{d}x \\
\frac{y}{x} &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = c_1 x + x^2
$$

which simplifies to

$$
y = x(x + c_1)
$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$
6 = 4 + 2c_1
$$

$$
c_1 = 1
$$

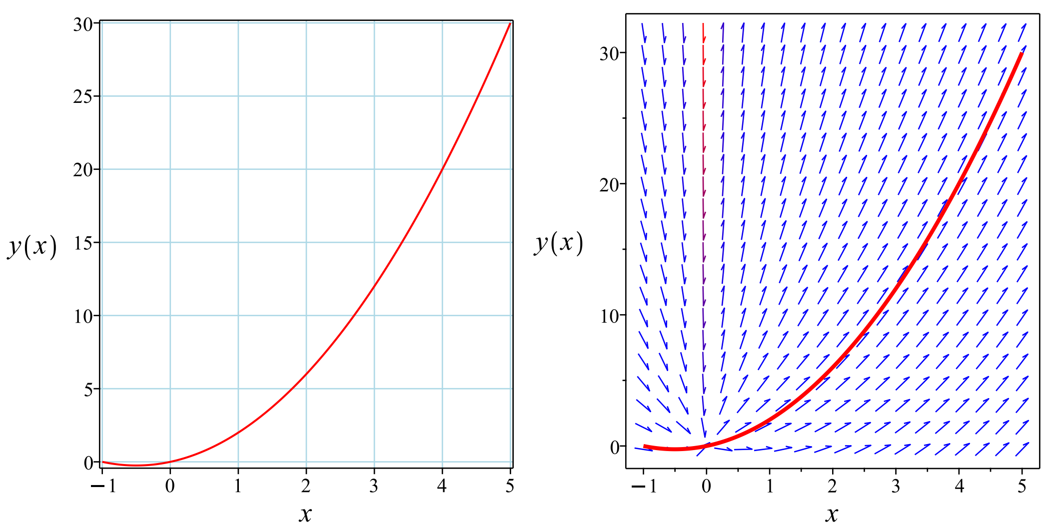
Substituting c_1 found above in the general solution gives

$$
y = x(x + 1)
$$

Summary

The solution(s) found are the following

$$
y = x(x + 1) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x(x+1)$$

Verified OK.

8.26.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + x(u'(x)\,x + u(x)) = x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 1\,\mathrm{d}x \\ &= c_2 + x \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(c_2 + x) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = 2c_2 + 4$$

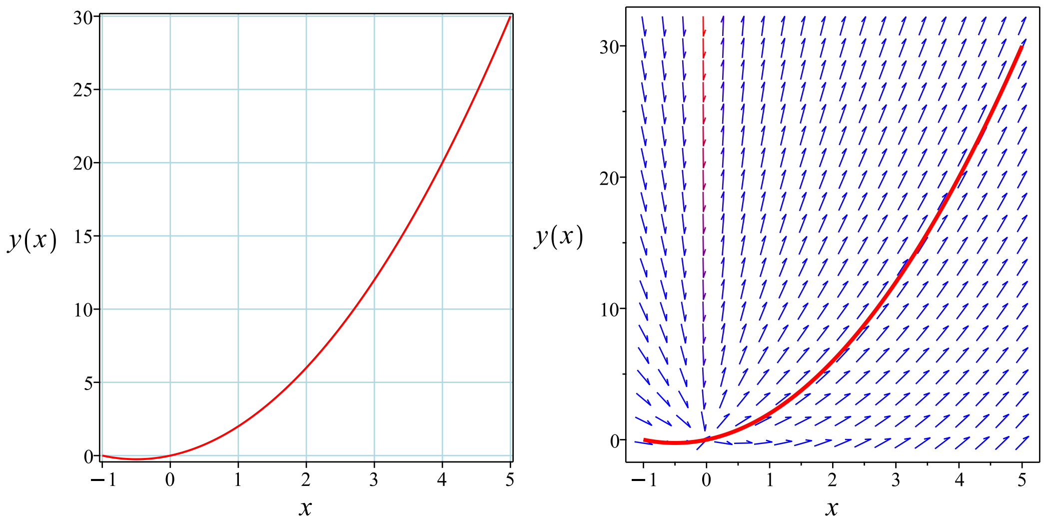
$$c_2 = 1$$

Substituting c_2 found above in the general solution gives

$$y = x(x+1)$$

Summary
The solution(s) found are the following

$$y = x(x+1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x(x+1)$$

Verified OK.

8.26.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 212: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = x + c_1$$

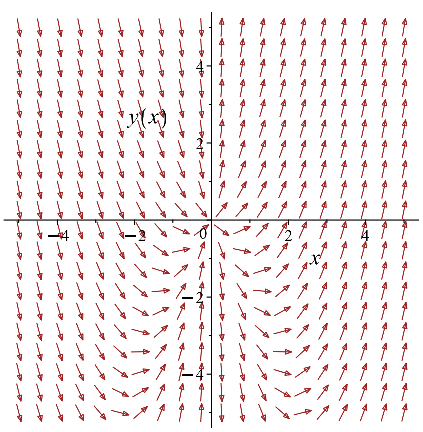
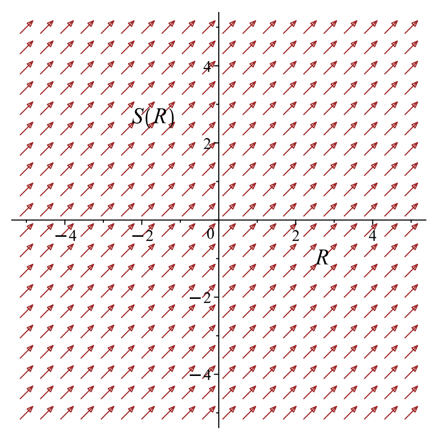
Which simplifies to

$$\frac{y}{x} = x + c_1$$

Which gives

$$y = x(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 1$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = 4 + 2c_1$$

$$c_1 = 1$$

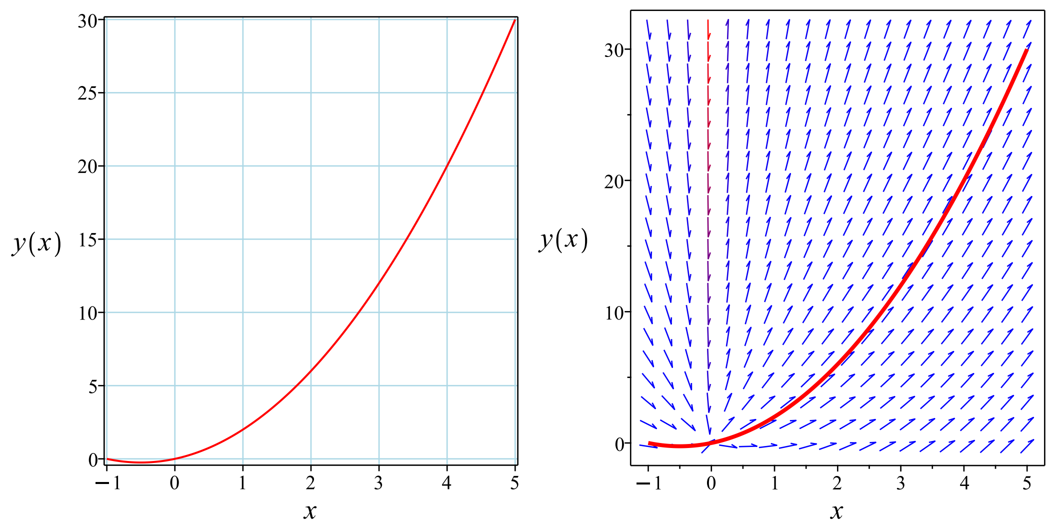
Substituting c_1 found above in the general solution gives

$$y = x(x+1)$$

Summary

The solution(s) found are the following

$$y = x(x+1) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x(x+1)$$

Verified OK.

8.26.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^2 + y\right)\mathrm{d}x \\ \left(-x^2 - y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2 - y \\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - y\right) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-1) - (1))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-x^2 - y)\\ &= \frac{-x^2 - y}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(x)\\ &= \frac{1}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^2 - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x^2 - y}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + \frac{y}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{y}{x}$$

The solution becomes

$$y = x(x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 6$ in the above solution gives an equation to solve for the constant of integration.

$$6 = 4 + 2c_1$$

$$c_1 = 1$$

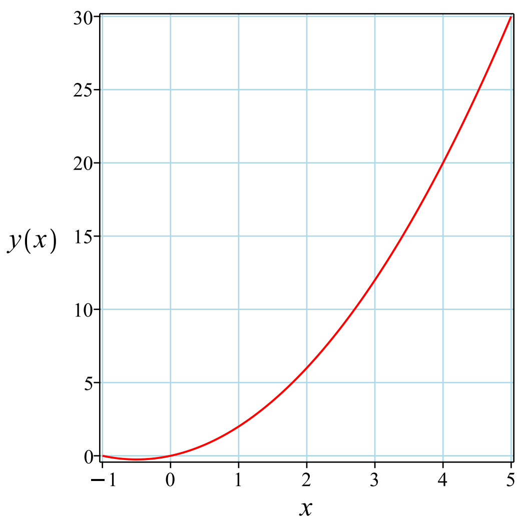
Substituting c_1 found above in the general solution gives

$$y = x(x + 1)$$

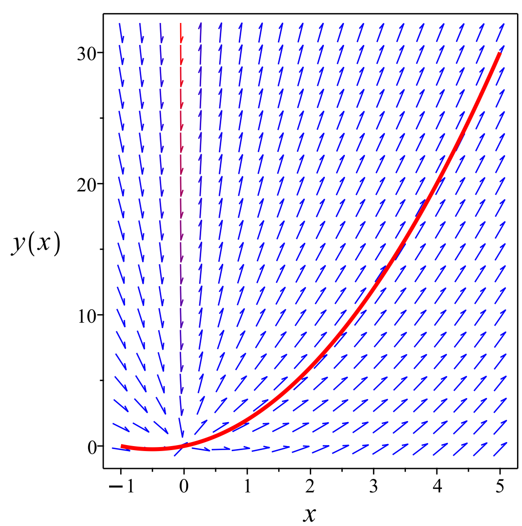
Summary

The solution(s) found are the following

$$y = x(x + 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x(x+1)$$

Verified OK.

8.26.6 Maple step by step solution

Let's solve

$[-y + xy' = x^2, y(2) = 6]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right)dx = \int \mu(x)\,xdx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,xdx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)xdx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int 1dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(x + c_1)$
- Use initial condition $y(2) = 6$

  $6 = 4 + 2c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = x(x + 1)$
- Solution to the IVP

  $y = x(x + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([x*diff(y(x),x)-y(x)=x^2,y(2) = 6],y(x), singsol=all)
```

$$y(x) = x(1 + x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 10

```
DSolve[{x*y'[x]-y[x]==x^2,{y[2]==6}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x + 1)$$

8.27 problem 27

8.27.1 Existence and uniqueness analysis 1511
8.27.2 Solving as second order linear constant coeff ode 1512
8.27.3 Solving using Kovacic algorithm 1516
8.27.4 Maple step by step solution 1521

Internal problem ID [4890]
Internal file name [`OUTPUT/4383_Sunday_June_05_2022_01_12_13_PM_57085905/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 27.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 6}$$

With initial conditions

$$[y(0) = 1, y'(0) = 4]$$

8.27.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -6 \\ F &= 6 \end{aligned}$$

Hence the ode is

$$y'' + y' - 6y = 6$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.27.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -6, f(x) = 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda \,\mathrm{e}^{\lambda x} - 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)} \\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{5}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-3x}\right) + (-1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-3x} - 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = 2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

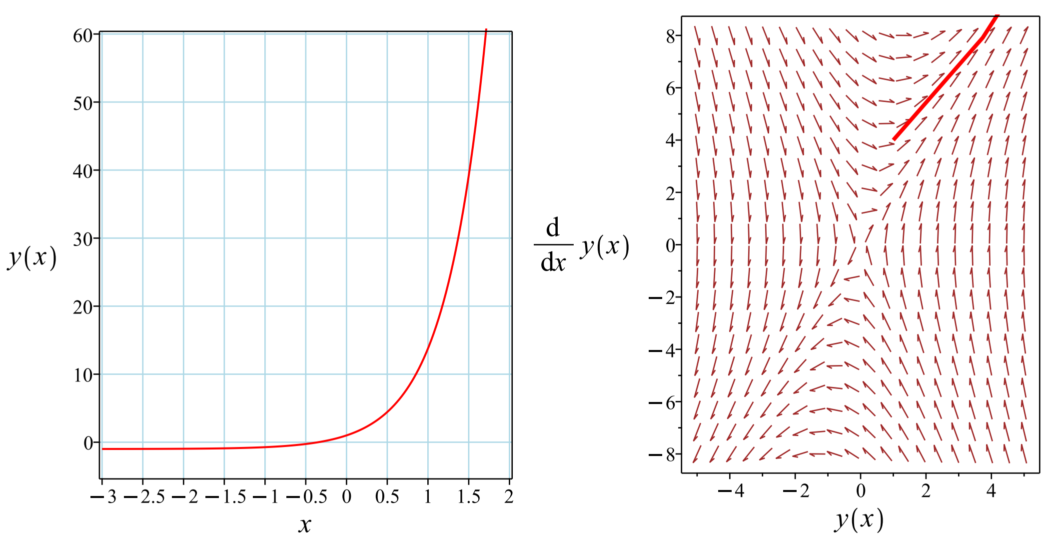
$$c_1 = 2$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -1 + 2\,e^{2x}$$

Summary
The solution(s) found are the following

$$y = -1 + 2\,e^{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -1 + 2\,e^{2x}$$

Verified OK.

8.27.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
y'' + y' - 6y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1\\
B &= 1 \qquad (3)\\
C &= -6
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{25}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 25\\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{25z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\,e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 215: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{5x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-3x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' - 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{2x}}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{2x}}{5}, e^{-3x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{2x}}{5}\right) + (-1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-3x} + \frac{c_2 e^{2x}}{5} - 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + \frac{c_2}{5} - 1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} + \frac{2c_2 e^{2x}}{5}$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -3c_1 + \frac{2c_2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

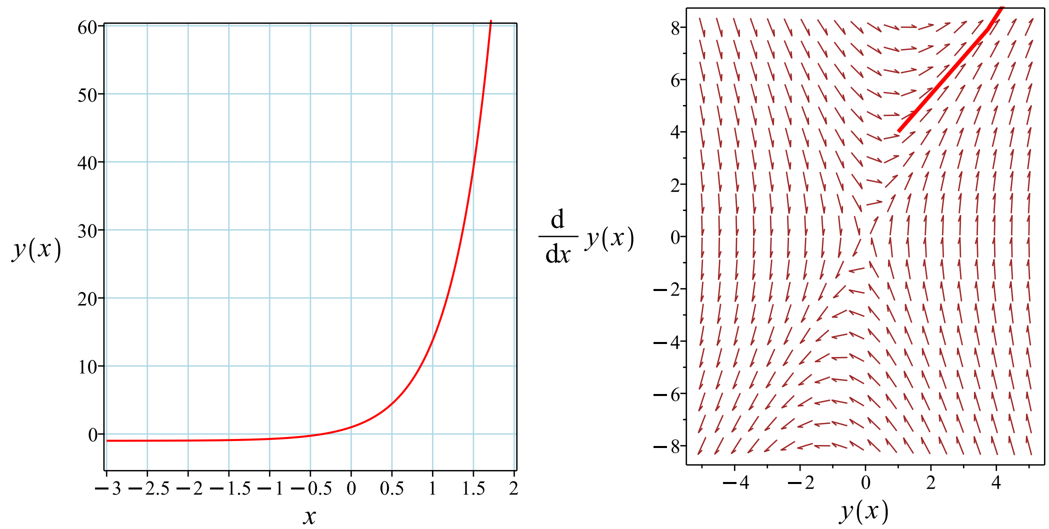
$$c_1 = 0$$
$$c_2 = 10$$

Substituting these values back in above solution results in

$$y = -1 + 2\,e^{2x}$$

Summary
The solution(s) found are the following

$$y = -1 + 2\,e^{2x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -1 + 2\,e^{2x}$$

Verified OK.

8.27.4 Maple step by step solution

Let's solve

$$\left[y'' + y' - 6y = 6, y(0) = 1, y'\Big|_{\{x=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2
  y''
- Characteristic polynomial of homogeneous ODE
  $r^2 + r - 6 = 0$
- Factor the characteristic polynomial
  $(r+3)(r-2) = 0$
- Roots of the characteristic polynomial
  $r = (-3, 2)$
- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 6\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{2x} \\ -3\,\mathrm{e}^{-3x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,\mathrm{e}^{-x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{6\left(\mathrm{e}^{5x}\left(\int \mathrm{e}^{-2x}dx\right) - \left(\int \mathrm{e}^{3x}dx\right)\right)\mathrm{e}^{-3x}}{5}$

  - Compute integrals

    $y_p(x) = -1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x} - 1$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x} - 1$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2 - 1$

  - Compute derivative of the solution

    $y' = -3c_1 \mathrm{e}^{-3x} + 2c_2 \mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 4$

    $4 = -3c_1 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = 2\}$
- Substitute constant values into general solution and simplify

  $y = -1 + 2\,\mathrm{e}^{2x}$

- Solution to the IVP

  $y = -1 + 2\,\mathrm{e}^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)+diff(y(x),x)-6*y(x)=6,y(0) = 1, D(y)(0) = 4],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{2x} - 1$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 14

```
DSolve[{y''[x]+y'[x]-6*y[x]==6,{y[0]==1,y'[0]==4}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{2x} - 1$$

8.28 problem 28

8.28.1 Solving as second order integrable as is ode 1525
8.28.2 Solving as second order ode missing x ode 1526
8.28.3 Solving as type second_order_integrable_as_is (not using ABC version) 1530
8.28.4 Solving as exact nonlinear second order ode ode 1531
8.28.5 Maple step by step solution 1534

Internal problem ID [4891]
Internal file name [`OUTPUT/4384_Sunday_June_05_2022_01_12_24_PM_52737621/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 8, Ordinary differential equations. Section 13. Miscellaneous problems. page 466
Problem number: 28.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_integrable_as_is", "second_order_ode_missing_x", "exact nonlinear second order ode"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x], [_2nd_order, _exact, _nonlinear], [
   _2nd_order, _reducible, _mu_x_y1], [_2nd_order, _reducible,
   _mu_xy]]
```

$$\boxed{yy'' + {y'}^2 = -4}$$

With initial conditions

$$[y(1) = 3, y'(1) = 0]$$

8.28.1 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(yy'' + y'^2\right) dx &= \int (-4)\, dx \\
y'y &= -4x + c_1
\end{aligned}
$$

Which is now solved for y. In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{-4x + c_1}{y}
\end{aligned}
$$

Where $f(x) = -4x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\, dy &= -4x + c_1\, dx \\
\int \frac{1}{\frac{1}{y}}\, dy &= \int -4x + c_1\, dx \\
\frac{y^2}{2} &= c_1 x - 2x^2 + c_2
\end{aligned}
$$

The solution is

$$
\frac{y^2}{2} - c_1 x + 2x^2 - c_2 = 0
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
\frac{y^2}{2} - c_1 x + 2x^2 - c_2 = 0 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
\frac{13}{2} - c_1 - c_2 = 0 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 - 8x}{2\sqrt{2c_1x - 4x^2 + 2c_2}}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{c_1 - 4}{\sqrt{2c_1 - 4 + 2c_2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 4 \\ c_2 &= \frac{5}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0$$

Verified OK.

