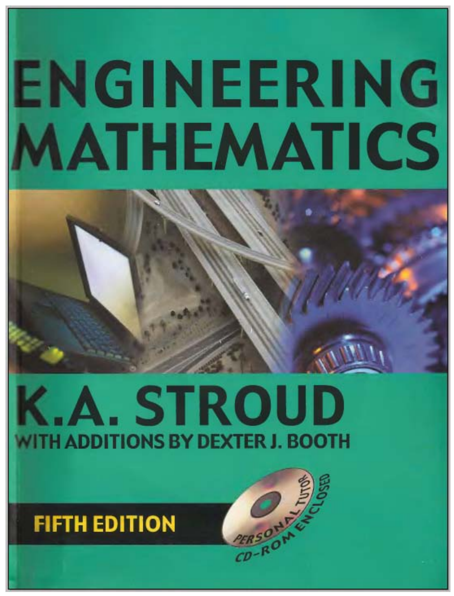
A Solution Manual For

Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001



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May 15, 2024

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1 Program 24. First order differential equations. Test excercise 24. page 1067

1.1 problem 1

Internal problem ID [5075]
Internal file name [`OUTPUT/4568_Sunday_June_05_2022_03_01_02_PM_43400769/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{xy' = x^2 + 2x - 3}$$

1.1.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^2 + 2x - 3}{x}\, \mathrm{d}x \\ &= \frac{x^2}{2} + 2x - 3\ln(x) + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2}{2} + 2x - 3\ln(x) + c_1 \tag{1}$$

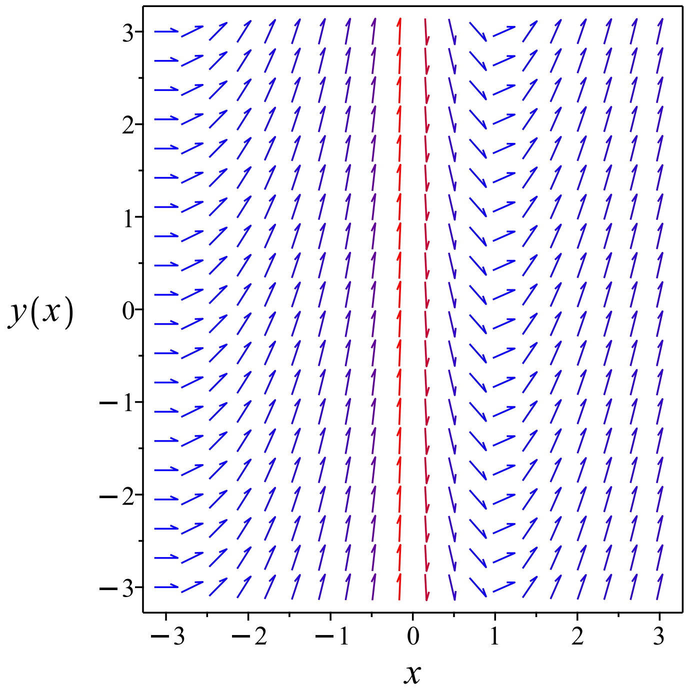


Figure 1: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^2}{2} + 2x - 3\ln(x) + c_1$$

Verified OK.

1.1.2 Maple step by step solution

Let's solve

$xy' = x^2 + 2x - 3$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{x^2+2x-3}{x}$

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{x^2+2x-3}{x} dx + c_1$

- Evaluate integral

$y = \frac{x^2}{2} + 2x - 3\ln(x) + c_1$

- Solve for y

  $y = \frac{x^2}{2} + 2x - 3\ln(x) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(x*diff(y(x),x)=x^2+2*x-3,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{2} + 2x - 3\ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
DSolve[x*y'[x]==x^2+2*x-3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{x^2}{2} + 2x - 3\log(x) + c_1$$

1.2 problem 2

Internal problem ID [5076]
Internal file name [`OUTPUT/4569_Sunday_June_05_2022_03_01_03_PM_13062409/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x+1)^2 y' - y^2 = 1}$$

1.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2+1}{(x+1)^2} \end{aligned}$$

Where $f(x) = \frac{1}{(x+1)^2}$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\, dy = \frac{1}{(x+1)^2}\, dx$$

$$\int \frac{1}{y^2+1}\, dy = \int \frac{1}{(x+1)^2}\, dx$$

$$\arctan(y) = -\frac{1}{x+1} + c_1$$

Which results in

$$y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right) \tag{1}$$

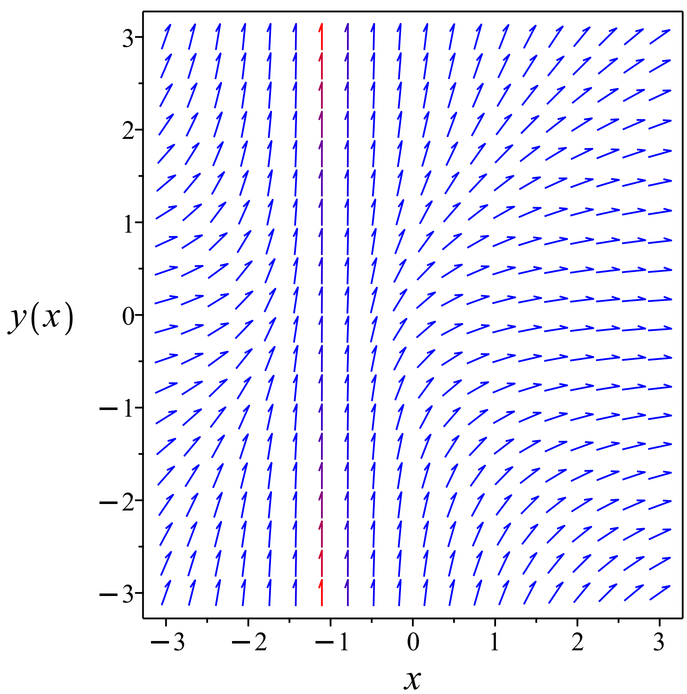


Figure 2: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right)$$

Verified OK.

1.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y^2+1}{(x+1)^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 2: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= (x+1)^2\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{(x+1)^2} dx\end{aligned}$$

Which results in

$$S = -\frac{1}{x+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2+1}{(x+1)^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{(x+1)^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x+1} = \arctan(y) + c_1$$

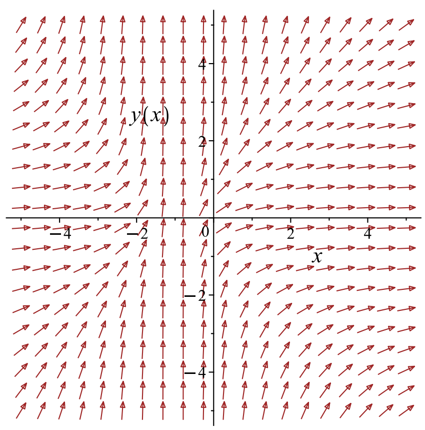
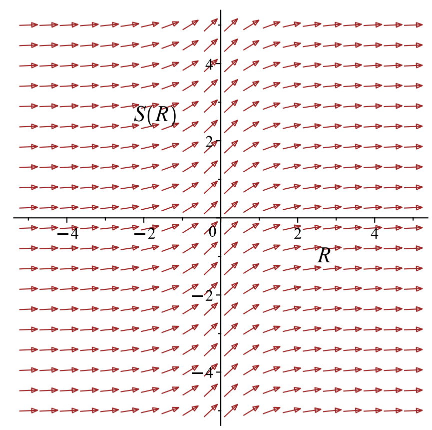
Which simplifies to

$$-\frac{1}{x+1} = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(\frac{c_1 x + c_1 + 1}{x+1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+1}{(x+1)^2}$ 	$R = y$ $S = -\frac{1}{x+1}$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{c_1 x + c_1 + 1}{x + 1}\right) \tag{1}$$

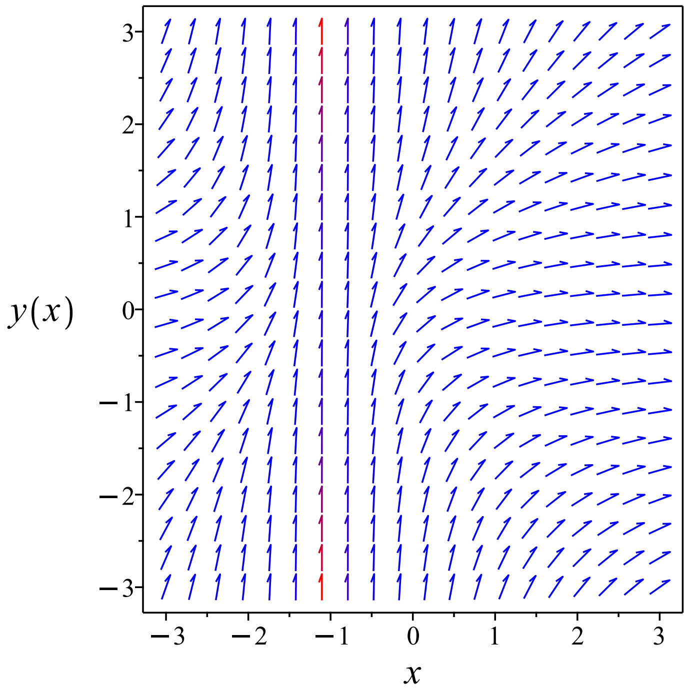


Figure 3: Slope field plot

Verification of solutions

$$y = -\tan\left(\frac{c_1x + c_1 + 1}{x + 1}\right)$$

Verified OK.

1.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{y^2+1}\right)\mathrm{d}y &= \left(\frac{1}{(x+1)^2}\right)\mathrm{d}x \\
\left(-\frac{1}{(x+1)^2}\right)\mathrm{d}x + \left(\frac{1}{y^2+1}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{1}{(x+1)^2} \\
N(x, y) &= \frac{1}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{(x+1)^2}\right) \\
&= 0
\end{aligned}
$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{(x+1)^2}\,\mathrm{d}x$$
$$\phi = \frac{1}{x+1} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$\frac{1}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2+1}\right)\mathrm{d}y$$
$$f(y) = \arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{1}{x+1} + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{1}{x+1} + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right) \tag{1}$$

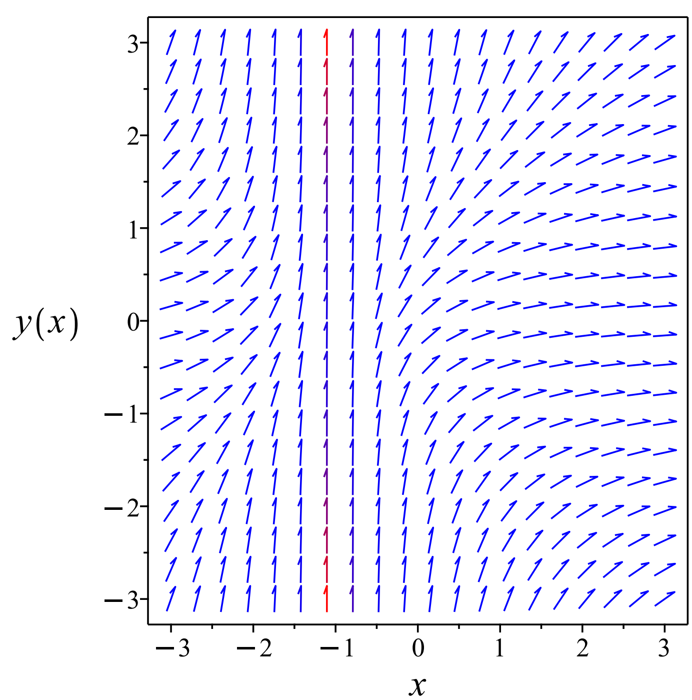


Figure 4: Slope field plot

Verification of solutions

$$
y=\tan\left(\frac{c_1x+c_1-1}{x+1}\right)
$$

Verified OK.

1.2.4 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= \frac{y^2+1}{(x+1)^2}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y'=\frac{y^2}{(x+1)^2}+\frac{1}{(x+1)^2}
$$

With Riccati ODE standard form

$$
y'=f_0(x)+f_1(x)y+f_2(x)y^2
$$

Shows that $f_0(x)=\frac{1}{(x+1)^2}$, $f_1(x)=0$ and $f_2(x)=\frac{1}{(x+1)^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2u}\\
&= \frac{-u'}{\frac{u}{(x+1)^2}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2u''(x)-(f_2'+f_1f_2)\,u'(x)+f_2^2f_0u(x)=0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2}{(x+1)^3}\\
f_1f_2 &= 0\\
f_2^2f_0 &= \frac{1}{(x+1)^6}
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{(x+1)^2} + \frac{2u'(x)}{(x+1)^3} + \frac{u(x)}{(x+1)^6} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{1}{x+1}\right) + c_2 \cos\left(\frac{1}{x+1}\right)$$

The above shows that

$$u'(x) = \frac{-c_1 \cos\left(\frac{1}{x+1}\right) + c_2 \sin\left(\frac{1}{x+1}\right)}{(x+1)^2}$$

Using the above in (1) gives the solution

$$y = -\frac{-c_1 \cos\left(\frac{1}{x+1}\right) + c_2 \sin\left(\frac{1}{x+1}\right)}{c_1 \sin\left(\frac{1}{x+1}\right) + c_2 \cos\left(\frac{1}{x+1}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3 \cos\left(\frac{1}{x+1}\right) - \sin\left(\frac{1}{x+1}\right)}{c_3 \sin\left(\frac{1}{x+1}\right) + \cos\left(\frac{1}{x+1}\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{c_3 \cos\left(\frac{1}{x+1}\right) - \sin\left(\frac{1}{x+1}\right)}{c_3 \sin\left(\frac{1}{x+1}\right) + \cos\left(\frac{1}{x+1}\right)} \tag{1}$$

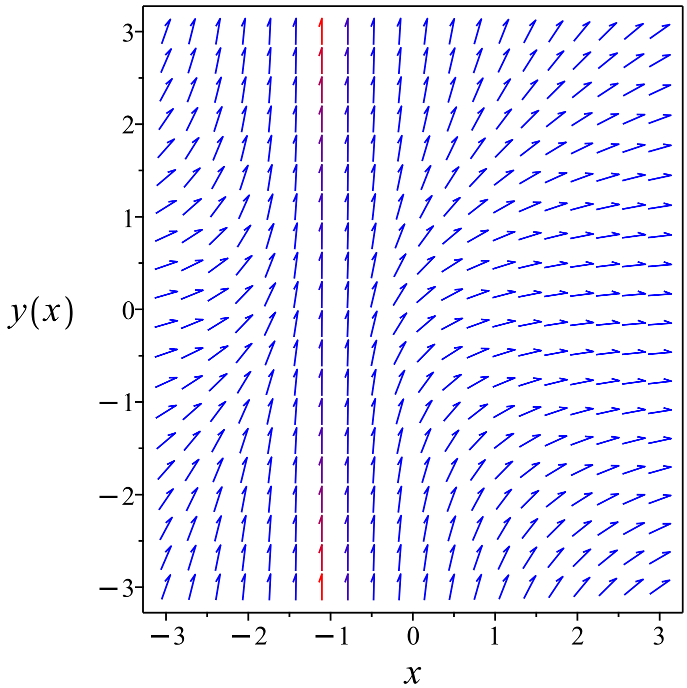


Figure 5: Slope field plot

Verification of solutions

$$y = \frac{c_3 \cos\left(\frac{1}{x+1}\right) - \sin\left(\frac{1}{x+1}\right)}{c_3 \sin\left(\frac{1}{x+1}\right) + \cos\left(\frac{1}{x+1}\right)}$$

Verified OK.

1.2.5 Maple step by step solution

Let's solve

$(x+1)^2 y' - y^2 = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = \frac{1}{(x+1)^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int \frac{1}{(x+1)^2} dx + c_1$

- Evaluate integral

  $\arctan(y) = -\frac{1}{x+1} + c_1$
- Solve for y

  $y = \tan\left(\frac{c_1 x + c_1 - 1}{x+1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 18

```
dsolve((1+x)^2*diff(y(x),x)=1+y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{-1 + c_1(x+1)}{x+1}\right)$$

✓ Solution by Mathematica

Time used: 0.264 (sec). Leaf size: 32

```
DSolve[(1+x)^2*y'[x]==1+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}
y(x) &\to -\tan\left(\frac{1}{x+1} - c_1\right) \\
y(x) &\to -i \\
y(x) &\to i
\end{aligned}$$

1.3 problem 3

Internal problem ID [5077]
Internal file name [`OUTPUT/4570_Sunday_June_05_2022_03_01_04_PM_96237136/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + 2y = e^{3x}}$$

1.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2$$
$$q(x) = e^{3x}$$

Hence the ode is

$$y' + 2y = e^{3x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dx} \\ &= e^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{3x}\right) \\ \frac{d}{dx}\left(e^{2x}y\right) &= \left(e^{2x}\right)\left(e^{3x}\right) \\ d\left(e^{2x}y\right) &= e^{5x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2x}y &= \int e^{5x}\, dx \\ e^{2x}y &= \frac{e^{5x}}{5} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$y = \frac{e^{-2x}e^{5x}}{5} + c_1 e^{-2x}$$

which simplifies to

$$y = \frac{\left(e^{5x} + 5c_1\right)e^{-2x}}{5}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\left(e^{5x} + 5c_1\right)e^{-2x}}{5} \tag{1}$$

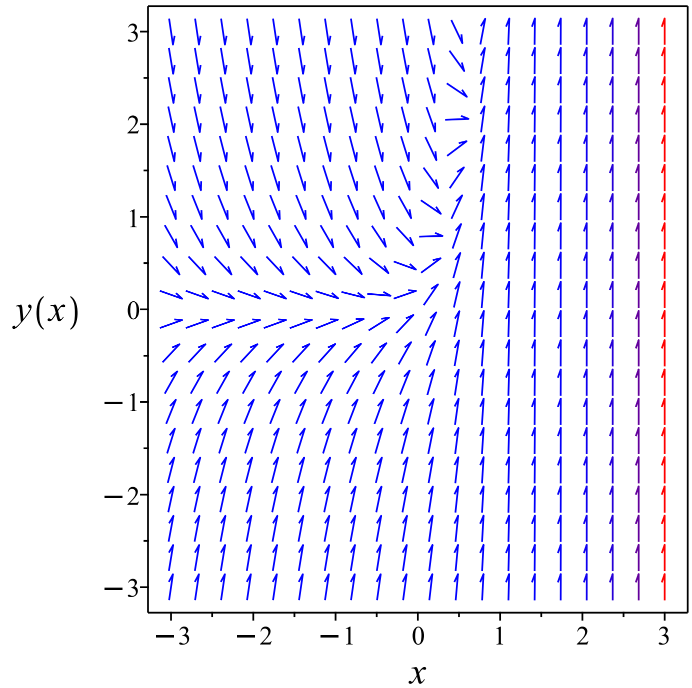


Figure 6: Slope field plot

Verification of solutions

$$y = \frac{(e^{5x} + 5c_1)\, e^{-2x}}{5}$$

Verified OK.

1.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2y + e^{3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 5: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-2x}} dy \end{aligned}$$

Which results in

$$S = e^{2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2y + e^{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2 e^{2x} y \\ S_y &= e^{2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{5x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{5R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{5R}}{5} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{2x} y = \frac{e^{5x}}{5} + c_1$$

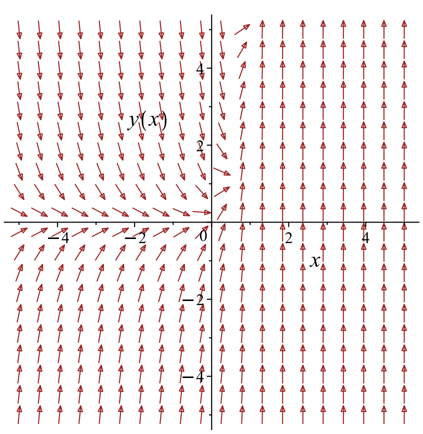
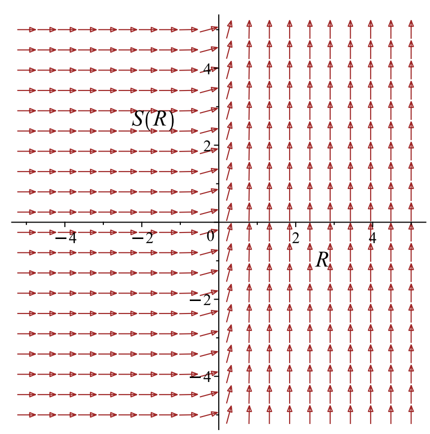
Which simplifies to

$$e^{2x} y = \frac{e^{5x}}{5} + c_1$$

Which gives

$$y = \frac{(e^{5x} + 5c_1) e^{-2x}}{5}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2y + e^{3x}$ 	$R = x$ $S = e^{2x} y$	$\frac{dS}{dR} = e^{5R}$ 

Summary

The solution(s) found are the following

$$y = \frac{(e^{5x} + 5c_1) e^{-2x}}{5} \tag{1}$$

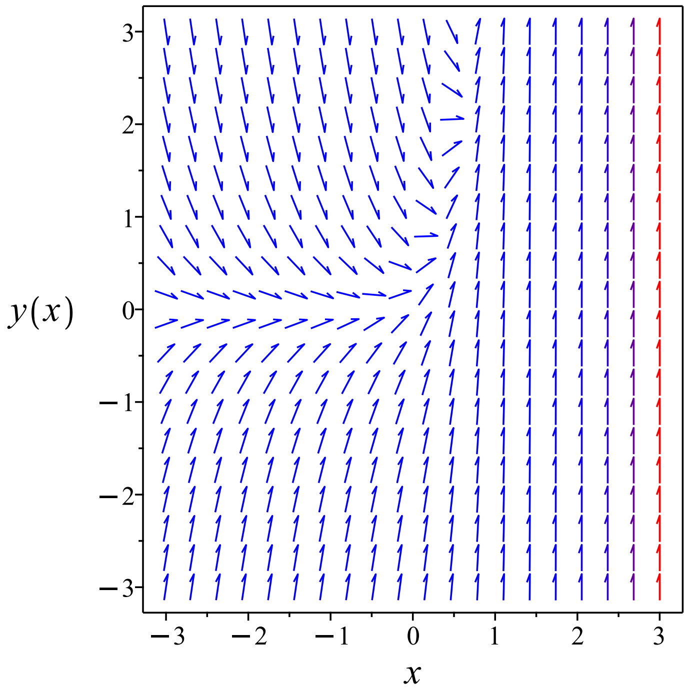


Figure 7: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{(\mathrm{e}^{5x} + 5c_1)\,\mathrm{e}^{-2x}}{5}$$

Verified OK.

1.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= \left(-2y + \mathrm{e}^{3x}\right) \mathrm{d}x \\
\left(2y - \mathrm{e}^{3x}\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - \mathrm{e}^{3x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2y - \mathrm{e}^{3x}\right) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((2) - (0)) \\
&= 2
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 2 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{2x} \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{2x}\left(2y - \mathrm{e}^{3x}\right) \\
&= \left(2y - \mathrm{e}^{3x}\right)\mathrm{e}^{2x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{2x}(1) \\
&= \mathrm{e}^{2x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(2y - \mathrm{e}^{3x}\right)\mathrm{e}^{2x}\right) + \left(\mathrm{e}^{2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \left(2y - \mathrm{e}^{3x}\right) \mathrm{e}^{2x} \, \mathrm{d}x \\
\phi &= -\frac{\mathrm{e}^{5x}}{5} + \mathrm{e}^{2x} y + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2x}$. Therefore equation (4) becomes

$$
\mathrm{e}^{2x} = \mathrm{e}^{2x} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\mathrm{e}^{5x}}{5} + \mathrm{e}^{2x} y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{\mathrm{e}^{5x}}{5} + \mathrm{e}^{2x} y
$$

The solution becomes

$$
y = \frac{\left(\mathrm{e}^{5x} + 5c_1\right) \mathrm{e}^{-2x}}{5}
$$

Summary

The solution(s) found are the following

$$y = \frac{(e^{5x} + 5c_1) e^{-2x}}{5} \tag{1}$$

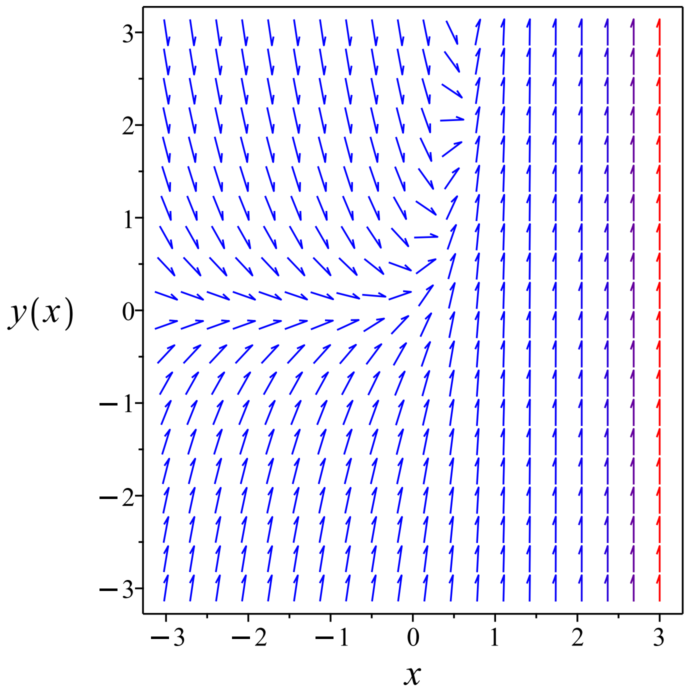


Figure 8: Slope field plot

Verification of solutions

$$y = \frac{(e^{5x} + 5c_1) e^{-2x}}{5}$$

Verified OK.

1.3.4 Maple step by step solution

Let's solve

$y' + 2y = e^{3x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -2y + e^{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + 2y = e^{3x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + 2y\right) = \mu(x)\, e^{3x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\, y\right)$

  $\mu(x)\left(y' + 2y\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 2\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = e^{2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}\left(\mu(x)\, y\right)\right) dx = \int \mu(x)\, e^{3x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, e^{3x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) e^{3x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{2x}$

  $y = \frac{\int e^{3x} e^{2x} dx + c_1}{e^{2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{e^{5x}}{5} + c_1}{e^{2x}}$

- Simplify

  $y = \frac{\left(e^{5x} + 5c_1\right) e^{-2x}}{5}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x)+2*y(x)=exp(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(e^{5x} + 5c_1) e^{-2x}}{5}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 23

```
DSolve[y'[x]+2*y[x]==Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{3x}}{5} + c_1 e^{-2x}$$

1.4 problem 4

Internal problem ID [5078]
Internal file name [`OUTPUT/4571_Sunday_June_05_2022_03_01_05_PM_98104833/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + xy' = x^2}$$

1.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = x$$

Hence the ode is

$$y' - \frac{y}{x} = x$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(x) \\
d\left(\frac{y}{x}\right) &= dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{x} &= \int dx \\
\frac{y}{x} &= x + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = c_1 x + x^2$$

which simplifies to

$$y = x(x + c_1)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x(x + c_1) \tag{1}$$

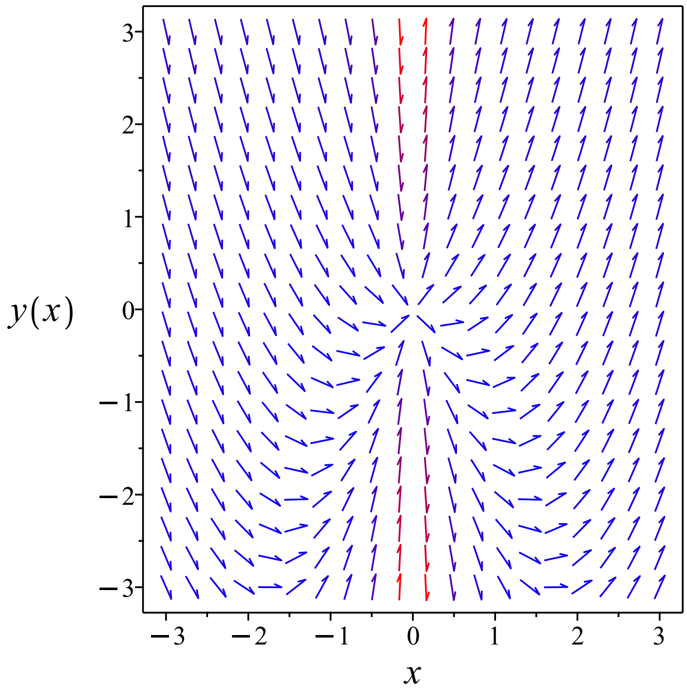


Figure 9: Slope field plot

<u>Verification of solutions</u>

$$y = x(x + c_1)$$

Verified OK.

1.4.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + x(u'(x)\,x + u(x)) = x^2$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int 1\, \mathrm{d}x \\ &= c_2 + x \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(c_2 + x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x(c_2 + x) \tag{1}$$

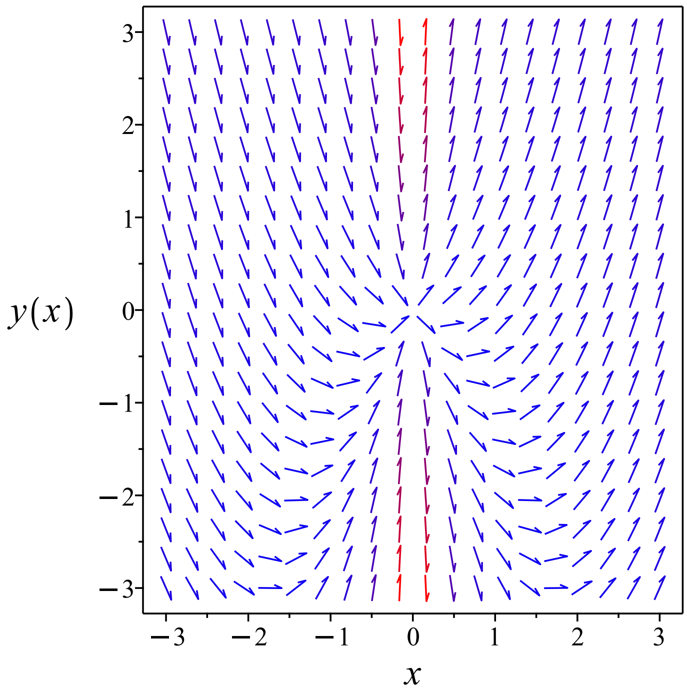


Figure 10: Slope field plot

Verification of solutions

$$y = x(c_2 + x)$$

Verified OK.

1.4.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 8: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{x} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -\frac{y}{x^2}\\ S_y &= \frac{1}{x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = x + c_1$$

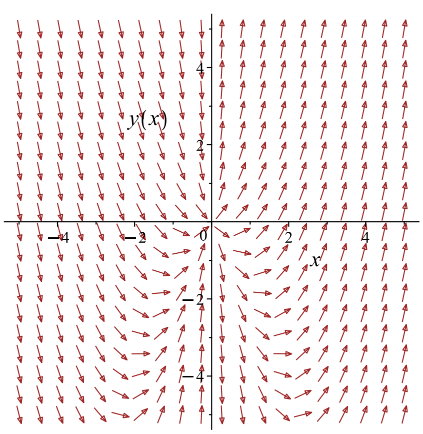
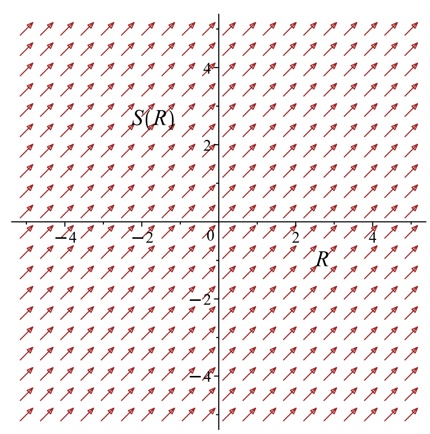
Which simplifies to

$$\frac{y}{x} = x + c_1$$

Which gives

$$y = x(x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = x(x + c_1) \tag{1}$$

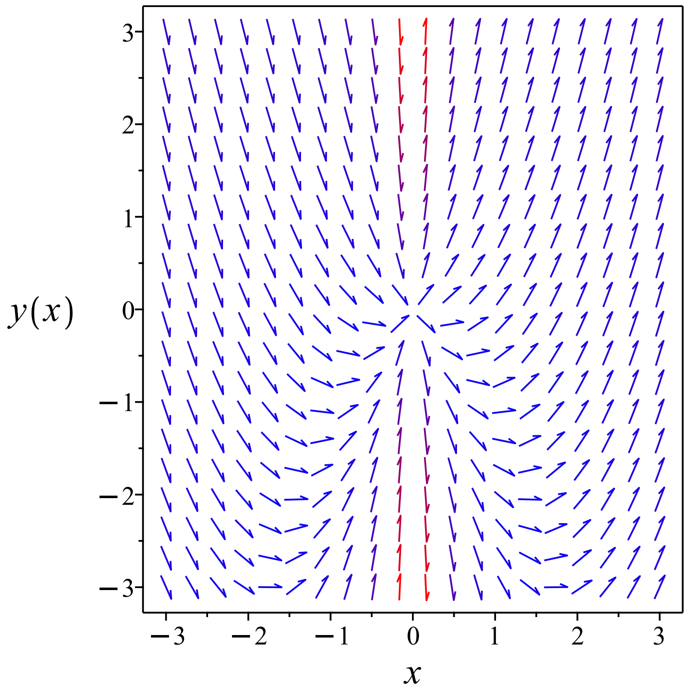


Figure 11: Slope field plot

Verification of solutions

$$y = x(x + c_1)$$

Verified OK.

1.4.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^2 + y\right)\mathrm{d}x\\ \left(-x^2 - y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 - y\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - y\right)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(-x^2 - y) \\
&= \frac{-x^2 - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^2 - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{-x^2-y}{x^2}\,\mathrm{d}x$$
$$\phi = -x + \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{y}{x}$$

The solution becomes

$$y = x(x + c_1)$$

Summary

The solution(s) found are the following

$$y = x(x + c_1) \tag{1}$$

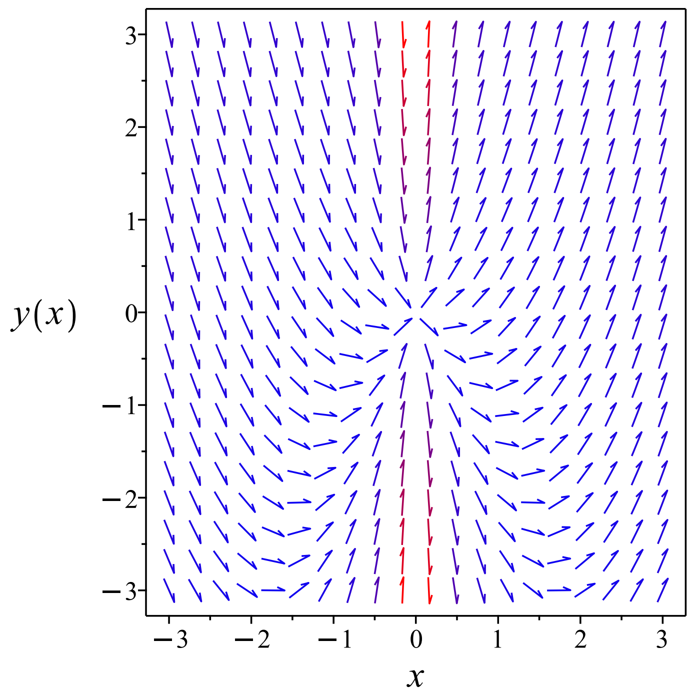


Figure 12: Slope field plot

Verification of solutions

$$y = x(x + c_1)$$

Verified OK.

1.4.5 Maple step by step solution

Let's solve

$-y + xy' = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\, x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int 1 dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(x + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve(x*diff(y(x),x)-y(x)=x^2,y(x), singsol=all)
```

$$y(x) = (x + c_1)\,x$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 11

```
DSolve[x*y'[x]-y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x + c_1)$$

1.5 problem 5

Internal problem ID [5079]
Internal file name [`OUTPUT/4572_Sunday_June_05_2022_03_01_06_PM_7648630/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{x^2y' = x^3 \sin(3x) + 4}$$

1.5.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{x^3 \sin(3x) + 4}{x^2} \, \mathrm{d}x \\ &= \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} - \frac{4}{x} + c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} - \frac{4}{x} + c_1 \tag{1}$$

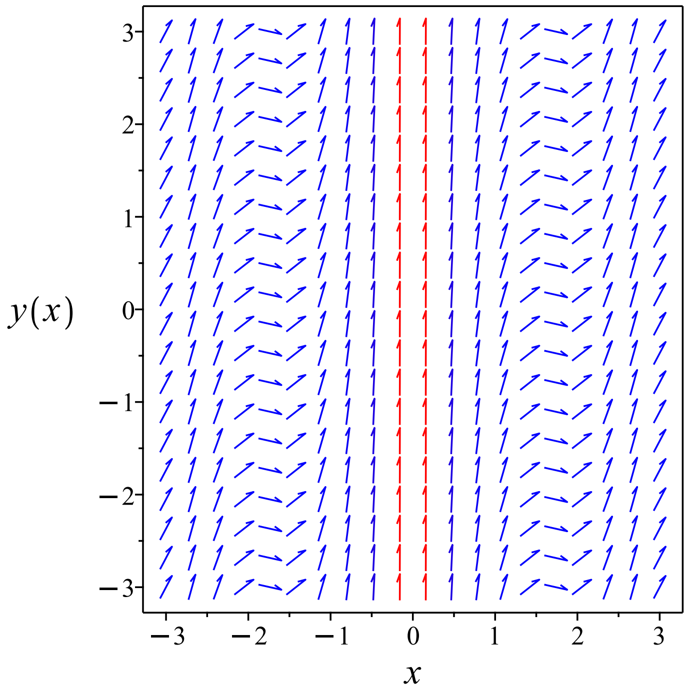


Figure 13: Slope field plot

Verification of solutions

$$y = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} - \frac{4}{x} + c_1$$

Verified OK.

1.5.2 Maple step by step solution

Let's solve

$x^2y' = x^3\sin(3x) + 4$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' = \frac{x^3\sin(3x)+4}{x^2}$

- Integrate both sides with respect to x

  $\int y'dx = \int \frac{x^3\sin(3x)+4}{x^2}dx + c_1$

- Evaluate integral

$$y = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} - \frac{4}{x} + c_1$$

- Solve for y

$$y = \frac{-3x^2\cos(3x) + x\sin(3x) + 9c_1x - 36}{9x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(x^2*diff(y(x),x)=x^3*sin(3*x)+4,y(x), singsol=all)
```

$$y(x) = \frac{\sin(3x)}{9} - \frac{x\cos(3x)}{3} - \frac{4}{x} + c_1$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 30

```
DSolve[x^2*y'[x]==x^3*Sin[3*x]+4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{4}{x} + \frac{1}{9}\sin(3x) - \frac{1}{3}x\cos(3x) + c_1$$

1.6 problem 6

Internal problem ID [5080]
Internal file name [`OUTPUT/4573_Sunday_June_05_2022_03_01_07_PM_98484249/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x \cos(y) y' - \sin(y) = 0}$$

1.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{\tan(y)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(y) = \tan(y)$. Integrating both sides gives

$$\frac{1}{\tan(y)}\, dy = \frac{1}{x}\, dx$$

$$\int \frac{1}{\tan(y)}\, dy = \int \frac{1}{x}\, dx$$

$$\ln(\sin(y)) = \ln(x) + c_1$$

Raising both side to exponential gives

$$\sin(y) = e^{\ln(x)+c_1}$$

Which simplifies to

$$\sin(y) = c_2 x$$

<u>Summary</u>
The solution(s) found are the following

$$y = \arcsin\left(c_2 x \,e^{c_1}\right) \tag{1}$$

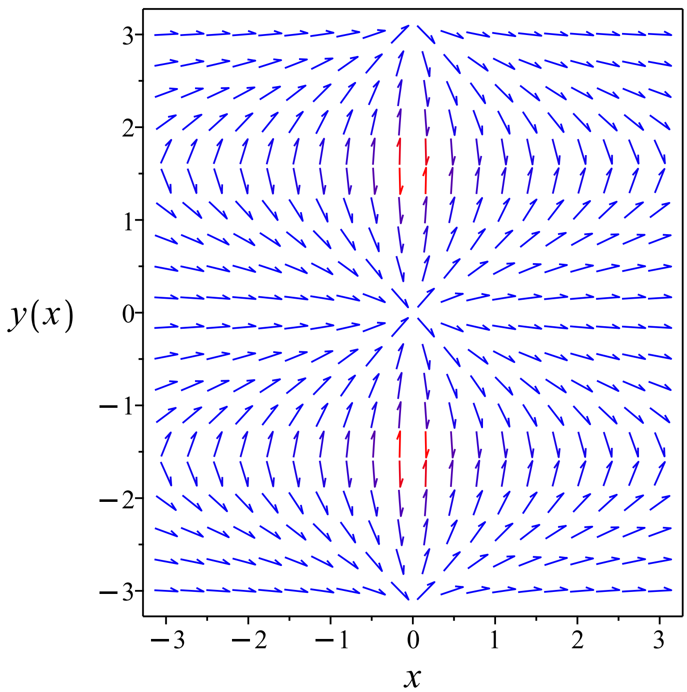


Figure 14: Slope field plot

<u>Verification of solutions</u>

$$y = \arcsin\left(c_2 x \,e^{c_1}\right)$$

Verified OK.

1.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(y)}{x\cos(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 12: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= x \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{x} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sin(y)}{x \cos(y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cot(y) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cot(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(\sin(R)) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(x) = \ln(\sin(y)) + c_1
$$

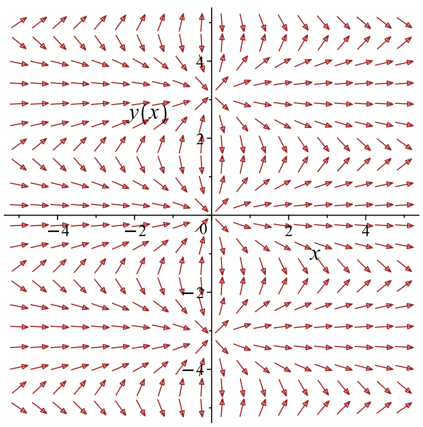
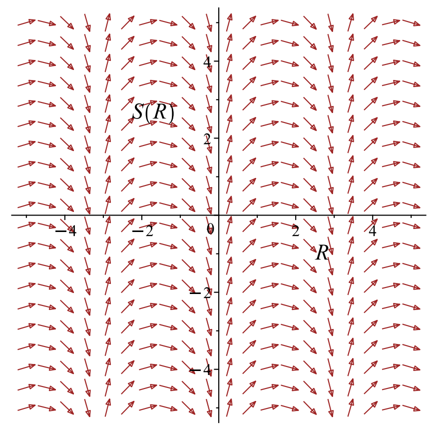
Which simplifies to

$$
\ln(x) = \ln(\sin(y)) + c_1
$$

Which gives

$$
y = \arcsin\left(x\,\mathrm{e}^{-c_1}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sin(y)}{x\cos(y)}$ 	$R = y$ $S = \ln(x)$	$\frac{dS}{dR} = \cot(R)$ 

Summary

The solution(s) found are the following

$$y = \arcsin\left(x\,e^{-c_1}\right) \tag{1}$$

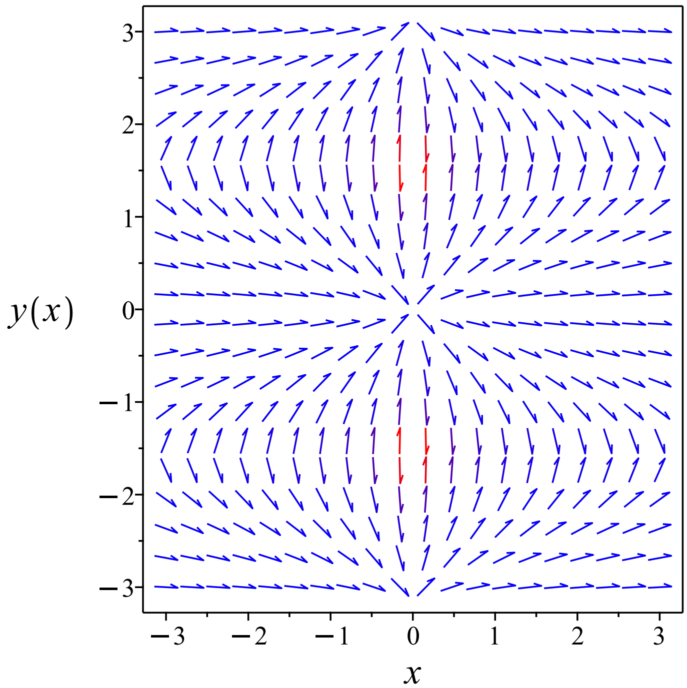


Figure 15: Slope field plot

Verification of solutions

$$y = \arcsin\left(x\,\mathrm{e}^{-c_1}\right)$$

Verified OK.

1.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{\cos(y)}{\sin(y)}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x \\
\left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{\cos(y)}{\sin(y)}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{x} \\
N(x,y) &= \frac{\cos(y)}{\sin(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\cos(y)}{\sin(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\end{aligned}$$
$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\cos(y)}{\sin(y)}$. Therefore equation (4) becomes

$$\frac{\cos(y)}{\sin(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{\cos(y)}{\sin(y)}\\ &= \cot(y)\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int (\cot(y))\,\mathrm{d}y$$
$$f(y) = \ln(\sin(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \ln(\sin(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \ln(\sin(y))$$

Summary

The solution(s) found are the following

$$-\ln(x) + \ln(\sin(y)) = c_1 \tag{1}$$

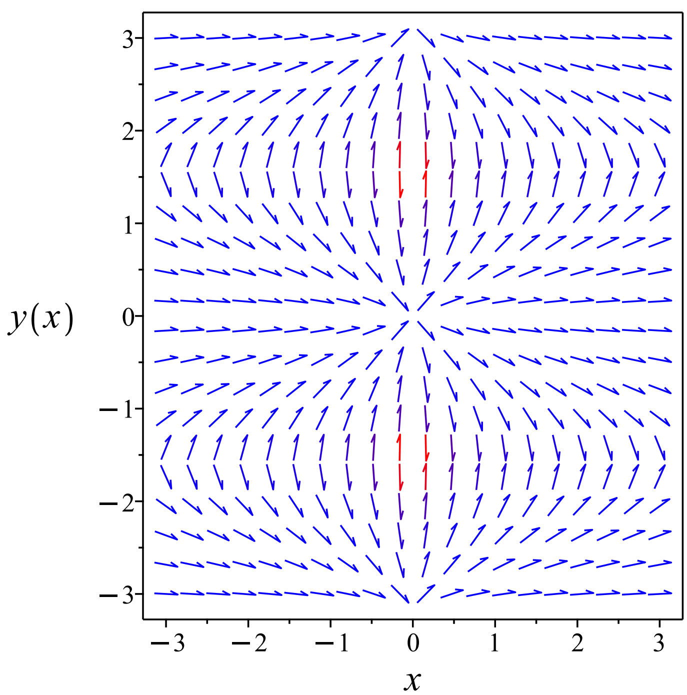


Figure 16: Slope field plot

Verification of solutions

$$-\ln(x)+\ln(\sin(y))=c_1$$

Verified OK.

1.6.4 Maple step by step solution

Let's solve

$x\cos(y)\,y'-\sin(y)=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'\cos(y)}{\sin(y)}=\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'\cos(y)}{\sin(y)}dx=\int \frac{1}{x}dx+c_1$

- Evaluate integral

  $\ln(\sin(y))=\ln(x)+c_1$

- Solve for y

  $y=\arcsin\left(x\,\mathrm{e}^{c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 8

```
dsolve(x*cos(y(x))*diff(y(x),x)-sin(y(x))=0,y(x), singsol=all)
```

$$y(x) = \arcsin(c_1 x)$$

✓ Solution by Mathematica

Time used: 9.024 (sec). Leaf size: 17

```
DSolve[x*Cos[y[x]]*y'[x]-Sin[y[x]]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin(e^{c_1} x)$$
$$y(x) \to 0$$

1.7 problem 7

Internal problem ID [5081]
Internal file name [`OUTPUT/4574_Sunday_June_05_2022_03_01_07_PM_69471112/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{(x^3 + xy^2)\, y' - 2y^3 = 0}$$

1.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(x^3 + x^3 u(x)^2)\,(u'(x)\, x + u(x)) - 2u(x)^3\, x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u^3 - u}{x\,(u^2+1)}
\end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u^3-u}{u^2+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^3-u}{u^2+1}}\, du = \frac{1}{x}\, dx$$

$$
\int \frac{1}{\frac{u^3-u}{u^2+1}}du = \int \frac{1}{x}dx
$$

$$
\ln(u+1)+\ln(u-1)-\ln(u) = \ln(x)+c_2
$$

Raising both side to exponential gives

$$
\mathrm{e}^{\ln(u+1)+\ln(u-1)-\ln(u)} = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u^2-1}{u} = c_3x
$$

The solution is

$$
\frac{u(x)^2-1}{u(x)} = c_3x
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\frac{x\left(\frac{y^2}{x^2}-1\right)}{y} = c_3x
$$

$$
\frac{y^2-x^2}{xy} = c_3x
$$

Summary

The solution(s) found are the following

$$
\frac{y^2-x^2}{xy} = c_3x \tag{1}
$$

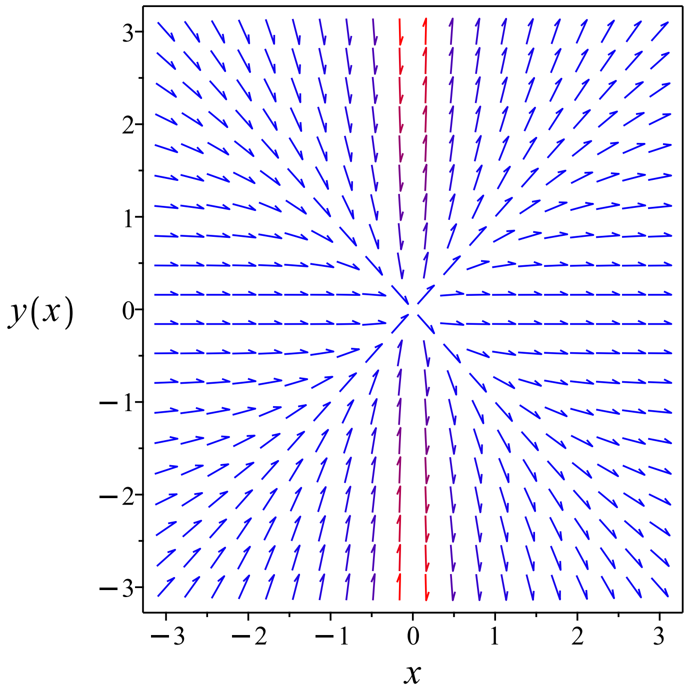


Figure 17: Slope field plot

Verification of solutions

$$\frac{y^2 - x^2}{xy} = c_3 x$$

Verified OK.

1.7.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{2y^3}{x\,(x^2 + y^2)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{2y^3(b_3 - a_2)}{x\,(x^2 + y^2)} - \frac{4y^6 a_3}{x^2\,(x^2 + y^2)^2} \\ &- \left(-\frac{2y^3}{x^2\,(x^2 + y^2)} - \frac{4y^3}{(x^2 + y^2)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(\frac{6y^2}{x\,(x^2 + y^2)} - \frac{4y^4}{x\,(x^2 + y^2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^6 b_2 - 4x^4 y^2 b_2 + 4x^3 y^3 a_2 - 4x^3 y^3 b_3 + 6x^2 y^4 a_3 - x^2 y^4 b_2 - 2y^6 a_3 - 6x^3 y^2 b_1 + 6x^2 y^3 a_1 - 2x\,y^4 b_1 + 2y^5 a_1}{x^2\,(x^2 + y^2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &x^6 b_2 - 4x^4 y^2 b_2 + 4x^3 y^3 a_2 - 4x^3 y^3 b_3 + 6x^2 y^4 a_3 - x^2 y^4 b_2 \\ &- 2y^6 a_3 - 6x^3 y^2 b_1 + 6x^2 y^3 a_1 - 2x\,y^4 b_1 + 2y^5 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2 v_1^3 v_2^3 + 6a_3 v_1^2 v_2^4 - 2a_3 v_2^6 + b_2 v_1^6 - 4b_2 v_1^4 v_2^2 - b_2 v_1^2 v_2^4 \\ &- 4b_3 v_1^3 v_2^3 + 6a_1 v_1^2 v_2^3 + 2a_1 v_2^5 - 6b_1 v_1^3 v_2^2 - 2b_1 v_1 v_2^4 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^6 - 4b_2 v_1^4 v_2^2 + (4a_2 - 4b_3) v_1^3 v_2^3 - 6b_1 v_1^3 v_2^2 + (6a_3 - b_2) v_1^2 v_2^4 + 6a_1 v_1^2 v_2^3 - 2b_1 v_1 v_2^4 - 2a_3 v_2^6 + 2a_1 v_2^5 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
b_2 &= 0 \\
2a_1 &= 0 \\
6a_1 &= 0 \\
-2a_3 &= 0 \\
-6b_1 &= 0 \\
-2b_1 &= 0 \\
-4b_2 &= 0 \\
4a_2 - 4b_3 &= 0 \\
6a_3 - b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{2y^3}{x\,(x^2+y^2)}\right)(x) \\
&= \frac{y\,x^2 - y^3}{x^2+y^2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y\,x^2-y^3}{x^2+y^2}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln(x+y) + \ln(y) - \ln(-x+y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{2y^3}{x\,(x^2+y^2)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{2x}{x^2 - y^2} \\
S_y &= -\frac{1}{x+y} + \frac{1}{y} + \frac{1}{x-y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{2}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{2}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -2\ln(R) + c_1 \tag{4}
$$

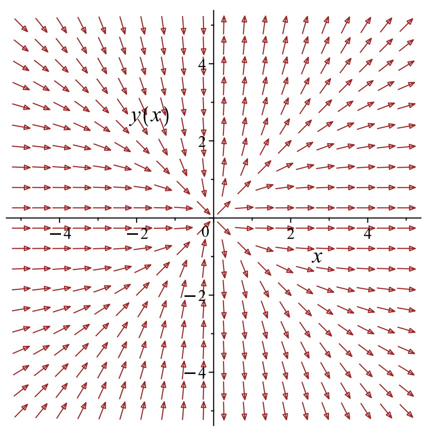
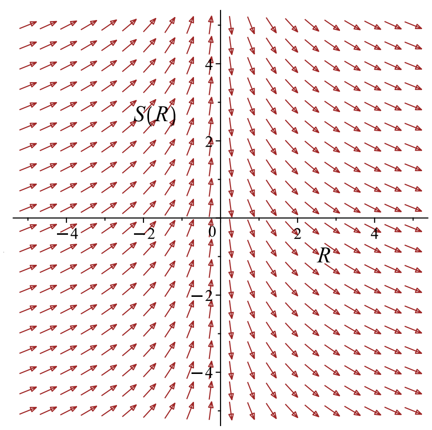
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-\ln(x+y) + \ln(y) - \ln(-x+y) = -2\ln(x) + c_1
$$

Which simplifies to

$$
-\ln(x+y) + \ln(y) - \ln(-x+y) = -2\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y^3}{x(x^2+y^2)}$ 	$R = x$ $S = -\ln(x+y) + \ln(y)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(x+y) + \ln(y) - \ln(-x+y) = -2\ln(x) + c_1 \tag{1}$$

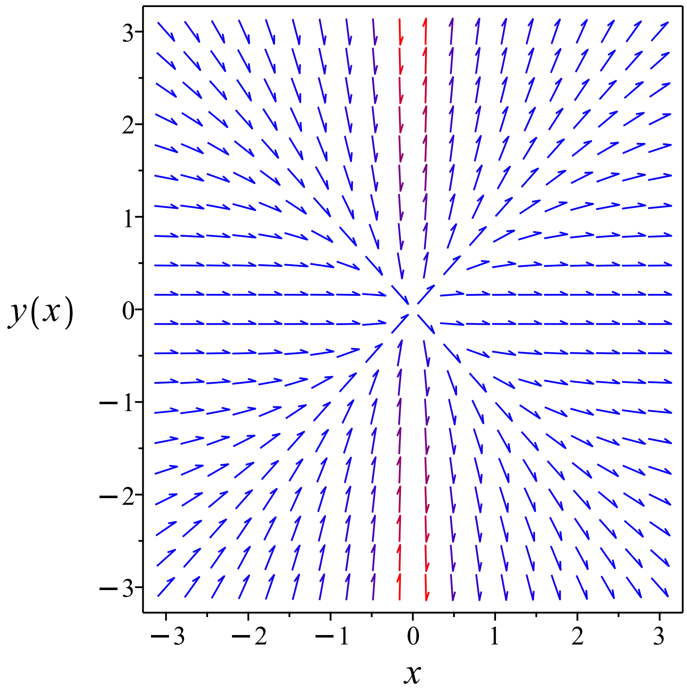


Figure 18: Slope field plot

Verification of solutions

$$-\ln(x+y)+\ln(y)-\ln(-x+y)=-2\ln(x)+c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 44

```
dsolve((x^3+x*y(x)^2)*diff(y(x),x)=2*y(x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{\left(-c_1 x + \sqrt{c_1^2 x^2 + 4}\right) x}{2}$$
$$y(x) = \frac{\left(c_1 x + \sqrt{c_1^2 x^2 + 4}\right) x}{2}$$

✓ Solution by Mathematica

Time used: 1.2 (sec). Leaf size: 83

```
DSolve[(x^3+x*y[x]^2)*y'[x]==2*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2} x \left(\sqrt{4 + e^{2c_1} x^2} + e^{c_1} x\right)$$
$$y(x) \to \frac{1}{2} x \left(\sqrt{4 + e^{2c_1} x^2} - e^{c_1} x\right)$$
$$y(x) \to 0$$
$$y(x) \to -x$$
$$y(x) \to x$$

1.8 problem 8

Internal problem ID [5082]
Internal file name [`OUTPUT/4575_Sunday_June_05_2022_03_01_10_PM_58866648/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2-1)\,y'+2xy=x}$$

1.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(-2y+1)}{x^2-1} \end{aligned}$$

Where $f(x)=\frac{x}{x^2-1}$ and $g(y)=-2y+1$. Integrating both sides gives

$$\frac{1}{-2y+1}\,dy=\frac{x}{x^2-1}\,dx$$

$$\int \frac{1}{-2y+1}\,dy = \int \frac{x}{x^2-1}\,dx$$

$$-\frac{\ln(-2y+1)}{2} = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{-2y+1}} = \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}+c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{-2y+1}} = c_2 \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}$$

Which simplifies to

$$y = \frac{\left(c_2^2(x-1)(x+1)\,\mathrm{e}^{2c_1}-1\right)\mathrm{e}^{-2c_1}}{2c_2^2(x-1)(x+1)}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_2^2(x-1)(x+1)\,\mathrm{e}^{2c_1}-1\right)\mathrm{e}^{-2c_1}}{2c_2^2(x-1)(x+1)} \tag{1}$$

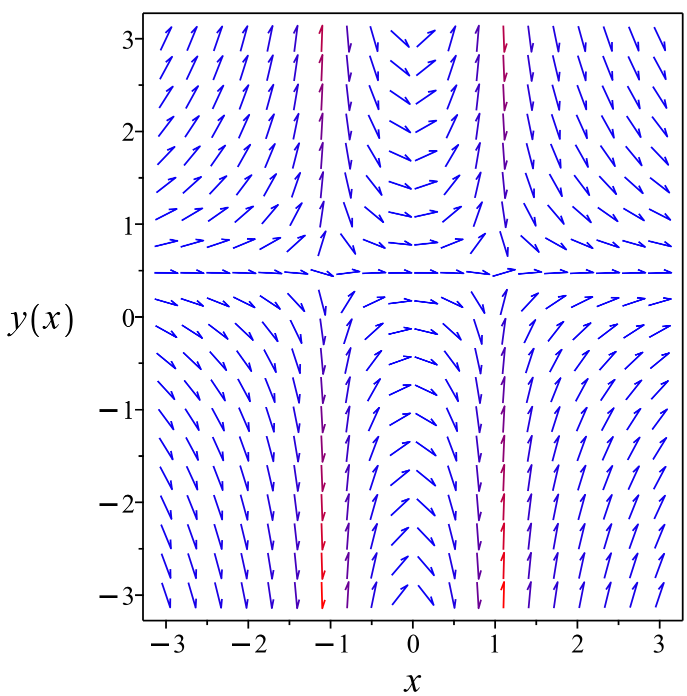


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{\left(c_2^2(x-1)\left(x+1\right)\mathrm{e}^{2c_1}-1\right)\mathrm{e}^{-2c_1}}{2c_2^2\left(x-1\right)\left(x+1\right)}$$

Verified OK.

1.8.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2x}{x^2-1} \\ q(x) &= \frac{x}{x^2-1} \end{aligned}$$

Hence the ode is

$$y' + \frac{2xy}{x^2-1} = \frac{x}{x^2-1}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2x}{x^2-1}dx} \\ &= \mathrm{e}^{\ln(x-1)+\ln(x+1)} \end{aligned}$$

Which simplifies to

$$\mu = x^2 - 1$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x}{x^2-1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2-1\right)y\right) &= \left(x^2-1\right)\left(\frac{x}{x^2-1}\right) \\ \mathrm{d}\left(\left(x^2-1\right)y\right) &= x\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \left(x^2-1\right)y &= \int x\,\mathrm{d}x \\ \left(x^2-1\right)y &= \frac{x^2}{2} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2 - 1$ results in

$$y = \frac{x^2}{2x^2 - 2} + \frac{c_1}{x^2 - 1}$$

which simplifies to

$$y = \frac{x^2 + 2c_1}{2x^2 - 2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x^2 - 2} \tag{1}$$

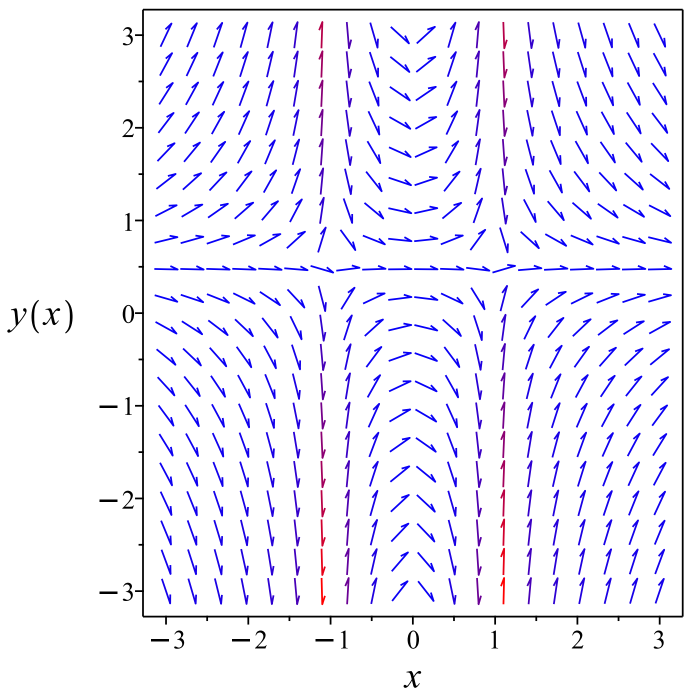


Figure 20: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x^2 - 2}$$

Verified OK.

1.8.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2xy + x}{x^2 - 1} \tag{1}$$

Which becomes

$$0 = (-x^2 + 1)\, dy + (-x(2y - 1))\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x^2 + 1)\, dy + (-x(2y - 1))\, dx = d\left(-\frac{x^2(2y - 1)}{2} + y\right)$$

Hence (2) becomes

$$0 = d\left(-\frac{x^2(2y - 1)}{2} + y\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{x^2 + 2c_1}{2x^2 - 2} + c_1$$

Summary
The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x^2 - 2} + c_1 \tag{1}$$

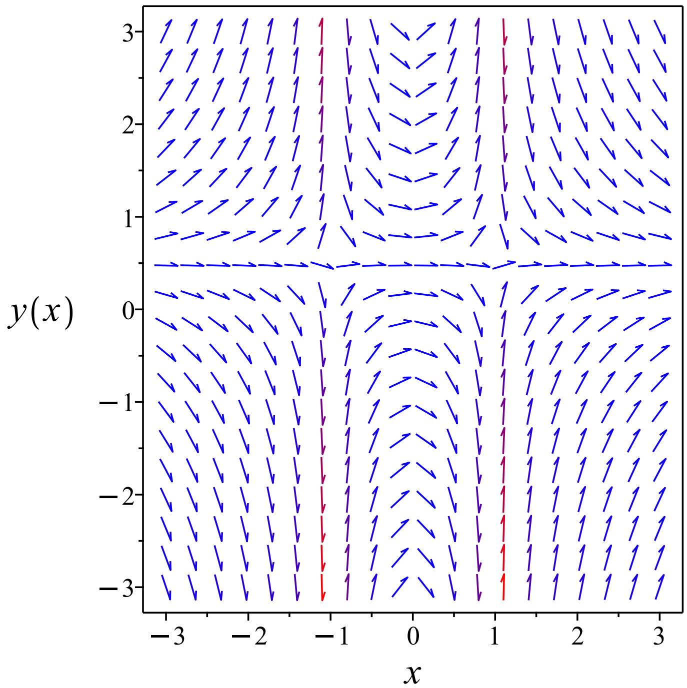


Figure 21: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x^2 - 2} + c_1$$

Verified OK.

1.8.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{x(2y-1)}{x^2-1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 15: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\ln(x-1)-\ln(x+1)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\ln(x-1)-\ln(x+1)}} dy \end{aligned}$$

Which results in

$$S = (x-1)(x+1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{x(2y-1)}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 - 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y x^2 - y = \frac{x^2}{2} + c_1$$

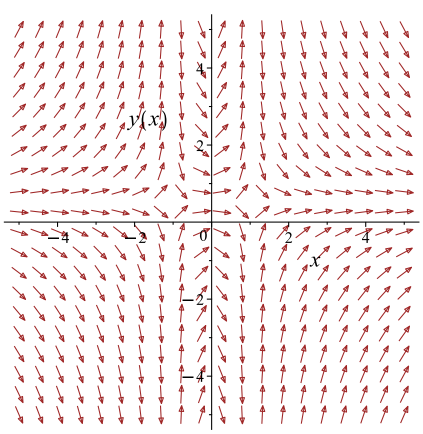
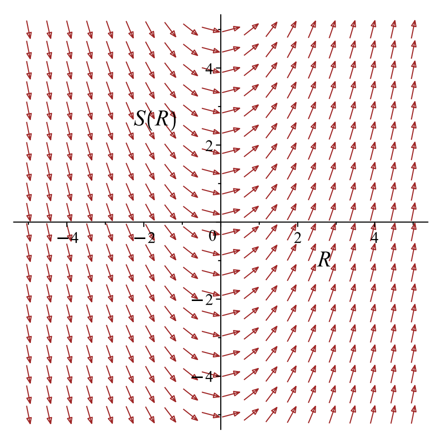
Which simplifies to

$$y x^2 - y = \frac{x^2}{2} + c_1$$

Which gives

$$y = \frac{x^2 + 2c_1}{2x^2 - 2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{x(2y-1)}{x^2-1}$

</td><td>$R = x$
$S = y\,x^2 - y$</td><td>$\frac{dS}{dR} = R$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{x^2 + 2c_1}{2x^2 - 2} \tag{1}$$

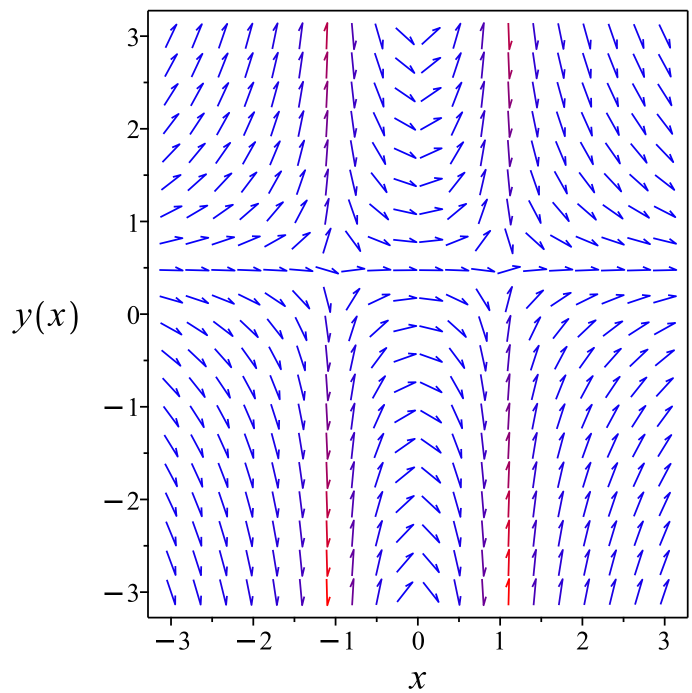


Figure 22: Slope field plot

Verification of solutions

$$y = \frac{x^2 + 2c_1}{2x^2 - 2}$$

Verified OK.

1.8.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{-2y+1}\right) \mathrm{d}y &= \left(\frac{x}{x^2-1}\right) \mathrm{d}x \\
\left(-\frac{x}{x^2-1}\right) \mathrm{d}x + \left(\frac{1}{-2y+1}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x}{x^2-1} \\
N(x, y) &= \frac{1}{-2y+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2-1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-2y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{x}{x^2-1} \,\mathrm{d}x \\
\phi &= -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-2y+1}$. Therefore equation (4) becomes

$$
\frac{1}{-2y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y-1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y-1}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(2y-1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(2y-1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2} - \frac{\ln(2y-1)}{2}$$

The solution becomes

$$y = \frac{x^2 + \mathrm{e}^{-2c_1} - 1}{2x^2 - 2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + \mathrm{e}^{-2c_1} - 1}{2x^2 - 2} \tag{1}$$

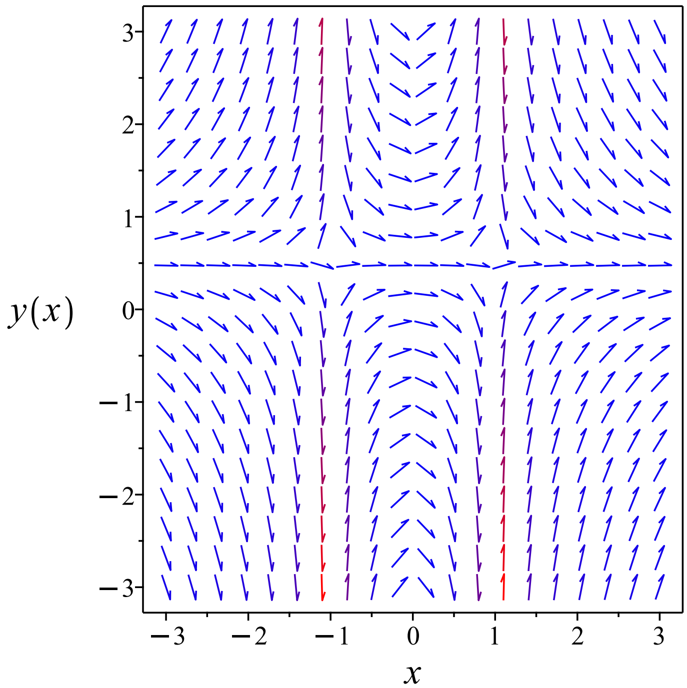


Figure 23: Slope field plot

Verification of solutions

$$y = \frac{x^2 + e^{-2c_1} - 1}{2x^2 - 2}$$

Verified OK.

1.8.6 Maple step by step solution

Let's solve

$(x^2 - 1) y' + 2xy = x$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int ((x^2 - 1) y' + 2xy) dx = \int x dx + c_1$

- Evaluate integral

  $(x^2 - 1) y = \frac{x^2}{2} + c_1$

- Solve for y

$$y = \frac{x^2+2c_1}{2(x^2-1)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve((x^2-1)*diff(y(x),x)+2*x*y(x)=x,y(x), singsol=all)
```

$$y(x) = \frac{x^2+2c_1}{2x^2-2}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 31

```
DSolve[(x^2-1)*y'[x]+2*x*y[x]==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2+2c_1}{2(x^2-1)}$$
$$y(x) \to \frac{1}{2}$$

1.9 problem 9

Internal problem ID [5083]
Internal file name [`OUTPUT/4576_Sunday_June_05_2022_03_01_11_PM_93920876/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y \tanh(x) = 2 \sinh(x)}$$

1.9.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tanh(x) \\ q(x) &= 2 \sinh(x) \end{aligned}$$

Hence the ode is

$$y' + y \tanh(x) = 2 \sinh(x)$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \tanh(x)dx} \\ &= \cosh(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2\sinh(x)\right) \\ \frac{d}{dx}(\cosh(x)\, y) &= (\cosh(x))\left(2\sinh(x)\right) \\ d(\cosh(x)\, y) &= \sinh(2x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\cosh(x)\, y &= \int \sinh(2x)\, dx \\ \cosh(x)\, y &= \frac{\cosh(2x)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cosh(x)$ results in

$$y = \frac{\operatorname{sech}(x)\cosh(2x)}{2} + c_1 \operatorname{sech}(x)$$

which simplifies to

$$y = \left(\cosh(x)^2 - \frac{1}{2} + c_1\right)\operatorname{sech}(x)$$

Summary

The solution(s) found are the following

$$y = \left(\cosh(x)^2 - \frac{1}{2} + c_1\right)\operatorname{sech}(x) \tag{1}$$

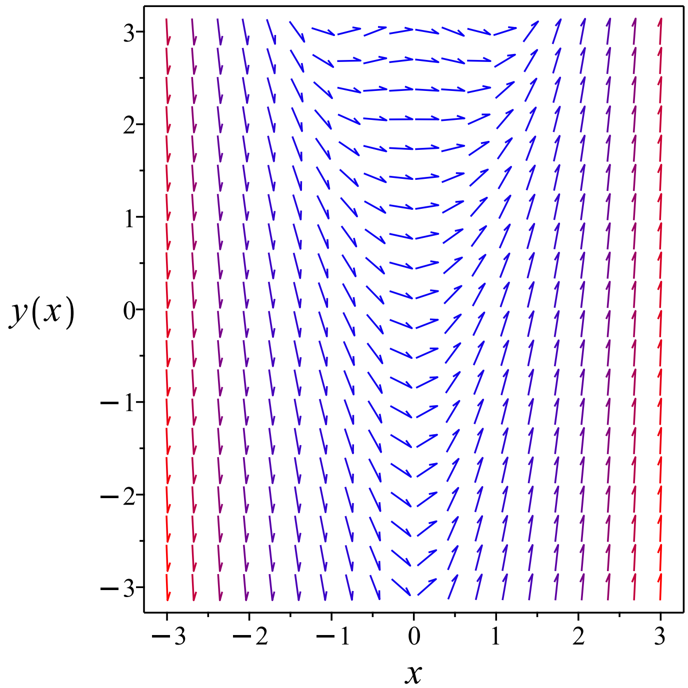


Figure 24: Slope field plot

Verification of solutions

$$y = \left(\cosh(x)^2 - \frac{1}{2} + c_1\right) \operatorname{sech}(x)$$

Verified OK.

1.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -y \tanh(x) + 2\sinh(x) \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 18: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cosh(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\cosh(x)}} dy \end{aligned}$$

Which results in

$$S = \cosh(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y \tanh(x) + 2 \sinh(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \sinh(x)\, y \\ S_y &= \cosh(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sinh(2x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sinh(2R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R,S. Integrating the above gives

$$S(R) = \frac{\cosh(2R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x,y coordinates. This results in

$$y\cosh(x) = \frac{\cosh(2x)}{2} + c_1$$

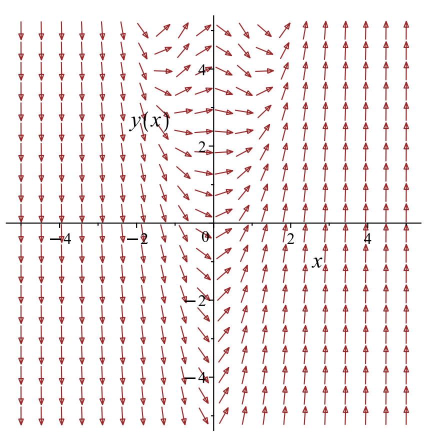
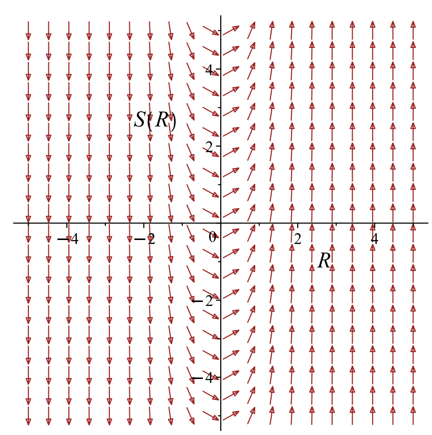
Which simplifies to

$$y\cosh(x) = \frac{\cosh(2x)}{2} + c_1$$

Which gives

$$y = \frac{\cosh(2x) + 2c_1}{2\cosh(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x,y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R,S)
$\frac{dy}{dx} = -y\tanh(x) + 2\sinh(x)$  	$R = x$ $S = \cosh(x)\, y$	$\frac{dS}{dR} = \sinh(2R)$ 

Summary
The solution(s) found are the following

$$y = \frac{\cosh(2x) + 2c_1}{2\cosh(x)} \tag{1}$$

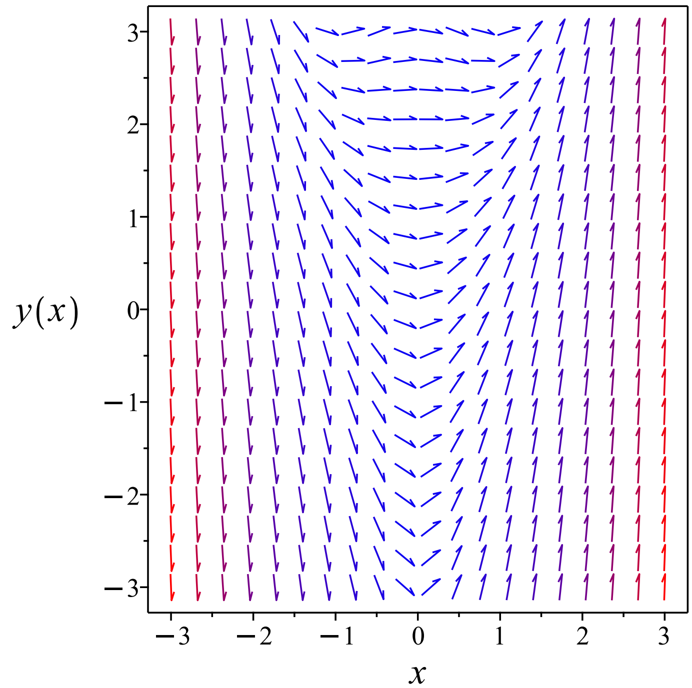


Figure 25: Slope field plot

Verification of solutions

$$y = \frac{\cosh(2x) + 2c_1}{2\cosh(x)}$$

Verified OK.

1.9.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y \tanh(x) + 2 \sinh(x)) \, \mathrm{d}x$$
$$(y \tanh(x) - 2 \sinh(x)) \, \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \tanh(x) - 2 \sinh(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \tanh(x) - 2 \sinh(x)) \\ &= \tanh(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\tanh(x)) - (0)) \\
&= \tanh(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \tanh(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cosh(x))} \\
&= \cosh(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cosh(x)(y \tanh(x) - 2\sinh(x)) \\
&= \sinh(x)(-2\cosh(x) + y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cosh(x)(1) \\
&= \cosh(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\sinh(x)(-2\cosh(x) + y)) + (\cosh(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \sinh(x) \left(-2\cosh(x) + y\right) \mathrm{d}x$$

$$\phi = \cosh(x) \left(-\cosh(x) + y\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cosh(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cosh(x)$. Therefore equation (4) becomes

$$\cosh(x) = \cosh(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \cosh(x) \left(-\cosh(x) + y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cosh(x) \left(-\cosh(x) + y\right)$$

The solution becomes

$$y = \frac{\cosh(x)^2 + c_1}{\cosh(x)}$$

Summary
The solution(s) found are the following

$$y = \frac{\cosh(x)^2 + c_1}{\cosh(x)} \tag{1}$$

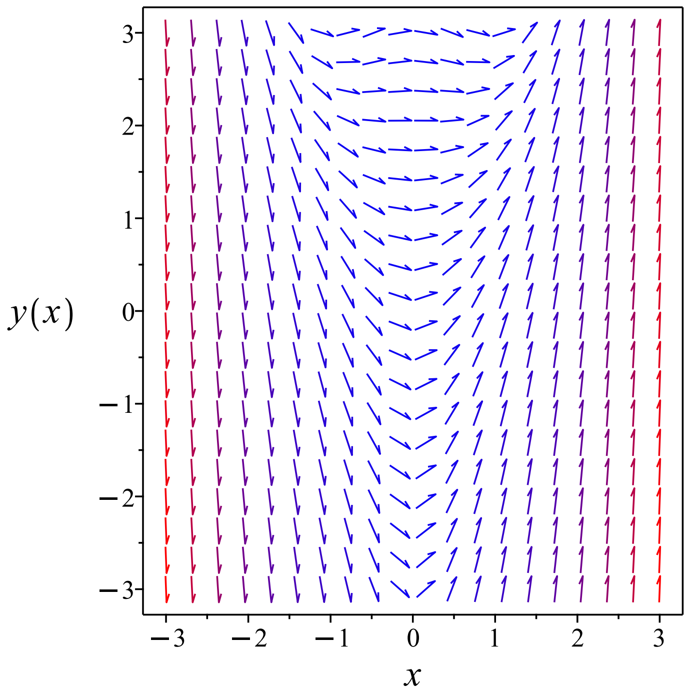


Figure 26: Slope field plot

Verification of solutions

$$y = \frac{\cosh(x)^2 + c_1}{\cosh(x)}$$

Verified OK.

1.9.4 Maple step by step solution

Let's solve

$y' + y\tanh(x) = 2\sinh(x)$

- Highest derivative means the order of the ODE is 1

y'

- Isolate the derivative

$y' = -y \tanh(x) + 2\sinh(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\tanh(x) = 2\sinh(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\tanh(x)\right) = 2\mu(x)\sinh(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + y\tanh(x)\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)\tanh(x)$

- Solve to find the integrating factor

  $\mu(x) = \cosh(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 2\mu(x)\sinh(x)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 2\mu(x)\sinh(x)\,dx + c_1$

- Solve for y

  $y = \frac{\int 2\mu(x)\sinh(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \cosh(x)$

  $y = \frac{\int 2\cosh(x)\sinh(x)dx + c_1}{\cosh(x)}$

- Evaluate the integrals on the rhs

  $y = \frac{\cosh(x)^2 + c_1}{\cosh(x)}$

- Simplify

  $y = \cosh(x) + c_1 \mathrm{sech}(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)*tanh(x)=2*sinh(x),y(x), singsol=all)
```

$$y(x) = \left(\cosh\left(x\right)^2 - \frac{1}{2} + c_1\right) \operatorname{sech}\left(x\right)$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 20

```
DSolve[y'[x]+y[x]*Tanh[x]==2*Sinh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\operatorname{sech}(x)(\cosh(2x) + 2c_1)$$

1.10 problem 10

Internal problem ID [5084]
Internal file name [`OUTPUT/4577_Sunday_June_05_2022_03_01_12_PM_65545073/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{xy' - 2y = \cos(x)\, x^3}$$

1.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = \cos(x)\, x^2$$

Hence the ode is

$$y' - \frac{2y}{x} = \cos(x)\, x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\cos(x)\, x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(\cos(x)\, x^2\right) \\ \mathrm{d}\left(\frac{y}{x^2}\right) &= \cos(x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2} &= \int \cos(x)\ \mathrm{d}x \\ \frac{y}{x^2} &= \sin(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$y = x^2 \sin(x) + c_1 x^2$$

which simplifies to

$$y = x^2(\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

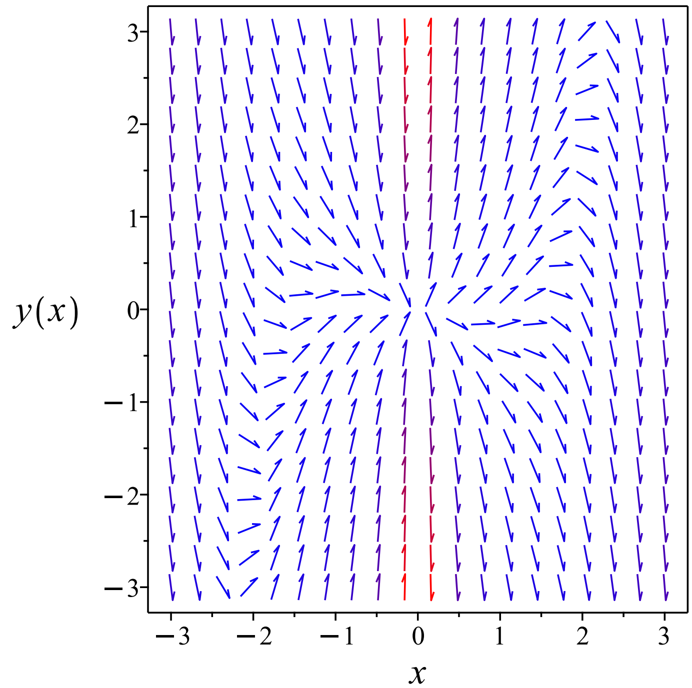


Figure 27: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

1.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2y + \cos(x)\, x^3}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 21: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x^2
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y + \cos(x) x^3}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = \sin(x) + c_1$$

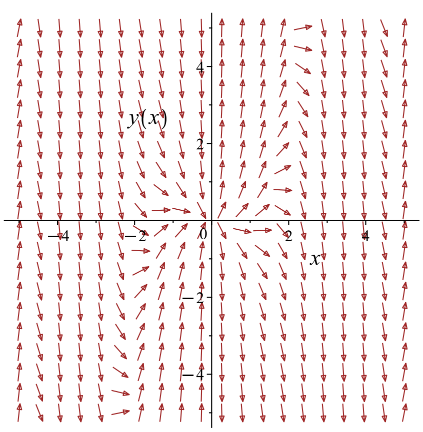
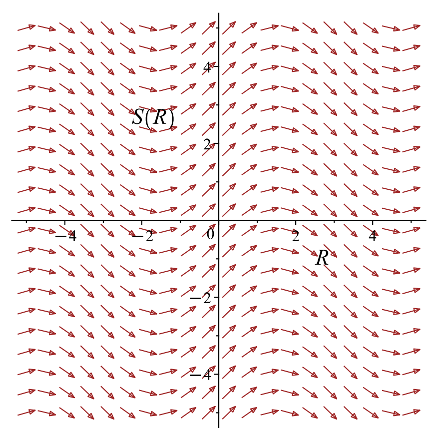
Which simplifies to

$$\frac{y}{x^2} = \sin(x) + c_1$$

Which gives

$$y = x^2(\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{2y+\cos(x)x^3}{x}$

</td><td>$R = x$
$S = \frac{y}{x^2}$</td><td>$\frac{dS}{dR} = \cos(R)$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

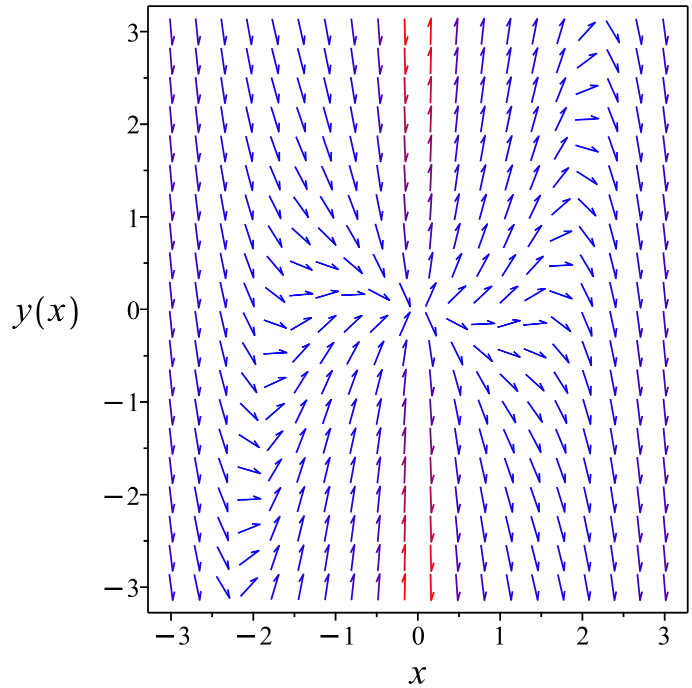


Figure 28: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

1.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(2y + \cos(x)\,x^3\right)\mathrm{d}x\\ \left(-2y - \cos(x)\,x^3\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -2y - \cos(x)\,x^3\\ N(x, y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2y - \cos(x)\,x^3\right)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-2) - (1)) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(-2y - \cos(x)\, x^3\right) \\
&= \frac{-2y - \cos(x)\, x^3}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(x) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2y - \cos(x)\, x^3}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \frac{-2y - \cos(x)\, x^3}{x^3} \, \mathrm{d}x \\
\phi &= -\sin(x) + \frac{y}{x^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$
\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\sin(x) + \frac{y}{x^2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\sin(x) + \frac{y}{x^2}
$$

The solution becomes

$$y = x^2(\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

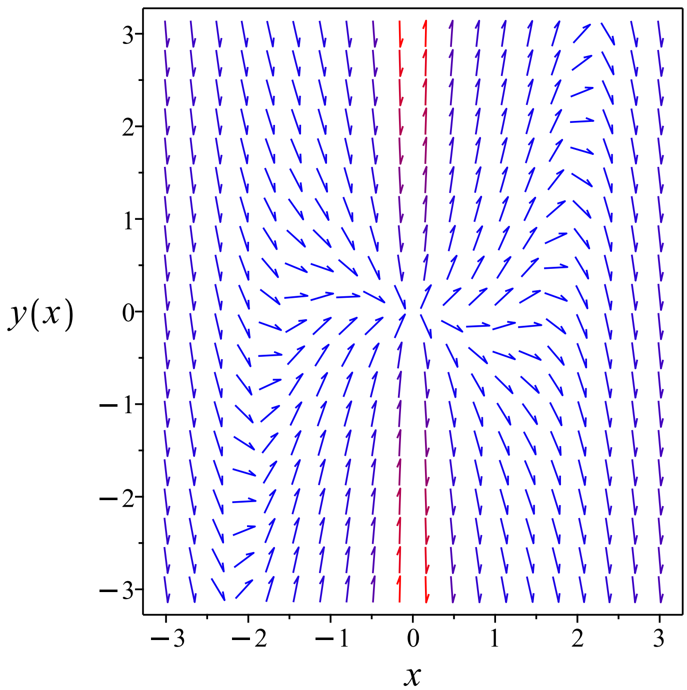


Figure 29: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

1.10.4 Maple step by step solution

Let's solve

$xy' - 2y = \cos(x)\, x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + \cos(x)\, x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = \cos(x)\, x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu(x)\cos(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\cos(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\cos(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2\left(\int \cos(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2(\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x)-2*y(x)=x^3*cos(x),y(x), singsol=all)
```

$$y(x) = (\sin(x) + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 14

```
DSolve[x*y'[x]-2*y[x]==x^3*Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(\sin(x) + c_1)$$

1.11 problem 11

Internal problem ID [5085]
Internal file name [`OUTPUT/4578_Sunday_June_05_2022_03_01_13_PM_85534098/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{y}{x} - y^3 = 0}$$

1.11.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(y^2x - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 24: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= y^3x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^3 x^2} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{2y^2 x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(y^2 x - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y^2 x^3} \\
S_y &= \frac{1}{y^3 x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

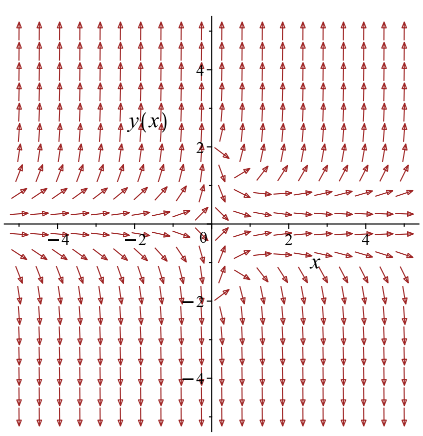
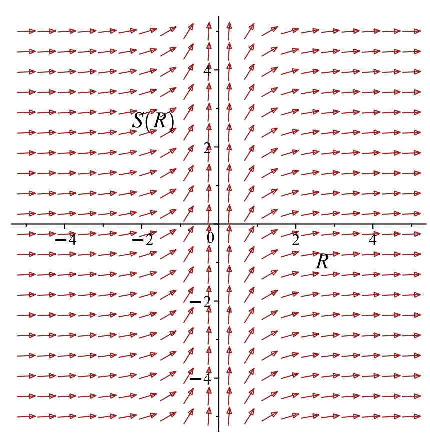
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2y^2x^2} = c_1 - \frac{1}{x}$$

Which simplifies to

$$-\frac{1}{2y^2x^2} = c_1 - \frac{1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y^2x-1)}{x}$ 	$R = x$ $S = -\frac{1}{2y^2x^2}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$-\frac{1}{2y^2x^2} = c_1 - \frac{1}{x} \tag{1}$$

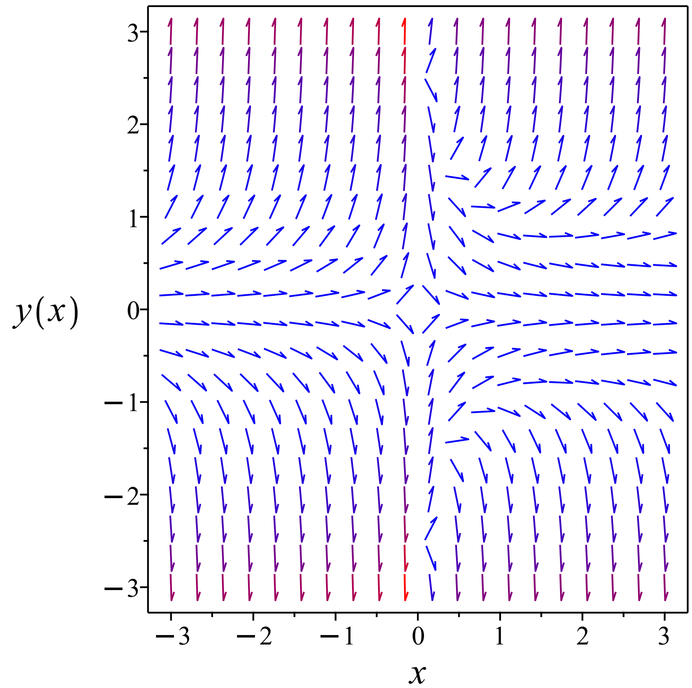


Figure 30: Slope field plot

Verification of solutions

$$-\frac{1}{2y^2x^2} = c_1 - \frac{1}{x}$$

Verified OK.