8.28.2 Solving as second order ode missing x ode

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p(y)$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dy}{dx}\frac{dp}{dy} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right)+p(y)^2=-4$$

Which is now solved as first order ode for $p(y)$. In canonical form the ODE is

$$\begin{aligned}p' &= F(y,p)\\ &= f(y)g(p)\\ &= -\frac{p^2+4}{yp}\end{aligned}$$

Where $f(y)=-\frac{1}{y}$ and $g(p)=\frac{p^2+4}{p}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{p^2+4}{p}}\,dp &= -\frac{1}{y}\,dy\\ \int\frac{1}{\frac{p^2+4}{p}}\,dp &= \int -\frac{1}{y}\,dy\\ \frac{\ln\left(p^2+4\right)}{2} &= -\ln\left(y\right)+c_1\end{aligned}$$

Raising both side to exponential gives

$$\sqrt{p^2+4}=\mathrm{e}^{-\ln(y)+c_1}$$

Which simplifies to

$$\sqrt{p^2+4}=\frac{c_2}{y}$$

Which can be simplified to become

$$\sqrt{p\left(y\right)^2+4}=\frac{c_2\mathrm{e}^{c_1}}{y}$$

The solution is

$$\sqrt{p\left(y\right)^2+4}=\frac{c_2\mathrm{e}^{c_1}}{y}$$

Initial conditions are used to solve for c_1. Substituting $y=3$ and $p=0$ in the above solution gives an equation to solve for the constant of integration.

$$2=\frac{c_2\mathrm{e}^{c_1}}{3}$$

$$c_1 = \ln\left(\frac{6}{c_2}\right)$$

Substituting c_1 found above in the general solution gives

$$\sqrt{p^2+4} = \frac{6}{y}$$

The above simplifies to

$$\sqrt{p^2+4}\, y - 6 = 0$$

Solving for $p(y)$ from the above gives

$$p(y) = \frac{2\sqrt{-y^2+9}}{y}$$
$$p(y) = -\frac{2\sqrt{-y^2+9}}{y}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{2\sqrt{-y^2+9}}{y}$$

Integrating both sides gives

$$\int \frac{y}{2\sqrt{-y^2+9}} dy = \int dx$$
$$\frac{(y-3)(y+3)}{2\sqrt{-y^2+9}} = x + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3 + 1$$

$$c_3 = -1$$

Substituting c_3 found above in the general solution gives

$$\frac{(y-3)(3+y)}{2\sqrt{-y^2+9}} = x - 1$$

The above simplifies to

$$-2x\sqrt{-y^2+9}+y^2+2\sqrt{-y^2+9}-9=0$$

For solution (2) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$y'=-\frac{2\sqrt{-y^2+9}}{y}$$

Integrating both sides gives

$$\int -\frac{y}{2\sqrt{-y^2+9}}dy=\int dx$$

$$-\frac{(y-3)\,(y+3)}{2\sqrt{-y^2+9}}=x+c_4$$

Initial conditions are used to solve for c_4. Substituting $x=1$ and $y=3$ in the above solution gives an equation to solve for the constant of integration.

$$0=1+c_4$$

$$c_4=-1$$

Substituting c_4 found above in the general solution gives

$$-\frac{(y-3)\,(3+y)}{2\sqrt{-y^2+9}}=x-1$$

The above simplifies to

$$-2x\sqrt{-y^2+9}-y^2+2\sqrt{-y^2+9}+9=0$$

Initial conditions are used to solve for the constants of integration.

Summary

The solution(s) found are the following

$$(-2x+2)\sqrt{-y^2+9}+y^2-9=0 \tag{1}$$

$$(-2x+2)\sqrt{-y^2+9}-y^2+9=0 \tag{2}$$

Verification of solutions

$$(-2x+2)\sqrt{-y^2+9}+y^2-9=0$$

Verified OK.

$$(-2x+2)\sqrt{-y^2+9}-y^2+9=0$$

Verified OK.

8.28.3 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$yy'' + y'^2 = -4$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(yy'' + y'^2\right) dx = \int (-4)\, dx$$
$$y'y = -4x + c_1$$

Which is now solved for y. In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{-4x + c_1}{y} \end{aligned}$$

Where $f(x) = -4x + c_1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{1}{y}}\, dy = -4x + c_1\, dx$$
$$\int \frac{1}{\frac{1}{y}}\, dy = \int -4x + c_1\, dx$$
$$\frac{y^2}{2} = c_1 x - 2x^2 + c_2$$

The solution is

$$\frac{y^2}{2} - c_1 x + 2x^2 - c_2 = 0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\frac{y^2}{2} - c_1 x + 2x^2 - c_2 = 0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$\frac{13}{2} - c_1 - c_2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{2c_1 - 8x}{2\sqrt{2c_1 x - 4x^2 + 2c_2}}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{c_1 - 4}{\sqrt{2c_1 - 4 + 2c_2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 4 \\ c_2 &= \frac{5}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} - 4x + 2x^2 - \frac{5}{2} = 0$$

Verified OK.

8.28.4 Solving as exact nonlinear second order ode ode

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$
\begin{aligned}
\frac{\partial a_2}{\partial y} &= \frac{\partial a_1}{\partial y'} \\
\frac{\partial a_2}{\partial x} &= \frac{\partial a_0}{\partial y'} \\
\frac{\partial a_1}{\partial x} &= \frac{\partial a_0}{\partial y}
\end{aligned}
$$

Looking at the the ode given we see that

$$
\begin{aligned}
a_2 &= y \\
a_1 &= y' \\
a_0 &= 4
\end{aligned}
$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$
\begin{aligned}
\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\
\int y\, dy' + \int y'\, dy + \int 4\, dx &= c_1
\end{aligned}
$$

Which results in

$$
2y'y + 4x = c_1
$$

Which is now solved In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{-2x + \frac{c_1}{2}}{y}
\end{aligned}
$$

Where $f(x) = -2x + \frac{c_1}{2}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{1}{y}}\, dy &= -2x + \frac{c_1}{2}\, dx \\
\int \frac{1}{\frac{1}{y}}\, dy &= \int -2x + \frac{c_1}{2}\, dx \\
\frac{y^2}{2} &= -x^2 + \frac{1}{2}c_1 x + c_2
\end{aligned}
$$

The solution is

$$\frac{y^2}{2} + x^2 - \frac{c_1 x}{2} - c_2 = 0$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$\frac{y^2}{2} + x^2 - \frac{c_1 x}{2} - c_2 = 0 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$\frac{11}{2} - \frac{c_1}{2} - c_2 = 0 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{-4x + c_1}{2\sqrt{c_1 x - 2x^2 + 2c_2}}$$

substituting $y' = 0$ and $x = 1$ in the above gives

$$0 = \frac{c_1 - 4}{2\sqrt{c_1 - 2 + 2c_2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = \frac{7}{2}$$

Substituting these values back in above solution results in

$$\frac{y^2}{2} + x^2 - 2x - \frac{7}{2} = 0$$

Summary

The solution(s) found are the following

$$\frac{y^2}{2} + x^2 - 2x - \frac{7}{2} = 0 \tag{1}$$

Verification of solutions

$$\frac{y^2}{2} + x^2 - 2x - \frac{7}{2} = 0$$

Warning, solution could not be verified

8.28.5 Maple step by step solution

Let's solve

$$\left[yy''+y'^2=-4, y(1)=3, y'\Big|_{\{x=1\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Define new dependent variable u

  $u(x)=y'$

- Compute y''

  $u'(x)=y''$

- Use chain rule on the lhs

  $y'\left(\frac{d}{dy}u(y)\right)=y''$

- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right)=y''$

- Make substitutions $y'=u(y)\,, y''=u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $yu(y)\left(\frac{d}{dy}u(y)\right)+u(y)^2=-4$

- Separate variables

  $\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2-4}=\frac{1}{y}$

- Integrate both sides with respect to y

  $\int\frac{\left(\frac{d}{dy}u(y)\right)u(y)}{-u(y)^2-4}dy=\int\frac{1}{y}dy+c_1$

- Evaluate integral

  $-\frac{\ln\left(u(y)^2+4\right)}{2}=\ln(y)+c_1$

- Solve for $u(y)$

  $\left\{u(y)=\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2y^2}}{\mathrm{e}^{c_1}y}, u(y)=-\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2y^2}}{\mathrm{e}^{c_1}y}\right\}$

- Solve 1st ODE for $u(y)$

  $u(y)=\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2y^2}}{\mathrm{e}^{c_1}y}$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = \frac{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}}{\mathrm{e}^{c_1} y}$$

- Separate variables

$$\frac{y' y}{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}} = \frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}} dx = \int \frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}}{4(\mathrm{e}^{c_1})^2} = \frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\,\mathrm{e}^{c_1}}, y = \frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\,\mathrm{e}^{c_1}}\right\}$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}}{\mathrm{e}^{c_1} y}$$

- Revert to original variables with substitution $u(y) = y', y = y$

$$y' = -\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}}{\mathrm{e}^{c_1} y}$$

- Separate variables

$$\frac{y' y}{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}} = -\frac{1}{\mathrm{e}^{c_1}}$$

- Integrate both sides with respect to x

$$\int \frac{y' y}{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}} dx = \int -\frac{1}{\mathrm{e}^{c_1}} dx + c_2$$

- Evaluate integral

$$-\frac{\sqrt{1-4(\mathrm{e}^{c_1})^2 y^2}}{4(\mathrm{e}^{c_1})^2} = -\frac{x}{\mathrm{e}^{c_1}} + c_2$$

- Solve for y

$$\left\{y = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2+32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\,\mathrm{e}^{c_1}}, y = \frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2+32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\,\mathrm{e}^{c_1}}\right\}$$

☐ Check validity of solution $y = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\mathrm{e}^{c_1}}$

  - Use initial condition $y(1) = 3$

$$3 = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2-16(\mathrm{e}^{c_1})^2}}{2\,\mathrm{e}^{c_1}}$$

- Compute derivative of the solution

$$y' = -\frac{-32(\mathrm{e}^{c_1})^3 c_2-32(\mathrm{e}^{c_1})^2 x}{4\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}\,\mathrm{e}^{c_1}}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

$$0 = -\frac{-32(\mathrm{e}^{c_1})^3 c_2-32(\mathrm{e}^{c_1})^2}{4\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2-16(\mathrm{e}^{c_1})^2}\,\mathrm{e}^{c_1}}$$

- Solve for c_1 and c_2

$$\{c_1 = -\ln(6) + \mathrm{I}\pi, c_2 = 6\}$$

- Substitute constant values into general solution and simplify

$$y = \sqrt{-4x^2+8x+5}$$

☐ Check validity of solution $y = \frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\mathrm{e}^{c_1}}$

- Use initial condition $y(1) = 3$

$$3 = \frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2-16(\mathrm{e}^{c_1})^2}}{2\,\mathrm{e}^{c_1}}$$

- Compute derivative of the solution

$$y' = \frac{-32(\mathrm{e}^{c_1})^3 c_2-32(\mathrm{e}^{c_1})^2 x}{4\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}\,\mathrm{e}^{c_1}}$$

- Use the initial condition $y'\Big|_{\{x=1\}} = 0$

$$0 = \frac{-32(\mathrm{e}^{c_1})^3 c_2-32(\mathrm{e}^{c_1})^2}{4\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2-32(\mathrm{e}^{c_1})^3 c_2-16(\mathrm{e}^{c_1})^2}\,\mathrm{e}^{c_1}}$$

- Solve for c_1 and c_2

$$\{c_1 = -\ln(6), c_2 = -6\}$$

- Substitute constant values into general solution and simplify

$$y = \sqrt{-4x^2+8x+5}$$

☐ Check validity of solution $y = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2+32(\mathrm{e}^{c_1})^3 c_2 x-16(\mathrm{e}^{c_1})^2 x^2}}{2\mathrm{e}^{c_1}}$

- Use initial condition $y(1) = 3$

$$3 = -\frac{\sqrt{1-16(\mathrm{e}^{c_1})^4 c_2^2+32(\mathrm{e}^{c_1})^3 c_2-16(\mathrm{e}^{c_1})^2}}{2\,\mathrm{e}^{c_1}}$$

- Compute derivative of the solution

$$y' = -\frac{32(e^{c_1})^3 c_2 - 32(e^{c_1})^2 x}{4\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2 x-16(e^{c_1})^2 x^2}\, e^{c_1}}$$

- Use the initial condition $y'\big|_{\{x=1\}} = 0$

$$0 = -\frac{32(e^{c_1})^3 c_2 - 32(e^{c_1})^2}{4\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2-16(e^{c_1})^2}\, e^{c_1}}$$

- Solve for c_1 and c_2

$$\{c_1 = -\ln(6) + I\pi, c_2 = -6\}$$

- Substitute constant values into general solution and simplify

$$y = \sqrt{-4x^2+8x+5}$$

□ Check validity of solution $y = \frac{\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2 x-16(e^{c_1})^2 x^2}}{2e^{c_1}}$

- Use initial condition $y(1) = 3$

$$3 = \frac{\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2-16(e^{c_1})^2}}{2\, e^{c_1}}$$

- Compute derivative of the solution

$$y' = \frac{32(e^{c_1})^3 c_2 - 32(e^{c_1})^2 x}{4\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2 x-16(e^{c_1})^2 x^2}\, e^{c_1}}$$

- Use the initial condition $y'\big|_{\{x=1\}} = 0$

$$0 = \frac{32(e^{c_1})^3 c_2 - 32(e^{c_1})^2}{4\sqrt{1-16(e^{c_1})^4 c_2^2+32(e^{c_1})^3 c_2-16(e^{c_1})^2}\, e^{c_1}}$$

- Solve for c_1 and c_2

$$\{c_1 = -\ln(6), c_2 = 6\}$$

- Substitute constant values into general solution and simplify

$$y = \sqrt{-4x^2+8x+5}$$

- Solution to the IVP

$$y = \sqrt{-4x^2+8x+5}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying a quadrature
<- quadrature successful
<- 2nd order, 2 integrating factors of the form mu(x,y) successful`
```

✓ Solution by Maple

Time used: 0.891 (sec). Leaf size: 16

```
dsolve([y(x)*diff(y(x),x$2)+diff(y(x),x)^2+4=0,y(1) = 3, D(y)(1) = 0],y(x), singsol=all)
```

$$y(x) = \sqrt{-4x^2 + 8x + 5}$$

✓ Solution by Mathematica

Time used: 31.559 (sec). Leaf size: 19

```
DSolve[{y[x]*y''[x]+y'[x]^2+4==0,{y[1]==3,y'[1]==0}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \sqrt{-4x^2 + 8x + 5}$$

9 Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564

9.1 problem 1, using series method 1540
9.2 problem 1, using elementary method 1547
9.3 problem 2, using series method 1561
9.4 problem 2, using elementary method 1570
9.5 problem 3, using series method 1585
9.6 problem 3, using elementary method 1591
9.7 problem 4, using series method 1605
9.8 problem 4, using elementary method 1615
9.9 problem 5, using series method 1625
9.10 problem 5, using elementary method 1635
9.11 problem 6, using series method 1645
9.12 problem 6, using elementary method 1655
9.13 problem 7, using series method 1664
9.14 problem 7, using elementary method 1676
9.15 problem 8, using series method 1695
9.16 problem 8, using elementary method 1710
9.17 problem 9, using series method 1724
9.18 problem 9, using elementary method 1732
9.19 problem 10, using series method 1744
9.20 problem 10, using elementary method 1754

9.1 problem 1, using series method

9.1.1 Solving as series ode 1540
9.1.2 Maple step by step solution 1545

Internal problem ID [4892]
Internal file name [`OUTPUT/4385_Sunday_June_05_2022_01_12_34_PM_42179273/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 1, using series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Regular singular point"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - xy - y = 0}$$

With the expansion point for the power series method at $x = 0$.

9.1.1 Solving as series ode

Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - \frac{(x+1)\,y}{x} = 0$$

Where

$$q(x) = -\frac{x+1}{x}$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = -1 - x$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \frac{(x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)}{x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(-1 - \frac{1}{x}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) - \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty}(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right) + \sum_{n=0}^{\infty}\left(-x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} - x^{n+r-1} a_n = 0$$

When $n=0$ the above becomes

$$r a_0 x^{-1+r} - x^{-1+r} a_0 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$-1+r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 1$$

We start by finding y_h. Replacing $r=1$ found above results in

$$\left(\sum_{n=0}^{\infty}(n+1)\, a_n x^{n}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n}\right) + \sum_{n=0}^{\infty}\left(-x^{n} a_n\right) = 0$$

For $1 \leq n$, the recurrence equation is

$$(n+1)\, a_n - a_{n-1} - a_n = 0 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$a_1 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = a_0$$

For $n = 2$ the recurrence equation gives

$$2a_2 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$3a_3 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_0}{6}$$

For $n = 4$ the recurrence equation gives

$$4a_4 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 5$ the recurrence equation gives

$$5a_5 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_0}{120}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= a_3 x^4 + a_2 x^3 + a_1 x^2 + a_0 x + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 x + a_0 x^2 + \frac{1}{2} a_0 x^3 + \frac{1}{6} a_0 x^4 + \frac{1}{24} a_0 x^5 + \frac{1}{120} a_0 x^6 + \dots
$$

Which can be written as

$$
y = x\left(a_0 + a_0 x + \frac{a_0 x^2}{2} + \frac{a_0 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_0 x^5}{120} + O(x^6)\, a_0\right)
$$

Collecting terms, the solution becomes

$$
y = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) a_0 \tag{3}
$$

Finally, since $r = 1$, then the solution becomes

$$
y = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) a_0 \tag{3}
$$

Therefore the homogeneous solution is

$$
y_h(x) = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) a_0
$$

At $x = 0$ the solution above becomes

$$
y = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) c_1
$$

Summary

The solution(s) found are the following

$$
y = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) c_1 \tag{1}
$$

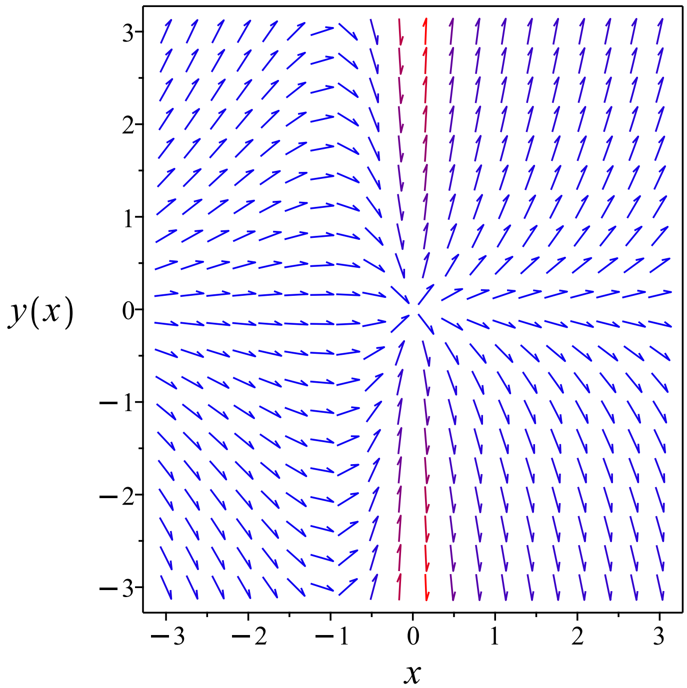


Figure 257: Slope field plot

Verification of solutions

$$y = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right) c_1$$

Verified OK.