1.11.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(y^2x - 1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= 1 \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{1}{y^2 x} + 1 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{w(x)}{x} + 1 \\ w' &= \frac{2w}{x} - 2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -2 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-2) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(-2) \\ \mathrm{d}\left(\frac{w}{x^2}\right) &= \left(-\frac{2}{x^2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -\frac{2}{x^2}\,\mathrm{d}x \\ \frac{w}{x^2} &= \frac{2}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1 x^2 + 2x$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = c_1 x^2 + 2x$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{x(c_1 x + 2)}} \\ y(x) &= -\frac{1}{\sqrt{x(c_1 x + 2)}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{x\,(c_1x+2)}} \tag{1}$$

$$y = -\frac{1}{\sqrt{x\,(c_1x+2)}} \tag{2}$$

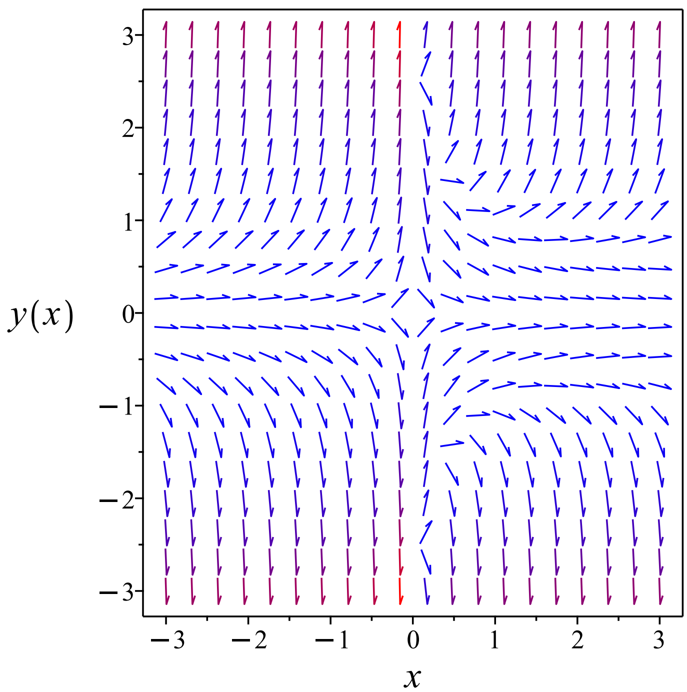


Figure 31: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{x\,(c_1x+2)}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{x\,(c_1x+2)}}$$

Verified OK.

1.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(y\left(y^2x-1\right)\right)\mathrm{d}x\\ \left(-y\left(y^2x-1\right)\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\left(y^2x-1\right)\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\left(y^2 x - 1\right)\right) \\
&= -3y^2 x + 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}\left(\left(-3y^2 x + 1\right) - (1)\right) \\
&= -3y^2
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y\left(y^2 x - 1\right)}\left((1) - \left(-3y^2 x + 1\right)\right) \\
&= -\frac{3yx}{y^2 x - 1}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(1) - (-3y^2x + 1)}{x\left(-y\left(y^2x - 1\right)\right) - y\left(x\right)} \\ &= -\frac{3}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{3}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \, \mathrm{d}t} \\ &= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-3\ln(t)} \\ &= \frac{1}{t^3} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^3y^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^3y^3}\left(-y\left(y^2x - 1\right)\right) \\ &= \frac{-y^2x + 1}{y^2x^3} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^3y^3}(x) \\ &= \frac{1}{y^3x^2} \end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y^2x+1}{y^2x^3}\right) + \left(\frac{1}{y^3x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y^2x+1}{y^2x^3}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{2y^2x-1}{2y^2x^2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2}{yx} - \frac{2y^2x-1}{y^3x^2} + f'(y) \tag{4}
$$

$$
= \frac{1}{y^3x^2} + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3x^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^3x^2} = \frac{1}{y^3x^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{2y^2x - 1}{2y^2x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{2y^2x - 1}{2y^2x^2}$$

Summary

The solution(s) found are the following

$$\frac{2xy^2 - 1}{2y^2x^2} = c_1 \tag{1}$$

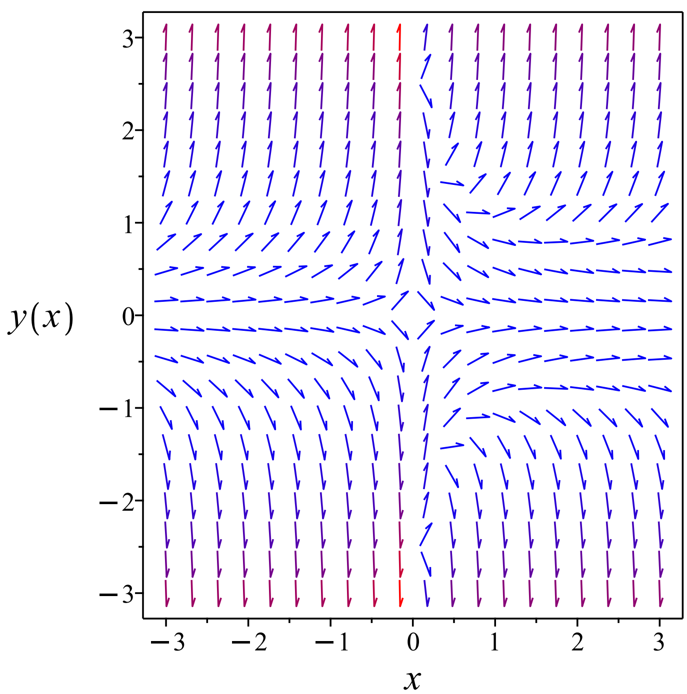


Figure 32: Slope field plot

Verification of solutions

$$\frac{2xy^2 - 1}{2y^2x^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x)+y(x)/x=y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{x\,(c_1x + 2)}}$$
$$y(x) = -\frac{1}{\sqrt{x\,(c_1x + 2)}}$$

✓ Solution by Mathematica

Time used: 0.375 (sec). Leaf size: 40

```
DSolve[y'[x]+y[x]/x==y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x(2 + c_1x)}}$$
$$y(x) \to \frac{1}{\sqrt{x(2 + c_1x)}}$$
$$y(x) \to 0$$

1.12 problem 12

Internal problem ID [5086]
Internal file name [`OUTPUT/4579_Sunday_June_05_2022_03_01_15_PM_92031906/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Test excercise 24. page 1067
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{xy' + 3y - y^2x^2 = 0}$$

1.12.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned}
y' &= \frac{y(y\,x^2 - 3)}{x} \\
y' &= \omega(x, y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2x^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^2 x^3} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{y\, x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(y\, x^2 - 3)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3}{y\, x^4} \\ S_y &= \frac{1}{y^2 x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx^3} = c_1 - \frac{1}{x}$$

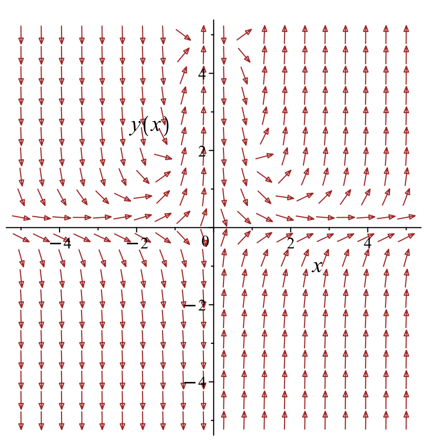
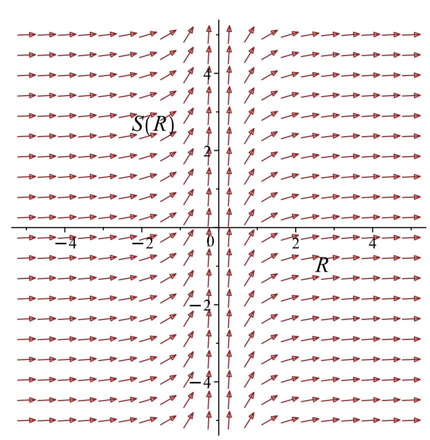
Which simplifies to

$$-\frac{1}{yx^3} = c_1 - \frac{1}{x}$$

Which gives

$$y = -\frac{1}{x^2 (c_1 x - 1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y(y\,x^2-3)}{x}$

</td><td>$R = x$
$S = -\frac{1}{y\,x^3}$</td><td>$\frac{dS}{dR} = \frac{1}{R^2}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = -\frac{1}{x^2 (c_1 x - 1)} \tag{1}$$

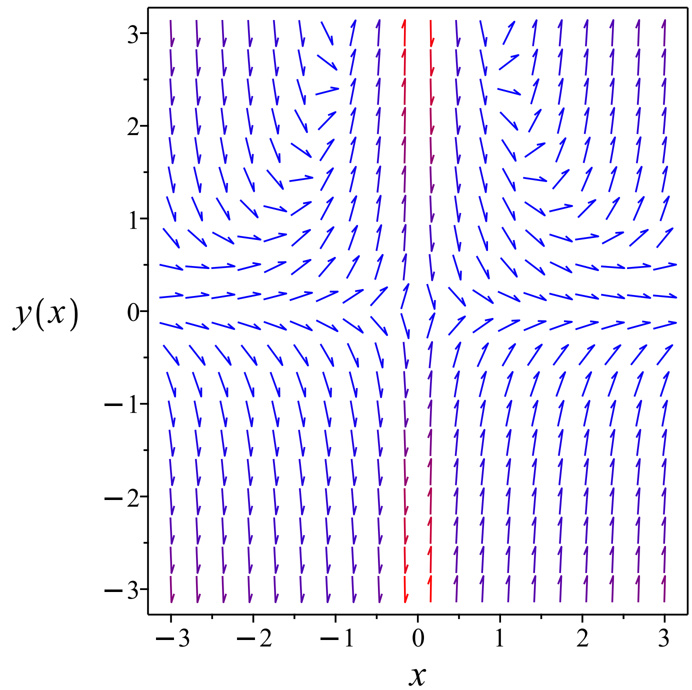


Figure 33: Slope field plot

Verification of solutions

$$y = -\frac{1}{x^2 (c_1 x - 1)}$$

Verified OK.

1.12.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(y\,x^2 - 3)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{3}{x} y + x y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{3}{x} \\ f_1(x) &= x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{3}{yx} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{3w(x)}{x} + x \\ w' &= \frac{3w}{x} - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= -x \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = -x$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)(-x) \\ \mathrm{d}\left(\frac{w}{x^3}\right) &= \left(-\frac{1}{x^2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x^3} &= \int -\frac{1}{x^2}\,\mathrm{d}x \\ \frac{w}{x^3} &= \frac{1}{x} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = c_1 x^3 + x^2$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = c_1 x^3 + x^2$$

Or

$$y = \frac{1}{c_1 x^3 + x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{c_1 x^3 + x^2} \tag{1}$$

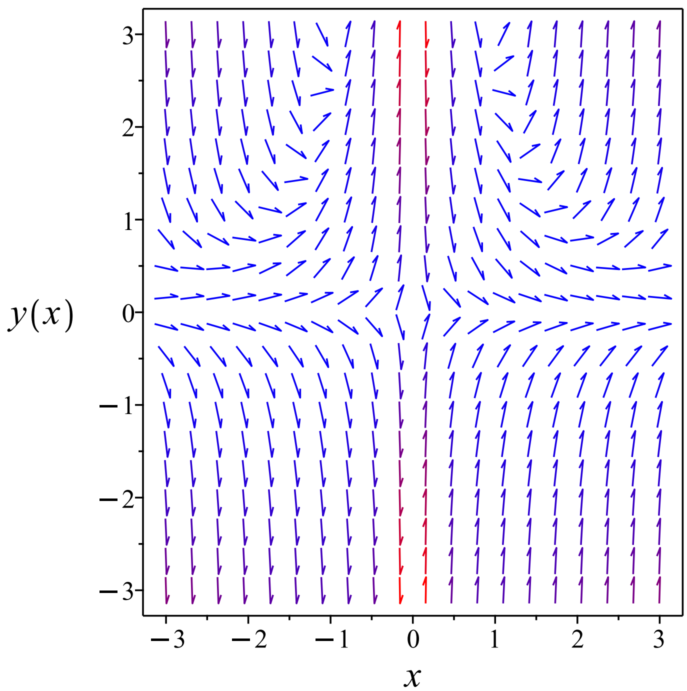


Figure 34: Slope field plot

Verification of solutions

$$y = \frac{1}{c_1 x^3 + x^2}$$

Verified OK.

1.12.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(y\,x^2-3)}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y^2x - \frac{3y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{3}{x}$ and $f_2(x) = x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2u} \\ &= \frac{-u'}{xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 1 \\ f_1f_2 &= -3 \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$xu''(x) + 2u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x}$$

The above shows that

$$u'(x) = -\frac{c_2}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{x^3 \left(c_1 + \frac{c_2}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{x^2 (c_3 x + 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 (c_3 x + 1)} \tag{1}$$

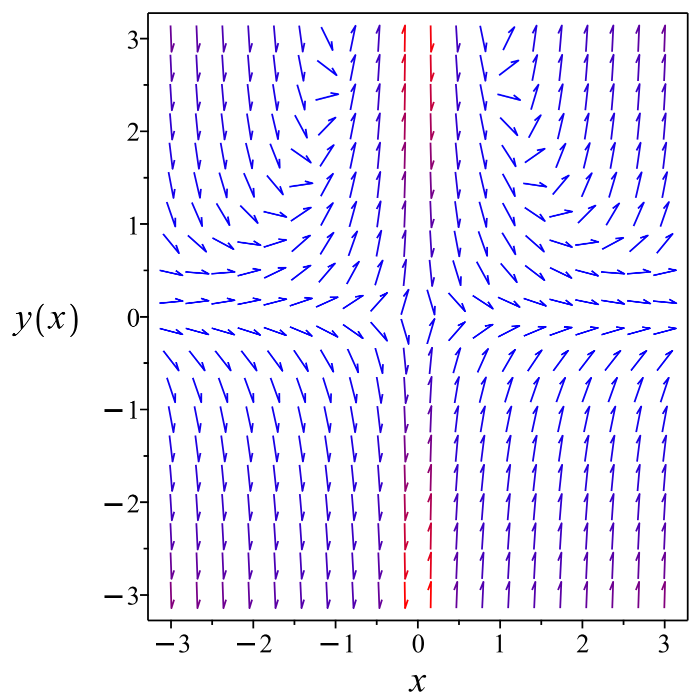


Figure 35: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 (c_3 x + 1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x*diff(y(x),x)+3*y(x)=x^2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{x^2 (c_1 x + 1)}$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 22

```
DSolve[x*y'[x]+3*y[x]==x^2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^2 + c_1 x^3}$$
$$y(x) \to 0$$

2 Program 24. First order differential equations. Further problems 24. page 1068

2.1 problem 1

Internal problem ID [5087]
Internal file name [`OUTPUT/4580_Sunday_June_05_2022_03_01_16_PM_65589201/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x(y-3)\,y'-4y=0}$$

2.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{4y}{x\,(y-3)} \end{aligned}$$

Where $f(x)=\frac{4}{x}$ and $g(y)=\frac{y}{y-3}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{y}{y-3}}\,dy &= \frac{4}{x}\,dx \\ \int \frac{1}{\frac{y}{y-3}}\,dy &= \int \frac{4}{x}\,dx \\ y-3\ln(y) &= 4\ln(x)+c_1 \end{aligned}$$

Which results in

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{4\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)-\frac{4\ln(x)}{3}-\frac{c_1}{3}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{4\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)-\frac{4\ln(x)}{3}-\frac{c_1}{3}} \tag{1}$$

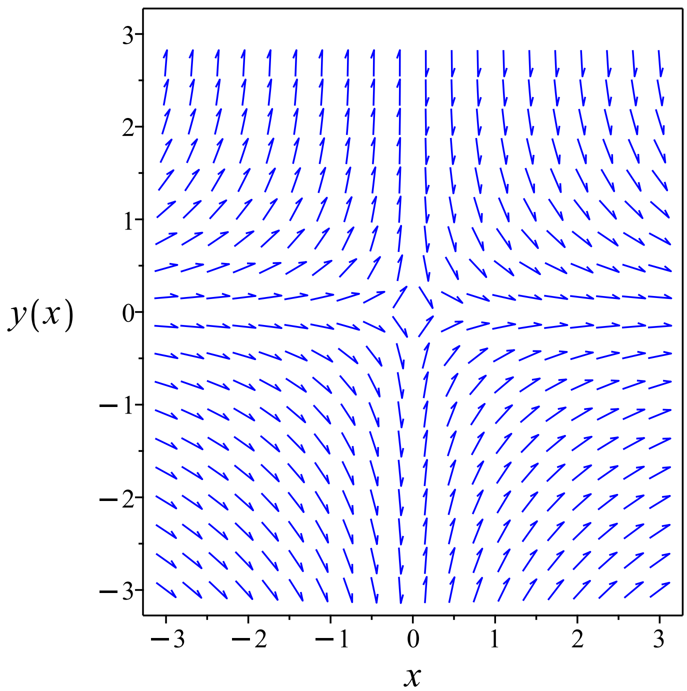


Figure 36: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-\frac{4\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)-\frac{4\ln(x)}{3}-\frac{c_1}{3}}$$

Verified OK.

2.1.2 Maple step by step solution

Let's solve

$x(y-3)\,y'-4y=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(y-3)}{y}=\frac{4}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'(y-3)}{y}dx=\int \frac{4}{x}dx+c_1$

- Evaluate integral

  $y-3\ln(y)=4\ln(x)+c_1$

- Solve for y

  $y=\mathrm{e}^{-\mathit{LambertW}\left(-\frac{\mathrm{e}^{-\frac{4\ln(x)}{3}-\frac{c_1}{3}}}{3}\right)-\frac{4\ln(x)}{3}-\frac{c_1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(x*(y(x)-3)*diff(y(x),x)=4*y(x),y(x), singsol=all)
```

$$y(x)=-3\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-\frac{4c_1}{3}}}{3x^{\frac{4}{3}}}\right)$$

✓ Solution by Mathematica

Time used: 13.068 (sec). Leaf size: 94

```
DSolve[x*(y[x]-3)*y'[x]==4*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3W\left(\frac{1}{3}\sqrt[3]{-\frac{e^{-c_1}}{x^4}}\right)$$
$$y(x) \to -3W\left(-\frac{1}{3}\sqrt[3]{-1}\sqrt[3]{-\frac{e^{-c_1}}{x^4}}\right)$$
$$y(x) \to -3W\left(\frac{1}{3}(-1)^{2/3}\sqrt[3]{-\frac{e^{-c_1}}{x^4}}\right)$$
$$y(x) \to 0$$

2.2 problem 2

Internal problem ID [5088]
Internal file name [`OUTPUT/4581_Sunday_June_05_2022_03_01_17_PM_19661026/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^3+1)\,y'-y x^2=0}$$

With initial conditions

$$[y(1)=2]$$

2.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-\frac{x^2}{x^3+1}\\ q(x)&=0\end{aligned}$$

Hence the ode is

$$y' - \frac{x^2 y}{x^3 + 1} = 0$$

The domain of $p(x) = -\frac{x^2}{x^3+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

2.2.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x^2 y}{x^3 + 1} \end{aligned}$$

Where $f(x) = \frac{x^2}{x^3+1}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\,dy &= \frac{x^2}{x^3 + 1}\,dx \\ \int \frac{1}{y}\,dy &= \int \frac{x^2}{x^3 + 1}\,dx \\ \ln(y) &= \frac{\ln(x^3 + 1)}{3} + c_1 \\ y &= e^{\frac{\ln(x^3+1)}{3} + c_1} \\ &= c_1 (x^3 + 1)^{\frac{1}{3}} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 2^{\frac{1}{3}} c_1$$

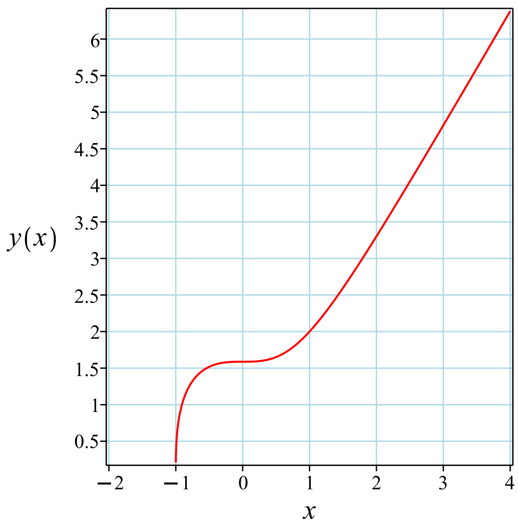
$$c_1 = 2^{\frac{2}{3}}$$

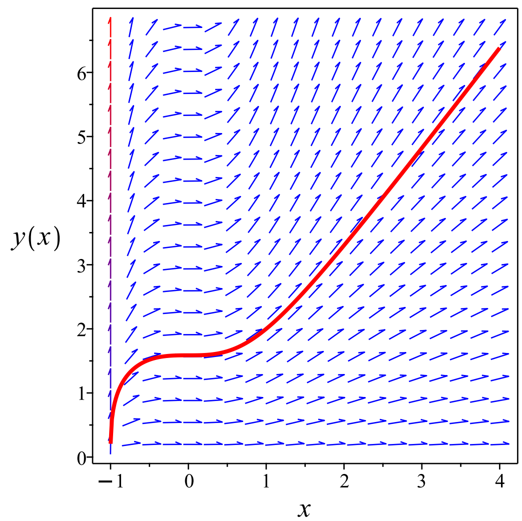
Substituting c_1 found above in the general solution gives

$$y = 2^{\frac{2}{3}} (x^3 + 1)^{\frac{1}{3}}$$

Summary
The solution(s) found are the following

$$y = 2^{\frac{2}{3}} \left(x^3+1\right)^{\frac{1}{3}} \tag{1}$$





(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2^{\frac{2}{3}} \left(x^3+1\right)^{\frac{1}{3}}$$

Verified OK.

2.2.3 Maple step by step solution

Let's solve

$\left[\left(x^3+1\right) y' - y x^2 = 0, y(1) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{x^2}{x^3+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{x^2}{x^3+1} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{\ln(x^3+1)}{3} + c_1$

- Solve for y

  $y = e^{\frac{\ln(x^3+1)}{3} + c_1}$

- Use initial condition $y(1) = 2$

  $2 = e^{\frac{\ln(2)}{3} + c_1}$

- Solve for c_1

  $c_1 = \frac{2\ln(2)}{3}$

- Substitute $c_1 = \frac{2\ln(2)}{3}$ into general solution and simplify

  $y = 2^{\frac{2}{3}}(x^3+1)^{\frac{1}{3}}$

- Solution to the IVP

  $y = 2^{\frac{2}{3}}(x^3+1)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 15

```
dsolve([(1+x^3)*diff(y(x),x)=x^2*y(x),y(1) = 2],y(x), singsol=all)
```

$$y(x) = 2^{\frac{2}{3}} \left(x^3 + 1\right)^{\frac{1}{3}}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 20

```
DSolve[{(1+x^3)*y'[x]==x^2*y[x],{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2^{2/3} \sqrt[3]{x^3 + 1}$$

2.3 problem 3

Internal problem ID [5089]
Internal file name [`OUTPUT/4582_Sunday_June_05_2022_03_01_18_PM_46227135/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(1+y)^2 y' = -x^3}$$

2.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{x^3}{(1+y)^2} \end{aligned}$$

Where $f(x) = -x^3$ and $g(y) = \frac{1}{(1+y)^2}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\frac{1}{(1+y)^2}}\,dy &= -x^3\,dx \\ \int \frac{1}{\frac{1}{(1+y)^2}}\,dy &= \int -x^3\,dx \\ \frac{(1+y)^3}{3} &= -\frac{x^4}{4} + c_1 \end{aligned}$$

Which results in

$$y = \frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{2} - 1$$

$$y = -\frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - 1$$

$$y = -\frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - 1$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{2} - 1 \tag{1}$$

$$y = -\frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - 1 \tag{2}$$

$$y = -\frac{\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\left(-6x^4+24c_1\right)^{\frac{1}{3}}}{4} - 1 \tag{3}$$

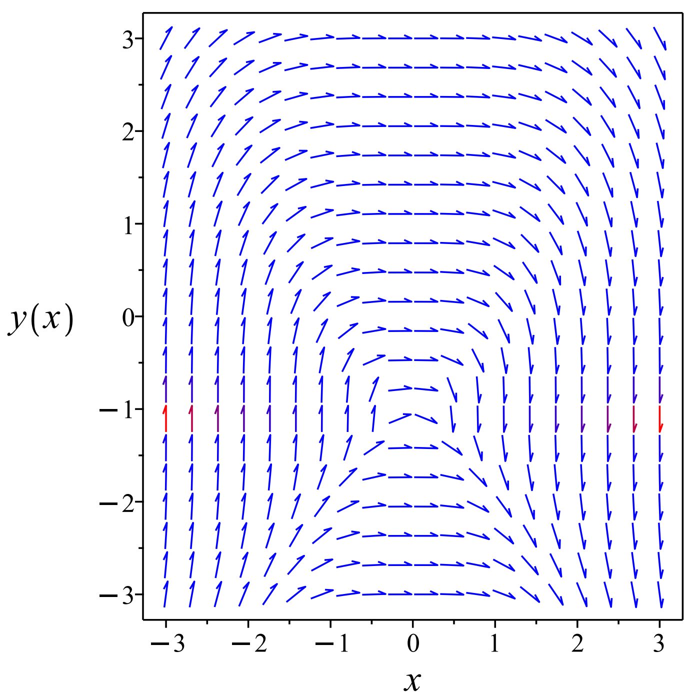


Figure 38: Slope field plot

Verification of solutions

$$y = \frac{(-6x^4 + 24c_1)^{\frac{1}{3}}}{2} - 1$$

Verified OK.

$$y = -\frac{(-6x^4 + 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(-6x^4 + 24c_1)^{\frac{1}{3}}}{4} - 1$$

Verified OK.

$$y = -\frac{(-6x^4 + 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(-6x^4 + 24c_1)^{\frac{1}{3}}}{4} - 1$$

Verified OK.

2.3.2 Maple step by step solution

Let's solve

$(1+y)^2\, y' = -x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (1+y)^2\, y' dx = \int -x^3 dx + c_1$

- Evaluate integral

  $\frac{(1+y)^3}{3} = -\frac{x^4}{4} + c_1$

- Solve for y

  $y = \frac{(-6x^4+24c_1)^{\frac{1}{3}}}{2} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 89

```
dsolve(x^3+(y(x)+1)^2*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{(-6x^4 - 24c_1)^{\frac{1}{3}}}{2} - 1$$

$$y(x) = -\frac{(-6x^4 - 24c_1)^{\frac{1}{3}}}{4} - \frac{i\sqrt{3}\,(-6x^4 - 24c_1)^{\frac{1}{3}}}{4} - 1$$

$$y(x) = -\frac{(-6x^4 - 24c_1)^{\frac{1}{3}}}{4} + \frac{i\sqrt{3}\,(-6x^4 - 24c_1)^{\frac{1}{3}}}{4} - 1$$

✓ Solution by Mathematica

Time used: 0.483 (sec). Leaf size: 110

```
DSolve[x^3+(y[x]+1)^2*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + \frac{\sqrt[3]{-3x^4 + 4 + 12c_1}}{2^{2/3}}$$

$$y(x) \to -1 + \frac{i\left(\sqrt{3} + i\right)\sqrt[3]{-3x^4 + 4 + 12c_1}}{2\ 2^{2/3}}$$

$$y(x) \to -1 - \frac{\left(1 + i\sqrt{3}\right)\sqrt[3]{-3x^4 + 4 + 12c_1}}{2\ 2^{2/3}}$$

2.4 problem 4

Internal problem ID [5090]
Internal file name [`OUTPUT/4583_Sunday_June_05_2022_03_01_19_PM_25170665/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\cos(y) + \left(1 + \mathrm{e}^{-x}\right)\sin(y)\, y' = 0}$$

With initial conditions

$$\left[y(0) = \frac{\pi}{4}\right]$$

2.4.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{\cos(y)}{(1 + \mathrm{e}^{-x})\sin(y)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = \frac{\pi}{4}$ is

$$\{-2i\pi\_Z92 - i\pi < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.4.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= -\frac{\cot(y)}{1 + \mathrm{e}^{-x}}
\end{aligned}$$

Where $f(x) = -\frac{1}{1+\mathrm{e}^{-x}}$ and $g(y) = \cot(y)$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\cot(y)}\, dy &= -\frac{1}{1 + \mathrm{e}^{-x}}\, dx \\
\int \frac{1}{\cot(y)}\, dy &= \int -\frac{1}{1 + \mathrm{e}^{-x}}\, dx \\
-\ln(\cos(y)) &= -\ln\left(1 + \mathrm{e}^{-x}\right) + \ln\left(\mathrm{e}^{-x}\right) + c_1
\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = \mathrm{e}^{-\ln(1+\mathrm{e}^{-x})+\ln(\mathrm{e}^{-x})+c_1}$$

Which simplifies to

$$\sec(y) = c_2 \mathrm{e}^{-\ln(1+\mathrm{e}^{-x})+\ln(\mathrm{e}^{-x})}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \frac{\pi}{2} - \arcsin\left(\frac{2\,\mathrm{e}^{-c_1}}{c_2}\right)$$

$$c_1 = -\frac{\ln\left(\frac{c_2^2}{8}\right)}{2}$$

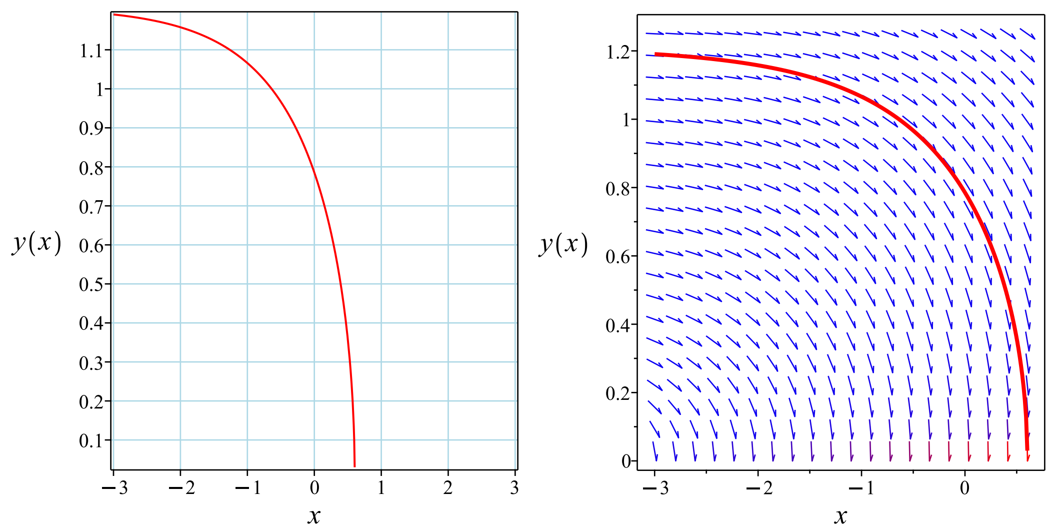
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1 + \mathrm{e}^{x})\sqrt{2}}{4}\right)$$

Summary

The solution(s) found are the following

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^{x})\sqrt{2}}{4}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\pi}{2} - \arcsin\left(\frac{(1+e^{x})\sqrt{2}}{4}\right)$$

Verified OK. `{positive}`

2.4.3 Maple step by step solution

Let's solve

$\left[\cos(y) + (1+e^{-x})\sin(y)\, y' = 0, y(0) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y' \sin(y)}{\cos(y)} = -\frac{1}{1+e^{-x}}$

- Integrate both sides with respect to x

$\int \frac{y' \sin(y)}{\cos(y)} dx = \int -\frac{1}{1+\mathrm{e}^{-x}} dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = -\ln(1+\mathrm{e}^{-x}) + \ln(\mathrm{e}^{-x}) + c_1$

- Solve for y

  $y = \arccos\left(\frac{\mathrm{e}^{-c_1+x}(1+\mathrm{e}^{x})}{\mathrm{e}^{x}}\right)$

- Use initial condition $y(0) = \frac{\pi}{4}$

  $\frac{\pi}{4} = \arccos(2\,\mathrm{e}^{-c_1})$

- Solve for c_1

  $c_1 = \frac{3\ln(2)}{2}$

- Substitute $c_1 = \frac{3\ln(2)}{2}$ into general solution and simplify

  $y = \arccos\left(\frac{(1+\mathrm{e}^{x})\sqrt{2}}{4}\right)$

- Solution to the IVP

  $y = \arccos\left(\frac{(1+\mathrm{e}^{x})\sqrt{2}}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.344 (sec). Leaf size: 14

```
dsolve([cos(y(x))+(1+exp(-x))*sin(y(x))*diff(y(x),x)=0,y(0) = 1/4*Pi],y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{\sqrt{2}\,(\mathrm{e}^{x}+1)}{4}\right)$$

✓ Solution by Mathematica

Time used: 50.086 (sec). Leaf size: 20

```
DSolve[{Cos[y[x]]+(1+Exp[-x])*Sin[y[x]]*y'[x]==0,{y[0]==Pi/4}},y[x],x,IncludeSingularSolutio
```

$$y(x) \to \arccos\left(\frac{e^x+1}{2\sqrt{2}}\right)$$

2.5 problem 5

Internal problem ID [5091]
Internal file name [`OUTPUT/4584_Sunday_June_05_2022_03_01_20_PM_83982119/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x^2(1+y)+y^2(x-1)\,y'=0}$$

2.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y)\\
&= f(x)g(y)\\
&= -\frac{x^2(1+y)}{y^2\,(x-1)}
\end{aligned}$$

Where $f(x)=-\frac{x^2}{x-1}$ and $g(y)=\frac{1+y}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{1+y}{y^2}}\,dy=-\frac{x^2}{x-1}\,dx$$

$$\int\frac{1}{\frac{1+y}{y^2}}\,dy=\int-\frac{x^2}{x-1}\,dx$$

$$\frac{y^2}{2}-y+\ln\left(1+y\right)=-\frac{x^2}{2}-x-\ln\left(x-1\right)+c_1$$

Which results in

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{2\_Z}-x^2-2\ln(x-1)+4\,\mathrm{e}^{\_Z}+2c_1-2\_Z-2x-3\right)} - 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{2\_Z}-x^2-2\ln(x-1)+4\,\mathrm{e}^{\_Z}+2c_1-2\_Z-2x-3\right)} - 1 \tag{1}$$

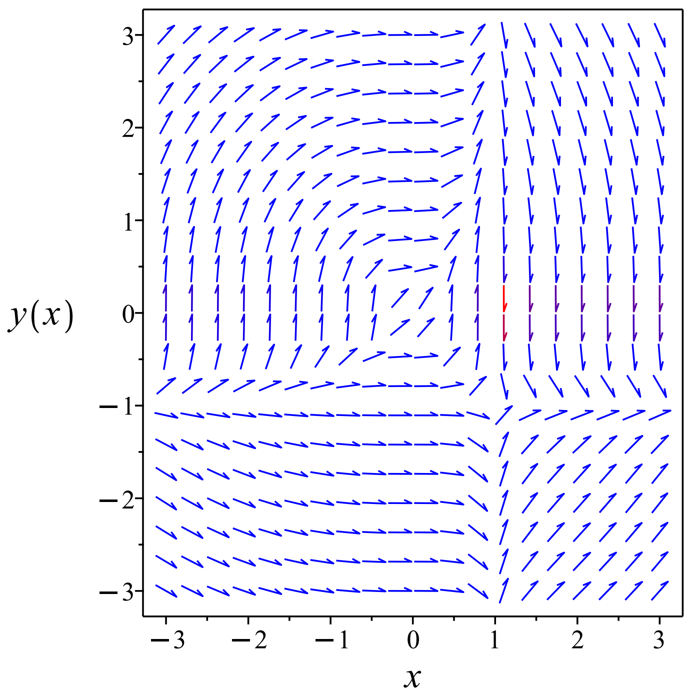


Figure 40: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\mathrm{RootOf}\left(-\mathrm{e}^{2\_Z}-x^2-2\ln(x-1)+4\,\mathrm{e}^{\_Z}+2c_1-2\_Z-2x-3\right)} - 1$$

Verified OK.

2.5.2 Maple step by step solution

Let's solve

$x^2(1+y)+y^2(x-1)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y^2}{1+y}=-\frac{x^2}{x-1}$

- Integrate both sides with respect to x

  $\int \frac{y'y^2}{1+y}dx=\int -\frac{x^2}{x-1}dx+c_1$

- Evaluate integral

  $\frac{y^2}{2}-y+\ln(1+y)=-\frac{x^2}{2}-x-\ln(x-1)+c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 30

```
dsolve(x^2*(y(x)+1)+y(x)^2*(x-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$\frac{x^2}{2}+x+\ln(x-1)+\frac{y(x)^2}{2}-y(x)+\ln(y(x)+1)+c_1=0$$

✓ Solution by Mathematica

Time used: 0.42 (sec). Leaf size: 56

```
DSolve[x^2*(y[x]+1)+y[x]^2*(x-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{InverseFunction}\left[\frac{1}{2}(\#1+1)^2 - 2(\#1+1) + \log(\#1+1)\&\right]\left[-\frac{x^2}{2} - x - \log(x-1) + \frac{3}{2} + c_1\right]$$

$$y(x) \to -1$$

2.6 problem 6

Internal problem ID [5092]
Internal file name [`OUTPUT/4585_Sunday_June_05_2022_03_01_21_PM_13981050/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(2y - x)\, y' - y = 2x}$$

2.6.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{align*} y' &= F(x,y) \\ &= \frac{y+2x}{-x+2y} \tag{1} \end{align*}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y - 2x$ and $N = x - 2y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= \frac{u+2}{2u-1} \\ \frac{du}{dx} &= \frac{\frac{u(x)+2}{2u(x)-1} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)+2}{2u(x)-1} - u(x)}{x} = 0$$

Or

$$2u'(x)\, xu(x) - u'(x)\, x + 2u(x)^2 - 2u(x) - 2 = 0$$

Or

$$-2 + x(2u(x) - 1)\, u'(x) + 2u(x)^2 - 2u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2(u^2 - u - 1)}{x\,(2u-1)}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2-u-1}{2u-1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u^2-u-1}{2u-1}}\,du &= -\frac{2}{x}\,dx \\ \int \frac{1}{\frac{u^2-u-1}{2u-1}}\,du &= \int -\frac{2}{x}\,dx \\ \ln\left(u^2 - u - 1\right) &= -2\ln(x) + c_2\end{aligned}$$

Raising both side to exponential gives

$$u^2 - u - 1 = e^{-2\ln(x) + c_2}$$

Which simplifies to

$$u^2 - u - 1 = \frac{c_3}{x^2}$$

Which simplifies to

$$u(x)^2 - u(x) - 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

The solution is

$$u(x)^2 - u(x) - 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$\frac{y^2}{x^2} - \frac{y}{x} - 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$y^2 - xy - x^2 = c_3 \mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$y^2 - xy - x^2 = c_3 \mathrm{e}^{c_2} \tag{1}$$

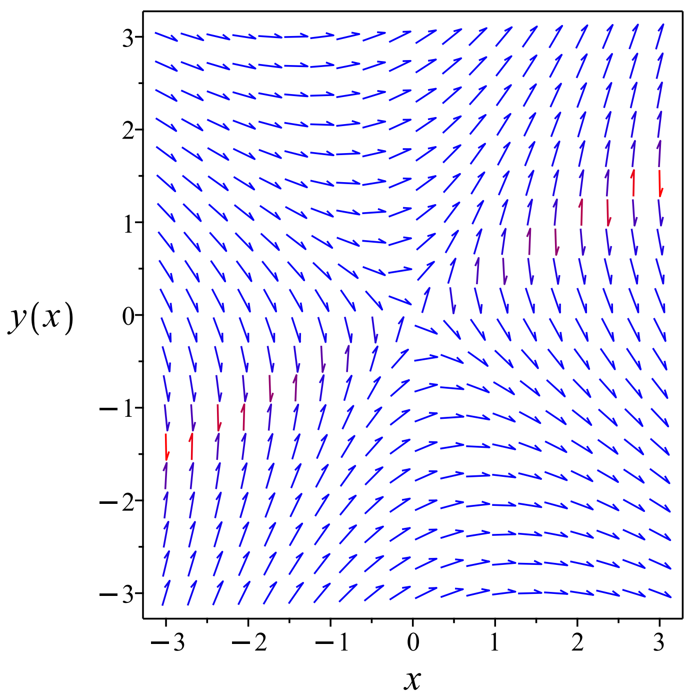


Figure 41: Slope field plot

Verification of solutions

$$y^2 - xy - x^2 = c_3 \mathrm{e}^{c_2}$$

Verified OK.

2.6.2 Maple step by step solution

Let's solve

$(2y - x) y' - y = 2x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $-1 = -1$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-y - 2x) \, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = -x^2 - xy + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative

  $-x + 2y = -x + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

$\frac{d}{dy}f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = -x^2 - xy + y^2$

- Substitute $F(x,y)$ into the solution of the ODE

  $-x^2 - xy + y^2 = c_1$

- Solve for y

  $\left\{y = \frac{x}{2} - \frac{\sqrt{5x^2+4c_1}}{2}, y = \frac{x}{2} + \frac{\sqrt{5x^2+4c_1}}{2}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 51

```
dsolve((2*y(x)-x)*diff(y(x),x)=2*x+y(x),y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{5c_1^2x^2+4}}{2c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{5c_1^2x^2+4}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.454 (sec). Leaf size: 102

```
DSolve[(2*y[x]-x)*y'[x]==2*x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x + \sqrt{5x^2 - 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(x - \sqrt{5}\sqrt{x^2}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{5}\sqrt{x^2} + x\right)$$

2.7 problem 7

Internal problem ID [5093]
Internal file name [`OUTPUT/4586_Sunday_June_05_2022_03_01_22_PM_75100719/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{xy + y^2 + (x^2 - xy)\, y' = 0}$$

2.7.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(x+y)}{x(-x+y)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y(x+y)$ and $N = x(x-y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u(u+1)}{u-1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)(u(x)+1)}{u(x)-1} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\frac{u(x)(u(x)+1)}{u(x)-1} - u(x)}{x} = 0
$$

Or

$$
u'(x)\, x u(x) - u'(x)\, x - 2u(x) = 0
$$

Or

$$
x(u(x) - 1)\, u'(x) - 2u(x) = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= \frac{2u}{x\,(u-1)}
\end{aligned}
$$

Where $f(x) = \frac{2}{x}$ and $g(u) = \frac{u}{u-1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u}{u-1}}\, du &= \frac{2}{x}\, dx \\
\int \frac{1}{\frac{u}{u-1}}\, du &= \int \frac{2}{x}\, dx \\
u - \ln(u) &= 2\ln(x) + c_2
\end{aligned}
$$

The solution is

$$
u(x) - \ln(u(x)) - 2\ln(x) - c_2 = 0
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
\frac{y}{x} - \ln\left(\frac{y}{x}\right) - 2\ln(x) - c_2 = 0
$$

Summary
The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - 2\ln(x) - c_2 = 0 \tag{1}$$

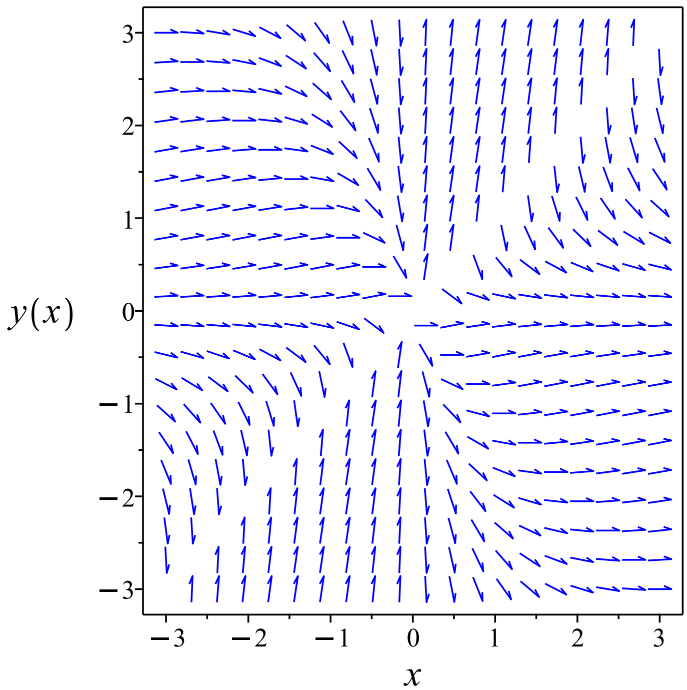


Figure 42: Slope field plot

Verification of solutions

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - 2\ln(x) - c_2 = 0$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 17

```
dsolve((x*y(x)+y(x)^2)+(x^2-x*y(x))*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(-\frac{e^{-2c_1}}{x^2}\right)x$$

✓ Solution by Mathematica

Time used: 2.801 (sec). Leaf size: 25

```
DSolve[(x*y[x]+y[x]^2)+(x^2-x*y[x])*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x^2}\right)$$
$$y(x) \to 0$$

2.8 problem 8

Internal problem ID [5094]
Internal file name [`OUTPUT/4587_Sunday_June_05_2022_03_01_23_PM_6639194/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y^3 - 3y'y^2x = -x^3}$$

2.8.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x^3+y^3}{3y^2x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = x^3+y^3$ and $N = 3y^2x$ are both homogeneous and of the same order $n = 3$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{1}{3u^2} + \frac{u}{3} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{1}{3u(x)^2} - \frac{2u(x)}{3}}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\frac{1}{3u(x)^2} - \frac{2u(x)}{3}}{x} = 0
$$

Or

$$
3u'(x)\,u(x)^2\,x + 2u(x)^3 - 1 = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2u^3 - 1}{3u^2x}
\end{aligned}
$$

Where $f(x) = -\frac{1}{3x}$ and $g(u) = \frac{2u^3-1}{u^2}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^3-1}{u^2}}\,du &= -\frac{1}{3x}\,dx \\
\int \frac{1}{\frac{2u^3-1}{u^2}}\,du &= \int -\frac{1}{3x}\,dx \\
\frac{\ln(2u^3-1)}{6} &= -\frac{\ln(x)}{3} + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
(2u^3 - 1)^{\frac{1}{6}} = \mathrm{e}^{-\frac{\ln(x)}{3} + c_2}
$$

Which simplifies to

$$
(2u^3 - 1)^{\frac{1}{6}} = \frac{c_3}{x^{\frac{1}{3}}}
$$

Which simplifies to

$$
(2u(x)^3 - 1)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{1}{3}}}
$$

The solution is

$$\left(2u(x)^{3}-1\right)^{\frac{1}{6}}=\frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Now u in the above solution is replaced back by y using $u=\frac{y}{x}$ which results in the solution

$$\left(\frac{2y^{3}}{x^{3}}-1\right)^{\frac{1}{6}}=\frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Which simplifies to

$$\left(-\frac{-2y^{3}+x^{3}}{x^{3}}\right)^{\frac{1}{6}}=\frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$\left(-\frac{-2y^{3}+x^{3}}{x^{3}}\right)^{\frac{1}{6}}=\frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{1}{3}}} \tag{1}$$

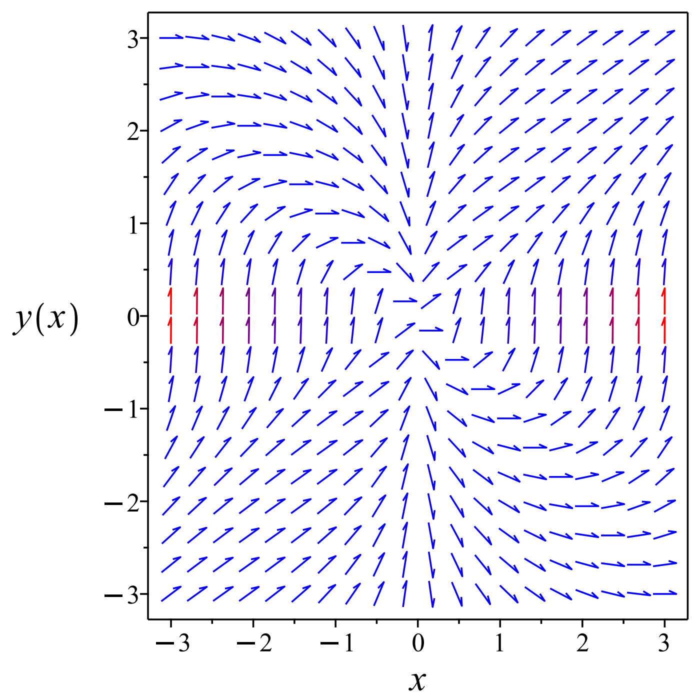


Figure 43: Slope field plot

Verification of solutions

$$\left(-\frac{-2y^3+x^3}{x^3}\right)^{\frac{1}{6}}=\frac{c_3\mathrm{e}^{c_2}}{x^{\frac{1}{3}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 74

```
dsolve((x^3+y(x)^3)=3*x*y(x)^2*diff(y(x),x),y(x), singsol=all)
```

$$y(x)=\frac{2^{\frac{2}{3}}\left(x(x^2+2c_1)\right)^{\frac{1}{3}}}{2}$$
$$y(x)=-\frac{2^{\frac{2}{3}}\left(x(x^2+2c_1)\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4}$$
$$y(x)=\frac{2^{\frac{2}{3}}\left(x(x^2+2c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.21 (sec). Leaf size: 90

```
DSolve[(x^3+y[x]^3)==3*x*y[x]^2*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to-\sqrt[3]{-\frac{1}{2}}\sqrt[3]{x}\sqrt[3]{x^2+2c_1}$$
$$y(x)\to\frac{\sqrt[3]{x}\sqrt[3]{x^2+2c_1}}{\sqrt[3]{2}}$$
$$y(x)\to\frac{(-1)^{2/3}\sqrt[3]{x}\sqrt[3]{x^2+2c_1}}{\sqrt[3]{2}}$$

2.9 problem 9

Internal problem ID [5095]
Internal file name [`OUTPUT/4588_Sunday_June_05_2022_03_01_24_PM_29063218/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y + (4y + 3x)\, y' = 3x}$$

2.9.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{-3x + y}{4y + 3x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 3x - y$ and $N = 4y + 3x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{-u+3}{4u+3}\\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{-u(x)+3}{4u(x)+3} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{-u(x)+3}{4u(x)+3} - u(x)}{x} = 0$$

Or

$$4u'(x)\, xu(x) + 3u'(x)\, x + 4u(x)^2 + 4u(x) - 3 = 0$$

Or

$$-3 + x(4u(x) + 3)\, u'(x) + 4u(x)^2 + 4u(x) = 0$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{4u^2+4u-3}{x\,(4u+3)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{4u^2+4u-3}{4u+3}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{4u^2+4u-3}{4u+3}}\, du &= -\frac{1}{x}\, dx\\ \int \frac{1}{\frac{4u^2+4u-3}{4u+3}}\, du &= \int -\frac{1}{x}\, dx\\ \frac{5\ln(2u-1)}{8} + \frac{3\ln(2u+3)}{8} &= -\ln(x) + c_2\end{aligned}$$

The above can be written as

$$\begin{aligned}\frac{5\ln(2u-1) + 3\ln(2u+3)}{8} &= -\ln(x) + c_2\\ 5\ln(2u-1) + 3\ln(2u+3) &= (8)(-\ln(x) + c_2)\\ &= -8\ln(x) + 8c_2\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{5\ln(2u-1)+3\ln(2u+3)} = \mathrm{e}^{-8\ln(x)+8c_2}$$

Which simplifies to

$$
\begin{aligned}
(2u-1)^5 (2u+3)^3 &= \frac{8c_2}{x^8} \\
&= \frac{c_3}{x^8}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{\mathrm{RootOf}\left(\_Z^8 + 4\_Z^7 - 8\_Z^6 - 28\_Z^5 + 50\_Z^4 + 44\_Z^3 - \frac{c_3 \mathrm{e}^{8c_2}}{x^8} - 144\_Z^2 + 108\_Z - 27\right)}{2}
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$
y = \frac{x\,\mathrm{RootOf}\left(\_Z^8 x^8 + 4\_Z^7 x^8 - 8\_Z^6 x^8 - 28\_Z^5 x^8 + 50\_Z^4 x^8 + 44\_Z^3 x^8 - 144\_Z^2 x^8 - c_3 \mathrm{e}^{8c_2} + 108\right.}{2}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x\,\mathrm{RootOf}\left(\_Z^8 x^8 + 4\_Z^7 x^8 - 8\_Z^6 x^8 - 28\_Z^5 x^8 + 50\_Z^4 x^8 + 44\_Z^3 x^8 - 144\_Z^2 x^8 - c_3 \mathrm{e}^{8c_2} + 108\_\right.}{2} \tag{1}
$$

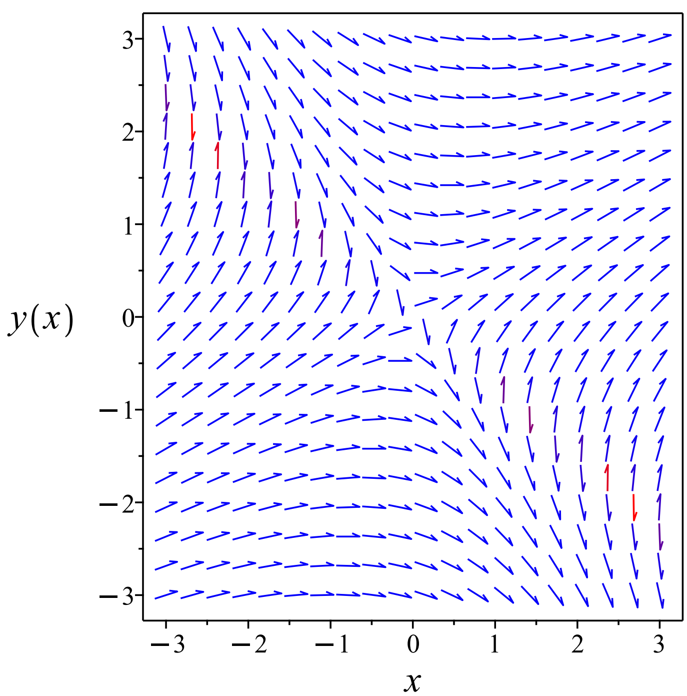


Figure 44: Slope field plot

Verification of solutions

$$y = \frac{x \,\mathrm{RootOf}\left(\_Z^8 x^8 + 4\_Z^7 x^8 - 8\_Z^6 x^8 - 28\_Z^5 x^8 + 50\_Z^4 x^8 + 44\_Z^3 x^8 - 144\_Z^2 x^8 - c_3 \mathrm{e}^{8c_2} + 108\_\right.}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.36 (sec). Leaf size: 278

```
dsolve(y(x)-3*x+(4*y(x)+3*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{-3x^8 c_1 \,\mathrm{RootOf}\left(\_Z^{64} c_1 x^8 + 12\_Z^{56} c_1 x^8 + 48\_Z^{48} c_1 x^8 + 64\_Z^{40} c_1 x^8 - 1\right)^{56} - 24x^8 c_1 \,\mathrm{RootOf}\left(\_Z^{64} c_1\right.}{2c_1 x^7 \,\mathrm{RootOf}\left(\_Z^{64} c_1 x^8 + 12\_Z^{56} c_1 x^8 + 48\_Z^{48} c_1 x^8 + 64\_Z^{40} c_1 x^8 - 1\right)^{40} \left(\mathrm{RootOf}\left(\_Z^{64} c_1 x^8\right.\right.}$$

✓ Solution by Mathematica

Time used: 5.296 (sec). Leaf size: 673

```
DSolve[y[x]-3*x+(4*y[x]+3*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 1\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 2\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 3\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 4\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 5\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 6\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 7\right]$$
$$y(x) \to \text{Root}\left[256\#1^8 + 512\#1^7 x - 512\#1^6 x^2 - 896\#1^5 x^3 + 800\#1^4 x^4 + 352\#1^3 x^5 - 576\#1^2 x^6 + 216\#1 x^7 - 27x^8 + e^{8c_1}\&, 8\right]$$

2.10 problem 10

Internal problem ID [5096]
Internal file name [`OUTPUT/4589_Sunday_June_05_2022_03_01_25_PM_72423458/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{(x^3 + 3xy^2)\, y' - y^3 - 3yx^2 = 0}$$

2.10.1 Solving as homogeneous ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(3x^2 + y^2)}{x\,(x^2 + 3y^2)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = y(3x^2 + y^2)$ and $N = x(x^2 + 3y^2)$ are both homogeneous and of the same order $n = 3$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u(u^2+3)}{3u^2+1} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)\left(u(x)^2+3\right)}{3u(x)^2+1} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{\frac{u(x)\left(u(x)^2+3\right)}{3u(x)^2+1} - u(x)}{x} = 0
$$

Or

$$
3u'(x)\, u(x)^2\, x + 2u(x)^3 + u'(x)\, x - 2u(x) = 0
$$

Or

$$
x\left(3u(x)^2 + 1\right) u'(x) + 2u(x)^3 - 2u(x) = 0
$$

Which is now solved as separable in $u(x)$. Which is now solved in $u(x)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{2(u^3-u)}{x\,(3u^2+1)}
\end{aligned}
$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^3-u}{3u^2+1}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{u^3-u}{3u^2+1}}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{\frac{u^3-u}{3u^2+1}}\, du &= \int -\frac{2}{x}\, dx \\
2\ln(u+1) + 2\ln(u-1) - \ln(u) &= -2\ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\mathrm{e}^{2\ln(u+1)+2\ln(u-1)-\ln(u)} = \mathrm{e}^{-2\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{u^4 - 2u^2 + 1}{u} = \frac{c_3}{x^2}
$$

Now u in the above solution is replaced back by y using $u = \frac{y}{x}$ which results in the solution

$$y = x \operatorname{RootOf}\left(x^2\_Z^4 - 2x^2\_Z^2 - \_Zc_3 + x^2\right)$$

<u>Summary</u>

The solution(s) found are the following

$$y = x \operatorname{RootOf}\left(x^2\_Z^4 - 2x^2\_Z^2 - \_Zc_3 + x^2\right) \tag{1}$$

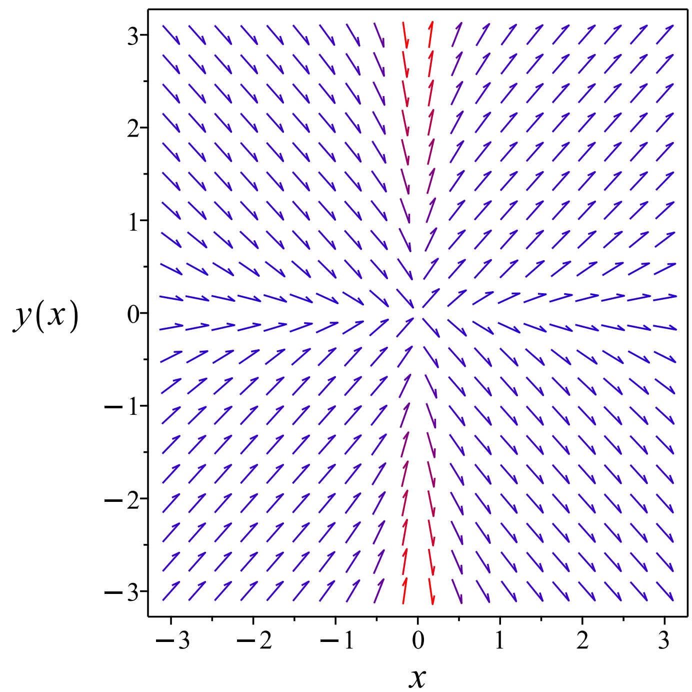


Figure 45: Slope field plot

<u>Verification of solutions</u>

$$y = x \operatorname{RootOf}\left(x^2\_Z^4 - 2x^2\_Z^2 - \_Zc_3 + x^2\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.079 (sec). Leaf size: 23

```
dsolve((x^3+3*x*y(x)^2)*diff(y(x),x)=y(x)^3+3*x^2*y(x),y(x), singsol=all)
```

$$y(x) = \operatorname{RootOf}\left(\_Z^4 c_1 x - c_1 x - \_Z\right)^2 x$$

✓ Solution by Mathematica

Time used: 60.142 (sec). Leaf size: 1659

```
DSolve[(x^3+3*x*y[x]^2)*y'[x]==y[x]^3+3*x^2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(-\sqrt{3}\sqrt{4x^2+\frac{16\sqrt[3]{2}x^4}{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+81e^{4c_1}x^4}}}+\frac{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8}}}{\sqrt[3]{2}}}\right.$$

$$-3\sqrt{\frac{8x^2}{3}-\frac{16\sqrt[3]{2}x^4}{3\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+81e^{4c_1}x^4}}}-\frac{}{\sqrt{4x^2+\frac{16\sqrt[3]{2}x^4}{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e}}}}}}$$

$$y(x) \to \frac{1}{6}\left(3\sqrt{\frac{8x^2}{3}-\frac{16\sqrt[3]{2}x^4}{3\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+81e^{4c_1}x^4}}}-\frac{}{\sqrt{4x^2+\frac{16\sqrt[3]{2}x^4}{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{76}}}}}}\right.$$

$$-\sqrt{3}\sqrt{4x^2+\frac{16\sqrt[3]{2}x^4}{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+81e^{4c_1}x^4}}}+\frac{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+8}}}{\sqrt[3]{2}}}$$

$$y(x) \to \frac{1}{6}\left(\sqrt{3}\sqrt{4x^2+\frac{16\sqrt[3]{2}x^4}{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+81e^{4c_1}x^4}}}+\frac{\sqrt[3]{128x^6+27e^{2c_1}x^2+3\sqrt{768e^{2c_1}x^8+}}}{\sqrt[3]{2}}}\right.$$

2.11 problem 11

Internal problem ID [5097]
Internal file name [OUTPUT/4590_Sunday_June_05_2022_03_01_26_PM_33328603/index.tex]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + xy' = x^3 + 3x^2 - 2x}$$

2.11.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= x^2 + 3x - 2 \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = x^2 + 3x - 2$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^2+3x-2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(x^2+3x-2\right) \\
\mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{x^2+3x-2}{x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{x^2+3x-2}{x}\,\mathrm{d}x \\
\frac{y}{x} &= \frac{x^2}{2}+3x-2\ln(x)+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
y = x\left(\frac{x^2}{2}+3x-2\ln(x)\right)+c_1 x
$$

which simplifies to

$$
y = \frac{x(x^2+6x-4\ln(x)+2c_1)}{2}
$$

Summary
The solution(s) found are the following

$$
y = \frac{x(x^2+6x-4\ln(x)+2c_1)}{2} \tag{1}
$$

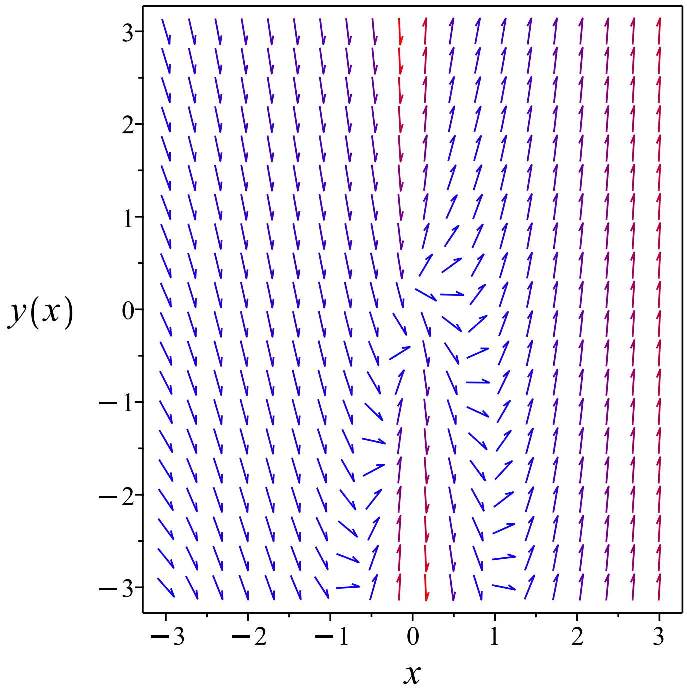


Figure 46: Slope field plot

Verification of solutions

$$y = \frac{x(x^2 + 6x - 4\ln(x) + 2c_1)}{2}$$

Verified OK.

2.11.2 Maple step by step solution

Let's solve

$-y + xy' = x^3 + 3x^2 - 2x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{y}{x} + x^2 + 3x - 2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = x^2 + 3x - 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-\frac{y}{x}\right)=\mu(x)\left(x^2+3x-2\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'-\frac{y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\left(x^2+3x-2\right)dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\left(x^2+3x-2\right)dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)(x^2+3x-2)dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{x}$

  $y=x\left(\int\frac{x^2+3x-2}{x}dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=x\left(\frac{x^2}{2}+3x-2\ln(x)+c_1\right)$
- Simplify

  $y=\frac{x(x^2+6x-4\ln(x)+2c_1)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve(x*diff(y(x),x)-y(x)=x^3+3*x^2-2*x,y(x), singsol=all)
```

$$y(x) = \frac{(x^2 + 6x - 4\ln(x) + 2c_1)\, x}{2}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 24

```
DSolve[x*y'[x]-y[x]==x^3+3*x^2-2*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\left(\frac{x^2}{2} + 3x - 2\log(x) + c_1\right)$$

2.12 problem 12

Internal problem ID [5098]
Internal file name [`OUTPUT/4591_Sunday_June_05_2022_03_01_27_PM_55050272/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\tan(x) = \sin(x)}$$

2.12.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \tan(x) \\ q(x) &= \sin(x) \end{aligned}$$

Hence the ode is

$$y' + y\tan(x) = \sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \tan(x)dx} \\ &= \frac{1}{\cos(x)} \end{aligned}$$

Which simplifies to

$$\mu = \sec(x)$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\sin(x)) \\
\frac{d}{dx}(\sec(x)\, y) &= (\sec(x))(\sin(x)) \\
d(\sec(x)\, y) &= \tan(x)\, dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sec(x)\, y &= \int \tan(x)\, dx \\
\sec(x)\, y &= -\ln(\cos(x)) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sec(x)$ results in

$$y = -\cos(x)\ln(\cos(x)) + \cos(x)\, c_1$$

which simplifies to

$$y = \cos(x)\,(-\ln(\cos(x)) + c_1)$$

Summary

The solution(s) found are the following

$$y = \cos(x)\,(-\ln(\cos(x)) + c_1) \tag{1}$$

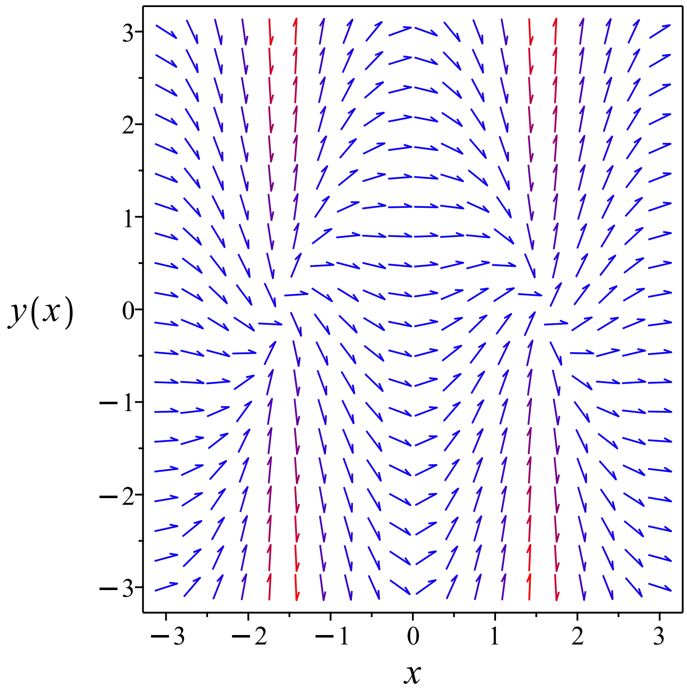


Figure 47: Slope field plot

Verification of solutions

$$y = \cos(x)\left(-\ln(\cos(x)) + c_1\right)$$

Verified OK.

2.12.2 Maple step by step solution

Let's solve

$y' + y\tan(x) = \sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -y\tan(x) + \sin(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y\tan(x) = \sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\tan(x)\right) = \mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y\tan(x))=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\mu(x)\tan(x)$
- Solve to find the integrating factor

  $\mu(x)=\frac{1}{\cos(x)}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\sin(x)\,dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\sin(x)\,dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)\sin(x)dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\frac{1}{\cos(x)}$

  $y=\cos(x)\left(\int\frac{\sin(x)}{\cos(x)}dx+c_1\right)$
- Evaluate the integrals on the rhs

  $y=\cos(x)\,(-\ln(\cos(x))+c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x)+y(x)*tan(x)=sin(x),y(x), singsol=all)
```

$$y(x)=(-\ln(\cos(x))+c_1)\cos(x)$$

✓ Solution by Mathematica

Time used: 0.058 (sec). Leaf size: 16

```
DSolve[y'[x]+y[x]*Tan[x]==Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)(-\log(\cos(x)) + c_1)$$

2.13 problem 13

Internal problem ID [5099]
Internal file name [`OUTPUT/4592_Sunday_June_05_2022_03_01_28_PM_86121755/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + xy' = \cos (x) x^3}$$

With initial conditions

$$[y(\pi) = 0]$$

2.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \cos (x) x^2$$

Hence the ode is

$$y' - \frac{y}{x} = \cos (x) x^2$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = \cos(x)\, x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

2.13.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\cos(x)\, x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\cos(x)\, x^2\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= (\cos(x)\, x)\ \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \cos(x)\, x\, \mathrm{d}x \\ \frac{y}{x} &= \sin(x)\, x + \cos(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x(\sin(x)\, x + \cos(x)) + c_1 x$$

which simplifies to

$$y = x(\sin(x)\, x + \cos(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \pi c_1 - \pi$$

$$c_1 = 1$$

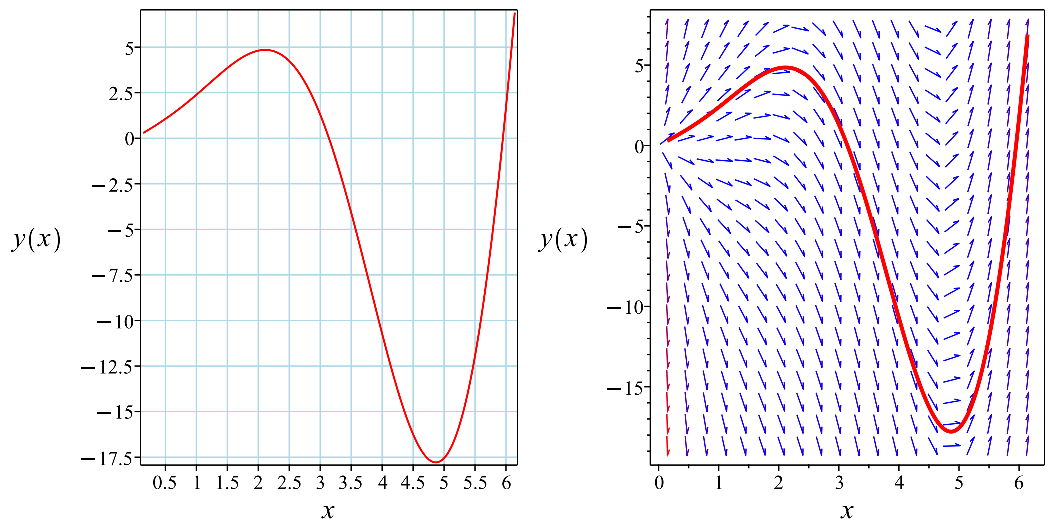
Substituting c_1 found above in the general solution gives

$$y = x(\sin(x)\, x + \cos(x) + 1)$$

Summary

The solution(s) found are the following

$$y = x(\sin(x)\, x + \cos(x) + 1) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x(\sin(x)\, x + \cos(x) + 1)$$

Verified OK.

2.13.3 Maple step by step solution

Let's solve

$[-y + xy' = \cos(x)\, x^3, y(\pi) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = \frac{y}{x} + \cos(x)\, x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = \cos(x)\, x^2$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu(x)\cos(x)\, x^2$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\cos(x)\, x^2 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\cos(x)\, x^2 dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)\cos(x) x^2 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \cos(x)\, x dx + c_1\right)$
- Evaluate the integrals on the rhs

  $y = x(\sin(x)\, x + \cos(x) + c_1)$
- Use initial condition $y(\pi) = 0$

  $0 = \pi(-1 + c_1)$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = x(\sin(x)\, x + \cos(x) + 1)$
- Solution to the IVP

  $y = x(\sin(x)\, x + \cos(x) + 1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([x*diff(y(x),x)-y(x)=x^3*cos(x),y(Pi) = 0],y(x), singsol=all)
```

$$y(x) = (\cos(x) + \sin(x)\, x + 1)\, x$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 15

```
DSolve[{x*y'[x]-y[x]==x^3*Cos[x],{y[Pi]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(x\sin(x) + \cos(x) + 1)$$

2.14 problem 14

Internal problem ID [5100]
Internal file name [`OUTPUT/4593_Sunday_June_05_2022_03_01_29_PM_28386368/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\, y' + 3xy = 5x}$$

With initial conditions

$$[y(1) = 2]$$

2.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3x}{x^2+1}$$
$$q(x) = \frac{5x}{x^2+1}$$

Hence the ode is

$$y' + \frac{3xy}{x^2+1} = \frac{5x}{x^2+1}$$

The domain of $p(x) = \frac{3x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{5x}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

2.14.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3x}{x^2+1}dx} \\ &= \left(x^2+1\right)^{\frac{3}{2}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{5x}{x^2+1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^2+1\right)^{\frac{3}{2}} y\right) &= \left(\left(x^2+1\right)^{\frac{3}{2}}\right)\left(\frac{5x}{x^2+1}\right) \\ \mathrm{d}\left(\left(x^2+1\right)^{\frac{3}{2}} y\right) &= \left(5\sqrt{x^2+1}\,x\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\left(x^2+1\right)^{\frac{3}{2}} y &= \int 5\sqrt{x^2+1}\,x\,\mathrm{d}x \\ \left(x^2+1\right)^{\frac{3}{2}} y &= \frac{5(x^2+1)^{\frac{3}{2}}}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \left(x^2+1\right)^{\frac{3}{2}}$ results in

$$y = \frac{5}{3} + \frac{c_1}{\left(x^2+1\right)^{\frac{3}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{5}{3} + \frac{c_1\sqrt{2}}{4}$$

$$c_1 = \frac{2\sqrt{2}}{3}$$

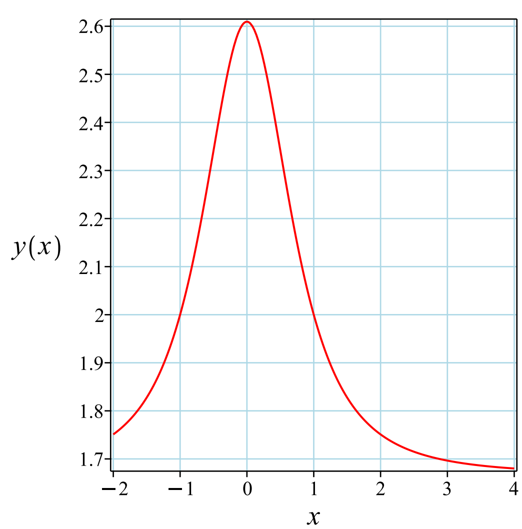
Substituting c_1 found above in the general solution gives

$$y = \frac{5(x^2+1)^{\frac{3}{2}} + 2\sqrt{2}}{3\,(x^2+1)^{\frac{3}{2}}}$$

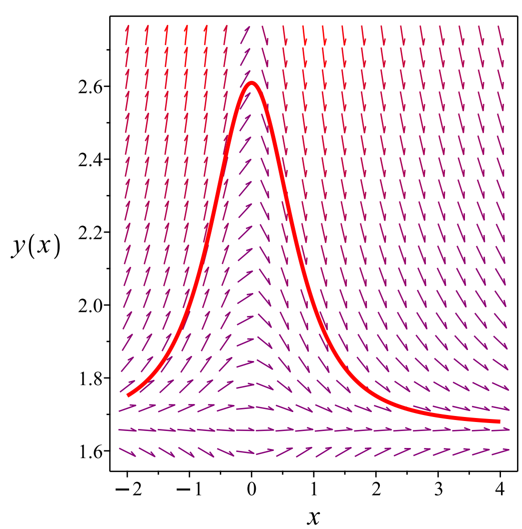
Summary

The solution(s) found are the following

$$y = \frac{5(x^2+1)^{\frac{3}{2}} + 2\sqrt{2}}{3\,(x^2+1)^{\frac{3}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5(x^2+1)^{\frac{3}{2}} + 2\sqrt{2}}{3\,(x^2+1)^{\frac{3}{2}}}$$

Verified OK.