9.1.2 Maple step by step solution

Let's solve

$y' - \frac{(x+1)y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{x+1}{x} dx + c_1$

- Evaluate integral

$$\ln (y) = x + \ln (x) + c_1$$

- Solve for y

$$y = \frac{x}{e^{-c_1 - x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
Order:=6;
dsolve(x*diff(y(x),x)=x*y(x)+y(x),y(x),type='series',x=0);
```

$$y(x) = c_1 x \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5\right) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 38

```
AsymptoticDSolveValue[x*y'[x]==x*y[x]+y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1\right)$$

9.2 problem 1, using elementary method

9.2.1 Solving as separable ode 1547
9.2.2 Solving as linear ode 1549
9.2.3 Solving as homogeneousTypeD2 ode 1550
9.2.4 Solving as first order ode lie symmetry lookup ode 1551
9.2.5 Solving as exact ode 1555
9.2.6 Maple step by step solution 1559

Internal problem ID [4893]
Internal file name [`OUTPUT/4386_Sunday_June_05_2022_01_12_43_PM_59467964/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 1, using elementary method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xy' - xy - y = 0}$$

9.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x+1)\,y}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{x+1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{x+1}{x}\,dx \\
\ln(y) &= x + \ln(x) + c_1 \\
y &= e^{x+\ln(x)+c_1} \\
&= c_1 e^{x+\ln(x)}
\end{aligned}
$$

Which simplifies to

$$y = c_1 x\,e^{x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\,e^{x} \tag{1}$$

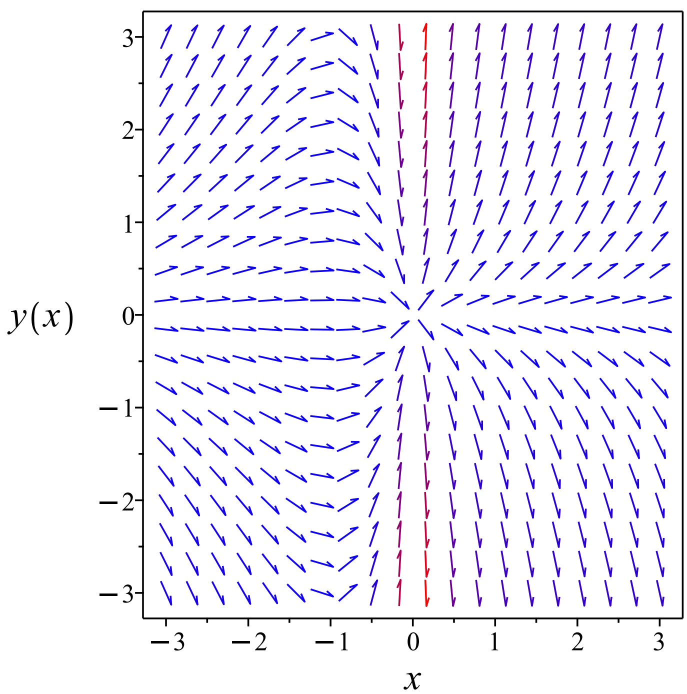


Figure 258: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 x\,e^{x}$$

Verified OK.

9.2.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{x+1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{(x+1)\,y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{x+1}{x}dx} \\ &= \mathrm{e}^{-x-\ln(x)} \end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{-x}}{x}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{-x} y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{\mathrm{e}^{-x} y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{-x}}{x}$ results in

$$y = c_1 x\,\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{x} \tag{1}$$

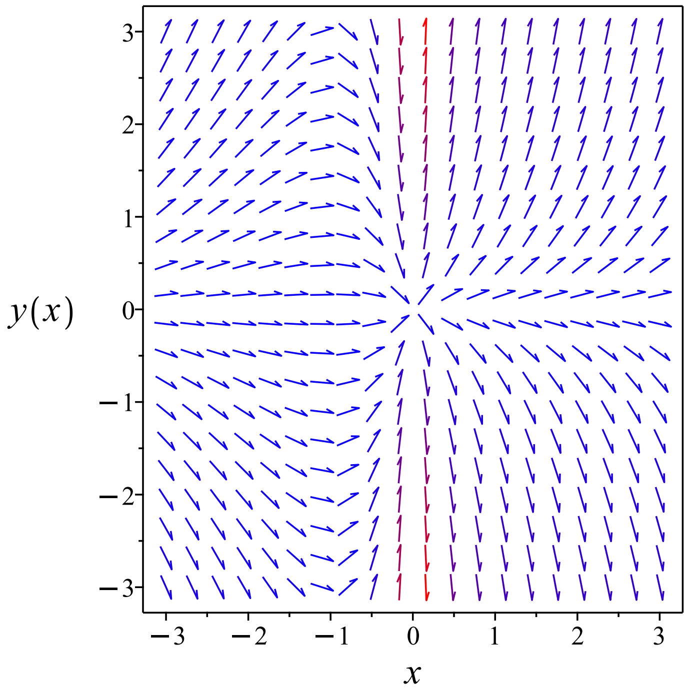


Figure 259: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{x}$$

Verified OK.

9.2.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(u'(x)\,x + u(x)) - x^2 u(x) - u(x)\,x = 0$$

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{u} du &= c_2 + x \\ \ln(u) &= c_2 + x \\ u &= \mathrm{e}^{c_2+x} \\ u &= c_2 \mathrm{e}^{x}\end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 x\,\mathrm{e}^{x}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x\,\mathrm{e}^{x} \tag{1}$$

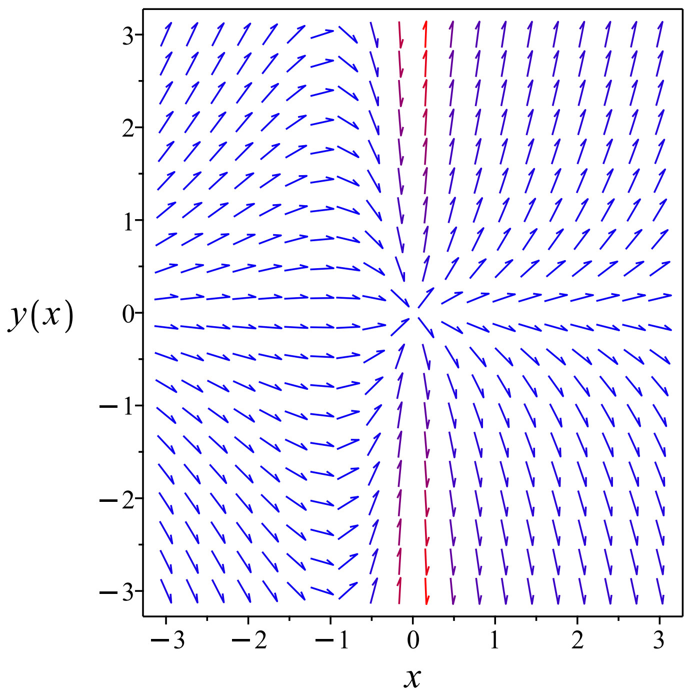


Figure 260: Slope field plot

Verification of solutions

$$y = c_2 x\,\mathrm{e}^{x}$$

Verified OK.

9.2.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{(x+1)\,y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x+\ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{x+\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{\mathrm{e}^{-x} y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(x+1)\, y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{\mathrm{e}^{-x} y(x+1)}{x^2} \\ S_y &= \frac{\mathrm{e}^{-x}}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{-x} y}{x} = c_1$$

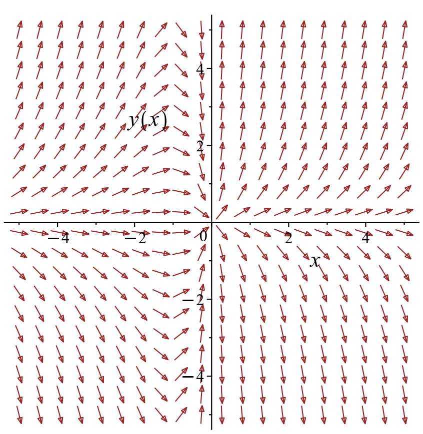
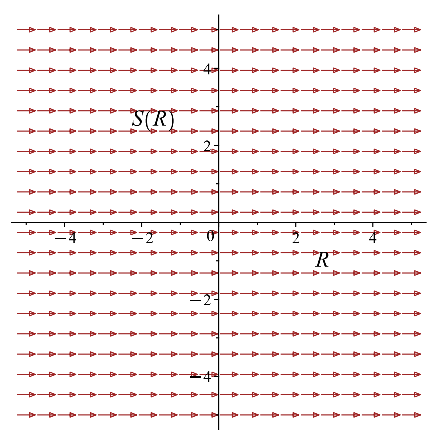
Which simplifies to

$$\frac{e^{-x} y}{x} = c_1$$

Which gives

$$y = c_1 x\, e^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(x+1)y}{x}$ 	$R = x$ $S = \frac{e^{-x} y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 x\, e^{x} \tag{1}$$

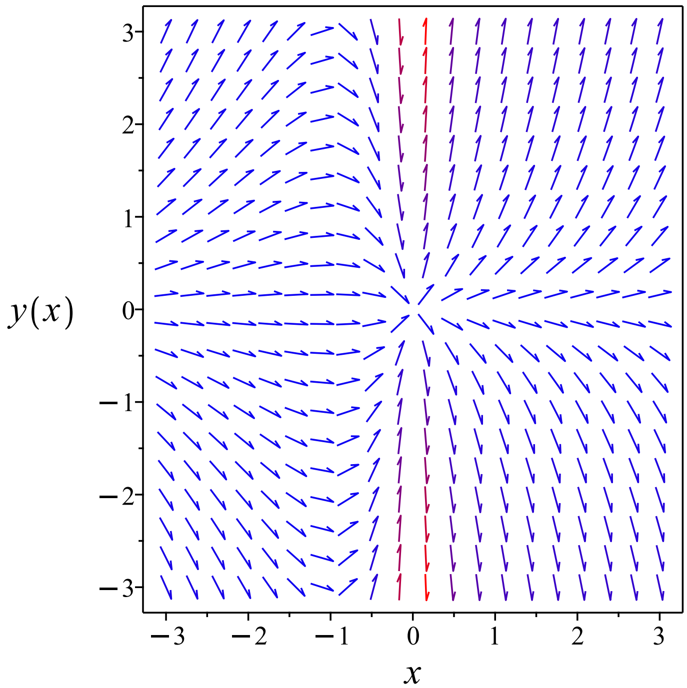


Figure 261: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{x}$$

Verified OK.

9.2.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{x+1}{x}\right)\mathrm{d}x \\
\left(-\frac{x+1}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x+1}{x} \\
N(x, y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x+1}{x} \, \mathrm{d}x \\
\phi &= -x - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{x+c_1} x$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x+c_1} x \tag{1}$$

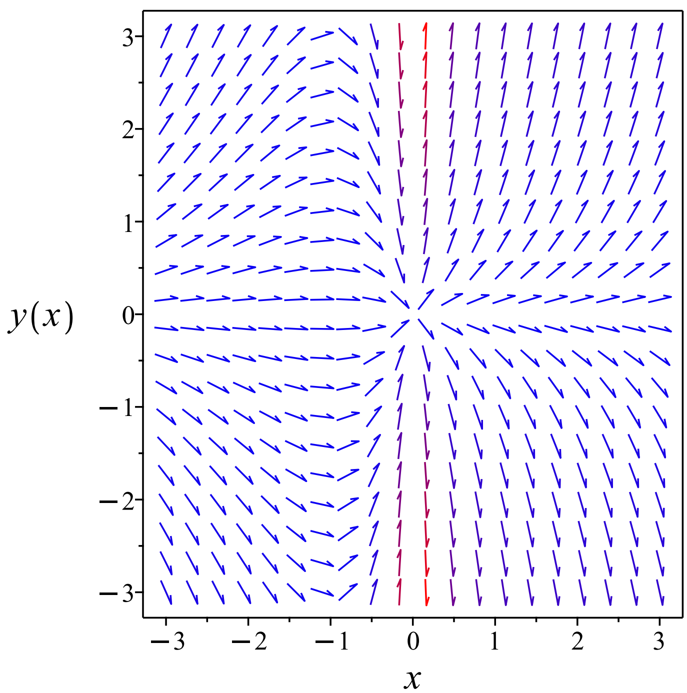


Figure 262: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x+c_1} x$$

Verified OK.

9.2.6 Maple step by step solution

Let's solve

$xy' - xy - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{x+1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = x + \ln(x) + c_1$

- Solve for y

  $y = \frac{x}{e^{-c_1 - x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)=x*y(x)+y(x),y(x), singsol=all)
```

$$y(x) = x\,e^{x}c_1$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 17

```
DSolve[x*y'[x]==x*y[x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^x x$$
$$y(x) \to 0$$

9.3 problem 2, using series method

9.3.1 Solving as series ode 1561
9.3.2 Maple step by step solution 1568

Internal problem ID [4894]
Internal file name [`OUTPUT/4387_Sunday_June_05_2022_01_12_53_PM_56613934/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 2, using series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Ordinary point", "first order ode series method. Taylor series method"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3yx^2 = 0}$$

With the expansion point for the power series method at $x = 0$.

9.3.1 Solving as series ode

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving first order ode. Let

$$y' = f(x, y)$$

Where $f(x, y)$ is analytic at expansion point x_0. We can always shift to $x_0 = 0$ if x_0 is not zero. So from now we assume $x_0 = 0$. Assume also that $y(x_0) = y_0$. Using Taylor series

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y''(x_0) + \cdots \\
&= y_0 + x f + \frac{x^2}{2} \frac{df}{dx}\bigg|_{x_0, y_0} + \frac{x^3}{3!} \frac{d^2 f}{dx^2}\bigg|_{x_0, y_0} + \cdots \\
&= y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f && (1) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y} \frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y)$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y) && (4) \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) F_0 && (5)
\end{aligned}
$$

For example, for $n = 1$ we see that

$$
\begin{aligned}
F_1 &= \frac{d}{dx}(F_0) \\
&= \frac{\partial}{\partial x} F_0 + \left(\frac{\partial F_0}{\partial y}\right) F_0 \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f
\end{aligned}
$$

Which is (1). And when $n = 2$

$$\begin{aligned}
F_2 &= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) F_0 \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right) f \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) f
\end{aligned}$$

Which is (2) and so on. Therefore (4,5) can be used from now on along with

$$y(x) = y_0 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} F_n|_{x_0,y_0} \tag{6}$$

Hence

$$\begin{aligned}
F_0 &= 3yx^2 \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} F_0 \\
&= \left(9x^4 + 6x\right) y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} F_1 \\
&= 3y\left(9x^6 + 18x^3 + 2\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} F_2 \\
&= \left(81x^8 + 324x^5 + 180x^2\right) y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} F_3 \\
&= \left(243x^{10} + 1620x^7 + 2160x^4 + 360x\right) y
\end{aligned}$$

And so on. Evaluating all the above at initial conditions $x(0) = 0$ and $y(0) = y(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= 0 \\
F_2 &= 6y(0) \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (6) and simplifying gives the solution as

$$
y = \left(x^3 + 1\right) y(0) + O\left(x^6\right)
$$

Since $x = 0$ is also an ordinary point, then standard power series can also be used. Writing the ODE as

$$
\begin{aligned}
y' + q(x)y &= p(x) \\
y' - 3yx^2 &= 0
\end{aligned}
$$

Where

$$
\begin{aligned}
q(x) &= -3x^2 \\
p(x) &= 0
\end{aligned}
$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
y' = \sum_{n=1}^{\infty} n a_n x^{n-1}
$$

Substituting the above back into the ode gives

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-3 x^{n+2} a_n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$\sum_{n=0}^{\infty} \left(-3 x^{n+2} a_n\right) = \sum_{n=2}^{\infty} \left(-3 a_{n-2} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \sum_{n=2}^{\infty} \left(-3 a_{n-2} x^n\right) = 0 \tag{3}$$

For $2 \leq n$, the recurrence equation is

$$(n+1) a_{n+1} - 3 a_{n-2} = 0 \tag{4}$$

Solving for a_{n+1}, gives

$$a_{n+1} = \frac{3 a_{n-2}}{n+1} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$3 a_3 - 3 a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = a_0$$

For $n = 3$ the recurrence equation gives

$$4a_4 - 3a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 4$ the recurrence equation gives

$$5a_5 - 3a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 5$ the recurrence equation gives

$$6a_6 - 3a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{2}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 x^3 + a_0 + \dots$$

Collecting terms, the solution becomes

$$y = (x^3 + 1)\, a_0 + O(x^6) \tag{3}$$

Summary

The solution(s) found are the following

$$y = (x^3 + 1)\, y(0) + O(x^6) \tag{1}$$
$$y = (x^3 + 1)\, c_1 + O(x^6) \tag{2}$$

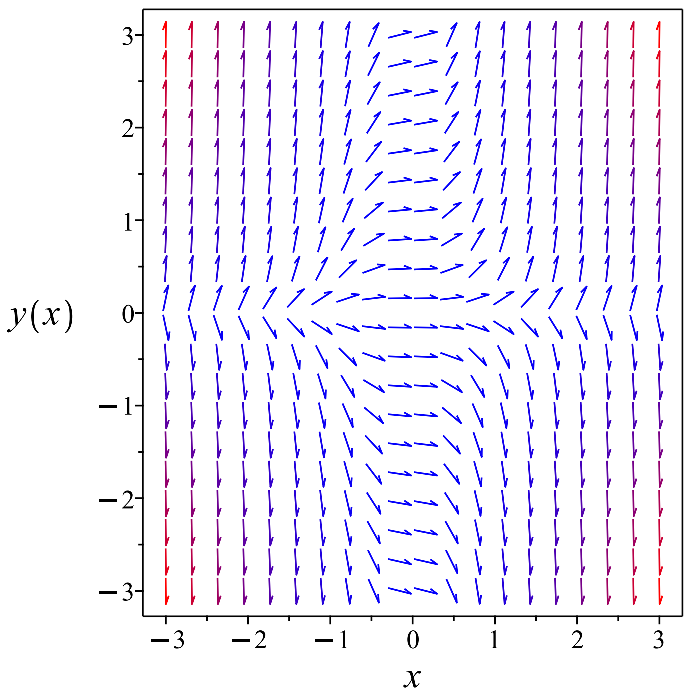


Figure 263: Slope field plot

Verification of solutions

$$y = (x^3 + 1)\, y(0) + O(x^6)$$

Verified OK.

$$y = (x^3 + 1)\, c_1 + O(x^6)$$

Verified OK.

9.3.2 Maple step by step solution

Let's solve

$y' - 3yx^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 3x^2dx + c_1$

- Evaluate integral

  $\ln(y) = x^3 + c_1$

- Solve for y

  $y = \mathrm{e}^{x^3+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
Order:=6;
dsolve(diff(y(x),x)=3*x^2*y(x),y(x),type='series',x=0);
```

$$y(x) = (x^3 + 1)\, y(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 11

```
AsymptoticDSolveValue[y'[x]==3*x^2*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1(x^3 + 1)$$

9.4 problem 2, using elementary method

9.4.1 Solving as separable ode 1570
9.4.2 Solving as linear ode 1572
9.4.3 Solving as homogeneousTypeD2 ode 1573
9.4.4 Solving as first order ode lie symmetry lookup ode 1575
9.4.5 Solving as exact ode 1579
9.4.6 Maple step by step solution 1583

Internal problem ID [4895]
Internal file name [`OUTPUT/4388_Sunday_June_05_2022_01_13_00_PM_48399041/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 2, using elementary method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 3yx^2 = 0}$$

9.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 3y\,x^2 \end{aligned}$$

Where $f(x) = 3x^2$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= 3x^2\,dx \\
\int \frac{1}{y}\,dy &= \int 3x^2\,dx \\
\ln(y) &= x^3 + c_1 \\
y &= \mathrm{e}^{x^3+c_1} \\
&= c_1 \mathrm{e}^{x^3}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x^3} \tag{1}$$

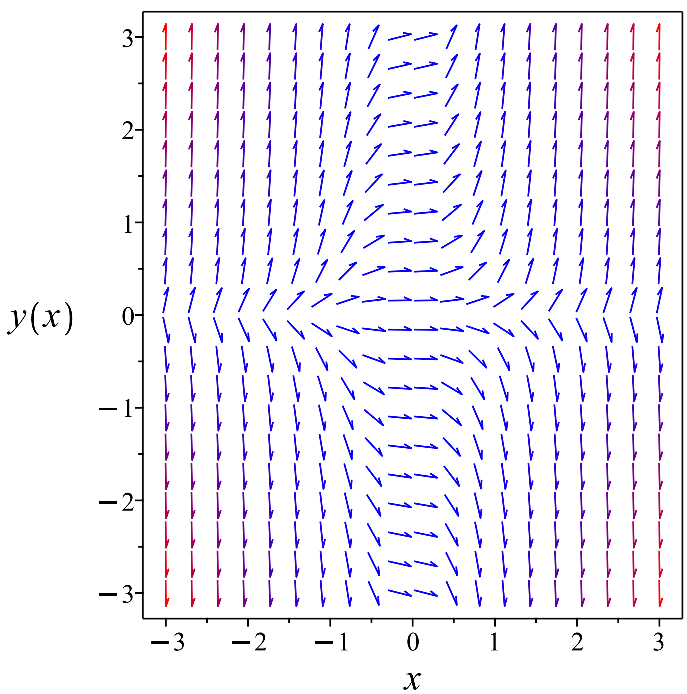


Figure 264: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x^3}$$

Verified OK.

9.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3x^2 \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - 3yx^2 = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -3x^2 dx} \\ &= e^{-x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-x^3} y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{-x^3} y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^3}$ results in

$$y = c_1 e^{x^3}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x^3} \tag{1}$$

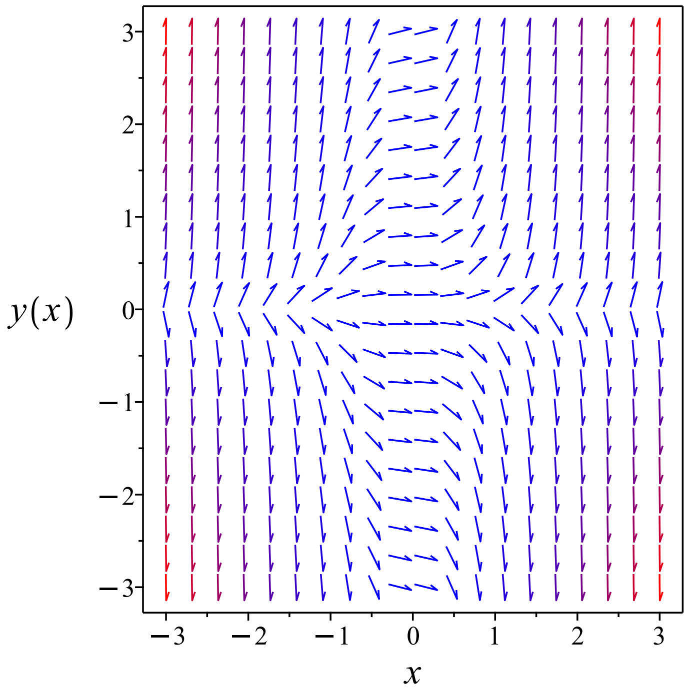


Figure 265: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{x^3}$$

Verified OK.

9.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - 3u(x)\, x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(3x^3-1)}{x} \end{aligned}$$

Where $f(x) = \frac{3x^3-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{3x^3-1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{3x^3-1}{x}\, dx \\
\ln(u) &= x^3 - \ln(x) + c_2 \\
u &= \mathrm{e}^{x^3-\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{x^3-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{x^3}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 \mathrm{e}^{x^3}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{x^3} \tag{1}
$$

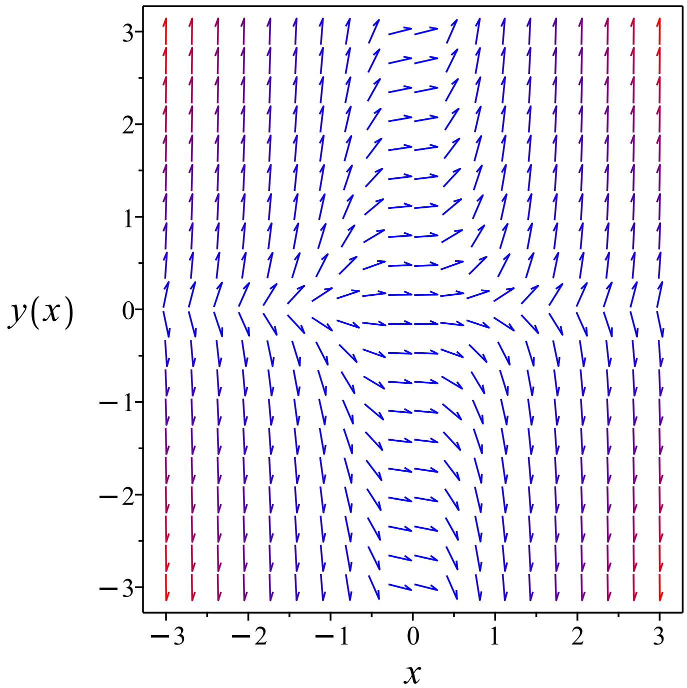


Figure 266: Slope field plot

Verification of solutions

$$y = c_2 e^{x^3}$$

Verified OK.

9.4.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3y x^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 223: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^3}} dy \end{aligned}$$

Which results in

$$S = e^{-x^3} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3y\, x^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -3x^2 e^{-x^3} y \\ S_y &= e^{-x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^3} y = c_1$$

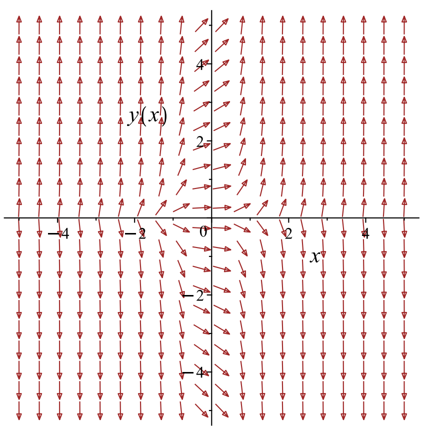
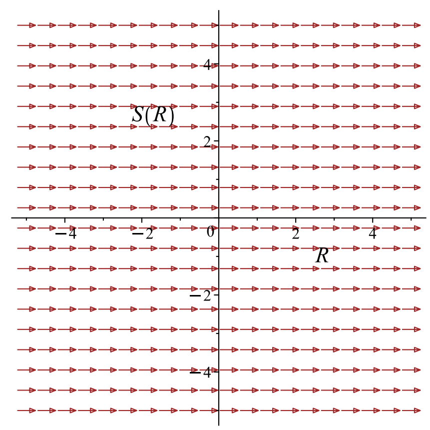
Which simplifies to

$$e^{-x^3} y = c_1$$

Which gives

$$y = c_1 e^{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3y\,x^2$ 	$R = x$ $S = e^{-x^3} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{x^3} \tag{1}$$

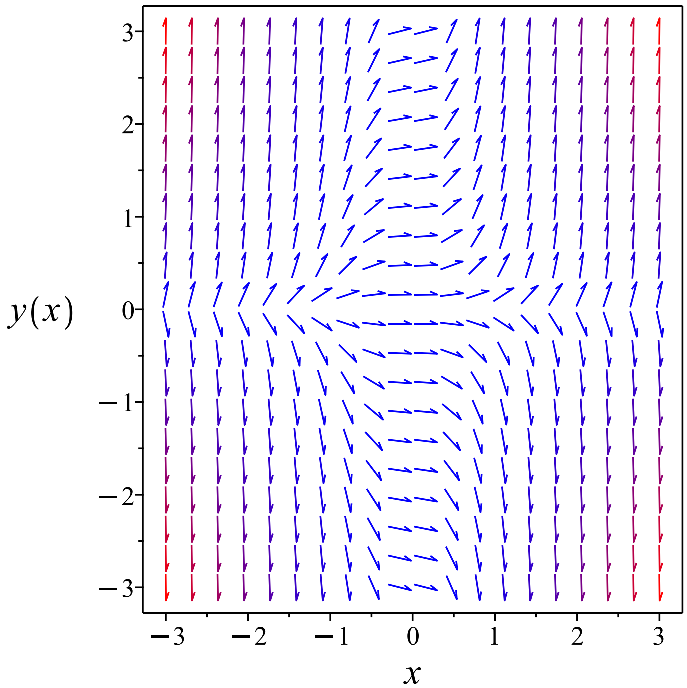


Figure 267: Slope field plot

Verification of solutions

$$y = c_1 e^{x^3}$$

Verified OK.

9.4.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{3y}\right) \mathrm{d}y &= \left(x^2\right) \mathrm{d}x \\
\left(-x^2\right) \mathrm{d}x + \left(\frac{1}{3y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= \frac{1}{3y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x \\
\phi &= -\frac{x^3}{3} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y}$. Therefore equation (4) becomes

$$
\frac{1}{3y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{3y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{3y}\right)\mathrm{d}y$$

$$f(y) = \frac{\ln(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{\ln(y)}{3}$$

The solution becomes

$$y = \mathrm{e}^{x^3+3c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^3+3c_1} \tag{1}$$

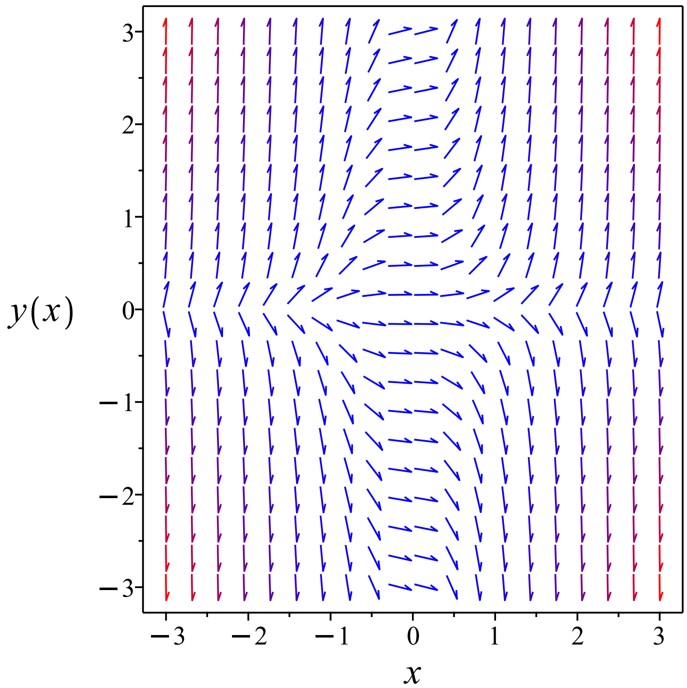


Figure 268: Slope field plot

Verification of solutions

$$y = e^{x^3+3c_1}$$

Verified OK.

9.4.6 Maple step by step solution

Let's solve

$y' - 3yx^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int 3x^2dx + c_1$

- Evaluate integral

$$\ln(y) = x^3 + c_1$$

- Solve for y

$$y = e^{x^3+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x)=3*x^2*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{x^3}$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[y'[x]==3*x^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x^3}$$
$$y(x) \to 0$$

9.5 problem 3, using series method

9.5.1 Solving as series ode 1585
9.5.2 Maple step by step solution 1589

Internal problem ID [4896]
Internal file name [`OUTPUT/4389_Sunday_June_05_2022_01_13_09_PM_21195494/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 3, using series method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first order ode series method. Regular singular point"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y + xy' = 0}$$

With the expansion point for the power series method at $x = 0$.

9.5.1 Solving as series ode

Writing the ODE as

$$y' + q(x)y = p(x)$$
$$y' - \frac{y}{x} = 0$$

Where

$$q(x) = -\frac{1}{x}$$
$$p(x) = 0$$

Next, the type of the expansion point $x = 0$ is determined. This point can be an ordinary point, a regular singular point (also called removable singularity), or irregular singular point (also called non-removable singularity or essential singularity). When $x = 0$ is an ordinary point, then the standard power series is used. If the point is a regular singular point, Frobenius series is used instead. Irregular singular point requires more advanced methods (asymptotic methods) and is not supported now. Hopefully this will be added in the future. $x = 0$ is called an ordinary point $q(x)$ has a Taylor series expansion around the point $x = 0$. $x = 0$ is called a regular singular point if $q(x)$ is not not analytic at $x = 0$ but $xq(x)$ has Taylor series expansion. And finally, $x = 0$ is an irregular singular point if the point is not ordinary and not regular singular. This is the most complicated case. Now the expansion point $x = 0$ is checked to see if it is an ordinary point or not.

Since $x = 0$ is not an ordinary point, we now check to see if it is a regular singular point. $xq(x) = -1$ has a Taylor series around $x = 0$. Since $x = 0$ is regular singular point, then Frobenius power series is used. Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Expanding the second term in (1) gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + -1 \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \frac{1}{x} \cdot \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n\,=0}^{\infty} \left(-x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} - x^{n+r-1} a_n = 0$$

When $n=0$ the above becomes

$$r a_0 x^{-1+r} - x^{-1+r} a_0 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(-1+r)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$-1+r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 1$$

We start by finding y_h. Replacing $r=1$ found above results in

$$\left(\sum_{n=0}^{\infty} (n+1)\, a_n x^{n}\right) + \sum_{n=0}^{\infty} \left(-x^{n} a_n\right) = 0$$

From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Therefore the homogeneous solution is

$$y_h(x) = a_0\left(x + O\left(x^6\right)\right)$$

At $x = 0$ the solution above becomes

$$y = c_1(x + O(x^6))$$

Summary

The solution(s) found are the following

$$y = c_1(x + O(x^6)) \tag{1}$$

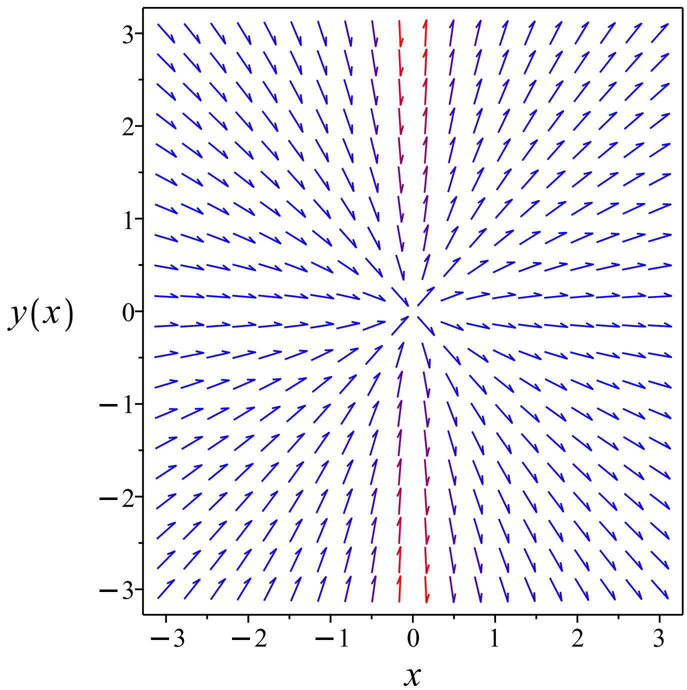


Figure 269: Slope field plot

Verification of solutions

$$y = c_1(x + O(x^6))$$

Verified OK.

9.5.2 Maple step by step solution

Let's solve

$y' - \frac{y}{x} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
Order:=6;
dsolve(x*diff(y(x),x)=y(x),y(x),type='series',x=0);
```

$$y(x) = c_1 x + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 7

```
AsymptoticDSolveValue[x*y'[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_1 x$$

9.6 problem 3, using elementary method

9.6.1 Solving as separable ode 1591
9.6.2 Solving as linear ode 1593
9.6.3 Solving as homogeneousTypeD2 ode 1594
9.6.4 Solving as first order ode lie symmetry lookup ode 1595
9.6.5 Solving as exact ode 1599
9.6.6 Maple step by step solution 1603

Internal problem ID [4897]
Internal file name [`OUTPUT/4390_Sunday_June_05_2022_01_13_18_PM_53028371/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 3, using elementary method.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{-y + xy' = 0}$$

9.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{x}\,dx \\
\ln(y) &= \ln(x) + c_1 \\
y &= \mathrm{e}^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 x \tag{1}$$

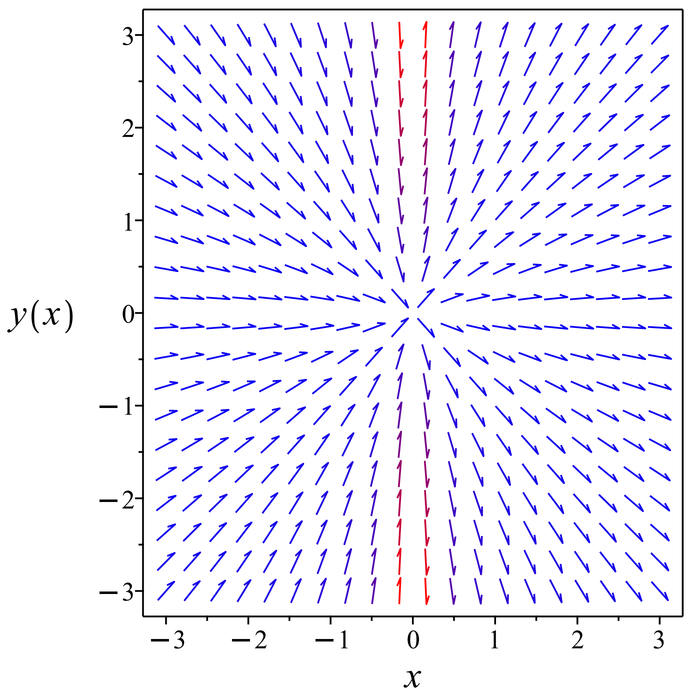


Figure 270: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

9.6.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x$$

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

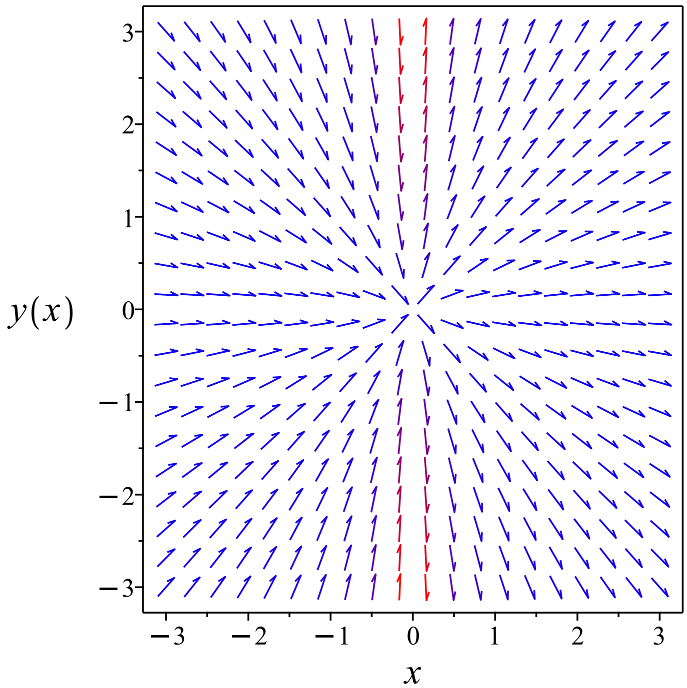


Figure 271: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

9.6.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + x(u'(x)\,x + u(x)) = 0$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0 \,\mathrm{d}x \\ &= c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= c_2 x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_2 x \tag{1}$$

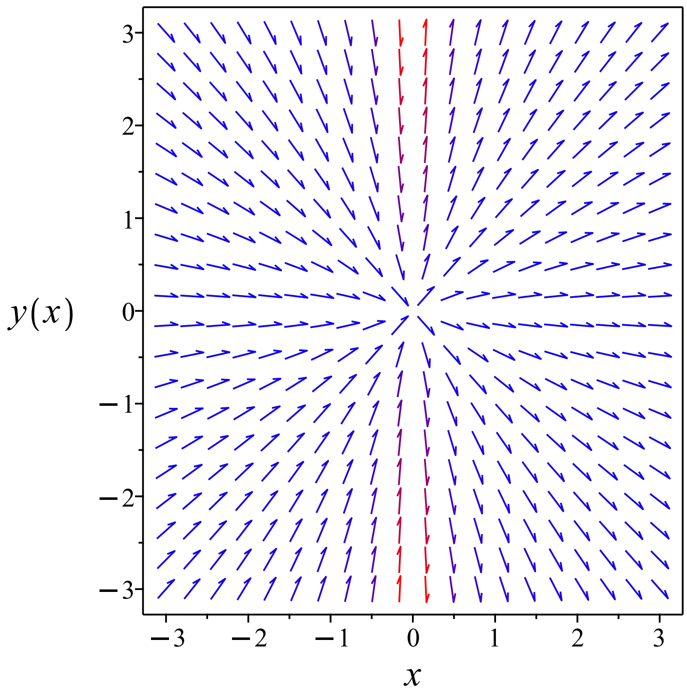


Figure 272: Slope field plot

Verification of solutions

$$y = c_2 x$$

Verified OK.