2.14.3 Maple step by step solution

Let's solve

$\left[\left(x^{2}+1\right) y^{\prime}+3 x y=5 x, y(1)=2\right]$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$
- Separate variables

  $\frac{y^{\prime}}{3 y-5}=-\frac{x}{x^{2}+1}$
- Integrate both sides with respect to x

  $\int \frac{y^{\prime}}{3 y-5} d x=\int-\frac{x}{x^{2}+1} d x+c_{1}$
- Evaluate integral

  $\frac{\ln (3 y-5)}{3}=-\frac{\ln \left(x^{2}+1\right)}{2}+c_{1}$
- Solve for y

  $y=\frac{\mathrm{e}^{-\frac{3 \ln \left(x^{2}+1\right)}{2}+3 c_{1}}}{3}+\frac{5}{3}$
- Use initial condition $y(1)=2$

  $2=\frac{\mathrm{e}^{-\frac{3 \ln (2)}{2}+3 c_{1}}}{3}+\frac{5}{3}$
- Solve for c_{1}

  $c_{1}=\frac{\ln (2)}{2}$
- Substitute $c_{1}=\frac{\ln (2)}{2}$ into general solution and simplify

  $y=\frac{5}{3}+\frac{2 \sqrt{2}}{3\left(x^{2}+1\right)^{\frac{3}{2}}}$
- Solution to the IVP

  $y=\frac{5}{3}+\frac{2 \sqrt{2}}{3\left(x^{2}+1\right)^{\frac{3}{2}}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 18

```
dsolve([(1+x^2)*diff(y(x),x)+3*x*y(x)=5*x,y(1) = 2],y(x), singsol=all)
```

$$y(x) = \frac{5}{3} + \frac{2\sqrt{2}}{3\left(x^2+1\right)^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 27

```
DSolve[{(1+x^2)*y'[x]+3*x*y[x]==5*x,{y[1]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2\sqrt{2}}{3\left(x^2+1\right)^{3/2}} + \frac{5}{3}$$

2.15 problem 15

Internal problem ID [5101]
Internal file name [`OUTPUT/4594_Sunday_June_05_2022_03_01_30_PM_42109241/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = 5\,\mathrm{e}^{\cos(x)}}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = -4\right]$$

2.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = 5\,\mathrm{e}^{\cos(x)}$$

Hence the ode is

$$y' + y\cot(x) = 5\,\mathrm{e}^{\cos(x)}$$

The domain of $p(x) = \cot(x)$ is

$$\{x < \pi\_Z94 \vee \pi\_Z94 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = 5\,\mathrm{e}^{\cos(x)}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

2.15.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \cot(x)dx} \\ &= \sin(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(5\,\mathrm{e}^{\cos(x)}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\sin(x)) &= (\sin(x))\left(5\,\mathrm{e}^{\cos(x)}\right) \\ \mathrm{d}(y\sin(x)) &= \left(5\,\mathrm{e}^{\cos(x)}\sin(x)\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\sin(x) &= \int 5\,\mathrm{e}^{\cos(x)}\sin(x)\,\mathrm{d}x \\ y\sin(x) &= -5\,\mathrm{e}^{\cos(x)} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = -5\,\mathrm{e}^{\cos(x)}\csc(x) + c_1\csc(x)$$

which simplifies to

$$y = \csc(x)\left(-5\,\mathrm{e}^{\cos(x)} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = -4$ in the above solution gives an equation to solve for the constant of integration.

$$-4 = -5 + c_1$$

$$c_1 = 1$$

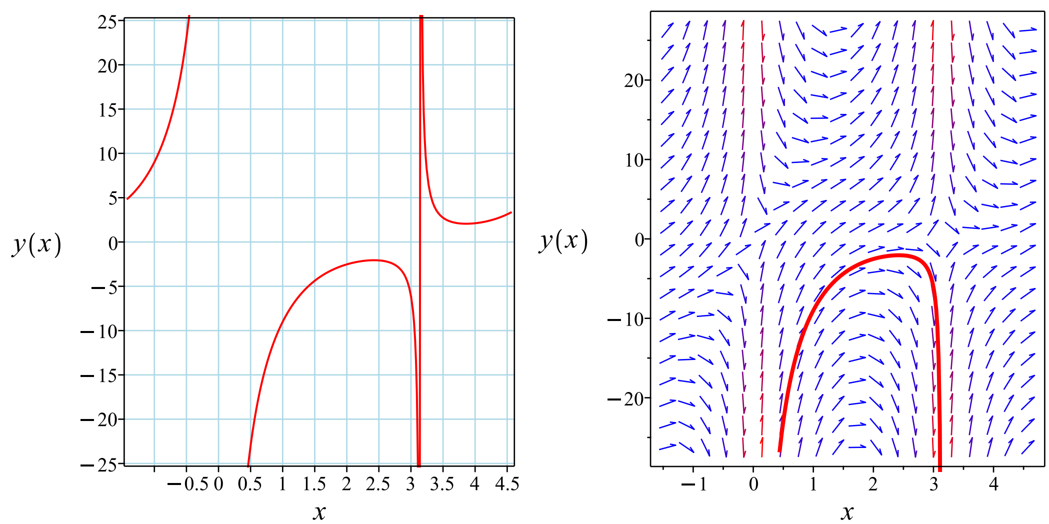
Substituting c_1 found above in the general solution gives

$$y = -5\,\mathrm{e}^{\cos(x)} \csc(x) + \csc(x)$$

Summary

The solution(s) found are the following

$$y = -5\,\mathrm{e}^{\cos(x)} \csc(x) + \csc(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -5\,\mathrm{e}^{\cos(x)} \csc(x) + \csc(x)$$

Verified OK.

2.15.3 Maple step by step solution

Let's solve

$\left[y' + y\cot(x) = 5\,\mathrm{e}^{\cos(x)}, y\left(\frac{\pi}{2}\right) = -4\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y \cot(x) + 5\,\mathrm{e}^{\cos(x)}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y \cot(x) = 5\,\mathrm{e}^{\cos(x)}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y \cot(x)) = 5\mu(x)\,\mathrm{e}^{\cos(x)}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y' + y \cot(x)) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x) \cot(x)$
- Solve to find the integrating factor

  $\mu(x) = \sin(x)$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int 5\mu(x)\,\mathrm{e}^{\cos(x)} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int 5\mu(x)\,\mathrm{e}^{\cos(x)} dx + c_1$
- Solve for y

  $y = \frac{\int 5\mu(x)\mathrm{e}^{\cos(x)} dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \sin(x)$

  $y = \frac{\int 5\,\mathrm{e}^{\cos(x)} \sin(x) dx + c_1}{\sin(x)}$
- Evaluate the integrals on the rhs

  $y = \frac{-5\,\mathrm{e}^{\cos(x)} + c_1}{\sin(x)}$
- Simplify

  $y = \csc(x)\left(-5\,\mathrm{e}^{\cos(x)} + c_1\right)$
- Use initial condition $y\left(\frac{\pi}{2}\right) = -4$

  $-4 = -5 + c_1$
- Solve for c_1

  $c_1 = 1$
- Substitute $c_1 = 1$ into general solution and simplify

  $y = -5\,\mathrm{e}^{\cos(x)} \csc(x) + \csc(x)$

- Solution to the IVP

  $y = -5\,e^{\cos(x)}\csc(x) + \csc(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+y(x)*cot(x)=5*exp(cos(x)),y(1/2*Pi) = -4],y(x), singsol=all)
```

$$y(x) = -5\,e^{\cos(x)}\csc(x) + \csc(x)$$

✓ Solution by Mathematica

Time used: 0.1 (sec). Leaf size: 16

```
DSolve[{y'[x]+y[x]*Cot[x]==5*Exp[Cos[x]],{y[Pi/2]==-4}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to \left(1 - 5e^{\cos(x)}\right)\csc(x)$$

2.16 problem 16

Internal problem ID [5102]
Internal file name [`OUTPUT/4595_Sunday_June_05_2022_03_01_31_PM_93079431/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(3x + 3y - 4)\,y' + y = -x}$$

2.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{x + y}{3x + 3y - 4} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
& b_2 - \frac{(x+y)(b_3-a_2)}{3x+3y-4} - \frac{(x+y)^2 a_3}{(3x+3y-4)^2} \\
& - \left(-\frac{1}{3x+3y-4} + \frac{3y+3x}{(3x+3y-4)^2}\right)(xa_2 + ya_3 + a_1) \\
& - \left(-\frac{1}{3x+3y-4} + \frac{3y+3x}{(3x+3y-4)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 + 9y^2b_2 - 3y^2b_3 - 8xa_2}{(3x+3y-4)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned}
& 3x^2a_2 - x^2a_3 + 9x^2b_2 - 3x^2b_3 + 6xya_2 - 2xya_3 + 18xyb_2 - 6xyb_3 + 3y^2a_2 - y^2a_3 \\
& + 9y^2b_2 - 3y^2b_3 - 8xa_2 - 28xb_2 + 4xb_3 - 4ya_2 - 4ya_3 - 24yb_2 - 4a_1 - 4b_1 + 16b_2 \\
& = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
& 3a_2v_1^2 + 6a_2v_1v_2 + 3a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - a_3v_2^2 + 9b_2v_1^2 \\
& + 18b_2v_1v_2 + 9b_2v_2^2 - 3b_3v_1^2 - 6b_3v_1v_2 - 3b_3v_2^2 - 8a_2v_1 - 4a_2v_2 \\
& - 4a_3v_2 - 28b_2v_1 - 24b_2v_2 + 4b_3v_1 - 4a_1 - 4b_1 + 16b_2 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(3a_2 - a_3 + 9b_2 - 3b_3)\, v_1^2 + (6a_2 - 2a_3 + 18b_2 - 6b_3)\, v_1 v_2 + (-8a_2 - 28b_2 + 4b_3)\, v_1 \\ + (3a_2 - a_3 + 9b_2 - 3b_3)\, v_2^2 + (-4a_2 - 4a_3 - 24b_2)\, v_2 - 4a_1 - 4b_1 + 16b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_1 - 4b_1 + 16b_2 &= 0 \\
-8a_2 - 28b_2 + 4b_3 &= 0 \\
-4a_2 - 4a_3 - 24b_2 &= 0 \\
3a_2 - a_3 + 9b_2 - 3b_3 &= 0 \\
6a_2 - 2a_3 + 18b_2 - 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_1 + 4b_2 \\
a_2 &= -3b_2 \\
a_3 &= -3b_2 \\
b_1 &= b_1 \\
b_2 &= b_2 \\
b_3 &= b_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 1 - \left(-\frac{x + y}{3x + 3y - 4}\right)(-1) \\
&= \frac{2x + 2y - 4}{3x + 3y - 4} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2x+2y-4}{3x+3y-4}} dy \end{aligned}$$

Which results in

$$S = \frac{3y}{2} + \ln (x + y - 2)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{3x + 3y - 4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{x + y - 2} \\ S_y &= \frac{3}{2} + \frac{1}{x + y - 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y}{2} + \ln(x + y - 2) = -\frac{x}{2} + c_1$$

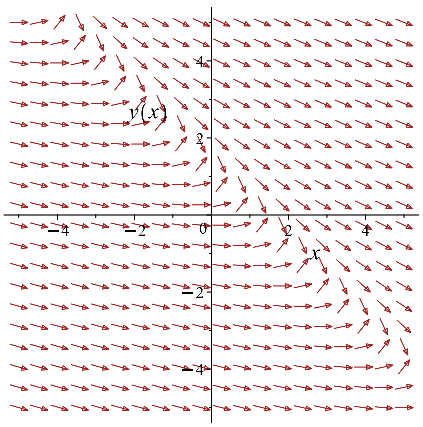
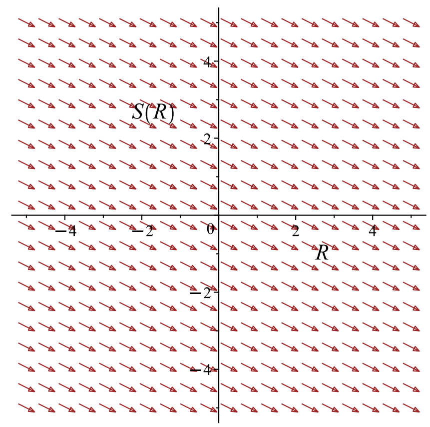
Which simplifies to

$$\frac{3y}{2} + \ln(x + y - 2) = -\frac{x}{2} + c_1$$

Which gives

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}$ 	$R = x$ $S = \frac{3y}{2} + \ln(x+y-2)$	$\frac{dS}{dR} = -\frac{1}{2}$ 

Summary

The solution(s) found are the following

$$y = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{x-3+c_1}}{2}\right)}{3} - x + 2 \tag{1}$$

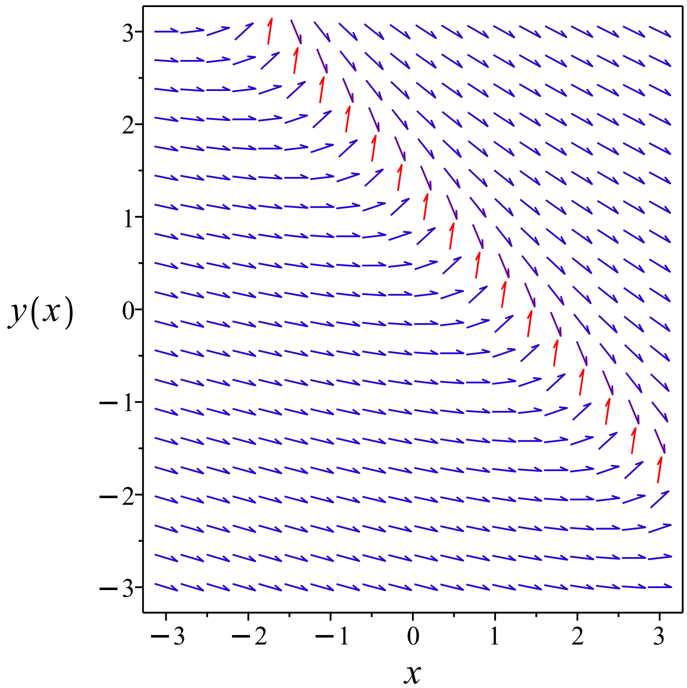


Figure 51: Slope field plot

Verification of solutions

$$y = \frac{2 \operatorname{LambertW}\left(\frac{3 \, e^{x-3+c_1}}{2}\right)}{3} - x + 2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 21

```
dsolve((3*x+3*y(x)-4)*diff(y(x),x)=-(x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{2\,\mathrm{LambertW}\left(\frac{3\,\mathrm{e}^{-3+x-c_1}}{2}\right)}{3} - x + 2$$

✓ Solution by Mathematica

Time used: 3.675 (sec). Leaf size: 33

```
DSolve[(3*x+3*y[x]-4)*y'[x]==-(x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}W\left(-e^{x-1+c_1}\right) - x + 2$$
$$y(x) \to 2 - x$$

2.17 problem 17

Internal problem ID [5103]
Internal file name [`OUTPUT/4596_Sunday_June_05_2022_03_01_32_PM_57694632/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[_rational, [_1st_order, `_with_symmetry_[F(x)*G(y),0]`], [
   _Abel, `2nd type`, `class B`]]
```

$$\boxed{-xy^2-\left(x+yx^2\right)y'=-x}$$

2.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(xy+1)\,\mathrm{d}y &= \left(-y^2+1\right)\mathrm{d}x\\ \left(y^2-1\right)\mathrm{d}x + (xy+1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y^2-1\\ N(x,y) &= xy+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2-1\right)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(xy+1)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{xy+1}((2y)-(y))\\ &= \frac{y}{xy+1}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y^2 - 1}((y) - (2y)) \\
&= -\frac{y}{y^2 - 1}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{y}{y^2-1} \,\mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(y-1)}{2} - \frac{\ln(1+y)}{2}} \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{1+y}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{1+y}}(y^2 - 1) \\
&= \frac{y^2 - 1}{\sqrt{y-1}\,\sqrt{1+y}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{1+y}}(xy + 1) \\
&= \frac{xy + 1}{\sqrt{y-1}\,\sqrt{1+y}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y^2 - 1}{\sqrt{y-1}\,\sqrt{1+y}}\right) + \left(\frac{xy + 1}{\sqrt{y-1}\,\sqrt{1+y}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y^2 - 1}{\sqrt{y-1}\,\sqrt{1+y}} \, \mathrm{d}x$$
$$\phi = \frac{(y^2 - 1)\, x}{\sqrt{y-1}\,\sqrt{1+y}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2yx}{\sqrt{y-1}\,\sqrt{1+y}} - \frac{(y^2-1)\, x}{2\,(y-1)^{\frac{3}{2}}\,\sqrt{1+y}} - \frac{(y^2-1)\, x}{2\sqrt{y-1}\,(1+y)^{\frac{3}{2}}} + f'(y) \tag{4}$$
$$= \frac{yx}{\sqrt{y-1}\,\sqrt{1+y}} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{xy+1}{\sqrt{y-1}\,\sqrt{1+y}}$. Therefore equation (4) becomes

$$\frac{xy+1}{\sqrt{y-1}\,\sqrt{1+y}} = \frac{yx}{\sqrt{y-1}\,\sqrt{1+y}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y-1}\,\sqrt{1+y}}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{\sqrt{y-1}\,\sqrt{1+y}} \right) \mathrm{d}y$$
$$f(y) = \frac{\sqrt{(y-1)(1+y)}\, \ln\left(y + \sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{1+y}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(y^2-1)\,x}{\sqrt{y-1}\,\sqrt{1+y}} + \frac{\sqrt{(y-1)\,(1+y)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{1+y}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(y^2-1)\,x}{\sqrt{y-1}\,\sqrt{1+y}} + \frac{\sqrt{(y-1)\,(1+y)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{1+y}}$$

Summary
The solution(s) found are the following

$$\frac{(y^2-1)\,x}{\sqrt{y-1}\,\sqrt{1+y}} + \frac{\sqrt{(y-1)\,(1+y)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{1+y}} = c_1 \tag{1}$$

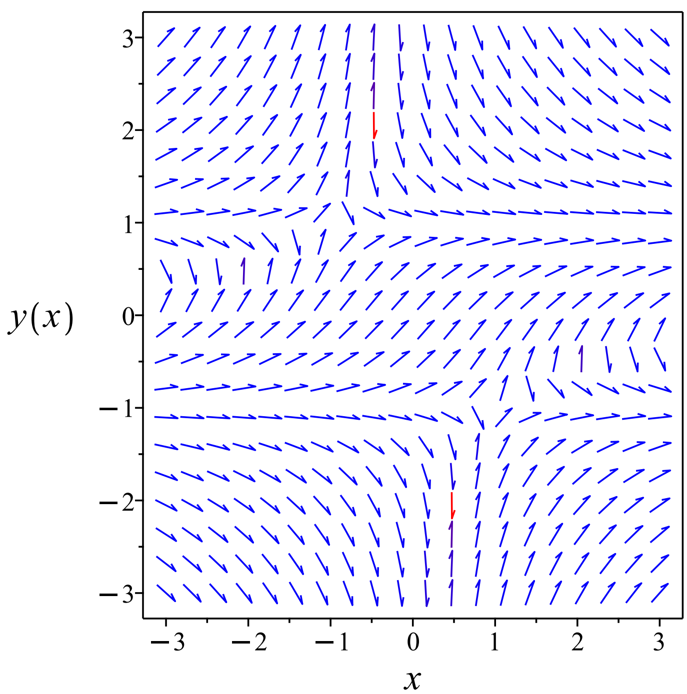


Figure 52: Slope field plot

Verification of solutions

$$\frac{(y^2-1)\,x}{\sqrt{y-1}\,\sqrt{1+y}}+\frac{\sqrt{(y-1)\,(1+y)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{1+y}}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 53

```
dsolve((x-x*y(x)^2)=(x+x^2*y(x))*diff(y(x),x),y(x), singsol=all)
```

$$x+\frac{\sqrt{y\left(x\right)^{2}-1}\,\ln\left(y(x)+\sqrt{y\left(x\right)^{2}-1}\right)}{\left(y\left(x\right)-1\right)\left(y\left(x\right)+1\right)}-\frac{c_1}{\sqrt{y\left(x\right)-1}\,\sqrt{y\left(x\right)+1}}=0$$

✓ Solution by Mathematica
Time used: 0.127 (sec). Leaf size: 55

```
DSolve[(x-x*y[x]^2)==(x+x^2*y[x])*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[x=-\frac{2\arctan\left(\frac{\sqrt{1-y(x)^2}}{y(x)+1}\right)}{\sqrt{1-y(x)^2}}+\frac{c_1}{\sqrt{1-y(x)^2}},y(x)\right]$$

2.18 problem 18

Internal problem ID [5104]
Internal file name [`OUTPUT/4597_Sunday_June_05_2022_03_01_34_PM_27322655/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{-y + (4y + x - 1)\, y' = 1 - x}$$

2.18.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{-X - x_0 + Y(X) + y_0 + 1}{4Y(X) + 4y_0 + X + x_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{-X + Y(X)}{4Y(X) + X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X,Y) \\
&= \frac{-X+Y}{4Y+X}
\end{aligned} \tag{1}
$$

An ode of the form $Y'=\frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X,Y)
$$

In this case, it can be seen that both $M=-X+Y$ and $N=4Y+X$ are both homogeneous and of the same order $n=1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u=\frac{Y}{X}$, or $Y=uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y=uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-1}{4u+1} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-1}{4u(X)+1} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{\frac{u(X)-1}{4u(X)+1} - u(X)}{X} = 0
$$

Or

$$
4\left(\frac{d}{dX}u(X)\right) X u(X) + \left(\frac{d}{dX}u(X)\right) X + 4u(X)^2 + 1 = 0
$$

Or

$$
1 + X(4u(X)+1)\left(\frac{d}{dX}u(X)\right) + 4u(X)^2 = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X,u) \\
&= f(X)g(u) \\
&= -\frac{4u^2+1}{X\,(4u+1)}
\end{aligned}
$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{4u^2+1}{4u+1}$. Integrating both sides gives

$$\frac{1}{\frac{4u^2+1}{4u+1}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{4u^2+1}{4u+1}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(4u^2+1)}{2} + \frac{\arctan(2u)}{2} = -\ln(X) + c_2$$

The solution is

$$\frac{\ln\left(4u(X)^2+1\right)}{2} + \frac{\arctan(2u(X))}{2} + \ln(X) - c_2 = 0$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{\ln\left(\frac{4Y(X)^2}{X^2}+1\right)}{2} + \frac{\arctan\left(\frac{2Y(X)}{X}\right)}{2} + \ln(X) - c_2 = 0$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{4Y(X)^2}{X^2}+1\right)}{2} + \frac{\arctan\left(\frac{2Y(X)}{X}\right)}{2} + \ln(X) - c_2 = 0$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$\frac{\ln\left(\frac{4y^2}{(x-1)^2}+1\right)}{2} + \frac{\arctan\left(\frac{2y}{x-1}\right)}{2} + \ln(x-1) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln\left(\frac{4y^2}{(x-1)^2}+1\right)}{2}+\frac{\arctan\left(\frac{2y}{x-1}\right)}{2}+\ln(x-1)-c_2=0 \tag{1}$$

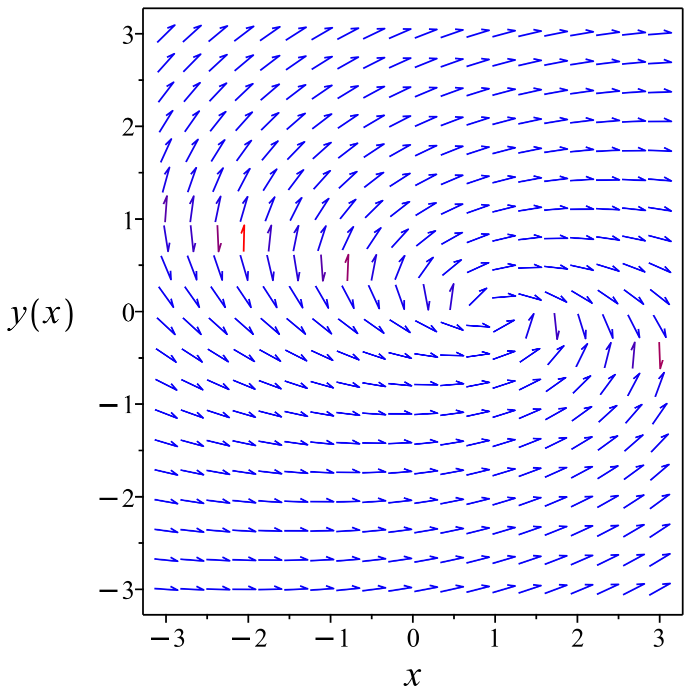


Figure 53: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{4y^2}{(x-1)^2}+1\right)}{2}+\frac{\arctan\left(\frac{2y}{x-1}\right)}{2}+\ln(x-1)-c_2=0$$

Verified OK.

2.18.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{-x+y+1}{4y+x-1}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 + \frac{(-x+y+1)(b_3-a_2)}{4y+x-1} - \frac{(-x+y+1)^2 a_3}{(4y+x-1)^2} \\ &- \left(-\frac{1}{4y+x-1} - \frac{-x+y+1}{(4y+x-1)^2}\right)(xa_2+ya_3+a_1) \\ &- \left(\frac{1}{4y+x-1} - \frac{4(-x+y+1)}{(4y+x-1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^2a_2 - x^2a_3 - 4x^2b_2 - x^2b_3 + 8xya_2 + 2xya_3 + 8xyb_2 - 8xyb_3 - 4y^2a_2 + 4y^2a_3 + 16y^2b_2 + 4y^2b_3 - 2xa_2}{(4y+x-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &x^2a_2 - x^2a_3 - 4x^2b_2 - x^2b_3 + 8xya_2 + 2xya_3 + 8xyb_2 - 8xyb_3 - 4y^2a_2 \\ &+ 4y^2a_3 + 16y^2b_2 + 4y^2b_3 - 2xa_2 + 2xa_3 - 5xb_1 + 3xb_2 + 2xb_3 \\ &+ 5ya_1 - 3ya_2 - 2ya_3 - 8yb_2 + 8yb_3 + a_2 - a_3 + 5b_1 + b_2 - b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&a_2v_1^2 + 8a_2v_1v_2 - 4a_2v_2^2 - a_3v_1^2 + 2a_3v_1v_2 + 4a_3v_2^2 - 4b_2v_1^2 + 8b_2v_1v_2 \\
&+ 16b_2v_2^2 - b_3v_1^2 - 8b_3v_1v_2 + 4b_3v_2^2 + 5a_1v_2 - 2a_2v_1 - 3a_2v_2 + 2a_3v_1 - 2a_3v_2 \\
&- 5b_1v_1 + 3b_2v_1 - 8b_2v_2 + 2b_3v_1 + 8b_3v_2 + a_2 - a_3 + 5b_1 + b_2 - b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(a_2 - a_3 - 4b_2 - b_3)\, v_1^2 + (8a_2 + 2a_3 + 8b_2 - 8b_3)\, v_1v_2 \\
&+ (-2a_2 + 2a_3 - 5b_1 + 3b_2 + 2b_3)\, v_1 + (-4a_2 + 4a_3 + 16b_2 + 4b_3)\, v_2^2 \\
&+ (5a_1 - 3a_2 - 2a_3 - 8b_2 + 8b_3)\, v_2 + a_2 - a_3 + 5b_1 + b_2 - b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-4a_2 + 4a_3 + 16b_2 + 4b_3 &= 0 \\
a_2 - a_3 - 4b_2 - b_3 &= 0 \\
8a_2 + 2a_3 + 8b_2 - 8b_3 &= 0 \\
5a_1 - 3a_2 - 2a_3 - 8b_2 + 8b_3 &= 0 \\
-2a_2 + 2a_3 - 5b_1 + 3b_2 + 2b_3 &= 0 \\
a_2 - a_3 + 5b_1 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -b_3 \\
a_2 &= b_3 \\
a_3 &= -4b_2 \\
b_1 &= -b_2 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= -4y \\
\eta &= x - 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= x - 1 - \left(\frac{-x + y + 1}{4y + x - 1}\right)(-4y) \\
&= \frac{x^2 + 4y^2 - 2x + 1}{4y + x - 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 + 4y^2 - 2x + 1}{4y + x - 1}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(x^2 + 4y^2 - 2x + 1\right)}{2} + \frac{2(x - 1)\arctan\left(\frac{8y}{4x - 4}\right)}{4x - 4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{-x + y + 1}{4y + x - 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x - y - 1}{x^2 + 4y^2 - 2x + 1} \\
S_y &= \frac{4y + x - 1}{x^2 + 4y^2 - 2x + 1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

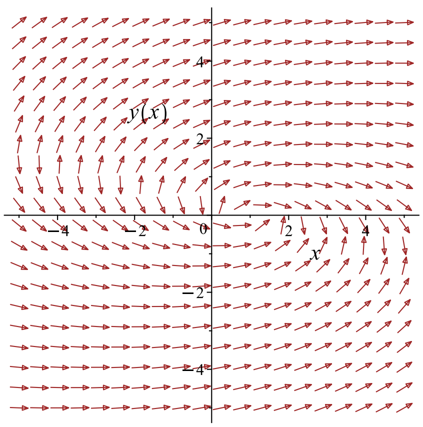
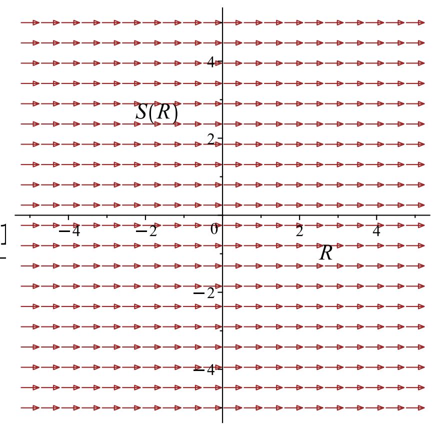
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(4y^2 + x^2 - 2x + 1)}{2} + \frac{\arctan\left(\frac{2y}{x-1}\right)}{2} = c_1
$$

Which simplifies to

$$
\frac{\ln(4y^2 + x^2 - 2x + 1)}{2} + \frac{\arctan\left(\frac{2y}{x-1}\right)}{2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+y+1}{4y+x-1}$ 	$R = x$ $S = \frac{\ln(x^2+4y^2-2x+1)}{2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(4y^2+x^2-2x+1)}{2} + \frac{\arctan\left(\frac{2y}{x-1}\right)}{2} = c_1 \tag{1}$$

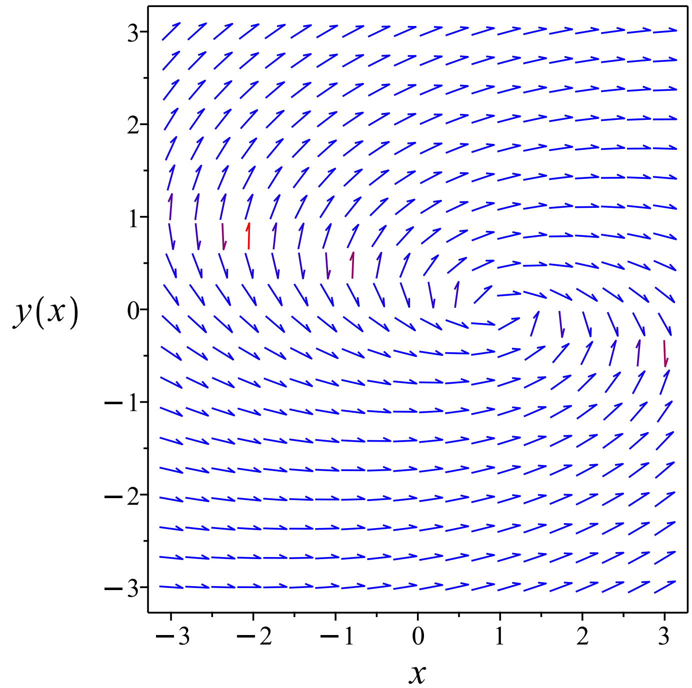


Figure 54: Slope field plot

Verification of solutions

$$\frac{\ln\left(4y^2 + x^2 - 2x + 1\right)}{2} + \frac{\arctan\left(\frac{2y}{x-1}\right)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 29

```
dsolve((x-y(x)-1)+(4*y(x)+x-1)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = -\frac{\tan\left(\mathrm{RootOf}\left(\ln\left(\sec\left(\_Z\right)^2\right) - \_Z + 2\ln\left(x-1\right) + 2c_1\right)\right)(x-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 58

```
DSolve[(x-y[x]-1)+(4*y[x]+x-1)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned}\text{Solve}\Bigg[&2\arctan\left(\frac{2y(x)-2x+2}{4y(x)+x-1}\right)\\&+2\log\left(\frac{4}{5}\left(\frac{4y(x)^2}{(x-1)^2}+1\right)\right)+4\log(x-1)+5c_1=0,y(x)\Bigg]\end{aligned}$$

2.19 problem 19

Internal problem ID [5105]
Internal file name [`OUTPUT/4598_Sunday_June_05_2022_03_01_35_PM_86667519/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeMapleC", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{3y + (7y - 3x + 3)\, y' = 7x - 7}$$

2.19.1 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 - 7X - 7x_0 + 7}{7Y(X) + 7y_0 - 3X - 3x_0 + 3}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) - 7X}{7Y(X) - 3X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{3Y - 7X}{7Y - 3X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 3Y - 7X$ and $N = -7Y + 3X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{-3u + 7}{7u - 3} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{\frac{-3u(X)+7}{7u(X)-3} - u(X)}{X} = 0$$

Or

$$7\left(\frac{d}{dX} u(X)\right) X u(X) - 3\left(\frac{d}{dX} u(X)\right) X + 7u(X)^2 - 7 = 0$$

Or

$$-7 + X(7u(X) - 3)\left(\frac{d}{dX} u(X)\right) + 7u(X)^2 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X) g(u) \\ &= -\frac{7(u^2 - 1)}{X(7u - 3)} \end{aligned}$$

Where $f(X) = -\frac{7}{X}$ and $g(u) = \frac{u^2-1}{7u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{7u-3}}\, du = -\frac{7}{X}\, dX$$
$$\int \frac{1}{\frac{u^2-1}{7u-3}}\, du = \int -\frac{7}{X}\, dX$$
$$2\ln(u-1) + 5\ln(u+1) = -7\ln(X) + c_2$$

Raising both side to exponential gives

$$e^{2\ln(u-1)+5\ln(u+1)} = e^{-7\ln(X)+c_2}$$

Which simplifies to

$$(u-1)^2 (u+1)^5 = \frac{c_3}{X^7}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \operatorname{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

Using the solution for $Y(X)$

$$Y(X) = \operatorname{RootOf}\left(X^7 + 3X^6\_Z + X^5\_Z^2 - 5X^4\_Z^3 - 5X^3\_Z^4 + X^2\_Z^5 + 3X\_Z^6 + \_Z^7 - c_3\right)$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x + 1$$

Then the solution in y becomes

$$y = \operatorname{RootOf}\left(\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 + (-5x^4+20x$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = \operatorname{RootOf}\Big(&\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 \\
&+ 35x^3-21x^2-c_3+7x-1\Big)
\end{aligned} \tag{1}
$$

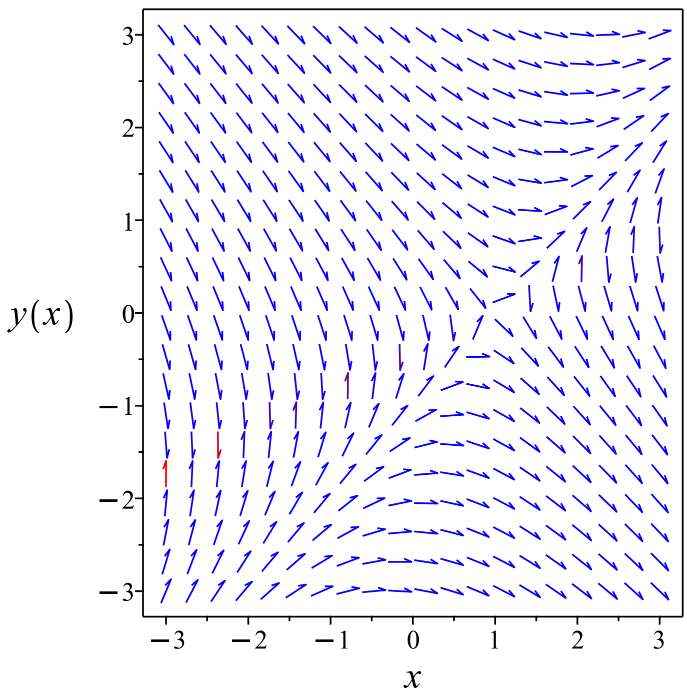


Figure 55: Slope field plot

Verification of solutions

$$
\begin{aligned}
y = \operatorname{RootOf}\Big(&\_Z^7 + (-3+3x)\_Z^6 + (x^2-2x+1)\_Z^5 + (-5x^3+15x^2-15x+5)\_Z^4 \\
&+ (-5x^4+20x^3-30x^2+20x-5)\_Z^3 + (x^5-5x^4+10x^3-10x^2+5x-1)\_Z^2 \\
&+ (3x^6-18x^5+45x^4-60x^3+45x^2-18x+3)\_Z + x^7-7x^6+21x^5-35x^4 \\
&+ 35x^3-21x^2-c_3+7x-1\Big)
\end{aligned}
$$

Verified OK.

2.19.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y-7x+7}{7y-3x+3} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(3y-7x+7)(b_3-a_2)}{7y-3x+3} - \frac{(3y-7x+7)^2 a_3}{(7y-3x+3)^2} \\
& - \left(\frac{7}{7y-3x+3} - \frac{3(3y-7x+7)}{(7y-3x+3)^2}\right)(x a_2 + y a_3 + a_1) \\
& - \left(-\frac{3}{7y-3x+3} + \frac{21y-49x+49}{(7y-3x+3)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 -}{}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 21x^2a_2 - 49x^2a_3 + 49x^2b_2 - 21x^2b_3 - 98xya_2 + 42xya_3 - 42xyb_2 + 98xyb_3 \\
& + 21y^2a_2 - 49y^2a_3 + 49y^2b_2 - 21y^2b_3 - 42xa_2 + 98xa_3 + 40xb_1 - 58xb_2 + 42xb_3 \\
& - 40ya_1 + 58ya_2 - 42ya_3 + 42yb_2 - 98yb_3 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &21a_2v_1^2 - 98a_2v_1v_2 + 21a_2v_2^2 - 49a_3v_1^2 + 42a_3v_1v_2 - 49a_3v_2^2 + 49b_2v_1^2 \\ &- 42b_2v_1v_2 + 49b_2v_2^2 - 21b_3v_1^2 + 98b_3v_1v_2 - 21b_3v_2^2 - 40a_1v_2 \\ &- 42a_2v_1 + 58a_2v_2 + 98a_3v_1 - 42a_3v_2 + 40b_1v_1 - 58b_2v_1 + 42b_2v_2 \\ &+ 42b_3v_1 - 98b_3v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_1^2 + (-98a_2 + 42a_3 - 42b_2 + 98b_3)\, v_1v_2 \\ &+ (-42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3)\, v_1 + (21a_2 - 49a_3 + 49b_2 - 21b_3)\, v_2^2 \\ &+ (-40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3)\, v_2 + 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -98a_2 + 42a_3 - 42b_2 + 98b_3 &= 0 \\ 21a_2 - 49a_3 + 49b_2 - 21b_3 &= 0 \\ -40a_1 + 58a_2 - 42a_3 + 42b_2 - 98b_3 &= 0 \\ -42a_2 + 98a_3 + 40b_1 - 58b_2 + 42b_3 &= 0 \\ 21a_2 - 49a_3 - 40b_1 + 9b_2 - 21b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -b_3 \\ a_2 &= b_3 \\ a_3 &= b_2 \\ b_1 &= -b_2 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= y \\
\eta &= x - 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= x - 1 - \left(-\frac{3y - 7x + 7}{7y - 3x + 3}\right)(y) \\
&= \frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 - 3y^2 - 6x + 3}{-7y + 3x - 3}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{5 \ln (x + y - 1)}{3} + \frac{2 \ln (-x + y + 1)}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y - 7x + 7}{7y - 3x + 3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{5}{3x - 3 + 3y} + \frac{2}{3x - 3y - 3} \\
S_y &= \frac{5}{3x - 3 + 3y} - \frac{2}{3x - 3y - 3}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

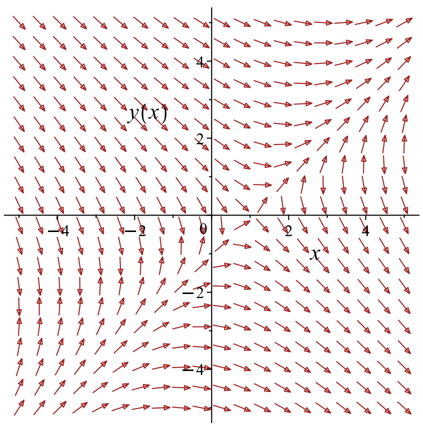
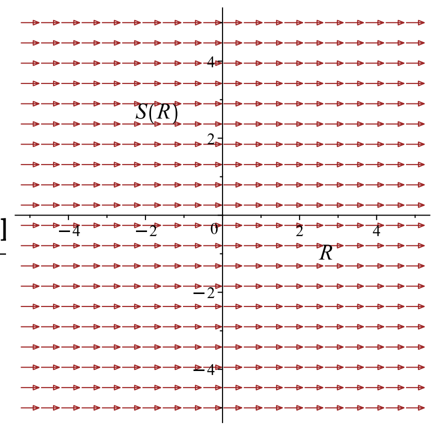
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{5\ln(x - 1 + y)}{3} + \frac{2\ln(-x + y + 1)}{3} = c_1$$

Which simplifies to

$$\frac{5\ln(x - 1 + y)}{3} + \frac{2\ln(-x + y + 1)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3}$
</td><td>$R = x$
$S = \frac{5\ln(x+y-1)}{3} + 2\mathrm{l}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{5\ln(x-1+y)}{3} + \frac{2\ln(-x+y+1)}{3} = c_1 \tag{1}$$

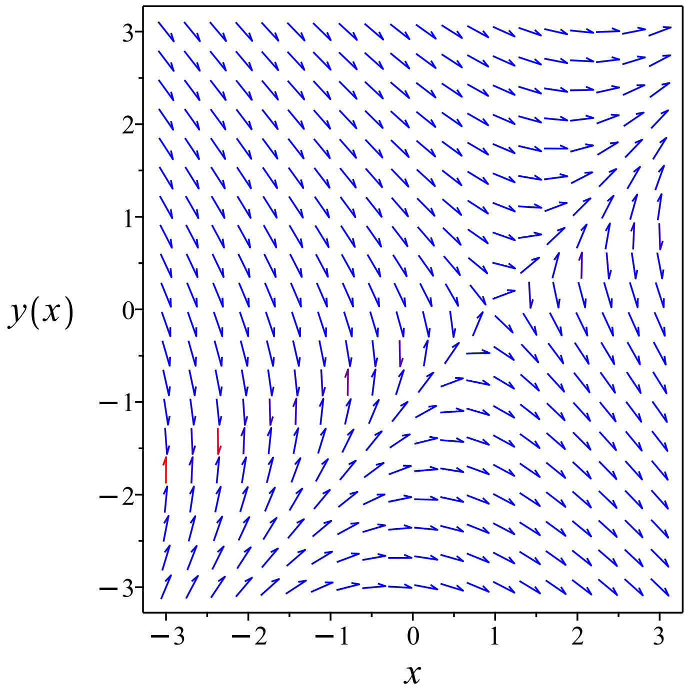


Figure 56: Slope field plot

Verification of solutions

$$\frac{5\ln\left(x-1+y\right)}{3}+\frac{2\ln\left(-x+y+1\right)}{3}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.422 (sec). Leaf size: 1814

```
dsolve((3*y(x)-7*x+7)+(7*y(x)-3*x+3)*diff(y(x),x)=0,y(x), singsol=all)
```

Expression too large to display

✓ Solution by Mathematica

Time used: 60.706 (sec). Leaf size: 7785

```
DSolve[(3*y[x]-7*x+7)+(7*y[x]-3*x+3)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

Too large to display

2.20 problem 20

Internal problem ID [5106]
Internal file name [`OUTPUT/4599_Sunday_June_05_2022_03_01_37_PM_51431466/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational]
```

$$\boxed{y(xy+1)+x\left(1+xy+y^2x^2\right)y'=0}$$

2.20.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}
y' &= -\frac{y(xy+1)}{x\left(y^2x^2+xy+1\right)} \\
y' &= \omega(x,y)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{y(xy+1)\,(b_3-a_2)}{x\,(y^2x^2+xy+1)} - \frac{y^2(xy+1)^2\,a_3}{x^2\,(y^2x^2+xy+1)^2} \\
&- \left(-\frac{y^2}{x\,(y^2x^2+xy+1)} + \frac{y(xy+1)}{x^2\,(y^2x^2+xy+1)}\right. \\
&\left.+ \frac{y(xy+1)\,(2y^2x+y)}{x\,(y^2x^2+xy+1)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{xy+1}{x\,(y^2x^2+xy+1)}\right. \\
&\left.- \frac{y}{y^2x^2+xy+1} + \frac{y(xy+1)\,(2y\,x^2+x)}{x\,(y^2x^2+xy+1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^6y^4b_2 + 2x^5y^3b_2 - x^4y^4a_2 - x^4y^4b_3 - 2x^3y^5a_3 - 2x^3y^4a_1 + 3x^4y^2b_2 - 2x^3y^3a_2 - 2x^3y^3b_3 - 5x^2y^4a_3 - 4x}{x^2\,(y^2x^2+xy+1)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&x^6y^4b_2 + 2x^5y^3b_2 - x^4y^4a_2 - x^4y^4b_3 - 2x^3y^5a_3 - 2x^3y^4a_1 \\
&+ 3x^4y^2b_2 - 2x^3y^3a_2 - 2x^3y^3b_3 - 5x^2y^4a_3 - 4x^2y^3a_1 + 4x^3yb_2 \\
&- 4x\,y^3a_3 + 2x^2yb_1 - 2x\,y^2a_1 + 2b_2x^2 - 2y^2a_3 + xb_1 - ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&b_2v_1^6v_2^4 - a_2v_1^4v_2^4 - 2a_3v_1^3v_2^5 + 2b_2v_1^5v_2^3 - b_3v_1^4v_2^4 - 2a_1v_1^3v_2^4 - 2a_2v_1^3v_2^3 \\
&- 5a_3v_1^2v_2^4 + 3b_2v_1^4v_2^2 - 2b_3v_1^3v_2^3 - 4a_1v_1^2v_2^3 - 4a_3v_1v_2^3 + 4b_2v_1^3v_2 \\
&- 2a_1v_1v_2^2 + 2b_1v_1^2v_2 - 2a_3v_2^2 + 2b_2v_1^2 - a_1v_2 + b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & b_2v_1^6v_2^4 + 2b_2v_1^5v_2^3 + (-a_2 - b_3)\, v_1^4v_2^4 + 3b_2v_1^4v_2^2 - 2a_3v_1^3v_2^5 - 2a_1v_1^3v_2^4 \\ & + (-2a_2 - 2b_3)\, v_1^3v_2^3 + 4b_2v_1^3v_2 - 5a_3v_1^2v_2^4 - 4a_1v_1^2v_2^3 + 2b_1v_1^2v_2 \\ & + 2b_2v_1^2 - 4a_3v_1v_2^3 - 2a_1v_1v_2^2 + b_1v_1 - 2a_3v_2^2 - a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ b_2 &= 0 \\ -4a_1 &= 0 \\ -2a_1 &= 0 \\ -a_1 &= 0 \\ -5a_3 &= 0 \\ -4a_3 &= 0 \\ -2a_3 &= 0 \\ 2b_1 &= 0 \\ 2b_2 &= 0 \\ 3b_2 &= 0 \\ 4b_2 &= 0 \\ -2a_2 - 2b_3 &= 0 \\ -a_2 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = -x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y(xy+1)}{x\,(y^2x^2+xy+1)}\right)(-x) \\
&= \frac{y^3x^2}{y^2x^2+xy+1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^3x^2}{y^2x^2+xy+1}} dy
\end{aligned}$$

Which results in

$$S = \ln(y) - \frac{1}{2y^2x^2} - \frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(xy+1)}{x\,(y^2x^2+xy+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{xy+1}{y^2x^3}\\
S_y &= \frac{y^2x^2+xy+1}{y^3x^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

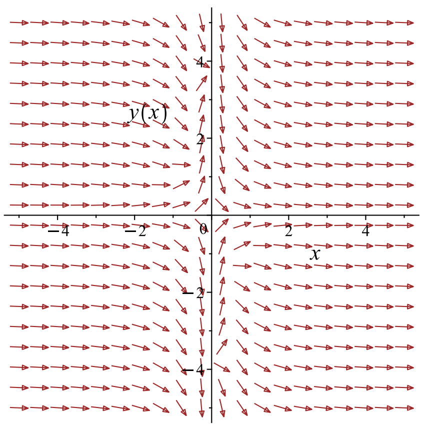
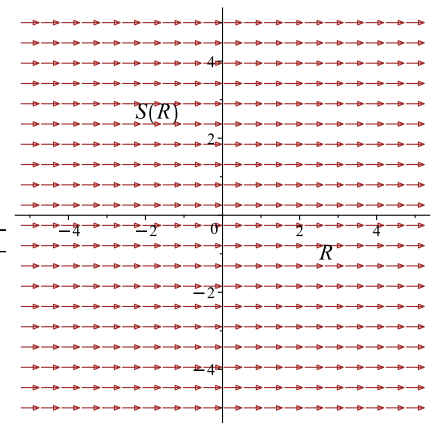
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\,y^2x^2-2xy-1}{2y^2x^2} = c_1$$

Which simplifies to

$$\frac{2\ln(y)\,y^2x^2-2xy-1}{2y^2x^2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(xy+1)}{x(y^2x^2+xy+1)}$ 	$R = x$ $S = \frac{2\ln(y)\,y^2x^2 - 2xy -}{2y^2x^2}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{2\ln(y)\,y^2x^2 - 2xy - 1}{2y^2x^2} = c_1 \tag{1}$$

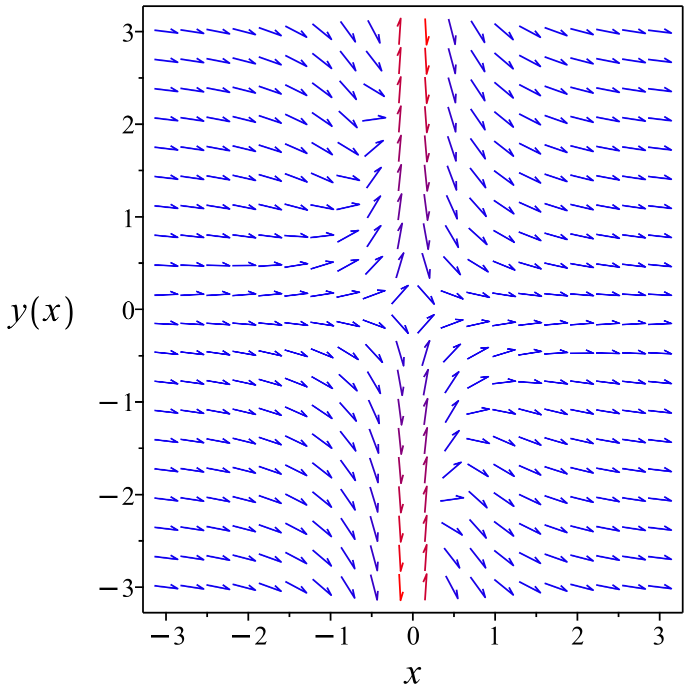


Figure 57: Slope field plot

Verification of solutions

$$\frac{2\ln(y)\, y^2x^2 - 2xy - 1}{2y^2x^2} = c_1$$

Verified OK.

2.20.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x\left(y^2x^2 + xy + 1\right)\right)\mathrm{d}y &= (-y(xy + 1))\,\mathrm{d}x\\ (y(xy + 1))\,\mathrm{d}x + \left(x\left(y^2x^2 + xy + 1\right)\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y(xy + 1)\\ N(x, y) &= x\left(y^2x^2 + xy + 1\right)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(xy + 1))\\ &= 2xy + 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(y^2x^2 + xy + 1\right)\right)\\ &= 3y^2x^2 + 2xy + 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\left(y^2x^2 + xy + 1\right)}\left((2xy+1) - \left(y^2x^2 + xy + 1 + x\left(2y^2x + y\right)\right)\right) \\
&= -\frac{3y^2x}{y^2x^2 + xy + 1}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{y\left(xy+1\right)}\left(\left(y^2x^2 + xy + 1 + x\left(2y^2x + y\right)\right) - (2xy+1)\right) \\
&= \frac{3y\,x^2}{xy+1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{\left(y^2x^2 + xy + 1 + x(2y^2x + y)\right) - (2xy+1)}{x\left(y\left(xy+1\right)\right) - y\left(x\left(y^2x^2 + xy + 1\right)\right)} \\
&= -\frac{3}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{3}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-3\ln(t)} \\ &= \frac{1}{t^3}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{x^3y^3}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^3y^3}(y(xy+1)) \\ &= \frac{xy+1}{y^2x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^3y^3}\left(x\left(y^2x^2+xy+1\right)\right) \\ &= \frac{y^2x^2+xy+1}{y^3x^2}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{xy+1}{y^2x^3}\right)+\left(\frac{y^2x^2+xy+1}{y^3x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{xy+1}{y^2x^3} \mathrm{d}x \\
\phi &= \frac{-2xy-1}{2y^2x^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{1}{y^2x} - \frac{-2xy-1}{y^3x^2} + f'(y) \qquad (4) \\
&= \frac{xy+1}{y^3x^2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2x^2+xy+1}{y^3x^2}$. Therefore equation (4) becomes

$$
\frac{y^2x^2+xy+1}{y^3x^2} = \frac{xy+1}{y^3x^2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{-2xy-1}{2y^2x^2} + \ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-2xy-1}{2y^2x^2} + \ln(y)$$

<u>Summary</u>
The solution(s) found are the following

$$\frac{-2xy-1}{2y^2x^2} + \ln(y) = c_1 \tag{1}$$

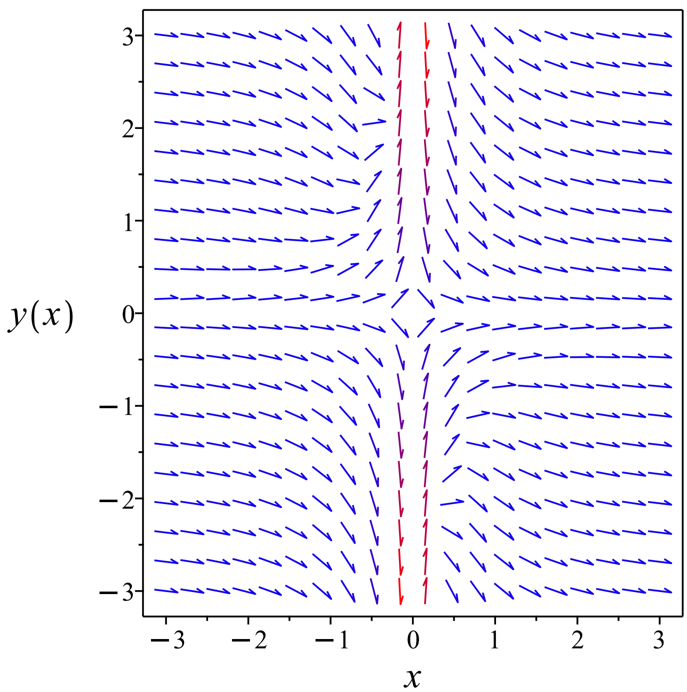


Figure 58: Slope field plot

<u>Verification of solutions</u>

$$\frac{-2xy-1}{2y^2x^2} + \ln(y) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 38

```
dsolve(y(x)*(x*y(x)+1)+x*(1+x*y(x)+x^2*y(x)^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{e^{\operatorname{RootOf}\left(-2\ln(x)e^{2\_Z}+2c_1e^{2\_Z}+2\_Ze^{2\_Z}-2\,e^{\_Z}-1\right)}}{x}$$

✓ Solution by Mathematica

Time used: 0.11 (sec). Leaf size: 30

```
DSolve[y[x]*(x*y[x]+1)+x*(1+x*y[x]+x^2*y[x]^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> T
```

$$\text{Solve}\left[\frac{-\frac{1}{2x^2}-\frac{y(x)}{x}}{y(x)^2}+\log(y(x))=c_1,y(x)\right]$$

2.21 problem 21

Internal problem ID [5107]
Internal file name [`OUTPUT/4600_Sunday_June_05_2022_03_01_38_PM_45463758/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y + y' - y^3 x = 0}$$

2.21.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x\,y^3 - y \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -1 \\
f_1(x) &= x \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{y^2} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2} \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -w(x) + x \\
w' &= 2w - 2x \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2 \\
q(x) &= -2x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2w(x) = -2x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-2)dx} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-2x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x} w\right) &= \left(\mathrm{e}^{-2x}\right)(-2x) \\
\mathrm{d}\left(\mathrm{e}^{-2x} w\right) &= \left(-2x\,\mathrm{e}^{-2x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-2x} w &= \int -2x\,\mathrm{e}^{-2x}\,\mathrm{d}x \\
\mathrm{e}^{-2x} w &= \frac{(1+2x)\,\mathrm{e}^{-2x}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$
w(x) = \frac{\mathrm{e}^{2x}(1+2x)\,\mathrm{e}^{-2x}}{2} + c_1 \mathrm{e}^{2x}
$$

which simplifies to

$$
w(x) = \frac{1}{2} + x + c_1 \mathrm{e}^{2x}
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = \frac{1}{2} + x + c_1 \mathrm{e}^{2x}
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \frac{2}{\sqrt{2 + 4c_1 \mathrm{e}^{2x} + 4x}} \\
y(x) &= -\frac{2}{\sqrt{2 + 4c_1 \mathrm{e}^{2x} + 4x}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2}{\sqrt{2 + 4c_1 \mathrm{e}^{2x} + 4x}} \tag{1}
$$

$$
y = -\frac{2}{\sqrt{2 + 4c_1 \mathrm{e}^{2x} + 4x}} \tag{2}
$$

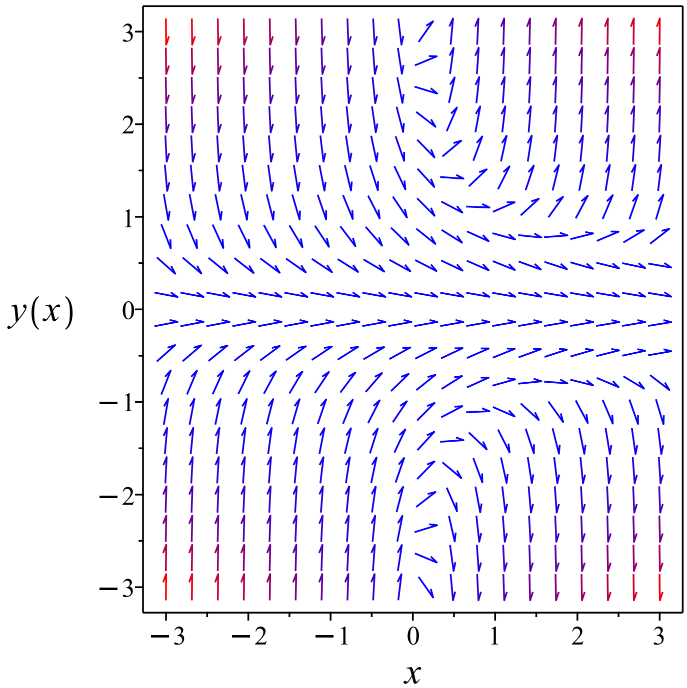


Figure 59: Slope field plot

Verification of solutions

$$y = \frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$

Verified OK.

$$y = -\frac{2}{\sqrt{2 + 4c_1 e^{2x} + 4x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 39

```
dsolve(diff(y(x),x)+y(x)=x*y(x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{2}{\sqrt{2 + 4\,\mathrm{e}^{2x} c_1 + 4x}}$$
$$y(x) = \frac{2}{\sqrt{2 + 4\,\mathrm{e}^{2x} c_1 + 4x}}$$

✓ Solution by Mathematica

Time used: 2.704 (sec). Leaf size: 50

```
DSolve[y'[x]+y[x]==x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x + c_1 e^{2x} + \frac{1}{2}}}$$
$$y(x) \to \frac{1}{\sqrt{x + c_1 e^{2x} + \frac{1}{2}}}$$
$$y(x) \to 0$$

2.22 problem 22

Internal problem ID [5108]
Internal file name [`OUTPUT/4601_Sunday_June_05_2022_03_01_39_PM_35987126/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{y + y' - y^4 \mathrm{e}^x = 0}$$

2.22.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y + y^4 \mathrm{e}^x \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -y + \mathrm{e}^x y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -1 \\
f_1(x) &= \mathrm{e}^x \\
n &= 4
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$
y' \frac{1}{y^4} = -\frac{1}{y^3} + \mathrm{e}^x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^3}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{3}{y^4} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{3} &= -w(x) + \mathrm{e}^x \\
w' &= 3w - 3\,\mathrm{e}^x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -3 \\
q(x) &= -3\,\mathrm{e}^x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 3w(x) = -3\,\mathrm{e}^x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int (-3)dx} \\
&= \mathrm{e}^{-3x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-3\,\mathrm{e}^{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-3x} w\right) &= \left(\mathrm{e}^{-3x}\right)\left(-3\,\mathrm{e}^{x}\right) \\
\mathrm{d}\left(\mathrm{e}^{-3x} w\right) &= \left(-3\,\mathrm{e}^{-2x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\mathrm{e}^{-3x} w &= \int -3\,\mathrm{e}^{-2x}\,\mathrm{d}x \\
\mathrm{e}^{-3x} w &= \frac{3\,\mathrm{e}^{-2x}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-3x}$ results in

$$
w(x) = \frac{3\,\mathrm{e}^{3x}\mathrm{e}^{-2x}}{2} + \mathrm{e}^{3x} c_1
$$

which simplifies to

$$
w(x) = \frac{3\,\mathrm{e}^{x}}{2} + \mathrm{e}^{3x} c_1
$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$
\frac{1}{y^3} = \frac{3\,\mathrm{e}^{x}}{2} + \mathrm{e}^{3x} c_1
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \frac{2^{\frac{1}{3}}\left(\mathrm{e}^{2x}\left(2c_1\mathrm{e}^{2x}+3\right)^2\right)^{\frac{1}{3}}\,\mathrm{e}^{-x}}{2c_1\mathrm{e}^{2x}+3} \\
y(x) &= \frac{2^{\frac{1}{3}}\left(\mathrm{e}^{2x}\left(2c_1\mathrm{e}^{2x}+3\right)^2\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)\mathrm{e}^{-x}}{4c_1\mathrm{e}^{2x}+6} \\
y(x) &= -\frac{\left(1+i\sqrt{3}\right)2^{\frac{1}{3}}\left(\mathrm{e}^{2x}\left(2c_1\mathrm{e}^{2x}+3\right)^2\right)^{\frac{1}{3}}\,\mathrm{e}^{-x}}{4c_1\mathrm{e}^{2x}+6}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x} + 3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{2c_1 \mathrm{e}^{2x} + 3} \tag{1}$$

$$y = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x} + 3)^2\right)^{\frac{1}{3}} (i\sqrt{3} - 1) \mathrm{e}^{-x}}{4c_1 \mathrm{e}^{2x} + 6} \tag{2}$$

$$y = -\frac{(1 + i\sqrt{3})\, 2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x} + 3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{4c_1 \mathrm{e}^{2x} + 6} \tag{3}$$

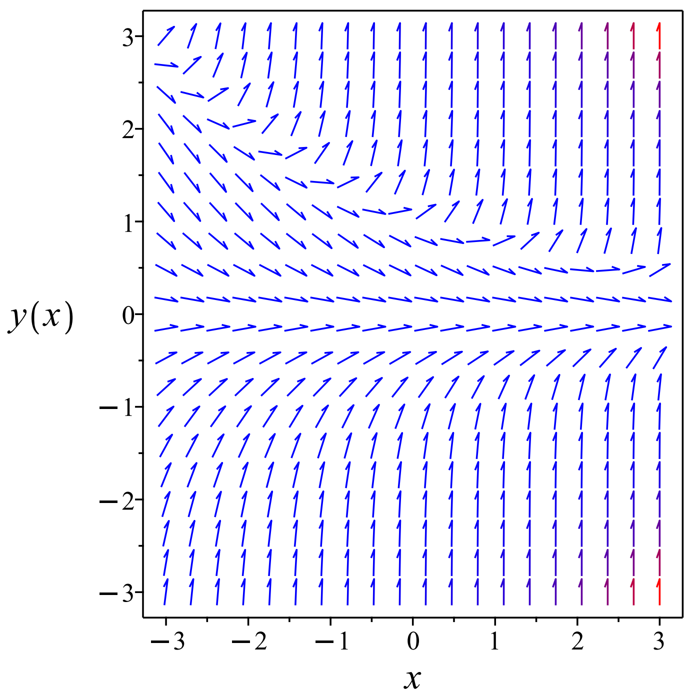


Figure 60: Slope field plot

Verification of solutions

$$y = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x}+3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{2c_1 \mathrm{e}^{2x}+3}$$

Verified OK.

$$y = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x}+3)^2\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right) \mathrm{e}^{-x}}{4c_1 \mathrm{e}^{2x}+6}$$

Verified OK.

$$y = -\frac{\left(1+i\sqrt{3}\right) 2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2c_1 \mathrm{e}^{2x}+3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{4c_1 \mathrm{e}^{2x}+6}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 138

```
dsolve(diff(y(x),x)+y(x)=y(x)^4*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2\,\mathrm{e}^{2x} c_1+3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{2\,\mathrm{e}^{2x} c_1+3}$$

$$y(x) = -\frac{\left(1+i\sqrt{3}\right) 2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2\,\mathrm{e}^{2x} c_1+3)^2\right)^{\frac{1}{3}} \mathrm{e}^{-x}}{4\,\mathrm{e}^{2x} c_1+6}$$

$$y(x) = \frac{2^{\frac{1}{3}} \left(\mathrm{e}^{2x} (2\,\mathrm{e}^{2x} c_1+3)^2\right)^{\frac{1}{3}} \left(i\sqrt{3}-1\right) \mathrm{e}^{-x}}{4\,\mathrm{e}^{2x} c_1+6}$$

✓ Solution by Mathematica

Time used: 4.751 (sec). Leaf size: 90

```
DSolve[y'[x]+y[x]==y[x]^4*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-2}}{\sqrt[3]{e^x(3+2c_1e^{2x})}}$$
$$y(x) \to \frac{1}{\sqrt[3]{\frac{3e^x}{2}+c_1e^{3x}}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{\frac{3e^x}{2}+c_1e^{3x}}}$$
$$y(x) \to 0$$

2.23 problem 23

Internal problem ID [5109]
Internal file name [`OUTPUT/4602_Sunday_June_05_2022_03_01_43_PM_1868891/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{2y' + y - y^3(x-1) = 0}$$

2.23.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{1}{2}y + \frac{1}{2}x\,y^3 - \frac{1}{2}y^3 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2}y + \frac{x}{2} - \frac{1}{2}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2} \\
f_1(x) &= \frac{x}{2} - \frac{1}{2} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{1}{2y^2} + \frac{x}{2} - \frac{1}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\frac{w(x)}{2} + \frac{x}{2} - \frac{1}{2} \\
w' &= w + 1 - x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -1 \\
q(x) &= 1 - x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - w(x) = 1 - x
$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(1-x) \\ \frac{d}{dx}(e^{-x} w) &= (e^{-x})(1-x) \\ d(e^{-x} w) &= (-(x-1)e^{-x})\ dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x} w &= \int -(x-1)e^{-x}\, dx \\ e^{-x} w &= x\,e^{-x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$w(x) = e^{x} x\,e^{-x} + c_1 e^{x}$$

which simplifies to

$$w(x) = x + c_1 e^{x}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = x + c_1 e^{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{x + c_1 e^{x}}} \\ y(x) &= -\frac{1}{\sqrt{x + c_1 e^{x}}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{x + c_1 e^{x}}} \tag{1}$$

$$y = -\frac{1}{\sqrt{x + c_1 e^{x}}} \tag{2}$$

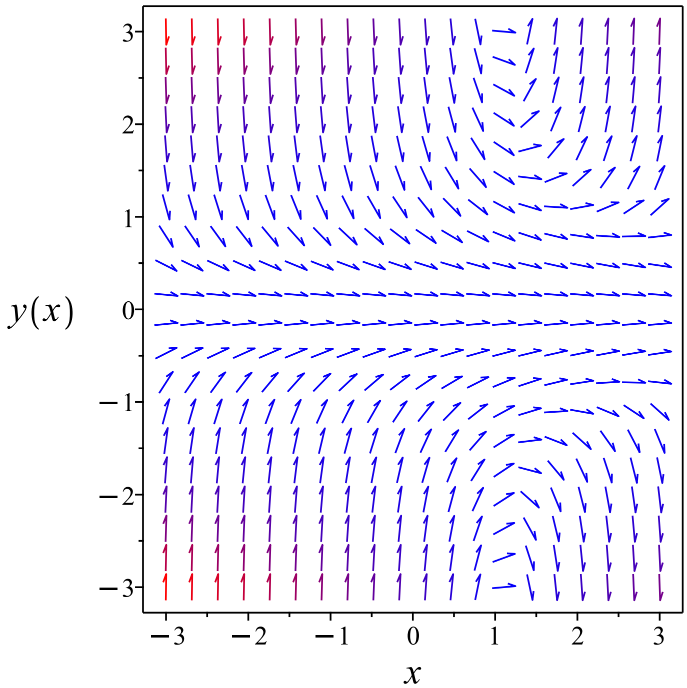


Figure 61: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{x + c_1 e^x}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{x + c_1 e^x}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(2*diff(y(x),x)+y(x)=y(x)^3*(x-1),y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{e^{x} c_1 + x}}$$
$$y(x) = -\frac{1}{\sqrt{e^{x} c_1 + x}}$$

✓ Solution by Mathematica

Time used: 2.721 (sec). Leaf size: 40

```
DSolve[2*y'[x]+y[x]==y[x]^3*(x-1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{x + c_1 e^x}}$$
$$y(x) \to \frac{1}{\sqrt{x + c_1 e^x}}$$
$$y(x) \to 0$$

2.24 problem 24

Internal problem ID [5110]
Internal file name [`OUTPUT/4603_Sunday_June_05_2022_03_01_45_PM_10672043/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' - 2y\tan(x) - \tan(x)^2 y^2 = 0}$$

2.24.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 2y\tan(x) + \tan(x)^2 y^2 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = 2\tan(x)\, y + \tan(x)^2 y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 2\tan(x) \\
f_1(x) &= \tan(x)^2 \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{2\tan(x)}{y} + \tan(x)^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= 2\tan(x)\, w(x) + \tan(x)^2 \\
w' &= -2\tan(x)\, w - \tan(x)^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2\tan(x) \\
q(x) &= -\tan(x)^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 2\tan(x)\, w(x) = -\tan(x)^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2\tan(x)dx} \\
&= \frac{1}{\cos(x)^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\tan(x)^2\right) \\
\frac{d}{dx}\left(\frac{w}{\cos(x)^2}\right) &= \left(\frac{1}{\cos(x)^2}\right)\left(-\tan(x)^2\right) \\
d\left(\frac{w}{\cos(x)^2}\right) &= \left(-\tan(x)^2\sec(x)^2\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{\cos(x)^2} &= \int -\tan(x)^2\sec(x)^2\,dx \\
\frac{w}{\cos(x)^2} &= -\frac{\tan(x)^3}{3} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\cos(x)^2}$ results in

$$
w(x) = -\frac{\cos(x)^2\tan(x)^3}{3} + c_1\cos(x)^2
$$

which simplifies to

$$
w(x) = \cos(x)^2\left(-\frac{\tan(x)^3}{3} + c_1\right)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = \cos(x)^2\left(-\frac{\tan(x)^3}{3} + c_1\right)
$$

Or

$$
y = \frac{1}{\cos(x)^2\left(-\frac{\tan(x)^3}{3} + c_1\right)}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{\cos(x)^2\left(-\frac{\tan(x)^3}{3} + c_1\right)} \tag{1}
$$

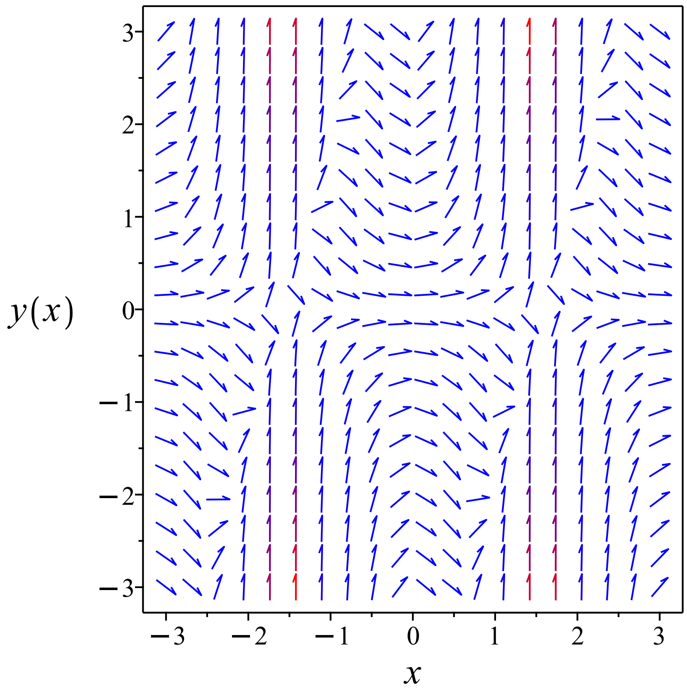


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{1}{\cos(x)^2 \left(-\frac{\tan(x)^3}{3} + c_1\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x)-2*y(x)*tan(x)=y(x)^2*tan(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{3\sec\left(x\right)^{2}}{\tan\left(x\right)^{3}-3c_{1}}$$

✓ Solution by Mathematica

Time used: 0.519 (sec). Leaf size: 31

```
DSolve[y'[x]-2*y[x]*Tan[x]==y[x]^2*Tan[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3}{-\sin^{2}(x)\tan(x)+3c_{1}\cos^{2}(x)}$$
$$y(x) \to 0$$

2.25 problem 25

Internal problem ID [5111]
Internal file name [`OUTPUT/4604_Sunday_June_05_2022_03_01_46_PM_93199263/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program :

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{y' + y\tan(x) - y^3\sec(x)^4 = 0}$$

2.25.1 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y\tan(x) + y^3\sec(x)^4 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\tan(x)\, y + \sec(x)^4\, y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\tan(x) \\
f_1(x) &= \sec(x)^4 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y' \frac{1}{y^3} = -\frac{\tan(x)}{y^2} + \sec(x)^4 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= -\tan(x)\, w(x) + \sec(x)^4 \\
w' &= 2\tan(x)\, w - 2\sec(x)^4
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -2\tan(x) \\
q(x) &= -2\sec(x)^4
\end{aligned}
$$

Hence the ode is

$$
w'(x) - 2\tan(x)\, w(x) = -2\sec(x)^4
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2\tan(x)dx} \\
&= \cos(x)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-2\sec(x)^4\right) \\
\frac{d}{dx}\left(\cos(x)^2 w\right) &= \left(\cos(x)^2\right)\left(-2\sec(x)^4\right) \\
d\left(\cos(x)^2 w\right) &= \left(-2\sec(x)^2\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\cos(x)^2 w &= \int -2\sec(x)^2 \, dx \\
\cos(x)^2 w &= -2\tan(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \cos(x)^2$ results in

$$
w(x) = -2\sec(x)^2 \tan(x) + c_1 \sec(x)^2
$$

which simplifies to

$$
w(x) = \sec(x)^2 (-2\tan(x) + c_1)
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = \sec(x)^2 (-2\tan(x) + c_1)
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{\cos(x) c_1 - 2\sin(x)} \\
y(x) &= \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{-\cos(x) c_1 + 2\sin(x)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{\cos(x) c_1 - 2\sin(x)} \tag{1}$$

$$y = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{-\cos(x) c_1 + 2\sin(x)} \tag{2}$$

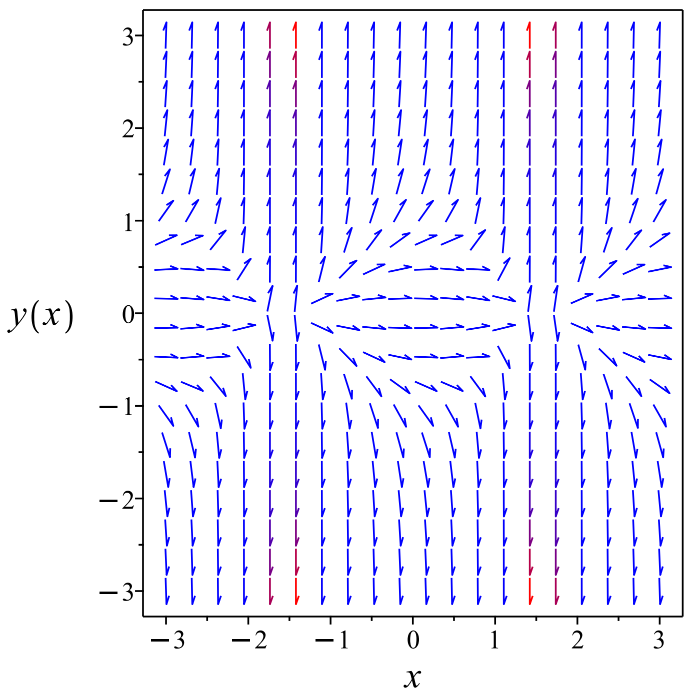


Figure 63: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{\cos(x) c_1 - 2\sin(x)}$$

Verified OK.

$$y = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{-\cos(x) c_1 + 2\sin(x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 68

```
dsolve(diff(y(x),x)+y(x)*tan(x)=y(x)^3*sec(x)^4,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{-\cos(x) c_1 + 2\sin(x)}$$
$$y(x) = \frac{\sqrt{\cos(x)^5 (\cos(x) c_1 - 2\sin(x))}\, \sec(x)}{\cos(x) c_1 - 2\sin(x)}$$

✓ Solution by Mathematica

Time used: 4.061 (sec). Leaf size: 48

```
DSolve[y'[x]+y[x]*Tan[x]==y[x]^3*Sec[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{\sec^2(x)(-2\tan(x) + c_1)}}$$
$$y(x) \to \frac{1}{\sqrt{\sec^2(x)(-2\tan(x) + c_1)}}$$
$$y(x) \to 0$$

2.26 problem 26

Internal problem ID [5112]
Internal file name [`OUTPUT/4605_Sunday_June_05_2022_03_01_49_PM_88824046/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(-x^2+1)\,y' - xy = 1}$$

2.26.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{x}{x^2-1}$$
$$q(x) = -\frac{1}{x^2-1}$$

Hence the ode is

$$y' + \frac{xy}{x^2-1} = -\frac{1}{x^2-1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x}{x^2-1}dx} \\
&= \mathrm{e}^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}}
\end{aligned}
$$

Which simplifies to

$$
\mu = \sqrt{x-1}\,\sqrt{x+1}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{1}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x-1}\,\sqrt{x+1}\,y\right) &= \left(\sqrt{x-1}\,\sqrt{x+1}\right)\left(-\frac{1}{x^2-1}\right) \\
\mathrm{d}\left(\sqrt{x-1}\,\sqrt{x+1}\,y\right) &= \left(-\frac{\sqrt{x-1}\,\sqrt{x+1}}{x^2-1}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{x-1}\,\sqrt{x+1}\,y &= \int -\frac{\sqrt{x-1}\,\sqrt{x+1}}{x^2-1}\,\mathrm{d}x \\
\sqrt{x-1}\,\sqrt{x+1}\,y &= -\frac{\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x-1}\,\sqrt{x+1}$ results in

$$
y = -\frac{\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}} \tag{1}
$$

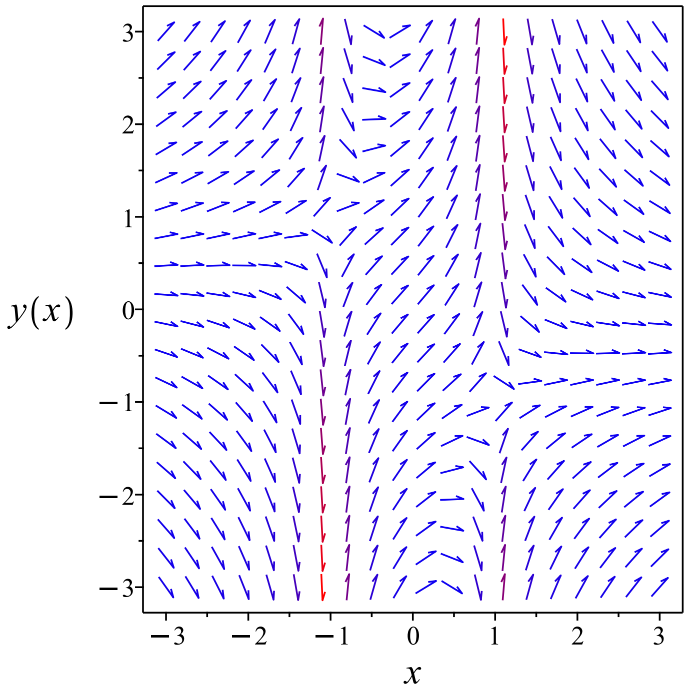


Figure 64: Slope field plot

Verification of solutions

$$y = -\frac{\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

Verified OK.

2.26.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{xy+1}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 39: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-\frac{\ln(x-1)}{2}-\frac{\ln(x+1)}{2}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\ln(\sqrt{x-1}) + \ln(\sqrt{x+1})} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{xy+1}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x-1}\,\sqrt{x+1}} \\ S_y &= \sqrt{x-1}\,\sqrt{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{\sqrt{x-1}\,\sqrt{x+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{\sqrt{R-1}\,\sqrt{R+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\sqrt{(R-1)(R+1)}\,\ln\left(R+\sqrt{R^2-1}\right)}{\sqrt{R-1}\,\sqrt{R+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\sqrt{x+1}\,\sqrt{x-1} = -\frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

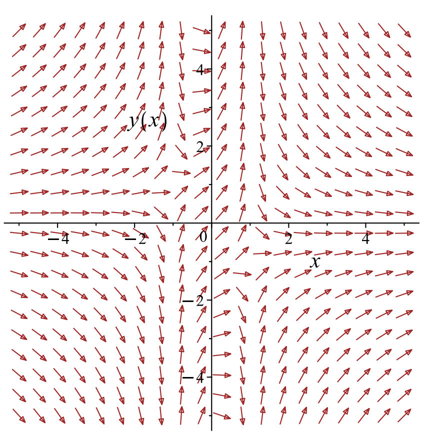
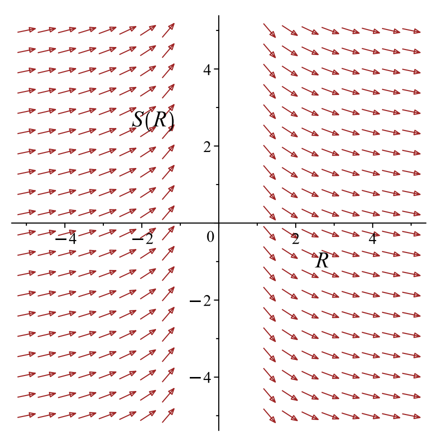
Which simplifies to

$$y\sqrt{x+1}\,\sqrt{x-1} = -\frac{\sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x-1}\,\sqrt{x+1}} + c_1$$

Which gives

$$y = \frac{c_1\sqrt{x-1}\,\sqrt{x+1} - \sqrt{(x-1)(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{x^2-1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+1}{x^2-1}$ 	$R = x$ $S = \sqrt{x-1}\,\sqrt{x+1}\,y$	$\frac{dS}{dR} = -\frac{1}{\sqrt{R-1}\,\sqrt{R+1}}$ 

Summary

The solution(s) found are the following

$$y = \frac{c_1\sqrt{x-1}\,\sqrt{x+1} - \sqrt{(x-1)\,(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{x^2-1} \tag{1}$$

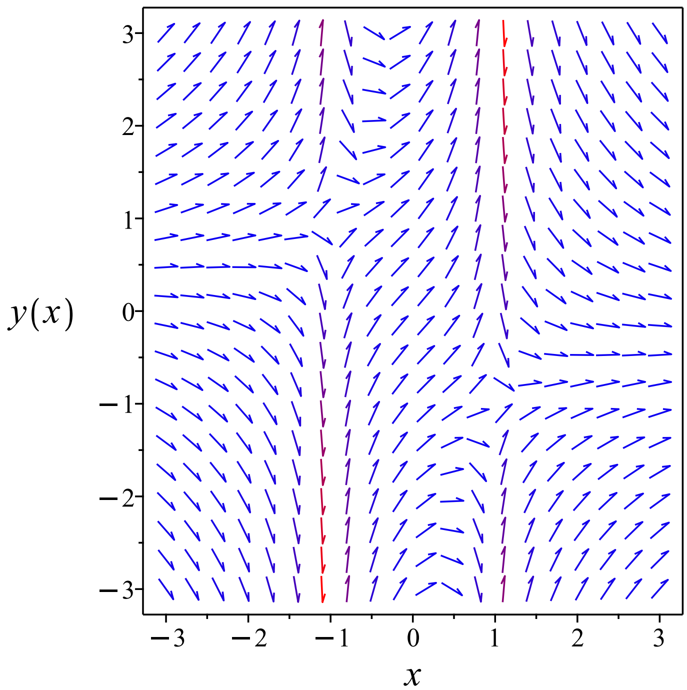


Figure 65: Slope field plot

Verification of solutions

$$y = \frac{c_1\sqrt{x-1}\,\sqrt{x+1} - \sqrt{(x-1)\,(x+1)}\,\ln\left(x+\sqrt{x^2-1}\right)}{x^2-1}$$

Verified OK.

2.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\\frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-x^2+1\right)\mathrm{d}y&=(xy+1)\,\mathrm{d}x\\(-xy-1)\,\mathrm{d}x+\left(-x^2+1\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-xy-1\\N(x,y)&=-x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-xy-1) \\
&= -x
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x^2+1) \\
&= -2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x^2-1}((-x)-(-2x)) \\
&= -\frac{x}{x^2-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{x}{x^2-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(x-1)}{2} - \frac{\ln(x+1)}{2}} \\
&= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}(-xy-1) \\
&= -\frac{xy+1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{x-1}\,\sqrt{x+1}}\left(-x^2+1\right) \\
&= \frac{-x^2+1}{\sqrt{x-1}\,\sqrt{x+1}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{xy+1}{\sqrt{x-1}\,\sqrt{x+1}}\right) + \left(\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{x+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{xy+1}{\sqrt{x-1}\,\sqrt{x+1}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\sqrt{x-1}\,\sqrt{x+1}\left(y\sqrt{x^2-1} + \ln\left(x+\sqrt{x^2-1}\right)\right)}{\sqrt{x^2-1}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -\sqrt{x-1}\,\sqrt{x+1} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{-x^2+1}{\sqrt{x-1}\,\sqrt{x+1}}$. Therefore equation (4) becomes

$$
\frac{-x^2+1}{\sqrt{x-1}\,\sqrt{x+1}} = -\sqrt{x-1}\,\sqrt{x+1} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{x-1}\,\sqrt{x+1}\,\left(y\sqrt{x^2-1} + \ln\left(x + \sqrt{x^2-1}\right)\right)}{\sqrt{x^2-1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{x-1}\,\sqrt{x+1}\,\left(y\sqrt{x^2-1} + \ln\left(x + \sqrt{x^2-1}\right)\right)}{\sqrt{x^2-1}}$$

The solution becomes

$$y = -\frac{\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x + \sqrt{x^2-1}\right) + c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{x+1}}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x + \sqrt{x^2-1}\right) + c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{x+1}} \tag{1}$$

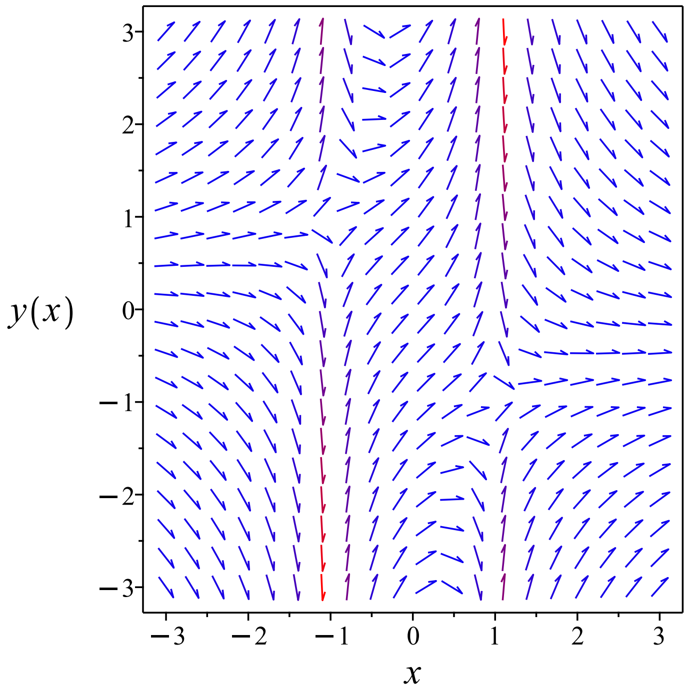


Figure 66: Slope field plot

Verification of solutions

$$y = -\frac{\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{x+1}}$$

Verified OK.

2.26.4 Maple step by step solution

Let's solve

$(-x^2+1)\,y'-xy=1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{xy}{x^2-1}-\frac{1}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{xy}{x^2-1}=-\frac{1}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{xy}{x^2-1}\right)=-\frac{\mu(x)}{x^2-1}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{xy}{x^2-1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{\mu(x)x}{x^2-1}$$

- Solve to find the integrating factor

$$\mu(x)=\sqrt{x-1}\,\sqrt{x+1}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int-\frac{\mu(x)}{x^2-1}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int-\frac{\mu(x)}{x^2-1}dx+c_1$$

- Solve for y

$$y=\frac{\int-\frac{\mu(x)}{x^2-1}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x)=\sqrt{x-1}\,\sqrt{x+1}$

$$y=\frac{\int-\frac{\sqrt{x-1}\,\sqrt{x+1}}{x^2-1}dx+c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

- Evaluate the integrals on the rhs

$$y=\frac{-\frac{\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}+c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

- Simplify

$$y=\frac{-\sqrt{x+1}\,\sqrt{x-1}\,\ln\left(x+\sqrt{x^2-1}\right)+c_1\sqrt{x^2-1}}{\sqrt{x^2-1}\,\sqrt{x-1}\,\sqrt{x+1}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 46

```
dsolve((1-x^2)*diff(y(x),x)=1+x*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{x^2-1}\,\ln\left(x+\sqrt{x^2-1}\right)}{(x-1)\,(x+1)} + \frac{c_1}{\sqrt{x-1}\,\sqrt{x+1}}$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 54

```
DSolve[(1-x^2)*y'[x]==1+x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log\left(1-\frac{x}{\sqrt{x^2-1}}\right) - \log\left(\frac{x}{\sqrt{x^2-1}}+1\right) + 2c_1}{2\sqrt{x^2-1}}$$

2.27 problem 27

Internal problem ID [5113]
Internal file name [`OUTPUT/4606_Sunday_June_05_2022_03_01_50_PM_99954190/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{xyy' - (x+1)\sqrt{y-1} = 0}$$

2.27.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x+1)\sqrt{y-1}}{yx} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(y) = \frac{\sqrt{y-1}}{y}$. Integrating both sides gives

$$\frac{1}{\frac{\sqrt{y-1}}{y}}\,dy = \frac{x+1}{x}\,dx$$

$$\int \frac{1}{\frac{\sqrt{y-1}}{y}}\,dy = \int \frac{x+1}{x}\,dx$$

$$\frac{2\sqrt{y-1}\,(y+2)}{3} = x+\ln(x)+c_1$$

The solution is

$$\frac{2\sqrt{y-1}\,(2+y)}{3} - x-\ln(x)-c_1 = 0$$

Summary

The solution(s) found are the following

$$\frac{2\sqrt{y-1}\,(2+y)}{3} - x-\ln(x)-c_1 = 0 \tag{1}$$

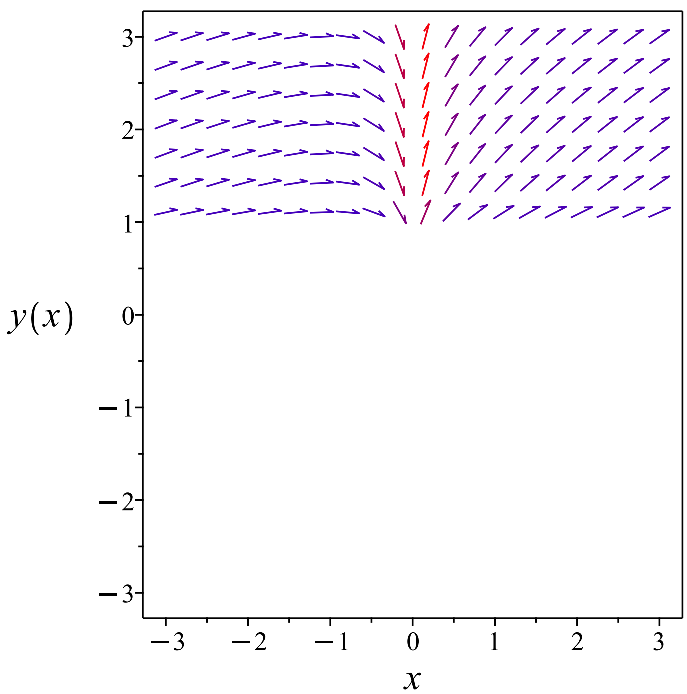


Figure 67: Slope field plot

Verification of solutions

$$\frac{2\sqrt{y-1}\,(2+y)}{3} - x-\ln(x)-c_1 = 0$$

Verified OK.

2.27.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{(x+1)\sqrt{y-1}}{yx} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 42: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x}{x+1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x}{x+1}} dx
\end{aligned}
$$

Which results in

$$
S = x + \ln (x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{(x+1)\sqrt{y-1}}{yx}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 1 + \frac{1}{x} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y}{\sqrt{y-1}} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R}{\sqrt{R-1}}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{2\sqrt{R-1}\,(R+2)}{3} + c_1 \tag{4}
$$

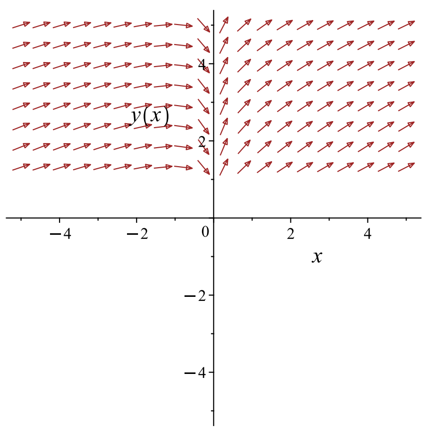
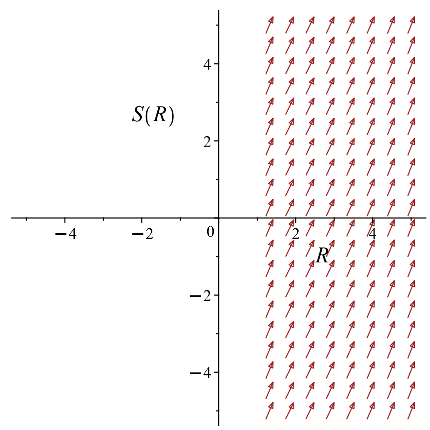
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x + \ln(x) = \frac{2\sqrt{y-1}\,(2+y)}{3} + c_1
$$

Which simplifies to

$$
x + \ln(x) = \frac{2\sqrt{y-1}\,(2+y)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(x+1)\sqrt{y-1}}{yx}$ 	$R = y$ $S = x + \ln(x)$	$\frac{dS}{dR} = \frac{R}{\sqrt{R-1}}$ 

Summary

The solution(s) found are the following

$$
x + \ln(x) = \frac{2\sqrt{y-1}\,(2+y)}{3} + c_1 \tag{1}
$$

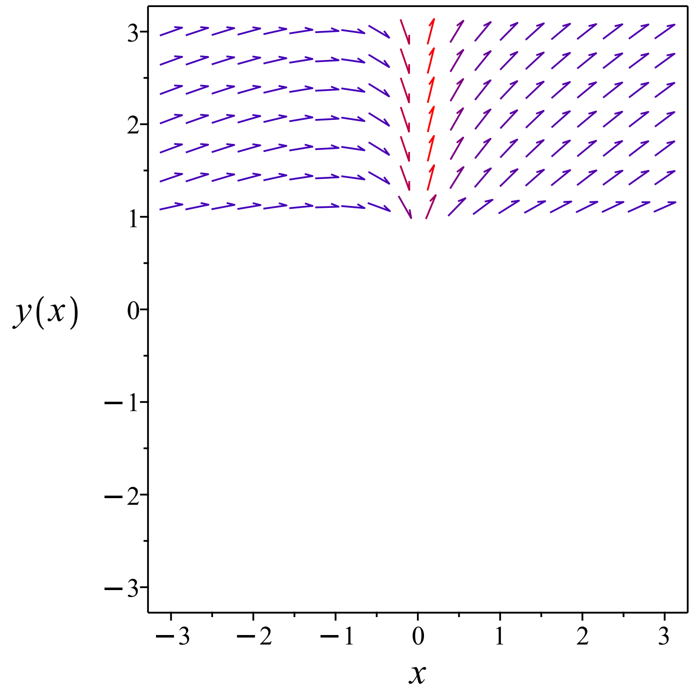


Figure 68: Slope field plot

<u>Verification of solutions</u>

$$x + \ln(x) = \frac{2\sqrt{y-1}\,(2+y)}{3} + c_1$$

Verified OK.