9.6.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 227: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = c_1$$

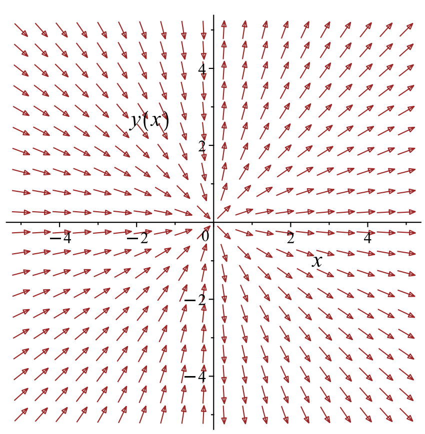
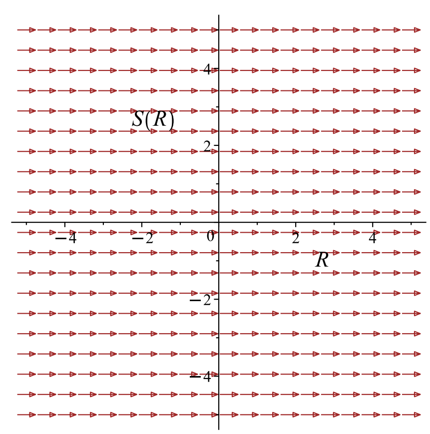
Which simplifies to

$$\frac{y}{x} = c_1$$

Which gives

$$y = c_1 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 x \tag{1}$$

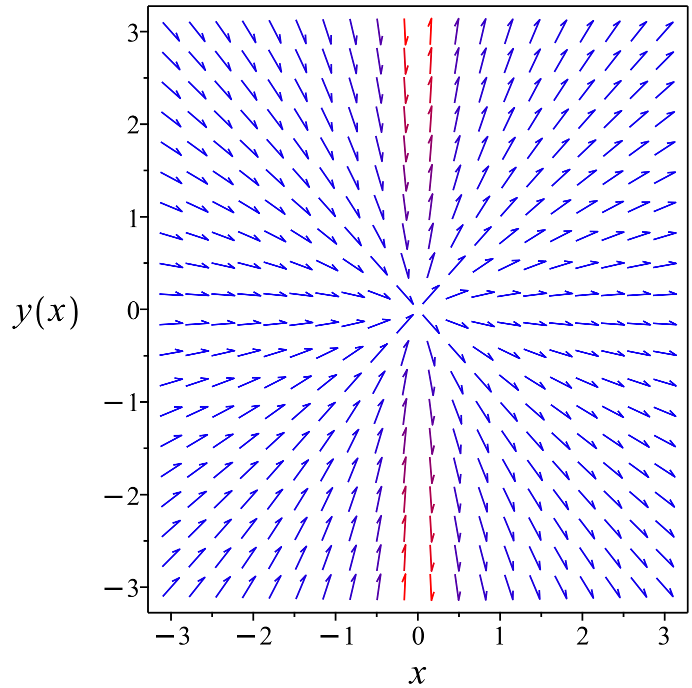


Figure 273: Slope field plot

Verification of solutions

$$y = c_1 x$$

Verified OK.

9.6.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x \\
\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\\ \phi &= -\ln(x) + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y}\right)\mathrm{d}y\\ f(y) &= \ln(y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(y)$$

The solution becomes

$$y = x\,\mathrm{e}^{c_1}$$

Summary
The solution(s) found are the following

$$y = x\,\mathrm{e}^{c_1} \tag{1}$$

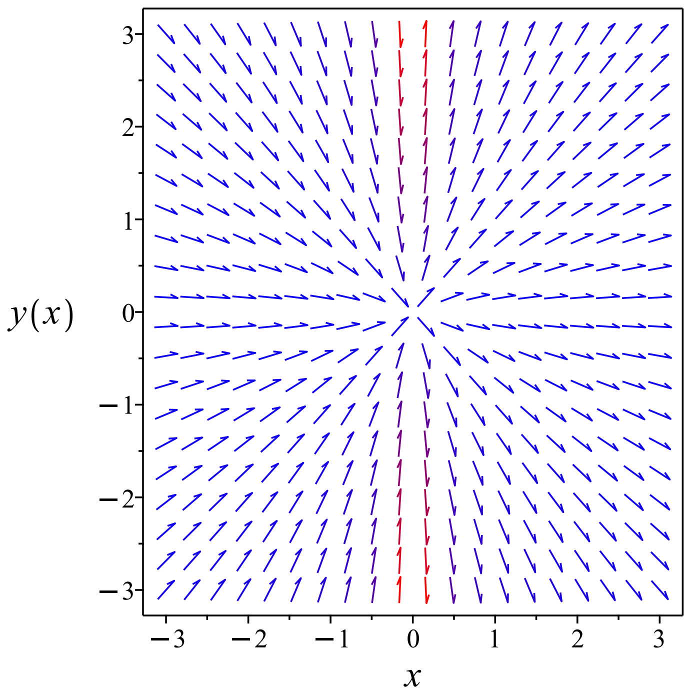


Figure 274: Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{c_1}$$

Verified OK.

9.6.6 Maple step by step solution

Let's solve

$-y + xy' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(x) + c_1$

- Solve for y

  $y = x\,\mathrm{e}^{c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 7

```
dsolve(x*diff(y(x),x)=y(x),y(x), singsol=all)
```

$$y(x) = c_1 x$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 14

```
DSolve[x*y'[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x$$
$$y(x) \to 0$$

9.7 problem 4, using series method

9.7.1 Maple step by step solution 1612

Internal problem ID [4898]
Internal file name [OUTPUT/4391_Sunday_June_05_2022_01_13_28_PM_22800546/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 4, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2}\, f|_{x_0,y_0,y_0'} + \frac{x^3}{3!}\, f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (369) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (370) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -4y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -4y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 16y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 16y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -64y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -4y(0) \\
F_1 &= -4y'(0) \\
F_2 &= 16y(0) \\
F_3 &= 16y'(0) \\
F_4 &= -64y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = -4\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \left(\sum_{n=0}^{\infty} 4 a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) + 4 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{4a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = -2a_0$$

For $n = 1$ the recurrence equation gives

$$6a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{2a_1}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{2a_0}{3}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{2a_1}{15}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + 4a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{4a_0}{45}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + 4a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = -\frac{4a_1}{315}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{2}{3} a_0 x^4 + \frac{2}{15} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) a_0 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6) \tag{2}$$

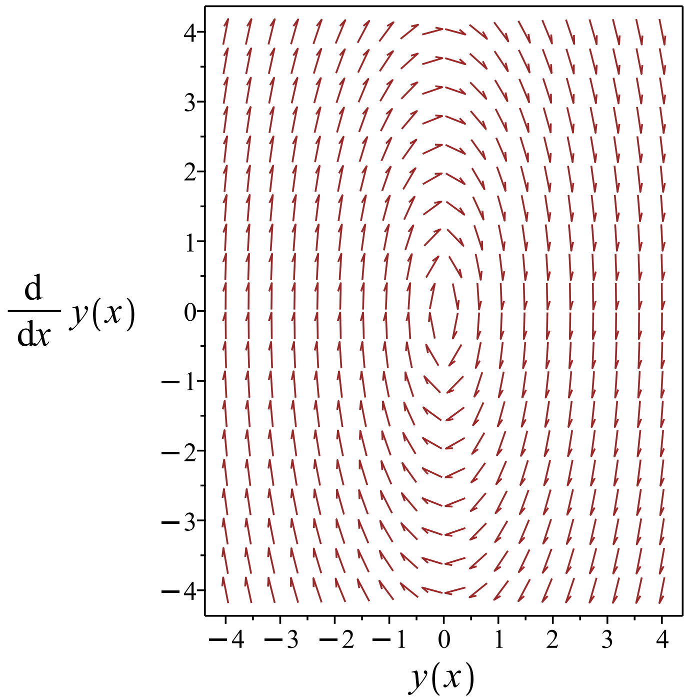


Figure 275: Slope field plot

Verification of solutions

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 - 2x^2 + \frac{2}{3}x^4\right) c_1 + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) c_2 + O(x^6)$$

Verified OK.

9.7.1 Maple step by step solution

Let's solve

$y'' = -4y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 4y = 0$

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)=-4*y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 - 2x^2 + \frac{2}{3}x^4\right) y(0) + \left(x - \frac{2}{3}x^3 + \frac{2}{15}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 40

```
AsymptoticDSolveValue[y''[x]==-4*y[x],y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{2x^5}{15} - \frac{2x^3}{3} + x\right) + c_1\left(\frac{2x^4}{3} - 2x^2 + 1\right)$$

9.8 problem 4, using elementary method

9.8.1 Solving as second order linear constant coeff ode 1615
9.8.2 Solving as second order ode can be made integrable ode 1617
9.8.3 Solving using Kovacic algorithm 1619
9.8.4 Maple step by step solution 1623

Internal problem ID [4899]
Internal file name [`OUTPUT/4392_Sunday_June_05_2022_01_13_35_PM_22116852/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 4, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\,e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) \tag{1}$$

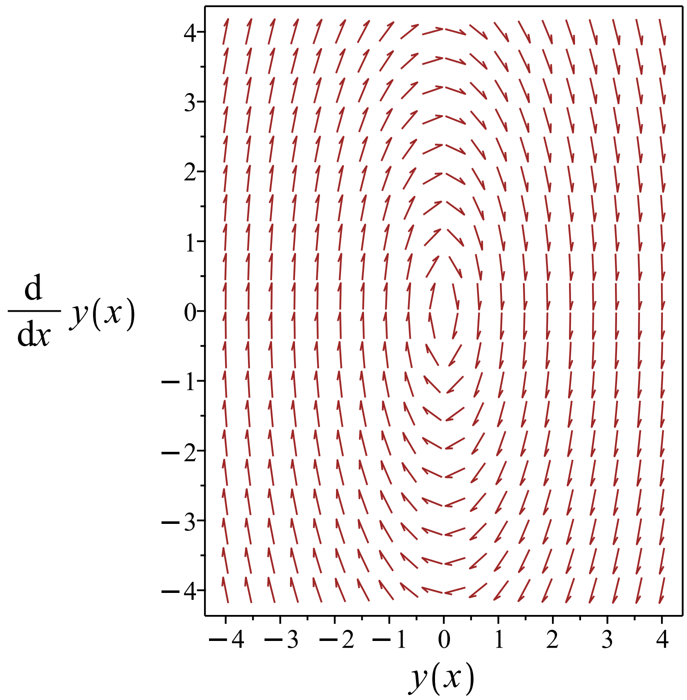


Figure 276: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Verified OK.

9.8.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 4y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 4y'y)\, dx = 0$$
$$\frac{y'^2}{2} + 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-4y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\int \frac{1}{\sqrt{-4y^2+2c_1}}dy = \int dx
$$

$$
\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_2 + x
$$

Solving equation (2)

Integrating both sides gives

$$
\int -\frac{1}{\sqrt{-4y^2+2c_1}}dy = \int dx
$$

$$
-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3
$$

Summary
The solution(s) found are the following

$$
\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_2 + x \tag{1}
$$

$$
-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3 \tag{2}
$$

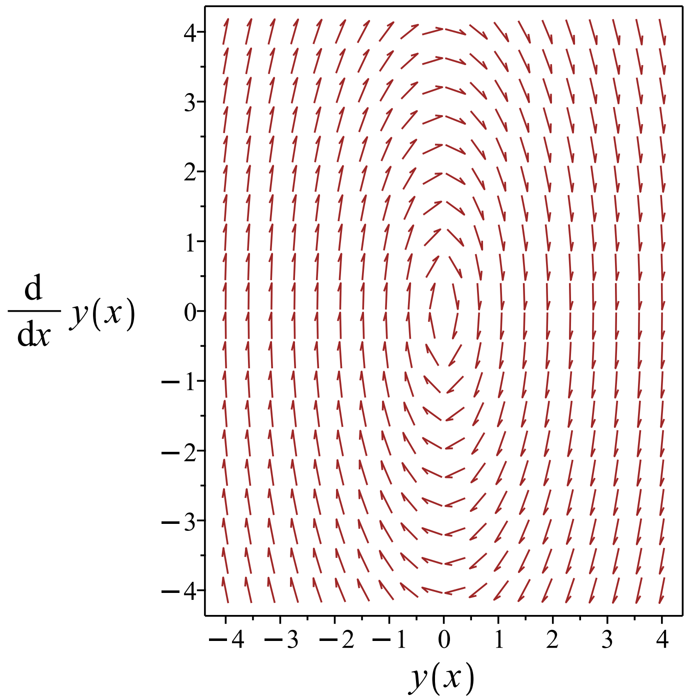


Figure 277: Slope field plot

Verification of solutions

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_2 + x$$

Verified OK.

$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_3$$

Verified OK.

9.8.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 4
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 231: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(2x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \tag{1}$$

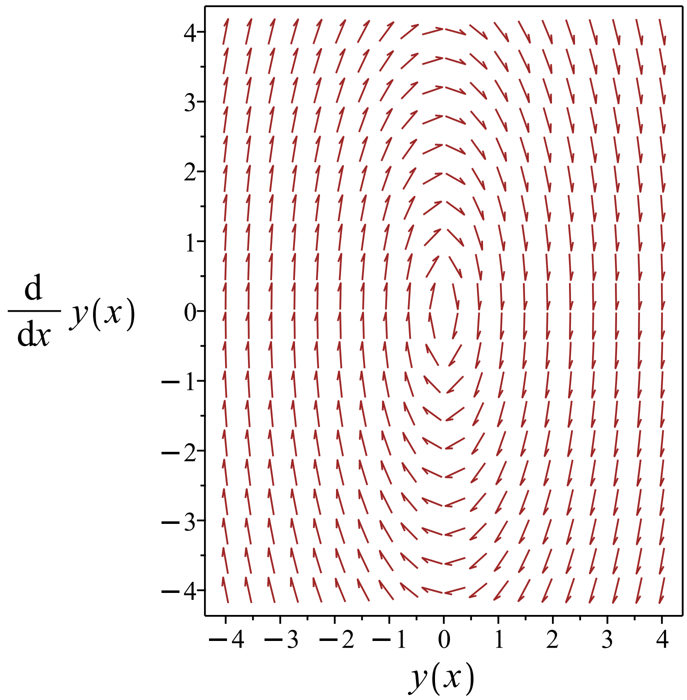


Figure 278: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

Verified OK.

9.8.4 Maple step by step solution

Let's solve

$y'' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

$$r = (-2\,\mathrm{I}, 2\,\mathrm{I})$$

- 1st solution of the ODE

$$y_1(x) = \cos(2x)$$

- 2nd solution of the ODE

$$y_2(x) = \sin(2x)$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)=-4*y(x),y(x), singsol=all)
```

$$y(x) = c_1 \sin(2x) + c_2 \cos(2x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 20

```
DSolve[y''[x]==-4*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(2x) + c_2 \sin(2x)$$

9.9 problem 5, using series method

9.9.1 Maple step by step solution 1632

Internal problem ID [4900]
Internal file name [OUTPUT/4393_Sunday_June_05_2022_01_13_43_PM_60071442/index.tex]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 5, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable", "second order series method. Ordinary point", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (375) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (376) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= y'(0) \\
F_2 &= y(0) \\
F_3 &= y'(0) \\
F_4 &= y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2) a_{n+2}(n+1) - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{720}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{24} a_0 x^4 + \frac{1}{120} a_1 x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{24} x^4\right) a_0 + \left(x + \frac{1}{6} x^3 + \frac{1}{120} x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{24} x^4\right) c_1 + \left(x + \frac{1}{6} x^3 + \frac{1}{120} x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \frac{1}{720} x^6\right) y(0) + \left(x + \frac{1}{6} x^3 + \frac{1}{120} x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + \frac{1}{2} x^2 + \frac{1}{24} x^4\right) c_1 + \left(x + \frac{1}{6} x^3 + \frac{1}{120} x^5\right) c_2 + O(x^6) \tag{2}$$

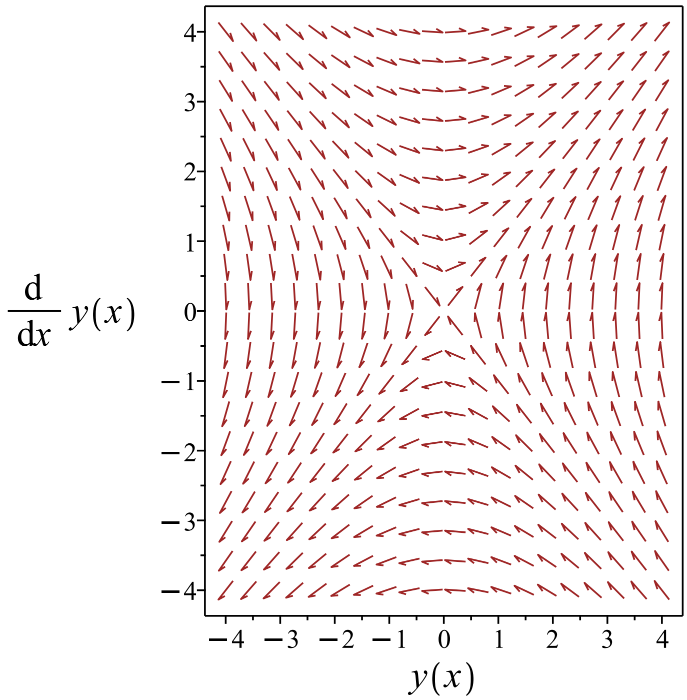


Figure 279: Slope field plot

Verification of solutions

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) y'(0) + O(x^6)$$

Verified OK.

$$y = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) c_1 + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Verified OK.

9.9.1 Maple step by step solution

Let's solve

$y'' = y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - y = 0$

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = e^{-x}$
- 2nd solution of the ODE

  $y_2(x) = e^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 e^{-x} + c_2 e^{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
Order:=6;
dsolve(diff(y(x),x$2)=y(x),y(x),type='series',x=0);
```

$$y(x) = \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4\right) y(0) + \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 42

```
AsymptoticDSolveValue[y''[x]==y[x],y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120}+\frac{x^3}{6}+x\right)+c_1\left(\frac{x^4}{24}+\frac{x^2}{2}+1\right)$$

9.10 problem 5, using elementary method

9.10.1 Solving as second order linear constant coeff ode 1635
9.10.2 Solving as second order ode can be made integrable ode 1637
9.10.3 Solving using Kovacic algorithm 1639
9.10.4 Maple step by step solution 1643

Internal problem ID [4901]
Internal file name [`OUTPUT/4394_Sunday_June_05_2022_01_13_50_PM_13358596/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 5, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

9.10.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1 \\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-x} \tag{1}$$

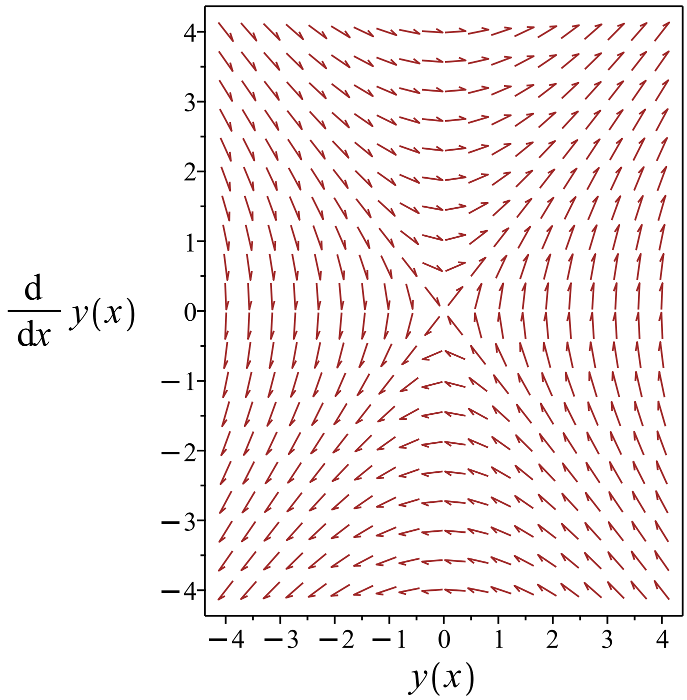


Figure 280: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-x}$$

Verified OK.

9.10.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - y'y = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - y'y)\, dx = 0$$
$$\frac{y'^2}{2} - \frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$\ln\left(y+\sqrt{y^2+2c_1}\right) = c_2 + x$$

Raising both side to exponential gives

$$y+\sqrt{y^2+2c_1} = e^{c_2+x}$$

Which simplifies to

$$y+\sqrt{y^2+2c_1} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2+2c_1}}dy = \int dx$$
$$-\ln\left(y+\sqrt{y^2+2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y+\sqrt{y^2+2c_1}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{y+\sqrt{y^2+2c_1}} = c_5 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(e^{2x}c_3^2 - 2c_1\right)e^{-x}}{2c_3} \tag{1}$$
$$y = -\frac{\left(2c_1 c_5^2 e^{2x} - 1\right)e^{-x}}{2c_5} \tag{2}$$

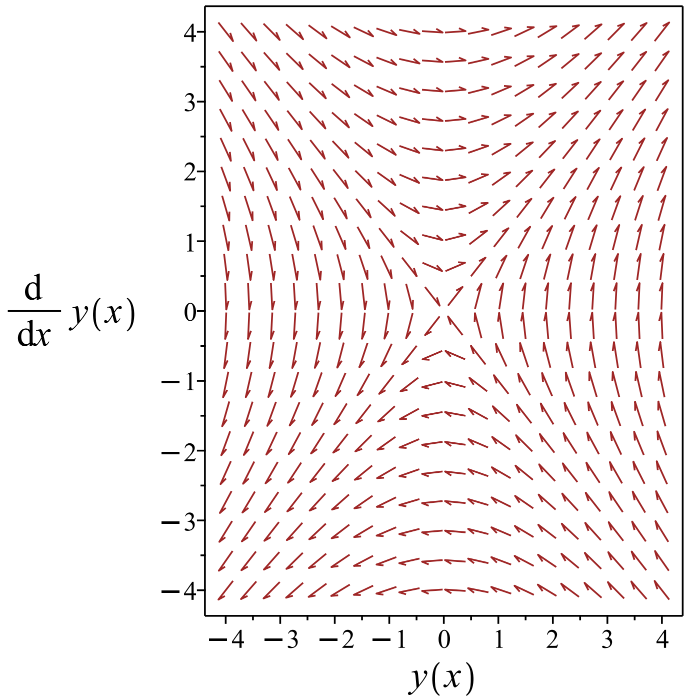


Figure 281: Slope field plot

Verification of solutions

$$y = \frac{(e^{2x}c_3^2 - 2c_1)\, e^{-x}}{2c_3}$$

Verified OK.

$$y = -\frac{(2c_1c_5^2e^{2x} - 1)\, e^{-x}}{2c_5}$$

Verified OK.

9.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 234: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\, dx \\ &= \mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

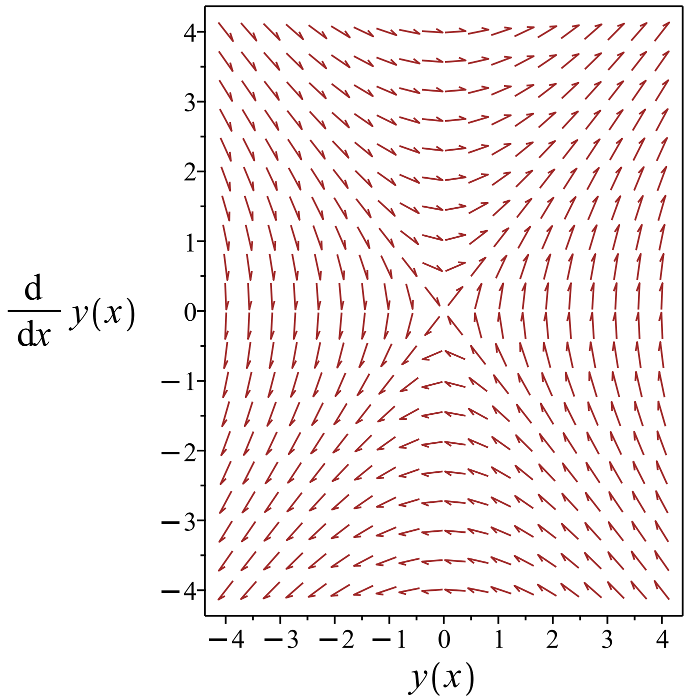


Figure 282: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + \frac{c_2 e^x}{2}$$

Verified OK.

9.10.4 Maple step by step solution

Let's solve

$y'' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 1 = 0$

- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$

- Roots of the characteristic polynomial

$$r = (-1, 1)$$

- 1st solution of the ODE

$$y_1(x) = e^{-x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-x} + c_2 e^{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)=y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[y''[x]==y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x} + c_2 e^{-x}$$

9.11 problem 6, using series method

9.11.1 Maple step by step solution 1653

Internal problem ID [4902]
Internal file name [`OUTPUT/4395_Sunday_June_05_2022_01_13_58_PM_22261923/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 6, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second_order_linear_constant_coeff", "second order series method. Ordinary point", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (381) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (382) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= 2y' - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 3y' - 2y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4y' - 3y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 5y' - 4y \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 6y' - 5y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y'(0) - y(0) \\
F_1 &= 3y'(0) - 2y(0) \\
F_2 &= 4y'(0) - 3y(0) \\
F_3 &= 5y'(0) - 4y(0) \\
F_4 &= 6y'(0) - 5y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5 - \frac{1}{144}x^6\right) y(0) \\
&+ \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 + \frac{1}{120}x^6\right) y'(0) + O(x^6)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 2\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n$$

$$\sum_{n=1}^{\infty} \left(-2n a_n x^{n-1}\right) = \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2}(n+1)\, x^n\right) + \sum_{n=0}^{\infty} \left(-2(n+1)\, a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{3}$$

For $0 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 2(n+1)\, a_{n+1} + a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{2n a_{n+1} - a_n + 2a_{n+1}}{(n+2)(n+1)} \\ &= -\frac{a_n}{(n+2)(n+1)} + \frac{(2n+2)\, a_{n+1}}{(n+2)(n+1)} \end{aligned} \tag{5}$$

For $n = 0$ the recurrence equation gives

$$2a_2 - 2a_1 + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = a_1 - \frac{a_0}{2}$$

For $n = 1$ the recurrence equation gives

$$6a_3 - 4a_2 + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = \frac{a_1}{2} - \frac{a_0}{3}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 6a_3 + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_1}{6} - \frac{a_0}{8}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 8a_4 + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{24} - \frac{a_0}{30}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 10a_5 + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_1}{120} - \frac{a_0}{144}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 12a_6 + a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{720} - \frac{a_0}{840}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(a_1 - \frac{a_0}{2}\right) x^2 + \left(\frac{a_1}{2} - \frac{a_0}{3}\right) x^3 + \left(\frac{a_1}{6} - \frac{a_0}{8}\right) x^4 + \left(\frac{a_1}{24} - \frac{a_0}{30}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) a_1 + O(x^6) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) c_2 + O(x^6)$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y &= \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}-\frac{1}{144}x^{6}\right) y(0) \\
&\quad + \left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}+\frac{1}{120}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned} \tag{1}
$$

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}\right) c_{1}+\left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}\right) c_{2}+O\left(x^{6}\right) \tag{2}
$$

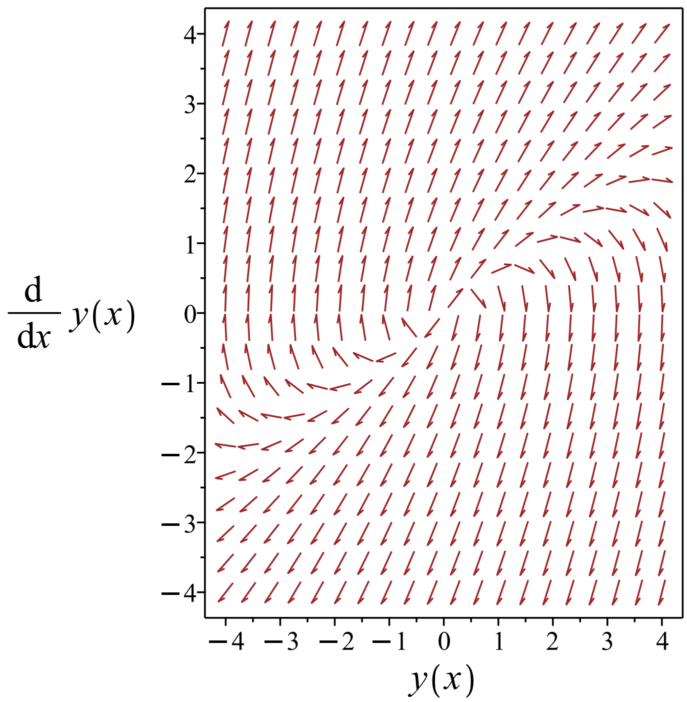


Figure 283: Slope field plot

Verification of solutions

$$
\begin{aligned}
y &= \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}-\frac{1}{144}x^{6}\right) y(0) \\
&\quad + \left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}+\frac{1}{120}x^{6}\right) y'(0)+O\left(x^{6}\right)
\end{aligned}
$$

Verified OK.

$$
y = \left(1-\frac{1}{2}x^{2}-\frac{1}{3}x^{3}-\frac{1}{8}x^{4}-\frac{1}{30}x^{5}\right) c_{1}+\left(x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}\right) c_{2}+O\left(x^{6}\right)
$$

Verified OK.

9.11.1 Maple step by step solution

Let's solve

$y'' = 2y' - y$

- Highest derivative means the order of the ODE is 2

  y''

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2y' + y = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^x x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 52

```
Order:=6;
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{30}x^5\right) y(0) + \left(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 66

```
AsymptoticDSolveValue[y''[x]-2*y'[x]+y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^5}{30} - \frac{x^4}{8} - \frac{x^3}{3} - \frac{x^2}{2} + 1\right) + c_2\left(\frac{x^5}{24} + \frac{x^4}{6} + \frac{x^3}{2} + x^2 + x\right)$$

9.12 problem 6, using elementary method

9.12.1 Solving as second order linear constant coeff ode 1655
9.12.2 Solving as linear second order ode solved by an integrating factor ode 1657
9.12.3 Solving using Kovacic algorithm 1658
9.12.4 Maple step by step solution 1662

Internal problem ID [4903]
Internal file name [`OUTPUT/4396_Sunday_June_05_2022_01_14_05_PM_92784334/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 6, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + y = 0}$$

9.12.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-2)^2 - (4)(1)(1)} \\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^x + c_2 e^x x \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^x + c_2 x\, e^x \tag{1}$$

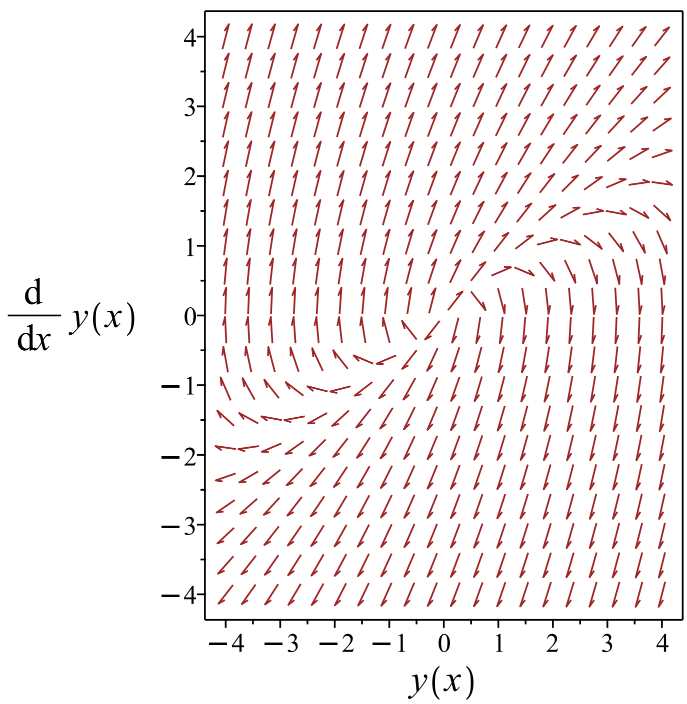


Figure 284: Slope field plot

Verification of solutions

$$y = c_1 e^x + c_2 x\, e^x$$

Verified OK.

9.12.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p \, dx} \\ &= e^{\int -2 \, dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-x} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-x} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x} \tag{1}$$

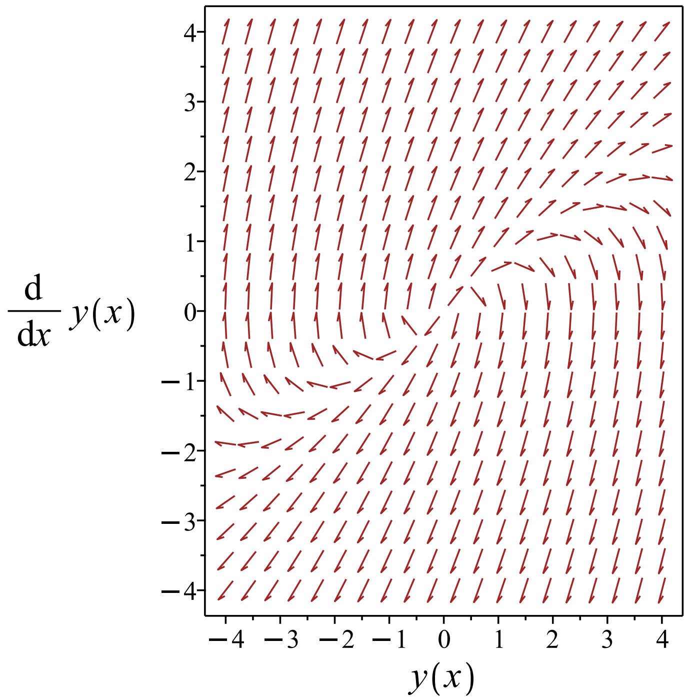


Figure 285: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x}$$

Verified OK.

9.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 237: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x \tag{1}
$$

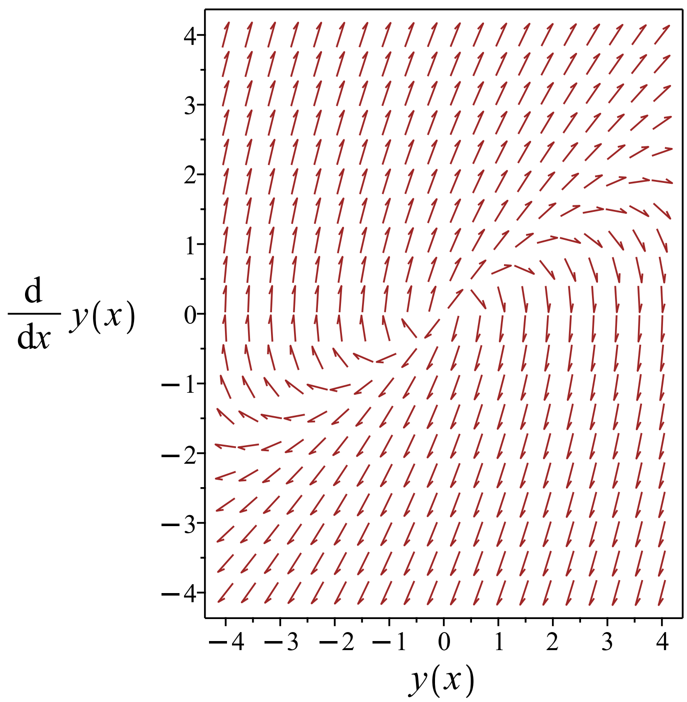


Figure 286: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^x + c_2 x \,\mathrm{e}^x
$$

Verified OK.

9.12.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = \mathrm{e}^x x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x + c_2 x\,\mathrm{e}^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[y''[x]-2*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x}(c_2 x + c_1)$$

9.13 problem 7, using series method

9.13.1 Maple step by step solution 1673

Internal problem ID [4904]
Internal file name [`OUTPUT/4397_Sunday_June_05_2022_01_14_13_PM_59075835/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 7, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 3xy' + 3y = 0}$$

With the expansion point for the power series method at $x = 0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 3xy' + 3y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Table 239: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{3}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 3xy' + 3y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\, (n+r-1)\, a_n x^{n+r-2}\right) - 3x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 3 \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\, (n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 3a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 3x^{n+r} a_n (n+r) + 3a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 3a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 3x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - 4r + 3 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 3$$
$$r_2 = 1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - 4r + 3\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= x^3 \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+3} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+1}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - 3a_n(n+r) + 3a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = 0 \tag{4}
$$

Which for the root $r = 3$ becomes

$$
a_n = 0 \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 3$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = 0
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^3\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^3\left(1 + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_2 \\
&= 0
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r \to r_2} 0 &= \lim_{r \to 1} 0 \\
&= 0
\end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n+1}
\end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$b_n(n + r)(n + r - 1) - 3b_n(n + r) + 3b_n = 0 \tag{4}$$

Which for for the root $r = 1$ becomes

$$b_n(n + 1)n - 3b_n(n + 1) + 3b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = 0 \tag{5}$$

Which for the root $r = 1$ becomes

$$b_n = 0 \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^3 \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \right) \\ &= x\left(1 + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^3 \left(1 + O(x^6)\right) + c_2 x\left(1 + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^3 \left(1 + O(x^6)\right) + c_2 x\left(1 + O(x^6)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 \left(1 + O(x^6)\right) + c_2 x\left(1 + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^3 \left(1 + O(x^6)\right) + c_2 x\left(1 + O(x^6)\right)$$

Verified OK.