2.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{y}{\sqrt{y-1}}\right) \mathrm{d}y &= \left(\frac{x+1}{x}\right) \mathrm{d}x \\
\left(-\frac{x+1}{x}\right) \mathrm{d}x + \left(\frac{y}{\sqrt{y-1}}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{x+1}{x} \\
N(x, y) &= \frac{y}{\sqrt{y-1}}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x+1}{x}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{\sqrt{y-1}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x+1}{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x - \ln(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{\sqrt{y-1}}$. Therefore equation (4) becomes

$$
\frac{y}{\sqrt{y-1}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{\sqrt{y-1}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{y}{\sqrt{y-1}}\right) \mathrm{d}y \\
f(y) &= \frac{2\sqrt{y-1}\,(y+2)}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \ln(x) + \frac{2\sqrt{y-1}\,(y+2)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \ln(x) + \frac{2\sqrt{y-1}\,(y+2)}{3}$$

Summary

The solution(s) found are the following

$$\frac{2\sqrt{y-1}\,(2+y)}{3} - x - \ln(x) = c_1 \tag{1}$$

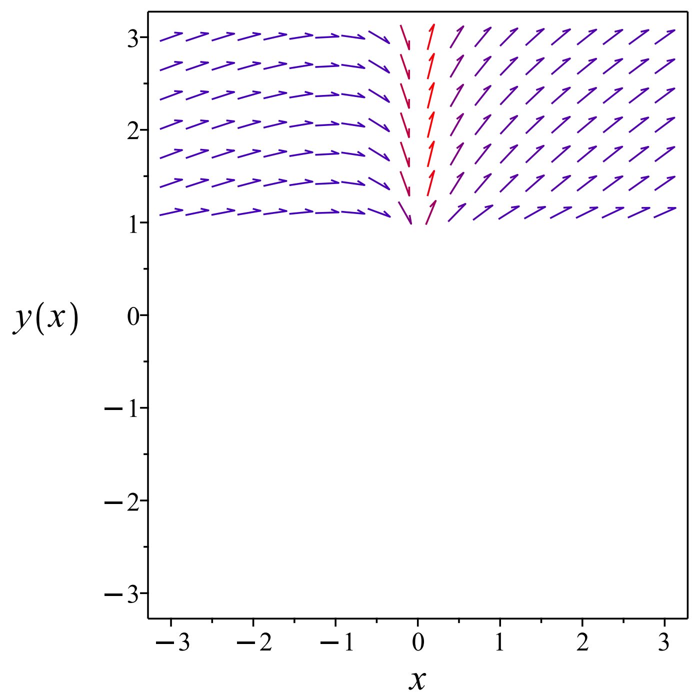


Figure 69: Slope field plot

Verification of solutions

$$\frac{2\sqrt{y-1}\,(2+y)}{3} - x - \ln(x) = c_1$$

Verified OK.

2.27.4 Maple step by step solution

Let's solve

$xyy' - (x+1)\sqrt{y-1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{\sqrt{y-1}} = \frac{x+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{\sqrt{y-1}}dx = \int \frac{x+1}{x}dx + c_1$

- Evaluate integral

  $\frac{2(y-1)^{\frac{3}{2}}}{3} + 2\sqrt{y-1} = x + \ln(x) + c_1$

- Solve for y

  $y = \left(\frac{\left(6\ln(x)+6c_1+6x+2\sqrt{16+9\ln(x)^2+18c_1\ln(x)+18\ln(x)x+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{2} - \frac{2}{\left(6\ln(x)+6c_1+6x+2\sqrt{16+9\ln(x)^2+18c_1}\right.}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(x*y(x)*diff(y(x),x)-(1+x)*sqrt(y(x)-1)=0,y(x), singsol=all)
```

$$\frac{(-2y(x)-4)\sqrt{y(x)-1}}{3} + x + c_1 + \ln(x) = 0$$

✓ Solution by Mathematica

Time used: 5.614 (sec). Leaf size: 582

```
DSolve[x*y[x]*y'[x]-(1+x)*Sqrt[y[x]-1]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(x)} + \frac{2}{\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(x)}} - 1$$

$$y(x) \to \frac{1}{4}i\left(\sqrt{3} + i\right)\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(} + \frac{-1 - i\sqrt{3}}{\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(x)}} - 1$$

$$y(x) \to -\frac{1}{4}i\left(\sqrt{3} - i\right)\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(} + \frac{-1 + i\sqrt{3}}{\sqrt[3]{9x^2 + 3\sqrt{(x + \log(x) + c_1)^2\left(9x^2 + 9\log^2(x) + 18c_1x + 18(x + c_1)\log(x) + 16 + 9c_1{}^2\right)} + 9\log^2(x)}} - 1$$

$$y(x) \to 1$$

2.28 problem 28

Internal problem ID [5114]
Internal file name [`OUTPUT/4607_Sunday_June_05_2022_03_01_51_PM_81823814/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{-2xy + 5y^2 - (x^2 + 2xy + y^2)\, y' = -x^2}$$

2.28.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-2x^2u(x) + 5u(x)^2\, x^2 - \left(x^2 + 2x^2u(x) + u(x)^2\, x^2\right)\left(u'(x)\, x + u(x)\right) = -x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{(u-1)^3}{x\,(u+1)^2} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{(u-1)^3}{(u+1)^2}$. Integrating both sides gives

$$\frac{1}{\frac{(u-1)^3}{(u+1)^2}}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{\frac{(u-1)^3}{(u+1)^2}}\,du = \int -\frac{1}{x}\,dx$$

$$\ln(u-1) - \frac{4}{u-1} - \frac{2}{(u-1)^2} = -\ln(x) + c_2$$

The solution is

$$\ln(u(x)-1) - \frac{4}{u(x)-1} - \frac{2}{(u(x)-1)^2} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}-1\right) - \frac{4}{\frac{y}{x}-1} - \frac{2}{\left(\frac{y}{x}-1\right)^2} + \ln(x) - c_2 = 0$$

$$\ln\left(\frac{-x+y}{x}\right) - \frac{4x}{-x+y} - \frac{2x^2}{(-x+y)^2} + \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\ln\left(\frac{-x+y}{x}\right) - \frac{4x}{-x+y} - \frac{2x^2}{(-x+y)^2} + \ln(x) - c_2 = 0 \tag{1}$$

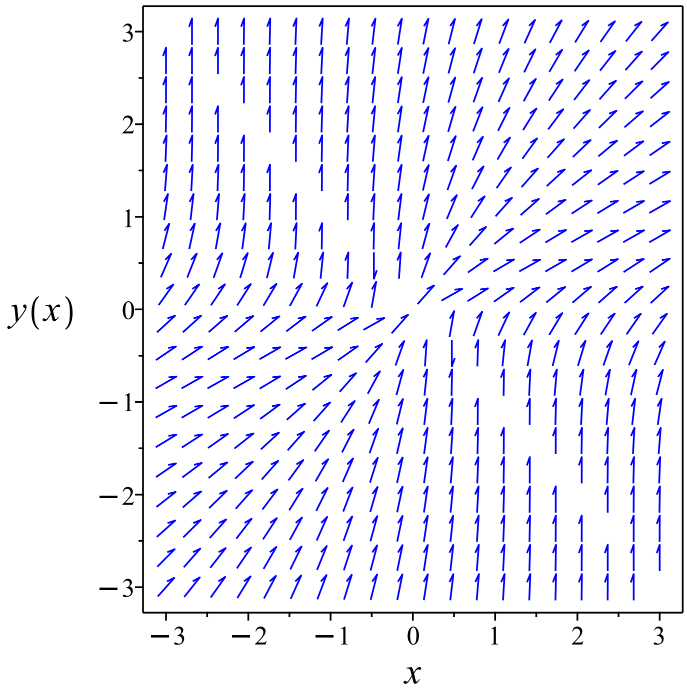


Figure 70: Slope field plot

Verification of solutions

$$\ln\left(\frac{-x+y}{x}\right)-\frac{4x}{-x+y}-\frac{2x^2}{(-x+y)^2}+\ln(x)-c_2=0$$

Verified OK.

2.28.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{x^2-2xy+5y^2}{x^2+2xy+y^2}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(x^2 - 2xy + 5y^2)(b_3 - a_2)}{x^2 + 2xy + y^2} - \frac{(x^2 - 2xy + 5y^2)^2 a_3}{(x^2 + 2xy + y^2)^2} - \left(\frac{-2y + 2x}{x^2 + 2xy + y^2} - \frac{(x^2 - 2xy + 5y^2)(2y + 2x)}{(x^2 + 2xy + y^2)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{-2x + 10y}{x^2 + 2xy + y^2} - \frac{(x^2 - 2xy + 5y^2)(2y + 2x)}{(x^2 + 2xy + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4a_2 + x^4a_3 - 5x^4b_2 - x^4b_3 + 4x^3ya_2 - 4x^3ya_3 + 4x^3yb_2 - 4x^3yb_3 - 6x^2y^2a_2 + 18x^2y^2a_3 + 6x^2y^2b_2 + 6}{}$$

$= 0$

Setting the numerator to zero gives

$$-x^4a_2 - x^4a_3 + 5x^4b_2 + x^4b_3 - 4x^3ya_2 + 4x^3ya_3 - 4x^3yb_2 + 4x^3yb_3 + 6x^2y^2a_2 - 18x^2y^2a_3 - 6x^2y^2b_2 - 6x^2y^2b_3 + 4x\,y^3a_2 + 28x\,y^3a_3 + 4x\,y^3b_2 - 4x\,y^3b_3 - 5y^4a_2 - 13y^4a_3 + y^4b_2 + 5y^4b_3 + 4x^3b_1 - 4x^2ya_1 - 8x^2yb_1 + 8x\,y^2a_1 - 12x\,y^2b_1 + 12y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2v_1^4 - 4a_2v_1^3v_2 + 6a_2v_1^2v_2^2 + 4a_2v_1v_2^3 - 5a_2v_2^4 - a_3v_1^4 + 4a_3v_1^3v_2 - 18a_3v_1^2v_2^2 + 28a_3v_1v_2^3 - 13a_3v_2^4 + 5b_2v_1^4 - 4b_2v_1^3v_2 - 6b_2v_1^2v_2^2 + 4b_2v_1v_2^3 + b_2v_2^4 + b_3v_1^4 + 4b_3v_1^3v_2 - 6b_3v_1^2v_2^2 - 4b_3v_1v_2^3 + 5b_3v_2^4 - 4a_1v_1^2v_2 + 8a_1v_1v_2^2 + 12a_1v_2^3 + 4b_1v_1^3 - 8b_1v_1^2v_2 - 12b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(-a_2 - a_3 + 5b_2 + b_3) v_1^4 + (-4a_2 + 4a_3 - 4b_2 + 4b_3) v_1^3 v_2 \\
&+ 4b_1 v_1^3 + (6a_2 - 18a_3 - 6b_2 - 6b_3) v_1^2 v_2^2 + (-4a_1 - 8b_1) v_1^2 v_2 \\
&+ (4a_2 + 28a_3 + 4b_2 - 4b_3) v_1 v_2^3 + (8a_1 - 12b_1) v_1 v_2^2 \\
&+ (-5a_2 - 13a_3 + b_2 + 5b_3) v_2^4 + 12a_1 v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
12a_1 &= 0 \\
4b_1 &= 0 \\
-4a_1 - 8b_1 &= 0 \\
8a_1 - 12b_1 &= 0 \\
-5a_2 - 13a_3 + b_2 + 5b_3 &= 0 \\
-4a_2 + 4a_3 - 4b_2 + 4b_3 &= 0 \\
-a_2 - a_3 + 5b_2 + b_3 &= 0 \\
4a_2 + 28a_3 + 4b_2 - 4b_3 &= 0 \\
6a_2 - 18a_3 - 6b_2 - 6b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{x^2 - 2xy + 5y^2}{x^2 + 2xy + y^2}\right)(x) \\
&= \frac{-x^3 + 3y\,x^2 - 3y^2x + y^3}{x^2 + 2xy + y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^3+3y\,x^2-3y^2x+y^3}{x^2+2xy+y^2}} dy
\end{aligned}$$

Which results in

$$S = -\frac{2x^2}{(-x+y)^2} + \ln(-x+y) - \frac{4x}{-x+y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 - 2xy + 5y^2}{x^2 + 2xy + y^2}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^2 - 2xy + 5y^2}{(x-y)^3} \\
S_y &= -\frac{(x+y)^2}{(x-y)^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{(x-y)^2 \ln(-x+y) + 2x(x-2y)}{(x-y)^2} = c_1
$$

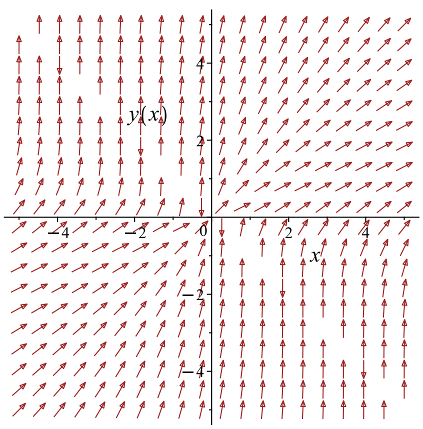
Which simplifies to

$$
\frac{(x-y)^2 \ln(-x+y) + 2x(x-2y)}{(x-y)^2} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr>
<th>Original ode in x, y coordinates</th>
<th>Canonical coordinates transformation</th>
<th>ODE in canonical coordinates (R, S)</th>
</tr>
<tr>
<td>

$\frac{dy}{dx} = \frac{x^2-2xy+5y^2}{x^2+2xy+y^2}$



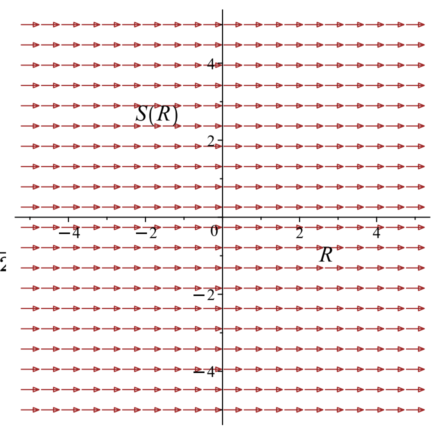
</td>
<td>

$R = x$

$S = \frac{(x-y)^2 \ln(-x+y)}{(x-y)^2}$

</td>
<td>

$\frac{dS}{dR} = 0$



</td>
</tr>
</table>

Summary

The solution(s) found are the following

$$\frac{(x-y)^2 \ln(-x+y) + 2x(x-2y)}{(x-y)^2} = c_1 \tag{1}$$

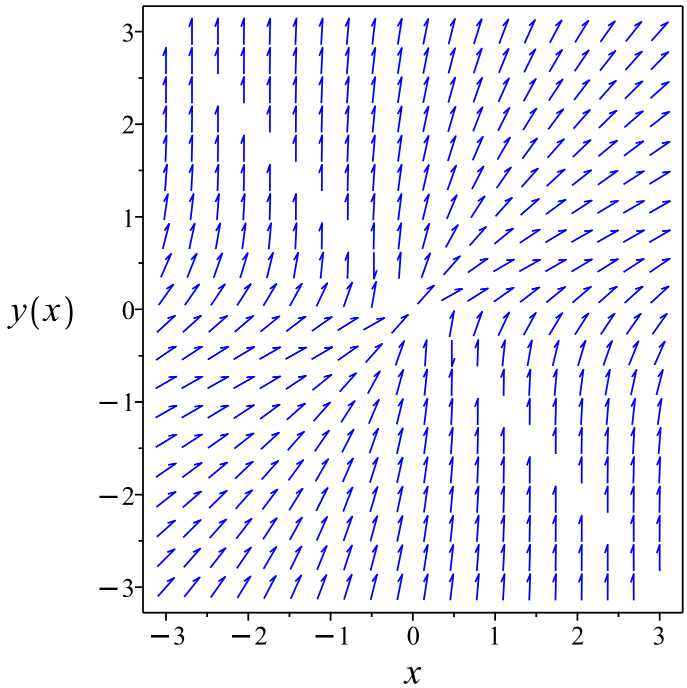


Figure 71: Slope field plot

Verification of solutions

$$\frac{(x-y)^2 \ln(-x+y) + 2x(x-2y)}{(x-y)^2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 35

```
dsolve((x^2-2*x*y(x)+5*y(x)^2)=(x^2+2*x*y(x)+y(x)^2)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = x\left(1 + e^{\mathrm{RootOf}\left(\ln(x)e^{2\_Z}+c_1e^{2\_Z}+\_Ze^{2\_Z}-4\,e^{\_Z}-2\right)}\right)$$

✓ Solution by Mathematica

Time used: 0.343 (sec). Leaf size: 41

```
DSolve[(x^2-2*x*y[x]+5*y[x]^2)==(x^2+2*x*y[x]+y[x]^2)*y'[x],y[x],x,IncludeSingularSolutions
```

$$\mathrm{Solve}\left[\frac{2-\frac{4y(x)}{x}}{\left(\frac{y(x)}{x}-1\right)^2}+\log\left(\frac{y(x)}{x}-1\right) = -\log(x)+c_1, y(x)\right]$$

2.29 problem 29

Internal problem ID [5115]
Internal file name [`OUTPUT/4608_Sunday_June_05_2022_03_01_52_PM_37720997/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y' - y\cot(x) - y^2 \sec(x)^2 = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = -1\right]$$

2.29.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= y\cot(x) + y^2 \sec(x)^2 \end{aligned}$$

The x domain of $f(x,y)$ when $y=-1$ is

$$\left\{-\infty \le x < \pi\_Z101, \pi\_Z101 < x < \frac{1}{2}\pi + \pi\_Z102, \frac{1}{2}\pi + \pi\_Z102 < x \le \infty\right\}$$

But the point $x_0 = \frac{\pi}{4}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.29.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\cot(x) + y^2\sec(x)^2 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 45: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{\sin(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^2}{\sin(x)}} dy
\end{aligned}$$

Which results in

$$S = -\frac{\sin(x)}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y\cot(x) + y^2 \sec(x)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\cos(x)}{y} \\
S_y &= \frac{\sin(x)}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sec(x)\tan(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sec(R)\tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sec(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sin(x)}{y} = \sec(x) + c_1$$

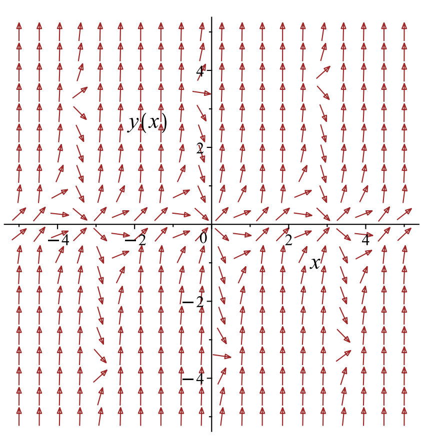
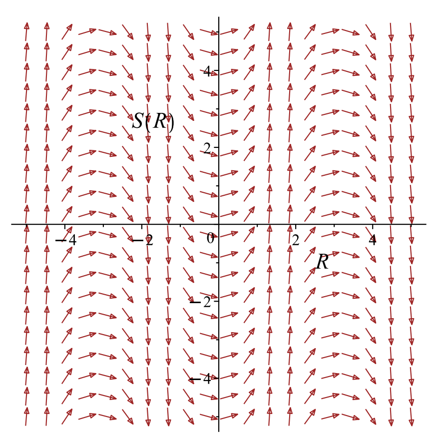
Which simplifies to

$$-\frac{\sin(x)}{y} = \sec(x) + c_1$$

Which gives

$$y = -\frac{\sin(x)}{\sec(x) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y\cot(x) + y^2 \sec(x)^2$ 	$R = x$ $S = -\frac{\sin(x)}{y}$	$\frac{dS}{dR} = \sec(R)\tan(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\sqrt{2}}{2\sqrt{2} + 2c_1}$$

$$c_1 = -\frac{\sqrt{2}}{2}$$

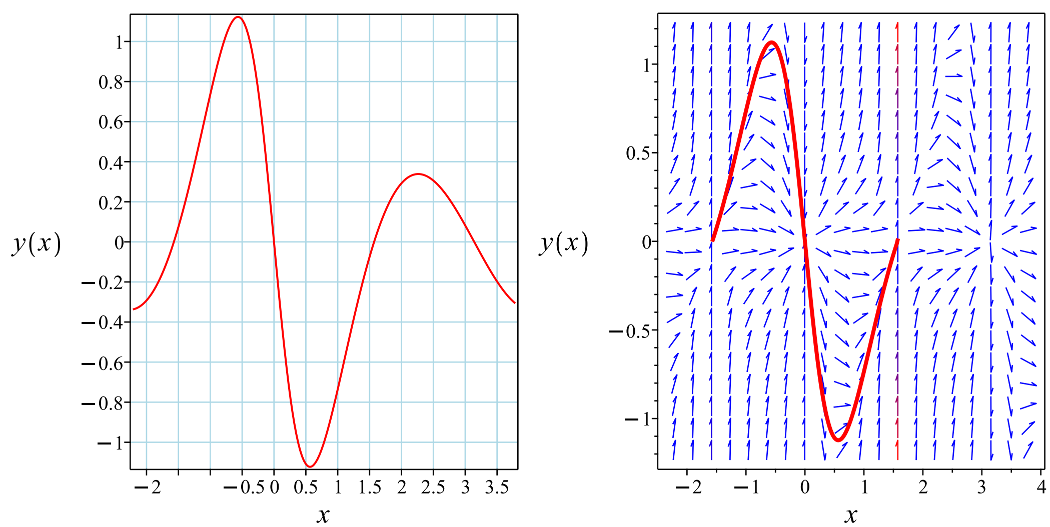
Substituting c_1 found above in the general solution gives

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}}$$

Verified OK.

2.29.3 Solving as bernoulli ode

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y)\\
&= y\cot(x) + y^2\sec(x)^2
\end{aligned}
$$

This is a Bernoulli ODE.

$$
y' = \cot(x)\,y + \sec(x)^2\,y^2 \tag{1}
$$

The standard Bernoulli ODE has the form

$$
y' = f_0(x)y + f_1(x)y^n \tag{2}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \cot(x)\\
f_1(x) &= \sec(x)^2\\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$
y'\frac{1}{y^2} = \frac{\cot(x)}{y} + \sec(x)^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n}\\
&= \frac{1}{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{1}{y^2}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-w'(x) &= \cot(x)\, w(x) + \sec(x)^2 \\
w' &= -\cot(x)\, w - \sec(x)^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \cot(x) \\
q(x) &= -\sec(x)^2
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \cot(x)\, w(x) = -\sec(x)^2
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \cot(x) dx} \\
&= \sin(x)
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\sec(x)^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)\, w) &= (\sin(x))\left(-\sec(x)^2\right) \\
\mathrm{d}(\sin(x)\, w) &= (-\sec(x)\tan(x))\, \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(x)\, w &= \int -\sec(x)\tan(x)\, \mathrm{d}x \\
\sin(x)\, w &= -\sec(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
w(x) = -\sec(x)\csc(x) + c_1 \csc(x)
$$

which simplifies to

$$
w(x) = \csc(x)\left(-\sec(x) + c_1\right)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \csc(x)\left(-\sec(x) + c_1\right)$$

Or

$$y = \frac{1}{\csc(x)\left(-\sec(x) + c_1\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{\sqrt{2}}{2\sqrt{2} - 2c_1}$$

$$c_1 = \frac{\sqrt{2}}{2}$$

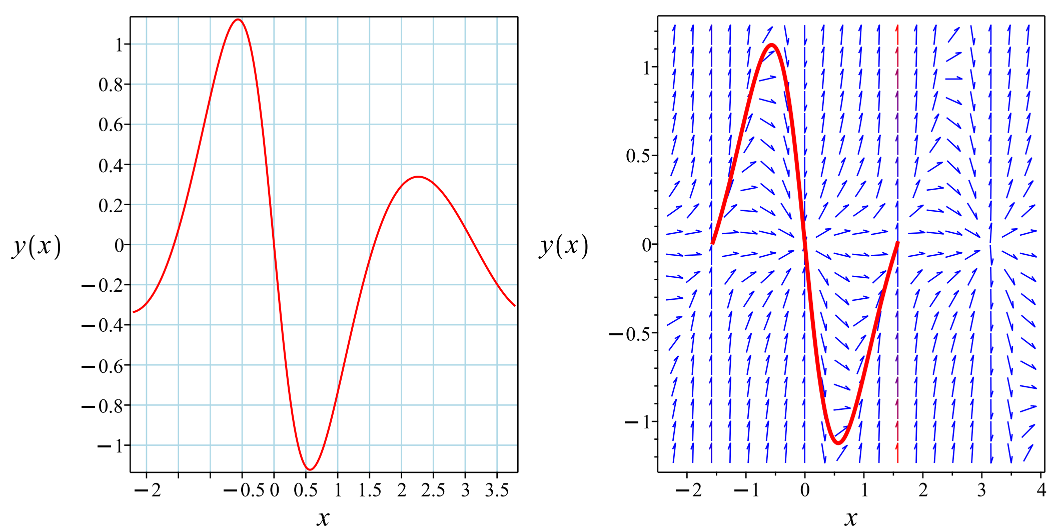
Substituting c_1 found above in the general solution gives

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{2 \sin (x)}{2 \sec (x) - \sqrt{2}}$$

Verified OK.

2.29.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= y \cot (x) + y^2 \sec (x)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = y \cot (x) + y^2 \sec (x)^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \cot (x)$ and $f_2(x) = \sec (x)^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\sec (x)^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2 \sec (x)^2 \tan (x) \\ f_1 f_2 &= \cot (x) \sec (x)^2 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\sec (x)^2 u''(x) - \left(2 \sec (x)^2 \tan (x) + \cot (x) \sec (x)^2\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x)=c_1+\sec(x)\,c_2
$$

The above shows that

$$
u'(x)=\sec(x)\tan(x)\,c_2
$$

Using the above in (1) gives the solution

$$
y=-\frac{\tan(x)\,c_2}{\sec(x)\,(c_1+\sec(x)\,c_2)}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1}=c_3$ the following solution

$$
y=-\frac{\sin(x)}{c_3+\sec(x)}
$$

Initial conditions are used to solve for c_3. Substituting $x=\frac{\pi}{4}$ and $y=-1$ in the above solution gives an equation to solve for the constant of integration.

$$
-1=-\frac{\sqrt{2}}{2c_3+2\sqrt{2}}
$$

$$
c_3=-\frac{\sqrt{2}}{2}
$$

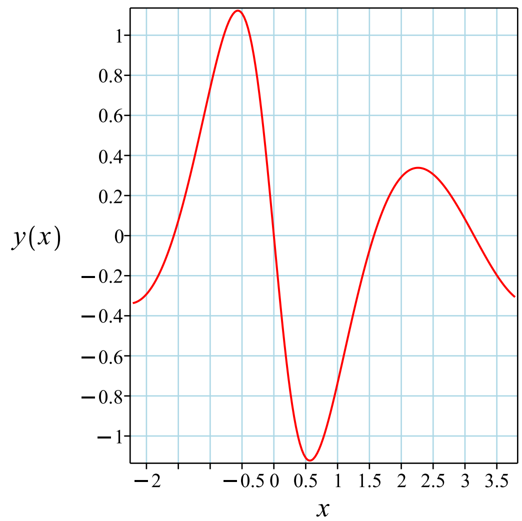
Substituting c_3 found above in the general solution gives

$$
y=-\frac{2\sin(x)}{2\sec(x)-\sqrt{2}}
$$

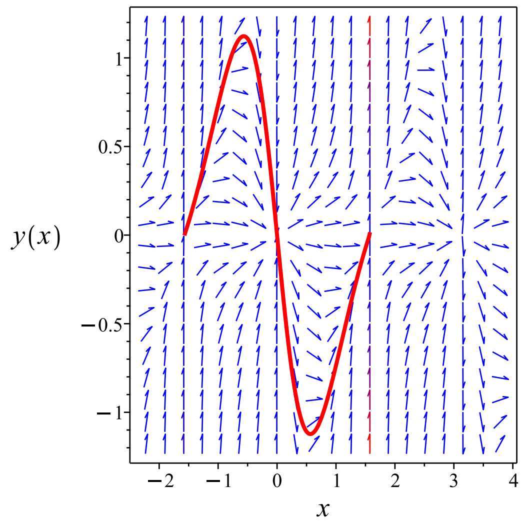
Summary

The solution(s) found are the following

$$
y=-\frac{2\sin(x)}{2\sec(x)-\sqrt{2}} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\sin(x)}{2\sec(x) - \sqrt{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.89 (sec). Leaf size: 18

```
dsolve([diff(y(x),x)-y(x)*cot(x)=y(x)^2*sec(x)^2,y(1/4*Pi) = -1],y(x), singsol=all)
```

$$y(x) = \frac{2\sin(x)}{\sqrt{2} - 2\sec(x)}$$

✓ Solution by Mathematica

Time used: 0.46 (sec). Leaf size: 22

```
DSolve[{y'[x]-y[x]*Cot[x]==y[x]^2*Sec[x]^2,{y[Pi/4]==-1}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{\sin(2x)}{\sqrt{2}\cos(x) - 2}$$

2.30 problem 30

Internal problem ID [5116]
Internal file name [`OUTPUT/4609_Sunday_June_05_2022_03_01_55_PM_74585404/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y + (x^2 - 4x)\, y' = 0}$$

2.30.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= -\frac{y}{x\,(-4 + x)} \end{aligned}$$

Where $f(x) = -\frac{1}{x(-4+x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= -\frac{1}{x\,(-4+x)}\,dx \\
\int \frac{1}{y}\,dy &= \int -\frac{1}{x\,(-4+x)}\,dx \\
\ln (y) &= \frac{\ln (x)}{4} - \frac{\ln (-4+x)}{4} + c_1 \\
y &= \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} + c_1} \\
&= c_1 \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4}}
\end{aligned}$$

Which simplifies to

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}} \tag{1}$$

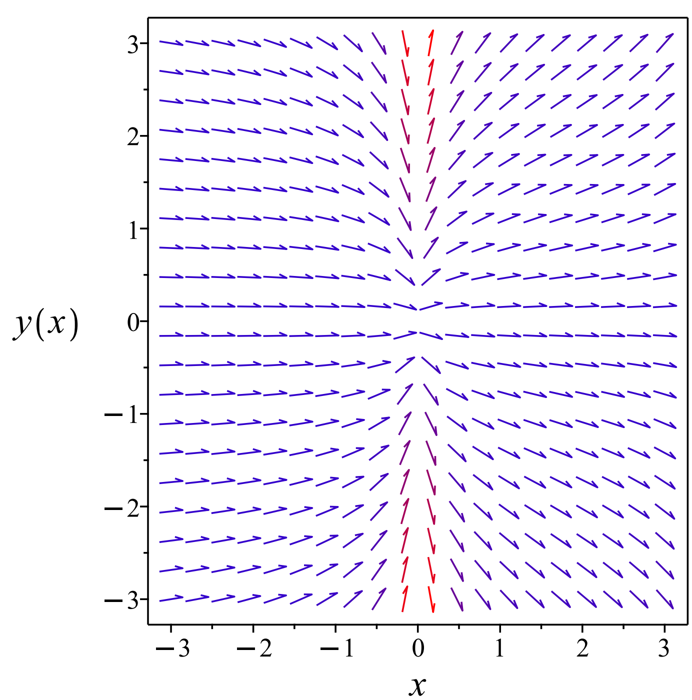


Figure 75: Slope field plot

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

Verified OK.

2.30.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x(-4+x)}$$
$$q(x) = 0$$

Hence the ode is

$$y' + \frac{y}{x(-4+x)} = 0$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int \frac{1}{x(-4+x)}dx}$$
$$= \mathrm{e}^{-\frac{\ln(x)}{4}+\frac{\ln(-4+x)}{4}}$$

Which simplifies to

$$\mu = \frac{(-4+x)^{\frac{1}{4}}}{x^{\frac{1}{4}}}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-4+x)^{\frac{1}{4}} y}{x^{\frac{1}{4}}}\right) = 0$$

Integrating gives

$$\frac{(-4+x)^{\frac{1}{4}} y}{x^{\frac{1}{4}}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{(-4+x)^{\frac{1}{4}}}{x^{\frac{1}{4}}}$ results in

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}} \tag{1}$$

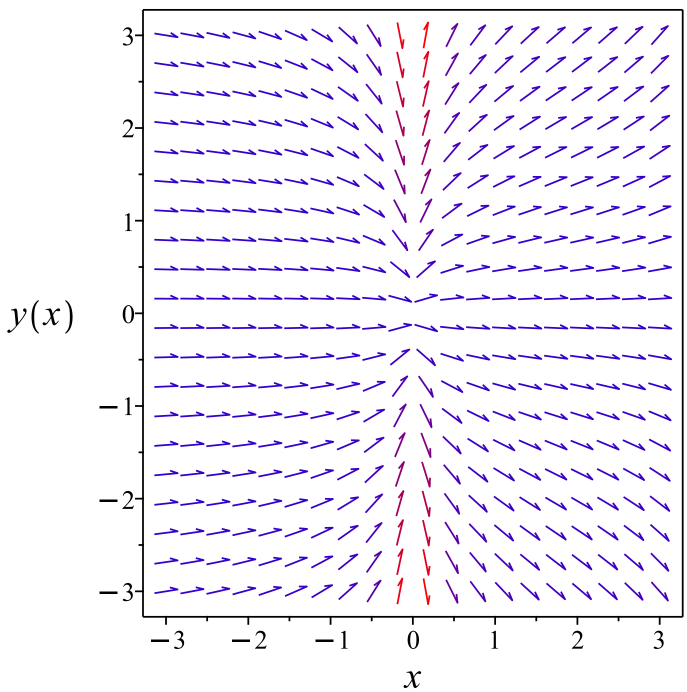


Figure 76: Slope field plot

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

Verified OK.

2.30.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x + \left(x^2 - 4x\right)\left(u'(x)\,x + u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x-3)}{x\,(-4+x)}
\end{aligned}$$

Where $f(x) = -\frac{x-3}{x(-4+x)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{x-3}{x\,(-4+x)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x-3}{x\,(-4+x)}\,dx \\
\ln(u) &= -\frac{3\ln(x)}{4} - \frac{\ln(-4+x)}{4} + c_2 \\
u &= \mathrm{e}^{-\frac{3\ln(x)}{4} - \frac{\ln(-4+x)}{4} + c_2} \\
&= c_2 \mathrm{e}^{-\frac{3\ln(x)}{4} - \frac{\ln(-4+x)}{4}}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2}{x^{\frac{3}{4}}\,(-4+x)^{\frac{1}{4}}}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= \frac{x^{\frac{1}{4}} c_2}{(-4+x)^{\frac{1}{4}}}
\end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^{\frac{1}{4}} c_2}{(-4+x)^{\frac{1}{4}}} \tag{1}$$

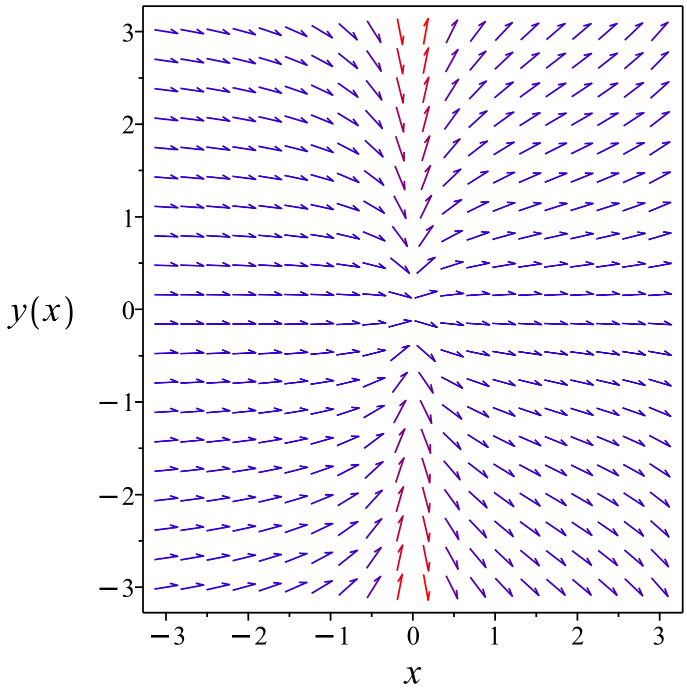


Figure 77: Slope field plot

Verification of solutions

$$y = \frac{x^{\frac{1}{4}} c_2}{(-4+x)^{\frac{1}{4}}}$$

Verified OK.

2.30.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y}{x(-4+x)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 47: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\ln\left(\frac{1}{x^{\frac{1}{4}}}\right) + \ln\left((-4+x)^{\frac{1}{4}}\right)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y}{x(-4+x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^{\frac{5}{4}} (-4+x)^{\frac{3}{4}}} \\ S_y &= \frac{(-4+x)^{\frac{1}{4}}}{x^{\frac{1}{4}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{(-4+x)^{\frac{1}{4}} y}{x^{\frac{1}{4}}} = c_1$$

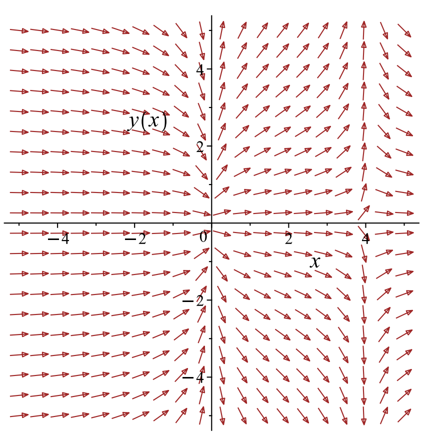
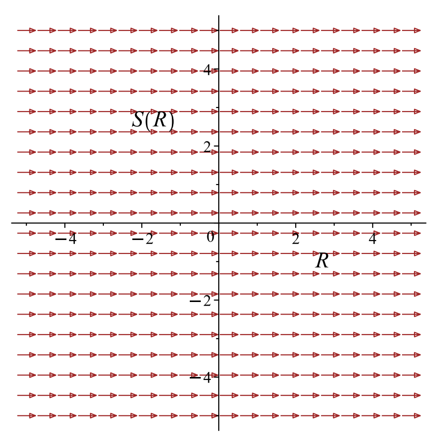
Which simplifies to

$$\frac{(-4+x)^{\frac{1}{4}} y}{x^{\frac{1}{4}}} = c_1$$

Which gives

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y}{x(-4+x)}$ 	$R = x$ $S = \frac{(-4+x)^{\frac{1}{4}} y}{x^{\frac{1}{4}}}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}} \tag{1}$$

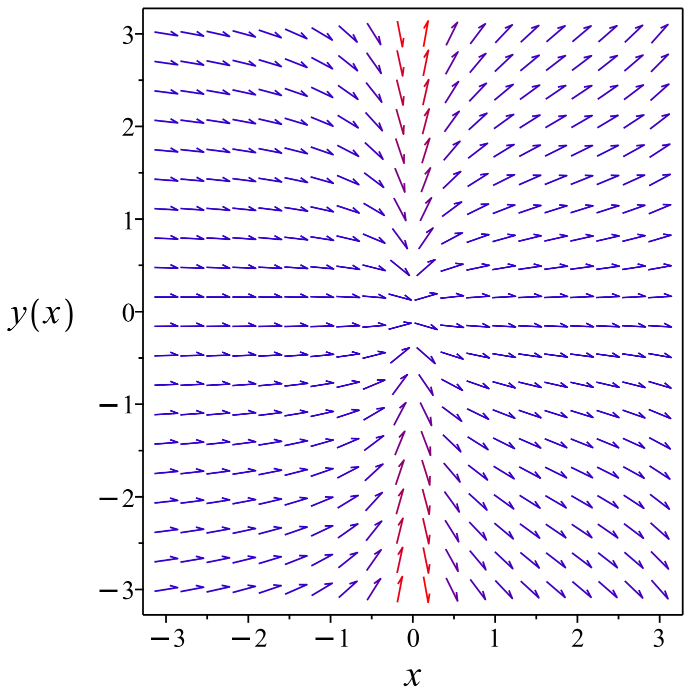


Figure 78: Slope field plot

Verification of solutions

$$y = \frac{c_1 x^{\frac{1}{4}}}{(-4+x)^{\frac{1}{4}}}$$

Verified OK.

2.30.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{1}{x\,(-4+x)}\right)\mathrm{d}x\\ \left(-\frac{1}{x\,(-4+x)}\right)\mathrm{d}x+\left(-\frac{1}{y}\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x\,(-4+x)}\\ N(x,y)&=-\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x\,(-4+x)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x\,(-4+x)}\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{\ln (x)}{4} - \frac{\ln (-4+x)}{4} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y}$. Therefore equation (4) becomes

$$-\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = -\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} - \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} - c_1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} - c_1} \tag{1}$$

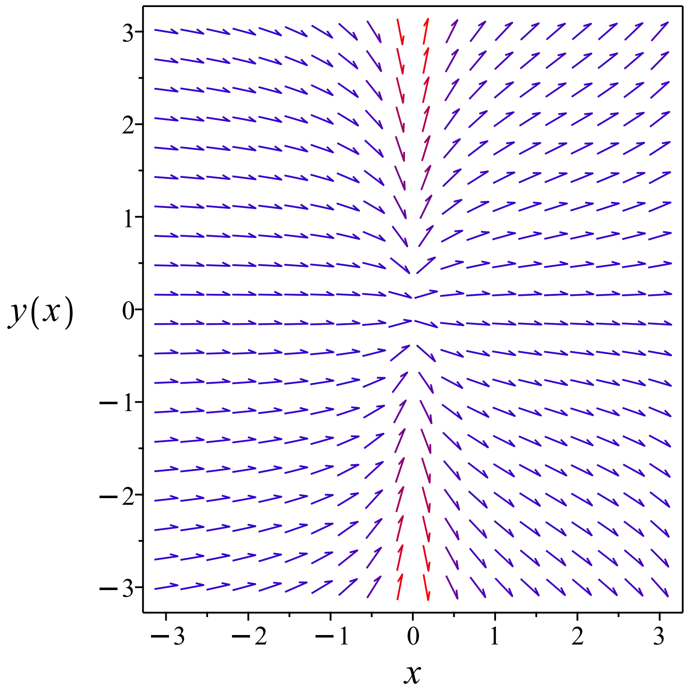


Figure 79: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{\ln(x)}{4}-\frac{\ln(-4+x)}{4}-c_1}$$

Verified OK.

2.30.6 Maple step by step solution

Let's solve

$y + (x^2 - 4x)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -\frac{1}{x^2-4x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -\frac{1}{x^2-4x} dx + c_1$

- Evaluate integral

$$\ln(y) = \frac{\ln(x)}{4} - \frac{\ln(-4+x)}{4} + c_1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)+(x^2-4*x)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^{\frac{1}{4}}}{(x-4)^{\frac{1}{4}}}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 27

```
DSolve[y[x]+(x^2-4*x)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1 \sqrt[4]{x}}{\sqrt[4]{4-x}}$$
$$y(x) \to 0$$

2.31 problem 31

Internal problem ID [5117]
Internal file name [`OUTPUT/4610_Sunday_June_05_2022_03_01_55_PM_57704246/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' - y\tan(x) = \cos(x) - 2\sin(x)\, x}$$

With initial conditions

$$\left[y\left(\frac{\pi}{6}\right) = 0\right]$$

2.31.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\tan(x) \\ q(x) &= \cos(x) - 2\sin(x)\, x \end{aligned}$$

Hence the ode is

$$y' - y\tan(x) = \cos(x) - 2\sin(x)\,x$$

The domain of $p(x) = -\tan(x)$ is

$$\left\{x < \frac{1}{2}\pi + \pi\_Z103 \vee \frac{1}{2}\pi + \pi\_Z103 < x\right\}$$

And the point $x_0 = \frac{\pi}{6}$ is inside this domain. The domain of $q(x) = \cos(x) - 2\sin(x)\,x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{6}$ is also inside this domain. Hence solution exists and is unique.

2.31.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\tan(x)dx}\\ &= \cos(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\cos(x) - 2\sin(x)\,x\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}(y\cos(x)) &= (\cos(x))\left(\cos(x) - 2\sin(x)\,x\right)\\ \mathrm{d}(y\cos(x)) &= \left((\cos(x) - 2\sin(x)\,x)\cos(x)\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\cos(x) &= \int (\cos(x) - 2\sin(x)\,x)\cos(x)\,\mathrm{d}x\\ y\cos(x) &= \cos(x)^2\,x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \cos(x)$ results in

$$y = \sec(x)\cos(x)^2\,x + c_1\sec(x)$$

which simplifies to

$$y = \cos(x)\,x + c_1\sec(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{6}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\sqrt{3}\,\pi}{12} + \frac{2c_1\sqrt{3}}{3}$$

$$c_1 = -\frac{\pi}{8}$$

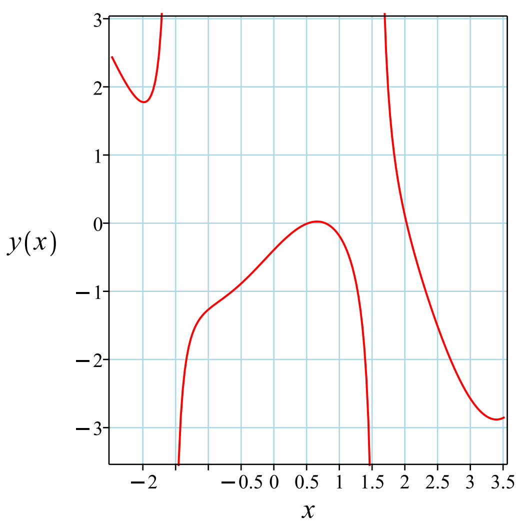
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\,\pi}{8}$$

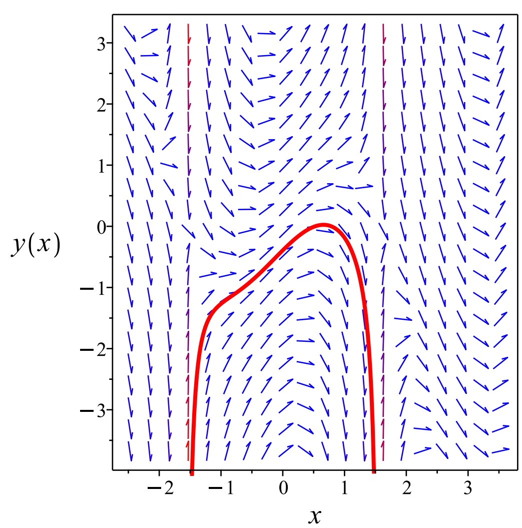
Summary

The solution(s) found are the following

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\,\pi}{8} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\,\pi}{8}$$

Verified OK.

2.31.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= y\tan(x) + \cos(x) - 2\sin(x)\,x \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 50: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\cos(x)}} dy
\end{aligned}
$$

Which results in

$$
S = y \cos(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = y \tan(x) + \cos(x) - 2 \sin(x) x
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -y \sin(x) \\
S_y &= \cos(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = (\cos(x) - 2\sin(x)\,x)\cos(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = (\cos(R) - 2\sin(R)\,R)\cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \cos(R)^2 R + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
y\cos(x) = \cos(x)^2 x + c_1
$$

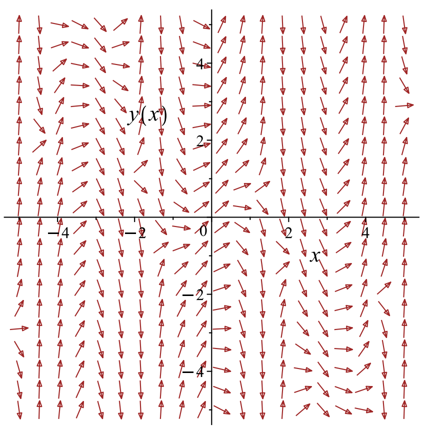
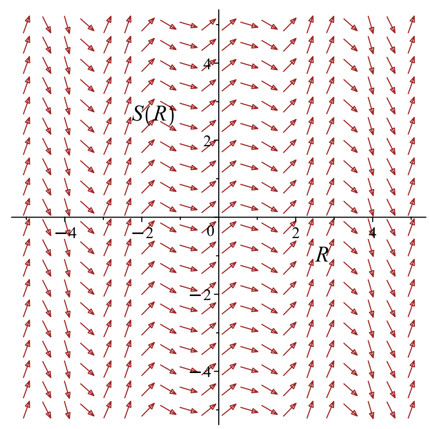
Which simplifies to

$$
y\cos(x) = \cos(x)^2 x + c_1
$$

Which gives

$$
y = \frac{\cos(x)^2 x + c_1}{\cos(x)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y\tan(x) + \cos(x) - 2\sin(x)\, x$ 	$R = x$ $S = y\cos(x)$	$\frac{dS}{dR} = (\cos(R) - 2\sin(R)\, R)\cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{6}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\sqrt{3}\,\pi}{12} + \frac{2c_1\sqrt{3}}{3}$$

$$c_1 = -\frac{\pi}{8}$$

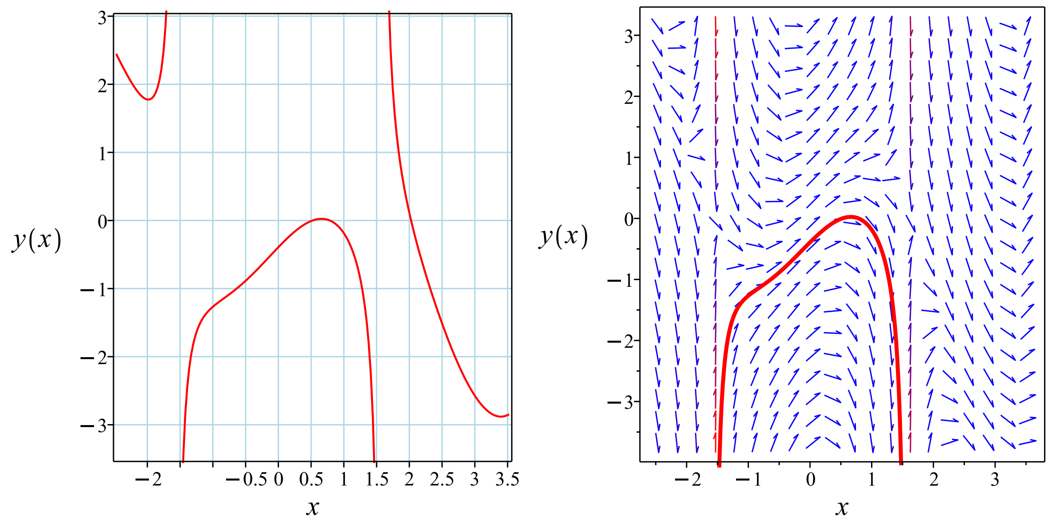
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\,\pi}{8}$$

Summary

The solution(s) found are the following

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\,\pi}{8} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\pi}{8}
$$

Verified OK.

2.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(y\tan(x)+\cos(x)-2\sin(x)\,x)\,\mathrm{d}x\\(-y\tan(x)-\cos(x)+2\sin(x)\,x)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\tan(x)-\cos(x)+2\sin(x)\,x\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-y\tan(x)-\cos(x)+2\sin(x)\,x)\\&=-\tan(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=1((-\tan(x))-(0))\\&=-\tan(x)\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int -\tan(x) \,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\cos(x))} \\
&= \cos(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \cos(x)\left(-y\tan(x) - \cos(x) + 2\sin(x)x\right) \\
&= \left(2\cos(x)x - y\right)\sin(x) - \cos(x)^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \cos(x)(1) \\
&= \cos(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\left(2\cos(x)x - y\right)\sin(x) - \cos(x)^2\right) + \left(\cos(x)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \left(2\cos(x)x - y\right)\sin(x) - \cos(x)^2 \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\cos(x)\left(\cos(x)x - y\right) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \cos(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \cos(x)$. Therefore equation (4) becomes

$$\cos(x) = \cos(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\cos(x)(\cos(x)x - y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\cos(x)(\cos(x)x - y)$$

The solution becomes

$$y = \frac{\cos(x)^2 x + c_1}{\cos(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{6}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\sqrt{3}\,\pi}{12} + \frac{2c_1\sqrt{3}}{3}$$

$$c_1 = -\frac{\pi}{8}$$

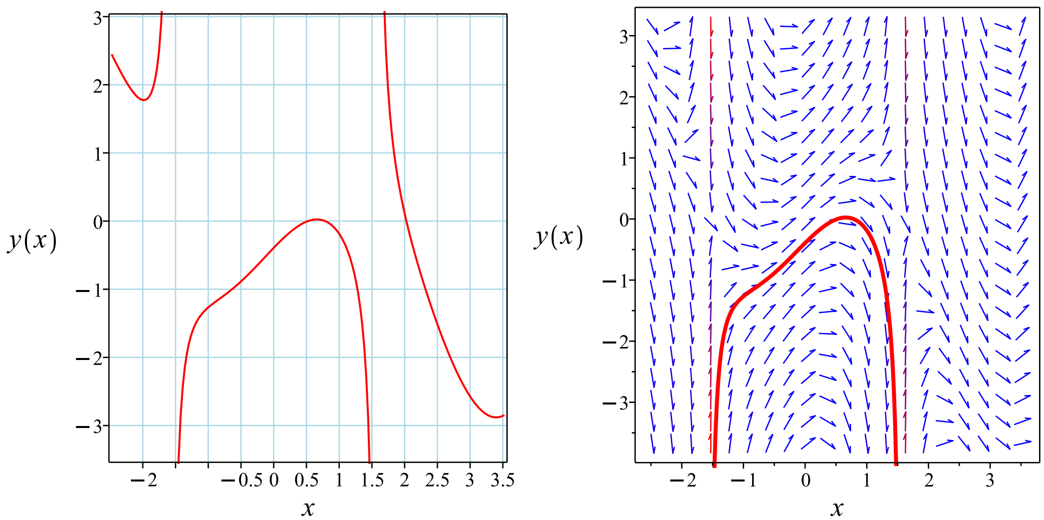
Substituting c_1 found above in the general solution gives

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\pi}{8}$$

Summary

The solution(s) found are the following

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\pi}{8} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \sec(x)\cos(x)^2 x - \frac{\sec(x)\pi}{8}$$

Verified OK.

2.31.5 Maple step by step solution

Let's solve

$\left[y' - y\tan(x) = \cos(x) - 2\sin(x)x, y\left(\frac{\pi}{6}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = y\tan(x) + \cos(x) - 2\sin(x)x$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y\tan(x) = \cos(x) - 2\sin(x)x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'-y\tan\left(x\right)\right)=\mu(x)\left(\cos\left(x\right)-2\sin\left(x\right)x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\,y\right)$

  $\mu(x)\left(y'-y\tan\left(x\right)\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)\tan\left(x\right)$

- Solve to find the integrating factor

  $\mu(x)=\cos\left(x\right)$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}\left(\mu(x)\,y\right)\right)dx=\int\mu(x)\left(\cos\left(x\right)-2\sin\left(x\right)x\right)dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\left(\cos\left(x\right)-2\sin\left(x\right)x\right)dx+c_1$

- Solve for y

  $y=\frac{\int\mu(x)(\cos(x)-2\sin(x)x)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\cos\left(x\right)$

  $y=\frac{\int(\cos(x)-2\sin(x)x)\cos(x)dx+c_1}{\cos(x)}$

- Evaluate the integrals on the rhs

  $y=\frac{\cos(x)^2x+c_1}{\cos(x)}$

- Simplify

  $y=\cos\left(x\right)x+c_1\sec\left(x\right)$

- Use initial condition $y\left(\frac{\pi}{6}\right)=0$

  $0=\frac{\sqrt{3}\,\pi}{12}+\frac{2c_1\sqrt{3}}{3}$

- Solve for c_1

  $c_1=-\frac{\pi}{8}$

- Substitute $c_1=-\frac{\pi}{8}$ into general solution and simplify

  $y=\cos\left(x\right)x-\frac{\sec(x)\pi}{8}$

- Solution to the IVP

  $y=\cos\left(x\right)x-\frac{\sec(x)\pi}{8}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)-y(x)*tan(x)=cos(x)-2*x*sin(x),y(1/6*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \cos(x)\, x - \frac{\pi \sec(x)}{8}$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 25

```
DSolve[{y'[x]-y[x]*Tan[x]==Cos[x]-2*x*Sin[x],{y[Pi/6]==0}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to \frac{1}{8}(4x + 4x\cos(2x) - \pi)\sec(x)$$

2.32 problem 32

Internal problem ID [5118]
Internal file name [`OUTPUT/4611_Sunday_June_05_2022_03_01_57_PM_6702736/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y' - \frac{2xy + y^2}{x^2 + 2xy} = 0}$$

2.32.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{2x^2u(x) + u(x)^2\,x^2}{x^2 + 2x^2u\,(x)} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u-1)}{x\,(2u+1)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u-1)}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u-1)}{2u+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u-1)}{2u+1}}\, du = \int -\frac{1}{x}\, dx$$

$$3\ln(u-1) - \ln(u) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{3\ln(u-1)-\ln(u)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{(u-1)^3}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{(u(x)-1)^3}{u(x)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\left(\frac{y}{x}-1\right)^3 x}{y} = \frac{c_3}{x}$$

$$\frac{(-x+y)^3}{x^2 y} = \frac{c_3}{x}$$

Which simplifies to

$$-\frac{(x-y)^3}{xy} = c_3$$

Summary

The solution(s) found are the following

$$-\frac{(x-y)^3}{xy} = c_3 \tag{1}$$

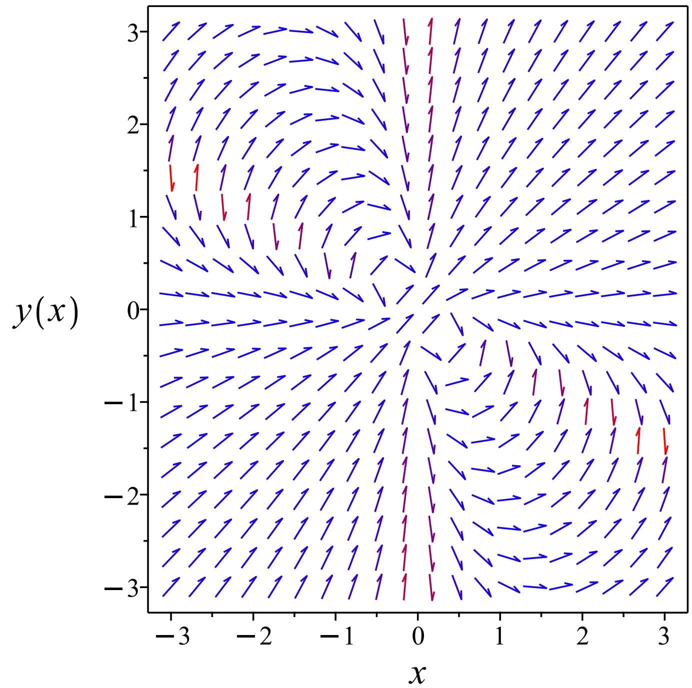


Figure 83: Slope field plot

Verification of solutions

$$-\frac{(x-y)^3}{xy} = c_3$$

Verified OK.

2.32.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y(y+2x)}{x(2y+x)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\frac{y(y+2x)\left(b_3-a_2\right)}{x\left(2y+x\right)}-\frac{y^2(y+2x)^2 a_3}{x^2\left(2y+x\right)^2} \\ & -\left(\frac{2y}{x\left(2y+x\right)}-\frac{y(y+2x)}{x^2\left(2y+x\right)}-\frac{y(y+2x)}{x\left(2y+x\right)^2}\right)\left(x a_2+y a_3+a_1\right) \\ & -\left(\frac{y+2x}{\left(2y+x\right)x}+\frac{y}{x\left(2y+x\right)}-\frac{2y(y+2x)}{x\left(2y+x\right)^2}\right)\left(x b_2+y b_3+b_1\right)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4 b_2-2x^3 y b_2+3x^2 y^2 a_2+2x^2 y^2 a_3-2x^2 y^2 b_2-3x^2 y^2 b_3+2x\,y^3 a_3-y^4 a_3+2x^3 b_1-2x^2 y a_1+2x^2 y b_1 -}{x^2\left(2y+x\right)^2}$$
$$=0$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4 b_2+2x^3 y b_2-3x^2 y^2 a_2-2x^2 y^2 a_3+2x^2 y^2 b_2+3x^2 y^2 b_3-2x\,y^3 a_3 \\ & +y^4 a_3-2x^3 b_1+2x^2 y a_1-2x^2 y b_1+2x\,y^2 a_1-2x\,y^2 b_1+2y^3 a_1=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -3a_2 v_1^2 v_2^2-2a_3 v_1^2 v_2^2-2a_3 v_1 v_2^3+a_3 v_2^4-b_2 v_1^4+2b_2 v_1^3 v_2+2b_2 v_1^2 v_2^2 \\ & +3b_3 v_1^2 v_2^2+2a_1 v_1^2 v_2+2a_1 v_1 v_2^2+2a_1 v_2^3-2b_1 v_1^3-2b_1 v_1^2 v_2-2b_1 v_1 v_2^2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&-b_2v_1^4 + 2b_2v_1^3v_2 - 2b_1v_1^3 + (-3a_2 - 2a_3 + 2b_2 + 3b_3)\, v_1^2v_2^2 \\ &+ (2a_1 - 2b_1)\, v_1^2v_2 - 2a_3v_1v_2^3 + (2a_1 - 2b_1)\, v_1v_2^2 + a_3v_2^4 + 2a_1v_2^3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}a_3 &= 0\\ 2a_1 &= 0\\ -2a_3 &= 0\\ -2b_1 &= 0\\ -b_2 &= 0\\ 2b_2 &= 0\\ 2a_1 - 2b_1 &= 0\\ -3a_2 - 2a_3 + 2b_2 + 3b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y(y+2x)}{x\,(2y+x)}\right)(x) \\
&= \frac{-xy+y^2}{2y+x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-xy+y^2}{2y+x}} dy
\end{aligned}
$$

Which results in

$$
S = -\ln(y) + 3\ln(-x+y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y(y+2x)}{x\,(2y+x)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{x - y} \\
S_y &= \frac{-2y - x}{y (x - y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln (R) + c_1 \tag{4}$$

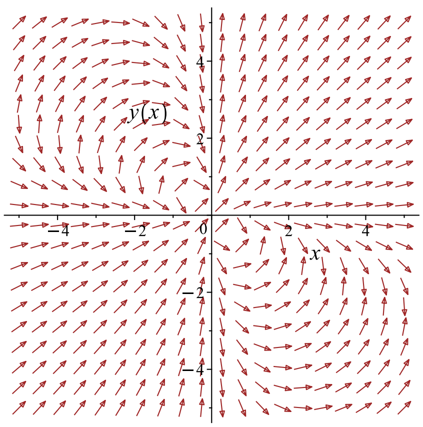
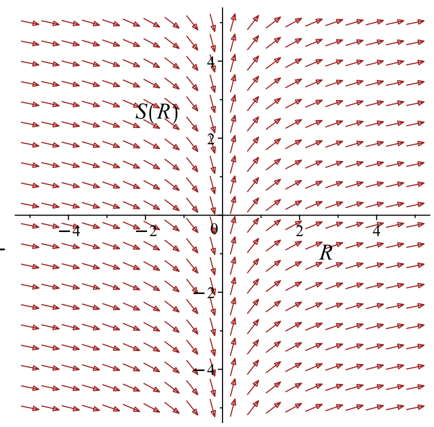
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln (y) + 3 \ln (-x + y) = \ln (x) + c_1$$

Which simplifies to

$$-\ln (y) + 3 \ln (-x + y) = \ln (x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y+2x)}{x(2y+x)}$ 	$R = x$ $S = -\ln(y) + 3\ln(-x +$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$-\ln(y) + 3\ln(-x + y) = \ln(x) + c_1 \tag{1}$$

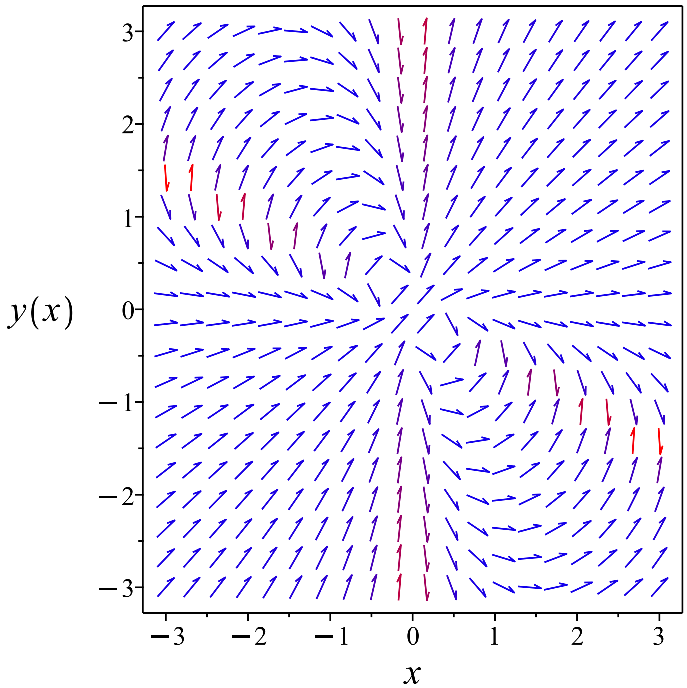


Figure 84: Slope field plot

Verification of solutions

$$-\ln(y)+3\ln(-x+y)=\ln(x)+c_1$$

Verified OK.

2.32.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(2y+x))\,\mathrm{d}y &= (y(y+2x))\,\mathrm{d}x\\ (-y(y+2x))\,\mathrm{d}x + (x(2y+x))\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y(y+2x)\\ N(x,y) &= x(2y+x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(y+2x))\\ &= -2y - 2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(2y+x))\\ &= 2y + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(2y+x)}\left((-2y-2x) - (2y+2x)\right) \\
&= \frac{-4y-4x}{x\,(2y+x)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{y\,(y+2x)}\left((2y+2x) - (-2y-2x)\right) \\
&= \frac{-4y-4x}{y\,(y+2x)}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(2y+2x) - (-2y-2x)}{x\,(-y\,(y+2x)) - y\,(x\,(2y+x))} \\
&= -\frac{4}{3yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{4}{3t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{4}{3t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{4\ln(t)}{3}}\\ &= \frac{1}{t^{\frac{4}{3}}}\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{(xy)^{\frac{4}{3}}}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{(xy)^{\frac{4}{3}}}(-y(y+2x))\\ &= -\frac{y+2x}{x\,(xy)^{\frac{1}{3}}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{(xy)^{\frac{4}{3}}}(x(2y+x))\\ &= \frac{2y+x}{y\,(xy)^{\frac{1}{3}}}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{y+2x}{x\,(xy)^{\frac{1}{3}}}\right) + \left(\frac{2y+x}{y\,(xy)^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\frac{y+2x}{x\,(xy)^{\frac{1}{3}}} \,\mathrm{d}x$$

$$\phi = -\frac{3(x-y)}{(xy)^{\frac{1}{3}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{3}{(xy)^{\frac{1}{3}}} + \frac{(x-y)\,x}{(xy)^{\frac{4}{3}}} + f'(y) \qquad (4)\\ &= \frac{2y+x}{y\,(xy)^{\frac{1}{3}}} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y+x}{y(xy)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{2y+x}{y\,(xy)^{\frac{1}{3}}} = \frac{2y+x}{y\,(xy)^{\frac{1}{3}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3(x-y)}{(xy)^{\frac{1}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3(x-y)}{(xy)^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$-\frac{3(x-y)}{(xy)^{\frac{1}{3}}}=c_1 \tag{1}$$

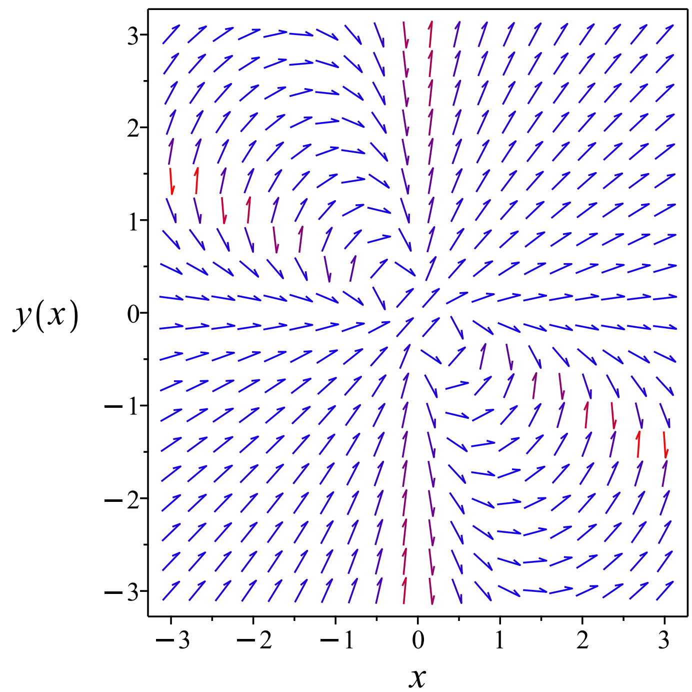


Figure 85: Slope field plot

Verification of solutions

$$-\frac{3(x-y)}{(xy)^{\frac{1}{3}}}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 356

```
dsolve(diff(y(x),x)=(2*x*y(x)+y(x)^2)/(x^2+2*x*y(x)),y(x), singsol=all)
```

$$y(x) = \frac{12^{\frac{1}{3}}\left(x\left(\sqrt{3}\,\sqrt{\frac{x(27c_1x-4)}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}}{6c_1}+\frac{x12^{\frac{2}{3}}}{6\left(x\left(\sqrt{3}\,\sqrt{\frac{x(27c_1x-4)}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}}+x$$

$$y(x) = \frac{-\frac{\left(i3^{\frac{5}{6}}+3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{2}{3}}}{6}+\left(2\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}+2^{\frac{1}{3}}\left(i3^{\frac{1}{6}}-\frac{3^{\frac{2}{3}}}{3}\right)\right)xc_1}{2\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}c_1}$$

$$y(x) = -\frac{-\frac{\left(i3^{\frac{5}{6}}-3^{\frac{1}{3}}\right)2^{\frac{2}{3}}\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{2}{3}}}{6}+\left(-2\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}+2^{\frac{1}{3}}\left(i3^{\frac{1}{6}}+\frac{3^{\frac{2}{3}}}{3}\right)\right)xc_1}{2\left(x\left(\sqrt{3}\,\sqrt{\frac{27c_1x^2-4x}{c_1}}+9x\right)c_1^2\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 56.42 (sec). Leaf size: 404

```
DSolve[y'[x]==(2*x*y[x]+y[x]^2)/(x^2+2*x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{\frac{2}{3}}e^{c_1}x}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-9e^{c_1}x^2}}+\frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-9e^{c_1}x^2}}{\sqrt[3]{2}3^{2/3}}+x$$

$$y(x) \to \frac{\left(1+i\sqrt{3}\right)e^{c_1}x}{2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-27e^{c_1}x^2}} + \frac{i\left(\sqrt{3}+i\right)\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-9e^{c_1}x^2}}{2\sqrt[3]{2}3^{2/3}}+x$$

$$y(x) \to \frac{\left(1-i\sqrt{3}\right)e^{c_1}x}{2^{2/3}\sqrt[3]{3\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-27e^{c_1}x^2}} - \frac{\left(1+i\sqrt{3}\right)\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3\left(27x+4e^{c_1}\right)}-9e^{c_1}x^2}}{2\sqrt[3]{2}3^{2/3}}+x$$

2.33 problem 33

Internal problem ID [5119]
Internal file name [`OUTPUT/4612_Sunday_June_05_2022_03_01_58_PM_27349160/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y'-x(1+y)=0}$$

2.33.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{x(1+y)}{x^2+1} \end{aligned}$$

Where $f(x)=\frac{x}{x^2+1}$ and $g(y)=1+y$. Integrating both sides gives

$$\frac{1}{1+y}\,dy = \frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{1+y}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$\ln(1+y) = \frac{\ln(x^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$1 + y = \mathrm{e}^{\frac{\ln(x^2+1)}{2} + c_1}$$

Which simplifies to

$$1 + y = c_2\sqrt{x^2+1}$$

Which simplifies to

$$y = c_2\sqrt{x^2+1}\,\mathrm{e}^{c_1} - 1$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_2\sqrt{x^2+1}\,\mathrm{e}^{c_1} - 1 \tag{1}$$

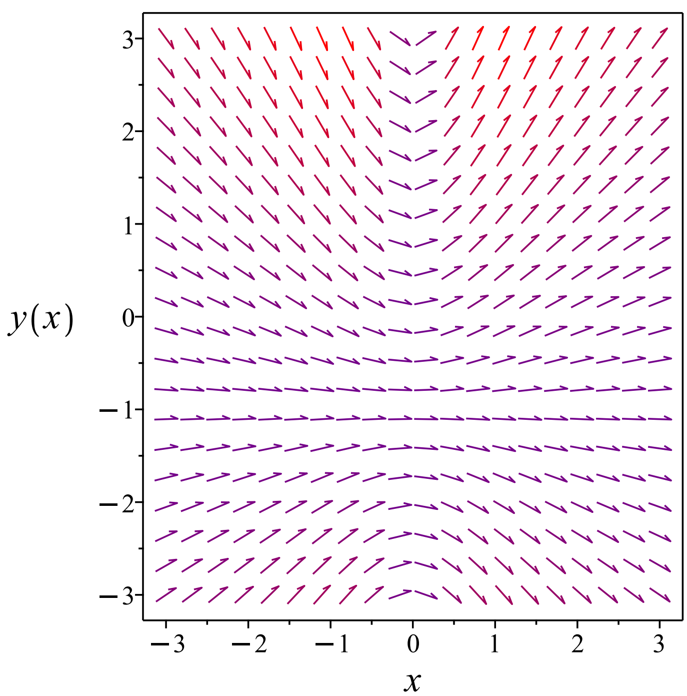


Figure 86: Slope field plot

<u>Verification of solutions</u>

$$y = c_2\sqrt{x^2+1}\,\mathrm{e}^{c_1} - 1$$

Verified OK.

2.33.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}
p(x) &= -\frac{x}{x^2+1} \\
q(x) &= \frac{x}{x^2+1}
\end{aligned}$$

Hence the ode is

$$y' - \frac{xy}{x^2+1} = \frac{x}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{x}{x^2+1}dx} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{x}{x^2+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x^2+1}}\right) &= \left(\frac{1}{\sqrt{x^2+1}}\right)\left(\frac{x}{x^2+1}\right) \\
\mathrm{d}\left(\frac{y}{\sqrt{x^2+1}}\right) &= \left(\frac{x}{(x^2+1)^{\frac{3}{2}}}\right)\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y}{\sqrt{x^2+1}} &= \int \frac{x}{(x^2+1)^{\frac{3}{2}}}\,\mathrm{d}x \\
\frac{y}{\sqrt{x^2+1}} &= -\frac{1}{\sqrt{x^2+1}} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\sqrt{x^2+1}}$ results in

$$y = -1 + c_1\sqrt{x^2+1}$$

Summary

The solution(s) found are the following

$$y = -1 + c_1\sqrt{x^2+1} \tag{1}$$

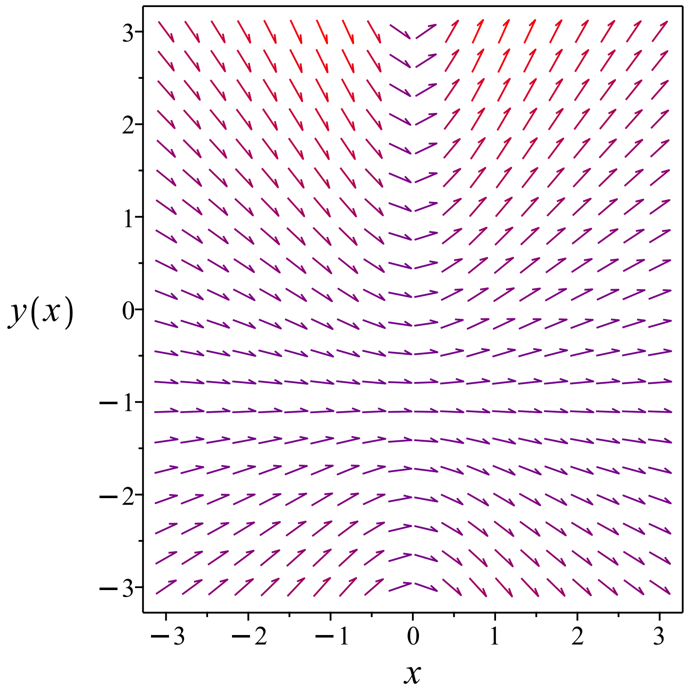


Figure 87: Slope field plot

Verification of solutions

$$y = -1 + c_1\sqrt{x^2+1}$$

Verified OK.

2.33.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x(1+y)}{x^2+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 53: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \sqrt{x^2+1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sqrt{x^2+1}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sqrt{x^2+1}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x(1+y)}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{yx}{(x^2+1)^{\frac{3}{2}}} \\ S_y &= \frac{1}{\sqrt{x^2+1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{(x^2+1)^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{(R^2+1)^{\frac{3}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{\sqrt{R^2+1}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} + c_1$$

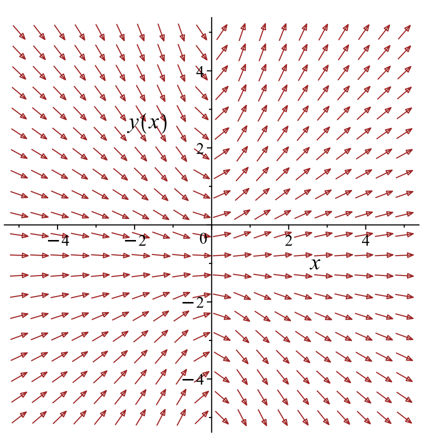
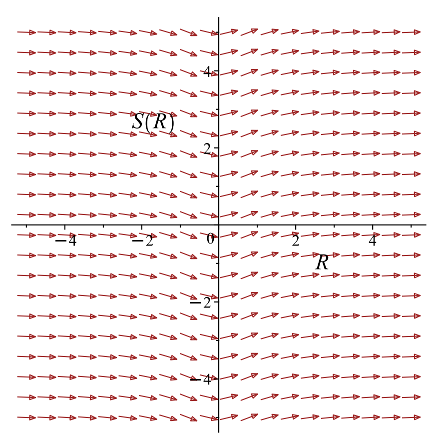
Which simplifies to

$$\frac{y}{\sqrt{x^2+1}} = -\frac{1}{\sqrt{x^2+1}} + c_1$$

Which gives

$$y = -1 + c_1\sqrt{x^2+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(1+y)}{x^2+1}$  	$R = x$ $S = \frac{y}{\sqrt{x^2+1}}$	$\frac{dS}{dR} = \frac{R}{(R^2+1)^{\frac{3}{2}}}$ 

Summary

The solution(s) found are the following

$$y = -1 + c_1\sqrt{x^2 + 1} \tag{1}$$

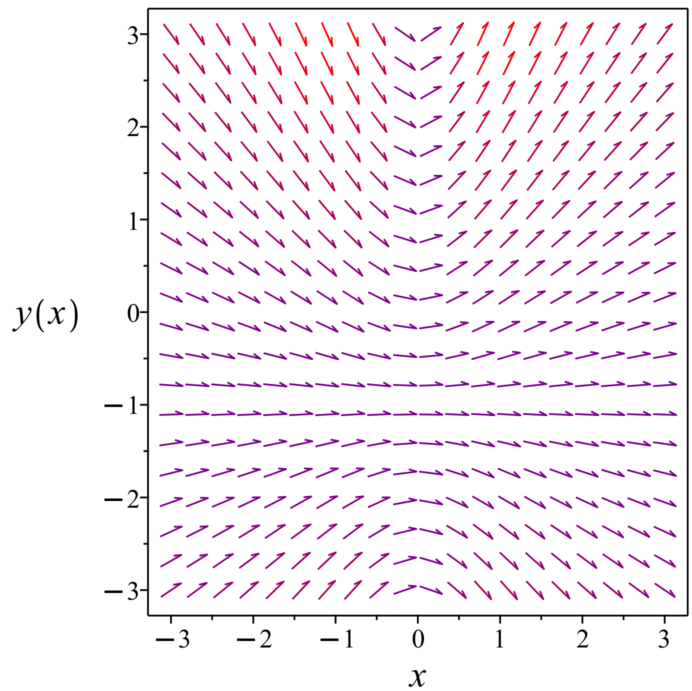


Figure 88: Slope field plot

Verification of solutions

$$y = -1 + c_1\sqrt{x^2 + 1}$$

Verified OK.

2.33.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1+y}\right)\mathrm{d}y&=\left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x+\left(\frac{1}{1+y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{x}{x^2+1}\\ N(x,y)&=\frac{1}{1+y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1+y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x\\ \phi &= -\frac{\ln(x^2+1)}{2} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1+y}$. Therefore equation (4) becomes

$$\frac{1}{1+y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{1+y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{1+y}\right)\mathrm{d}y\\ f(y) &= \ln(1+y) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \ln(1+y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \ln(1+y)$$

The solution becomes

$$y = e^{\frac{\ln(x^2+1)}{2}+c_1} - 1$$

Summary

The solution(s) found are the following

$$y = e^{\frac{\ln(x^2+1)}{2}+c_1} - 1 \tag{1}$$

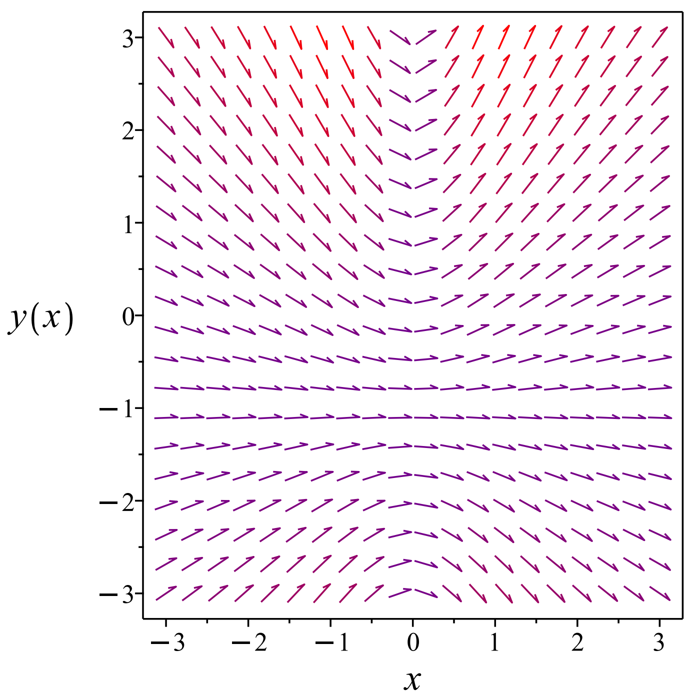


Figure 89: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(x^2+1)}{2}+c_1} - 1$$

Verified OK.

2.33.5 Maple step by step solution

Let's solve

$(x^2+1)\,y' - x(1+y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = \frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y}dx = \int \frac{x}{x^2+1}dx + c_1$

- Evaluate integral

  $\ln(1+y) = \frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = e^{\frac{\ln(x^2+1)}{2}+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((1+x^2)*diff(y(x),x)=x*(1+y(x)),y(x), singsol=all)
```