9.13.1 Maple step by step solution

Let's solve

$x^2 y'' - 3xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

y''

- Isolate 2nd derivative

$y'' = \frac{3y'}{x} - \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{3y'}{x} + \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

$x^2 y'' - 3xy' + 3y = 0$

- Make a change of variables

$t = \ln(x)$

□ Substitute the change of variables back into the ODE

- ○ Calculate the 1st derivative of y with respect to x , using the chain rule

$y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- ○ Compute derivative

$y' = \frac{\frac{d}{dt} y(t)}{x}$

- ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

$y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

- ○ Compute derivative

$y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2 \left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) - 3 \frac{d}{dt} y(t) + 3y(t) = 0$

- Simplify

$\frac{d^2}{dt^2} y(t) - 4 \frac{d}{dt} y(t) + 3y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3)=0$
- Roots of the characteristic polynomial

  $r=(1,3)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{3t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^t+c_2\mathrm{e}^{3t}$
- Change variables back using $t=\ln(x)$

  $y=c_2x^3+c_1x$
- Simplify

  $y=x(c_2x^2+c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
Order:=6;
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+3*y(x)=0,y(x),type='series',x=0);
```

$$y(x)=c_1x^3\left(1+\mathrm{O}\left(x^6\right)\right)+c_2x\left(-2+\mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.004 (sec). Leaf size: 14

```
AsymptoticDSolveValue[x^2*y''[x]-3*x*y'[x]+3*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^3 + c_1 x$$

9.14 problem 7, using elementary method

9.14.1 Solving as second order euler ode ode 1677
9.14.2 Solving as second order change of variable on x method 2 ode . 1678
9.14.3 Solving as second order change of variable on x method 1 ode . 1680
9.14.4 Solving as second order change of variable on y method 2 ode . 1682
9.14.5 Solving as second order ode non constant coeff transformation on B ode 1684
9.14.6 Solving using Kovacic algorithm 1687
9.14.7 Maple step by step solution 1692

Internal problem ID [4905]
Internal file name [`OUTPUT/4398_Sunday_June_05_2022_01_14_21_PM_41606990/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 7, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - 3xy' + 3y = 0}$$

9.14.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 3 = 0$$

Or

$$r^2 - 4r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^3 + c_1 x$$

Summary
The solution(s) found are the following

$$y = c_2 x^3 + c_1 x \tag{1}$$

Verification of solutions

$$y = c_2 x^3 + c_1 x$$

Verified OK.

9.14.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{3}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\ &= \int e^{3\ln(x)}\, dx \\ &= \int x^3 dx \\ &= \frac{x^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{3}{x^2}}{x^6} \\ &= \frac{3}{x^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{3}{x^8} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{4}$$
$$r_2 = \frac{3}{4}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{4}} + c_2 \tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\sqrt{2}\,(x^4)^{\frac{1}{4}} \left(c_2 \sqrt{x^4} + 2c_1\right)}{4}$$

Verified OK.

9.14.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - 3xy' + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{3}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{3}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{4c\sqrt{3}}{3}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\,\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1 \cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2 \sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{3}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = \frac{((ic_2 + c_1)\,x^2 - ic_2 + c_1)\,x}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{((ic_2 + c_1)\,x^2 - ic_2 + c_1)\,x}{2} \tag{1}
$$

Verification of solutions

$$
y = \frac{((ic_2 + c_1)\,x^2 - ic_2 + c_1)\,x}{2}
$$

Verified OK.

9.14.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$
x^2y'' - 3xy' + 3y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= \frac{3}{x^2}
\end{aligned}
$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$
v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}
$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{3}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \\
&= c_2 x^3 - \frac{1}{2} c_1 x
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x^3 \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x^3
$$

Verified OK.

9.14.5 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=x^2\\
B&=-3x\\
C&=3\\
F&=0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=\left(x^2\right)(0)+(-3x)(-3)+(3)(-3x)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-3x^3v''+\left(3x^2\right)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
-3x^2(u'(x)\,x-u(x))=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=\frac{u}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln(u) &= \ln(x) + c_1 \\
u &= e^{\ln(x)+c_1} \\
&= c_1 x
\end{aligned}
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1 x
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int c_1 x \,\mathrm{d}x \\
&= \frac{c_1 x^2}{2} + c_2
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-3x)\left(\frac{c_1 x^2}{2} + c_2\right) \\
&= -\frac{3x(c_1 x^2 + 2c_2)}{2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{3x(c_1 x^2 + 2c_2)}{2} \tag{1}
$$

Verification of solutions

$$
y = -\frac{3x(c_1 x^2 + 2c_2)}{2}
$$

Verified OK.

9.14.6 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x^2y'' - 3xy' + 3y = 0 \tag{1}\\
Ay'' + By' + Cy = 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= x^2\\
B &= -3x \tag{3}\\
C &= 3
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 3\\
t &= 4x^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 241: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = \frac{3}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{3}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2\left(x\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x + \frac{1}{2}c_2 x^3 \tag{1}
$$

Verification of solutions

$$
y = c_1 x + \frac{1}{2}c_2 x^3
$$

Verified OK.

9.14.7 Maple step by step solution

Let's solve

$x^2y'' - 3xy' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{3y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{3y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 3xy' + 3y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 3y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 3y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3)=0$
- Roots of the characteristic polynomial

  $r=(1,3)$
- 1st solution of the ODE

  $y_1(t)=\mathrm{e}^t$
- 2nd solution of the ODE

  $y_2(t)=\mathrm{e}^{3t}$
- General solution of the ODE

  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions

  $y(t)=c_1\mathrm{e}^t+c_2\mathrm{e}^{3t}$
- Change variables back using $t=\ln(x)$

  $y=c_2x^3+c_1x$
- Simplify

  $y=x(c_2x^2+c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x)=x(c_2x^2+c_1)$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]-3*x*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(c_2x^2 + c_1)$$

9.15 problem 8, using series method

9.15.1 Maple step by step solution 1706

Internal problem ID [4906]
Internal file name [`OUTPUT/4399_Sunday_June_05_2022_01_14_29_PM_57104482/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 8, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Regular singular point. Difference is integer"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+2x)\,y''-2(x+1)\,y'+2y=0}$$

With the expansion point for the power series method at $x=0$.

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x^2+2x)\,y''+(-2-2x)\,y'+2y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$\begin{aligned}p(x)&=-\frac{2(x+1)}{x\,(x+2)}\\ q(x)&=\frac{2}{x\,(x+2)}\end{aligned}$$

Table 243: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{2(x+1)}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

$q(x) = \frac{2}{x(x+2)}$	
singularity	type
$x = -2$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$y''x(x+2) + (-2-2x)\,y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$\begin{aligned}
y' &= \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1} \\
y'' &= \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) x(x+2) \\
&+ (-2-2x)\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n(n+r)(n+r-1)\right)\\+\sum_{n=0}^{\infty}\left(-2x^{n+r} a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-2(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)=0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)=\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r} a_n(n+r)\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r}=\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\,x^{n+r-1}\right)\\+\left(\sum_{n=0}^{\infty} 2x^{n+r-1} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)\,x^{n+r-1}\right)\\+\sum_{n=0}^{\infty}\left(-2(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r-1} a_n(n+r)(n+r-1)-2(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$2x^{-1+r} a_0 r(-1+r)-2r a_0 x^{-1+r}=0$$

Or

$$\left(2x^{-1+r} r(-1+r) - 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2r\, x^{-1+r}(-2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 2$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$2r\, x^{-1+r}(-2+r) = 0$$

Solving for r gives the roots of the indicial equation as Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^2\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+2}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ &- 2a_{n-1}(n+r-1) - 2a_n(n+r) + 2a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r-3)\, a_{n-1}}{2\,(n+r)} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = -\frac{(n-1)\, a_{n-1}}{2n+4} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{2-r}{2+2r}$$

Which for the root $r = 2$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 - 3r + 2}{4\,(1+r)\,(2+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{(r^2 - 3r + 2)\, r}{8\,(2 + r)\,(1 + r)\,(3 + r)}$$

Which for the root $r = 2$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(r^2 - 3r + 2)\, r}{16\,(3 + r)\,(4 + r)\,(2 + r)}$$

Which for the root $r = 2$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0
a_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{(r^2 - 3r + 2)\, r}{32\,(4 + r)\,(5 + r)\,(3 + r)}$$

Which for the root $r = 2$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2-r}{2+2r}$	0
a_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	0
a_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0
a_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0
a_5	$-\frac{(r^2-3r+2)r}{32(4+r)(5+r)(3+r)}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x^2\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\big) \\
&= x^2\big(1 + O(x^6)\big)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= \frac{r^2 - 3r + 2}{4\,(1+r)\,(2+r)} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{r^2 - 3r + 2}{4\,(1+r)\,(2+r)} &= \lim_{r\to 0} \frac{r^2 - 3r + 2}{4\,(1+r)\,(2+r)} \\ &= \frac{1}{4} \end{aligned}$$

The limit is $\frac{1}{4}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^n \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} & b_{n-1}(n+r-1)\,(n+r-2) + 2b_n(n+r)\,(n+r-1) \\ & - 2b_{n-1}(n+r-1) - 2(n+r)\,b_n + 2b_{n-1} = 0 \end{aligned} \tag{4}$$

Which for for the root $r = 0$ becomes

$$b_{n-1}(n-1)\,(n-2) + 2b_n n(n-1) - 2b_{n-1}(n-1) - 2nb_n + 2b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{(n+r-3)\,b_{n-1}}{2\,(n+r)} \tag{5}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{(n-3)\,b_{n-1}}{2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{-2+r}{2(1+r)}$$

Which for the root $r = 0$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2 - 3r + 2}{4(1+r)(2+r)}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{4}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{(r^2 - 3r + 2)\, r}{8(2+r)(1+r)(3+r)}$$

Which for the root $r = 0$ becomes

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{(r^2 - 3r + 2)\, r}{16\,(3 + r)\,(4 + r)\,(2 + r)}$$

Which for the root $r = 0$ becomes

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0
b_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{(r^2 - 3r + 2)\, r}{32\,(4 + r)\,(5 + r)\,(3 + r)}$$

Which for the root $r = 0$ becomes

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{2-r}{2+2r}$	1
b_2	$\frac{r^2-3r+2}{4(1+r)(2+r)}$	$\frac{1}{4}$
b_3	$-\frac{(r^2-3r+2)r}{8(2+r)(1+r)(3+r)}$	0
b_4	$\frac{(r^2-3r+2)r}{16(3+r)(4+r)(2+r)}$	0
b_5	$-\frac{(r^2-3r+2)r}{32(4+r)(5+r)(3+r)}$	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= 1 + x + \frac{x^2}{4} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 (1 + O(x^6)) + c_2 \left(1 + x + \frac{x^2}{4} + O(x^6)\right)$$

Verified OK.

9.15.1 Maple step by step solution

Let's solve

$y''x(x+2)+(-2-2x)\,y'+2y=0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y}{x(x+2)}+\frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-\frac{2(x+1)y'}{x(x+2)}+\frac{2y}{x(x+2)}=0$

□ Check to see if x_0 is a regular singular point

  - Define functions

    $\left[P_2(x)=-\frac{2(x+1)}{x(x+2)},P_3(x)=\frac{2}{x(x+2)}\right]$

  - $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

    $\left((x+2)\cdot P_2(x)\right)\Big|_{x=-2}=-1$

  - $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

    $\left((x+2)^2\cdot P_3(x)\right)\Big|_{x=-2}=0$

  - $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $y''x(x+2)+(-2-2x)\,y'+2y=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(u^2-2u)\left(\frac{d^2}{du^2}y(u)\right)+(2-2u)\left(\frac{d}{du}y(u)\right)+2y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$

- Shift index using $k- >k+1-m$

  $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, u^{k+r-2+m}$

- Shift index using $k- >k+2-m$

  $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, u^{k+r}$

  Rewrite ODE with series expansions

  $-2a_0 r(-2+r)\, u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right) u^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$

- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\, a_{k+1} + a_k(k+r-2))(k+r-1) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$

- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$

- Apply recursion relation for $k = 0$

  $a_1 = -a_0$

- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$

- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$

- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$

- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$

- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$

- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

  $\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Combine solutions and rename parameters

  $\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 28

```
Order:=6;
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + \mathrm{O}\left(x^6\right)\right) c_1 x^2 + c_2 \left(-2 - 2x - \frac{1}{2}x^2 + \mathrm{O}\left(x^6\right)\right)$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 23

```
AsymptoticDSolveValue[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2 x^2 + c_1 \left(\frac{x^2}{4} + x + 1\right)$$

9.16 problem 8, using elementary method

9.16.1 Solving as second order change of variable on y method 2 ode . 1710
9.16.2 Solving as second order ode non constant coeff transformation on B ode 1713
9.16.3 Solving using Kovacic algorithm 1715
9.16.4 Maple step by step solution 1720

Internal problem ID [4907]
Internal file name [`OUTPUT/4400_Sunday_June_05_2022_01_14_37_PM_26311791/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 8, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{\left(x^2+2x\right)y''-2(x+1)\,y'+2y=0}$$

9.16.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$\left(x^2+2x\right)y''+(-2-2x)\,y'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=\frac{-2-2x}{x\,(x+2)}$$
$$q(x)=\frac{2}{x\,(x+2)}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n(-2-2x)}{x^2\,(x+2)} + \frac{2}{x\,(x+2)} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{4}{x} + \frac{-2-2x}{x\,(x+2)}\right) v'(x) &= 0 \\ v''(x) + \frac{(2x+6)\,v'(x)}{x\,(x+2)} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(2x+6)\,u(x)}{x\,(x+2)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u(x+3)}{x\,(x+2)} \end{aligned}$$

Where $f(x) = -\frac{2(x+3)}{x(x+2)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2(x+3)}{x\,(x+2)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2(x+3)}{x\,(x+2)}\,dx \\
\ln(u) &= -3\ln(x) + \ln(x+2) + c_1 \\
u &= \mathrm{e}^{-3\ln(x)+\ln(x+2)+c_1} \\
&= c_1\mathrm{e}^{-3\ln(x)+\ln(x+2)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(\frac{1}{x^2} + \frac{2}{x^3}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(-\frac{1}{x} - \frac{1}{x^2}\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(c_1\left(-\frac{1}{x} - \frac{1}{x^2}\right) + c_2\right)x^2 \\
&= (-1-x)\,c_1 + c_2x^2
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(c_1\left(-\frac{1}{x} - \frac{1}{x^2}\right) + c_2\right)x^2 \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \left(c_1\left(-\frac{1}{x} - \frac{1}{x^2}\right) + c_2\right)x^2
$$

Verified OK.

9.16.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}$$

And now the original ode becomes

$$\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right)v' + (AB'' + BB' + CB)\,v &= 0 \qquad (1)
\end{aligned}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned}
A &= x^2 + 2x \\
B &= -2 - 2x \\
C &= 2 \\
F &= 0
\end{aligned}$$

The above shows that for this ode

$$\begin{aligned}
AB'' + BB' + CB &= \left(x^2 + 2x\right)(0) + (-2 - 2x)(-2) + (2)(-2 - 2x) \\
&= 0
\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x(x+2)(x+1)v''+(4)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-2x^3-6x^2-4x\right)u'(x)+4u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{2u}{x\left(x^2+3x+2\right)}\end{aligned}$$

Where $f(x)=\frac{2}{x(x^2+3x+2)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{2}{x\left(x^2+3x+2\right)}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{2}{x\left(x^2+3x+2\right)}\,dx\\ \ln(u) &= -2\ln(x+1)+\ln(x)+\ln(x+2)+c_1\\ u &= \mathrm{e}^{-2\ln(x+1)+\ln(x)+\ln(x+2)+c_1}\\ &= c_1\mathrm{e}^{-2\ln(x+1)+\ln(x)+\ln(x+2)}\end{aligned}$$

Which simplifies to

$$u(x)=c_1\left(\frac{x^2}{(x+1)^2}+\frac{2x}{(x+1)^2}\right)$$

The ode for v now becomes

$$\begin{aligned}v' &= u\\ &= c_1\left(\frac{x^2}{(x+1)^2}+\frac{2x}{(x+1)^2}\right)\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}v(x) &= \int\frac{c_1x(x+2)}{(x+1)^2}\,\mathrm{d}x\\ &= c_1\left(x+\frac{1}{x+1}\right)+c_2\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-2-2x)\left(c_1\left(x+\frac{1}{x+1}\right)+c_2\right) \\
&= \left(-2x^2-2x-2\right)c_1-2c_2(x+1)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-2x^2-2x-2\right)c_1-2c_2(x+1) \tag{1}
$$

Verification of solutions

$$
y = \left(-2x^2-2x-2\right)c_1-2c_2(x+1)
$$

Verified OK.

9.16.3 Solving using Kovacic algorithm

Writing the ode as

$$
\left(x^2+2x\right)y''+(-2-2x)\,y'+2y=0 \tag{1}
$$
$$
Ay''+By'+Cy=0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2+2x \\
B &= -2-2x \\
C &= 2
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2 + 2x)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= (x^2 + 2x)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{(x^2 + 2x)^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 245: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2 + 2x)^2$. There is a pole at $x = 0$ of order 2. There is a pole at $x = -2$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4x} + \frac{3}{4x^2} + \frac{3}{4(x+2)} + \frac{3}{4(x+2)^2}$$

For the pole at $x = -2$ let b be the coefficient of $\frac{1}{(x+2)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2 + 2x)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
-2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} + (-)\left(0\right) \\
&= -\frac{1}{2\left(x+2\right)} + \frac{3}{2x} \\
&= \frac{x+3}{x\left(x+2\right)}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)(0) + \left(\left(\frac{1}{2(x+2)^2} - \frac{3}{2x^2}\right) + \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right)^2 - \left(\frac{3}{(x^2+2x)^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int \left(-\frac{1}{2(x+2)} + \frac{3}{2x}\right) dx} \\ &= \frac{x^{\frac{3}{2}}}{\sqrt{x+2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2-2x}{x^2+2x}\, dx} \\ &= z_1 e^{\frac{\ln(x(x+2))}{2}} \\ &= z_1\left(\sqrt{x(x+2)}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{x^{\frac{3}{2}}\sqrt{x(x+2)}}{\sqrt{x+2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2-2x}{x^2+2x}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x(x+2))}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{-1-x}{x^2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\right) + c_2 \left(\frac{x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}}\left(\frac{-1-x}{x^2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-1-x)}{\sqrt{x}\,\sqrt{x+2}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^{\frac{3}{2}}\sqrt{x\,(x+2)}}{\sqrt{x+2}} + \frac{c_2\sqrt{x\,(x+2)}\,(-1-x)}{\sqrt{x}\,\sqrt{x+2}}$$

Verified OK.

9.16.4 Maple step by step solution

Let's solve

$(x^2+2x)\,y'' + (-2-2x)\,y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y}{x(x+2)} + \frac{2(x+1)y'}{x(x+2)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

$y'' - \frac{2(x+1)y'}{x(x+2)} + \frac{2y}{x(x+2)} = 0$

- ☐ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = -\frac{2(x+1)}{x(x+2)}, P_3(x) = \frac{2}{x(x+2)}\right]$
  - $(x+2) \cdot P_2(x)$ is analytic at $x = -2$

    $\left((x+2) \cdot P_2(x)\right)\Big|_{x=-2} = -1$
  - $(x+2)^2 \cdot P_3(x)$ is analytic at $x = -2$

    $\left((x+2)^2 \cdot P_3(x)\right)\Big|_{x=-2} = 0$
  - $x = -2$is a regular singular point

    Check to see if x_0 is a regular singular point

    $x_0 = -2$
- Multiply by denominators

  $y''x(x+2) + (-2-2x)\, y' + 2y = 0$
- Change variables using $x = u - 2$ so that the regular singular point is at $u = 0$

  $(u^2 - 2u)\left(\frac{d^2}{du^2}y(u)\right) + (2-2u)\left(\frac{d}{du}y(u)\right) + 2y(u) = 0$
- Assume series solution for $y(u)$

  $y(u) = \sum_{k=0}^{\infty} a_k u^{k+r}$
- ☐ Rewrite ODE with series expansions
  - Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..1$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$
  - Shift index using $k- >k+1-m$

    $u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$
  - Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..2$

    $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,u^{k+r}$$

Rewrite ODE with series expansions

$$-2a_0 r(-2+r)\,u^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-2a_{k+1}(k+1+r)(k+r-1) + a_k(k+r-1)(k+r-2)\right)u^{k+r}\right)$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-2r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- Each term in the series must be 0, giving the recursion relation

  $((-2k-2r-2)\,a_{k+1} + a_k(k+r-2))\,(k+r-1) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k(k+r-2)}{2(k+1+r)}$
- Recursion relation for $r = 0$; series terminates at $k = 2$

  $a_{k+1} = \frac{a_k(k-2)}{2(k+1)}$
- Apply recursion relation for $k = 0$

  $a_1 = -a_0$
- Apply recursion relation for $k = 1$

  $a_2 = -\frac{a_1}{4}$
- Express in terms of a_0

  $a_2 = \frac{a_0}{4}$
- Terminating series solution of the ODE for $r = 0$. Use reduction of order to find the second li

  $y(u) = a_0 \cdot \left(1 - u + \frac{1}{4}u^2\right)$
- Revert the change of variables $u = x + 2$

  $\left[y = \frac{a_0 x^2}{4}\right]$
- Recursion relation for $r = 2$

  $a_{k+1} = \frac{a_k k}{2(k+3)}$
- Solution for $r = 2$

  $\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$

- Revert the change of variables $u = x + 2$

$$\left[y = \sum_{k=0}^{\infty} a_k (x+2)^{k+2}, a_{k+1} = \frac{a_k k}{2(k+3)}\right]$$

- Combine solutions and rename parameters

$$\left[y = \frac{a_0 x^2}{4} + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k+2}\right), b_{k+1} = \frac{b_k k}{2(k+3)}\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve((x^2+2*x)*diff(y(x),x$2)-2*(x+1)*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 x^2 + c_2 x + c_2$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 19

```
DSolve[(x^2+2*x)*y''[x]-2*(x+1)*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 x^2 - c_2 (x+1)$$

9.17 problem 9, using series method

Internal problem ID [4908]
Internal file name [`OUTPUT/4401_Sunday_June_05_2022_01_14_47_PM_25797457/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 9, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_2", "second order series method. Taylor series method", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

With the expansion point for the power series method at $x=0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using

Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0, y_0, y_0'} + \frac{x^3}{3!} f'\big|_{x_0, y_0, y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0, y_0, y_0'}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (398) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (399) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') && (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{2xy' - 2y}{x^2 + 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= 0 \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 0 \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= 0 \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 0
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -2y(0) \\
F_1 &= 0 \\
F_2 &= 0 \\
F_3 &= 0 \\
F_4 &= 0
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(-x^2 + 1\right) y(0) + xy'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series The ode is normalized to be

$$
\left(x^2 + 1\right) y'' - 2xy' + 2y = 0
$$

Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}\\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\end{aligned}$$

Substituting the above back into the ode gives

$$(x^2+1)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)-2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+2\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)+\sum_{n=1}^{\infty}(-2n a_n x^n)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} x^n a_n n(n-1)\right)+\left(\sum_{n=0}^{\infty}(n+2) a_{n+2}(n+1) x^n\right)\\ &+\sum_{n=1}^{\infty}(-2n a_n x^n)+\left(\sum_{n=0}^{\infty} 2a_n x^n\right)=0\end{aligned} \tag{3}$$

$n=0$ gives

$$2a_2+2a_0=0$$

$$a_2 = -a_0$$

For $2 \leq n$, the recurrence equation is

$$n a_n (n-1) + (n+2) a_{n+2} (n+1) - 2 n a_n + 2 a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$a_{n+2} = -\frac{a_n (n^2 - 3n + 2)}{(n+2)(n+1)} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12 a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = 0$$

For $n = 3$ the recurrence equation gives

$$2 a_3 + 20 a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = 0$$

For $n = 4$ the recurrence equation gives

$$6 a_4 + 30 a_6 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = 0$$

For $n = 5$ the recurrence equation gives

$$12 a_5 + 42 a_7 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = 0$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = -a_0 x^2 + a_1 x + a_0 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(-x^2 + 1\right) a_0 + a_1 x + O\left(x^6\right) \tag{3}
$$

At $x = 0$ the solution above becomes

$$
y = \left(-x^2 + 1\right) c_1 + c_2 x + O\left(x^6\right)
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \left(-x^2 + 1\right) y(0) + x y'(0) + O\left(x^6\right) \tag{1}
$$

$$
y = \left(-x^2 + 1\right) c_1 + c_2 x + O\left(x^6\right) \tag{2}
$$

<u>Verification of solutions</u>

$$
y = \left(-x^2 + 1\right) y(0) + x y'(0) + O\left(x^6\right)
$$

Verified OK.

$$
y = \left(-x^2 + 1\right) c_1 + c_2 x + O\left(x^6\right)
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
Order:=6;
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = y(0) + D(y)(0)\,x - y(0)\,x^2$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 18

```
AsymptoticDSolveValue[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_1(1 - x^2) + c_2 x$$

9.18 problem 9, using elementary method

9.18.1 Solving as second order change of variable on y method 2 ode . 1732
9.18.2 Solving as second order ode non constant coeff transformation on B ode 1735
9.18.3 Solving using Kovacic algorithm 1737

Internal problem ID [4909]
Internal file name [`OUTPUT/4402_Sunday_June_05_2022_01_14_54_PM_59355640/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 9, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{(x^2+1)\,y''-2xy'+2y=0}$$

9.18.1 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$(x^2+1)\,y''-2xy'+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\,y'+q(x)\,y=0 \tag{2}$$

Where

$$p(x)=-\frac{2x}{x^2+1}$$
$$q(x)=\frac{2}{x^2+1}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2+1} + \frac{2}{x^2+1} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \left(\frac{2}{x} - \frac{2x}{x^2+1}\right) v'(x) = 0$$
$$v''(x) + \frac{2v'(x)}{x\,(x^2+1)} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x\,(x^2+1)} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x\,(x^2+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x(x^2+1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^2+1\right)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^2+1\right)}\,dx \\
\ln\left(u\right) &= \ln\left(x^2+1\right) - 2\ln\left(x\right) + c_1 \\
u &= \mathrm{e}^{\ln\left(x^2+1\right)-2\ln\left(x\right)+c_1} \\
&= c_1 \mathrm{e}^{\ln\left(x^2+1\right)-2\ln\left(x\right)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(1+\frac{1}{x^2}\right)
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1\left(x-\frac{1}{x}\right) + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(c_1\left(x-\frac{1}{x}\right)+c_2\right)x \\
&= c_1x^2 + c_2x - c_1
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \left(c_1\left(x-\frac{1}{x}\right)+c_2\right)x \tag{1}
$$

<u>Verification of solutions</u>

$$
y = \left(c_1\left(x-\frac{1}{x}\right)+c_2\right)x
$$

Verified OK.

9.18.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 + 1 \\ B &= -2x \\ C &= 2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2 + 1\right)(0) + (-2x)(-2) + (2)(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x\left(x^{2}+1\right)v''+\left(-4\right)v'=0$$

Now by applying $v'=u$ the above becomes

$$\left(-2x^{3}-2x\right)u'(x)-4u(x)=0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u}{x\left(x^{2}+1\right)}
\end{aligned}$$

Where $f(x)=-\frac{2}{x(x^2+1)}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\int \frac{1}{u}\,du &= \int -\frac{2}{x\left(x^{2}+1\right)}\,dx\\
\ln\left(u\right) &= \ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}\\
u &= \mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)+c_{1}}\\
&= c_{1}\mathrm{e}^{\ln\left(x^{2}+1\right)-2\ln\left(x\right)}
\end{aligned}$$

Which simplifies to

$$u(x)=c_{1}\left(1+\frac{1}{x^{2}}\right)$$

The ode for v now becomes

$$\begin{aligned}
v' &= u\\
&= c_{1}\left(1+\frac{1}{x^{2}}\right)
\end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned}
v(x) &= \int \frac{\left(x^{2}+1\right)c_{1}}{x^{2}}\,\mathrm{d}x\\
&= c_{1}\left(x-\frac{1}{x}\right)+c_{2}
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= Bv \\
&= (-2x)\left(c_1\left(x-\frac{1}{x}\right)+c_2\right) \\
&= -2c_1x^2-2c_2x+2c_1
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -2c_1x^2-2c_2x+2c_1 \tag{1}
$$

Verification of solutions

$$
y = -2c_1x^2-2c_2x+2c_1
$$

Verified OK.

9.18.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{aligned}
(x^2+1)\,y''-2xy'+2y &= 0 \qquad (1)\\
Ay''+By'+Cy &= 0 \qquad (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2+1 \\
B &= -2x \qquad (3)\\
C &= 2
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{\left(x^2+1\right)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= \left(x^2+1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{3}{\left(x^2+1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (x^2+1)^2$. There is a pole at $x = i$ of order 2. There is a pole at $x = -i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4(x-i)^2} + \frac{3}{4(x+i)^2} + \frac{3i}{4(x-i)} - \frac{3i}{4(x+i)}$$

For the pole at $x = i$ let b be the coefficient of $\frac{1}{(x-i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

For the pole at $x = -i$ let b be the coefficient of $\frac{1}{(x+i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= 0 \\
\alpha_\infty^{-} &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{3}{(x^2+1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
i	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
$-i$	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^- + \alpha_{c_2}^+\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} + (-)(0) \\ &= -\frac{1}{2(x-i)} + \frac{3}{2(x+i)} \\ &= \frac{x-2i}{x^2+1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)(0) + \left(\left(\frac{1}{2\,(x-i)^2} - \frac{3}{2\,(x+i)^2}\right) + \left(-\frac{1}{2\,(x-i)} + \frac{3}{2\,(x+i)}\right)^2 - \left(-\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-i)} + \frac{3}{2(x+i)}\right) dx} \\
&= \frac{(x^2+1)^{\frac{3}{2}}}{(ix+1)^2}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2+1}\, dx} \\
&= z_1 e^{\frac{\ln\left(x^2+1\right)}{2}} \\
&= z_1\left(\sqrt{x^2+1}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{(x^2+1)^2}{(ix+1)^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2+1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x^2+1)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{x}{(x+i)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{(x^2+1)^2}{(ix+1)^2}\right) + c_2 \left(\frac{(x^2+1)^2}{(ix+1)^2}\left(-\frac{x}{(x+i)^2}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2\,x}{(x-i)^2\,(x+i)^2} \tag{1}
$$

Verification of solutions

$$
y = \frac{c_1 (x^2+1)^2}{(ix+1)^2} + \frac{c_2 (x^2+1)^2\,x}{(x-i)^2\,(x+i)^2}
$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((x^2+1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_2x^2 + c_1x - c_2$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 21

```
DSolve[(x^2+1)*y''[x]-2*x*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2x - c_1(x - i)^2$$

9.19 problem 10, using series method

9.19.1 Maple step by step solution 1751

Internal problem ID [4910]
Internal file name [`OUTPUT/4403_Sunday_June_05_2022_01_15_05_PM_22804024/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 10, using series method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"second order series method. Ordinary point", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor", "second order series method. Taylor series method"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + (4x^2 - 2)\, y = 0}$$

With the expansion point for the power series method at $x = 0$.

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using

Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f\big|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'\big|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (404) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (405) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\ \ \vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -4yx^2 + 4xy' + 2y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -16yx^3 + 12x^2y' + 6y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \left(32x^3 + 48x\right) y' - 48y\left(x^4 + x^2 - \frac{1}{4}\right) \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -128yx^5 + 80y'x^4 - 320yx^3 + 240x^2y' + 60y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(192x^5 + 960x^3 + 720x\right) y' - 320\left(x^6 + \frac{9}{2}x^4 + \frac{9}{4}x^2 - \frac{3}{8}\right) y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 2y(0) \\
F_1 &= 6y'(0) \\
F_2 &= 12y(0) \\
F_3 &= 60y'(0) \\
F_4 &= 120y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary, we can also solve this using standard power series Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = -4\left(\sum_{n=0}^{\infty} a_n x^n\right) x^2 + 4x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^n\right) \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \left(\sum_{n=0}^{\infty} 4x^{n+2} a_n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$
$$\sum_{n=0}^{\infty} 4x^{n+2} a_n = \sum_{n=2}^{\infty} 4a_{n-2} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \sum_{n=1}^{\infty} (-4n\, x^n a_n) + \left(\sum_{n=2}^{\infty} 4a_{n-2} x^n\right) + \sum_{n=0}^{\infty} (-2a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - 2a_0 = 0$$

$$a_2 = a_0$$

$n = 1$ gives

$$6a_3 - 6a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = a_1$$

For $2 \leq n$, the recurrence equation is

$$(n+2)\, a_{n+2}(n+1) - 4na_n + 4a_{n-2} - 2a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{4na_n + 2a_n - 4a_{n-2}}{(n+2)\,(n+1)} \\ &= \frac{2(2n+1)\, a_n}{(n+2)\,(n+1)} - \frac{4a_{n-2}}{(n+2)\,(n+1)} \end{aligned} \tag{5}$$

For $n = 2$ the recurrence equation gives

$$12a_4 - 10a_2 + 4a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 - 14a_3 + 4a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{a_1}{2}$$

For $n = 4$ the recurrence equation gives

$$30a_6 - 18a_4 + 4a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{6}$$

For $n = 5$ the recurrence equation gives

$$42a_7 - 22a_5 + 4a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{a_1}{6}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1x + a_0x^2 + a_1x^3 + \frac{1}{2}a_0x^4 + \frac{1}{2}a_1x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) a_0 + \left(x + x^3 + \frac{1}{2}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6)$$

Summary

The solution(s) found are the following

$$y = \left(1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) y'(0) + O(x^6) \tag{1}$$

$$y = \left(1 + x^2 + \frac{1}{2}x^4\right) c_1 + \left(x + x^3 + \frac{1}{2}x^5\right) c_2 + O(x^6) \tag{2}$$

Verification of solutions

$$y = \left(1+x^2+\frac{1}{2}x^4+\frac{1}{6}x^6\right)y(0)+\left(x+x^3+\frac{1}{2}x^5\right)y'(0)+O(x^6)$$

Verified OK.

$$y = \left(1+x^2+\frac{1}{2}x^4\right)c_1+\left(x+x^3+\frac{1}{2}x^5\right)c_2+O(x^6)$$

Verified OK.

9.19.1 Maple step by step solution

Let's solve

$y'' = -4yx^2+4xy'+2y$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = (-4x^2+2)\,y+4xy'$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''-4xy'+(4x^2-2)\,y = 0$

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- >k-m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k\, x^k$

  ◦ Convert y'' to series expansion

$$y'' = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k- >k+2$

  $$y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^{k}$$

  Rewrite ODE with series expansions

  $$2a_2 - 2a_0 + (6a_3 - 6a_1)\, x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)\,(k+1) - 2a_k(2k+1) + 4a_{k-2})\, x^k\right) = 0$$

- The coefficients of each power of x must be 0

  $$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

  $$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

  $$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- >k+2$

  $$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

  $$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2k a_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
Order:=6;
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x),type='series',x=0);
```

$$y(x) = \left(1 + x^2 + \frac{1}{2}x^4\right) y(0) + \left(x + x^3 + \frac{1}{2}x^5\right) D(y)(0) + O(x^6)$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 34

```
AsymptoticDSolveValue[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{2} + x^3 + x\right) + c_1\left(\frac{x^4}{2} + x^2 + 1\right)$$

9.20 problem 10, using elementary method

9.20.1 Solving as linear second order ode solved by an integrating factor ode . . . 1754
9.20.2 Solving as second order change of variable on y method 1 ode . 1755
9.20.3 Solving using Kovacic algorithm . . . 1757
9.20.4 Maple step by step solution . . . 1760

Internal problem ID [4911]
Internal file name [`OUTPUT/4404_Sunday_June_05_2022_01_15_12_PM_41070150/index.tex`]

Book: Mathematical Methods in the Physical Sciences. third edition. Mary L. Boas. John Wiley. 2006
Section: Chapter 12, Series Solutions of Differential Equations. Section 1. Miscellaneous problems. page 564
Problem number: 10, using elementary method.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_1", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4xy' + \left(4x^2 - 2\right) y = 0}$$

9.20.1 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -4x$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -4x\, dx} \\ &= \mathrm{e}^{-x^2} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{-x^2} y\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-x^2} y\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-x^2} y\right) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{\mathrm{e}^{-x^2}}
$$

Or

$$
y = c_1 x\, \mathrm{e}^{x^2} + c_2 \mathrm{e}^{x^2}
$$

Summary
The solution(s) found are the following

$$
y = c_1 x\, \mathrm{e}^{x^2} + c_2 \mathrm{e}^{x^2} \tag{1}
$$

Verification of solutions

$$
y = c_1 x\, \mathrm{e}^{x^2} + c_2 \mathrm{e}^{x^2}
$$

Verified OK.

9.20.2 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$
y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -4x \\
q(x) &= 4x^2 - 2
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= 4x^2 - 2 - \frac{(-4x)'}{2} - \frac{(-4x)^2}{4} \\
&= 4x^2 - 2 - \frac{(-4)}{2} - \frac{(16x^2)}{4} \\
&= 4x^2 - 2 - (-2) - 4x^2 \\
&= 0
\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}
z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\
&= e^{-\int \frac{-4x}{2}} \\
&= \mathrm{e}^{x^2}
\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, \mathrm{e}^{x^2} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x)\, \mathrm{e}^{x^2} = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned}
y &= v(x)\, z(x) \\
&= (c_1 x + c_2)\, (z(x))
\end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{x^2}$$

Hence (7) becomes

$$y = \mathrm{e}^{x^2}(c_1 x + c_2)$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{x^2}(c_1 x + c_2) \tag{1}$$

Verification of solutions

$$y = \mathrm{e}^{x^2}(c_1 x + c_2)$$

Verified OK.

9.20.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4xy' + (4x^2 - 2)\, y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4x \\ C &= 4x^2 - 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 0$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 249: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{1}\,dx} \\
&= z_1 e^{x^2} \\
&= z_1\left(\mathrm{e}^{x^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x^2}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x^2}\right) + c_2\left(\mathrm{e}^{x^2}(x)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x^2} + c_2 x\,\mathrm{e}^{x^2} \tag{1}$$

Verification of solutions

$$y = c_1 e^{x^2} + c_2 x \, e^{x^2}$$

Verified OK.

9.20.4 Maple step by step solution

Let's solve

$y'' - 4xy' + (4x^2 - 2) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Assume series solution for y

  $y = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y$ to series expansion for $m = 0..2$

  $x^m \cdot y = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k- > k - m$

  $x^m \cdot y = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $x \cdot y'$ to series expansion

  $x \cdot y' = \sum_{k=0}^{\infty} a_k k \, x^k$

  ◦ Convert y'' to series expansion

  $y'' = \sum_{k=2}^{\infty} a_k k(k-1) \, x^{k-2}$

  ◦ Shift index using $k- > k + 2$

  $y'' = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) \, x^k$

  Rewrite ODE with series expansions

  $2a_2 - 2a_0 + (6a_3 - 6a_1) x + \left(\sum_{k=2}^{\infty} (a_{k+2}(k+2)(k+1) - 2a_k(2k+1) + 4a_{k-2}) x^k \right) = 0$

- The coefficients of each power of x must be 0

$$[2a_2 - 2a_0 = 0, 6a_3 - 6a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\{a_2 = a_0, a_3 = a_1\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} - 4a_k k - 2a_k + 4a_{k-2} = 0$$

- Shift index using $k- > k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} - 4a_{k+2}(k+2) - 2a_{k+2} + 4a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{2(2ka_{k+2} - 2a_k + 5a_{k+2})}{k^2 + 7k + 12}, a_2 = a_0, a_3 = a_1\right]$$

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)-4*x*diff(y(x),x)+(4*x^2-2)*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{x^2}(c_2 x + c_1)$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[y''[x]-4*x*y'[x]+(4*x^2-2)*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(c_2x + c_1)$$