$$y(x) = \sqrt{x^2+1}\,c_1 - 1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 24

```
DSolve[(1+x^2)*y'[x]==x*(1+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + c_1\sqrt{x^2+1}$$
$$y(x) \to -1$$

2.34 problem 34

Internal problem ID [5120]
Internal file name [`OUTPUT/4613_Sunday_June_05_2022_03_01_59_PM_61874419/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{xy' + 2y = 3x - 1}$$

With initial conditions

$$[y(2) = 1]$$

2.34.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{3x - 1}{x}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{3x - 1}{x}$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{3x-1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

2.34.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3x - 1}{x}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\, x^2\right) &= \left(x^2\right)\left(\frac{3x - 1}{x}\right) \\ \mathrm{d}\left(y\, x^2\right) &= \left(3x^2 - x\right)\, \mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\, x^2 &= \int 3x^2 - x\, \mathrm{d}x \\ y\, x^2 &= x^3 - \frac{1}{2}x^2 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{x^3 - \frac{1}{2}x^2}{x^2} + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4} + \frac{3}{2}$$

$$c_1 = -2$$

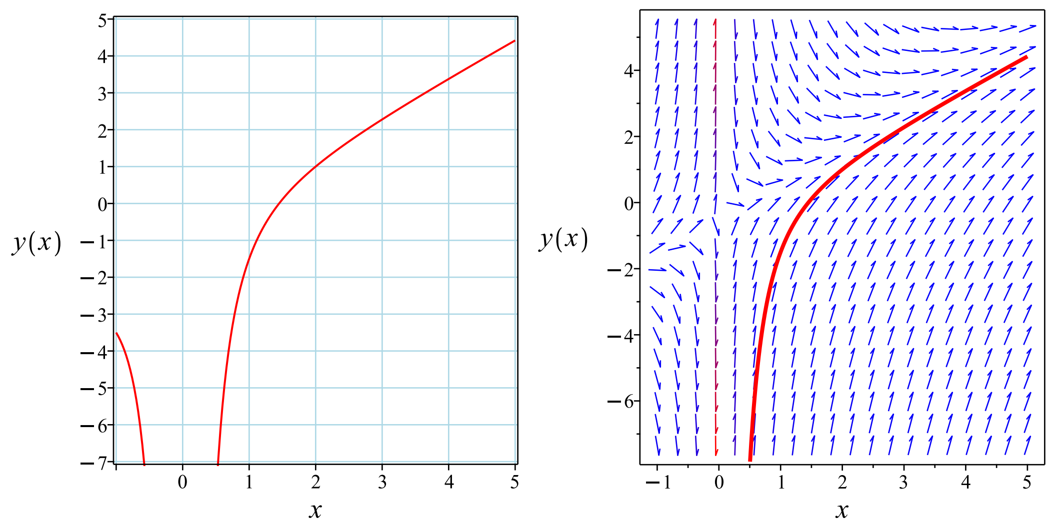
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{2x^3 - x^2 - 4}{2x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

Verified OK.

2.34.3 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) + 2y_0 - 3X - 3x_0 + 1}{X + x_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 0\\ y_0 &= -\frac{1}{2}\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) - 3X}{X}$$

In canonical form, the ODE is

$$\begin{aligned}Y' &= F(X,Y)\\ &= -\frac{2Y - 3X}{X} \qquad (1)\end{aligned}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX, t^nY) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -2Y+3X$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -2u + 3\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-3u(X) + 3}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{-3u(X) + 3}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X) - 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= \frac{-3u+3}{X}\end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = -3u + 3$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-3u+3}\,du &= \frac{1}{X}\,dX\\ \int \frac{1}{-3u+3}\,du &= \int \frac{1}{X}\,dX\\ -\frac{\ln(u-1)}{3} &= \ln(X) + c_2\end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{(u-1)^{\frac{1}{3}}} = e^{\ln(X)+c_2}$$

Which simplifies to

$$\frac{1}{(u-1)^{\frac{1}{3}}} = c_3X$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{(c_3^3 e^{3c_2}X^3 + 1)\,e^{-3c_2}}{X^2c_3^3}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{(c_3^3 e^{3c_2}X^3 + 1)\,e^{-3c_2}}{X^2c_3^3}$$

And replacing back terms in the above solution using

$$\begin{aligned}Y &= y + y_0\\ X &= x + x_0\end{aligned}$$

Or

$$Y = y - \frac{1}{2}$$
$$X = x$$

Then the solution in y becomes

$$y + \frac{1}{2} = \frac{\left(c_3^3 \mathrm{e}^{3c_2} x^3 + 1\right) \mathrm{e}^{-3c_2}}{x^2 c_3^3}$$

Initial conditions are used to solve for c_2. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = \frac{8\,\mathrm{e}^{-3c_2} \mathrm{e}^{3c_2} c_3^3 + \mathrm{e}^{-3c_2}}{4c_3^3}$$

$$c_2 = -\frac{\ln\left(-2c_3^3\right)}{3}$$

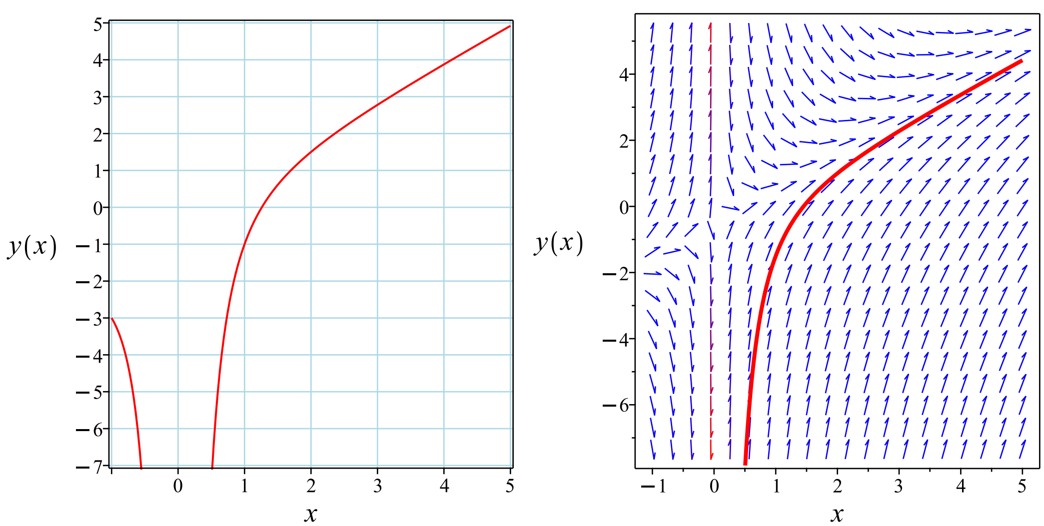
Substituting c_2 found above in the general solution gives

$$y = \frac{x^3 - 2}{x^2}$$

Summary

The solution(s) found are the following

$$y + \frac{1}{2} = \frac{x^3 - 2}{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y + \frac{1}{2} = \frac{x^3 - 2}{x^2}$$

Verified OK.

2.34.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y - 3x + 1}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 56: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y - 3x + 1}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2xy \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 - x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2 - R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 - \frac{1}{2}R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^2 = x^3 - \frac{1}{2}x^2 + c_1$$

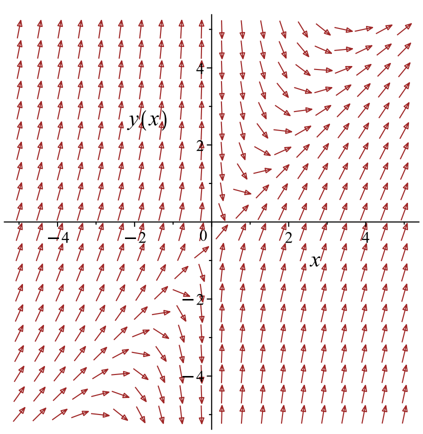
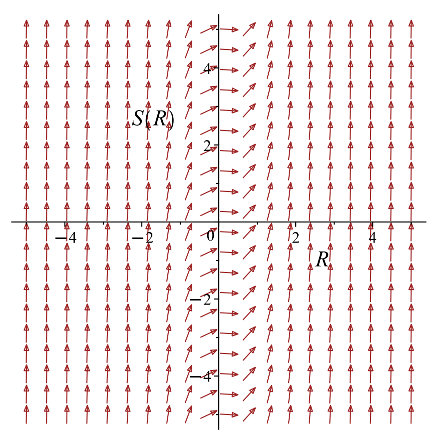
Which simplifies to

$$yx^2 = x^3 - \frac{1}{2}x^2 + c_1$$

Which gives

$$y = \frac{2x^3 - x^2 + 2c_1}{2x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{2y-3x+1}{x}$

</td><td>$R = x$
$S = y\,x^2$</td><td>$\frac{dS}{dR} = 3R^2 - R$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4} + \frac{3}{2}$$

$$c_1 = -2$$

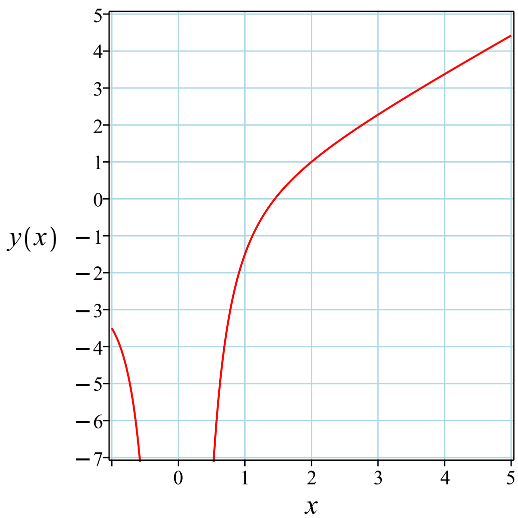
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

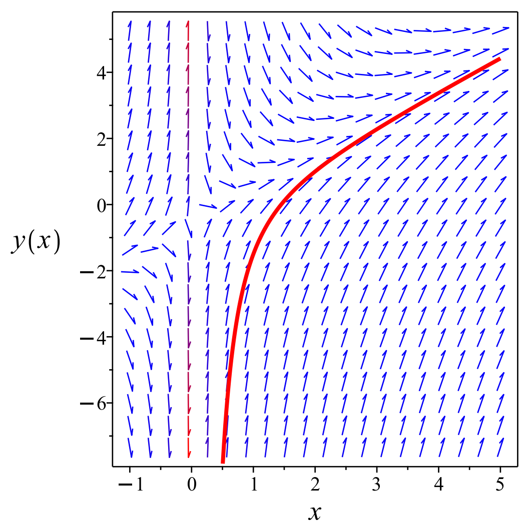
Summary

The solution(s) found are the following

$$y = \frac{2x^3 - x^2 - 4}{2x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

Verified OK.

2.34.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-2y+3x-1)\,\mathrm{d}x\\ (2y-3x+1)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y-3x+1\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - 3x + 1)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((2) - (1))\\ &= \frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{1}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)}\\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x(2y - 3x + 1)\\ &= x(2y - 3x + 1)\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x) \\
&= x^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(x(2y-3x+1)) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int x(2y-3x+1)\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2(2x-2y-1)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$
x^2 = x^2 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2(2x - 2y - 1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2(2x - 2y - 1)}{2}$$

The solution becomes

$$y = \frac{2x^3 - x^2 + 2c_1}{2x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{4} + \frac{3}{2}$$

$$c_1 = -2$$

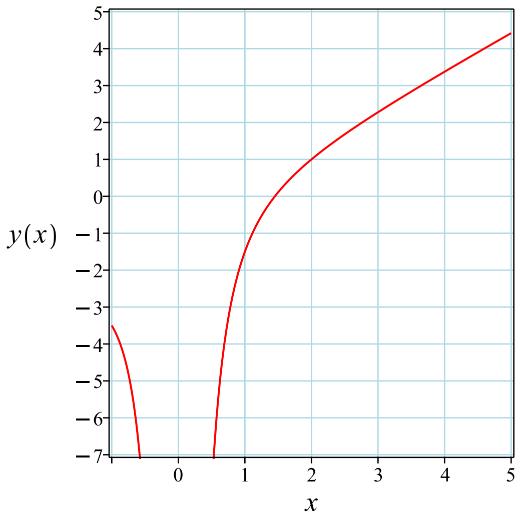
Substituting c_1 found above in the general solution gives

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

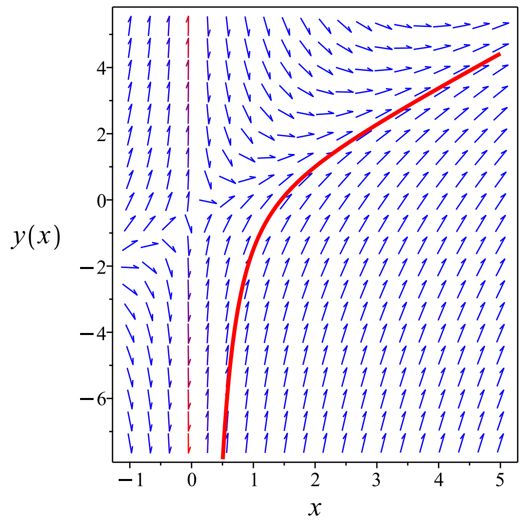
Summary

The solution(s) found are the following

$$y = \frac{2x^3 - x^2 - 4}{2x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^3 - x^2 - 4}{2x^2}$$

Verified OK.

2.34.6 Maple step by step solution

Let's solve

$[xy' + 2y = 3x - 1, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{3x-1}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{3x-1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)(3x-1)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)(3x-1)}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)(3x-1)}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)(3x-1)}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int (3x-1)x dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{x^3 - \frac{1}{2}x^2 + c_1}{x^2}$

- Use initial condition $y(2) = 1$

  $1 = \frac{c_1}{4} + \frac{3}{2}$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $y = \frac{x^3 - \frac{1}{2}x^2 - 2}{x^2}$

- Solution to the IVP

  $y = \frac{x^3 - \frac{1}{2}x^2 - 2}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([x*diff(y(x),x)+2*y(x)=3*x-1,y(2) = 1],y(x), singsol=all)
```

$$y(x) = x - \frac{1}{2} - \frac{2}{x^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 15

```
DSolve[{x*y'[x]+2*y[x]==3*x-1,{y[2]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{x^2} + x - \frac{1}{2}$$

2.35 problem 35

Internal problem ID [5121]
Internal file name [`OUTPUT/4614_Sunday_June_05_2022_03_02_00_PM_29432663/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{x^2y' - y^2 + xyy' = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.35.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y^2}{x(x+y)} \end{aligned}$$

The x domain of $f(x,y)$ when $y=1$ is

$$\{-\infty \leq x < -1, -1 < x < 0, 0 < x \leq \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{y < -1 \lor -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y^2}{x\,(x+y)}\right)\\ &= \frac{2y}{x\,(x+y)} - \frac{y^2}{x\,(x+y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty \le x < -1, -1 < x < 0, 0 < x \le \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{y < -1 \lor -1 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.35.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2(u'(x)\,x + u(x)) - u(x)^2\,x^2 + x^2u(x)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x, u)\\ &= f(x)g(u)\\ &= -\frac{u}{x\,(u+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u}{u+1}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{u}{u+1}}\,du &= -\frac{1}{x}\,dx\\ \int \frac{1}{\frac{u}{u+1}}\,du &= \int -\frac{1}{x}\,dx\\ u + \ln(u) &= -\ln(x) + c_2\end{aligned}$$

The solution is

$$u(x) + \ln(u(x)) + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$
$$\frac{y}{x} + \ln\left(\frac{y}{x}\right) + \ln(x) - c_2 = 0$$

Substituting initial conditions and solving for c_2 gives $c_2 = 1$. Hence the solution becomes

Summary

The solution(s) found are the following

$$\frac{y}{x} + \ln\left(\frac{y}{x}\right) + \ln(x) - 1 = 0 \tag{1}$$

Verification of solutions

$$\frac{y}{x} + \ln\left(\frac{y}{x}\right) + \ln(x) - 1 = 0$$

Verified OK.

2.35.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y^2}{x(x+y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2+\frac{y^2(b_3-a_2)}{x(x+y)}-\frac{y^4a_3}{x^2(x+y)^2}-\left(-\frac{y^2}{x^2(x+y)}-\frac{y^2}{x(x+y)^2}\right)(xa_2+ya_3+a_1) \\
&-\left(\frac{2y}{x(x+y)}-\frac{y^2}{x(x+y)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{x^4b_2+x^2y^2a_2-x^2y^2b_3+2x\,y^3a_3-2x^2yb_1+2x\,y^2a_1-x\,y^2b_1+y^3a_1}{x^2(x+y)^2}=0
$$

Setting the numerator to zero gives

$$
x^4b_2+x^2y^2a_2-x^2y^2b_3+2x\,y^3a_3-2x^2yb_1+2x\,y^2a_1-x\,y^2b_1+y^3a_1=0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2\}
$$

The above PDE (6E) now becomes

$$
a_2v_1^2v_2^2+2a_3v_1v_2^3+b_2v_1^4-b_3v_1^2v_2^2+2a_1v_1v_2^2+a_1v_2^3-2b_1v_1^2v_2-b_1v_1v_2^2=0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2\}
$$

Equation (7E) now becomes

$$
b_2v_1^4+(-b_3+a_2)\,v_1^2v_2^2-2b_1v_1^2v_2+2a_3v_1v_2^3+(2a_1-b_1)\,v_1v_2^2+a_1v_2^3=0 \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
b_2 &= 0 \\
2a_3 &= 0 \\
-2b_1 &= 0 \\
2a_1 - b_1 &= 0 \\
-b_3 + a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y^2}{x\,(x+y)}\right)(x) \\
&= \frac{yx}{x+y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{yx}{x+y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) + \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x(x+y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{x+y}{xy} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x \ln (y) + y}{x} = c_1$$

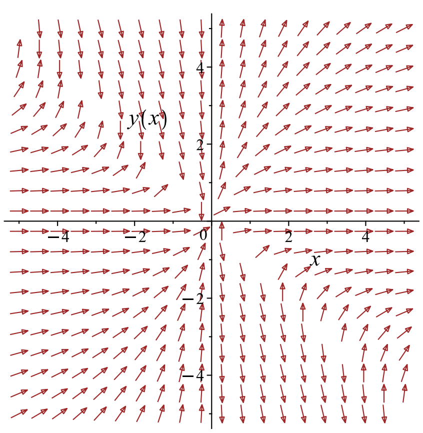
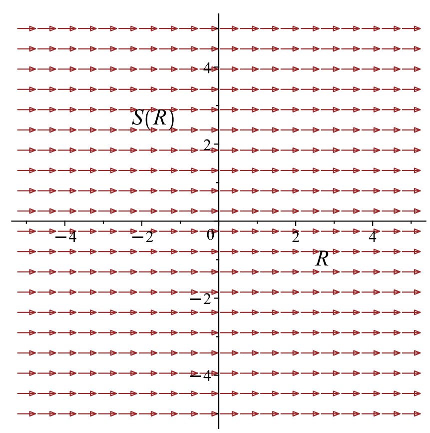
Which simplifies to

$$\frac{x \ln (y) + y}{x} = c_1$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(x+y)}$ 	$R = x$ $S = \frac{\ln (y) x + y}{x}$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \operatorname{LambertW}\left(\mathrm{e}^{c_1}\right)$$

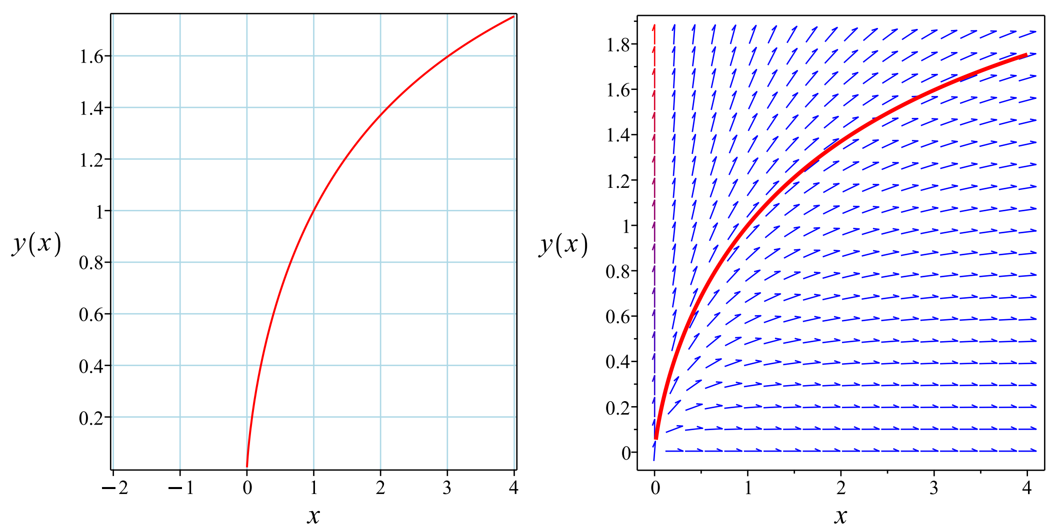
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = x \,\mathrm{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

Summary
The solution(s) found are the following

$$y = x \,\mathrm{LambertW}\left(\frac{\mathrm{e}}{x}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x \,\mathrm{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

Verified OK.

2.35.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2+xy)\,\mathrm{d}y&=(y^2)\,\mathrm{d}x\\ (-y^2)\,\mathrm{d}x+(x^2+xy)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y^2\\ N(x,y)&=x^2+xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y^2) \\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 + xy) \\ &= y + 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = -y^2$ and $N = xy + x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= -\frac{y}{x^2} \\ N &= \frac{xy + x^2}{x^2 y}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x^2+xy}{x^2y}\right)\mathrm{d}y &= \left(\frac{y}{x^2}\right)\mathrm{d}x\\ \left(-\frac{y}{x^2}\right)\mathrm{d}x + \left(\frac{x^2+xy}{x^2y}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{y}{x^2}\\ N(x,y) &= \frac{x^2+xy}{x^2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right)\\ &= -\frac{1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2+xy}{x^2y}\right)\\ &= -\frac{1}{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{y}{x^2}\,\mathrm{d}x$$
$$\phi = \frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2+xy}{x^2y}$. Therefore equation (4) becomes

$$\frac{x^2+xy}{x^2y} = \frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right)\mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) + \frac{y}{x}$$

The solution becomes

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \operatorname{LambertW}\left(\mathrm{e}^{c_1}\right)$$

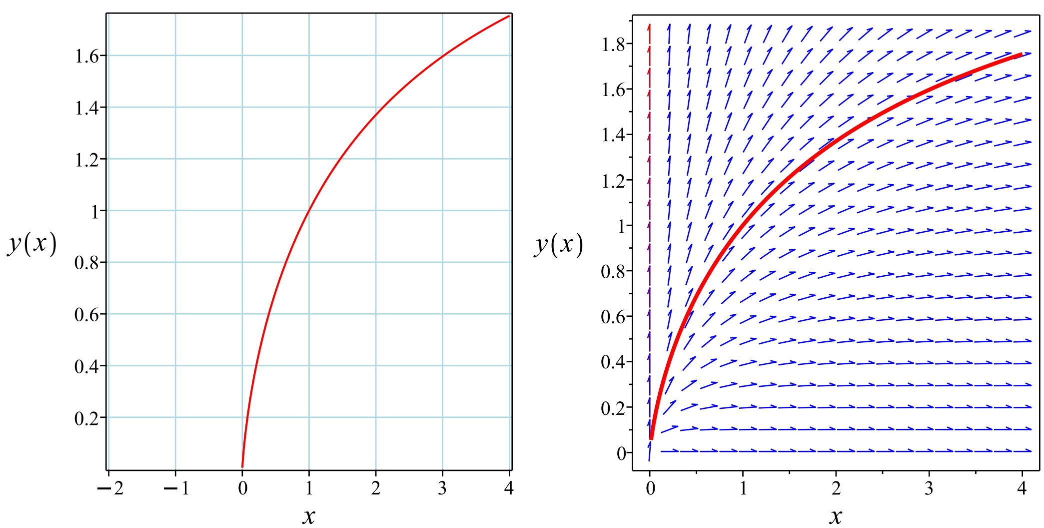
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = x \operatorname{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = x \operatorname{LambertW}\left(\frac{\mathrm{e}}{x}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = x \operatorname{LambertW}\left(\frac{\mathrm{e}}{x}\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.375 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x)=y(x)^2-x*y(x)*diff(y(x),x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \text{LambertW}\left(\frac{\mathrm{e}}{x}\right) x$$

✓ Solution by Mathematica

Time used: 2.335 (sec). Leaf size: 13

```
DSolve[{x^2*y'[x]==y[x]^2-x*y[x]*y'[x],{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to xW\left(\frac{e}{x}\right)$$

2.36 problem 36

Internal problem ID [5122]
Internal file name [`OUTPUT/4615_Sunday_June_05_2022_03_02_02_PM_20895447/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - e^{3x-2y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.36.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= e^{3x-2y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^{3x-2y}\right) \\ &= -2\,\mathrm{e}^{3x-2y}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.36.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \mathrm{e}^{3x}\mathrm{e}^{-2y}\end{aligned}$$

Where $f(x) = \mathrm{e}^{3x}$ and $g(y) = \mathrm{e}^{-2y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{-2y}}\,dy &= \mathrm{e}^{3x}\,dx \\ \int \frac{1}{\mathrm{e}^{-2y}}\,dy &= \int \mathrm{e}^{3x}\,dx \\ \frac{\mathrm{e}^{2y}}{2} &= \frac{\mathrm{e}^{3x}}{3} + c_1\end{aligned}$$

Which results in

$$y = \frac{\ln\left(\frac{2\,\mathrm{e}^{3x}}{3} + 2c_1\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln (2)}{2} - \frac{\ln (3)}{2} + \frac{\ln (1 + 3c_1)}{2}$$

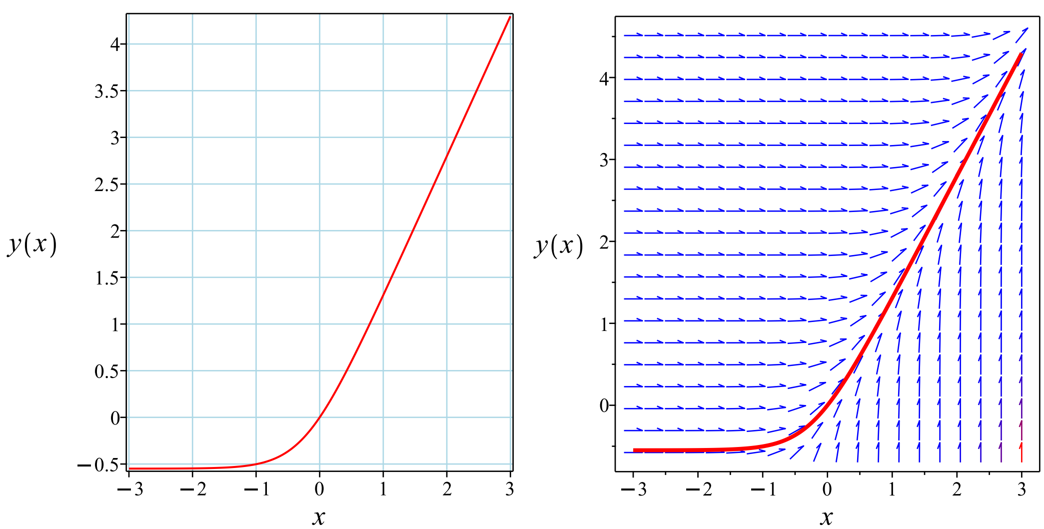
$$c_1 = \frac{1}{6}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{\ln (3)}{2} + \frac{\ln (2\,\mathrm{e}^{3x} + 1)}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -\frac{\ln (3)}{2} + \frac{\ln (2\,\mathrm{e}^{3x} + 1)}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\ln (3)}{2} + \frac{\ln (2\,\mathrm{e}^{3x} + 1)}{2}$$

Verified OK.

2.36.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = e^{3x-2y} \tag{1}$$

And using the substitution $u = e^{2y}$ then

$$u' = 2y' e^{2y}$$

The above shows that

$$\begin{aligned} y' &= \frac{u'(x)\, e^{-2y}}{2} \\ &= \frac{u'(x)}{2u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{2u} = \frac{e^{3x}}{u}$$

The above simplifies to

$$u'(x) = 2\,e^{3x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int 2\,e^{3x}\, dx \\ &= \frac{2\,e^{3x}}{3} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^{2y}$ gives

$$\begin{aligned} y &= \frac{\ln(u(x))}{2} \\ &= \frac{\ln\left(\frac{2\,e^{3x}}{3} + c_1\right)}{2} \\ &= -\frac{\ln(3)}{2} + \frac{\ln(2\,e^{3x} + 3c_1)}{2} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{\ln(3)}{2} + \frac{\ln(2 + 3c_1)}{2}$$

$$c_1 = \frac{1}{3}$$

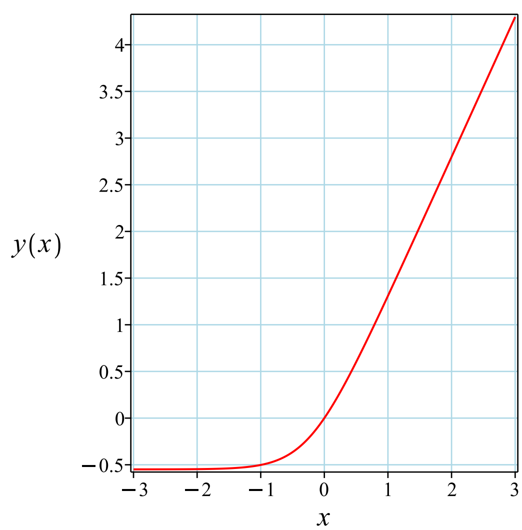
Substituting c_1 found above in the general solution gives

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,e^{3x} + 1)}{2}$$

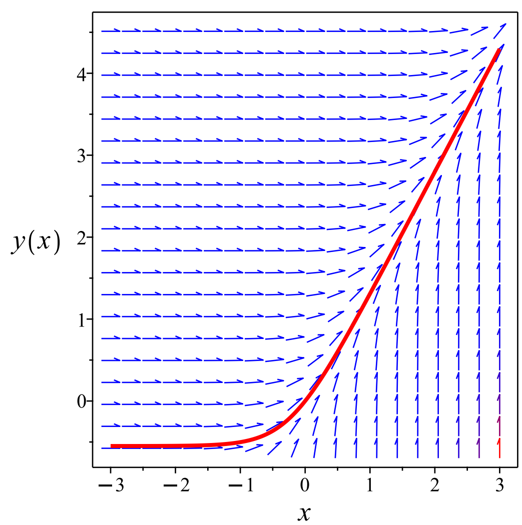
Summary

The solution(s) found are the following

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,e^{3x} + 1)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,e^{3x} + 1)}{2}$$

Verified OK.

2.36.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \mathrm{e}^{3x-2y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 59: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \mathrm{e}^{-3x} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\mathrm{e}^{-3x}} dx \end{aligned}$$

Which results in

$$S = \frac{\mathrm{e}^{3x}}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^{3x-2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \mathrm{e}^{3x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{3x}}{3} = \frac{e^{2y}}{2} + c_1$$

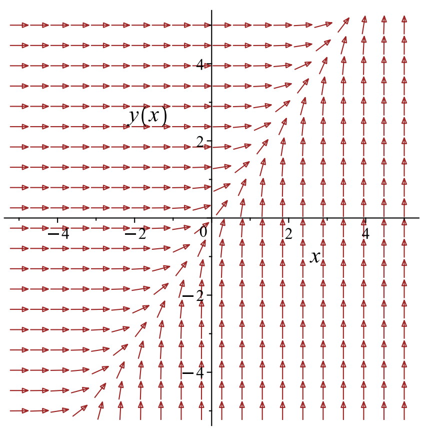
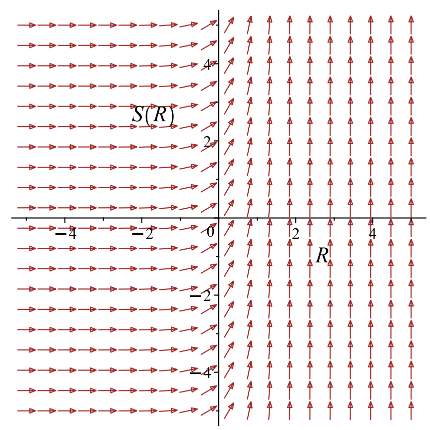
Which simplifies to

$$\frac{e^{3x}}{3} = \frac{e^{2y}}{2} + c_1$$

Which gives

$$y = \frac{\ln\left(\frac{2\,e^{3x}}{3} - 2c_1\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \mathrm{e}^{3x-2y}$ 	$R = y$ $S = \frac{\mathrm{e}^{3x}}{3}$	$\frac{dS}{dR} = \mathrm{e}^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2} - \frac{\ln(3)}{2} + \frac{\ln(1 - 3c_1)}{2}$$

$$c_1 = -\frac{1}{6}$$

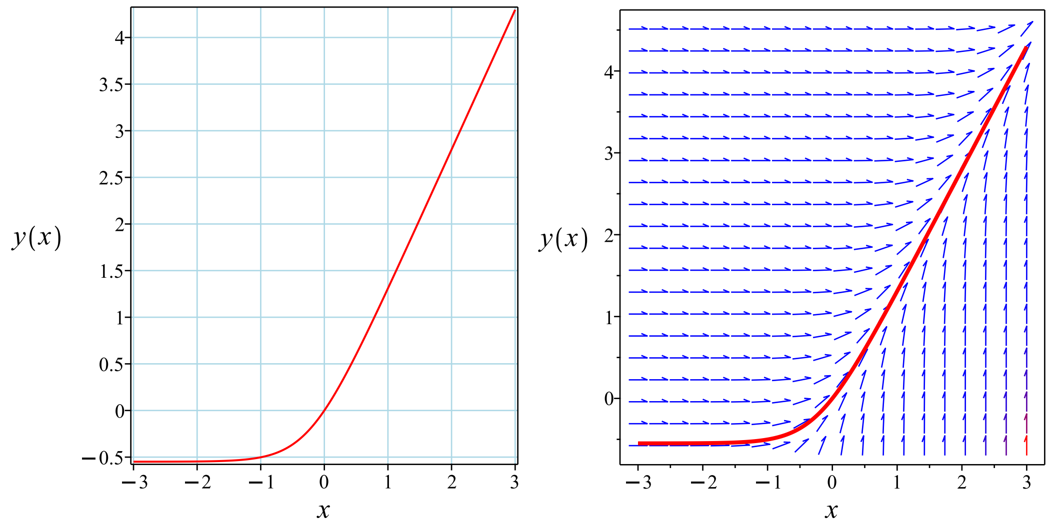
Substituting c_1 found above in the general solution gives

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,\mathrm{e}^{3x} + 1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,\mathrm{e}^{3x} + 1)}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\ln(3)}{2} + \frac{\ln(2\,e^{3x}+1)}{2}$$

Verified OK.

2.36.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\mathrm{e}^{2y}\right)\mathrm{d}y&=\left(\mathrm{e}^{3x}\right)\mathrm{d}x\\ \left(-\mathrm{e}^{3x}\right)\mathrm{d}x+\left(\mathrm{e}^{2y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\mathrm{e}^{3x}\\ N(x,y)&=\mathrm{e}^{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\mathrm{e}^{3x}\right)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\mathrm{e}^{2y}\right)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -\mathrm{e}^{3x} \,\mathrm{d}x$$

$$\phi = -\frac{\mathrm{e}^{3x}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{2y}$. Therefore equation (4) becomes

$$\mathrm{e}^{2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\mathrm{e}^{2y}\right) \mathrm{d}y$$

$$f(y) = \frac{\mathrm{e}^{2y}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\mathrm{e}^{3x}}{3} + \frac{\mathrm{e}^{2y}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\mathrm{e}^{3x}}{3} + \frac{\mathrm{e}^{2y}}{2}$$

The solution becomes

$$y = \frac{\ln\left(\frac{2\,\mathrm{e}^{3x}}{3}+2c_1\right)}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{\ln(2)}{2}-\frac{\ln(3)}{2}+\frac{\ln(1+3c_1)}{2}$$

$$c_1 = \frac{1}{6}$$

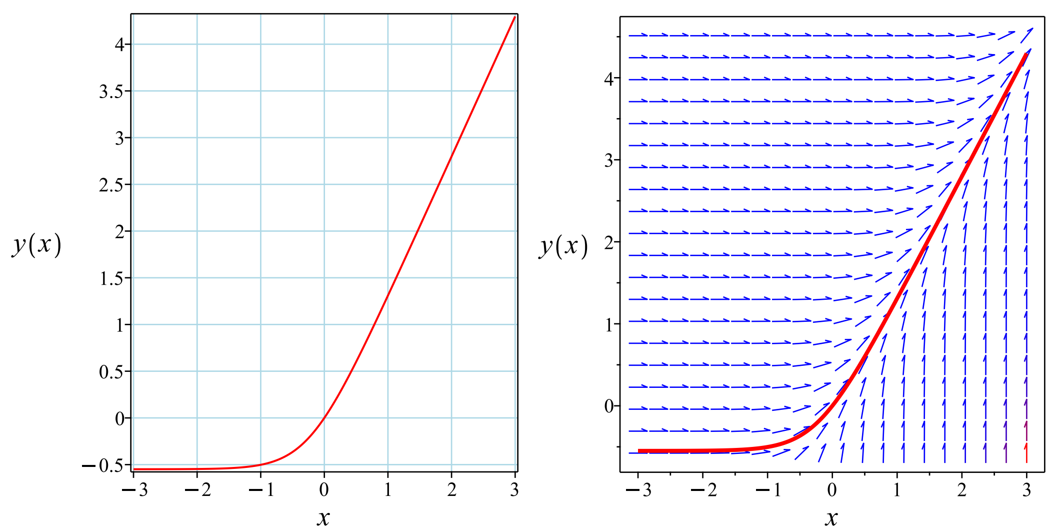
Substituting c_1 found above in the general solution gives

$$y = -\frac{\ln(3)}{2}+\frac{\ln(2\,\mathrm{e}^{3x}+1)}{2}$$

Summary

The solution(s) found are the following

$$y = -\frac{\ln(3)}{2}+\frac{\ln(2\,\mathrm{e}^{3x}+1)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\frac{\ln(3)}{2}+\frac{\ln(2\,\mathrm{e}^{3x}+1)}{2}$$

Verified OK.

2.36.6 Maple step by step solution

Let's solve

$\left[y' - \mathrm{e}^{3x-2y} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y'(\mathrm{e}^{y})^{2} = (\mathrm{e}^{x})^{3}$

- Integrate both sides with respect to x

  $\int y'(\mathrm{e}^{y})^{2} dx = \int (\mathrm{e}^{x})^{3} dx + c_1$

- Evaluate integral

  $\frac{(\mathrm{e}^{y})^{2}}{2} = \frac{(\mathrm{e}^{x})^{3}}{3} + c_1$

- Solve for y

  $y = \frac{\ln\left(\frac{2(\mathrm{e}^{x})^{3}}{3} + 2c_1\right)}{2}$

- Use initial condition $y(0) = 0$

  $0 = \frac{\ln\left(\frac{2}{3} + 2c_1\right)}{2}$

- Solve for c_1

  $c_1 = \frac{1}{6}$

- Substitute $c_1 = \frac{1}{6}$ into general solution and simplify

  $y = -\frac{\ln(3)}{2} + \frac{\ln(2\,\mathrm{e}^{3x} + 1)}{2}$

- Solution to the IVP

  $y = -\frac{\ln(3)}{2} + \frac{\ln(2\,\mathrm{e}^{3x} + 1)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 15

```
dsolve([diff(y(x),x)=exp(3*x-2*y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{\ln (3)}{2} + \frac{\ln (1 + 2\,e^{3x})}{2}$$

✓ Solution by Mathematica

Time used: 0.881 (sec). Leaf size: 23

```
DSolve[{y'[x]==Exp[3*x-2*y[x]],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} \log \left(\frac{1}{3}\left(2e^{3x} + 1\right)\right)$$

2.37 problem 37

Internal problem ID [5123]
Internal file name [`OUTPUT/4616_Sunday_June_05_2022_03_02_03_PM_46396852/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{x} = \sin(2x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{4}\right) = 2\right]$$

2.37.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \sin(2x)$$

Hence the ode is

$$y' + \frac{y}{x} = \sin(2x)$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \frac{\pi}{4}$ is inside this domain. The domain of $q(x) = \sin(2x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{4}$ is also inside this domain. Hence solution exists and is unique.

2.37.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x} dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\sin(2x)) \\ \frac{d}{dx}(xy) &= (x)(\sin(2x)) \\ d(xy) &= (x\sin(2x))\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xy &= \int x\sin(2x)\, dx \\ xy &= \frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{\frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2}}{x} + \frac{c_1}{x}$$

which simplifies to

$$y = \frac{-2x\cos(2x) + \sin(2x) + 4c_1}{4x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1 + 4c_1}{\pi}$$

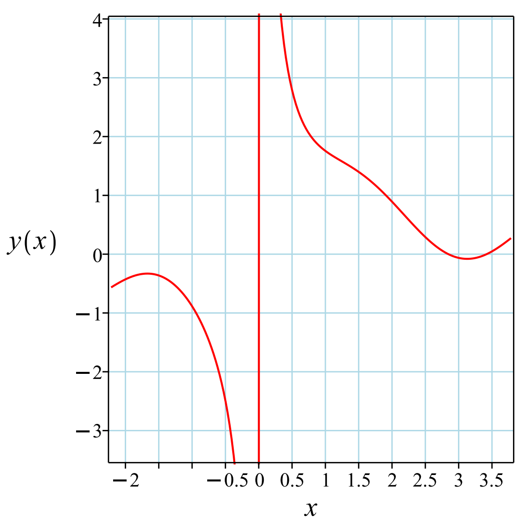
$$c_1 = \frac{\pi}{2} - \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

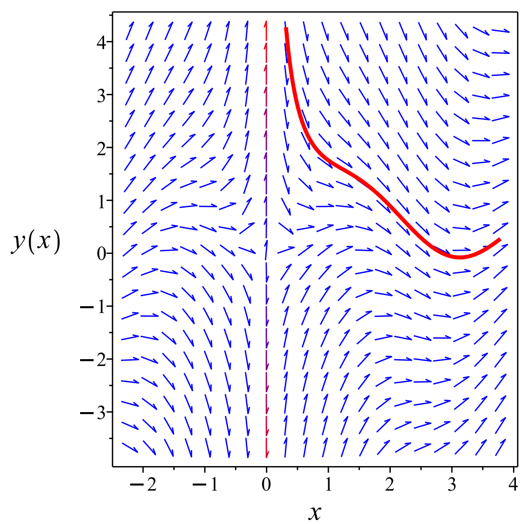
$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

Summary
The solution(s) found are the following

$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

Verified OK.

2.37.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{-y + x\sin(2x)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 62: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = xy
$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{-y + x\sin(2x)}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = x \sin(2x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = R \sin(2R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\sin(2R)}{4} - \frac{R\cos(2R)}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
xy = \frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2} + c_1
$$

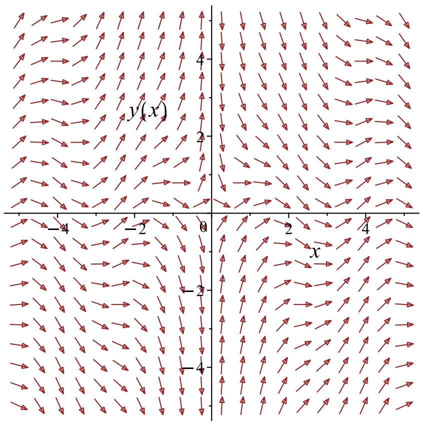
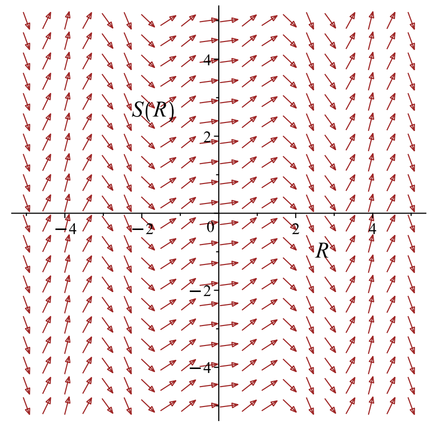
Which simplifies to

$$
xy = \frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2} + c_1
$$

Which gives

$$
y = \frac{-2x\cos(2x) + \sin(2x) + 4c_1}{4x}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+x\sin(2x)}{x}$ 	$R = x$ $S = xy$	$\frac{dS}{dR} = R\sin(2R)$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1 + 4c_1}{\pi}$$

$$c_1 = \frac{\pi}{2} - \frac{1}{4}$$

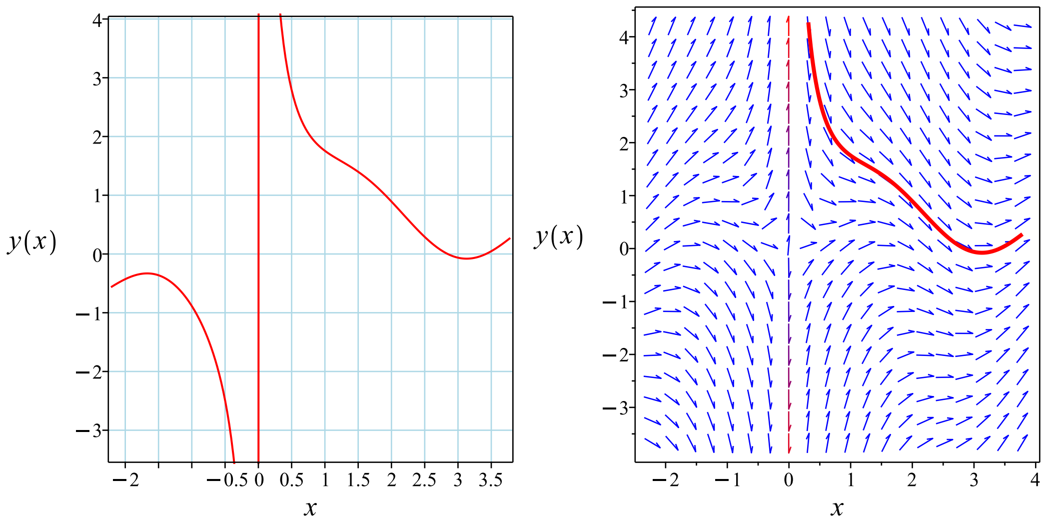
Substituting c_1 found above in the general solution gives

$$y = \frac{-2x\cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{-2x\cos(2x) + \sin(2x) + 2\pi - 1}{4x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-2x\cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

Verified OK.

2.37.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= (-y + x\sin(2x))\,\mathrm{d}x\\ (y - x\sin(2x))\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y - x\sin(2x)\\ N(x, y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - x\sin(2x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int y - x \sin(2x) \mathrm{d}x$$
$$\phi = xy - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = xy - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = xy - \frac{\sin(2x)}{4} + \frac{x\cos(2x)}{2}$$

The solution becomes

$$y = \frac{-2x\cos(2x) + \sin(2x) + 4c_1}{4x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{4}$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{1 + 4c_1}{\pi}$$

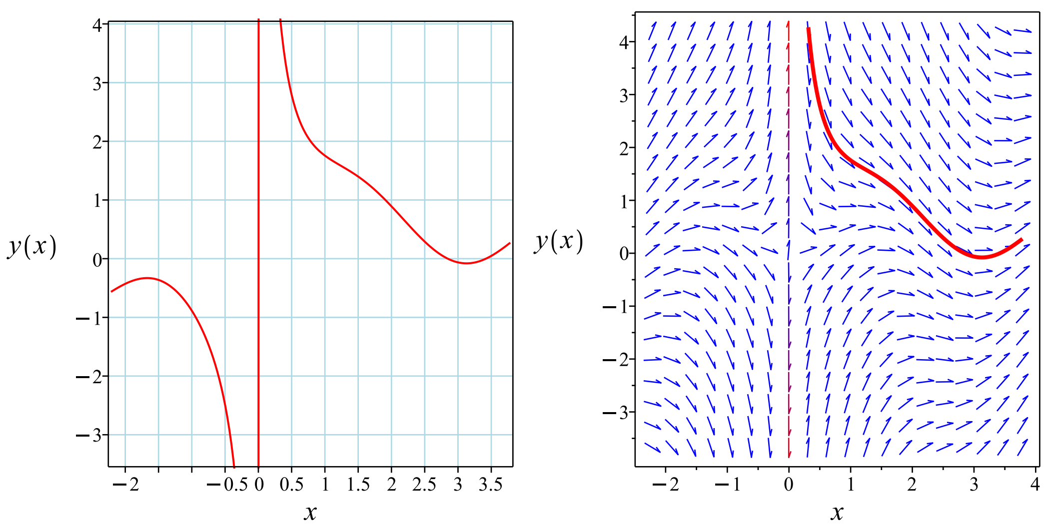
$$c_1 = \frac{\pi}{2} - \frac{1}{4}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-2x \cos(2x) + \sin(2x) + 2\pi - 1}{4x}$$

Verified OK.

2.37.5 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{x} = \sin\left(2x\right), y\left(\frac{\pi}{4}\right) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \sin\left(2x\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \sin\left(2x\right)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\sin\left(2x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\, y\right)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}\left(\mu(x)\, y\right)\right) dx = \int \mu(x)\sin\left(2x\right) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin\left(2x\right) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(2x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int x\sin(2x)dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\frac{\sin(2x)}{4} - \frac{x\cos(2x)}{2} + c_1}{x}$

- Simplify

$y = \frac{-2x\cos(2x)+\sin(2x)+4c_1}{4x}$

- Use initial condition $y\left(\frac{\pi}{4}\right) = 2$

  $2 = \frac{1+4c_1}{\pi}$

- Solve for c_1

  $c_1 = \frac{\pi}{2} - \frac{1}{4}$

- Substitute $c_1 = \frac{\pi}{2} - \frac{1}{4}$ into general solution and simplify

  $y = \frac{-2x\cos(2x)+\sin(2x)+2\pi-1}{4x}$

- Solution to the IVP

  $y = \frac{-2x\cos(2x)+\sin(2x)+2\pi-1}{4x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(y(x),x)+1/x*y(x)=sin(2*x),y(1/4*Pi) = 2],y(x), singsol=all)
```

$$y(x) = \frac{-2x\cos(2x) + 2\pi + \sin(2x) - 1}{4x}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 28

```
DSolve[{y'[x]+1/x*y[x]==Sin[2*x],{y[Pi/4]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(2x) - 2x\cos(2x) + 2\pi - 1}{4x}$$

2.38 problem 38

Internal problem ID [5124]
Internal file name [`OUTPUT/4617_Sunday_June_05_2022_03_02_04_PM_19127363/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y^2 + x^2y' - xyy' = 0}$$

2.38.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)^2\,x^2 + x^2(u'(x)\,x + u(x)) - x^2u(x)\,(u'(x)\,x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u}{x\,(u-1)} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{u}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u}{u-1}}\, du = \frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u}{u-1}}\, du = \int \frac{1}{x}\, dx$$

$$u - \ln(u) = \ln(x) + c_2$$

The solution is

$$u(x) - \ln(u(x)) - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y}{x} - \ln\left(\frac{y}{x}\right) - \ln(x) - c_2 = 0 \tag{1}$$

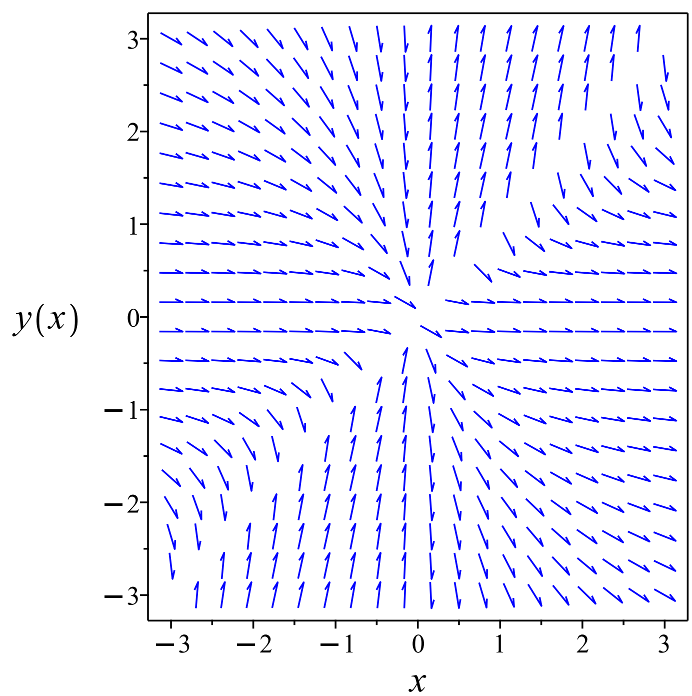


Figure 103: Slope field plot

Verification of solutions

$$\frac{y}{x}-\ln\left(\frac{y}{x}\right)-\ln\left(x\right)-c_2=0$$

Verified OK.

2.38.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned}y' &= \frac{y^2}{x\left(-x+y\right)}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2+\frac{y^2(b_3-a_2)}{x\left(-x+y\right)}-\frac{y^4a_3}{x^2\left(-x+y\right)^2}\\ &-\left(-\frac{y^2}{x^2\left(-x+y\right)}+\frac{y^2}{x\left(-x+y\right)^2}\right)\left(xa_2+ya_3+a_1\right)\\ &-\left(\frac{2y}{x\left(-x+y\right)}-\frac{y^2}{x\left(-x+y\right)^2}\right)\left(xb_2+yb_3+b_1\right)=0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4b_2-x^2y^2a_2+x^2y^2b_3-2x\,y^3a_3+2x^2yb_1-2x\,y^2a_1-x\,y^2b_1+y^3a_1}{x^2\left(x-y\right)^2}=0$$

Setting the numerator to zero gives

$$x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 + b_2 v_1^4 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 + a_1 v_2^3 + 2b_1 v_1^2 v_2 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^4 + (b_3 - a_2)\, v_1^2 v_2^2 + 2b_1 v_1^2 v_2 - 2a_3 v_1 v_2^3 + (-2a_1 - b_1)\, v_1 v_2^2 + a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -2a_3 &= 0 \\ 2b_1 &= 0 \\ -2a_1 - b_1 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y^2}{x\,(-x+y)}\right)(x) \\
&= \frac{yx}{x-y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{yx}{x-y}} dy
\end{aligned}
$$

Which results in

$$
S = \ln(y) - \frac{y}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2}{x(-x+y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2} \\ S_y &= \frac{x-y}{xy} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x \ln(y) - y}{x} = c_1$$

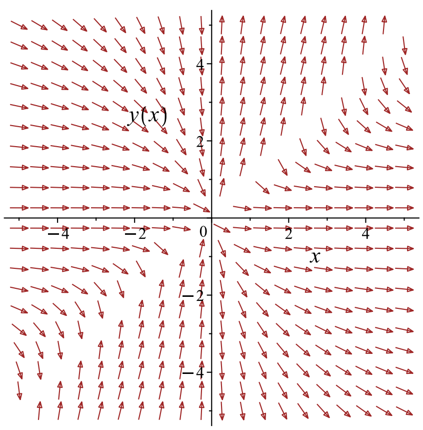
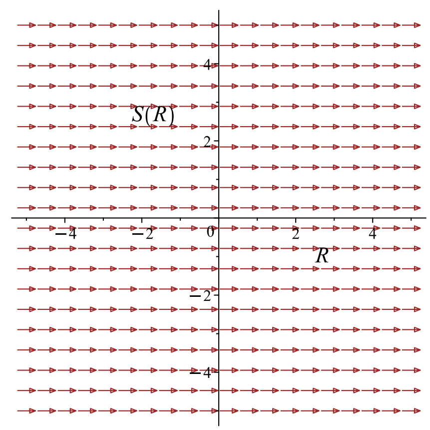
Which simplifies to

$$\frac{x \ln(y) - y}{x} = c_1$$

Which gives

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(-x+y)}$ 	$R = x$ $S = \frac{\ln(y)\,x - y}{x}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1} \tag{1}$$

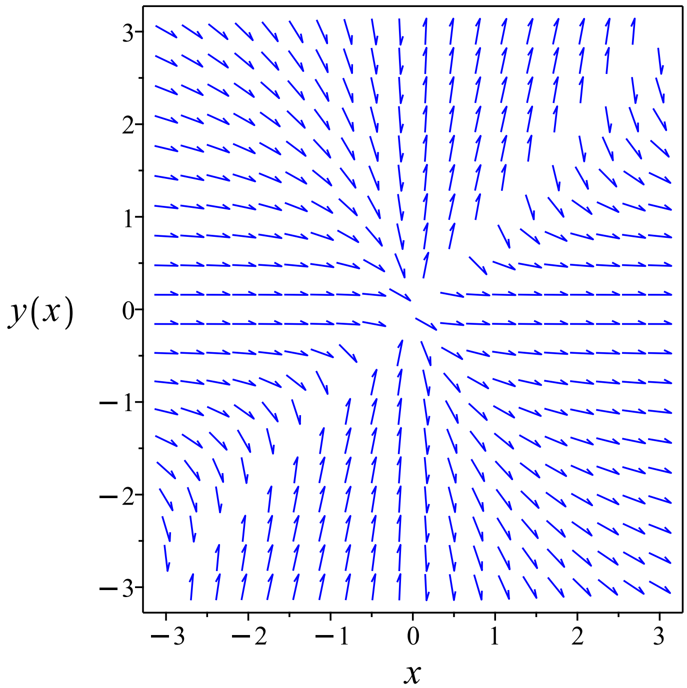


Figure 104: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

Verified OK.

2.38.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x^2 - xy\right)\mathrm{d}y &= \left(-y^2\right)\mathrm{d}x \\
\left(y^2\right)\mathrm{d}x + \left(x^2 - xy\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y^2 \\
N(x, y) &= x^2 - xy
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^2\right) \\
&= 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 - xy\right) \\
&= 2x - y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y^2$ and $N = x^2 - xy$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{y}{x^2} \\
N &= \frac{x^2 - xy}{x^2 y}
\end{aligned}
$$

To solve an ode of the form

$$
M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx} \phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$\left(\frac{x^2 - xy}{x^2 y}\right) \mathrm{d}y = \left(-\frac{y}{x^2}\right) \mathrm{d}x$$

$$\left(\frac{y}{x^2}\right) \mathrm{d}x + \left(\frac{x^2 - xy}{x^2 y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \frac{y}{x^2}$$
$$N(x, y) = \frac{x^2 - xy}{x^2 y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y}{x^2}\right) \\ &= \frac{1}{x^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x^2 - xy}{x^2 y}\right) \\ &= \frac{1}{x^2} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{y}{x^2} \,\mathrm{d}x$$
$$\phi = -\frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2-xy}{x^2 y}$. Therefore equation (4) becomes

$$\frac{x^2 - xy}{x^2 y} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(y) - \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(y) - \frac{y}{x}$$

The solution becomes

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1} \tag{1}$$

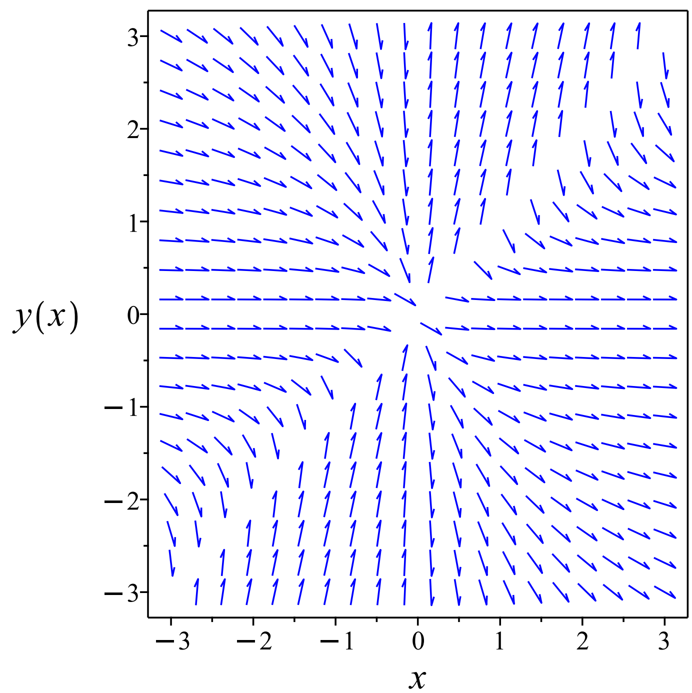


Figure 105: Slope field plot

Verification of solutions

$$y = e^{-\operatorname{LambertW}\left(-\frac{e^{c_1}}{x}\right)+c_1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve(y(x)^2+x^2*diff(y(x),x)=x*y(x)*diff(y(x),x),y(x), singsol=all)
```

$$y(x) = -x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

✓ Solution by Mathematica

Time used: 2.23 (sec). Leaf size: 25

```
DSolve[y[x]^2+x^2*y'[x]==x*y[x]*y'[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -xW\left(-\frac{e^{-c_1}}{x}\right)$$
$$y(x) \to 0$$

2.39 problem 39

Internal problem ID [5125]
Internal file name [`OUTPUT/4618_Sunday_June_05_2022_03_02_05_PM_59597733/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _Bernoulli]
```

$$\boxed{2xyy' + y^2 = x^2}$$

2.39.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2u(x)\,(u'(x)\,x + u(x)) + u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{3u^2-1}{2ux}
\end{aligned}$$

Where $f(x) = -\frac{1}{2x}$ and $g(u) = \frac{3u^2-1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{3u^2-1}{u}}\, du &= -\frac{1}{2x}\, dx \\
\int \frac{1}{\frac{3u^2-1}{u}}\, du &= \int -\frac{1}{2x}\, dx \\
\frac{\ln\left(3u^2-1\right)}{6} &= -\frac{\ln\left(x\right)}{2} + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(3u^2-1\right)^{\frac{1}{6}} = \mathrm{e}^{-\frac{\ln(x)}{2}+c_2}
$$

Which simplifies to

$$
\left(3u^2-1\right)^{\frac{1}{6}} = \frac{c_3}{\sqrt{x}}
$$

Which simplifies to

$$
\left(3u(x)^2-1\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}
$$

The solution is

$$
\left(3u(x)^2-1\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{3y^2}{x^2}-1\right)^{\frac{1}{6}} &= \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}} \\
\left(\frac{3y^2-x^2}{x^2}\right)^{\frac{1}{6}} &= \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}
\end{aligned}
$$

Which simplifies to

$$
\left(-\frac{-3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}
$$

Summary

The solution(s) found are the following

$$
\left(-\frac{-3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}} \tag{1}
$$

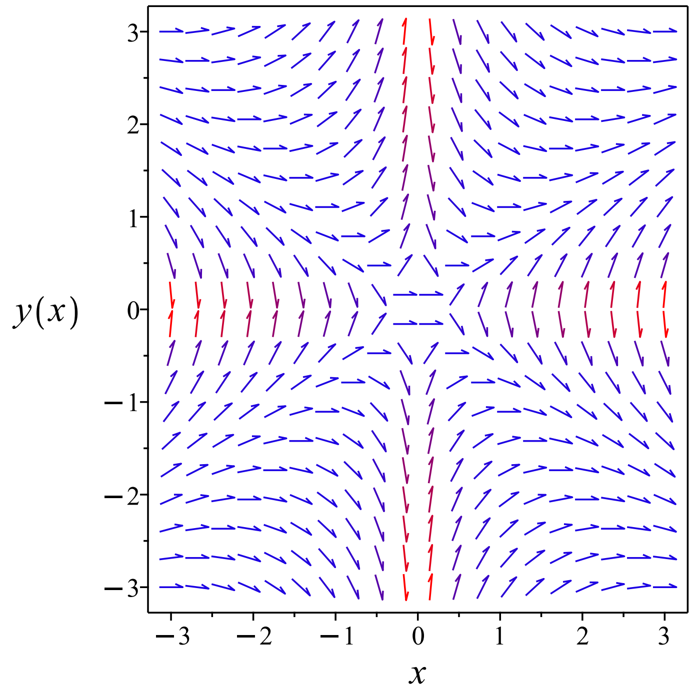


Figure 106: Slope field plot

Verification of solutions

$$\left(-\frac{-3y^2+x^2}{x^2}\right)^{\frac{1}{6}} = \frac{c_3 \mathrm{e}^{c_2}}{\sqrt{x}}$$

Verified OK.

2.39.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{-x^2+y^2}{2xy} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 65: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{yx}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2 x}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{-x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= xy \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{6} + c_1 \tag{4}$$

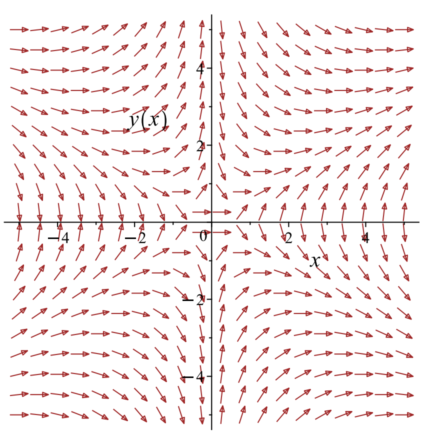
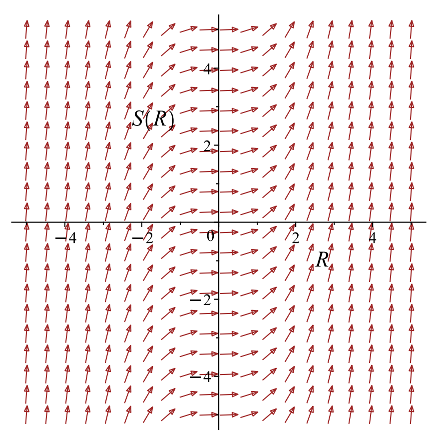
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x y^2}{2} = \frac{x^3}{6} + c_1$$

Which simplifies to

$$\frac{x y^2}{2} = \frac{x^3}{6} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2 x}{2}$	$\frac{dS}{dR} = \frac{R^2}{2}$ 

Summary

The solution(s) found are the following

$$\frac{x y^2}{2} = \frac{x^3}{6} + c_1 \tag{1}$$

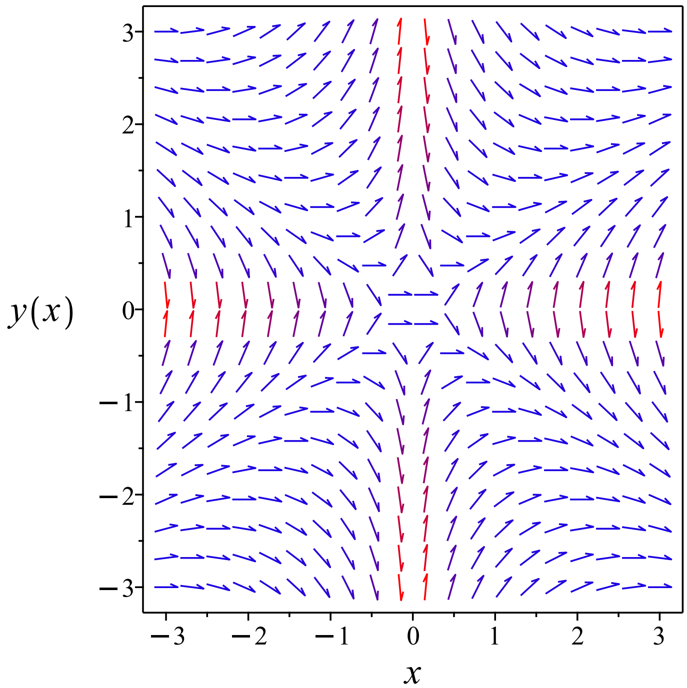


Figure 107: Slope field plot

Verification of solutions

$$\frac{xy^2}{2} = \frac{x^3}{6} + c_1$$

Verified OK.

2.39.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-x^2+y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= \frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{2x} + \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= -\frac{w(x)}{2x} + \frac{x}{2} \\
w' &= -\frac{w}{x} + x \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}(xw) &= (x)(x) \\ d(xw) &= x^2\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int x^2\,dx \\ xw &= \frac{x^3}{3} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = \frac{x^2}{3} + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = \frac{x^2}{3} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x} \\ y(x) &= -\frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x} \tag{1}$$

$$y = -\frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x} \tag{2}$$

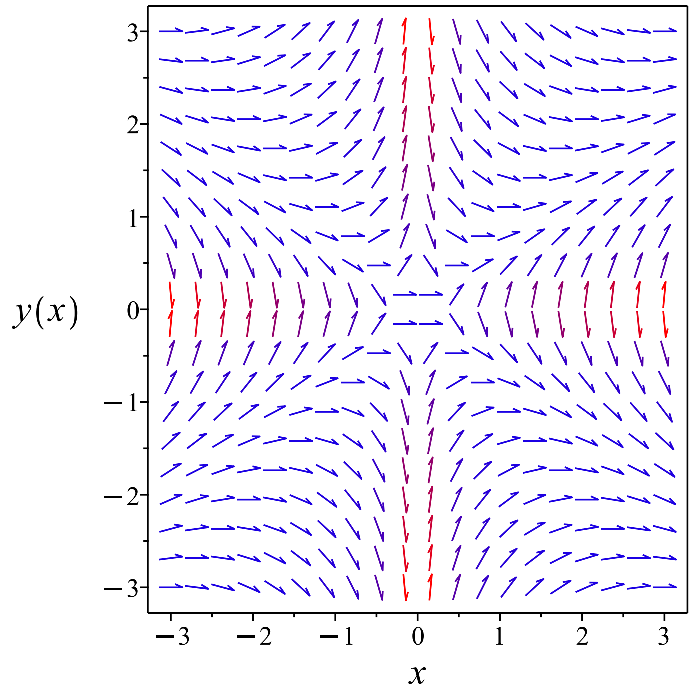


Figure 108: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{3}\,\sqrt{x\,(x^3 + 3c_1)}}{3x}$$

Verified OK.

$$y = -\frac{\sqrt{3}\,\sqrt{x\,(x^3 + 3c_1)}}{3x}$$

Verified OK.

2.39.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2xy)\,\mathrm{d}y&=\left(x^2-y^2\right)\mathrm{d}x\\ \left(-x^2+y^2\right)\mathrm{d}x+(2xy)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2+y^2\\ N(x,y)&=2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^2+y^2\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2xy)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x^2+y^2\,\mathrm{d}x\end{aligned}$$
$$\phi = -\frac{1}{3}x^3 + y^2x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2xy + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2xy$. Therefore equation (4) becomes

$$2xy = 2xy + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{3}x^3 + y^2x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{3}x^3 + y^2 x$$

Summary
The solution(s) found are the following

$$-\frac{x^3}{3} + x y^2 = c_1 \tag{1}$$

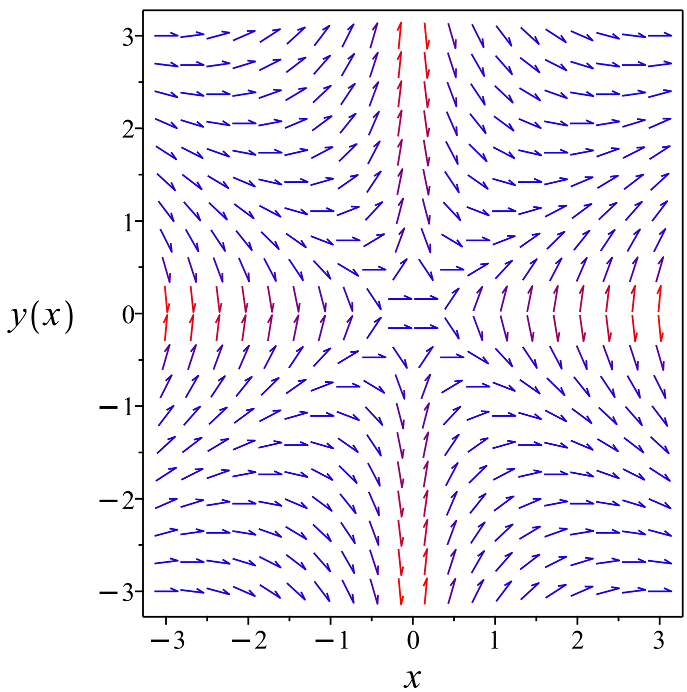


Figure 109: Slope field plot

Verification of solutions

$$-\frac{x^3}{3} + x y^2 = c_1$$

Verified OK.

2.39.5 Maple step by step solution

Let's solve

$2xyy' + y^2 = x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $2y = 2y$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (-x^2 + y^2) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = -\frac{x^3}{3} + y^2 x + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2xy = 2xy + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 0$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = -\frac{1}{3}x^3 + y^2x$$

- Substitute $F(x,y)$ into the solution of the ODE

$$-\frac{1}{3}x^3 + y^2x = c_1$$

- Solve for y

$$\left\{y = -\frac{\sqrt{3}\,\sqrt{x(x^3+3c_1)}}{3x}, y = \frac{\sqrt{3}\,\sqrt{x(x^3+3c_1)}}{3x}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 45

```
dsolve(2*x*y(x)*diff(y(x),x)=x^2-y(x)^2,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x}$$
$$y(x) = \frac{\sqrt{3}\,\sqrt{x\,(x^3+3c_1)}}{3x}$$

✓ Solution by Mathematica

Time used: 0.2 (sec). Leaf size: 56

```
DSolve[2*x*y[x]*y'[x]==x^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{x^3+3c_1}}{\sqrt{3}\sqrt{x}}$$

2.40 problem 40

Internal problem ID [5126]
Internal file name [`OUTPUT/4619_Sunday_June_05_2022_03_02_07_PM_33169822/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{-2y + x + 1}{2x - 4y} = 0}$$

With initial conditions

$$[y(1) = 1]$$

2.40.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{2y - x - 1}{-2x + 4y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{x < 2 \lor 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{2y - x - 1}{-2x + 4y}\right)\\ &= \frac{1}{-x + 2y} - \frac{2y - x - 1}{(-x + 2y)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{x < 2 \vee 2 < x\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\left\{y < \frac{1}{2} \vee \frac{1}{2} < y\right\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.40.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2y + x + 1}{2x - 4y} \tag{1}$$

Which becomes

$$(-4y)\, dy = (-2x)\, dy + (-2y + x + 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-2x)\, dy + (-2y + x + 1)\, dx = d\left(\frac{1}{2}x^2 - 2xy + x\right)$$

Hence (2) becomes

$$(-4y)\, dy = d\left(\frac{1}{2}x^2 - 2xy + x\right)$$

Integrating both sides gives gives these solutions

$$
\begin{aligned}
y &= \frac{x}{2}+\frac{\sqrt{-2c_1-2x}}{2}+c_1 \\
y &= \frac{x}{2}-\frac{\sqrt{-2c_1-2x}}{2}+c_1
\end{aligned}
$$

Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$
1=\frac{1}{2}-\frac{\sqrt{-2c_1-2}}{2}+c_1
$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x=1$ and $y=1$ in the above solution gives an equation to solve for the constant of integration.

$$
1=\frac{1}{2}+\frac{\sqrt{-2c_1-2}}{2}+c_1
$$

$$
c_1=\frac{1}{4}-\frac{i\sqrt{11}}{4}
$$

Substituting c_1 found above in the general solution gives

$$
y=\frac{x}{2}+\frac{\sqrt{-2+2i\sqrt{11}-8x}}{4}+\frac{1}{4}-\frac{i\sqrt{11}}{4}
$$

Summary

The solution(s) found are the following

$$
y=\frac{x}{2}+\frac{\sqrt{-2+2i\sqrt{11}-8x}}{4}+\frac{1}{4}-\frac{i\sqrt{11}}{4} \tag{1}
$$

Verification of solutions

$$
y=\frac{x}{2}+\frac{\sqrt{-2+2i\sqrt{11}-8x}}{4}+\frac{1}{4}-\frac{i\sqrt{11}}{4}
$$

Verified OK.

2.40.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2y - x - 1}{-2x + 4y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \frac{(2y - x - 1)(b_3 - a_2)}{-2x + 4y} - \frac{(2y - x - 1)^2 a_3}{4(-x + 2y)^2} \\
& - \left(-\frac{1}{2(-x + 2y)} + \frac{2y - x - 1}{2(-x + 2y)^2} \right) (x a_2 + y a_3 + a_1) \\
& - \left(\frac{1}{-x + 2y} - \frac{2y - x - 1}{(-x + 2y)^2} \right) (x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^2 a_2 + x^2 a_3 - 4x^2 b_2 - 2x^2 b_3 - 8xy a_2 - 4xy a_3 + 16xy b_2 + 8xy b_3 + 8y^2 a_2 + 4y^2 a_3 - 16y^2 b_2 - 8y^2 b_3 +}{4(x - 2y)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -2x^2 a_2 - x^2 a_3 + 4x^2 b_2 + 2x^2 b_3 + 8xy a_2 + 4xy a_3 - 16xy b_2 - 8xy b_3 - 8y^2 a_2 \\
& - 4y^2 a_3 + 16y^2 b_2 + 8y^2 b_3 - 2x a_3 - 4x b_2 + 2x b_3 + 4y a_2 + 6y a_3 - 8y b_3 + 2a_1 - a_3 \\
& - 4b_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_2v_1^2 + 8a_2v_1v_2 - 8a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 - 4a_3v_2^2 + 4b_2v_1^2 \\ &- 16b_2v_1v_2 + 16b_2v_2^2 + 2b_3v_1^2 - 8b_3v_1v_2 + 8b_3v_2^2 + 4a_2v_2 \\ &- 2a_3v_1 + 6a_3v_2 - 4b_2v_1 + 2b_3v_1 - 8b_3v_2 + 2a_1 - a_3 - 4b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-2a_2 - a_3 + 4b_2 + 2b_3)\, v_1^2 + (8a_2 + 4a_3 - 16b_2 - 8b_3)\, v_1v_2 + (-2a_3 - 4b_2 + 2b_3)\, v_1 \\ &+ (-8a_2 - 4a_3 + 16b_2 + 8b_3)\, v_2^2 + (4a_2 + 6a_3 - 8b_3)\, v_2 + 2a_1 - a_3 - 4b_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 - a_3 - 4b_1 &= 0 \\ 4a_2 + 6a_3 - 8b_3 &= 0 \\ -2a_3 - 4b_2 + 2b_3 &= 0 \\ -8a_2 - 4a_3 + 16b_2 + 8b_3 &= 0 \\ -2a_2 - a_3 + 4b_2 + 2b_3 &= 0 \\ 8a_2 + 4a_3 - 16b_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_2 + 2b_1 - 4b_2 \\ a_2 &= a_2 \\ a_3 &= 2a_2 - 8b_2 \\ b_1 &= b_1 \\ b_2 &= b_2 \\ b_3 &= 2a_2 - 6b_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 2 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= 1 - \left(\frac{2y - x - 1}{-2x + 4y}\right)(2) \\
&= -\frac{1}{x - 2y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{1}{x - 2y}} dy
\end{aligned}
$$

Which results in

$$
S = -xy + y^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y - x - 1}{-2x + 4y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y \\ S_y &= -x + 2y \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{x}{2} - \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{2} - \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{4}R^2 - \frac{1}{2}R + c_1 \tag{4}$$

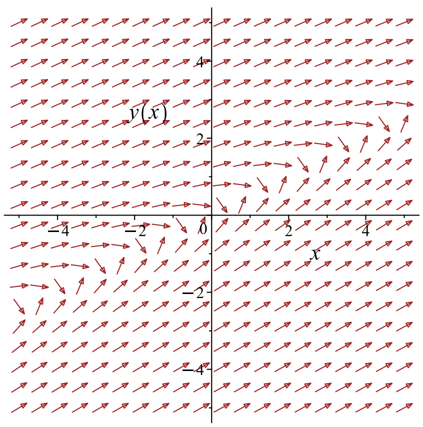
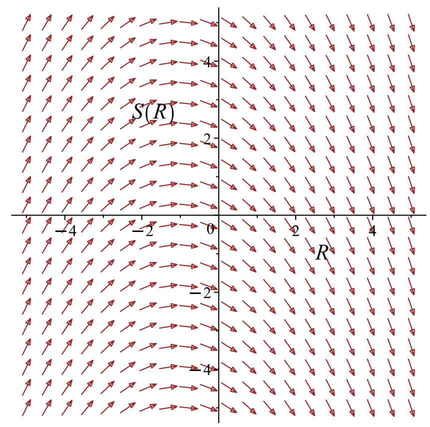
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-y(x - y) = -\frac{1}{4}x^2 - \frac{1}{2}x + c_1$$

Which simplifies to

$$-y(x - y) = -\frac{1}{4}x^2 - \frac{1}{2}x + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{2y-x-1}{-2x+4y}$
</td><td>$R = x$
$S = -y(x-y)$</td><td>$\frac{dS}{dR} = -\frac{R}{2} - \frac{1}{2}$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\frac{3}{4} + c_1$$

$$c_1 = \frac{3}{4}$$

Substituting c_1 found above in the general solution gives

$$-y(x-y) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}$$

Summary

The solution(s) found are the following

$$-y(x-y) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4} \tag{1}$$

Verification of solutions

$$-y(x-y) = -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{3}{4}$$

Verified OK.

2.40.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-2x + 4y)\,\mathrm{d}y &= (2y - x - 1)\,\mathrm{d}x\\ (-2y + x + 1)\,\mathrm{d}x + (-2x + 4y)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2y + x + 1\\ N(x,y) &= -2x + 4y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2y + x + 1)\\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-2x + 4y)\\ &= -2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -2y + x + 1 \,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(x - 4y + 2)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -2x + 4y$. Therefore equation (4) becomes

$$-2x + 4y = -2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (4y)\,\mathrm{d}y$$
$$f(y) = 2y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x-4y+2)}{2} + 2y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x-4y+2)}{2} + 2y^2$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{x(x-4y+2)}{2} + 2y^2 = \frac{3}{2}$$

Summary
The solution(s) found are the following

$$\frac{x^2}{2} - 2xy + 2y^2 + x = \frac{3}{2} \tag{1}$$

Verification of solutions

$$\frac{x^2}{2} - 2xy + 2y^2 + x = \frac{3}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1/2, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.234 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)=(x-2*y(x)+1)/(2*x-4*y(x)),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x}{2} + \frac{\sqrt{-2x+3}}{2}$$

✓ Solution by Mathematica

Time used: 0.115 (sec). Leaf size: 24

```
DSolve[{y'[x]==(x-2*y[x]+1)/(2*x-4*y[x]),{y[1]==1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x - i\sqrt{2x-3}\right)$$

2.41 problem 41

Internal problem ID [5127]
Internal file name [`OUTPUT/4620_Sunday_June_05_2022_03_02_08_PM_70738348/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(-x^3+1)\,y'+yx^2=x^2(-x^3+1)}$$

2.41.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{x^2}{x^3-1}$$
$$q(x)=x^2$$

Hence the ode is

$$y'-\frac{x^2y}{x^3-1}=x^2$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{x^2}{x^3-1}dx} \\
&= \frac{1}{\left(x^3-1\right)^{\frac{1}{3}}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\left(x^3-1\right)^{\frac{1}{3}}}\right) &= \left(\frac{1}{\left(x^3-1\right)^{\frac{1}{3}}}\right)\left(x^2\right) \\
\mathrm{d}\left(\frac{y}{\left(x^3-1\right)^{\frac{1}{3}}}\right) &= \left(\frac{x^2}{\left(x^3-1\right)^{\frac{1}{3}}}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\left(x^3-1\right)^{\frac{1}{3}}} &= \int \frac{x^2}{\left(x^3-1\right)^{\frac{1}{3}}}\,\mathrm{d}x \\
\frac{y}{\left(x^3-1\right)^{\frac{1}{3}}} &= \frac{(x-1)\left(x^2+x+1\right)}{2\left(x^3-1\right)^{\frac{1}{3}}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{\left(x^3-1\right)^{\frac{1}{3}}}$ results in

$$
y = \frac{(x-1)\left(x^2+x+1\right)}{2} + c_1\left(x^3-1\right)^{\frac{1}{3}}
$$

which simplifies to

$$
y = \frac{x^3}{2} - \frac{1}{2} + c_1\left(x^3-1\right)^{\frac{1}{3}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3}{2} - \frac{1}{2} + c_1\left(x^3-1\right)^{\frac{1}{3}} \tag{1}
$$

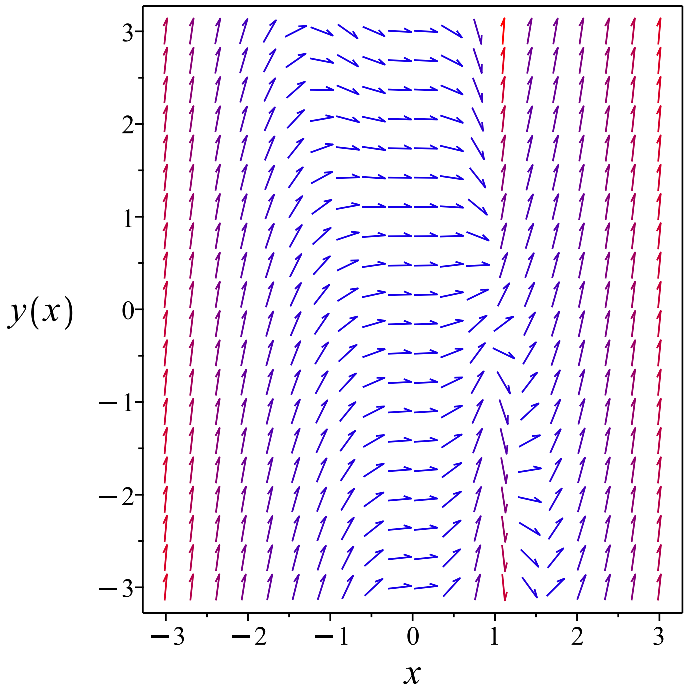


Figure 110: Slope field plot

<u>Verification of solutions</u>

$$
y = \frac{x^3}{2} - \frac{1}{2} + c_1 (x^3 - 1)^{\frac{1}{3}}
$$

Verified OK.

2.41.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^2(x^3 + y - 1)}{x^3 - 1} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 68: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \left(x^3 - 1\right)^{\frac{1}{3}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(x^3-1)^{\frac{1}{3}}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(x^3-1)^{\frac{1}{3}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2(x^3+y-1)}{x^3-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y\, x^2}{(x^3-1)^{\frac{4}{3}}} \\ S_y &= \frac{1}{(x^3-1)^{\frac{1}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{(x^3-1)^{\frac{1}{3}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^2}{(R^3-1)^{\frac{1}{3}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{(R-1)(R^2+R+1)}{2(R^3-1)^{\frac{1}{3}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{(x^3-1)^{\frac{1}{3}}} = \frac{(x-1)(x^2+x+1)}{2(x^3-1)^{\frac{1}{3}}} + c_1$$

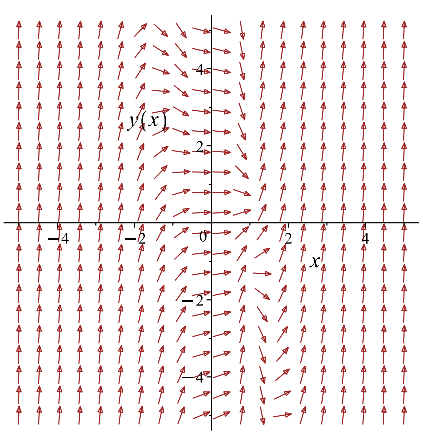
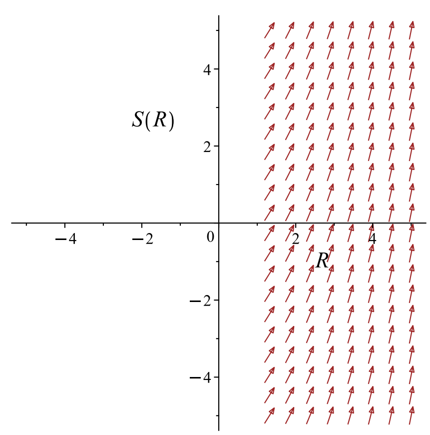
Which simplifies to

$$-\frac{x^3 + 2c_1(x^3-1)^{\frac{1}{3}} - 2y - 1}{2(x^3-1)^{\frac{1}{3}}} = 0$$

Which gives

$$y = \frac{x^3}{2} - \frac{1}{2} + c_1(x^3-1)^{\frac{1}{3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2(x^3+y-1)}{x^3-1}$ 	$R = x$ $S = \frac{y}{(x^3-1)^{\frac{1}{3}}}$	$\frac{dS}{dR} = \frac{R^2}{(R^3-1)^{\frac{1}{3}}}$ 

Summary

The solution(s) found are the following

$$y = \frac{x^3}{2} - \frac{1}{2} + c_1 \left(x^3 - 1\right)^{\frac{1}{3}} \tag{1}$$

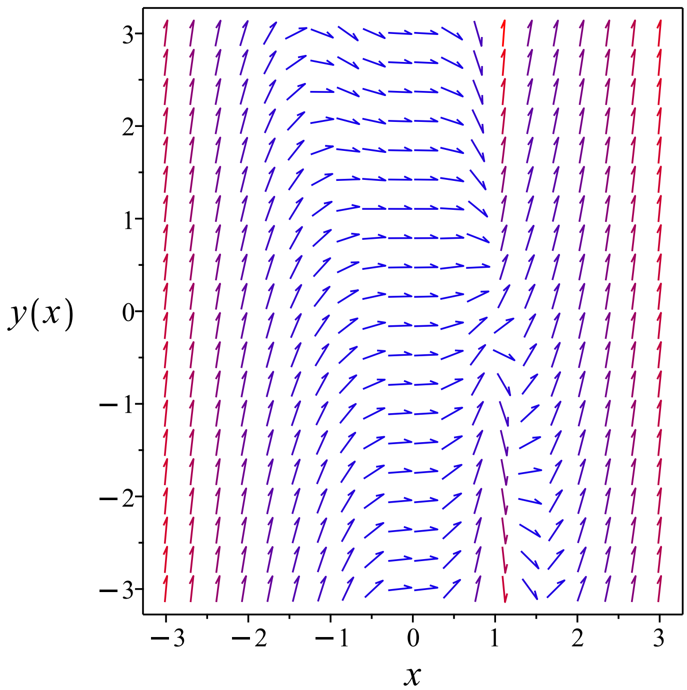


Figure 111: Slope field plot

Verification of solutions

$$y = \frac{x^3}{2} - \frac{1}{2} + c_1 \left(x^3 - 1\right)^{\frac{1}{3}}$$

Verified OK.

2.41.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(-x^3 + 1\right) \mathrm{d}y &= \left(-y\, x^2 + x^2\left(-x^3 + 1\right)\right) \mathrm{d}x \\
\left(y\, x^2 - x^2\left(-x^3 + 1\right)\right) \mathrm{d}x + \left(-x^3 + 1\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y\, x^2 - x^2\left(-x^3 + 1\right) \\
N(x, y) &= -x^3 + 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y\, x^2 - x^2\left(-x^3 + 1\right)\right) \\
&= x^2
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^3+1\right)\\
&= -3x^2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= -\frac{1}{x^3-1}\left(\left(x^2\right)-\left(-3x^2\right)\right)\\
&= -\frac{4x^2}{x^3-1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x}\\
&= e^{\int -\frac{4x^2}{x^3-1}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{4\ln\left(x^3-1\right)}{3}}\\
&= \frac{1}{\left(x^3-1\right)^{\frac{4}{3}}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M\\
&= \frac{1}{\left(x^3-1\right)^{\frac{4}{3}}}\left(y\,x^2 - x^2\left(-x^3+1\right)\right)\\
&= \frac{x^2(x^3+y-1)}{\left(x^3-1\right)^{\frac{4}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N\\
&= \frac{1}{\left(x^3-1\right)^{\frac{4}{3}}}\left(-x^3+1\right)\\
&= -\frac{1}{\left(x^3-1\right)^{\frac{1}{3}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(\frac{x^2(x^3+y-1)}{(x^3-1)^{\frac{4}{3}}}\right) + \left(-\frac{1}{(x^3-1)^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{x^2(x^3+y-1)}{(x^3-1)^{\frac{4}{3}}}\,\mathrm{d}x$$

$$\phi = \frac{x^3-2y-1}{2\,(x^3-1)^{\frac{1}{3}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{(x^3-1)^{\frac{1}{3}}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{(x^3-1)^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$-\frac{1}{(x^3-1)^{\frac{1}{3}}} = -\frac{1}{(x^3-1)^{\frac{1}{3}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^3 - 2y - 1}{2\left(x^3 - 1\right)^{\frac{1}{3}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^3 - 2y - 1}{2\left(x^3 - 1\right)^{\frac{1}{3}}}$$

The solution becomes

$$y = \frac{x^3}{2} - c_1\left(x^3 - 1\right)^{\frac{1}{3}} - \frac{1}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3}{2} - c_1\left(x^3 - 1\right)^{\frac{1}{3}} - \frac{1}{2} \tag{1}$$

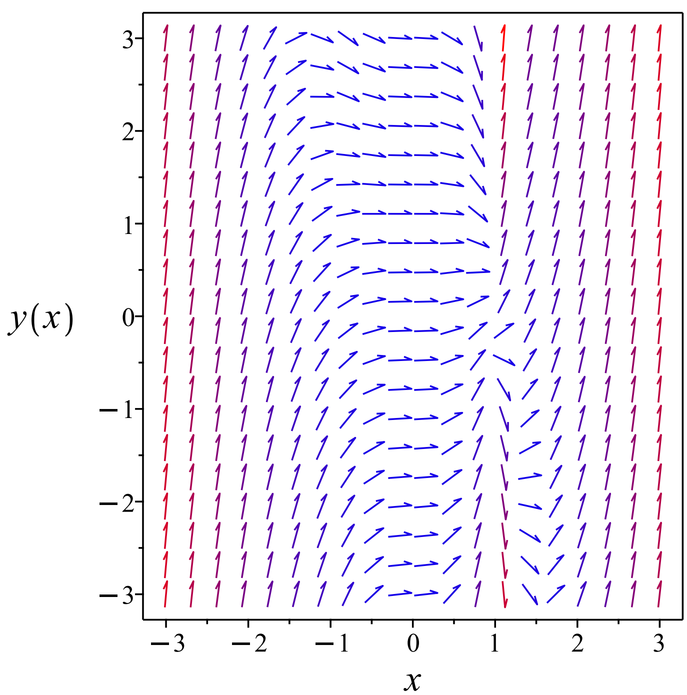


Figure 112: Slope field plot

Verification of solutions

$$y = \frac{x^3}{2} - c_1\left(x^3 - 1\right)^{\frac{1}{3}} - \frac{1}{2}$$

Verified OK.

2.41.4 Maple step by step solution

Let's solve

$(-x^3 + 1)\, y' + yx^2 = x^2(-x^3 + 1)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{x^2 y}{x^3 - 1} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{x^2 y}{x^3 - 1} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{x^2 y}{x^3 - 1}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{x^2 y}{x^3 - 1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x) x^2}{x^3 - 1}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{((x-1)(x^2+x+1))^{\frac{1}{3}}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{((x-1)(x^2+x+1))^{\frac{1}{3}}}$

  $$y = ((x-1)(x^2+x+1))^{\frac{1}{3}} \left(\int \frac{x^2}{((x-1)(x^2+x+1))^{\frac{1}{3}}} dx + c_1 \right)$$

- Evaluate the integrals on the rhs

  $$y = ((x-1)(x^2+x+1))^{\frac{1}{3}} \left(\frac{(x-1)(x^2+x+1)}{2((x-1)(x^2+x+1))^{\frac{1}{3}}} + c_1 \right)$$

- Simplify

  $$y = \frac{x^3}{2} - \frac{1}{2} + c_1((x-1)(x^2+x+1))^{\frac{1}{3}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((1-x^3)*diff(y(x),x)+x^2*y(x)=x^2*(1-x^3),y(x), singsol=all)
```

$$y(x) = \frac{x^3}{2} - \frac{1}{2} + (x^3-1)^{\frac{1}{3}} c_1$$

✓ Solution by Mathematica

Time used: 0.054 (sec). Leaf size: 27

```
DSolve[(1-x^3)*y'[x]+x^2*y[x]==x^2*(1-x^3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(x^3 + 2c_1\sqrt[3]{x^3-1} - 1\right)$$

2.42 problem 42

Internal problem ID [5128]
Internal file name [`OUTPUT/4621_Sunday_June_05_2022_03_02_09_PM_95706010/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \frac{y}{x} = \sin(x)}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = 0\right]$$

2.42.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \sin(x)$$

Hence the ode is

$$y' + \frac{y}{x} = \sin(x)$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = \frac{\pi}{2}$ is inside this domain. The domain of $q(x) = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \frac{\pi}{2}$ is also inside this domain. Hence solution exists and is unique.

2.42.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(\sin(x)) \\ \frac{d}{dx}(xy) &= (x)(\sin(x)) \\ d(xy) &= (\sin(x)\, x)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}xy &= \int \sin(x)\, x\, dx \\ xy &= -\cos(x)\, x + \sin(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$y = \frac{-\cos(x)\, x + \sin(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2 + 2c_1}{\pi}$$

$$c_1 = -1$$

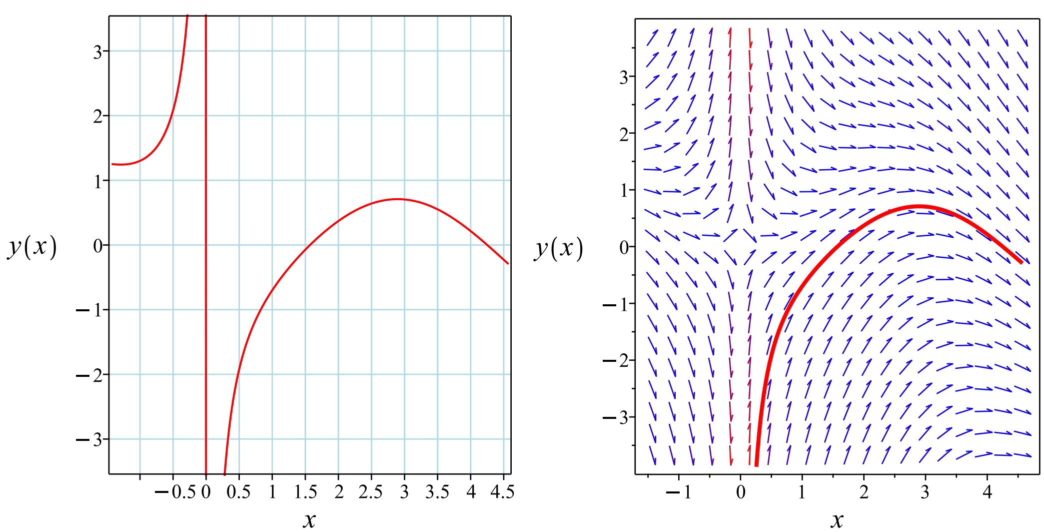
Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x}$$

Verified OK.

2.42.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\sin(x)\,x - y}{x} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 71: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = xy
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sin(x)\,x - y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \sin(x)\, x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \sin(R)\, R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) - \cos(R)\, R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$xy = -\cos(x)\, x + \sin(x) + c_1$$

Which simplifies to

$$xy = -\cos(x)\, x + \sin(x) + c_1$$

Which gives

$$y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

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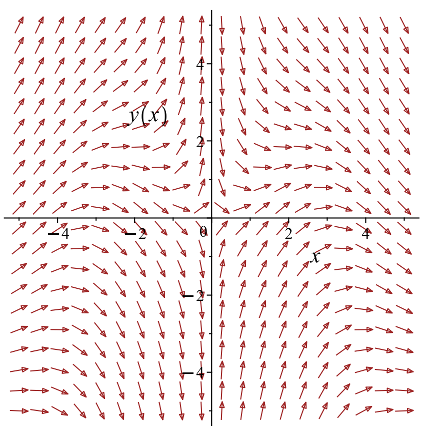
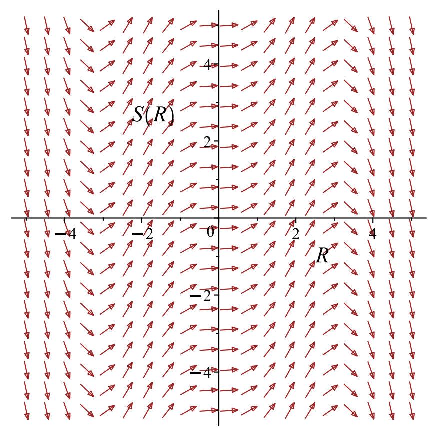
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{\sin(x)x-y}{x}$
</td><td>$R = x$
$S = xy$</td><td>$\frac{dS}{dR} = \sin(R) R$
</td></tr>
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Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2 + 2c_1}{\pi}$$

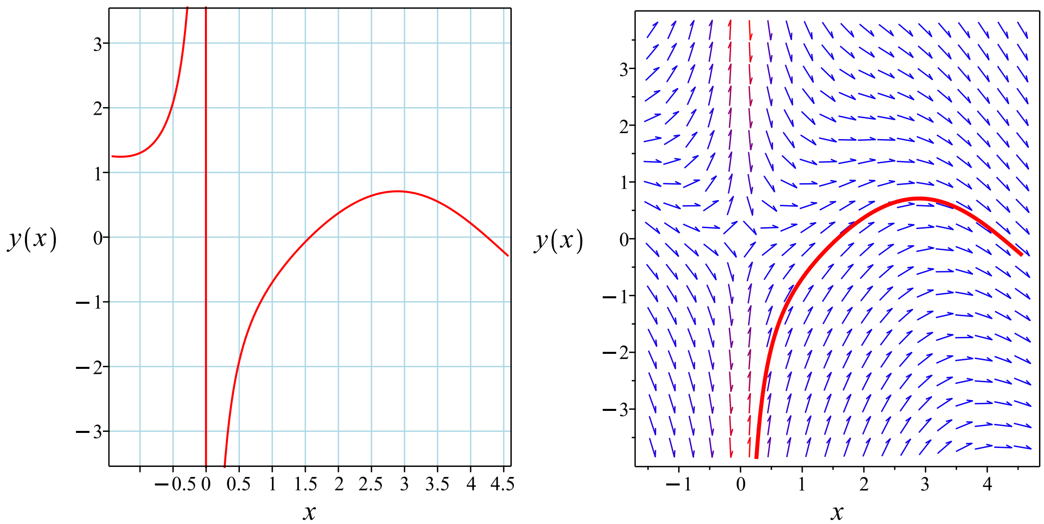
$$c_1 = -1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x)x + \sin(x) - 1}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{-\cos(x)x + \sin(x) - 1}{x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-\cos(x)\,x + \sin(x) - 1}{x}$$

Verified OK.

2.42.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(\sin(x)\,x-y)\,\mathrm{d}x\\(-\sin(x)\,x+y)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sin(x)\,x+y\\N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\sin(x)\,x+y)\\&=1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\&=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\sin(x)\, x + y \, \mathrm{d}x \\
\phi &= xy - \sin(x) + \cos(x)\, x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$
x = x + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = xy - \sin(x) + \cos(x)\, x + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = xy - \sin(x) + \cos(x)\, x
$$

The solution becomes

$$
y = \frac{-\cos(x)\, x + \sin(x) + c_1}{x}
$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{2 + 2c_1}{\pi}$$

$$c_1 = -1$$

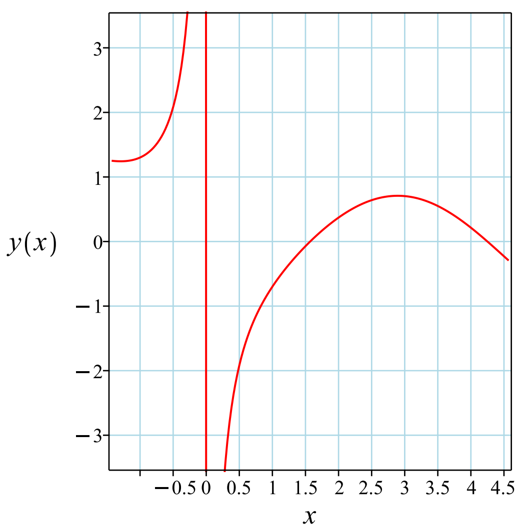
Substituting c_1 found above in the general solution gives

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x}$$

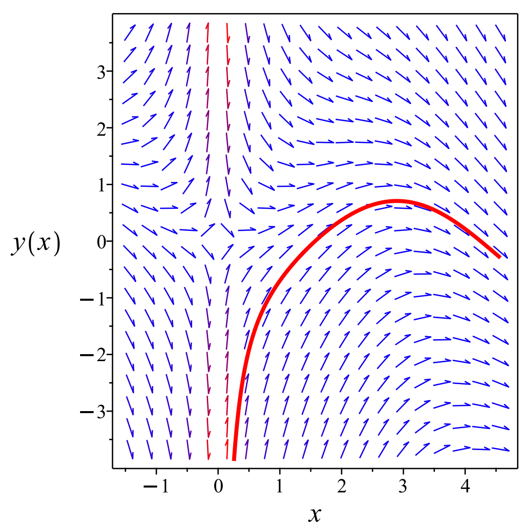
Summary

The solution(s) found are the following

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{-\cos(x)\, x + \sin(x) - 1}{x}$$

Verified OK.

2.42.5 Maple step by step solution

Let's solve

$\left[y' + \frac{y}{x} = \sin\left(x\right), y\left(\frac{\pi}{2}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \sin\left(x\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \sin\left(x\right)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu(x)\sin\left(x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\sin\left(x\right) dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\sin\left(x\right) dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int \sin(x)x dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{-\cos(x)x + \sin(x) + c_1}{x}$

- Use initial condition $y\left(\frac{\pi}{2}\right) = 0$

  $0 = \frac{2(c_1+1)}{\pi}$

- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $y = \frac{-\cos(x)x+\sin(x)-1}{x}$
- Solution to the IVP

  $y = \frac{-\cos(x)x+\sin(x)-1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(y(x),x)+y(x)/x=sin(x),y(1/2*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sin(x) - \cos(x)x - 1}{x}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 18

```
DSolve[{y'[x]+y[x]/x==Sin[x],{y[Pi/2]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x) - x\cos(x) - 1}{x}$$

2.43 problem 43

Internal problem ID [5129]
Internal file name [`OUTPUT/4622_Sunday_June_05_2022_03_02_11_PM_39630131/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 43.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + xy^2 = -x}$$

With initial conditions

$$[y(1) = 0]$$

2.43.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -y^2x - x \end{aligned}$$

The x domain of $f(x, y)$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(-y^2 x - x) \\ &= -2xy\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.43.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(-y^2 - 1)\end{aligned}$$

Where $f(x) = x$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\begin{aligned}\frac{1}{-y^2 - 1}\,dy &= x\,dx \\ \int \frac{1}{-y^2 - 1}\,dy &= \int x\,dx \\ -\arctan(y) &= \frac{x^2}{2} + c_1\end{aligned}$$

Which results in

$$y = -\tan\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan\left(\frac{1}{2} + c_1\right)$$

$$c_1 = -\frac{1}{2}$$

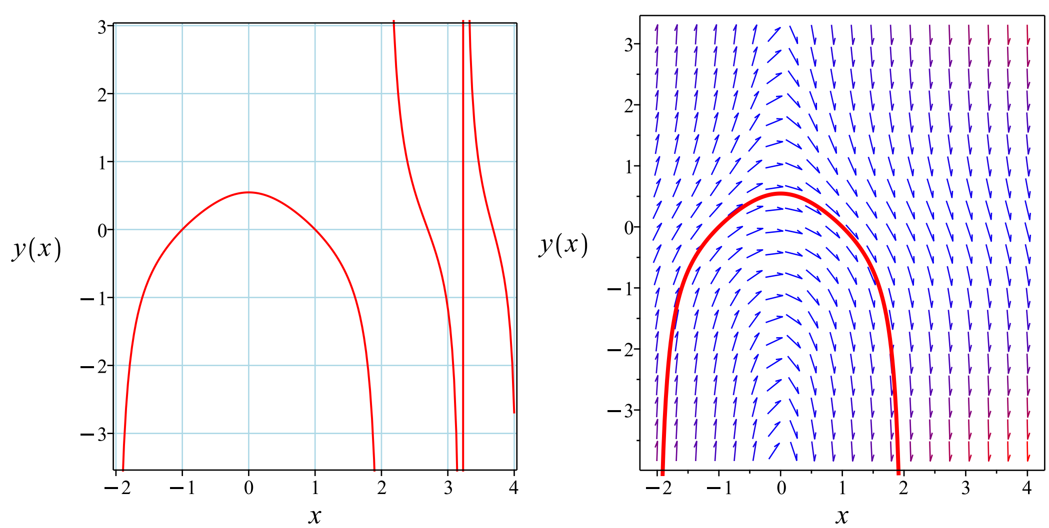
Substituting c_1 found above in the general solution gives

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

Verified OK.

2.43.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y^2x - x\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 74: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -y^2 x - x
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\arctan(y) + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\arctan(y) + c_1$$

Which gives

$$y = \tan\left(-\frac{x^2}{2} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

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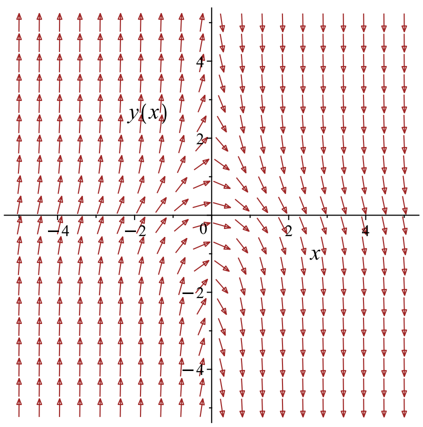</td>
<td>$R = y$
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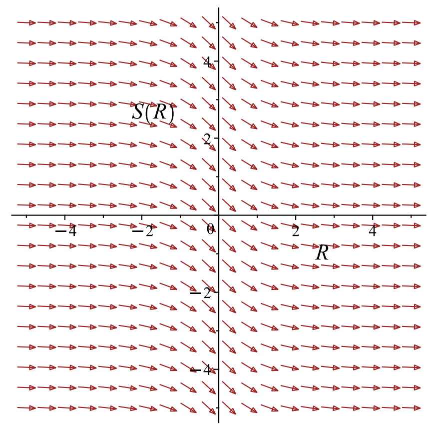
<td>$\frac{dS}{dR} = -\frac{1}{R^2+1}$
</td>
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Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \tan\left(-\frac{1}{2} + c_1\right)$$

$$c_1 = \frac{1}{2}$$

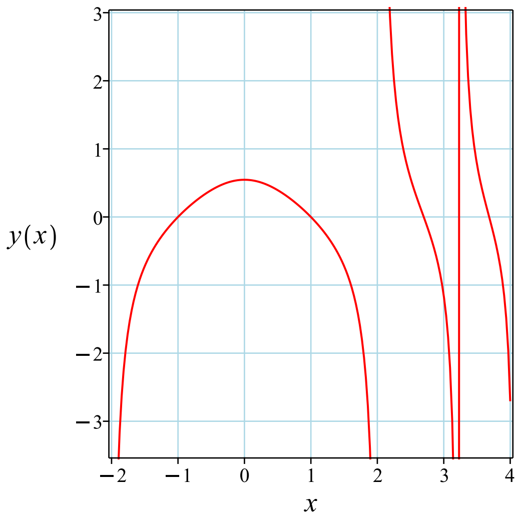
Substituting c_1 found above in the general solution gives

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

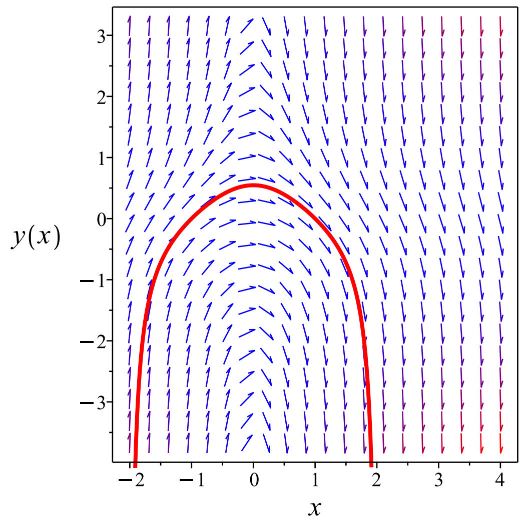
Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

Verified OK.

2.43.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-y^2 - 1}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{-y^2 - 1}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x \\ N(x, y) &= \frac{1}{-y^2 - 1} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y^2 - 1}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -x\,\mathrm{d}x$$
$$\phi=-\frac{x^2}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{-y^2-1}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=-\frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(-\frac{1}{y^2+1}\right)\mathrm{d}y$$
$$f(y)=-\arctan(y)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \arctan(y)$$

The solution becomes

$$y = -\tan\left(\frac{x^2}{2} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\tan\left(\frac{1}{2} + c_1\right)$$

$$c_1 = -\frac{1}{2}$$

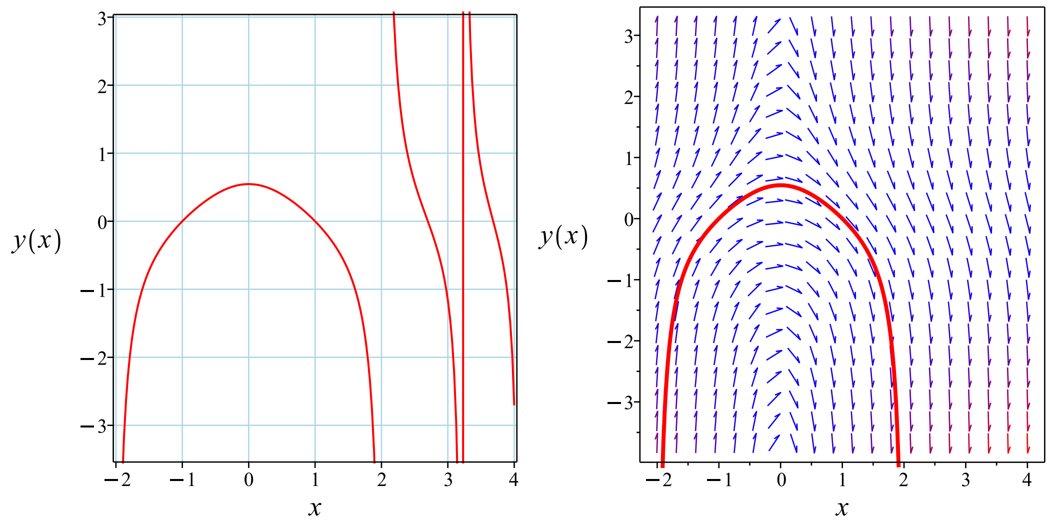
Substituting c_1 found above in the general solution gives

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

Verified OK.

2.43.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -y^2x - x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2x - x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -x$, $f_1(x) = 0$ and $f_2(x) = -x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -1 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= -x^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-xu''(x) + u'(x) - x^3u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)$$

The above shows that

$$u'(x) = x\left(c_1 \cos\left(\frac{x^2}{2}\right) - c_2 \sin\left(\frac{x^2}{2}\right)\right)$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \cos\left(\frac{x^2}{2}\right) - c_2 \sin\left(\frac{x^2}{2}\right)}{c_1 \sin\left(\frac{x^2}{2}\right) + c_2 \cos\left(\frac{x^2}{2}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\sin\left(\frac{x^2}{2}\right) + c_3 \cos\left(\frac{x^2}{2}\right)}{c_3 \sin\left(\frac{x^2}{2}\right) + \cos\left(\frac{x^2}{2}\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{c_3 \cos\left(\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right)}{c_3 \sin\left(\frac{1}{2}\right) + \cos\left(\frac{1}{2}\right)}$$

$$
c_3 = \frac{\sin\left(\frac{1}{2}\right)}{\cos\left(\frac{1}{2}\right)}
$$

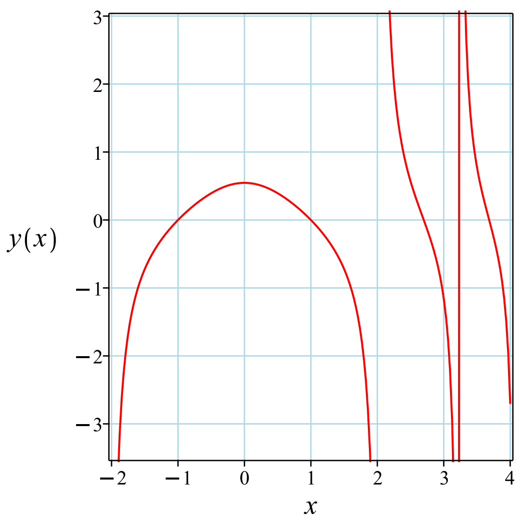
Substituting c_3 found above in the general solution gives

$$
y = \frac{\cos\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) - \sin\left(\frac{x^2}{2}\right)}{\sin\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) + \cos\left(\frac{x^2}{2}\right)}
$$

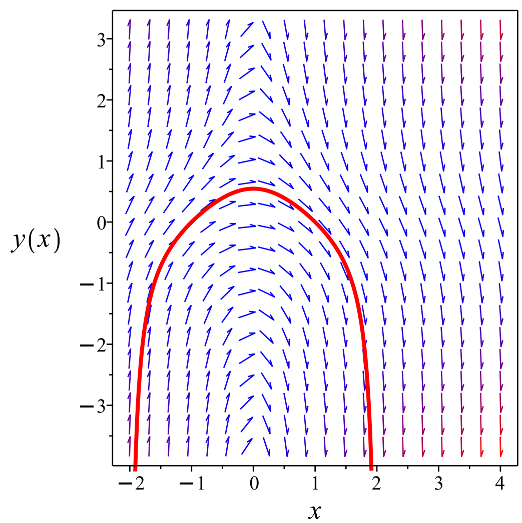
Summary

The solution(s) found are the following

$$
y = \frac{\cos\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) - \sin\left(\frac{x^2}{2}\right)}{\sin\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) + \cos\left(\frac{x^2}{2}\right)} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \frac{\cos\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) - \sin\left(\frac{x^2}{2}\right)}{\sin\left(\frac{x^2}{2}\right)\tan\left(\frac{1}{2}\right) + \cos\left(\frac{x^2}{2}\right)}
$$

Verified OK.

2.43.6 Maple step by step solution

Let's solve

$[y' + x y^2 = -x, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = -x$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int -x dx + c_1$

- Evaluate integral

  $\arctan(y) = -\frac{x^2}{2} + c_1$

- Solve for y

  $y = \tan\left(-\frac{x^2}{2} + c_1\right)$

- Use initial condition $y(1) = 0$

  $0 = \tan\left(-\frac{1}{2} + c_1\right)$

- Solve for c_1

  $c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$

- Solution to the IVP

  $y = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 14

```
dsolve([diff(y(x),x)+x+x*y(x)^2=0,y(1) = 0],y(x), singsol=all)
```

$$y(x) = -\tan\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 17

```
DSolve[{y'[x]+x+x*y[x]^2==0,{y[1]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{1}{2}(1 - x^2)\right)$$

2.44 problem 44

Internal problem ID [5130]
Internal file name [`OUTPUT/4623_Sunday_June_05_2022_03_02_12_PM_91376671/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 44.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + \left(\frac{1}{x} - \frac{2x}{-x^2+1}\right) y = \frac{1}{-x^2+1}}$$

2.44.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-3x^2+1}{x^3-x}$$
$$q(x) = -\frac{1}{x^2-1}$$

Hence the ode is

$$y' - \frac{(-3x^2+1)\,y}{x^3-x} = -\frac{1}{x^2-1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{-3x^2+1}{x^3-x}dx} \\
&= x(x^2-1)
\end{aligned}
$$

Which simplifies to

$$
\mu = x^3 - x
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(-\frac{1}{x^2-1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^3-x\right)y\right) &= \left(x^3-x\right)\left(-\frac{1}{x^2-1}\right) \\
\mathrm{d}\left(\left(x^3-x\right)y\right) &= (-x)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\left(x^3-x\right)y &= \int -x\,\mathrm{d}x \\
\left(x^3-x\right)y &= -\frac{x^2}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3 - x$ results in

$$
y = -\frac{x^2}{2\left(x^3-x\right)} + \frac{c_1}{x^3-x}
$$

which simplifies to

$$
y = \frac{-x^2 + 2c_1}{2x^3 - 2x}
$$

Summary
The solution(s) found are the following

$$
y = \frac{-x^2 + 2c_1}{2x^3 - 2x} \tag{1}
$$

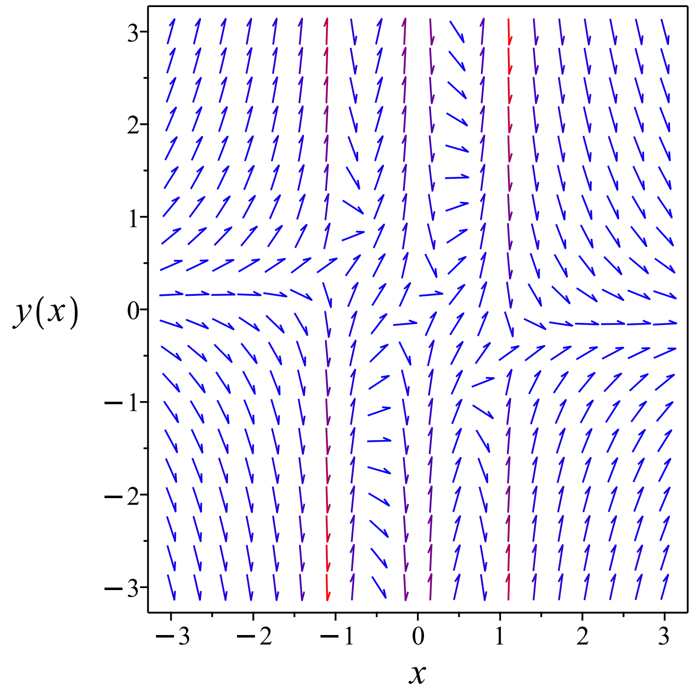


Figure 120: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x^2 + 2c_1}{2x^3 - 2x}$$

Verified OK.

2.44.2 Solving as differentialType ode

Writing the ode as

$$y' = -\left(\frac{1}{x} - \frac{2x}{-x^2 + 1}\right) y + \frac{1}{-x^2 + 1} \tag{1}$$

Which becomes

$$0 = \left(-x^3 + x\right) dy + \left(-3y\, x^2 - x + y\right) dx \tag{2}$$

But the RHS is complete differential because

$$\left(-x^3 + x\right) dy + \left(-3y\, x^2 - x + y\right) dx = d\left(-y\, x^3 - \frac{1}{2}x^2 + xy\right)$$

Hence (2) becomes

$$0 = d\left(-y\,x^3 - \frac{1}{2}x^2 + xy\right)$$

Integrating both sides gives gives these solutions

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)} + c_1 \tag{1}$$

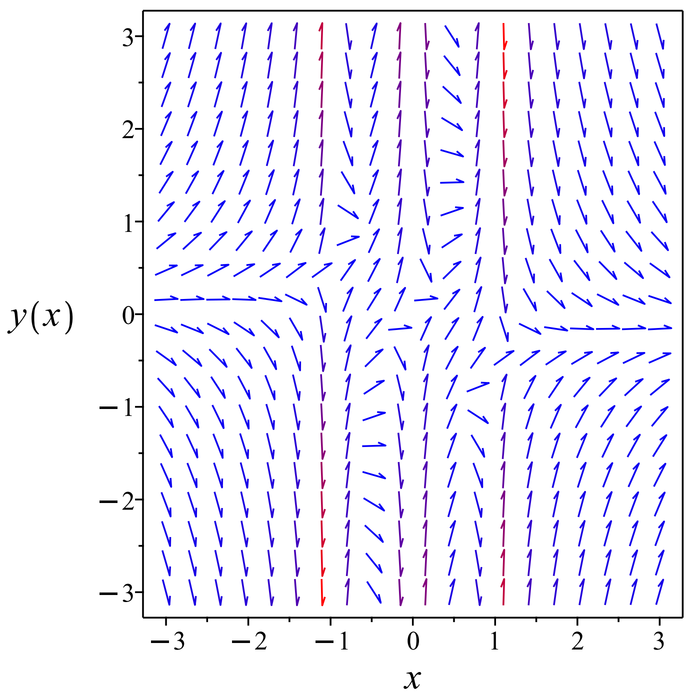


Figure 121: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)} + c_1$$

Verified OK.

2.44.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{3y\,x^2 + x - y}{x\,(x^2-1)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 77: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x\left(x^{2}-1\right)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x(x^{2}-1)}} dy
\end{aligned}
$$

Which results in

$$
S = x\left(x^{2}-1\right)y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{3y\,x^{2}+x-y}{x\left(x^{2}-1\right)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3y\,x^2 - y \\
S_y &= x^3 - x
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{R^2}{2} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\left(x^3 - x\right) y = -\frac{x^2}{2} + c_1
$$

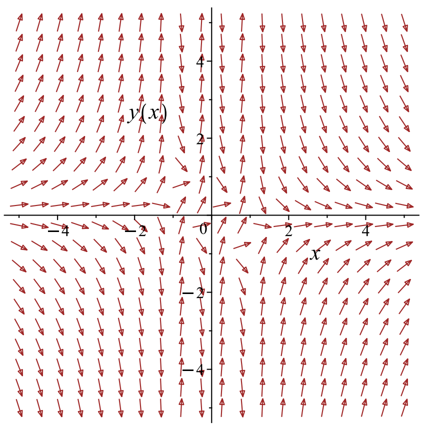
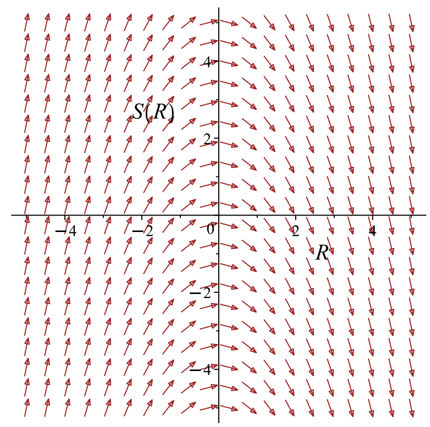
Which simplifies to

$$
\left(x^3 - x\right) y = -\frac{x^2}{2} + c_1
$$

Which gives

$$
y = \frac{-x^2 + 2c_1}{2x\left(x^2 - 1\right)}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y\,x^2+x-y}{x(x^2-1)}$ 	$R = x$ $S = \left(x^3 - x\right) y$	$\frac{dS}{dR} = -R$ 

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x\left(x^2 - 1\right)} \tag{1}$$

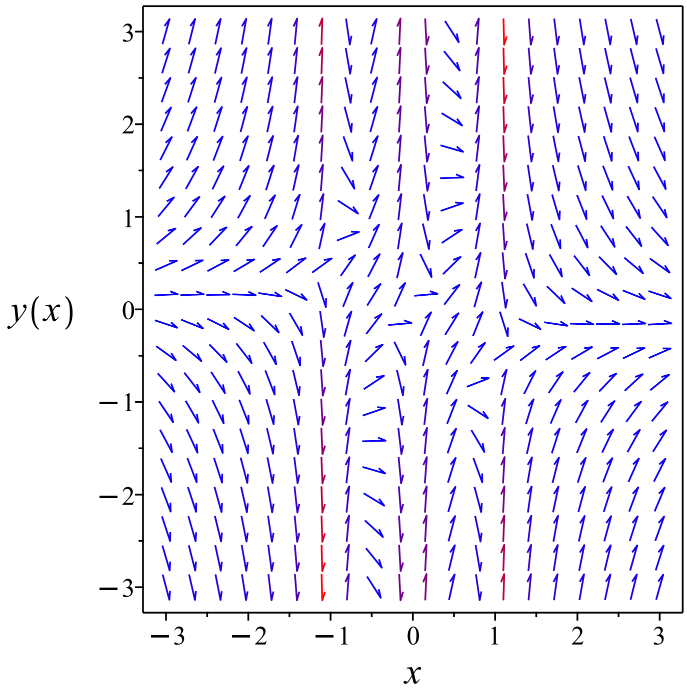


Figure 122: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-x^2 + 2c_1}{2x(x^2 - 1)}$$

Verified OK.

2.44.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(x\left(x^2-1\right)\right) \mathrm{d}y &= \left(-3y\, x^2 - x + y\right) \mathrm{d}x \\
\left(3y\, x^2 + x - y\right) \mathrm{d}x + \left(x\left(x^2-1\right)\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 3y\, x^2 + x - y \\
N(x, y) &= x\left(x^2-1\right)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y\, x^2 + x - y\right) \\
&= 3x^2 - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x\left(x^2-1\right)\right) \\
&= 3x^2 - 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int 3y\, x^2 + x - y \,\mathrm{d}x \\
\phi &= \frac{x(2y\, x^2 + x - 2y)}{2} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{x(2x^2 - 2)}{2} + f'(y) && (4) \\
&= x^3 - x + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x(x^2 - 1)$. Therefore equation (4) becomes

$$
x\left(x^2 - 1\right) = x^3 - x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{x(2y\, x^2 + x - 2y)}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(2y\,x^2 + x - 2y)}{2}$$

The solution becomes

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)}$$

Summary

The solution(s) found are the following

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)} \tag{1}$$

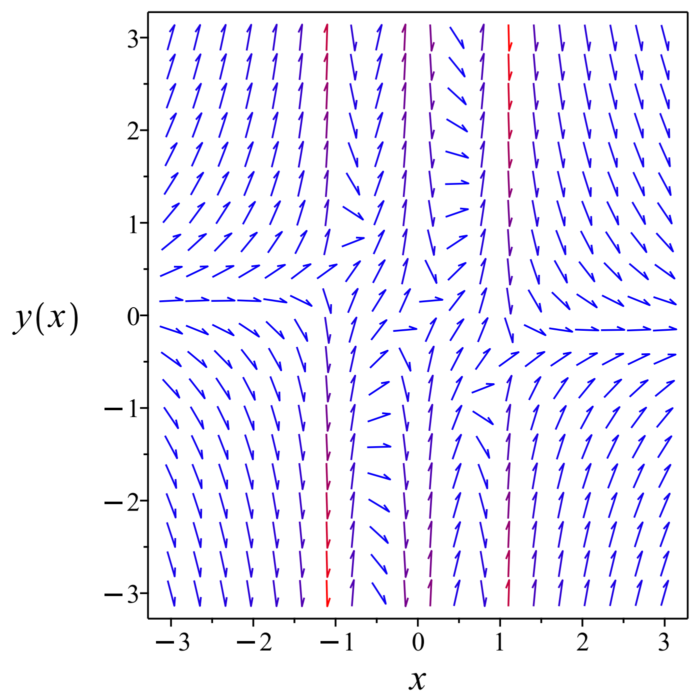


Figure 123: Slope field plot

Verification of solutions

$$y = \frac{-x^2 + 2c_1}{2x\,(x^2 - 1)}$$

Verified OK.

2.44.5 Maple step by step solution

Let's solve

$y' + \left(\frac{1}{x} - \frac{2x}{-x^2+1}\right) y = \frac{1}{-x^2+1}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(3x^2-1)y}{x(x^2-1)} - \frac{1}{x^2-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(3x^2-1)y}{x(x^2-1)} = -\frac{1}{x^2-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{(3x^2-1)y}{x(x^2-1)}\right) = -\frac{\mu(x)}{x^2-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{(3x^2-1)y}{x(x^2-1)}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(3x^2-1)}{x(x^2-1)}$

- Solve to find the integrating factor

  $\mu(x) = x(x-1)(x+1)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\frac{\mu(x)}{x^2-1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int -\frac{\mu(x)}{x^2-1} dx + c_1$

- Solve for y

  $y = \frac{\int -\frac{\mu(x)}{x^2-1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x(x-1)(x+1)$

  $y = \frac{\int -\frac{x(x-1)(x+1)}{x^2-1} dx + c_1}{x(x-1)(x+1)}$

- Evaluate the integrals on the rhs

$$y = \frac{-\frac{x^2}{2}+c_1}{x(x-1)(x+1)}$$

- Simplify

$$y = \frac{-x^2+2c_1}{2x^3-2x}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x)+(1/x-(2*x)/(1-x^2))*y(x)=1/(1-x^2),y(x), singsol=all)
```

$$y(x) = \frac{-x^2+2c_1}{2x^3-2x}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 25

```
DSolve[y'[x]+(1/x-(2*x)/(1-x^2))*y[x]==1/(1-x^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2+2c_1}{2x-2x^3}$$

2.45 problem 45

Internal problem ID [5131]
Internal file name [`OUTPUT/4624_Sunday_June_05_2022_03_02_13_PM_99733561/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 45.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(x^2+1)\,y'+xy=(x^2+1)^{\frac{3}{2}}}$$

2.45.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{x}{x^2+1}$$
$$q(x)=\sqrt{x^2+1}$$

Hence the ode is

$$y'+\frac{xy}{x^2+1}=\sqrt{x^2+1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int \frac{x}{x^2+1}dx} \\
&= \sqrt{x^2+1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\sqrt{x^2+1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x^2+1}\, y\right) &= \left(\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right) \\
\mathrm{d}\left(\sqrt{x^2+1}\, y\right) &= \left(x^2+1\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sqrt{x^2+1}\, y &= \int x^2+1\,\mathrm{d}x \\
\sqrt{x^2+1}\, y &= \frac{1}{3}x^3+x+c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sqrt{x^2+1}$ results in

$$
y = \frac{\frac{1}{3}x^3+x}{\sqrt{x^2+1}} + \frac{c_1}{\sqrt{x^2+1}}
$$

which simplifies to

$$
y = \frac{x^3+3c_1+3x}{3\sqrt{x^2+1}}
$$

Summary

The solution(s) found are the following

$$
y = \frac{x^3+3c_1+3x}{3\sqrt{x^2+1}} \tag{1}
$$

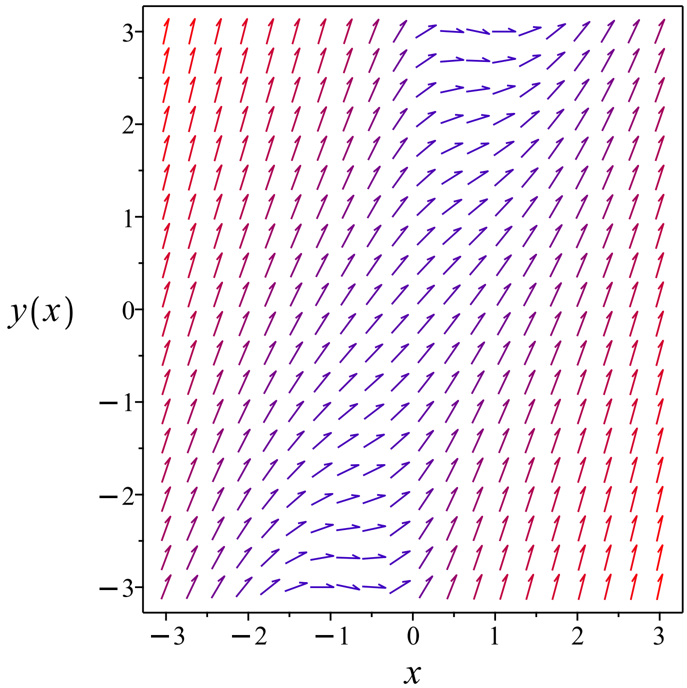


Figure 124: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2+1}}$$

Verified OK.

2.45.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-xy + (x^2+1)^{\frac{3}{2}}}{x^2+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 80: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x^2+1}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{x^2+1}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-xy + (x^2+1)^{\frac{3}{2}}}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{yx}{\sqrt{x^2+1}} \\ S_y &= \sqrt{x^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 + 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 + 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{1}{3}R^3 + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x^2+1}\, y = \frac{1}{3}x^3 + x + c_1$$

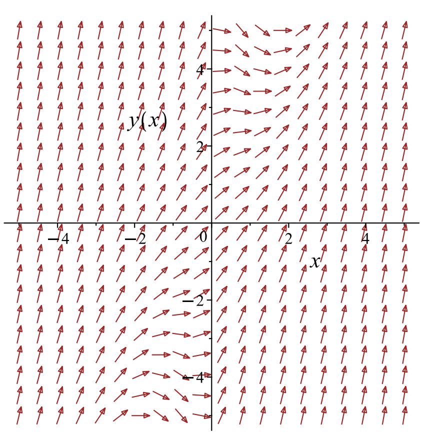
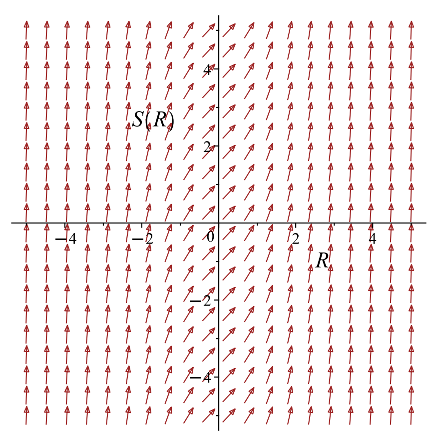
Which simplifies to

$$\sqrt{x^2+1}\, y = \frac{1}{3}x^3 + x + c_1$$

Which gives

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2+1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-xy+(x^2+1)^{\frac{3}{2}}}{x^2+1}$ 	$R = x$ $S = \sqrt{x^2+1}\, y$	$\frac{dS}{dR} = R^2 + 1$ 

Summary
The solution(s) found are the following

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2 + 1}} \tag{1}$$

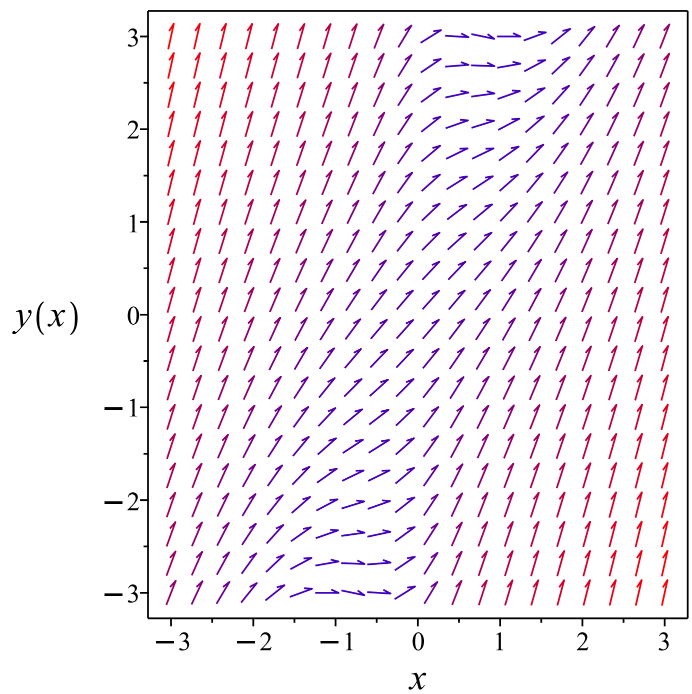


Figure 125: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2 + 1}}$$

Verified OK.

2.45.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2+1)\,\mathrm{d}y &= \left(-xy + (x^2+1)^{\frac{3}{2}}\right)\mathrm{d}x\\ \left(-(x^2+1)^{\frac{3}{2}} + xy\right)\mathrm{d}x + (x^2+1)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -(x^2+1)^{\frac{3}{2}} + xy\\ N(x,y) &= x^2+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-(x^2+1)^{\frac{3}{2}} + xy\right)\\ &= x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2+1)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^2+1}((x) - (2x))\\ &= -\frac{x}{x^2+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{x}{x^2+1} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(x^2+1)}{2}}\\ &= \frac{1}{\sqrt{x^2+1}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{x^2+1}}\left(-(x^2+1)^{\frac{3}{2}} + xy\right)\\ &= -\frac{\sqrt{x^2+1}\,x^2 - xy + \sqrt{x^2+1}}{\sqrt{x^2+1}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{x^2+1}}\left(x^2+1\right) \\
&= \sqrt{x^2+1}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{\sqrt{x^2+1}\,x^2 - xy + \sqrt{x^2+1}}{\sqrt{x^2+1}}\right) + \left(\sqrt{x^2+1}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sqrt{x^2+1}\,x^2 - xy + \sqrt{x^2+1}}{\sqrt{x^2+1}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} - x + \sqrt{x^2+1}\,y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \sqrt{x^2+1} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sqrt{x^2+1}$. Therefore equation (4) becomes

$$
\sqrt{x^2+1} = \sqrt{x^2+1} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - x + \sqrt{x^2+1}\, y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - x + \sqrt{x^2+1}\, y$$

The solution becomes

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2+1}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2+1}} \tag{1}$$

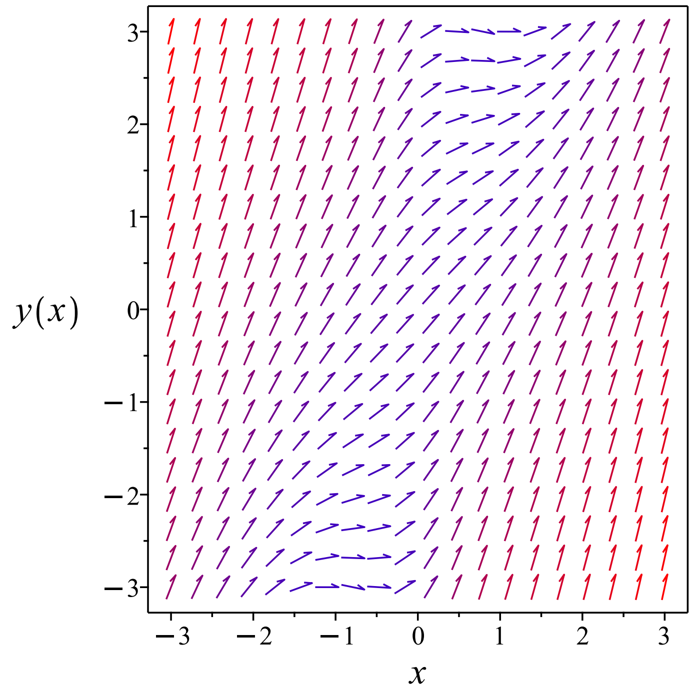


Figure 126: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2 + 1}}$$

Verified OK.

2.45.4 Maple step by step solution

Let's solve

$(x^2 + 1) y' + xy = (x^2 + 1)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{xy}{x^2+1} + \sqrt{x^2 + 1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{xy}{x^2+1} = \sqrt{x^2 + 1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'+\frac{xy}{x^2+1}\right)=\mu(x)\sqrt{x^2+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{xy}{x^2+1}\right)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)x}{x^2+1}$
- Solve to find the integrating factor

  $\mu(x)=\sqrt{x^2+1}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\sqrt{x^2+1}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\sqrt{x^2+1}dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)\sqrt{x^2+1}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\sqrt{x^2+1}$

  $y=\frac{\int(x^2+1)dx+c_1}{\sqrt{x^2+1}}$
- Evaluate the integrals on the rhs

  $y=\frac{\frac{1}{3}x^3+x+c_1}{\sqrt{x^2+1}}$
- Simplify

  $y=\frac{x^3+3c_1+3x}{3\sqrt{x^2+1}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve((1+x^2)*diff(y(x),x)+x*y(x)=(1+x^2)^(3/2),y(x), singsol=all)
```

$$y(x) = \frac{x^3 + 3c_1 + 3x}{3\sqrt{x^2 + 1}}$$

✓ Solution by Mathematica

Time used: 0.065 (sec). Leaf size: 29

```
DSolve[(1+x^2)*y'[x]+x*y[x]==(1+x^2)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 + 3x + 3c_1}{3\sqrt{x^2 + 1}}$$

2.46 problem 46

Internal problem ID [5132]
Internal file name [`OUTPUT/4625_Sunday_June_05_2022_03_02_14_PM_59311949/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 46.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{x(1+y^2)-y(x^2+1)\,y'=0}$$

2.46.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(y^2+1)}{y\,(x^2+1)} \end{aligned}$$

Where $f(x)=\frac{x}{x^2+1}$ and $g(y)=\frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\,dy = \frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\, dy = \int \frac{x}{x^2+1}\, dx$$

$$\frac{\ln(y^2+1)}{2} = \frac{\ln(x^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{\frac{\ln(x^2+1)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2\sqrt{x^2+1}$$

Which simplifies to

$$\sqrt{1+y^2} = c_2\sqrt{x^2+1}\, e^{c_1}$$

The solution is

$$\sqrt{1+y^2} = c_2\sqrt{x^2+1}\, e^{c_1}$$

Summary

The solution(s) found are the following

$$\sqrt{1+y^2} = c_2\sqrt{x^2+1}\, e^{c_1} \tag{1}$$

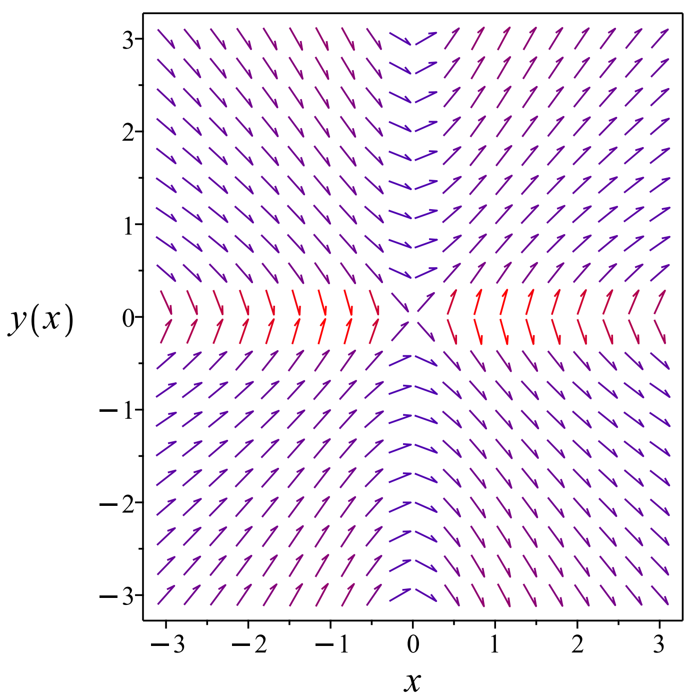


Figure 127: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = c_2\sqrt{x^2+1}\,\mathrm{e}^{c_1}$$

Verified OK.

2.46.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x\left(1+u(x)^2\,x^2\right) - u(x)\,x\left(x^2+1\right)\left(u'(x)\,x+u(x)\right) = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u^2-1}{ux\,(x^2+1)}
\end{aligned}$$

Where $f(x) = -\frac{1}{x(x^2+1)}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{u^2-1}{u}}\,du &= -\frac{1}{x\,(x^2+1)}\,dx \\
\int \frac{1}{\frac{u^2-1}{u}}\,du &= \int -\frac{1}{x\,(x^2+1)}\,dx \\
\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} &= \frac{\ln(x^2+1)}{2} - \ln(x) + c_2
\end{aligned}$$

The above can be written as

$$\begin{aligned}
\left(\frac{1}{2}\right)\left(\ln(u-1)+\ln(u+1)\right) &= \frac{\ln(x^2+1)}{2} - \ln(x) + 2c_2 \\
\ln(u-1)+\ln(u+1) &= (2)\left(\frac{\ln(x^2+1)}{2} - \ln(x) + 2c_2\right) \\
&= \ln\left(x^2+1\right) - 2\ln(x) + 4c_2
\end{aligned}$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)} = \mathrm{e}^{\ln(x^2+1)-2\ln(x)+2c_2}$$

Which simplifies to

$$\begin{aligned}
u^2-1 &= 2c_2\mathrm{e}^{\ln(x^2+1)-2\ln(x)} \\
&= c_3\mathrm{e}^{\ln(x^2+1)-2\ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x)^2 - 1 = c_3\left(1 + \frac{1}{x^2}\right)$$

The solution is

$$u(x)^2 - 1 = c_3\left(1 + \frac{1}{x^2}\right)$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} - 1 = c_3\left(1 + \frac{1}{x^2}\right)$$
$$\frac{y^2}{x^2} - 1 = c_3\left(1 + \frac{1}{x^2}\right)$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x^2} - 1 = c_3\left(1 + \frac{1}{x^2}\right) \tag{1}$$

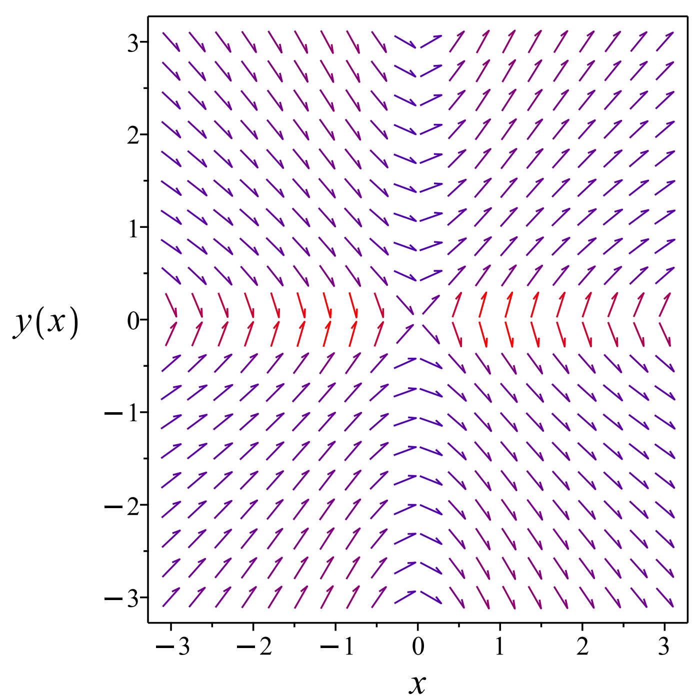


Figure 128: Slope field plot

Verification of solutions

$$\frac{y^2}{x^2} - 1 = c_3\left(1 + \frac{1}{x^2}\right)$$

Verified OK.

2.46.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{x(y^2+1)}{y\,(x^2+1)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 83: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2+1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2+1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln (x^2 + 1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x(y^2 + 1)}{y (x^2 + 1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x}{x^2 + 1} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2 + 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2 + 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

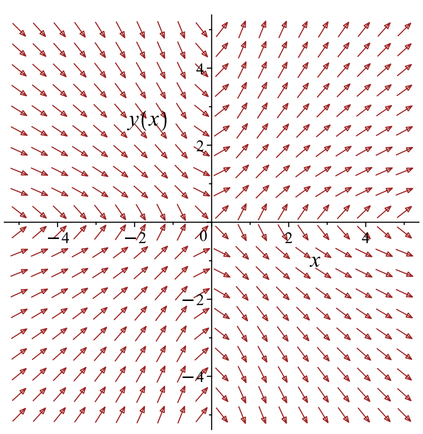
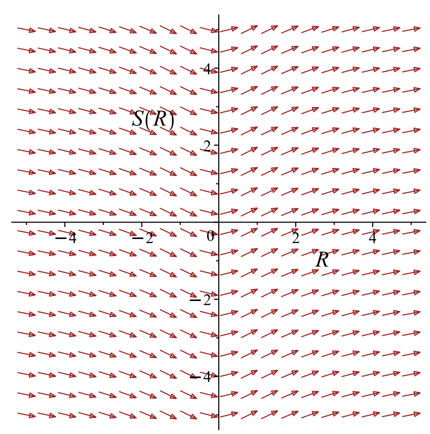
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2+1)}$ 	$R = y$ $S = \frac{\ln(x^2+1)}{2}$	$\frac{dS}{dR} = \frac{R}{R^2+1}$ 

Summary
The solution(s) found are the following

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

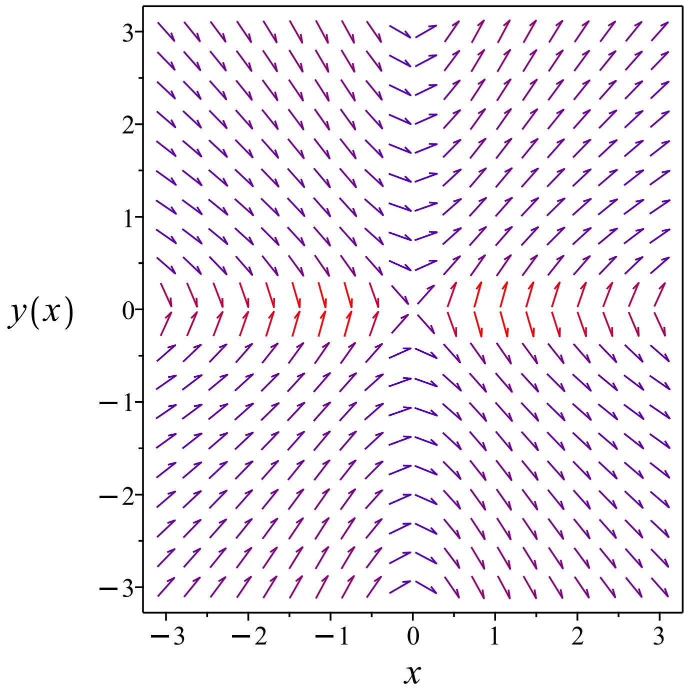


Figure 129: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+1)}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

2.46.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x(y^2+1)}{y\,(x^2+1)} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{x}{x^2+1}y + \frac{x}{x^2+1}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{x}{x^2+1} \\ f_1(x) &= \frac{x}{x^2+1} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{x\,y^2}{x^2+1} + \frac{x}{x^2+1} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= \frac{xw(x)}{x^2+1} + \frac{x}{x^2+1} \\ w' &= \frac{2xw}{x^2+1} + \frac{2x}{x^2+1} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2x}{x^2+1} \\ q(x) &= \frac{2x}{x^2+1} \end{aligned}$$

Hence the ode is

$$w'(x)-\frac{2xw(x)}{x^2+1}=\frac{2x}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int -\frac{2x}{x^2+1}dx}\\&=\frac{1}{x^2+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w)&=(\mu)\left(\frac{2x}{x^2+1}\right)\\\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2+1}\right)&=\left(\frac{1}{x^2+1}\right)\left(\frac{2x}{x^2+1}\right)\\\mathrm{d}\left(\frac{w}{x^2+1}\right)&=\left(\frac{2x}{(x^2+1)^2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2+1}&=\int\frac{2x}{(x^2+1)^2}\,\mathrm{d}x\\\frac{w}{x^2+1}&=-\frac{1}{x^2+1}+c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu=\frac{1}{x^2+1}$ results in

$$w(x)=-1+c_1\left(x^2+1\right)$$

which simplifies to

$$w(x)=c_1x^2+c_1-1$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2=c_1x^2+c_1-1$$

Solving for y gives

$$\begin{aligned}y(x)&=\sqrt{c_1x^2+c_1-1}\\y(x)&=-\sqrt{c_1x^2+c_1-1}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \sqrt{c_1 x^2 + c_1 - 1} \tag{1}$$

$$y = -\sqrt{c_1 x^2 + c_1 - 1} \tag{2}$$

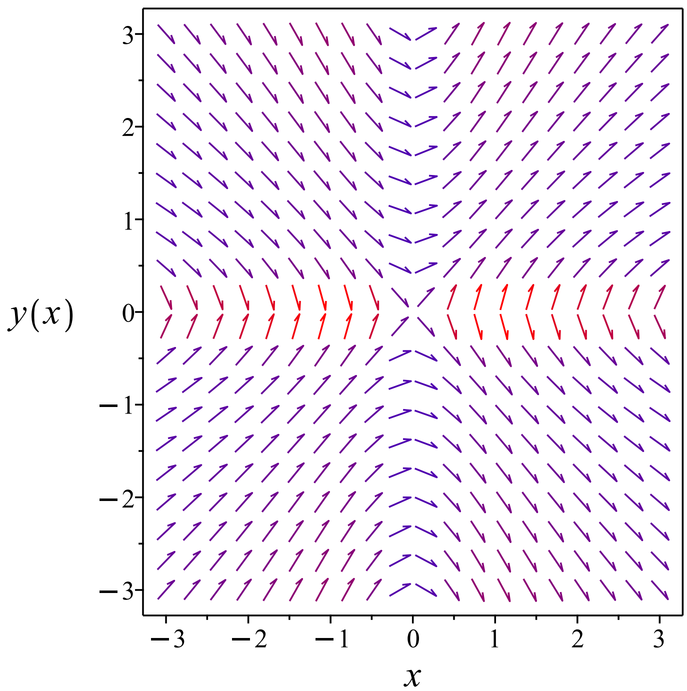


Figure 130: Slope field plot

Verification of solutions

$$y = \sqrt{c_1 x^2 + c_1 - 1}$$

Verified OK.

$$y = -\sqrt{c_1 x^2 + c_1 - 1}$$

Verified OK.

2.46.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=\left(\frac{x}{x^2+1}\right)\mathrm{d}x\\ \left(-\frac{x}{x^2+1}\right)\mathrm{d}x+\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{x^2+1} \\
N(x,y) &= \frac{y}{y^2+1}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{x^2+1}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{y}{y^2+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{y}{y^2+1}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y^2+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+1)}{2} + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

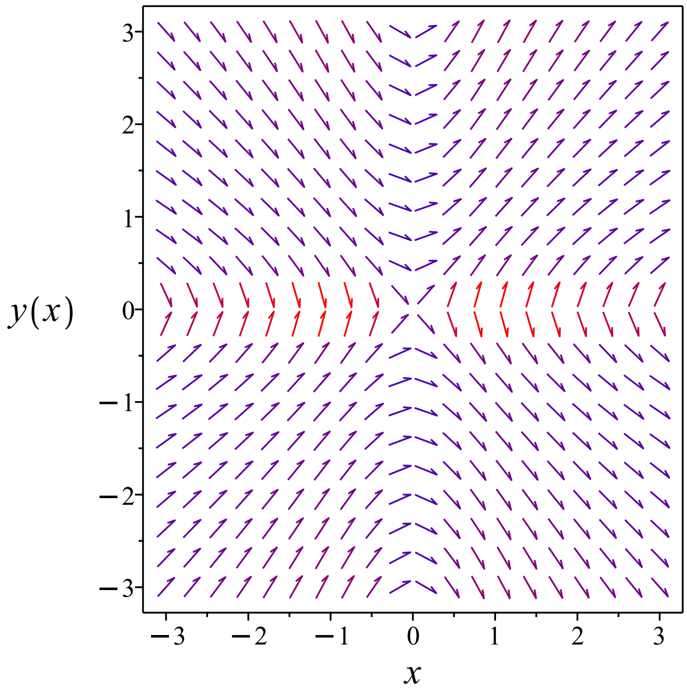


Figure 131: Slope field plot

Verification of solutions

$$-\frac{\ln\left(x^{2}+1\right)}{2}+\frac{\ln\left(1+y^{2}\right)}{2}=c_{1}$$

Verified OK.

2.46.6 Maple step by step solution

Let's solve

$x(1+y^2)-y(x^2+1)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'y}{1+y^2}=\frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'y}{1+y^2}dx=\int \frac{x}{x^2+1}dx+c_1$

- Evaluate integral

$$\frac{\ln(1+y^2)}{2} = \frac{\ln(x^2+1)}{2} + c_1$$

- Solve for y

$$\left\{y = \frac{\sqrt{-\mathrm{e}^{-2c_1}\left(-x^2+\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}, y = -\frac{\sqrt{-\mathrm{e}^{-2c_1}\left(-x^2+\mathrm{e}^{-2c_1}-1\right)}}{\mathrm{e}^{-2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(x*(1+y(x)^2)-y(x)*(1+x^2)*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = \sqrt{c_1x^2 + c_1 - 1}$$
$$y(x) = -\sqrt{c_1x^2 + c_1 - 1}$$

✓ Solution by Mathematica

Time used: 0.499 (sec). Leaf size: 61

```
DSolve[x*(1+y[x]^2)-y[x]*(1+x^2)*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-1 + e^{2c_1}\left(x^2+1\right)}$$
$$y(x) \to \sqrt{-1 + e^{2c_1}\left(x^2+1\right)}$$
$$y(x) \to -i$$
$$y(x) \to i$$

2.47 problem 47

Internal problem ID [5133]
Internal file name [`OUTPUT/4626_Sunday_June_05_2022_03_02_15_PM_54703841/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 47.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\frac{r\tan(\theta)\, r'}{a^2 - r^2} = 1}$$

With initial conditions

$$\left[r\left(\frac{\pi}{4}\right) = 0\right]$$

2.47.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$
\begin{aligned}
r' &= f(\theta, r) \\
&= -\frac{-a^2 + r^2}{\tan(\theta)\, r}
\end{aligned}
$$

$f(\theta, r)$ is not defined at $r = 0$ therefore existence and uniqueness theorem do not apply.

2.47.2 Solving as separable ode

In canonical form the ODE is

$$
\begin{aligned}
r' &= F(\theta, r) \\
&= f(\theta) g(r) \\
&= -\frac{-a^2 + r^2}{\tan(\theta)\, r}
\end{aligned}
$$

Where $f(\theta) = -\frac{1}{\tan(\theta)}$ and $g(r) = \frac{-a^2+r^2}{r}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{-a^2+r^2}{r}}\, dr &= -\frac{1}{\tan(\theta)}\, d\theta \\
\int \frac{1}{\frac{-a^2+r^2}{r}}\, dr &= \int -\frac{1}{\tan(\theta)}\, d\theta \\
\frac{\ln(-a^2 + r^2)}{2} &= -\ln(\sin(\theta)) + c_1
\end{aligned}
$$

Raising both side to exponential gives

$$
\sqrt{-a^2 + r^2} = e^{-\ln(\sin(\theta)) + c_1}
$$

Which simplifies to

$$
\sqrt{-a^2 + r^2} = \frac{c_2}{\sin(\theta)}
$$

Which can be simplified to become

$$
\sqrt{r^2 - a^2} = \frac{c_2 e^{c_1}}{\sin(\theta)}
$$

The solution is

$$\sqrt{r^2-a^2} = \frac{c_2 \mathrm{e}^{c_1}}{\sin(\theta)}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{4}$ and $r = 0$ in the above solution gives an equation to solve for the constant of integration.

$$i \operatorname{csgn}(ia)\, a = \sqrt{2}\, c_2 \mathrm{e}^{c_1}$$

$$c_1 = \frac{\ln\left(-\frac{\operatorname{csgn}(ia)^2 a^2}{2c_2^2}\right)}{2}$$

Substituting c_1 found above in the general solution gives

$$\sqrt{-a^2+r^2} = \frac{c_2\sqrt{-\frac{2a^2}{c_2^2}}}{2\sin(\theta)}$$

The above simplifies to

$$2\sqrt{-a^2+r^2}\,\sin(\theta) - c_2\sqrt{-\frac{2a^2}{c_2^2}} = 0$$

Solving for r from the above gives

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Summary

The solution(s) found are the following

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{1}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{2}$$

Verification of solutions

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

2.47.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
r' &= -\frac{-a^2 + r^2}{\tan(\theta)\, r} \\
r' &= \omega(\theta, r)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_\theta + \omega(\eta_r - \xi_\theta) - \omega^2 \xi_r - \omega_\theta \xi - \omega_r \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 86: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(\theta, r) &= -\tan(\theta) \\
\eta(\theta, r) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(\theta, r) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{d\theta}{\xi} = \frac{dr}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial \theta} + \eta \frac{\partial}{\partial r}\right) S(\theta, r) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = r
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} d\theta \\
&= \int \frac{1}{-\tan(\theta)} d\theta
\end{aligned}
$$

Which results in

$$
S = -\ln(\sin(\theta))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_\theta + \omega(\theta, r) S_r}{R_\theta + \omega(\theta, r) R_r} \tag{2}
$$

Where in the above $R_\theta, R_r, S_\theta, S_r$ are all partial derivatives and $\omega(\theta, r)$ is the right hand side of the original ode given by

$$
\omega(\theta, r) = -\frac{-a^2 + r^2}{\tan(\theta) r}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_\theta &= 0 \\
R_r &= 1 \\
S_\theta &= -\cot(\theta) \\
S_r &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{r}{a^2 - r^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for θ, r in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{R}{-R^2 + a^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln (R^2 - a^2)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to θ, r coordinates. This results in

$$-\ln (\sin (\theta)) = \frac{\ln (r^2 - a^2)}{2} + c_1$$

Which simplifies to

$$-\ln (\sin (\theta)) = \frac{\ln (r^2 - a^2)}{2} + c_1$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{4}$ and $r = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\ln (2)}{2} = \frac{\ln (-a^2)}{2} + c_1$$

$$c_1 = -\frac{\ln (-a^2)}{2} + \frac{\ln (2)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\ln (\sin (\theta)) = \frac{\ln (-a^2 + r^2)}{2} - \frac{\ln (-a^2)}{2} + \frac{\ln (2)}{2}$$

Solving for r from the above gives

$$r = \frac{a\sqrt{2}\, \sqrt{-\cos (2\theta)}\, \csc (\theta)}{2}$$
$$r = -\frac{a\sqrt{2}\, \sqrt{-\cos (2\theta)}\, \csc (\theta)}{2}$$

Summary

The solution(s) found are the following

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{1}$$

$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{2}$$

Verification of solutions

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

2.47.4 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} r' &= F(\theta, r) \\ &= -\frac{-a^2 + r^2}{\tan(\theta)\, r} \end{aligned}$$

This is a Bernoulli ODE.

$$r' = -\frac{1}{\tan(\theta)} r + \frac{a^2}{\tan(\theta)} \frac{1}{r} \tag{1}$$

The standard Bernoulli ODE has the form

$$r' = f_0(\theta) r + f_1(\theta) r^n \tag{2}$$

The first step is to divide the above equation by r^n which gives

$$\frac{r'}{r^n} = f_0(\theta) r^{1-n} + f_1(\theta) \tag{3}$$

The next step is use the substitution $w = r^{1-n}$ in equation (3) which generates a new ODE in $w(\theta)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $r(\theta)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(\theta) &= -\frac{1}{\tan(\theta)} \\
f_1(\theta) &= \frac{a^2}{\tan(\theta)} \\
n &= -1
\end{aligned}$$

Dividing both sides of ODE (1) by $r^n = \frac{1}{r}$ gives

$$r'r = -\frac{r^2}{\tan(\theta)} + \frac{a^2}{\tan(\theta)} \tag{4}$$

Let

$$\begin{aligned}
w &= r^{1-n} \\
&= r^2 \qquad (5)
\end{aligned}$$

Taking derivative of equation (5) w.r.t θ gives

$$w' = 2rr' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
\frac{w'(\theta)}{2} &= -\frac{w(\theta)}{\tan(\theta)} + \frac{a^2}{\tan(\theta)} \\
w' &= -\frac{2w}{\tan(\theta)} + \frac{2a^2}{\tan(\theta)} \qquad (7)
\end{aligned}$$

The above now is a linear ODE in $w(\theta)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(\theta) + p(\theta)w(\theta) = q(\theta)$$

Where here

$$\begin{aligned}
p(\theta) &= 2\cot(\theta) \\
q(\theta) &= 2a^2\cot(\theta)
\end{aligned}$$

Hence the ode is

$$w'(\theta) + 2\cot(\theta)\, w(\theta) = 2a^2\cot(\theta)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int 2\cot(\theta)d\theta} \\
&= \sin(\theta)^2
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}(\mu w) &= (\mu)\left(2a^2\cot(\theta)\right) \\
\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin(\theta)^2 w\right) &= \left(\sin(\theta)^2\right)\left(2a^2\cot(\theta)\right) \\
\mathrm{d}\left(\sin(\theta)^2 w\right) &= \left(a^2\sin(2\theta)\right)\,\mathrm{d}\theta
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(\theta)^2 w &= \int a^2\sin(2\theta)\,\mathrm{d}\theta \\
\sin(\theta)^2 w &= -\frac{a^2\cos(2\theta)}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(\theta)^2$ results in

$$
w(\theta) = -\frac{\csc(\theta)^2 a^2\cos(2\theta)}{2} + c_1\csc(\theta)^2
$$

Replacing w in the above by r^2 using equation (5) gives the final solution.

$$
r^2 = -\frac{\csc(\theta)^2 a^2\cos(2\theta)}{2} + c_1\csc(\theta)^2
$$

Which is simplified to

$$
r^2 = -\left(a^2\cos(\theta)^2 - \frac{a^2}{2} - c_1\right)\csc(\theta)^2
$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{4}$ and $r = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = 2c_1
$$

$$
c_1 = 0
$$

Substituting c_1 found above in the general solution gives

$$
r^2 = -\csc(\theta)^2 a^2\cos(\theta)^2 + \frac{a^2\csc(\theta)^2}{2}
$$

Solving for r from the above gives

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2}$$

Summary

The solution(s) found are the following

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2} \tag{1}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2} \tag{2}$$

Verification of solutions

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2}$$

Verified OK. `{positive}`

$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos\left(2\theta\right)}\,\csc\left(\theta\right)}{2}$$

Verified OK. `{positive}`

2.47.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(\theta, r)\,\mathrm{d}\theta + N(\theta, r)\,\mathrm{d}r = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{r}{-a^2+r^2}\right)\mathrm{d}r &= \left(\frac{1}{\tan(\theta)}\right)\mathrm{d}\theta\\ \left(-\frac{1}{\tan(\theta)}\right)\mathrm{d}\theta + \left(-\frac{r}{-a^2+r^2}\right)\mathrm{d}r &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(\theta, r) &= -\frac{1}{\tan(\theta)}\\ N(\theta, r) &= -\frac{r}{-a^2+r^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial r} &= \frac{\partial}{\partial r}\left(-\frac{1}{\tan(\theta)}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial \theta} &= \frac{\partial}{\partial \theta}\left(-\frac{r}{-a^2+r^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial \theta}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(\theta, r)$

$$\frac{\partial \phi}{\partial \theta} = M \tag{1}$$

$$\frac{\partial \phi}{\partial r} = N \tag{2}$$

Integrating (1) w.r.t. θ gives

$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int M \, \mathrm{d}\theta$$

$$\int \frac{\partial \phi}{\partial \theta} \, \mathrm{d}\theta = \int -\frac{1}{\tan(\theta)} \, \mathrm{d}\theta$$

$$\phi = -\ln(\sin(\theta)) + f(r) \tag{3}$$

Where $f(r)$ is used for the constant of integration since ϕ is a function of both θ and r. Taking derivative of equation (3) w.r.t r gives

$$\frac{\partial \phi}{\partial r} = 0 + f'(r) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial r} = -\frac{r}{-a^2+r^2}$. Therefore equation (4) becomes

$$-\frac{r}{-a^2 + r^2} = 0 + f'(r) \tag{5}$$

Solving equation (5) for $f'(r)$ gives

$$f'(r) = \frac{r}{a^2 - r^2}$$

Integrating the above w.r.t r gives

$$\int f'(r) \, \mathrm{d}r = \int \left(\frac{r}{a^2 - r^2}\right) \mathrm{d}r$$

$$f(r) = -\frac{\ln(-a^2 + r^2)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(r)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(\theta)) - \frac{\ln(-a^2 + r^2)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(\theta)) - \frac{\ln(-a^2 + r^2)}{2}$$

Initial conditions are used to solve for c_1. Substituting $\theta = \frac{\pi}{4}$ and $r = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{\ln(-a^2)}{2} + \frac{\ln(2)}{2} = c_1$$

$$c_1 = -\frac{\ln(-a^2)}{2} + \frac{\ln(2)}{2}$$

Substituting c_1 found above in the general solution gives

$$-\ln(\sin(\theta)) - \frac{\ln(-a^2 + r^2)}{2} = -\frac{\ln(-a^2)}{2} + \frac{\ln(2)}{2}$$

Solving for r from the above gives

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Summary
The solution(s) found are the following

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{1}$$
$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2} \tag{2}$$

Verification of solutions

$$r = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

$$r = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

Verified OK. `{positive}`

2.47.6 Maple step by step solution

Let's solve

$\left[\frac{r\tan(\theta)r'}{a^2-r^2} = 1, r\left(\frac{\pi}{4}\right) = 0\right]$

- Highest derivative means the order of the ODE is 1

  r'

- Separate variables

  $\frac{r'r}{a^2-r^2} = \frac{1}{\tan(\theta)}$

- Integrate both sides with respect to θ

  $\int \frac{r'r}{a^2-r^2}d\theta = \int \frac{1}{\tan(\theta)}d\theta + c_1$

- Evaluate integral

  $-\frac{\ln(r^2-a^2)}{2} = \ln(\sin(\theta)) + c_1$

- Solve for r

  $\left\{r = \frac{\sqrt{a^2\sin(\theta)^2(\mathrm{e}^{c_1})^2+1}}{\mathrm{e}^{c_1}\sin(\theta)}, r = -\frac{\sqrt{a^2\sin(\theta)^2(\mathrm{e}^{c_1})^2+1}}{\mathrm{e}^{c_1}\sin(\theta)}\right\}$

- Use initial condition $r\left(\frac{\pi}{4}\right) = 0$

  $0 = \frac{\sqrt{\frac{a^2(\mathrm{e}^{c_1})^2}{2}+1}\,\sqrt{2}}{\mathrm{e}^{c_1}}$

- Solve for c_1

  $c_1 = \frac{\ln\left(-\frac{2}{a^2}\right)}{2}$

- Substitute $c_1 = \frac{\ln\left(-\frac{2}{a^2}\right)}{2}$ into general solution and simplify

$$r = \frac{\sqrt{2}\,\sqrt{\cos(2\theta)}\,\csc(\theta)}{2\sqrt{-\frac{1}{a^2}}}$$

- Use initial condition $r\left(\frac{\pi}{4}\right) = 0$

$$0 = -\frac{\sqrt{\frac{a^2 (e^{c_1})^2}{2} + 1}\,\sqrt{2}}{e^{c_1}}$$

- Solve for c_1

$$c_1 = \frac{\ln\left(-\frac{2}{a^2}\right)}{2}$$

- Substitute $c_1 = \frac{\ln\left(-\frac{2}{a^2}\right)}{2}$ into general solution and simplify

$$r = -\frac{\sqrt{2}\,\sqrt{\cos(2\theta)}\,\csc(\theta)}{2\sqrt{-\frac{1}{a^2}}}$$

- Solutions to the IVP

$$\left\{r = -\frac{\sqrt{2}\,\sqrt{\cos(2\theta)}\,\csc(\theta)}{2\sqrt{-\frac{1}{a^2}}}, r = \frac{\sqrt{2}\,\sqrt{\cos(2\theta)}\,\csc(\theta)}{2\sqrt{-\frac{1}{a^2}}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.188 (sec). Leaf size: 39

```
dsolve([r(theta)*tan(theta)/(a^2-r(theta)^2)*diff(r(theta),theta)=1,r(1/4*Pi) = 0],r(theta),
```

$$r(\theta) = -\frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$
$$r(\theta) = \frac{a\sqrt{2}\,\sqrt{-\cos(2\theta)}\,\csc(\theta)}{2}$$

✓ Solution by Mathematica

Time used: 0.149 (sec). Leaf size: 51

```
DSolve[{r[\[Theta]]*Tan[\[Theta]]/(a^2-r[\[Theta]]^2)*r'[\[Theta]]==1,{r[Pi/4]==0}},r[\[Thet
```

$$r(\theta) \to -\sqrt{\frac{a^2\cos(2\theta)}{\cos(2\theta)-1}}$$
$$r(\theta) \to \sqrt{\frac{a^2\cos(2\theta)}{\cos(2\theta)-1}}$$

2.48 problem 48

Internal problem ID [5134]
Internal file name [`OUTPUT/4627_Sunday_June_05_2022_03_02_17_PM_78923587/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 48.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y' + y\cot(x) = \cos(x)}$$

With initial conditions

$$[y(0) = 0]$$

2.48.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cot(x)$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' + y\cot(x) = \cos(x)$$

The domain of $p(x) = \cot(x)$ is

$$\{x < \pi\_Z104 \vee \pi\_Z104 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

2.48.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \cot(x)dx} \\ &= \sin(x)\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\cos(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}(y\sin(x)) &= (\sin(x))(\cos(x)) \\ \mathrm{d}(y\sin(x)) &= \left(\frac{\sin(2x)}{2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}y\sin(x) &= \int \frac{\sin(2x)}{2}\,\mathrm{d}x \\ y\sin(x) &= -\frac{\cos(2x)}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$y = -\frac{\csc(x)\cos(2x)}{4} + c_1\csc(x)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = \frac{2\cos(x)\cot(x) + 4y - \csc(x)}{4\csc(x)}$$

Using given initial conditions results in $c_1 = \frac{1}{4}$ Hence the solution is

$$y = -\frac{\csc(x)\cos(2x)}{4} + \frac{\csc(x)}{4}$$

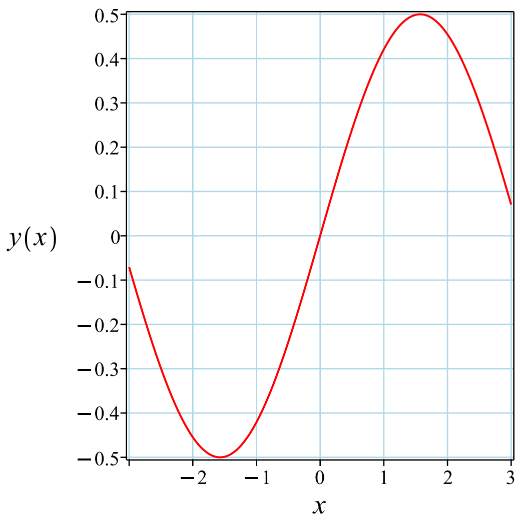
Therefore the solution is

$$y = -\frac{\csc(x)\cos(2x)}{4} + \frac{\csc(x)}{4}$$

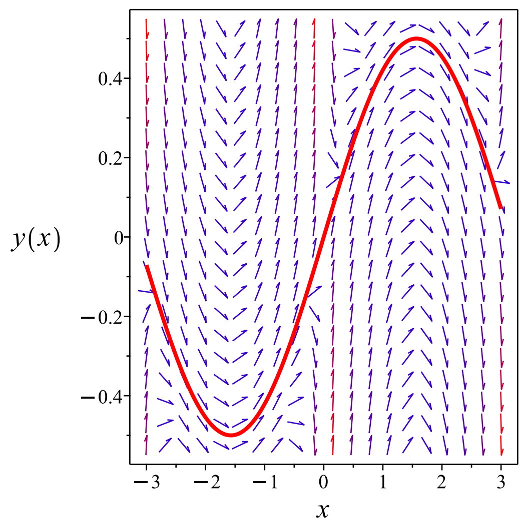
<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{\csc(x)\cos(2x)}{4} + \frac{\csc(x)}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{\csc(x)\cos(2x)}{4} + \frac{\csc(x)}{4}$$

Verified OK.

2.48.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -y\cot(x) + \cos(x) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 89: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sin(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sin(x)}} dy
\end{aligned}
$$

Which results in

$$
S = y \sin(x)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -y \cot(x) + \cos(x)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \cos(x) \\
S_y &= \sin(x)
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

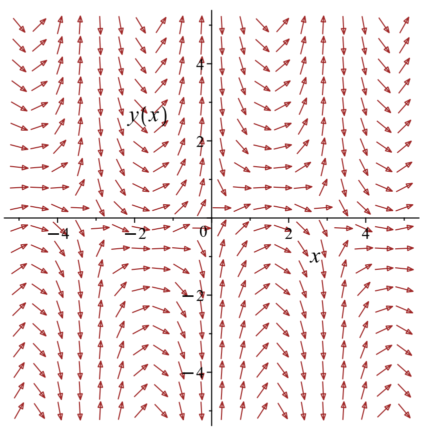
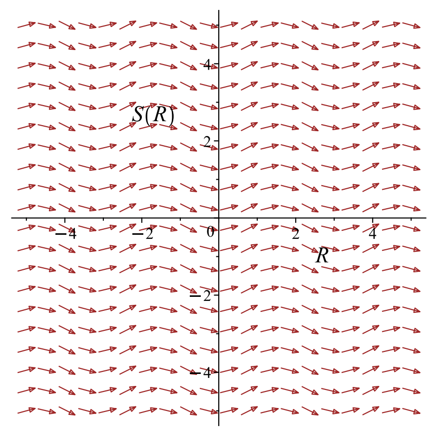
Which simplifies to

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

Which gives

$$y = -\frac{\cos(2x) - 4c_1}{4\sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + \cos(x)$ 	$R = x$ $S = y\sin(x)$	$\frac{dS}{dR} = \frac{\sin(2R)}{2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = \frac{\cos(x)^2}{2} + y\sin(x) - \frac{1}{4}$$

Using given initial conditions results in $c_1 = \frac{1}{4}$ Hence the solution is

$$y = -\frac{-1 + \cos(2x)}{4\sin(x)}$$

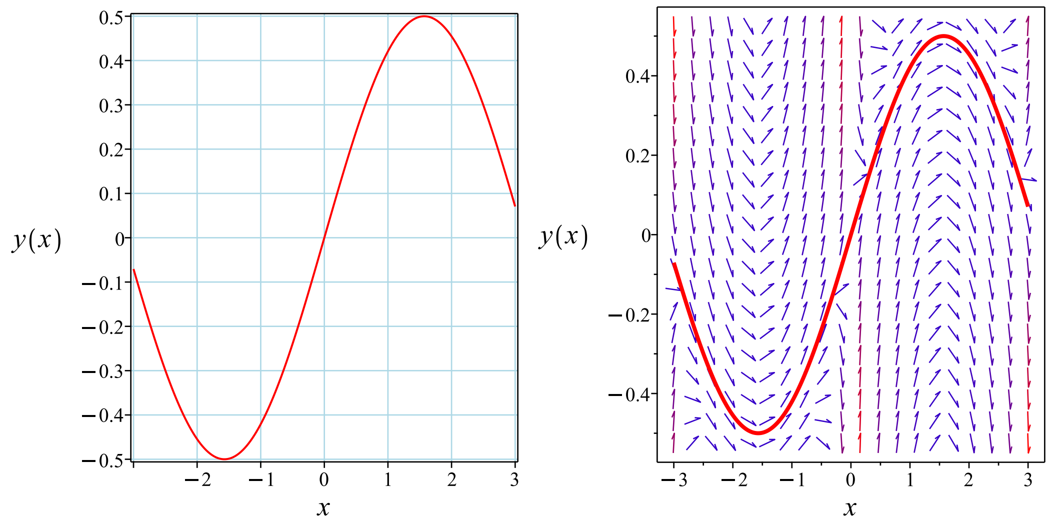
Therefore the solution is

$$y = -\frac{-1 + \cos(2x)}{4\sin(x)}$$

Summary

The solution(s) found are the following

$$y = -\frac{-1 + \cos(2x)}{4\sin(x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{-1 + \cos(2x)}{4 \sin(x)}$$

Verified OK.

2.48.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= (-y\cot(x) + \cos(x))\,\mathrm{d}x \\
(y\cot(x) - \cos(x))\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= y\cot(x) - \cos(x) \\
N(x,y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y\cot(x) - \cos(x)) \\
&= \cot(x)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cot(x)) - (0)) \\
&= \cot(x)
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \cot(x) \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(\sin(x))} \\ &= \sin(x)\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sin(x)\,(y\cot(x) - \cos(x)) \\ &= \cos(x)\,(-\sin(x) + y)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sin(x)\,(1) \\ &= \sin(x)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (\cos(x)\,(-\sin(x) + y)) + (\sin(x))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \cos(x)\,(-\sin(x) + y) \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\sin(x)\,(\sin(x) - 2y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(x)(\sin(x) - 2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(x)(\sin(x) - 2y)}{2}$$

The solution becomes

$$y = \frac{\sin(x)^2 + 2c_1}{2\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration. Solving for c_1 gives

$$c_1 = -\frac{\sin(x)^2}{2} + y\sin(x)$$

Using given initial conditions results in $c_1 = 0$ Hence the solution is

$$y = \frac{\sin(x)}{2}$$

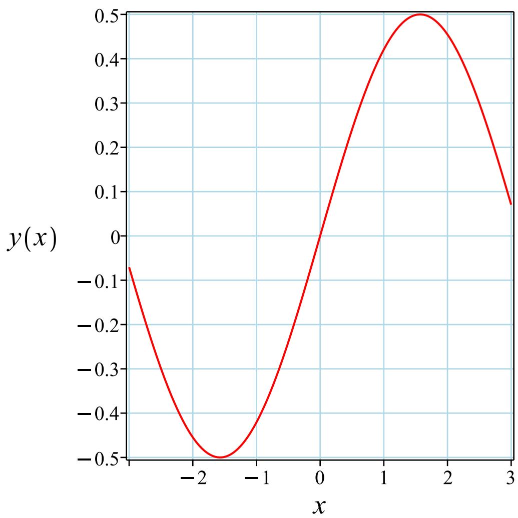
Therefore the solution is

$$y = \frac{\sin(x)}{2}$$

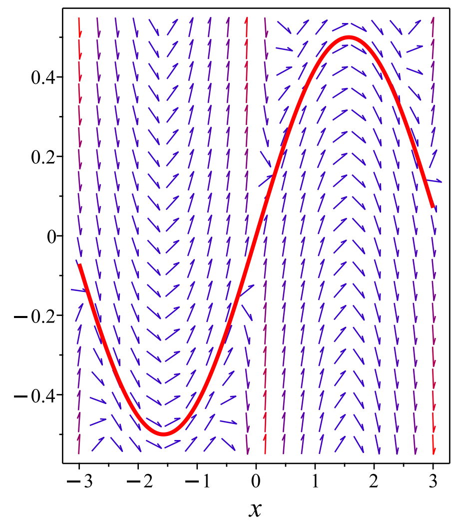
Summary

The solution(s) found are the following

$$y = \frac{\sin(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(x)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.157 (sec). Leaf size: 8

```
dsolve([diff(y(x),x)+y(x)*cot(x)=cos(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{\sin(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.103 (sec). Leaf size: 11

```
DSolve[{y'[x]+y[x]*Cot[x]==Cos[x],{y[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x)}{2}$$

2.49 problem 49

Internal problem ID [5135]
Internal file name [`OUTPUT/4628_Sunday_June_05_2022_03_02_18_PM_63767691/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 24. First order differential equations. Further problems 24. page 1068
Problem number: 49.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y' + \frac{y}{x} - x y^2 = 0}$$

2.49.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(y\,x^2 - 1)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^2 x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^2 x} dy
\end{aligned}$$

Which results in

$$S = -\frac{1}{yx}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(y\,x^2 - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y\,x^2} \\
S_y &= \frac{1}{y^2 x}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{yx} = x + c_1$$

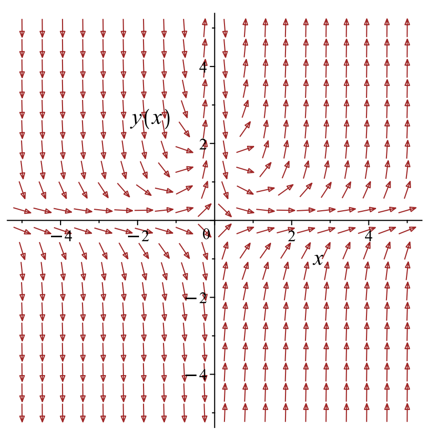
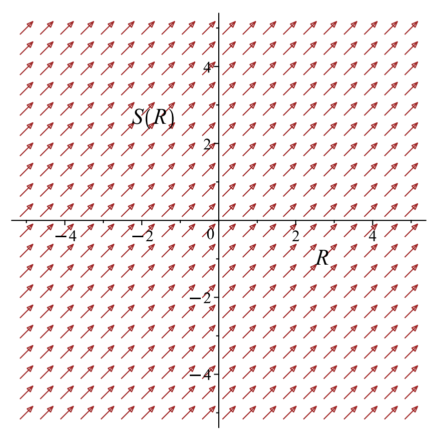
Which simplifies to

$$-\frac{1}{yx} = x + c_1$$

Which gives

$$y = -\frac{1}{x\,(x + c_1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(y\,x^2-1)}{x}$ 	$R = x$ $S = -\frac{1}{yx}$	$\frac{dS}{dR} = 1$ 

Summary
The solution(s) found are the following

$$y = -\frac{1}{x(x+c_1)} \tag{1}$$

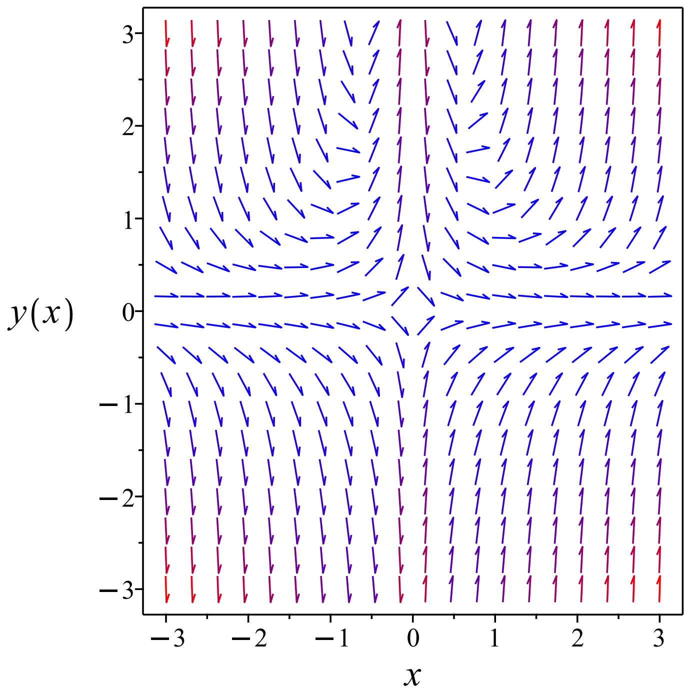


Figure 135: Slope field plot

Verification of solutions

$$y = -\frac{1}{x(x+c_1)}$$

Verified OK.

2.49.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(y\,x^2-1)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + xy^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= x \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{1}{yx} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{w(x)}{x} + x \\ w' &= \frac{w}{x} - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= -x
\end{aligned}
$$

Hence the ode is

$$
w'(x) - \frac{w(x)}{x} = -x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(-x) \\
\mathrm{d}\left(\frac{w}{x}\right) &= -1\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{w}{x} &= \int -1\,\mathrm{d}x \\
\frac{w}{x} &= -x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$
w(x) = c_1 x - x^2
$$

which simplifies to

$$
w(x) = x(-x + c_1)
$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$
\frac{1}{y} = x(-x + c_1)
$$

Or

$$y = \frac{1}{x\left(-x+c_1\right)}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{x\left(-x+c_1\right)} \tag{1}$$

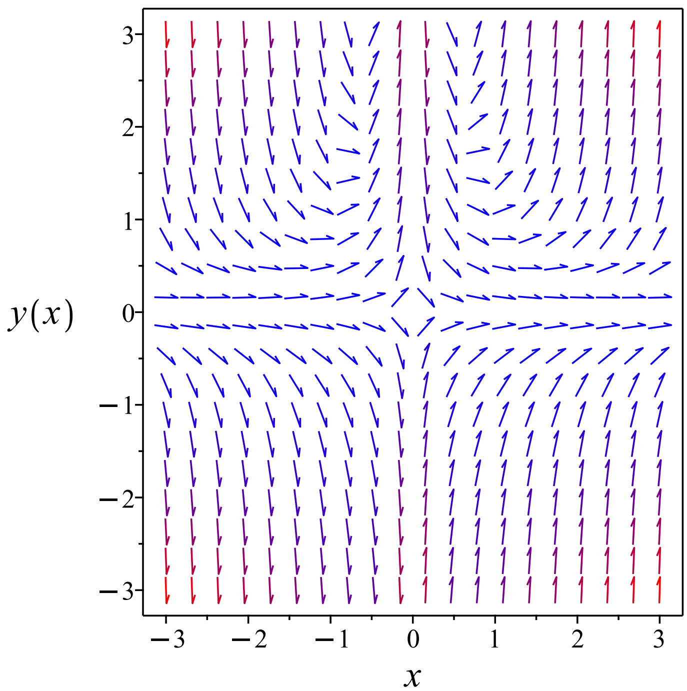


Figure 136: Slope field plot

Verification of solutions

$$y = \frac{1}{x\left(-x+c_1\right)}$$

Verified OK.

2.49.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(y\left(y\,x^2-1\right)\right)\mathrm{d}x\\ \left(-y\left(y\,x^2-1\right)\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\left(y\,x^2-1\right)\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\left(y\,x^{2}-1\right)\right)\\ &= -2y\,x^{2}+1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}\left(\left(-2y\,x^{2}+1\right)-(1)\right)\\ &= -2xy\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y\left(y\,x^{2}-1\right)}\left((1)-\left(-2y\,x^{2}+1\right)\right)\\ &= -\frac{2x^{2}}{y\,x^{2}-1}\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(1) - (-2y\,x^2 + 1)}{x\,(-y\,(y\,x^2 - 1)) - y\,(x)} \\
&= -\frac{2}{yx}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{2}{t}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(-\frac{2}{t}\right) \mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(t)} \\
&= \frac{1}{t^2}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{1}{y^2 x^2}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^2 x^2}\big(-y\big(y\,x^2 - 1\big)\big) \\
&= \frac{-y\,x^2 + 1}{y\,x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2 x^2}(x) \\
&= \frac{1}{y^2 x}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-y\,x^2+1}{y\,x^2}\right) + \left(\frac{1}{y^2 x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-y\,x^2+1}{y\,x^2}\,\mathrm{d}x \\
\phi &= \frac{-y\,x^2-1}{xy} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= -\frac{x}{y} - \frac{-y\,x^2-1}{x\,y^2} + f'(y) \qquad (4) \\
&= \frac{1}{y^2 x} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2 x}$. Therefore equation (4) becomes

$$
\frac{1}{y^2 x} = \frac{1}{y^2 x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-y\,x^2 - 1}{xy} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-y\,x^2 - 1}{xy}$$

The solution becomes

$$y = -\frac{1}{x\,(x + c_1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{x\,(x + c_1)} \tag{1}$$

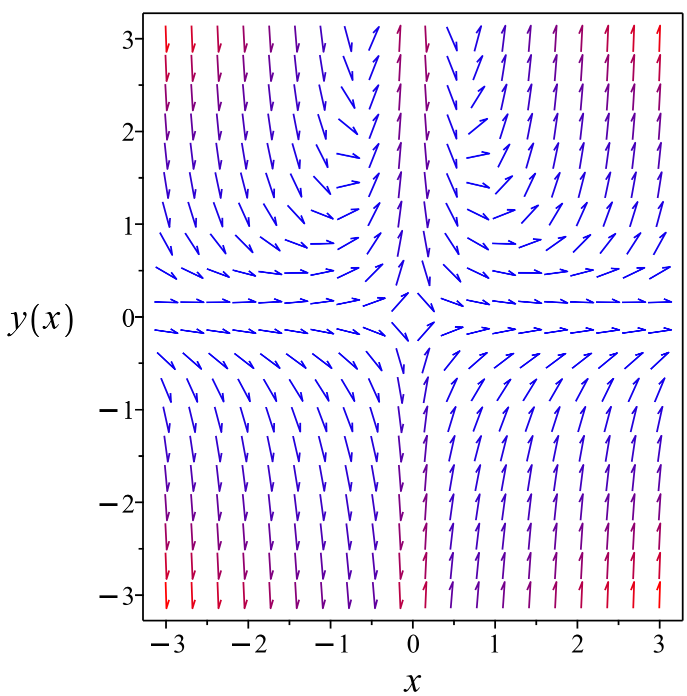


Figure 137: Slope field plot

Verification of solutions

$$
y = -\frac{1}{x (x + c_1)}
$$

Verified OK.

2.49.4 Solving as riccati ode

In canonical form the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{y(y\,x^2 - 1)}{x}
\end{aligned}
$$

This is a Riccati ODE. Comparing the ODE to solve

$$
y' = -\frac{y}{x} + y^2 x
$$

With Riccati ODE standard form

$$
y' = f_0(x) + f_1(x)y + f_2(x)y^2
$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = x$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{xu} \qquad (1)
\end{aligned}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\,u'(x) + f_2^2 f_0 u(x) = 0 \qquad (2)
$$

But

$$
\begin{aligned}
f_2' &= 1 \\
f_1 f_2 &= -1 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
xu''(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 x + c_2$$

The above shows that

$$u'(x) = c_1$$

Using the above in (1) gives the solution

$$y = -\frac{c_1}{x\,(c_1 x + c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{c_3}{x\,(c_3 x + 1)}$$

Summary

The solution(s) found are the following

$$y = -\frac{c_3}{x\,(c_3 x + 1)} \tag{1}$$

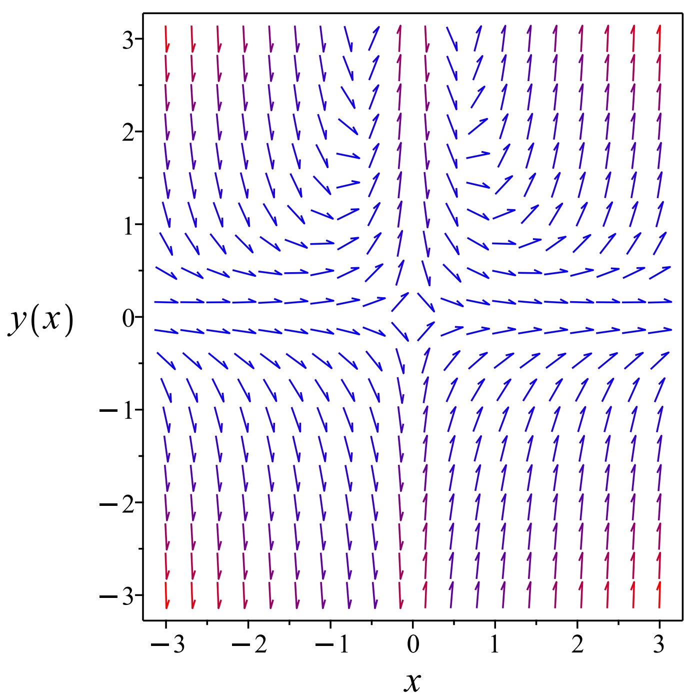


Figure 138: Slope field plot

Verification of solutions

$$y = -\frac{c_3}{x(c_3x+1)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x)+y(x)/x=x*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{(-x+c_1)x}$$

✓ Solution by Mathematica

Time used: 0.135 (sec). Leaf size: 23

```
DSolve[y'[x]+y[x]/x==x*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x^2-c_1x}$$
$$y(x) \to 0$$

3 Program 25. Second order differential equations. Test Excercise 25. page 1093

3.1 problem 1

Internal problem ID [5136]
Internal file name [`OUTPUT/4629_Sunday_June_05_2022_03_02_19_PM_13527249/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y' - 2y = 8}$$

3.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 8$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\
&= \frac{1}{2} \pm \frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{3}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(2)x} + c_2 e^{(-1)x}
\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 = 8$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-x}\right) + (-4) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-x} - 4 \tag{1}$$

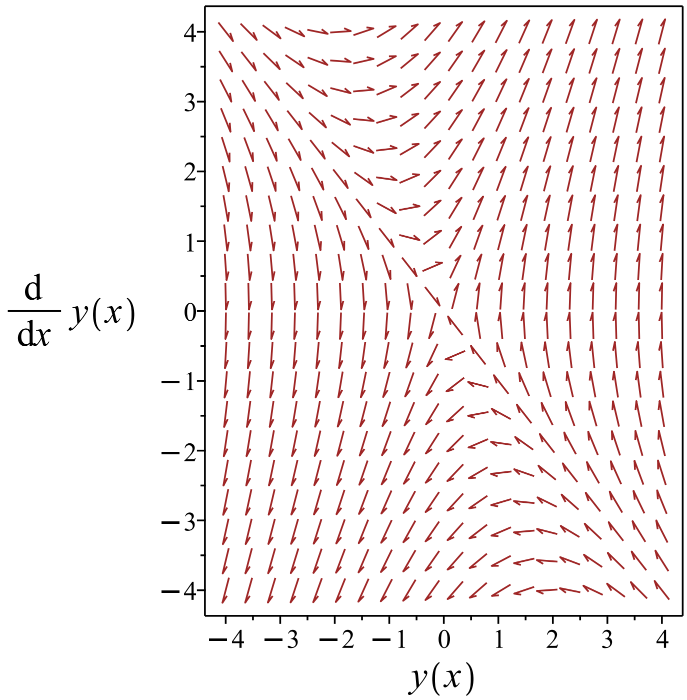


Figure 139: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-x} - 4$$

Verified OK.

3.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 93: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-1}{1} \, dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{3}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 = 8$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -4$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}\right) + (-4) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} - 4 \tag{1}$$

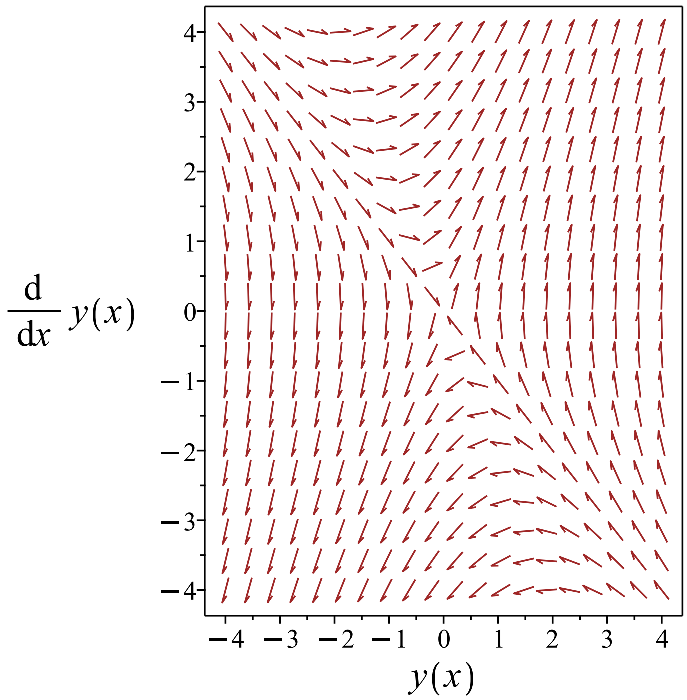


Figure 140: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + \frac{c_2 e^{2x}}{3} - 4$$

Verified OK.

3.1.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 8$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 8\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-x} & \mathrm{e}^{2x} \\ -\mathrm{e}^{-x} & 2\,\mathrm{e}^{2x} \end{array}\right]$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,\mathrm{e}^{x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{8\,\mathrm{e}^{-x}\left(\int \mathrm{e}^{x} dx\right)}{3} + \frac{8\,\mathrm{e}^{2x}\left(\int \mathrm{e}^{-2x} dx\right)}{3}$

- ○ Compute integrals

  $y_p(x) = -4$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{2x} - 4$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=8,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{2x} c_1 - 4$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 23

```
DSolve[y''[x]-y'[x]-2*y[x]==8,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x} + c_2 e^{2x} - 4$$

3.2 problem 2

Internal problem ID [5137]
Internal file name [`OUTPUT/4630_Sunday_June_05_2022_03_02_20_PM_58316311/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 4y = 10\,\mathrm{e}^{3x}}$$

3.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 10\,\mathrm{e}^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\,\mathrm{e}^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \mathrm{e}^{3x} = 10\,\mathrm{e}^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}\right) + \left(2\,\mathrm{e}^{3x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} + 2\,\mathrm{e}^{3x} \tag{1}$$

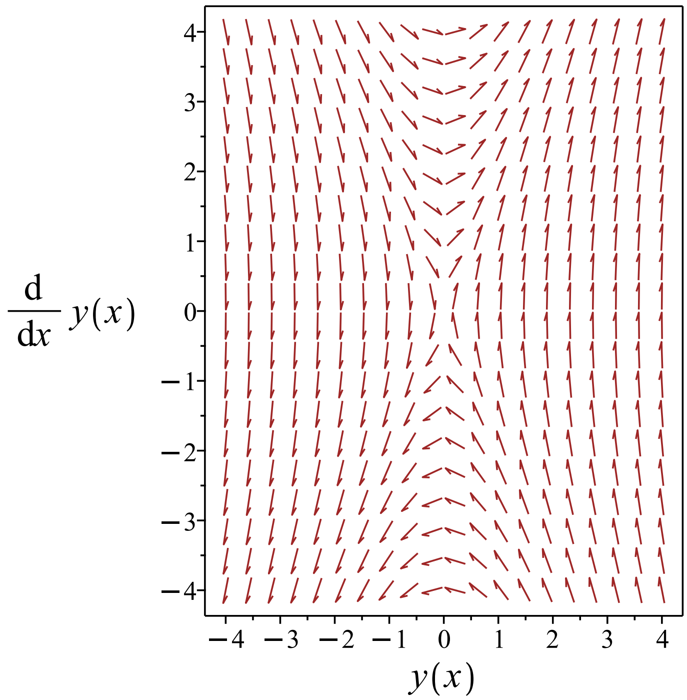


Figure 141: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{2x} + c_2 e^{-2x} + 2\,e^{3x}$$

Verified OK.

3.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 95: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= e^{-2x}
\end{aligned}$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= e^{-2x} \int \frac{1}{e^{-4x}}\,dx \\
&= e^{-2x}\left(\frac{e^{4x}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10\,\mathrm{e}^{3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{3x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{4}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{3x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 e^{3x} = 10\, e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\, e^{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(2\, e^{3x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + 2\, e^{3x} \tag{1}$$

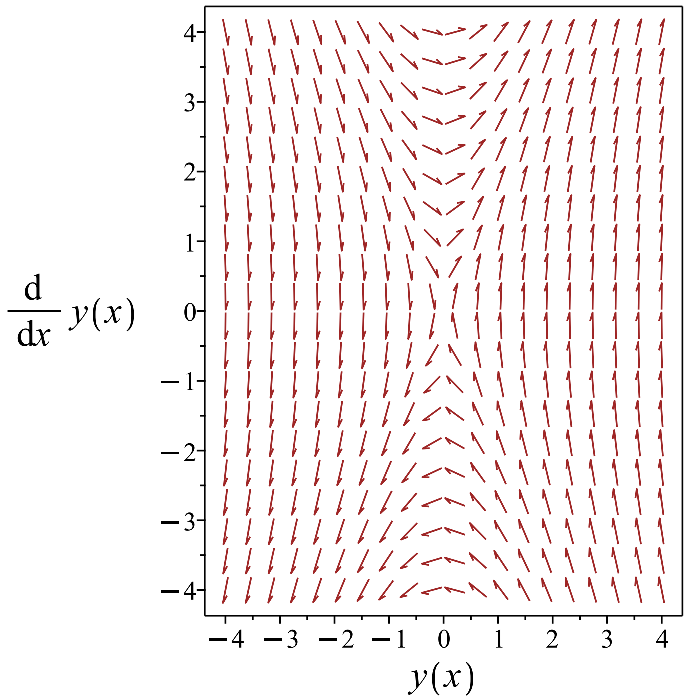


Figure 142: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + 2\,e^{3x}$$

Verified OK.

3.2.3 Maple step by step solution

Let's solve

$y'' - 4y = 10\,e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 10\,\mathrm{e}^{3x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{5\,\mathrm{e}^{-2x} \left(\int \mathrm{e}^{5x} dx\right)}{2} + \frac{5\,\mathrm{e}^{2x} \left(\int \mathrm{e}^{x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = 2\,\mathrm{e}^{3x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + 2\,\mathrm{e}^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)-4*y(x)=10*exp(3*x),y(x), singsol=all)
```

$$y(x) = \left(2\,\mathrm{e}^{5x} + \mathrm{e}^{4x}c_1 + c_2\right)\mathrm{e}^{-2x}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 29

```
DSolve[y''[x]-4*y[x]==10*Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(2e^{5x} + c_1e^{4x} + c_2\right)$$

3.3 problem 3

Internal problem ID [5138]
Internal file name [`OUTPUT/4631_Sunday_June_05_2022_03_02_21_PM_69647471/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y' + y = \mathrm{e}^{-2x}}$$

3.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = \mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 e^{-x} + c_2 x\, e^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + x\, e^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{-x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} = e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + x\, e^{-x} c_2\right) + \left(e^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + e^{-2x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + e^{-2x} \tag{1}$$

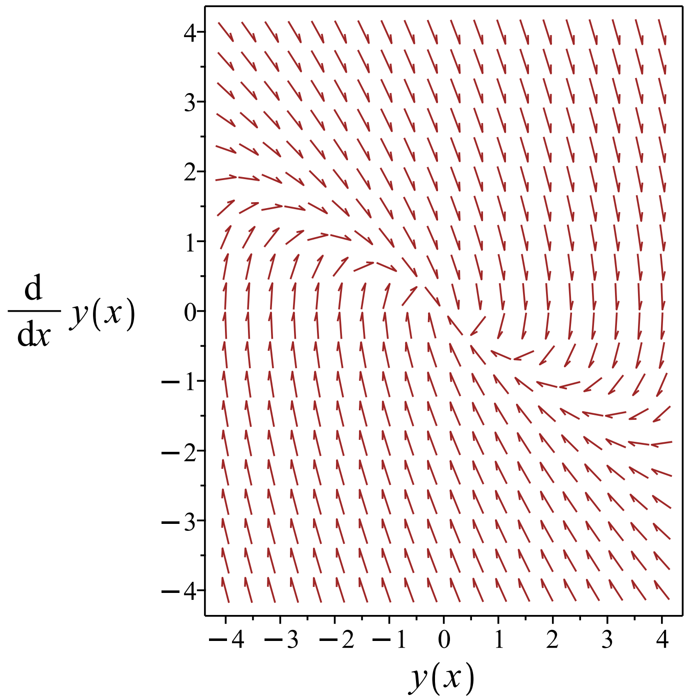


Figure 143: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^{-2x}$$

Verified OK.

3.3.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\, dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = e^{x}e^{-2x}$$
$$(y\,e^{x})'' = e^{x}e^{-2x}$$

Integrating once gives

$$(y\,e^{x})' = -e^{-x} + c_1$$

Integrating again gives

$$(y\,e^{x}) = c_1 x + e^{-x} + c_2$$

Hence the solution is

$$y = \frac{c_1 x + e^{-x} + c_2}{e^{x}}$$

Or

$$y = c_1 x\,e^{-x} + c_2 e^{-x} + e^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{-x} + c_2 e^{-x} + e^{-2x} \tag{1}$$

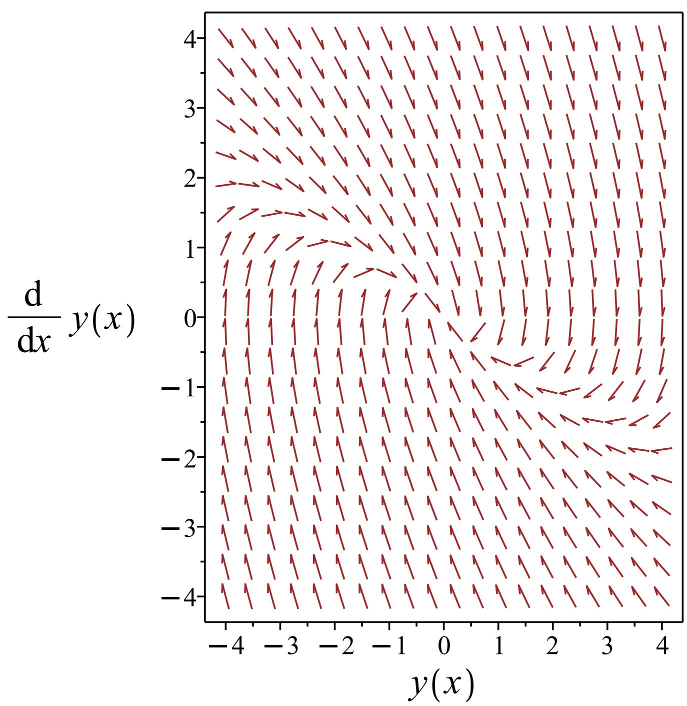


Figure 144: Slope field plot

Verification of solutions

$$y = c_1 x \, \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} + \mathrm{e}^{-2x}$$

Verified OK.

3.3.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 97: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{\left(y_1\right)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{\left(y_1\right)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \mathrm{e}^{-2x} = \mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2\right) + \left(\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-x}(c_2 x + c_1) + e^{-2x}$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2 x + c_1) + e^{-2x} \tag{1}$$

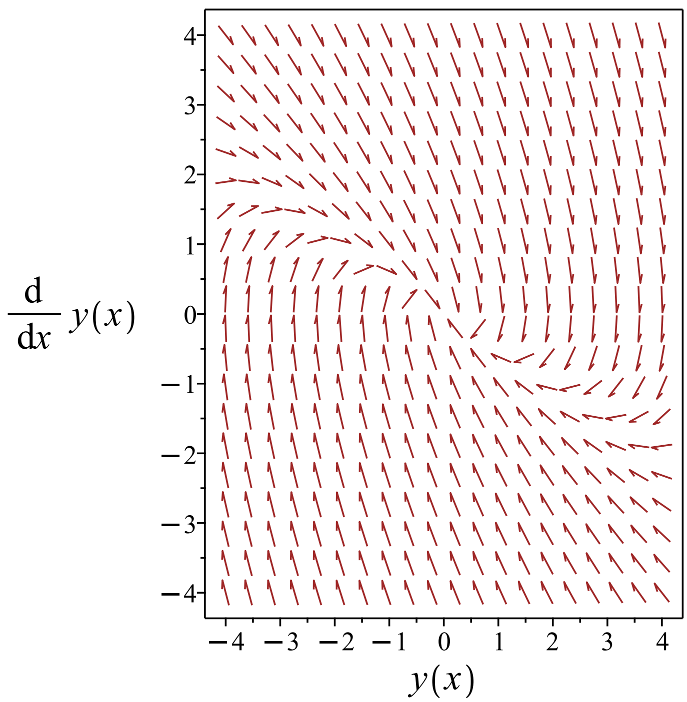


Figure 145: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2 x + c_1) + e^{-2x}$$

Verified OK.

3.3.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = \mathrm{e}^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \mathrm{e}^{-2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \mathrm{e}^{-x}\left(-\left(\int x\,\mathrm{e}^{-x}dx\right) + \left(\int \mathrm{e}^{-x}dx\right)x\right)$

- Compute integrals

  $y_p(x) = e^{-2x}$

- Substitute particular solution into general solution to ODE

  $y = x\,e^{-x}c_2 + c_1 e^{-x} + e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=exp(-2*x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,e^{-x} + e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.029 (sec). Leaf size: 24

```
DSolve[y''[x]+2*y'[x]+y[x]==Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}(1 + e^x(c_2 x + c_1))$$

3.4 problem 4

Internal problem ID [5139]
Internal file name [`OUTPUT/4632_Sunday_June_05_2022_03_02_22_PM_3862751/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 25y = 5x^2 + x}$$

3.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 25, f(x) = 5x^2 + x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 25y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 25\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(25)} \\ &= \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5i \\ \lambda_2 &= -5i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5i \\ \lambda_2 &= -5i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(5x) + c_2 \sin(5x))$$

Or

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(5x) + c_2 \sin(5x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(5x), \sin(5x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25A_3x^2 + 25A_2x + 25A_1 + 2A_3 = 5x^2 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{125}, A_2 = \frac{1}{25}, A_3 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(5x) + c_2 \sin(5x)) + \left(\frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125} \tag{1}$$

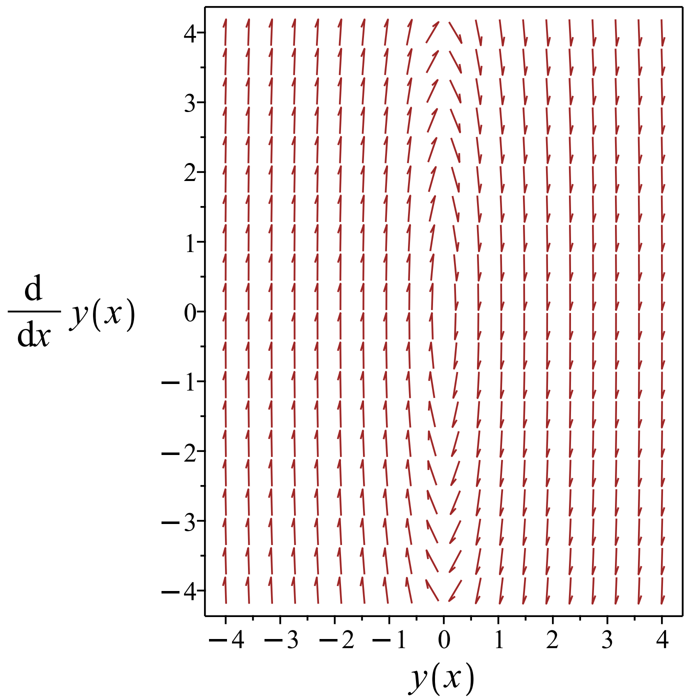


Figure 146: Slope field plot

Verification of solutions

$$y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$$

Verified OK.

3.4.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 25y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(5x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(5x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(5x) \int \frac{1}{\cos(5x)^2}\, dx \\
&= \cos(5x) \left(\frac{\tan(5x)}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(5x)) + c_2\left(\cos(5x)\left(\frac{\tan(5x)}{5}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 25y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(5x)}{5}, \cos(5x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25A_3x^2 + 25A_2x + 25A_1 + 2A_3 = 5x^2 + x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{125}, A_2 = \frac{1}{25}, A_3 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(5x) + \frac{c_2\sin(5x)}{5}\right) + \left(\frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(5x) + \frac{c_2\sin(5x)}{5} + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125} \tag{1}$$

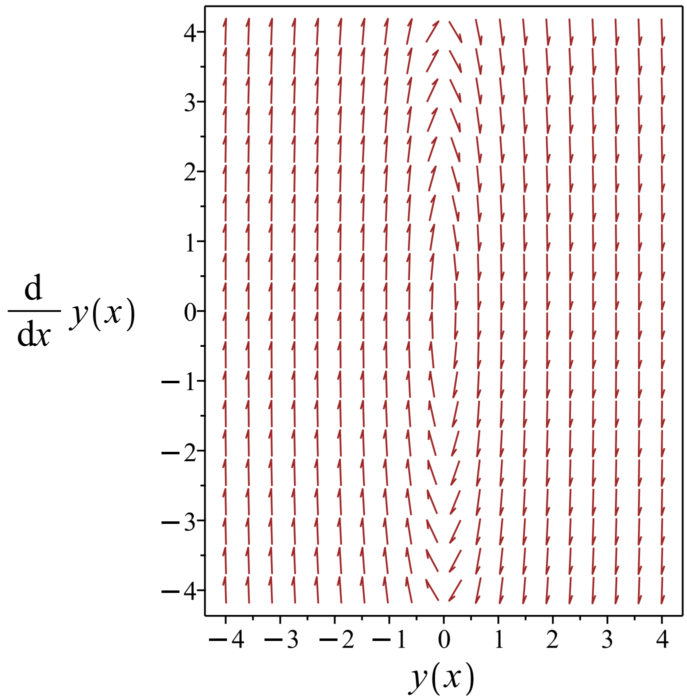


Figure 147: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$$

Verified OK.

3.4.3 Maple step by step solution

Let's solve

$y'' + 25y = 5x^2 + x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

$r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(5x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(5x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(5x) + c_2 \sin(5x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 5x^2 + x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(5x) & \sin(5x) \\ -5\sin(5x) & 5\cos(5x) \end{array}\right]$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 5$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(5x)\left(\int \sin(5x)(5x^2+x)dx\right)}{5} + \frac{\sin(5x)\left(\int \cos(5x)(5x^2+x)dx\right)}{5}$

  ◦ Compute integrals

  $y_p(x) = \frac{1}{5}x^2 + \frac{1}{25}x - \frac{2}{125}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(5x) + c_2 \sin(5x) + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+25*y(x)=5*x^2+x,y(x), singsol=all)
```

$$y(x) = \sin(5x) c_2 + \cos(5x) c_1 + \frac{x^2}{5} + \frac{x}{25} - \frac{2}{125}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 34

```
DSolve[y''[x]+25*y[x]==5*x^2+x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{125}(25x^2 + 5x - 2) + c_1 \cos(5x) + c_2 \sin(5x)$$

3.5 problem 5

Internal problem ID [5140]
Internal file name [`OUTPUT/4633_Sunday_June_05_2022_03_02_23_PM_47570725/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + y = 4\sin(x)}$$

3.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 1, f(x) = 4\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)}\\ &= 1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$y = c_1 e^{x} + c_2 e^{x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 x\, e^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x} x, e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 \sin(x) - 2A_2 \cos(x) = 4 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 \cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \mathrm{e}^{x} + c_2 x \,\mathrm{e}^{x}) + (2 \cos(x)) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x} (c_2 x + c_1) + 2 \cos(x)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x} (c_2 x + c_1) + 2 \cos(x) \tag{1}$$

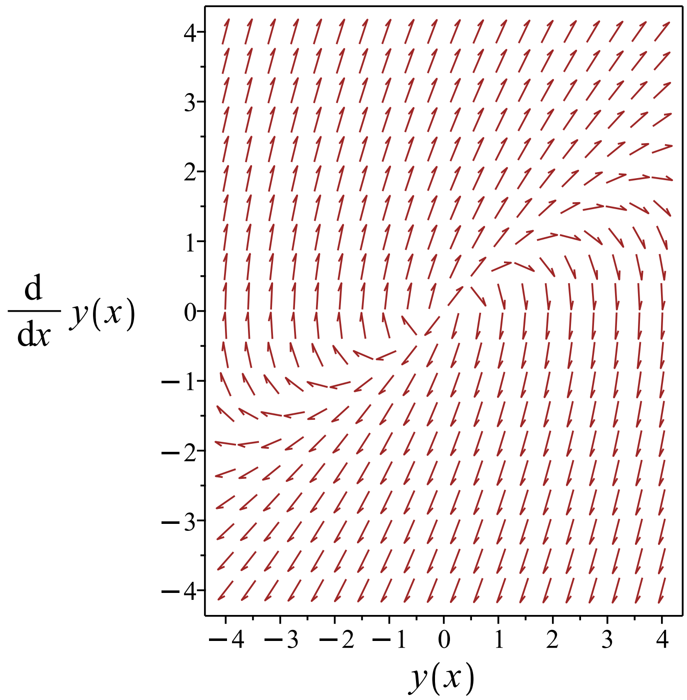


Figure 148: Slope field plot

Verification of solutions

$$y = e^{x}(c_2 x + c_1) + 2\cos(x)$$

Verified OK.

3.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -2\, dx} \\ &= e^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 4\,\mathrm{e}^{-x}\sin(x)$$
$$\left(\mathrm{e}^{-x}y\right)'' = 4\,\mathrm{e}^{-x}\sin(x)$$

Integrating once gives

$$\left(\mathrm{e}^{-x}y\right)' = -2\,\mathrm{e}^{-x}(\sin(x)+\cos(x)) + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-x}y\right) = c_1x + 2\,\mathrm{e}^{-x}\cos(x) + c_2$$

Hence the solution is

$$y = \frac{c_1x + 2\,\mathrm{e}^{-x}\cos(x) + c_2}{\mathrm{e}^{-x}}$$

Or

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + 2\cos(x)$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{x} + c_2\mathrm{e}^{x} + 2\cos(x) \tag{1}$$

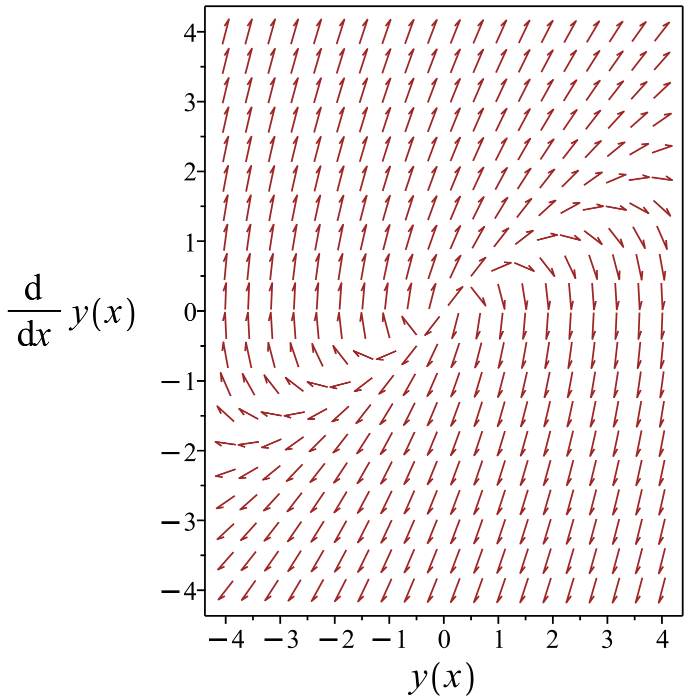


Figure 149: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{x} + c_2 \mathrm{e}^{x} + 2 \cos(x)$$

Verified OK.

3.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(x))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - 2y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^x + c_2 x\, \mathrm{e}^x
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
4\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^x x, \mathrm{e}^x\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
2A_1 \sin(x) - 2A_2 \cos(x) = 4\sin(x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 e^x + c_2 x\, e^x) + (2\cos(x)) \end{aligned}$$

Which simplifies to

$$y = e^x (c_2 x + c_1) + 2\cos(x)$$

Summary

The solution(s) found are the following

$$y = e^x (c_2 x + c_1) + 2\cos(x) \tag{1}$$

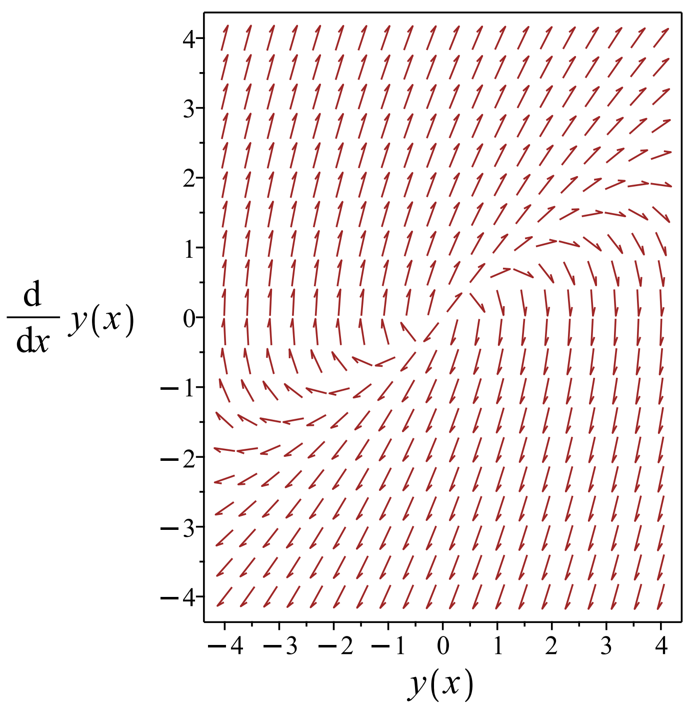


Figure 150: Slope field plot

Verification of solutions

$$y = e^x(c_2x + c_1) + 2\cos(x)$$

Verified OK.

3.5.4 Maple step by step solution

Let's solve

$y'' - 2y' + y = 4\sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r-1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^x x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 x\, e^x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\sin(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^x x \\ e^x & e^x x + e^x \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 4\,e^x\left(-\left(\int x\,e^{-x}\sin(x)\,dx\right) + x\left(\int e^{-x}\sin(x)\,dx\right)\right)$

- Compute integrals

  $y_p(x) = 2\cos(x)$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,e^x + c_1 e^x + 2\cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+y(x)=4*sin(x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,e^x + 2\cos(x)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 21

```
DSolve[y''[x]-2*y'[x]+y[x]==4*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\cos(x) + e^x(c_2 x + c_1)$$

3.6 problem 6

Internal problem ID [5141]
Internal file name [`OUTPUT/4634_Sunday_June_05_2022_03_02_24_PM_54977219/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y' + 5y = 2\,\mathrm{e}^{-2x}}$$

With initial conditions

$$[y(0) = 1, y'(0) = -2]$$

3.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 4 \\ q(x) &= 5 \\ F &= 2\,\mathrm{e}^{-2x} \end{aligned}$$

Hence the ode is

$$y'' + 4y' + 5y = 2\,\mathrm{e}^{-2x}$$

The domain of $p(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 5$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2\,\mathrm{e}^{-2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

3.6.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 5, f(x) = 2\,\mathrm{e}^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 5\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2x}(\cos(x)\, c_1 + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x} \cos(x), \sin(x) e^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 e^{-2x} = 2\, e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\, e^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2x}(\cos(x) c_1 + c_2 \sin(x))\right) + \left(2\, e^{-2x}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + 2\, e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = 2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\,e^{-2x}(\cos(x)\,c_1 + c_2\sin(x)) + e^{-2x}(-\sin(x)\,c_1 + c_2\cos(x)) - 4\,e^{-2x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -2c_1 + c_2 - 4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

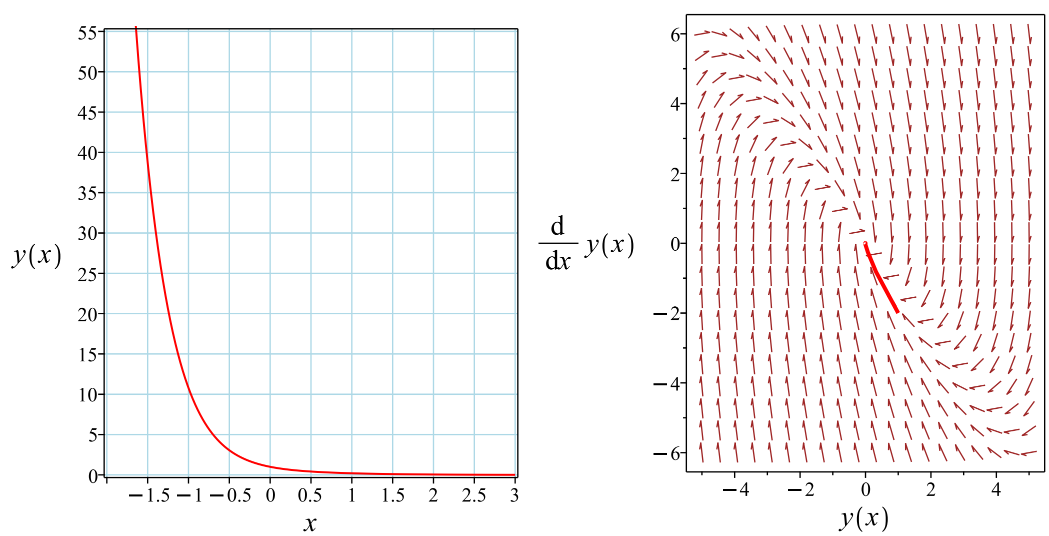
$$y = -e^{-2x}\cos(x) + 2\,e^{-2x}$$

Which simplifies to

$$y = -e^{-2x}(-2 + \cos(x))$$

Summary
The solution(s) found are the following

$$y = -e^{-2x}(-2 + \cos(x)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -e^{-2x}(-2 + \cos(x))$$

Verified OK.

3.6.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' + 4y' + 5y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 4 \tag{3}\\
C &= 5
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -1 \\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = -z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-2x} \\
&= z_1 \left(\mathrm{e}^{-2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x} \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-4x}}{(y_1)^2}\, dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x} \cos(x)\right) + c_2 \left(\mathrm{e}^{-2x} \cos(x) (\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-2x} \cos(x)\, c_1 + \mathrm{e}^{-2x} \sin(x)\, c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}\cos(x), \sin(x)\,\mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\mathrm{e}^{-2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-2x} = 2\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2\,\mathrm{e}^{-2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-2x}\cos(x)\,c_1 + \mathrm{e}^{-2x}\sin(x)\,c_2\right) + \left(2\,\mathrm{e}^{-2x}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(\cos(x)\,c_1 + c_2\sin(x)) + 2\,\mathrm{e}^{-2x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + 2 e^{-2x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = 2 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2 e^{-2x}(\cos(x) c_1 + c_2 \sin(x)) + e^{-2x}(-\sin(x) c_1 + c_2 \cos(x)) - 4 e^{-2x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = -2c_1 + c_2 - 4 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 0$$

Substituting these values back in above solution results in

$$y = -e^{-2x} \cos(x) + 2 e^{-2x}$$

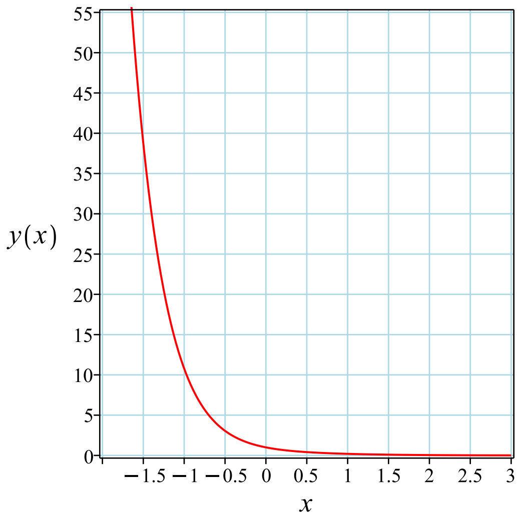
Which simplifies to

$$y = -e^{-2x}(-2 + \cos(x))$$

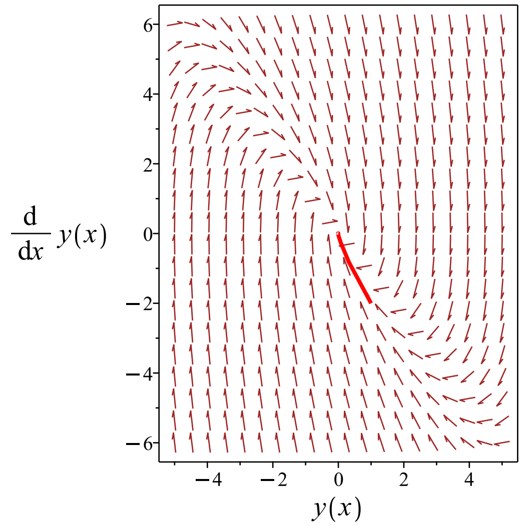
Summary

The solution(s) found are the following

$$y = -e^{-2x}(-2 + \cos(x)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{-2x}(-2 + \cos(x))$$

Verified OK.

3.6.4 Maple step by step solution

Let's solve

$$\left[y'' + 4y' + 5y = 2\,e^{-2x}, y(0) = 1, y'\Big|_{\{x=0\}} = -2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-2x} \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x) \mathrm{e}^{-2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{-2x} \cos(x) c_1 + \mathrm{e}^{-2x} \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\,\mathrm{e}^{-2x}\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} \cos(x) & \sin(x) \mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} \cos(x) - \sin(x) \mathrm{e}^{-2x} & \mathrm{e}^{-2x} \cos(x) - 2 \sin(x) \mathrm{e}^{-2x} \end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,\mathrm{e}^{-2x} \left(\cos(x) \left(\int \sin(x) dx\right) - \sin(x) \left(\int \cos(x) dx\right)\right)$
- Compute integrals

  $y_p(x) = 2\,\mathrm{e}^{-2x}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-2x} \cos(x) c_1 + \mathrm{e}^{-2x} \sin(x) c_2 + 2\,\mathrm{e}^{-2x}$

□ Check validity of solution $y = \mathrm{e}^{-2x} \cos(x) c_1 + \mathrm{e}^{-2x} \sin(x) c_2 + 2\mathrm{e}^{-2x}$

- Use initial condition $y(0) = 1$

  $1 = 2 + c_1$
- Compute derivative of the solution

  $y' = -2\,\mathrm{e}^{-2x} \cos(x) c_1 - \mathrm{e}^{-2x} \sin(x) c_1 - 2\,\mathrm{e}^{-2x} \sin(x) c_2 + \mathrm{e}^{-2x} \cos(x) c_2 - 4\,\mathrm{e}^{-2x}$
- Use the initial condition $y'\Big|_{\{x=0\}} = -2$

  $-2 = -2c_1 + c_2 - 4$

- Solve for c_1 and c_2

  $\{c_1 = -1, c_2 = 0\}$

- Substitute constant values into general solution and simplify

  $y = -\mathrm{e}^{-2x}(-2 + \cos(x))$

- Solution to the IVP

  $y = -\mathrm{e}^{-2x}(-2 + \cos(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+4*diff(y(x),x)+5*y(x)=2*exp(-2*x),y(0) = 1, D(y)(0) = -2],y(x), sings
```

$$y(x) = -\mathrm{e}^{-2x}(\cos(x) - 2)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 16

```
DSolve[{y''[x]+4*y'[x]+5*y[x]==2*Exp[-2*x],{y[0]==1,y'[0]==-2}},y[x],x,IncludeSingularSoluti
```

$$y(x) \to -e^{-2x}(\cos(x) - 2)$$

3.7 problem 7

Internal problem ID [5142]
Internal file name [`OUTPUT/4635_Sunday_June_05_2022_03_02_25_PM_2136530/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{3y'' - 2y' - y = 2x - 3}$$

3.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 3, B = -2, C = -1, f(x) = 2x - 3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$3y'' - 2y' - y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 3, B = -2, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$3\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} - \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$3\lambda^2 - 2\lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = -2, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(3)} \pm \frac{1}{(2)(3)}\sqrt{-2^2 - (4)(3)(-1)}\\ &= \frac{1}{3} \pm \frac{2}{3}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{3} + \frac{2}{3}$$

$$\lambda_2 = \frac{1}{3} - \frac{2}{3}$$

Which simplifies to

$$\lambda_1 = 1$$

$$\lambda_2 = -\frac{1}{3}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$y = c_1 e^{(1)x} + c_2 e^{\left(-\frac{1}{3}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-\frac{x}{3}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-\frac{x}{3}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{x}, e^{-\frac{x}{3}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 x - A_1 - 2A_2 = 2x - 3$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{x} + c_2 e^{-\frac{x}{3}}\right) + (-2x + 7) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{x} + c_2 e^{-\frac{x}{3}} - 2x + 7 \tag{1}$$

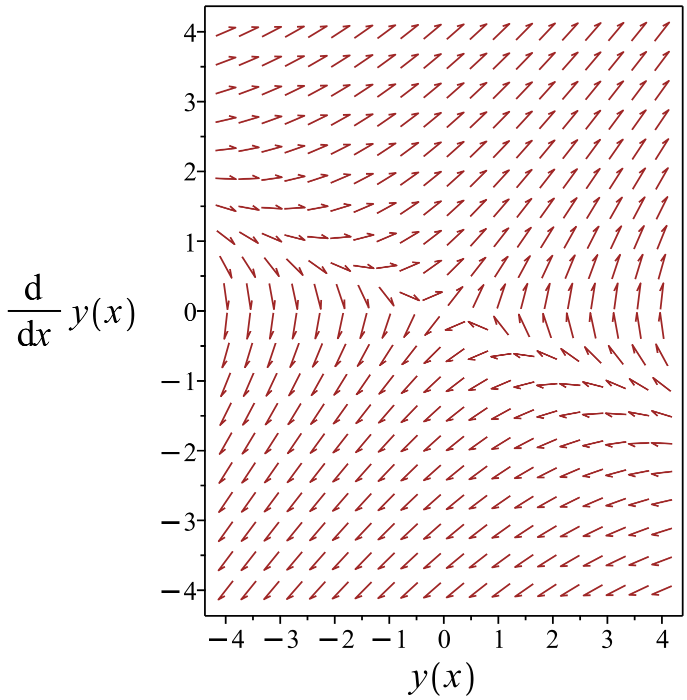


Figure 153: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-\frac{x}{3}} - 2x + 7$$

Verified OK.

3.7.2 Solving using Kovacic algorithm

Writing the ode as

$$3y'' - 2y' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= -2 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{9} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 9 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{4z(x)}{9} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{4}{9}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{2x}{3}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{3}\,dx} \\
&= z_1 e^{\frac{x}{3}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{3}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{3}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{3}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{2x}{3}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{3\,\mathrm{e}^{\frac{4x}{3}}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{x}{3}} \left(\frac{3\,\mathrm{e}^{\frac{4x}{3}}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
3y'' - 2y' - y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{3c_2 \mathrm{e}^{x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{3\,\mathrm{e}^{x}}{4}, \mathrm{e}^{-\frac{x}{3}}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2x - A_1 - 2A_2 = 2x - 3$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 7, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x + 7$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{3c_2 \mathrm{e}^{x}}{4}\right) + (-2x + 7) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x}{3}} + \frac{3c_2 \mathrm{e}^{x}}{4} - 2x + 7 \tag{1}$$

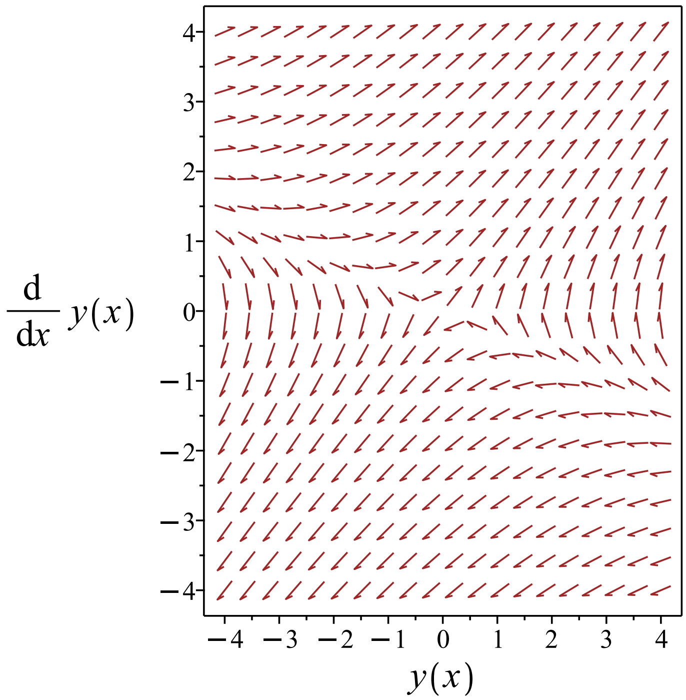


Figure 154: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{3}} + \frac{3c_2 e^{x}}{4} - 2x + 7$$

Verified OK.

3.7.3 Maple step by step solution

Let's solve

$3y'' - 2y' - y = 2x - 3$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{3} + \frac{y}{3} + \frac{2x}{3} - 1$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{3} - \frac{y}{3} = \frac{2x}{3} - 1$

- Characteristic polynomial of homogeneous ODE

$r^2 - \frac{2}{3}r - \frac{1}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+1)(r-1)}{3} = 0$

- Roots of the characteristic polynomial

  $r = \left(1, -\frac{1}{3}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-\frac{x}{3}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{2x}{3} - 1\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^x & \mathrm{e}^{-\frac{x}{3}} \\ \mathrm{e}^x & -\frac{\mathrm{e}^{-\frac{x}{3}}}{3} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = -\frac{4\,\mathrm{e}^{\frac{2x}{3}}}{3}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{\mathrm{e}^x\left(\int (2x-3)\mathrm{e}^{-x}dx\right)}{4} - \frac{\mathrm{e}^{-\frac{x}{3}}\left(\int (2x-3)\mathrm{e}^{\frac{x}{3}}dx\right)}{4}$

  - Compute integrals

    $y_p(x) = -2x + 7$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{-\frac{x}{3}} - 2x + 7$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(3*diff(y(x),x$2)-2*diff(y(x),x)-y(x)=2*x-3,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{3}} c_2 + e^x c_1 - 2x + 7$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 26

```
DSolve[3*y''[x]-2*y'[x]-y[x]==2*x-3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2x + c_1 e^{-x/3} + c_2 e^x + 7$$

3.8 problem 8

Internal problem ID [5143]
Internal file name [`OUTPUT/4636_Sunday_June_05_2022_03_02_26_PM_85407740/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Test Excercise 25. page 1093
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 8y = 8\,\mathrm{e}^{4x}}$$

3.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 8, f(x) = 8\,\mathrm{e}^{4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=-6, C=8$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda\,\mathrm{e}^{\lambda x} + 8\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A=1, B=-6, C=8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-6^2 - (4)\,(1)\,(8)}\\ &= 3 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 3+1\\ \lambda_2 &= 3-1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= 2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\,\mathrm{e}^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{2x}, \mathrm{e}^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{4x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{4x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1\mathrm{e}^{4x} = 8\,\mathrm{e}^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4x\,\mathrm{e}^{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{4x} + c_2\mathrm{e}^{2x}\right) + \left(4x\,\mathrm{e}^{4x}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{2x} + 4x e^{4x} \tag{1}$$

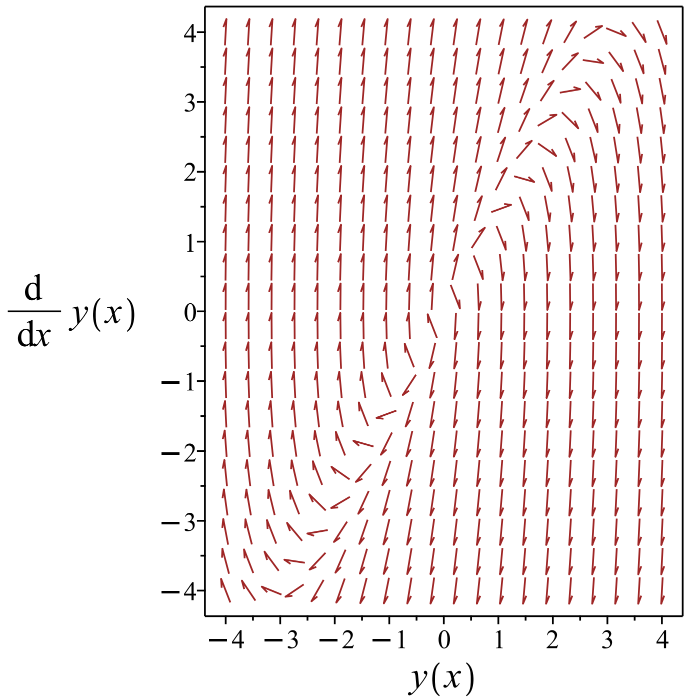


Figure 155: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{2x} + 4x e^{4x}$$

Verified OK.

3.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1\left(\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{e^{2x}}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{2x}\right) + c_2 \left(e^{2x} \left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + \frac{c_2 e^{4x}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = \frac{e^{4x}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & \frac{e^{4x}}{2} \\ \frac{d}{dx}\left(e^{2x}\right) & \frac{d}{dx}\left(\frac{e^{4x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & \frac{e^{4x}}{2} \\ 2\,e^{2x} & 2\,e^{4x} \end{vmatrix}$$

Therefore

$$W = \left(e^{2x}\right)\left(2\,e^{4x}\right) - \left(\frac{e^{4x}}{2}\right)\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = e^{2x} e^{4x}$$

Which simplifies to

$$W = e^{6x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4\,e^{8x}}{e^{6x}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\,e^{2x} dx$$

Hence

$$u_1 = -2\,e^{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8\,e^{2x} e^{4x}}{e^{6x}}\,dx$$

Which simplifies to

$$u_2 = \int 8 dx$$

Hence

$$u_2 = 8x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -2\,e^{4x} + 4x\,e^{4x}$$

Which simplifies to

$$y_p(x) = (4x-2)\,e^{4x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + \frac{c_2 e^{4x}}{2}\right) + \left((4x-2)\,e^{4x}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + \frac{c_2 \mathrm{e}^{4x}}{2} + (4x - 2)\,\mathrm{e}^{4x} \tag{1}$$

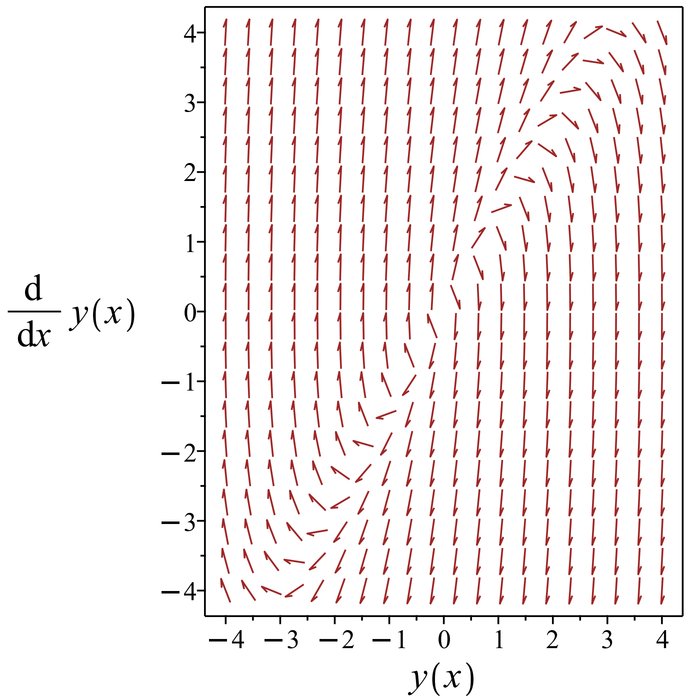


Figure 156: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{2x} + \frac{c_2 \mathrm{e}^{4x}}{2} + (4x - 2)\,\mathrm{e}^{4x}$$

Verified OK.

3.8.3 Maple step by step solution

Let's solve

$y'' - 6y' + 8y = 8\,\mathrm{e}^{4x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 6r + 8 = 0$

- Factor the characteristic polynomial

  $(r-2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (2, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 8\,e^{4x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \left[\begin{array}{cc} e^{2x} & e^{4x} \\ 2\,e^{2x} & 4\,e^{4x} \end{array}\right]$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2\,e^{6x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\,e^{2x}\left(\int e^{2x}dx\right) + 4\,e^{4x}\left(\int 1dx\right)$

  - Compute integrals

    $y_p(x) = (4x-2)\,e^{4x}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + c_2 e^{4x} + (4x-2)\,e^{4x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+8*y(x)=8*exp(4*x),y(x), singsol=all)
```

$$y(x) = \frac{(8x + c_1 - 4)\,\mathrm{e}^{4x}}{2} + c_2\mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 27

```
DSolve[y''[x]-6*y'[x]+8*y[x]==8*Exp[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{2x} + e^{4x}(4x - 2 + c_2)$$

4 Program 25. Second order differential equations. Further problems 25. page 1094

4.1 problem 1

Internal problem ID [5144]
Internal file name [`OUTPUT/4637_Sunday_June_05_2022_03_02_27_PM_83965471/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' - 7y' - 4y = e^{3x}}$$

4.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = -7, C = -4, f(x) = e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' - 7y' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -7, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 \mathrm{e}^{\lambda x} - 7\lambda\, \mathrm{e}^{\lambda x} - 4\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 7\lambda - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -7, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-7^2 - (4)\,(2)\,(-4)}\\ &= \frac{7}{4} \pm \frac{9}{4}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{7}{4} + \frac{9}{4}\\ \lambda_2 &= \frac{7}{4} - \frac{9}{4}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -\frac{1}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{\left(-\frac{1}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{4x}, e^{-\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{-\frac{x}{2}}\right) + \left(-\frac{e^{3x}}{7}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-\frac{x}{2}} - \frac{e^{3x}}{7} \tag{1}$$

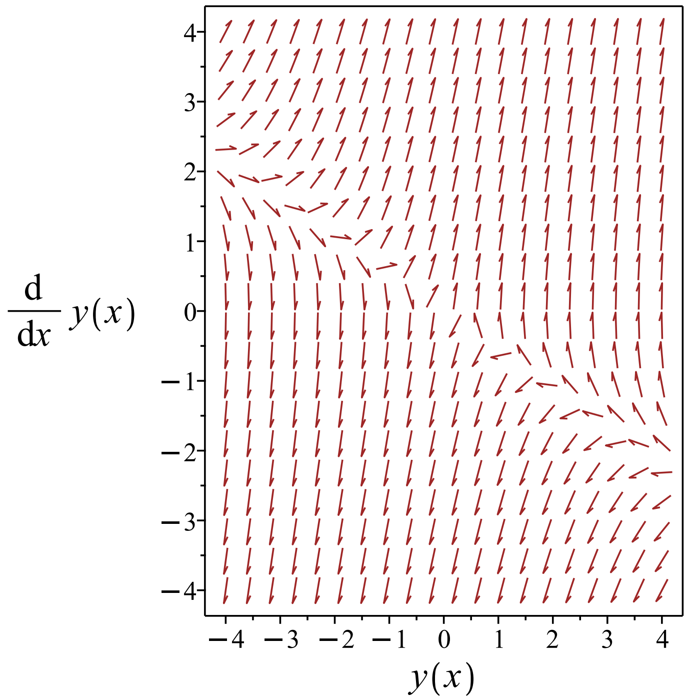


Figure 157: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{4x} + c_2 e^{-\frac{x}{2}} - \frac{e^{3x}}{7}$$

Verified OK.

4.1.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 7y' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -7 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{81}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 81 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{81z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{81}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{9x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-7}{2}\,dx} \\
&= z_1 e^{\frac{7x}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{7x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{7x}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{\frac{9x}{2}}}{9}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \left(\frac{2\,\mathrm{e}^{\frac{9x}{2}}}{9}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
2y'' - 7y' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + \frac{2c_2 \mathrm{e}^{4x}}{9}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{3x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{3x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{2\,\mathrm{e}^{4x}}{9}, \mathrm{e}^{-\frac{x}{2}}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{3x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{3x}}{7}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} + \frac{2c_2 e^{4x}}{9}\right) + \left(-\frac{e^{3x}}{7}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + \frac{2c_2 e^{4x}}{9} - \frac{e^{3x}}{7} \tag{1}$$

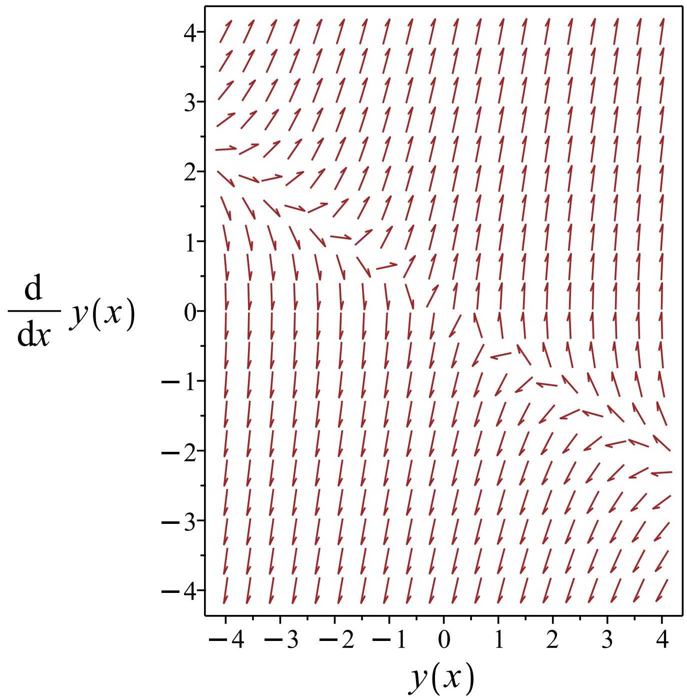


Figure 158: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + \frac{2c_2 e^{4x}}{9} - \frac{e^{3x}}{7}$$

Verified OK.

4.1.3 Maple step by step solution

Let's solve

$2y'' - 7y' - 4y = e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{7y'}{2} + 2y + \frac{e^{3x}}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{7y'}{2} - 2y = \frac{e^{3x}}{2}$

- Characteristic polynomial of homogeneous ODE

$r^2 - \frac{7}{2}r - 2 = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(r-4)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(4, -\frac{1}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{4x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-\frac{x}{2}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{\mathrm{e}^{3x}}{2}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{4x} & \mathrm{e}^{-\frac{x}{2}} \\ 4\,\mathrm{e}^{4x} & -\frac{\mathrm{e}^{-\frac{x}{2}}}{2} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = -\frac{9\,\mathrm{e}^{\frac{7x}{2}}}{2}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^{4x}\left(\int \mathrm{e}^{-x}dx\right)}{9} - \frac{\mathrm{e}^{-\frac{x}{2}}\left(\int \mathrm{e}^{\frac{7x}{2}}dx\right)}{9}$

  ◦ Compute integrals

  $y_p(x) = -\frac{\mathrm{e}^{3x}}{7}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-\frac{x}{2}} - \frac{\mathrm{e}^{3x}}{7}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(2*diff(y(x),x$2)-7*diff(y(x),x)-4*y(x)=exp(3*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x}{2}} c_2 + \mathrm{e}^{4x} c_1 - \frac{\mathrm{e}^{3x}}{7}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 33

```
DSolve[2*y''[x]-7*y'[x]-4*y[x]==Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{3x}}{7} + c_1 e^{-x/2} + c_2 e^{4x}$$

4.2 problem 2

Internal problem ID [5145]
Internal file name [`OUTPUT/4638_Sunday_June_05_2022_03_02_28_PM_90662562/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 6y' + 9y = 54x + 18}$$

4.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 9, f(x) = 54x + 18$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-6)^2 - (4)\,(1)\,(9)}\\ &= 3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -3$. Therefore the solution is

$$y = c_1 e^{3x} + c_2 e^{3x} x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x} c_1 + c_2 x\, e^{3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{3x} x, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_2x + 9A_1 - 6A_2 = 54x + 18$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = 6]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{3x}c_1 + c_2x\,\mathrm{e}^{3x}\right) + (6x + 6) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{3x}(c_2x + c_1) + 6x + 6$$

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_2x + c_1) + 6x + 6 \tag{1}$$

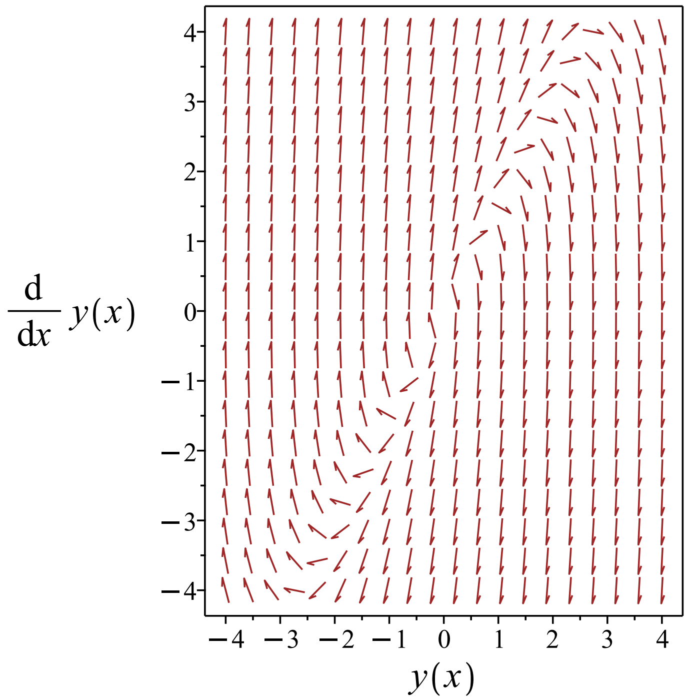


Figure 159: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{3x}(c_2x + c_1) + 6x + 6$$

Verified OK.

4.2.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -6\, dx} \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= \mathrm{e}^{-3x}(54x+18) \\
\left(\mathrm{e}^{-3x}y\right)'' &= \mathrm{e}^{-3x}(54x+18)
\end{aligned}
$$

Integrating once gives

$$\left(\mathrm{e}^{-3x}y\right)' = (-12-18x)\,\mathrm{e}^{-3x} + c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-3x}y\right) = (6x+6)\,\mathrm{e}^{-3x} + c_1x + c_2$$

Hence the solution is

$$y = \frac{(6x+6)\,\mathrm{e}^{-3x} + c_1x + c_2}{\mathrm{e}^{-3x}}$$

Or

$$y = c_1x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} + 6x + 6$$

Summary

The solution(s) found are the following

$$y = c_1x\,\mathrm{e}^{3x} + c_2\mathrm{e}^{3x} + 6x + 6 \tag{1}$$

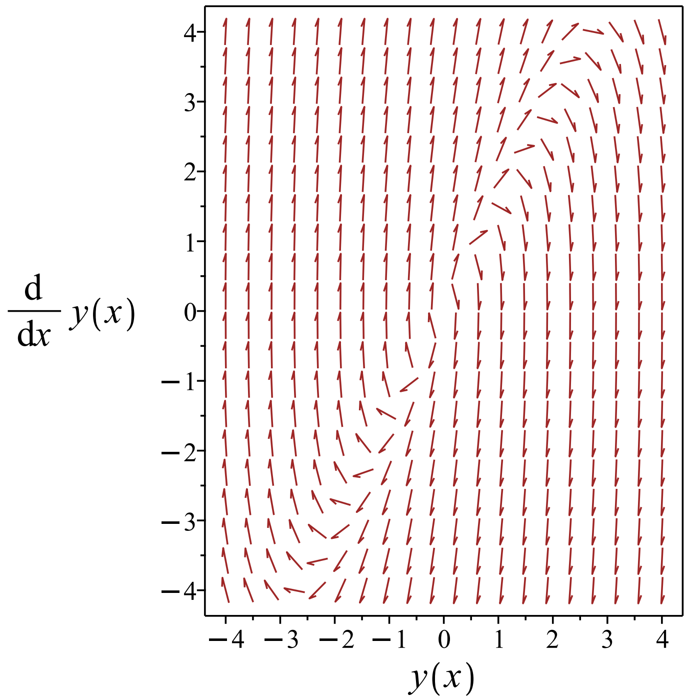


Figure 160: Slope field plot

Verification of solutions

$$y = c_1 x\,e^{3x} + c_2 e^{3x} + 6x + 6$$

Verified OK.

4.2.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1\left(\mathrm{e}^{3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{3x}\right) + c_2\left(\mathrm{e}^{3x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{3x} c_1 + c_2 x\, \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{3x} x, \mathrm{e}^{3x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
9A_2 x + 9A_1 - 6A_2 = 54x + 18
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 6, A_2 = 6]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 6x + 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 x\, e^{3x}\right) + (6x + 6) \end{aligned}$$

Which simplifies to

$$y = e^{3x}(c_2 x + c_1) + 6x + 6$$

Summary
The solution(s) found are the following

$$y = e^{3x}(c_2 x + c_1) + 6x + 6 \tag{1}$$

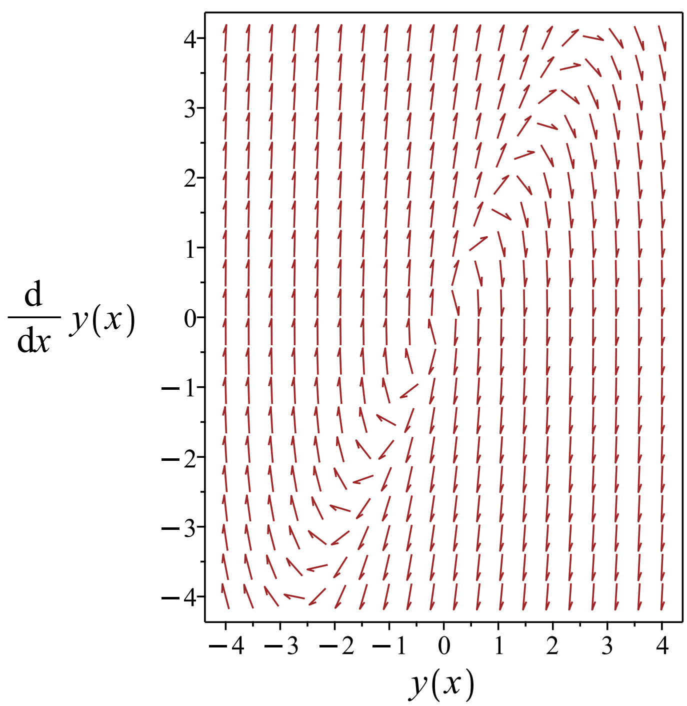


Figure 161: Slope field plot

Verification of solutions

$$y = e^{3x}(c_2 x + c_1) + 6x + 6$$

Verified OK.

4.2.4 Maple step by step solution

Let's solve

$y'' - 6y' + 9y = 54x + 18$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r-3)^2 = 0$

- Root of the characteristic polynomial

  $r = 3$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = e^{3x} x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{3x} c_1 + c_2 x\, e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 54x + 18\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{3x} & e^{3x} x \\ 3\,e^{3x} & 3\,e^{3x} x + e^{3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{6x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = 18\, e^{3x}\left(-\left(\int (3x^2 + x)\, e^{-3x} dx\right) + \left(\int e^{-3x}(3x + 1)\, dx\right) x\right)$
- Compute integrals

  $y_p(x) = 6x + 6$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{3x} + e^{3x} c_1 + 6x + 6$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+9*y(x)=54*x+18,y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\, e^{3x} + 6x + 6$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 27

```
DSolve[y''[x]-6*y'[x]+9*y[x]==54*x+18,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{3x} + x\left(6 + c_2 e^{3x}\right) + 6$$

4.3 problem 3

Internal problem ID [5146]
Internal file name [`OUTPUT/4639_Sunday_June_05_2022_03_02_29_PM_29408844/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 5y' + 6y = 100 \sin(4x)}$$

4.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -5, C = 6, f(x) = 100 \sin(4x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 5y' + 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -5, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 5\lambda\,\mathrm{e}^{\lambda x} + 6\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -5, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-5^2 - (4)\,(1)\,(6)} \\
&= \frac{5}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{5}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{5}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(2)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{2x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100 \sin(4x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(4x) + A_2 \sin(4x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-10A_1 \cos(4x) - 10A_2 \sin(4x) + 20A_1 \sin(4x) - 20A_2 \cos(4x) = 100 \sin(4x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4 \cos(4x) - 2 \sin(4x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{2x}\right) + \left(4 \cos(4x) - 2 \sin(4x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{2x} + 4 \cos(4x) - 2 \sin(4x) \tag{1}$$

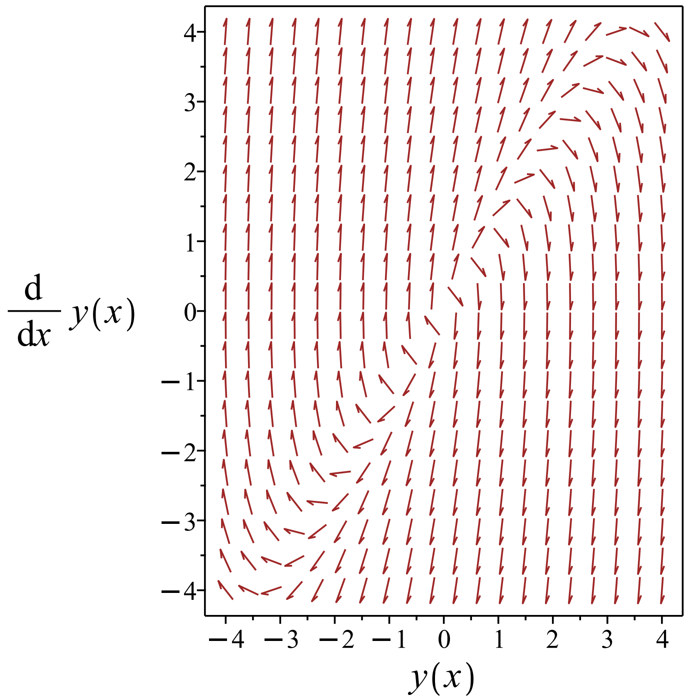


Figure 162: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2 e^{2x} + 4\cos(4x) - 2\sin(4x)$$

Verified OK.

4.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 5y' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -5 \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-5}{1}\,dx} \\
&= z_1 e^{\frac{5x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{5x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-5}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{5x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x} \left(\mathrm{e}^{x}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 5y' + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
100 \sin(4x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(4x), \sin(4x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{2x}, \mathrm{e}^{3x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(4x) + A_2 \sin(4x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-10A_1 \cos(4x) - 10A_2 \sin(4x) + 20A_1 \sin(4x) - 20A_2 \cos(4x) = 100 \sin(4x)
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 4\cos(4x) - 2\sin(4x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{3x}\right) + \left(4\cos(4x) - 2\sin(4x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{3x} + 4\cos(4x) - 2\sin(4x) \tag{1}$$

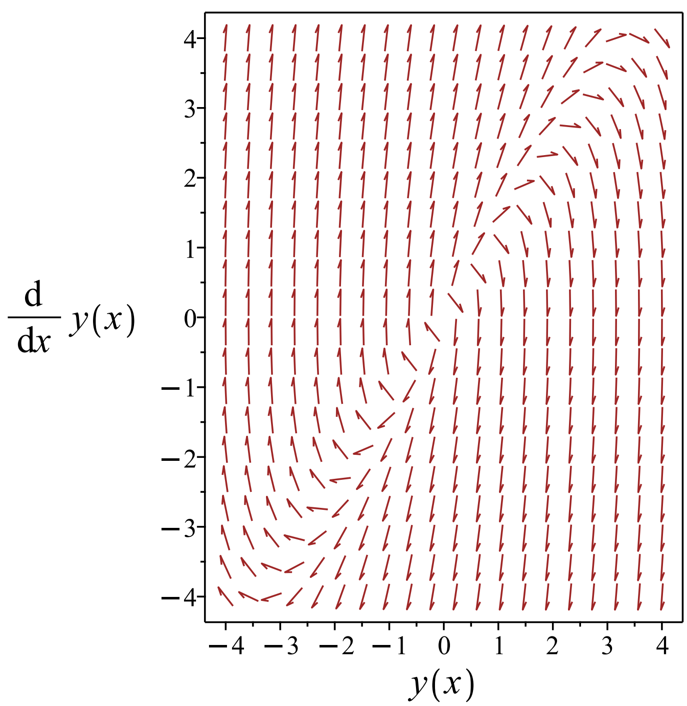


Figure 163: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{3x} + 4\cos(4x) - 2\sin(4x)$$

Verified OK.

4.3.3 Maple step by step solution

Let's solve

$y'' - 5y' + 6y = 100 \sin (4x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r - 3) = 0$

- Roots of the characteristic polynomial

  $r = (2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 100 \sin (4x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{2x} & \mathrm{e}^{3x} \\ 2\,\mathrm{e}^{2x} & 3\,\mathrm{e}^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{5x}$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -100\,\mathrm{e}^{2x} \left(\int \sin (4x)\, \mathrm{e}^{-2x} dx\right) + 100\,\mathrm{e}^{3x} \left(\int \sin (4x)\, \mathrm{e}^{-3x} dx\right)$

- Compute integrals

  $y_p(x) = 4\cos(4x) - 2\sin(4x)$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{2x} + c_2 e^{3x} + 4\cos(4x) - 2\sin(4x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-5*diff(y(x),x)+6*y(x)=100*sin(4*x),y(x), singsol=all)
```

$$y(x) = e^{3x}c_2 + e^{2x}c_1 - 2\sin(4x) + 4\cos(4x)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 33

```
DSolve[y''[x]-5*y'[x]+6*y[x]==100*Sin[4*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -2\sin(4x) + 4\cos(4x) + e^{2x}(c_2 e^x + c_1)$$

4.4 problem 4

Internal problem ID [5147]
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Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' + y = 4\sinh(x)}$$

4.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = 1, f(x) = 4\sinh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)} \\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{-x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + x\, \mathrm{e}^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \mathrm{e}^{-x} \\ y_2 &= x\, \mathrm{e}^{-x}\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right) - \left(x\,e^{-x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\,e^{-x}\sinh(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\sinh(x)\,x\,e^{x}dx$$

Hence

$$u_1 = -2\sinh(x)\, x\cosh(x) + x^2 + \cosh(x)^2 - 2x\cosh(x)^2 + \cosh(x)\sinh(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,\mathrm{e}^{-x}\sinh(x)}{\mathrm{e}^{-2x}}\, dx$$

Which simplifies to

$$u_2 = \int 4\,\mathrm{e}^{x}\sinh(x)\, dx$$

Hence

$$u_2 = 2\cosh(x)\sinh(x) - 2x + 2\cosh(x)^2$$

Which simplifies to

$$\begin{aligned} u_1 &= (1-2x)\cosh(x)^2 + (1-2x)\sinh(x)\cosh(x) + x^2 + x \\ u_2 &= 2\cosh(x)\sinh(x) - 2x + 2\cosh(x)^2 \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & \left((1-2x)\cosh(x)^2 + (1-2x)\sinh(x)\cosh(x) + x^2 + x\right)\mathrm{e}^{-x} \\ & + \left(2\cosh(x)\sinh(x) - 2x + 2\cosh(x)^2\right) x\,\mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}c_2\right) + \left(\mathrm{e}^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right)$$

Summary

The solution(s) found are the following

$$y = e^{-x}(c_2x + c_1) + e^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right) \tag{1}$$

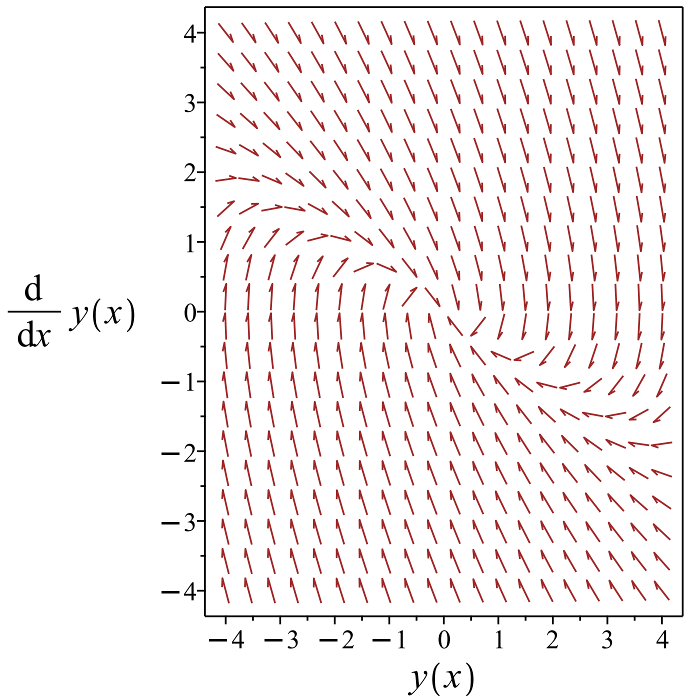


Figure 164: Slope field plot

Verification of solutions

$$y = e^{-x}(c_2x + c_1) + e^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right)$$

Verified OK.

4.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 4\,\mathrm{e}^{x}\sinh(x) \\
(y\,\mathrm{e}^{x})'' &= 4\,\mathrm{e}^{x}\sinh(x)
\end{aligned}
$$

Integrating once gives

$$
(y\,\mathrm{e}^{x})' = 2\cosh(x)\sinh(x) - 2x + 2\cosh(x)^{2} + c_1
$$

Integrating again gives

$$
(y\,\mathrm{e}^{x}) = \cosh(x)^{2} + \cosh(x)\sinh(x) - x(x - c_1 - 1) + c_2
$$

Hence the solution is

$$
y = \frac{\cosh(x)^{2} + \cosh(x)\sinh(x) - x(x - c_1 - 1) + c_2}{\mathrm{e}^{x}}
$$

Or

$$
y = c_1 x\,\mathrm{e}^{-x} - x^{2}\mathrm{e}^{-x} + \mathrm{e}^{-x}\cosh(x)^{2} + \sinh(x)\,\mathrm{e}^{-x}\cosh(x) + c_2\mathrm{e}^{-x} + x\,\mathrm{e}^{-x}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x\,\mathrm{e}^{-x} - x^{2}\mathrm{e}^{-x} + \mathrm{e}^{-x}\cosh(x)^{2} + \sinh(x)\,\mathrm{e}^{-x}\cosh(x) + c_2\mathrm{e}^{-x} + x\,\mathrm{e}^{-x} \tag{1}
$$

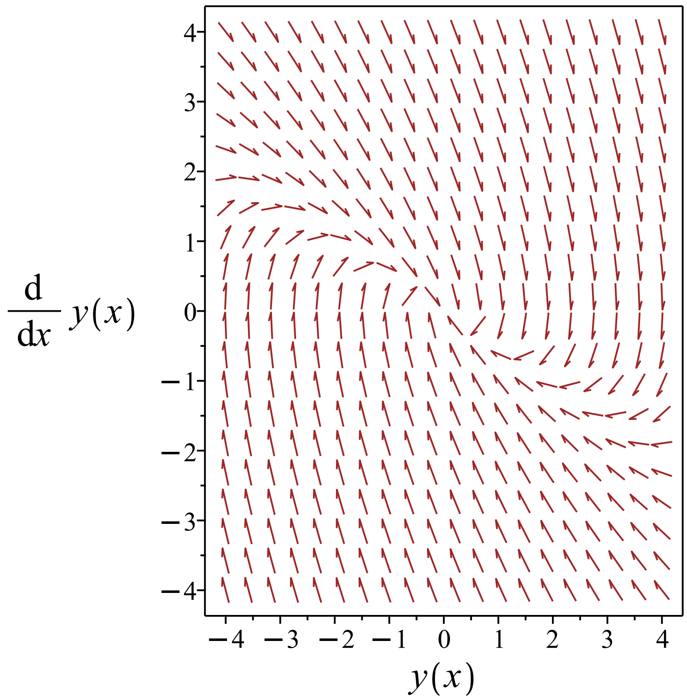


Figure 165: Slope field plot

Verification of solutions

$$y = c_1 x\, \mathrm{e}^{-x} - x^2 \mathrm{e}^{-x} + \mathrm{e}^{-x} \cosh(x)^2 + \sinh(x)\, \mathrm{e}^{-x} \cosh(x) + c_2 \mathrm{e}^{-x} + x\, \mathrm{e}^{-x}$$

Verified OK.

4.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-x} \\
y_2 &= x\,\mathrm{e}^{-x}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ \frac{d}{dx}\left(e^{-x}\right) & \frac{d}{dx}\left(x\,e^{-x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-x} & x\,e^{-x} \\ -e^{-x} & e^{-x} - x\,e^{-x} \end{vmatrix}$$

Therefore

$$W = \left(e^{-x}\right)\left(e^{-x} - x\,e^{-x}\right) - \left(x\,e^{-x}\right)\left(-e^{-x}\right)$$

Which simplifies to

$$W = e^{-2x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\,e^{-x}\sinh(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 4\sinh(x)\,x\,e^{x}\,dx$$

Hence

$$u_1 = -2\sinh(x)\,x\cosh(x) + x^2 + \cosh(x)^2 - 2x\cosh(x)^2 + \cosh(x)\sinh(x) + x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,e^{-x}\sinh(x)}{e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int 4\,\mathrm{e}^{x} \sinh\left(x\right) dx$$

Hence

$$u_2 = 2\cosh\left(x\right)\sinh\left(x\right) - 2x + 2\cosh\left(x\right)^{2}$$

Which simplifies to

$$\begin{aligned} u_1 &= (1-2x)\cosh\left(x\right)^{2} + (1-2x)\sinh\left(x\right)\cosh\left(x\right) + x^{2} + x \\ u_2 &= 2\cosh\left(x\right)\sinh\left(x\right) - 2x + 2\cosh\left(x\right)^{2} \end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &\left((1-2x)\cosh\left(x\right)^{2} + (1-2x)\sinh\left(x\right)\cosh\left(x\right) + x^{2} + x\right)\mathrm{e}^{-x} \\ &+ \left(2\cosh\left(x\right)\sinh\left(x\right) - 2x + 2\cosh\left(x\right)^{2}\right) x\,\mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_p(x) = \mathrm{e}^{-x}\left(\cosh\left(x\right)\sinh\left(x\right) + \cosh\left(x\right)^{2} - x^{2} + x\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2\right) + \left(\mathrm{e}^{-x}\left(\cosh\left(x\right)\sinh\left(x\right) + \cosh\left(x\right)^{2} - x^{2} + x\right)\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\cosh\left(x\right)\sinh\left(x\right) + \cosh\left(x\right)^{2} - x^{2} + x\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\cosh\left(x\right)\sinh\left(x\right) + \cosh\left(x\right)^{2} - x^{2} + x\right) \tag{1}$$

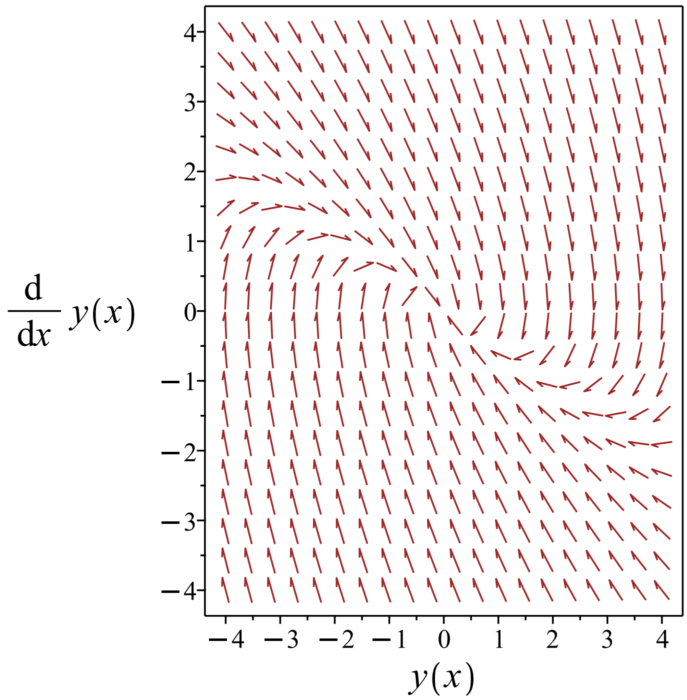


Figure 166: Slope field plot

<u>Verification of solutions</u>

$$y = \mathrm{e}^{-x}(c_2 x + c_1) + \mathrm{e}^{-x}\left(\cosh(x)\sinh(x) + \cosh(x)^2 - x^2 + x\right)$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$y'' + 2y' + y = 4\sinh(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r+1)^2 = 0$

- Root of the characteristic polynomial

$r = -1$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\sinh(x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & x\,\mathrm{e}^{-x} \\ -\mathrm{e}^{-x} & \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 4\,\mathrm{e}^{-x}\left(-\left(\int \sinh(x)\, x\,\mathrm{e}^{x} dx\right) + x\left(\int \mathrm{e}^{x} \sinh(x)\, dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{\mathrm{e}^{-x}(-2x^2+2x+1+\sinh(2x)+\cosh(2x))}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-x} + x\,\mathrm{e}^{-x} c_2 + \frac{\mathrm{e}^{-x}(-2x^2+2x+1+\sinh(2x)+\cosh(2x))}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)+y(x)=4*sinh(x),y(x), singsol=all)
```

$$y(x) = \frac{(-2x^2 + (2c_1 + 2)\, x + 2c_2 + 1)\, \mathrm{e}^{-x}}{2} + \frac{\mathrm{e}^x}{2}$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 31

```
DSolve[y''[x]+2*y'[x]+y[x]==4*Sinh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^x}{2} + e^{-x}(-x^2 + c_2 x + c_1)$$

4.5 problem 5

Internal problem ID [5148]
Internal file name [`OUTPUT/4641_Sunday_June_05_2022_03_02_31_PM_72953435/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{y'' + y' - 2y = 2\cosh(2x)}$$

4.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = -2, f(x) = 2\cosh(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(-2)}\\ &= -\frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{3}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \mathrm{e}^{x} \\ y_2 &= \mathrm{e}^{-2x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}(\mathrm{e}^{x}) & \frac{d}{dx}(\mathrm{e}^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{x} & \mathrm{e}^{-2x} \\ \mathrm{e}^{x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{x})\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)(\mathrm{e}^{x})$$

Which simplifies to

$$W = -3\,\mathrm{e}^{x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -3\,\mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{2\,\mathrm{e}^{-2x}\cosh\left(2x\right)}{-3\,\mathrm{e}^{-x}}d x\right)$$

Which simplifies to

$$u_1 = -\left(\int -\frac{2\,\mathrm{e}^{-x}\cosh\left(2x\right)}{3}d x\right)$$

Hence

$$u_1 = \frac{\sinh\left(x\right)}{3}+\frac{\sinh\left(3x\right)}{9}+\frac{\cosh\left(x\right)}{3}-\frac{\cosh\left(3x\right)}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{x}\cosh\left(2x\right)}{-3\,\mathrm{e}^{-x}}d x$$

Which simplifies to

$$u_2 = \int -\frac{2\cosh\left(2x\right)\mathrm{e}^{2x}}{3}d x$$

Hence

$$u_2 = -\frac{\cosh\left(2x\right)^{2}}{6}-\frac{\cosh\left(2x\right)\sinh\left(2x\right)}{6}-\frac{x}{3}$$

Which simplifies to

$$\begin{aligned}
u_1 &= \frac{\sinh\left(x\right)}{3}+\frac{\sinh\left(3x\right)}{9}+\frac{\cosh\left(x\right)}{3}-\frac{\cosh\left(3x\right)}{9}\\
u_2 &= -\frac{x}{3}-\frac{\cosh\left(4x\right)}{12}-\frac{1}{12}-\frac{\sinh\left(4x\right)}{12}
\end{aligned}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) = {} & \left(\frac{\sinh\left(x\right)}{3}+\frac{\sinh\left(3x\right)}{9}+\frac{\cosh\left(x\right)}{3}-\frac{\cosh\left(3x\right)}{9}\right)\mathrm{e}^{x}\\
& +\left(-\frac{x}{3}-\frac{\cosh\left(4x\right)}{12}-\frac{1}{12}-\frac{\sinh\left(4x\right)}{12}\right)\mathrm{e}^{-2x}
\end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{e^{-2x}\left(\left(\frac{4\cosh(x)^3}{3} - \frac{4\sinh(x)\cosh(x)^2}{3} - 2\cosh(x) - \frac{2\sinh(x)}{3}\right)e^{3x} + 2\cosh(x)^4 + 2\cosh(x)^3\sinh(x) - 2\operatorname{co}\right.}{3}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 e^x + c_2 e^{-2x}\right)$$
$$+ \left(-\frac{e^{-2x}\left(\left(\frac{4\cosh(x)^3}{3} - \frac{4\sinh(x)\cosh(x)^2}{3} - 2\cosh(x) - \frac{2\sinh(x)}{3}\right)e^{3x} + 2\cosh(x)^4 + 2\cosh(x)^3\sinh(x) -\right.}{3}\right.$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^x + c_2 e^{-2x} \tag{1}$$
$$-\frac{e^{-2x}\left(\left(\frac{4\cosh(x)^3}{3} - \frac{4\sinh(x)\cosh(x)^2}{3} - 2\cosh(x) - \frac{2\sinh(x)}{3}\right)e^{3x} + 2\cosh(x)^4 + 2\cosh(x)^3\sinh(x) - 2\operatorname{co}\right.}{3}$$

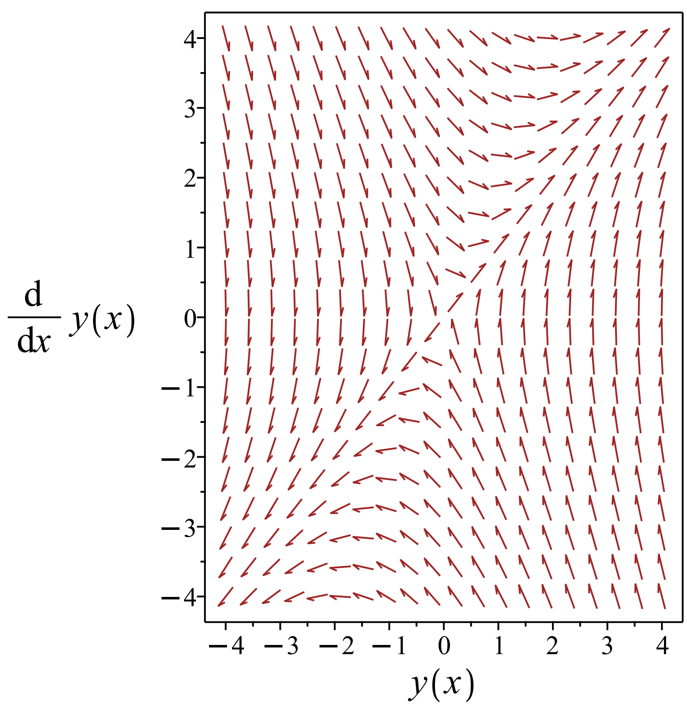


Figure 167: Slope field plot

Verification of solutions

$$y = c_1\mathrm{e}^{x} + c_2\mathrm{e}^{-2x} - \frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^3}{3} - \frac{4\sinh(x)\cosh(x)^2}{3} - 2\cosh(x) - \frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x} + 2\cosh(x)^4 + 2\cosh(x)^3\sinh(x) - 2\cos}{3}$$

Verified OK.

4.5.2 Solving using Kovacic algorithm

Writing the ode as

$$\begin{align*} y'' + y' - 2y &= 0 \tag{1}\\ Ay'' + By' + Cy &= 0 \tag{2} \end{align*}$$

Comparing (1) and (2) shows that

$$\begin{align*} A &= 1\\ B &= 1 \tag{3}\\ C &= -2 \end{align*}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{align*} r &= \frac{s}{t} \tag{5}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{align*}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{align*} s &= 9\\ t &= 4 \end{align*}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'-2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{x}}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{x}}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{x}}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \\ -2\,\mathrm{e}^{-2x} & \frac{\mathrm{e}^{x}}{3} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-2x}\right)\left(\frac{\mathrm{e}^{x}}{3}\right) - \left(\frac{\mathrm{e}^{x}}{3}\right)\left(-2\,\mathrm{e}^{-2x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{-x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\,\mathrm{e}^{x}\cosh(2x)}{3}}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\cosh(2x)\,\mathrm{e}^{2x}}{3} dx$$

Hence

$$u_1 = -\frac{\cosh(2x)^2}{6} - \frac{\cosh(2x)\sinh(2x)}{6} - \frac{x}{3}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\,\mathrm{e}^{-2x}\cosh(2x)}{\mathrm{e}^{-x}}\,dx$$

Which simplifies to

$$u_2 = \int 2\,\mathrm{e}^{-x}\cosh(2x)\,dx$$

Hence

$$u_2 = \sinh(x) + \frac{\sinh(3x)}{3} + \cosh(x) - \frac{\cosh(3x)}{3}$$

Which simplifies to

$$
\begin{aligned}
u_1 &= -\frac{x}{3}-\frac{\cosh\left(4x\right)}{12}-\frac{1}{12}-\frac{\sinh\left(4x\right)}{12}\\
u_2 &= \sinh\left(x\right)+\frac{\sinh\left(3x\right)}{3}+\cosh\left(x\right)-\frac{\cosh\left(3x\right)}{3}
\end{aligned}
$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(-\frac{x}{3}-\frac{\cosh\left(4x\right)}{12}-\frac{1}{12}-\frac{\sinh\left(4x\right)}{12}\right)\mathrm{e}^{-2x}\\
& +\frac{\left(\sinh\left(x\right)+\frac{\sinh(3x)}{3}+\cosh\left(x\right)-\frac{\cosh(3x)}{3}\right)\mathrm{e}^{x}}{3}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
& y_p(x) = \\
& -\frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^{3}}{3}-\frac{4\sinh(x)\cosh(x)^{2}}{3}-2\cosh\left(x\right)-\frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x}+2\cosh\left(x\right)^{4}+2\cosh\left(x\right)^{3}\sinh\left(x\right)-2\,\mathrm{co}\ldots}{3}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= \left(c_1\mathrm{e}^{-2x}+\frac{c_2\mathrm{e}^{x}}{3}\right)\\
&\quad +\left(-\frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^{3}}{3}-\frac{4\sinh(x)\cosh(x)^{2}}{3}-2\cosh\left(x\right)-\frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x}+2\cosh\left(x\right)^{4}+2\cosh\left(x\right)^{3}\sinh\left(x\right)-\ldots}{3}\right.
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
\begin{aligned}
y = {} & c_1\mathrm{e}^{-2x}+\frac{c_2\mathrm{e}^{x}}{3}\\
& -\frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^{3}}{3}-\frac{4\sinh(x)\cosh(x)^{2}}{3}-2\cosh\left(x\right)-\frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x}+2\cosh\left(x\right)^{4}+2\cosh\left(x\right)^{3}\sinh\left(x\right)-2\,\mathrm{co}\ldots}{3}
\end{aligned} \tag{1}
$$

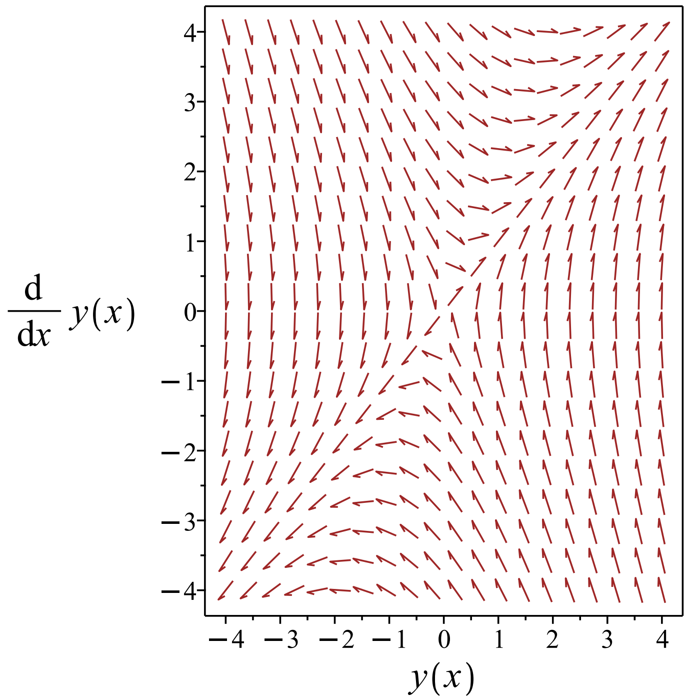


Figure 168: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{x}}{3} - \frac{e^{-2x}\left(\left(\frac{4\cosh(x)^3}{3} - \frac{4\sinh(x)\cosh(x)^2}{3} - 2\cosh(x) - \frac{2\sinh(x)}{3}\right)e^{3x} + 2\cosh(x)^4 + 2\cosh(x)^3\sinh(x) - 2\cos}{3}$$

Verified OK.

4.5.3 Maple step by step solution

Let's solve

$y'' + y' - 2y = 2\cosh(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r - 2 = 0$

- Factor the characteristic polynomial

$$(r+2)(r-1)=0$$

- Roots of the characteristic polynomial

$$r=(-2,1)$$

- 1st solution of the homogeneous ODE

$$y_1(x)=\mathrm{e}^{-2x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x)=\mathrm{e}^{x}$$

- General solution of the ODE

$$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{x}+y_p(x)$$

☐ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=2\cosh(2x)\right]$$

- ○ Wronskian of solutions of the homogeneous equation

$$W(y_1(x),y_2(x))=\left[\begin{array}{cc} \mathrm{e}^{-2x} & \mathrm{e}^{x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{x} \end{array}\right]$$

- ○ Compute Wronskian

$$W(y_1(x),y_2(x))=3\,\mathrm{e}^{-x}$$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x)=-\frac{2\left(-\mathrm{e}^{3x}\left(\int \mathrm{e}^{-x}\cosh(2x)dx\right)+\int \cosh(2x)\mathrm{e}^{2x}dx\right)\mathrm{e}^{-2x}}{3}$$

- ○ Compute integrals

$$y_p(x)=-\frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^3}{3}-\frac{4\sinh(x)\cosh(x)^2}{3}-2\cosh(x)-\frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x}+2\cosh(x)^4+2\cosh(x)^3\sinh(x)-2\cosh(x)^2-\cosh(x)\,\mathrm{si}}{3}$$

- Substitute particular solution into general solution to ODE

$$y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{x}-\frac{\mathrm{e}^{-2x}\left(\left(\frac{4\cosh(x)^3}{3}-\frac{4\sinh(x)\cosh(x)^2}{3}-2\cosh(x)-\frac{2\sinh(x)}{3}\right)\mathrm{e}^{3x}+2\cosh(x)^4+2\cosh(x)^3\sinh(x)-2\cosh(}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)+diff(y(x),x)-2*y(x)=2*cosh(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(9\,\mathrm{e}^{4x} + 36\,\mathrm{e}^{3x}c_2 + 36c_1 - 12x - 7)\,\mathrm{e}^{-2x}}{36}$$

✓ Solution by Mathematica

Time used: 0.047 (sec). Leaf size: 39

```
DSolve[y''[x]+y'[x]-2*y[x]==2*Cosh[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{36}e^{-2x}\left(-12x + 9e^{4x} + 36c_2e^{3x} - 4 + 36c_1\right)$$

4.6 problem 6

Internal problem ID [5149]
Internal file name [`OUTPUT/4642_Sunday_June_05_2022_03_02_32_PM_21488523/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' + 10y = 20 - e^{2x}}$$

4.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = 10, f(x) = 20 - e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda\, e^{\lambda x} + 10\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(10)}\\ &= \frac{1}{2} \pm \frac{i\sqrt{39}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{39}}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{39}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{39}}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{39}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{39}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{39}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{39}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$20 - e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right), e^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_2 e^{2x} + 10A_1 = 20 - e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - \frac{e^{2x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{39}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right)\right) + \left(2 - \frac{e^{2x}}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{39}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right) + 2 - \frac{e^{2x}}{12} \tag{1}$$

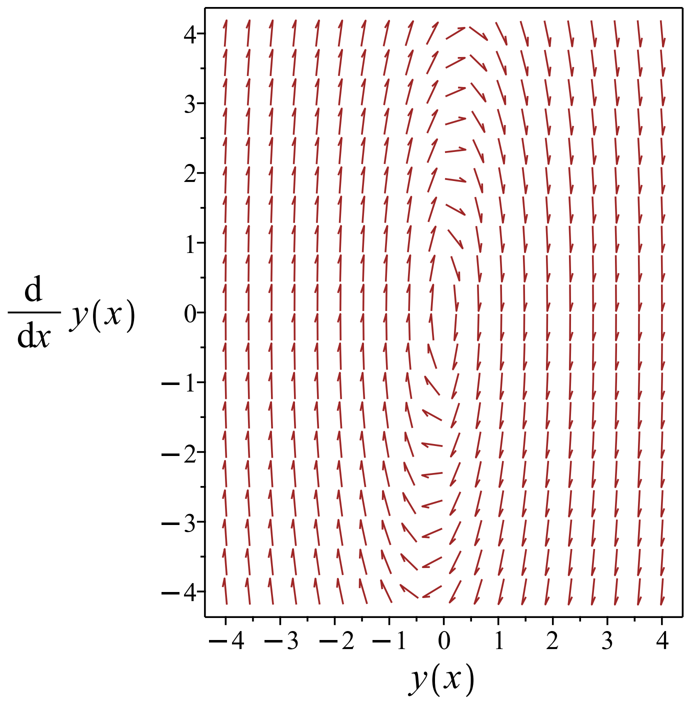


Figure 169: Slope field plot

Verification of solutions

$$y = e^{\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{39}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{39}\, x}{2}\right)\right) + 2 - \frac{e^{2x}}{12}$$

Verified OK.

4.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-39}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -39 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{39z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 119: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{39}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{39}\, x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\
&= z_1 e^{\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{39}\,\tan\left(\frac{\sqrt{39}\,x}{2}\right)}{39}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\,x}{2}\right) \left(\frac{2\sqrt{39}\,\tan\left(\frac{\sqrt{39}\,x}{2}\right)}{39}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' + 10y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) c_1 + \frac{2c_2 e^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\, x}{2}\right)}{39}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$20 - e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right), \frac{2\, e^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\, x}{2}\right)}{39}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 e^{2x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_2 e^{2x} + 10A_1 = 20 - e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 2, A_2 = -\frac{1}{12}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2 - \frac{e^{2x}}{12}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\,x}{2}\right) c_1 + \frac{2c_2 e^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\,x}{2}\right)}{39}\right) + \left(2 - \frac{e^{2x}}{12}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\,x}{2}\right) c_1 + \frac{2c_2 e^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\,x}{2}\right)}{39} + 2 - \frac{e^{2x}}{12} \tag{1}$$

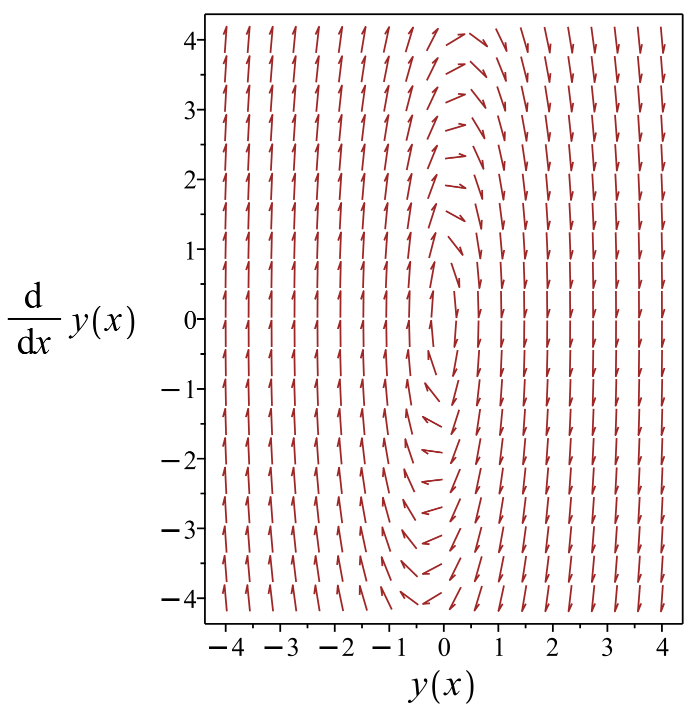


Figure 170: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) c_1 + \frac{2 c_2 \mathrm{e}^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\, x}{2}\right)}{39} + 2 - \frac{\mathrm{e}^{2x}}{12}$$

Verified OK.

4.6.3 Maple step by step solution

Let's solve

$y'' - y' + 10y = 20 - \mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{1 \pm (\sqrt{-39})}{2}$

- Roots of the characteristic polynomial

  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{39}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{39}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) c_1 + \mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 20 - \mathrm{e}^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) & e^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right) \\ \frac{e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right)}{2} - \frac{e^{\frac{x}{2}} \sqrt{39}\, \sin\left(\frac{\sqrt{39}\, x}{2}\right)}{2} & \frac{e^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right)}{2} + \frac{e^{\frac{x}{2}} \sqrt{39}\, \cos\left(\frac{\sqrt{39}\, x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \frac{\sqrt{39}\, e^{x}}{2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = \frac{2\, e^{\frac{x}{2}} \sqrt{39}\, \left(\cos\left(\frac{\sqrt{39}\, x}{2}\right) \left(\int \sin\left(\frac{\sqrt{39}\, x}{2}\right) \left(-20\, e^{-\frac{x}{2}} + e^{\frac{3x}{2}}\right) dx\right) - \sin\left(\frac{\sqrt{39}\, x}{2}\right) \left(\int \cos\left(\frac{\sqrt{39}\, x}{2}\right) \left(-20\, e^{-\frac{x}{2}} + e^{\frac{3x}{2}}\right) dx\right)\right)}{39}$$

- Compute integrals

  $$y_p(x) = 2 - \frac{e^{2x}}{12}$$

- Substitute particular solution into general solution to ODE

  $$y = e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) c_1 + e^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right) c_2 - \frac{e^{2x}}{12} + 2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 38

```
dsolve(diff(y(x),x$2)-diff(y(x),x)+10*y(x)=20-exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{\frac{x}{2}} \sin\left(\frac{\sqrt{39}\, x}{2}\right) c_2 + e^{\frac{x}{2}} \cos\left(\frac{\sqrt{39}\, x}{2}\right) c_1 + 2 - \frac{e^{2x}}{12}$$

✓ Solution by Mathematica

Time used: 1.291 (sec). Leaf size: 58

```
DSolve[y''[x]-y'[x]+10*y[x]==20-Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^{2x}}{12} + c_2 e^{x/2} \cos\left(\frac{\sqrt{39}x}{2}\right) + c_1 e^{x/2} \sin\left(\frac{\sqrt{39}x}{2}\right) + 2$$

4.7 problem 7

Internal problem ID [5150]
Internal file name [`OUTPUT/4643_Sunday_June_05_2022_03_02_33_PM_22831220/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 4y = 2\cos(x)^2}$$

4.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 4, C = 4, f(x) = 2\cos(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(4)^2 - (4)(1)(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-8A_2 \sin(2x) + 8A_3 \cos(2x) + 4A_1 = 2\cos(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4} + \frac{\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x \, \mathrm{e}^{-2x}\right) + \left(\frac{1}{4} + \frac{\sin(2x)}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8} \tag{1}$$

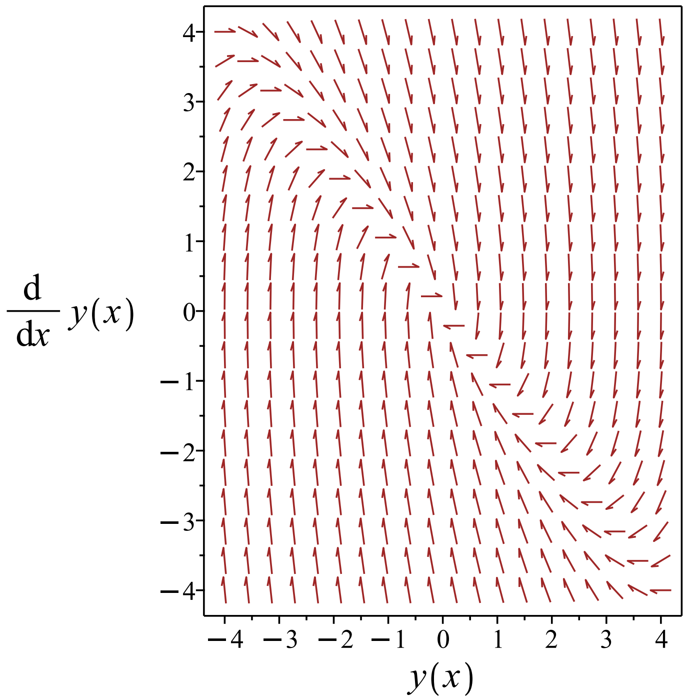


Figure 171: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8}$$

Verified OK.

4.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int 4\,dx}\\ &= e^{2x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 2\,\mathrm{e}^{2x}\cos(x)^2 \\
\left(\mathrm{e}^{2x}y\right)'' &= 2\,\mathrm{e}^{2x}\cos(x)^2
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{2x}y\right)' = \frac{(2+\cos(2x)+\sin(2x))\,\mathrm{e}^{2x}}{4} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{2x}y\right) = \frac{(2+\sin(2x))\,\mathrm{e}^{2x}}{8} + c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{\frac{(2+\sin(2x))\mathrm{e}^{2x}}{8} + c_1 x + c_2}{\mathrm{e}^{2x}}
$$

Or

$$
y = \frac{\cos(x)\sin(x)}{4} + c_1 x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} + \frac{1}{4}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\cos(x)\sin(x)}{4} + c_1 x\,\mathrm{e}^{-2x} + c_2\mathrm{e}^{-2x} + \frac{1}{4} \tag{1}
$$

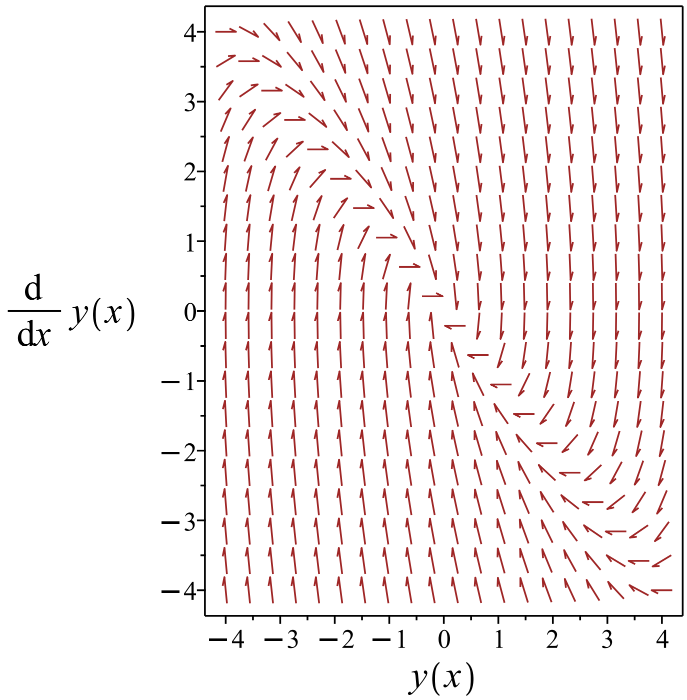


Figure 172: Slope field plot

Verification of solutions

$$y = \frac{\cos(x)\sin(x)}{4} + c_1 x\,\mathrm{e}^{-2x} + c_2 \mathrm{e}^{-2x} + \frac{1}{4}$$

Verified OK.

4.7.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 121: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dx} \\ &= z_1 e^{-2x} \\ &= z_1\left(\mathrm{e}^{-2x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-4x}}{(y_1)^2}\,dx \\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
2 \cos(x)^2
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}, \{\cos(2x), \sin(2x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{x \,\mathrm{e}^{-2x}, \mathrm{e}^{-2x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-8A_2 \sin(2x) + 8A_3 \cos(2x) + 4A_1 = 2 \cos(x)^2
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}, A_2 = 0, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4} + \frac{\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 x \,\mathrm{e}^{-2x}\right) + \left(\frac{1}{4} + \frac{\sin(2x)}{8}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2x}(c_2 x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8} \tag{1}$$

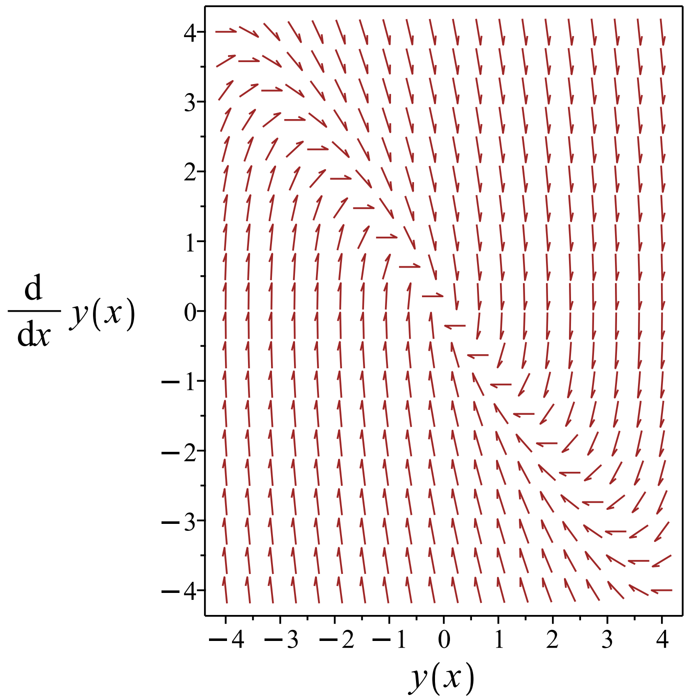


Figure 173: Slope field plot

Verification of solutions

$$y = e^{-2x}(c_2x + c_1) + \frac{1}{4} + \frac{\sin(2x)}{8}$$

Verified OK.

4.7.4 Maple step by step solution

Let's solve

$y'' + 4y' + 4y = 2\cos(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r + 2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 x\,\mathrm{e}^{-2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\cos(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & x\,\mathrm{e}^{-2x} \\ -2\,\mathrm{e}^{-2x} & \mathrm{e}^{-2x} - 2x\,\mathrm{e}^{-2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = 2\,\mathrm{e}^{-2x}\left(-\left(\int x\cos(x)^2\,\mathrm{e}^{2x}dx\right) + x\left(\int \mathrm{e}^{2x}\cos(x)^2\,dx\right)\right)$

  - Compute integrals

    $y_p(x) = \frac{1}{4} + \frac{\sin(2x)}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\,\mathrm{e}^{-2x} + c_1 \mathrm{e}^{-2x} + \frac{\sin(2x)}{8} + \frac{1}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+4*diff(y(x),x)+4*y(x)=2*cos(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{1}{4} + (c_1 x + c_2)\,\mathrm{e}^{-2x} + \frac{\sin(2x)}{8}$$

✓ Solution by Mathematica

Time used: 0.132 (sec). Leaf size: 29

```
DSolve[y''[x]+4*y'[x]+4*y[x]==2*Cos[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{8}\left(\sin(2x) + 8e^{-2x}(c_2 x + c_1) + 2\right)$$

4.8 problem 8

Internal problem ID [5151]
Internal file name [`OUTPUT/4644_Sunday_June_05_2022_03_02_35_PM_94342933/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 3y = x + e^{2x}}$$

4.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 3, f(x) = x + e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\lambda\,\mathrm{e}^{\lambda x} + 3\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)} \\
&= 2 \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= 2 + 1 \\
\lambda_2 &= 2 - 1
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= 1
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(1)x}
\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}, \{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{2x} + A_2 + A_3 x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 e^{2x} - 4A_3 + 3A_2 + 3A_3 x = x + e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{4}{9}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -e^{2x} + \frac{4}{9} + \frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{x}\right) + \left(-e^{2x} + \frac{4}{9} + \frac{x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x} c_1 + c_2 e^{x} - e^{2x} + \frac{4}{9} + \frac{x}{3} \tag{1}$$

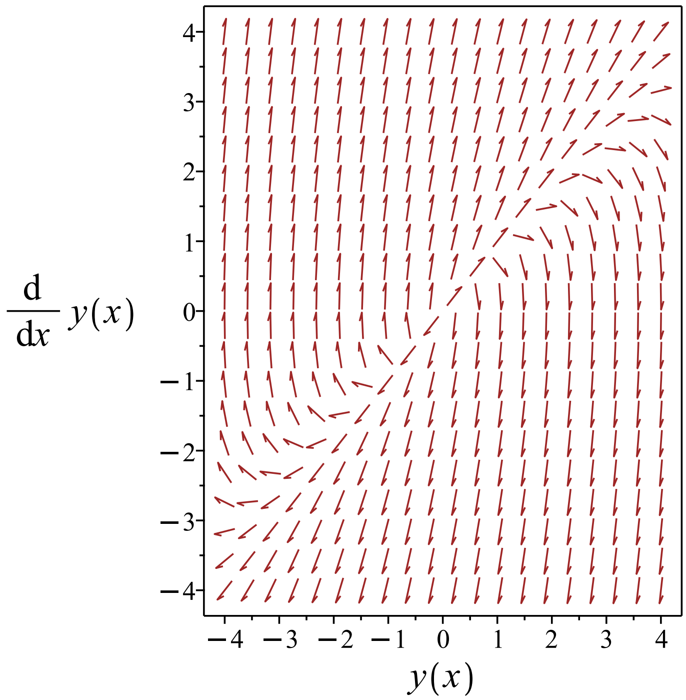


Figure 174: Slope field plot

Verification of solutions

$$y = e^{3x} c_1 + c_2 e^x - e^{2x} + \frac{4}{9} + \frac{x}{3}$$

Verified OK.

4.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 123: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\ &= z_1 e^{2x} \\ &= z_1\left(\mathrm{e}^{2x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^x + \frac{c_2\mathrm{e}^{3x}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x + \mathrm{e}^{2x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{2x}\}, \{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{3x}}{2}, \mathrm{e}^x\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\mathrm{e}^{2x} + A_2 + A_3x
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \mathrm{e}^{2x} - 4A_3 + 3A_2 + 3A_3 x = x + \mathrm{e}^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{4}{9}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\mathrm{e}^{2x} + \frac{4}{9} + \frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2}\right) + \left(-\mathrm{e}^{2x} + \frac{4}{9} + \frac{x}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{x} + \frac{c_2 \mathrm{e}^{3x}}{2} - \mathrm{e}^{2x} + \frac{4}{9} + \frac{x}{3} \tag{1}$$

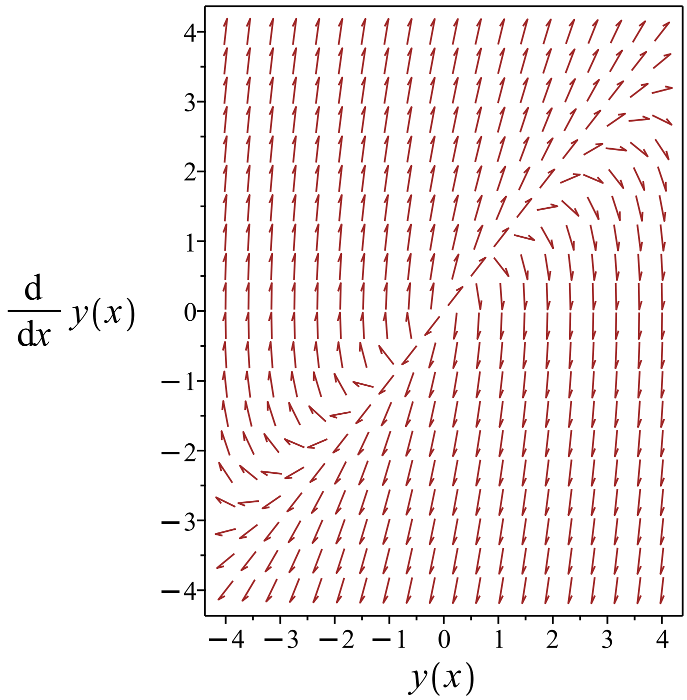


Figure 175: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^x + \frac{c_2 e^{3x}}{2} - e^{2x} + \frac{4}{9} + \frac{x}{3}$$

Verified OK.

4.8.3 Maple step by step solution

Let's solve

$y'' - 4y' + 3y = x + e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

$r = (1, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^x + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x + e^{2x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^x & e^{3x} \\ e^x & 3\,e^{3x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,e^{4x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^x\left(\int (x\,e^{-x}+e^x)dx\right)}{2} + \frac{e^{3x}\left(\int (x+e^{2x})e^{-3x}dx\right)}{2}$

- ○ Compute integrals

  $y_p(x) = -e^{2x} + \frac{4}{9} + \frac{x}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^x + c_2 e^{3x} - e^{2x} + \frac{4}{9} + \frac{x}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=x+exp(2*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x} c_2 + \mathrm{e}^{3x} c_1 - \mathrm{e}^{2x} + \frac{x}{3} + \frac{4}{9}$$

✓ Solution by Mathematica

Time used: 0.129 (sec). Leaf size: 35

```
DSolve[y''[x]-4*y'[x]+3*y[x]==x+Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{3} - e^{2x} + c_1 e^{x} + c_2 e^{3x} + \frac{4}{9}$$

4.9 problem 9

Internal problem ID [5152]
Internal file name [`OUTPUT/4645_Sunday_June_05_2022_03_02_35_PM_57848089/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2y' + 3y = x^2 - 1}$$

4.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 3, f(x) = x^2 - 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(3)}\\ &= 1 \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{2}\\ \lambda_2 &= 1 - i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\sqrt{2}\\ \lambda_2 &= -i\sqrt{2} + 1\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x}\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^x \cos\left(x\sqrt{2}\right), e^x \sin\left(x\sqrt{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3 x^2 + 3A_2 x - 4x A_3 + 3A_1 - 2A_2 + 2A_3 = x^2 - 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{7}{27}, A_2 = \frac{4}{9}, A_3 = \frac{1}{3} \right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^2 + \frac{4}{9}x - \frac{7}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^x \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) \right) \right) + \left(\frac{1}{3}x^2 + \frac{4}{9}x - \frac{7}{27} \right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^x \left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right) \right) + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27} \tag{1}$$

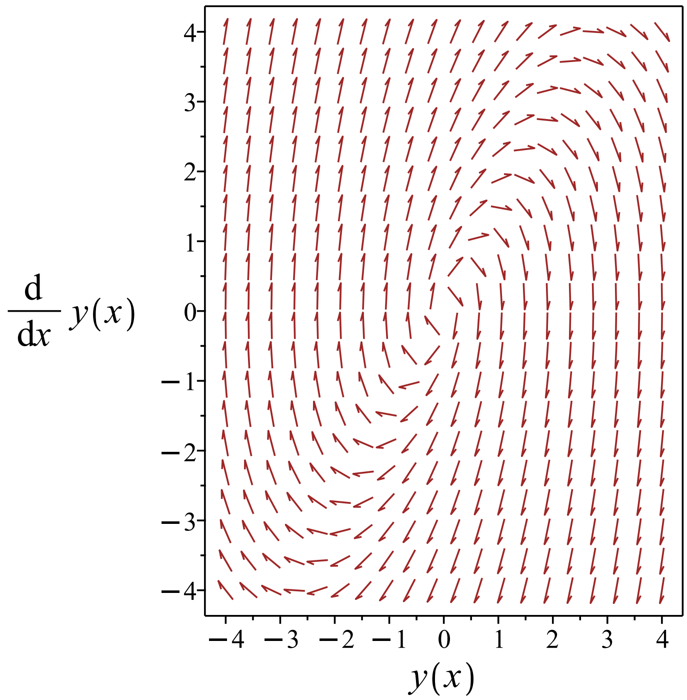


Figure 176: Slope field plot

<u>Verification of solutions</u>

$$y = e^{x}\left(c_1 \cos\left(x\sqrt{2}\right) + c_2 \sin\left(x\sqrt{2}\right)\right) + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27}$$

Verified OK.

4.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 125: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(x\sqrt{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1\left(\mathrm{e}^{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}\cos\left(x\sqrt{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{2}\tan\left(x\sqrt{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{x} \cos\left(x\sqrt{2}\right)\right) + c_2\left(\mathrm{e}^{x} \cos\left(x\sqrt{2}\right)\left(\frac{\sqrt{2}\,\tan\left(x\sqrt{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos\left(x\sqrt{2}\right) \mathrm{e}^{x} c_1 + \frac{c_2 \mathrm{e}^{x} \sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x^2 + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x, x^2\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{x} \cos\left(x\sqrt{2}\right), \frac{\mathrm{e}^{x} \sqrt{2}\,\sin\left(x\sqrt{2}\right)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_3 x^2 + A_2 x + A_1
$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_3x^2 + 3A_2x - 4xA_3 + 3A_1 - 2A_2 + 2A_3 = x^2 - 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{7}{27}, A_2 = \frac{4}{9}, A_3 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{3}x^2 + \frac{4}{9}x - \frac{7}{27}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(x\sqrt{2}\right) e^x c_1 + \frac{c_2 e^x \sqrt{2}\, \sin\left(x\sqrt{2}\right)}{2}\right) + \left(\frac{1}{3}x^2 + \frac{4}{9}x - \frac{7}{27}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(x\sqrt{2}\right) e^x c_1 + \frac{c_2 e^x \sqrt{2}\, \sin\left(x\sqrt{2}\right)}{2} + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27} \tag{1}$$

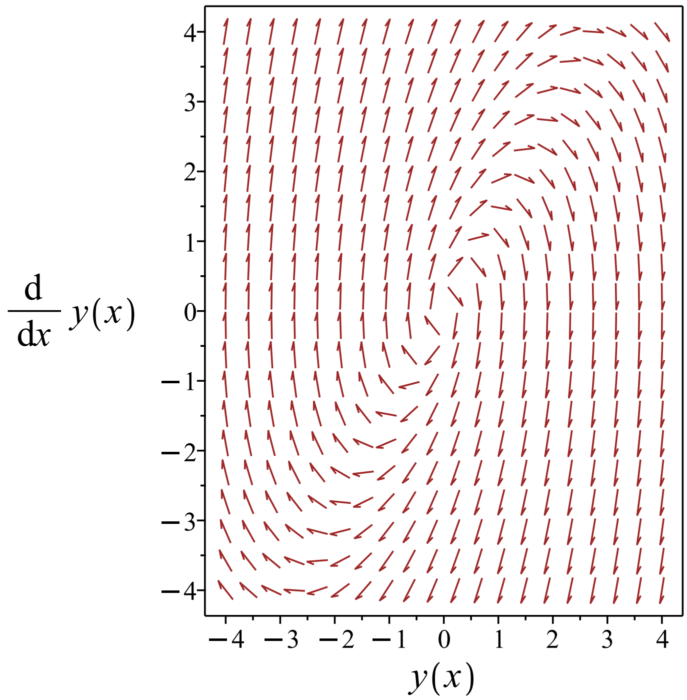


Figure 177: Slope field plot

<u>Verification of solutions</u>

$$y = \cos\left(x\sqrt{2}\right) e^{x} c_1 + \frac{c_2 e^{x} \sqrt{2}\, \sin\left(x\sqrt{2}\right)}{2} + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27}$$

Verified OK.

4.9.3 Maple step by step solution

Let's solve

$y'' - 2y' + 3y = x^2 - 1$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 3 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r = \left(1+\mathrm{I}\sqrt{2}, -\mathrm{I}\sqrt{2}+1\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{x} \cos\left(x\sqrt{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x} \sin\left(x\sqrt{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(x\sqrt{2}\right) \mathrm{e}^{x} c_1 + \sin\left(x\sqrt{2}\right) \mathrm{e}^{x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x^2-1\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{x}\cos\left(x\sqrt{2}\right) & \mathrm{e}^{x}\sin\left(x\sqrt{2}\right) \\ \mathrm{e}^{x}\cos\left(x\sqrt{2}\right) - \mathrm{e}^{x}\sqrt{2}\,\sin\left(x\sqrt{2}\right) & \mathrm{e}^{x}\sin\left(x\sqrt{2}\right) + \mathrm{e}^{x}\sqrt{2}\,\cos\left(x\sqrt{2}\right) \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}\,\mathrm{e}^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $$y_p(x) = -\frac{\mathrm{e}^{x}\sqrt{2}\left(\cos\left(x\sqrt{2}\right)\left(\int \mathrm{e}^{-x}(x^2-1)\sin\left(x\sqrt{2}\right)dx\right) - \sin\left(x\sqrt{2}\right)\left(\int \mathrm{e}^{-x}(x^2-1)\cos\left(x\sqrt{2}\right)dx\right)\right)}{2}$$

  - Compute integrals

    $y_p(x) = \frac{1}{3}x^2 + \frac{4}{9}x - \frac{7}{27}$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(x\sqrt{2}\right) \mathrm{e}^{x} c_1 + \sin\left(x\sqrt{2}\right) \mathrm{e}^{x} c_2 + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+3*y(x)=x^2-1,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x} \sin\left(\sqrt{2}\,x\right) c_2 + \mathrm{e}^{x} \cos\left(\sqrt{2}\,x\right) c_1 + \frac{x^2}{3} + \frac{4x}{9} - \frac{7}{27}$$

✓ Solution by Mathematica
Time used: 0.023 (sec). Leaf size: 48

```
DSolve[y''[x]-2*y'[x]+3*y[x]==x^2-1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{27}\left(9x^2 + 12x - 7\right) + c_2 e^x \cos\left(\sqrt{2}x\right) + c_1 e^x \sin\left(\sqrt{2}x\right)$$

4.10 problem 10

Internal problem ID [5153]
Internal file name [`OUTPUT/4646_Sunday_June_05_2022_03_02_37_PM_51493109/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 9y = e^{3x} + \sin(x)}$$

4.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -9, f(x) = e^{3x} + \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-9)}\\ &= \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3\\ \lambda_2 &= -3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(3)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{3x} c_1 + c_2 \mathrm{e}^{-3x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x} + \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3x}, e^{3x}\}$$

Since e^{3x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{3x}x\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{3x} x + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{3x} - 10A_2 \cos(x) - 10A_3 \sin(x) = e^{3x} + \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}, A_2 = 0, A_3 = -\frac{1}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x} x}{6} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x} c_1 + c_2 e^{-3x}\right) + \left(\frac{e^{3x} x}{6} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}c_1 + c_2 e^{-3x} + \frac{e^{3x}x}{6} - \frac{\sin(x)}{10} \tag{1}$$

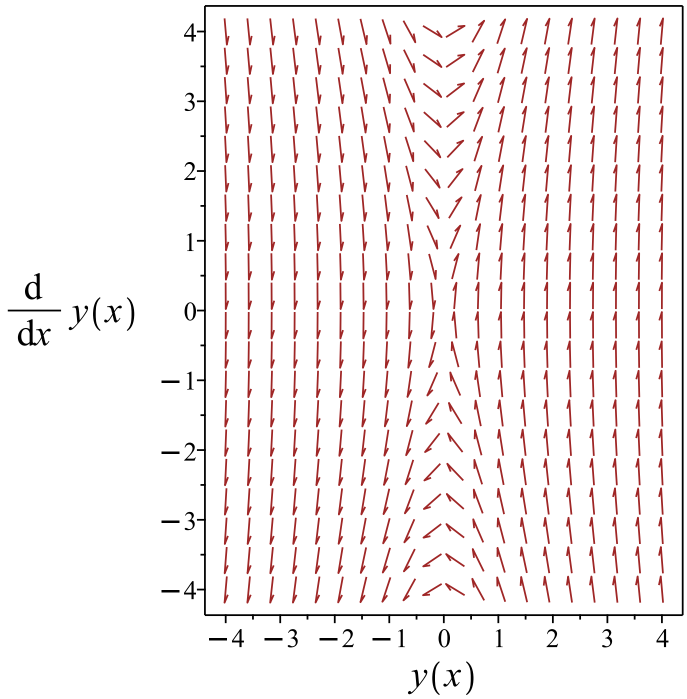


Figure 178: Slope field plot

Verification of solutions

$$y = e^{3x}c_1 + c_2 e^{-3x} + \frac{e^{3x}x}{6} - \frac{\sin(x)}{10}$$

Verified OK.

4.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -9
\end{aligned}
\tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 127: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= e^{-3x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-3x} \int \frac{1}{\mathrm{e}^{-6x}}\,dx \\ &= \mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-3x}$$
$$y_2 = \frac{\mathrm{e}^{3x}}{6}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ \frac{d}{dx}\left(\mathrm{e}^{-3x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{3x}}{6}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{6} \\ -3\,\mathrm{e}^{-3x} & \frac{\mathrm{e}^{3x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{-3x}\right)\left(\frac{\mathrm{e}^{3x}}{2}\right) - \left(\frac{\mathrm{e}^{3x}}{6}\right)\left(-3\,\mathrm{e}^{-3x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{-3x}\mathrm{e}^{3x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{3x}(e^{3x}+\sin(x))}{6}}{1}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{3x}(e^{3x} + \sin(x))}{6} dx$$

Hence

$$u_1 = -\frac{e^{6x}}{36} + \frac{e^{3x}\cos(x)}{60} - \frac{e^{3x}\sin(x)}{20}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-3x}(e^{3x} + \sin(x))}{1}\, dx$$

Which simplifies to

$$u_2 = \int \left(1 + \sin(x)\, e^{-3x}\right) dx$$

Hence

$$u_2 = x - \frac{\cos(x)\, e^{-3x}}{10} - \frac{3\sin(x)\, e^{-3x}}{10}$$

Which simplifies to

$$u_1 = \frac{(-3\sin(x) + \cos(x))\, e^{3x}}{60} - \frac{e^{6x}}{36}$$

$$u_2 = \frac{(-\cos(x) - 3\sin(x))\, e^{-3x}}{10} + x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(-3\sin(x) + \cos(x))\, e^{3x}}{60} - \frac{e^{6x}}{36}\right) e^{-3x} + \frac{\left(\frac{(-\cos(x)-3\sin(x))e^{-3x}}{10} + x\right) e^{3x}}{6}$$

Which simplifies to

$$y_p(x) = \frac{(-1+6x)\,e^{3x}}{36} - \frac{\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^{3x}}{6}\right) + \left(\frac{(-1+6x)\,e^{3x}}{36} - \frac{\sin(x)}{10}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} + \frac{(-1+6x)\,e^{3x}}{36} - \frac{\sin(x)}{10} \tag{1}$$

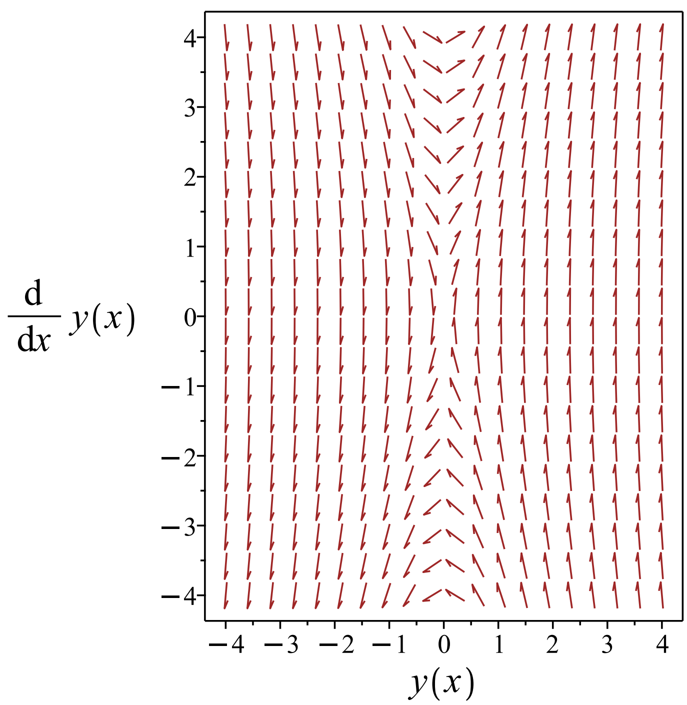


Figure 179: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^{3x}}{6} + \frac{(-1+6x)\,e^{3x}}{36} - \frac{\sin(x)}{10}$$

Verified OK.

4.10.3 Maple step by step solution

Let's solve

$y'' - 9y = e^{3x} + \sin(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r-3)(r+3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-3x} + c_2 e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{3x} + \sin(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-3x} & e^{3x} \\ -3\,e^{-3x} & 3\,e^{3x} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 6$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{e^{-3x} \left(\int e^{3x} \left(e^{3x}+\sin(x)\right) dx\right)}{6} + \frac{e^{3x} \left(\int \left(1+\sin(x) e^{-3x}\right) dx\right)}{6}$$

- Compute integrals

$$y_p(x) = \frac{(-1+6x) e^{3x}}{36} - \frac{\sin(x)}{10}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 e^{-3x} + c_2 e^{3x} + \frac{(-1+6x) e^{3x}}{36} - \frac{\sin(x)}{10}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)-9*y(x)=exp(3*x)+sin(x),y(x), singsol=all)
```

$$y(x) = \frac{(-1 + 6x + 36c_2) e^{3x}}{36} + e^{-3x} c_1 - \frac{\sin(x)}{10}$$

✓ Solution by Mathematica

Time used: 0.126 (sec). Leaf size: 37

```
DSolve[y''[x]-9*y[x]==Exp[3*x]+Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sin(x)}{10} + e^{3x} \left(\frac{x}{6} - \frac{1}{36} + c_1\right) + c_2 e^{-3x}$$

4.11 problem 12

Internal problem ID [5154]
Internal file name [`OUTPUT/4647_Sunday_June_05_2022_03_02_38_PM_24001607/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x'' + 4x' + 3x = e^{-3t}}$$

With initial conditions

$$\left[x(0) = \frac{1}{2}, x'(0) = -2\right]$$

4.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 3 \\ F &= e^{-3t} \end{aligned}$$

Hence the ode is

$$x'' + 4x' + 3x = e^{-3t}$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = e^{-3t}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 4, C = 3, f(t) = e^{-3t}$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x' + 3x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(3)} \\ &= -2 \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + 1 \\ \lambda_2 &= -2 - 1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ x &= c_1 e^{(-1)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$x = \mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t} c_1 + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{-3t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-t}\}$$

Since e^{-3t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\,e^{-3t}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$x_p = A_1 t\,e^{-3t}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 e^{-3t} = e^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t\,e^{-3t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-t}c_1 + c_2 e^{-3t}\right) + \left(-\frac{t\,e^{-3t}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-t}c_1 + c_2 e^{-3t} - \frac{t\,e^{-3t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{2}$ and $t = 0$ in the above gives

$$\frac{1}{2} = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\mathrm{e}^{-t} c_1 - 3c_2 \mathrm{e}^{-3t} - \frac{\mathrm{e}^{-3t}}{2} + \frac{3t\,\mathrm{e}^{-3t}}{2}$$

substituting $x' = -2$ and $t = 0$ in the above gives

$$-2 = -c_1 - 3c_2 - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= \frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

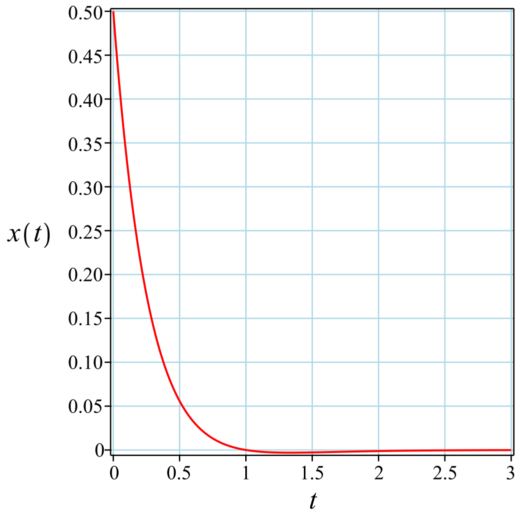
$$x = \frac{\mathrm{e}^{-3t}}{2} - \frac{t\,\mathrm{e}^{-3t}}{2}$$

Which simplifies to

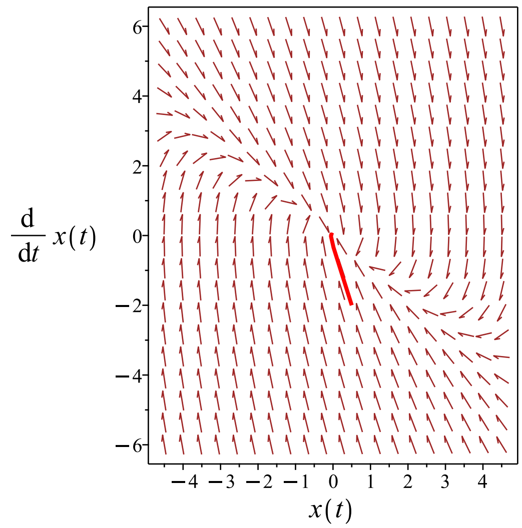
$$x = -\frac{\mathrm{e}^{-3t}(t-1)}{2}$$

Summary
The solution(s) found are the following

$$x = -\frac{\mathrm{e}^{-3t}(t-1)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = -\frac{e^{-3t}(t-1)}{2}$$

Verified OK.

4.11.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 3x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 129: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt}\\ &= z_1 e^{-2t}\\ &= z_1\left(\mathrm{e}^{-2t}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(x_1)^2}\,dt\\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\,dt\\ &= x_1\left(\frac{\mathrm{e}^{2t}}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1\left(\mathrm{e}^{-3t}\right) + c_2\left(\mathrm{e}^{-3t}\left(\frac{\mathrm{e}^{2t}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' + 4x' + 3x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\mathrm{e}^{-3t} + \frac{c_2\mathrm{e}^{-t}}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\mathrm{e}^{-3t}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{-3t}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{-t}}{2}, \mathrm{e}^{-3t}\right\}
$$

Since e^{-3t} is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$
[\{t\,\mathrm{e}^{-3t}\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
x_p = A_1 t\,\mathrm{e}^{-3t}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \mathrm{e}^{-3t} = \mathrm{e}^{-3t}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{t\,\mathrm{e}^{-3t}}{2}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-3t} + \frac{c_2 \mathrm{e}^{-t}}{2}\right) + \left(-\frac{t\,\mathrm{e}^{-3t}}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-3t} + \frac{c_2 \mathrm{e}^{-t}}{2} - \frac{t\,\mathrm{e}^{-3t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{2}$ and $t = 0$ in the above gives

$$\frac{1}{2} = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \mathrm{e}^{-3t} - \frac{c_2 \mathrm{e}^{-t}}{2} - \frac{\mathrm{e}^{-3t}}{2} + \frac{3t\,\mathrm{e}^{-3t}}{2}$$

substituting $x' = -2$ and $t = 0$ in the above gives

$$-2 = -3c_1 - \frac{c_2}{2} - \frac{1}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{1}{2} \\
c_2 &= 0
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = \frac{e^{-3t}}{2} - \frac{t\,e^{-3t}}{2}
$$

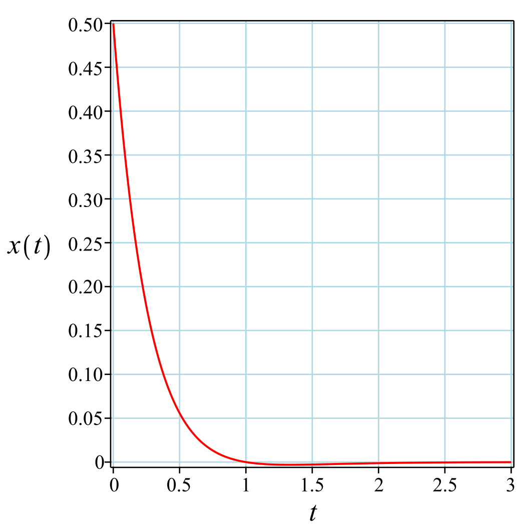
Which simplifies to

$$
x = -\frac{e^{-3t}(t-1)}{2}
$$

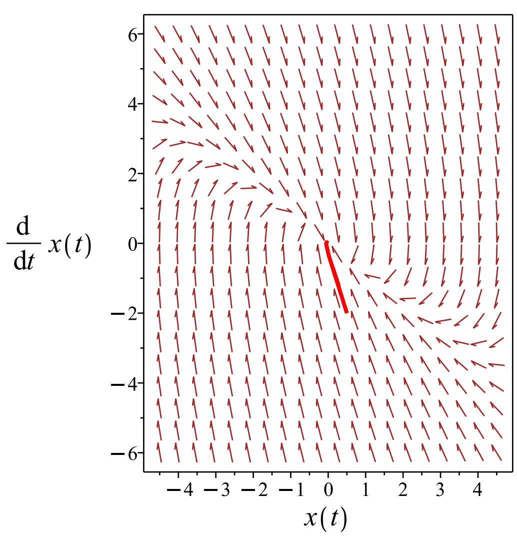
Summary

The solution(s) found are the following

$$
x = -\frac{e^{-3t}(t-1)}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
x = -\frac{e^{-3t}(t-1)}{2}
$$

Verified OK.

4.11.4 Maple step by step solution

Let's solve

$\left[x''+4x'+3x=\mathrm{e}^{-3t}, x(0)=\frac{1}{2}, x'\Big|_{\{t=0\}}=-2\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+4r+3=0$

- Factor the characteristic polynomial

  $(r+3)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-3,-1)$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{-3t}$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{-t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{-3t}+c_2\mathrm{e}^{-t}+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=\mathrm{e}^{-3t}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t))=\begin{bmatrix}\mathrm{e}^{-3t} & \mathrm{e}^{-t}\\ -3\,\mathrm{e}^{-3t} & -\mathrm{e}^{-t}\end{bmatrix}$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t))=2\,\mathrm{e}^{-4t}$

  ◦ Substitute functions into equation for $x_p(t)$

$x_p(t) = -\frac{\mathrm{e}^{-3t}\left(\int 1dt\right)}{2} + \frac{\mathrm{e}^{-t}\left(\int \mathrm{e}^{-2t}dt\right)}{2}$

- ○ Compute integrals

  $x_p(t) = -\frac{\mathrm{e}^{-3t}(2t+1)}{4}$

- Substitute particular solution into general solution to ODE

  $x = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} - \frac{\mathrm{e}^{-3t}(2t+1)}{4}$

□ Check validity of solution $x = c_1\mathrm{e}^{-3t} + c_2\mathrm{e}^{-t} - \frac{\mathrm{e}^{-3t}(2t+1)}{4}$

- ○ Use initial condition $x(0) = \frac{1}{2}$

  $\frac{1}{2} = c_1 + c_2 - \frac{1}{4}$

- ○ Compute derivative of the solution

  $x' = -3c_1\mathrm{e}^{-3t} - c_2\mathrm{e}^{-t} + \frac{3\,\mathrm{e}^{-3t}(2t+1)}{4} - \frac{\mathrm{e}^{-3t}}{2}$

- ○ Use the initial condition $x'\Big|_{\{t=0\}} = -2$

  $-2 = -3c_1 - c_2 + \frac{1}{4}$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = \frac{3}{4}, c_2 = 0\right\}$

- ○ Substitute constant values into general solution and simplify

  $x = -\frac{\mathrm{e}^{-3t}(t-1)}{2}$

- Solution to the IVP

  $x = -\frac{\mathrm{e}^{-3t}(t-1)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(x(t),t$2)+4*diff(x(t),t)+3*x(t)=exp(-3*t),x(0) = 1/2, D(x)(0) = -2],x(t), sings
```

$$x(t) = -\frac{e^{-3t}(t-1)}{2}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 17

```
DSolve[{x''[t]+4*x'[t]+3*x[t]==Exp[-3*t],{x[0]==1/2,x'[0]==-2}},x[t],t,IncludeSingularSoluti
```

$$x(t) \to -\frac{1}{2}e^{-3t}(t-1)$$

4.12 problem 13

Internal problem ID [5155]
Internal file name [`OUTPUT/4648_Sunday_June_05_2022_03_02_39_PM_45092347/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y' + 5y = 6\sin(t)}$$

4.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 4, C = 5, f(t) = 6\sin(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' + 4y' + 5y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\lambda \,\mathrm{e}^{\lambda t} + 5\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{4^2 - (4)\,(1)\,(5)} \\
&= -2 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -2 + i \\
\lambda_2 &= -2 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$y = e^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$6\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}\cos(t), e^{-2t}\sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(t) + A_2\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1\cos(t) + 4A_2\sin(t) - 4A_1\sin(t) + 4A_2\cos(t) = 6\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{4}, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-2t}(c_1\cos(t) + c_2\sin(t))\right) + \left(-\frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4} \tag{1}$$

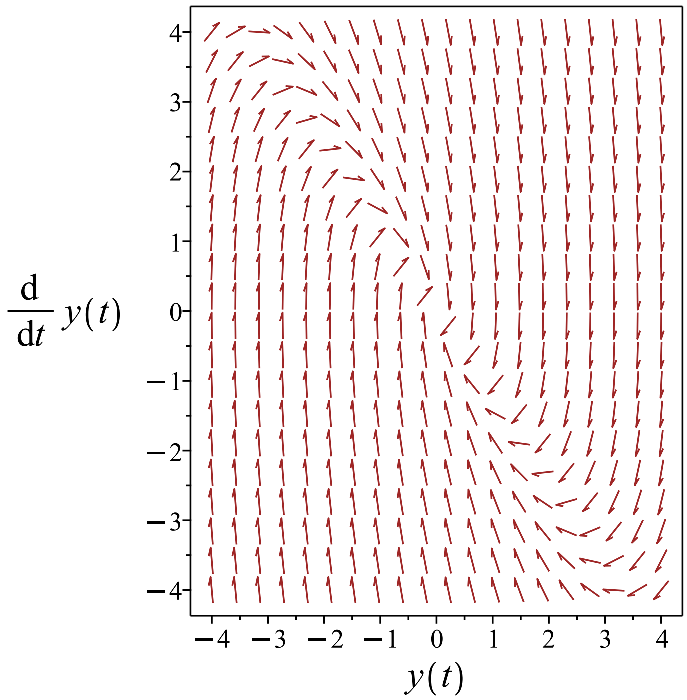


Figure 182: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(t) + c_2 \sin(t)) - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

Verified OK.

4.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 131: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\, dt} \\ &= z_1 e^{-2t} \\ &= z_1 \left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2t} \cos(t)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{4}{1}\, dt}}{(y_1)^2}\, dt \\ &= y_1 \int \frac{e^{-4t}}{(y_1)^2}\, dt \\ &= y_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2t}\cos(t)\right) + c_2\left(\mathrm{e}^{-2t}\cos(t)\left(\tan(t)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$
y'' + 4y' + 5y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \cos(t)\,\mathrm{e}^{-2t} c_1 + \sin(t)\,\mathrm{e}^{-2t} c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
6\sin(t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(t), \sin(t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}\cos(t), \mathrm{e}^{-2t}\sin(t)\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos(t) + A_2\sin(t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
4A_1\cos(t) + 4A_2\sin(t) - 4A_1\sin(t) + 4A_2\cos(t) = 6\sin(t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{4}, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos(t)\,\mathrm{e}^{-2t}c_1 + \sin(t)\,\mathrm{e}^{-2t}c_2\right) + \left(-\frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}\right) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4} \tag{1}$$

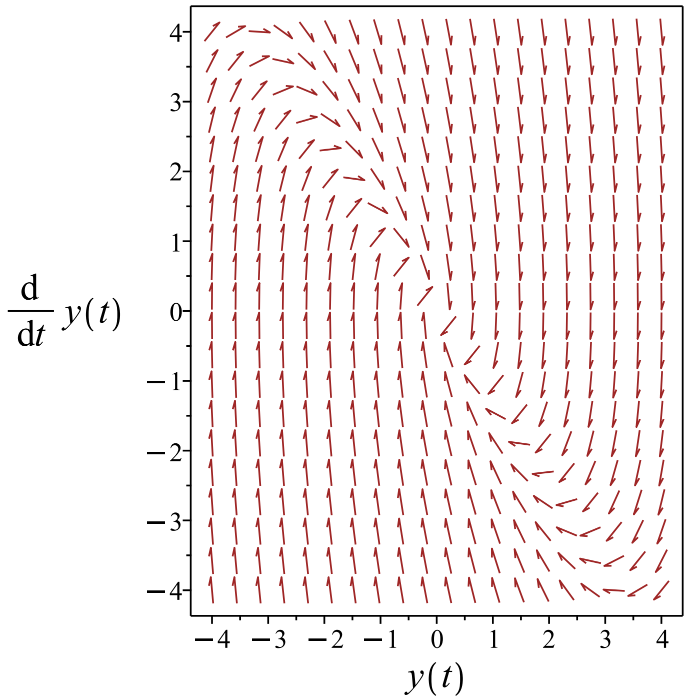


Figure 183: Slope field plot

Verification of solutions

$$y = e^{-2t}(c_1 \cos(t) + c_2 \sin(t)) - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

Verified OK.

4.12.3 Maple step by step solution

Let's solve

$y'' + 4y' + 5y = 6\sin(t)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-4)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

$r = (-2 - \mathrm{I}, -2 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(t) = \mathrm{e}^{-2t} \cos(t)$

- 2nd solution of the homogeneous ODE

  $y_2(t) = \mathrm{e}^{-2t} \sin(t)$

- General solution of the ODE

  $y = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos(t)\, \mathrm{e}^{-2t} c_1 + \sin(t)\, \mathrm{e}^{-2t} c_2 + y_p(t)$

☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = 6 \sin(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} \cos(t) & \mathrm{e}^{-2t} \sin(t) \\ -2\, \mathrm{e}^{-2t} \cos(t) - \mathrm{e}^{-2t} \sin(t) & -2\, \mathrm{e}^{-2t} \sin(t) + \mathrm{e}^{-2t} \cos(t) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = \mathrm{e}^{-4t}$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = 3\, \mathrm{e}^{-2t} \left(-2 \cos(t) \left(\int \sin(t)^2 \mathrm{e}^{2t} dt\right) + \sin(t) \left(\int \sin(2t)\, \mathrm{e}^{2t} dt\right)\right)$

  - Compute integrals

    $y_p(t) = -\frac{3 \cos(t)}{4} + \frac{3 \sin(t)}{4}$

- Substitute particular solution into general solution to ODE

  $y = \sin(t)\, \mathrm{e}^{-2t} c_2 + \cos(t)\, \mathrm{e}^{-2t} c_1 + \frac{3 \sin(t)}{4} - \frac{3 \cos(t)}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(t),t$2)+4*diff(y(t),t)+5*y(t)=6*sin(t),y(t), singsol=all)
```

$$y(t) = \mathrm{e}^{-2t} \sin(t)\, c_2 + \mathrm{e}^{-2t} \cos(t)\, c_1 - \frac{3\cos(t)}{4} + \frac{3\sin(t)}{4}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 36

```
DSolve[y''[t]+4*y'[t]+5*y[t]==6*Sin[t],y[t],t,IncludeSingularSolutions -> True]
```

$$y(t) \to \left(-\frac{3}{4} + c_2 e^{-2t}\right) \cos(t) + \left(\frac{3}{4} + c_1 e^{-2t}\right) \sin(t)$$

4.13 problem 14

Internal problem ID [5156]
Internal file name [`OUTPUT/4649_Sunday_June_05_2022_03_02_40_PM_6305597/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' - 3x' + 2x = \sin(t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

4.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= -3 \\ q(t) &= 2 \\ F &= \sin(t) \end{aligned}$$

Hence the ode is

$$x'' - 3x' + 2x = \sin(t)$$

The domain of $p(t) = -3$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = -3, C = 2, f(t) = \sin(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' - 3x' + 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} - 3\lambda\, \mathrm{e}^{\lambda t} + 2\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)}\\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(2)t} + c_2 e^{(1)t}$$

Or

$$x = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{2t} + c_2 \mathrm{e}^{t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t) + A_2 \sin(t) + 3A_1 \sin(t) - 3A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{2t} + c_2 e^t\right) + \left(\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{2t} + c_2 e^t + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{3}{10} \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2c_1 e^{2t} + c_2 e^t - \frac{3\sin(t)}{10} + \frac{\cos(t)}{10}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 2c_1 + c_2 + \frac{1}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{5}$$
$$c_2 = -\frac{1}{2}$$

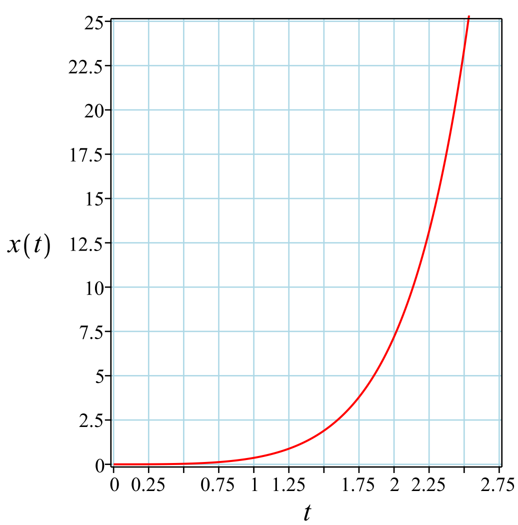
Substituting these values back in above solution results in

$$x = \frac{e^{2t}}{5} - \frac{e^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

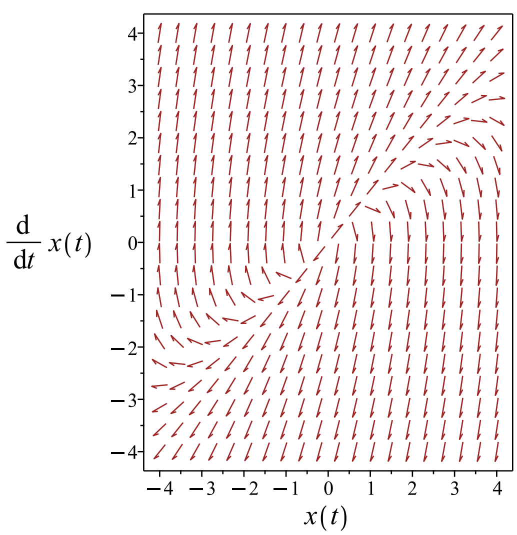
Summary

The solution(s) found are the following

$$x = \frac{e^{2t}}{5} - \frac{e^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{2t}}{5} - \frac{e^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

4.13.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' - 3x' + 2x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 133: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-3}{1} \, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^t
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{-3}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{3t}}{(x_1)^2}\, dt \\
&= x_1 \left(\mathrm{e}^t\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^t\right) + c_2 \left(\mathrm{e}^t \left(\mathrm{e}^t\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' - 3x' + 2x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^t, e^{2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(t) + A_2 \sin(t) + 3A_1 \sin(t) - 3A_2 \cos(t) = \sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^t + c_2 e^{2t}\right) + \left(\frac{3\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{3}{10} \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1 \mathrm{e}^t + 2c_2 \mathrm{e}^{2t} - \frac{3\sin(t)}{10} + \frac{\cos(t)}{10}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_1 + 2c_2 + \frac{1}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{1}{2} \\ c_2 &= \frac{1}{5} \end{aligned}$$

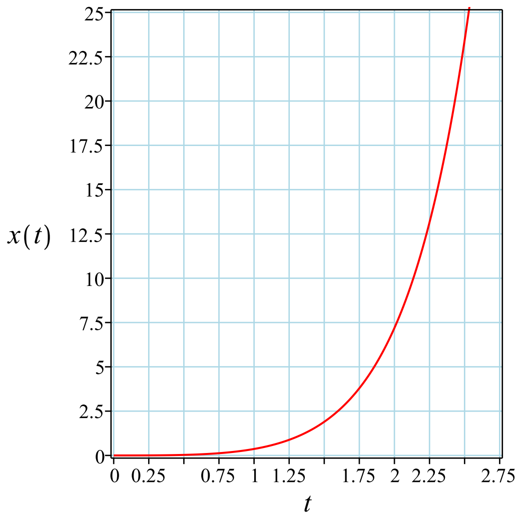
Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{2t}}{5} - \frac{\mathrm{e}^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

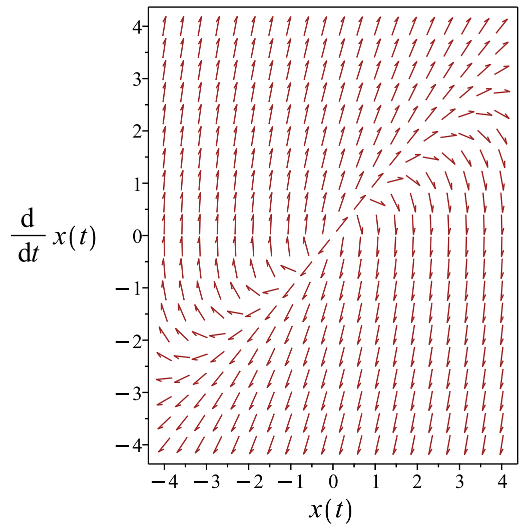
Summary

The solution(s) found are the following

$$x = \frac{\mathrm{e}^{2t}}{5} - \frac{\mathrm{e}^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{2t}}{5} - \frac{e^{t}}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

4.13.4 Maple step by step solution

Let's solve

$$\left[x'' - 3x' + 2x = \sin(t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the homogeneous ODE

$x_1(t) = \mathrm{e}^t$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t) \left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t) \left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \sin(t)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^t & \mathrm{e}^{2t} \\ \mathrm{e}^t & 2\,\mathrm{e}^{2t} \end{bmatrix}$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = \mathrm{e}^{3t}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\mathrm{e}^t \left(\int \sin(t)\, \mathrm{e}^{-t} dt\right) + \mathrm{e}^{2t} \left(\int \mathrm{e}^{-2t} \sin(t)\, dt\right)$

- Compute integrals

  $x_p(t) = \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

- Substitute particular solution into general solution to ODE

  $x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

☐ Check validity of solution $x = c_1 \mathrm{e}^t + c_2 \mathrm{e}^{2t} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

- Use initial condition $x(0) = 0$

  $0 = c_1 + c_2 + \frac{3}{10}$

- Compute derivative of the solution

  $x' = c_1 \mathrm{e}^t + 2c_2 \mathrm{e}^{2t} - \frac{3\sin(t)}{10} + \frac{\cos(t)}{10}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = c_1 + 2c_2 + \frac{1}{10}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{1}{2}, c_2 = \frac{1}{5}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{e^{2t}}{5} - \frac{e^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

- Solution to the IVP

  $x = \frac{e^{2t}}{5} - \frac{e^t}{2} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)-3*diff(x(t),t)+2*x(t)=sin(t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all
```

$$x(t) = \frac{e^{2t}}{5} + \frac{3\cos(t)}{10} + \frac{\sin(t)}{10} - \frac{e^t}{2}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 27

```
DSolve[{x''[t]-3*x'[t]+2*x[t]==Sin[t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to \frac{1}{10}\left(e^t(2e^t - 5) + \sin(t) + 3\cos(t)\right)$$

4.14 problem 15

Internal problem ID [5157]
Internal file name [OUTPUT/4650_Sunday_June_05_2022_03_02_41_PM_86276180/index.tex]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = 3\sin(x)}$$

With initial conditions

$$\left[y(0) = -\frac{9}{10}, y'(0) = -\frac{7}{10}\right]$$

4.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 2 \\ F &= 3\sin(x) \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = 3\sin(x)$$

The domain of $p(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 3\sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 3\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\
&= -\frac{3}{2} \pm \frac{1}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{1}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -1 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1 \cos(x) + A_2 \sin(x) - 3A_1 \sin(x) + 3A_2 \cos(x) = 3 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{9}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(-\frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{9}{10}$ and $x = 0$ in the above gives

$$-\frac{9}{10} = c_1 + c_2 - \frac{9}{10} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} - 2c_2 \mathrm{e}^{-2x} + \frac{9\sin(x)}{10} + \frac{3\cos(x)}{10}$$

substituting $y' = -\frac{7}{10}$ and $x = 0$ in the above gives

$$-\frac{7}{10} = -c_1 - 2c_2 + \frac{3}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

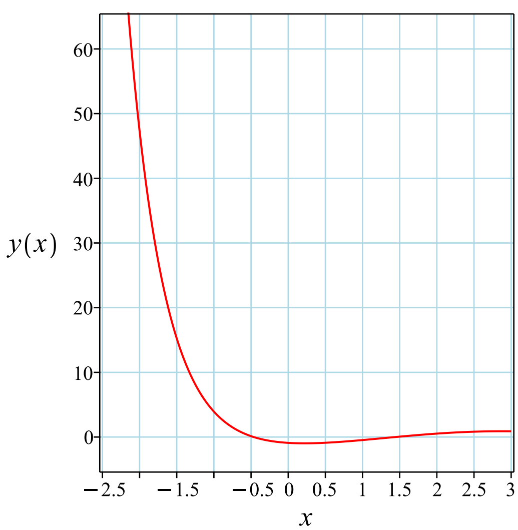
$$c_1 = -1$$
$$c_2 = 1$$

Substituting these values back in above solution results in

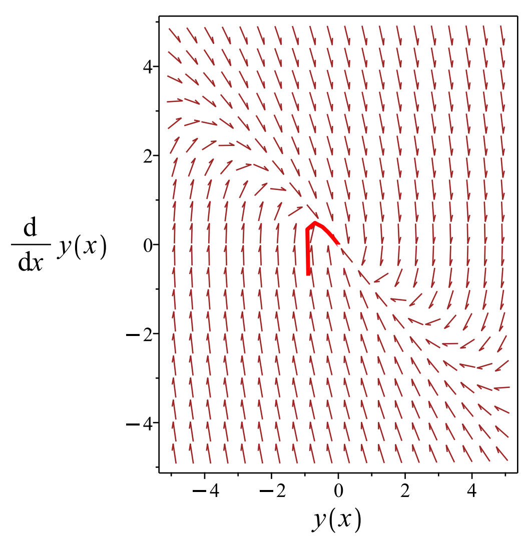
$$y = -\mathrm{e}^{-x} + \mathrm{e}^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Summary
The solution(s) found are the following

$$y = -\mathrm{e}^{-x} + \mathrm{e}^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{-x} + e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Verified OK.

4.14.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 135: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+3y'+2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x),\sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x},\mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1\cos(x)+A_2\sin(x)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\cos(x)+A_2\sin(x)-3A_1\sin(x)+3A_2\cos(x)=3\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{9}{10},A_2=\frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=-\frac{9\cos(x)}{10}+\frac{3\sin(x)}{10}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(-\frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -\frac{9}{10}$ and $x = 0$ in the above gives

$$
-\frac{9}{10} = c_1 + c_2 - \frac{9}{10} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} - c_2 \mathrm{e}^{-x} + \frac{9\sin(x)}{10} + \frac{3\cos(x)}{10}
$$

substituting $y' = -\frac{7}{10}$ and $x = 0$ in the above gives

$$
-\frac{7}{10} = -2c_1 - c_2 + \frac{3}{10} \tag{2A}
$$

Equations $\{1\text{A},2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -1
\end{aligned}
$$

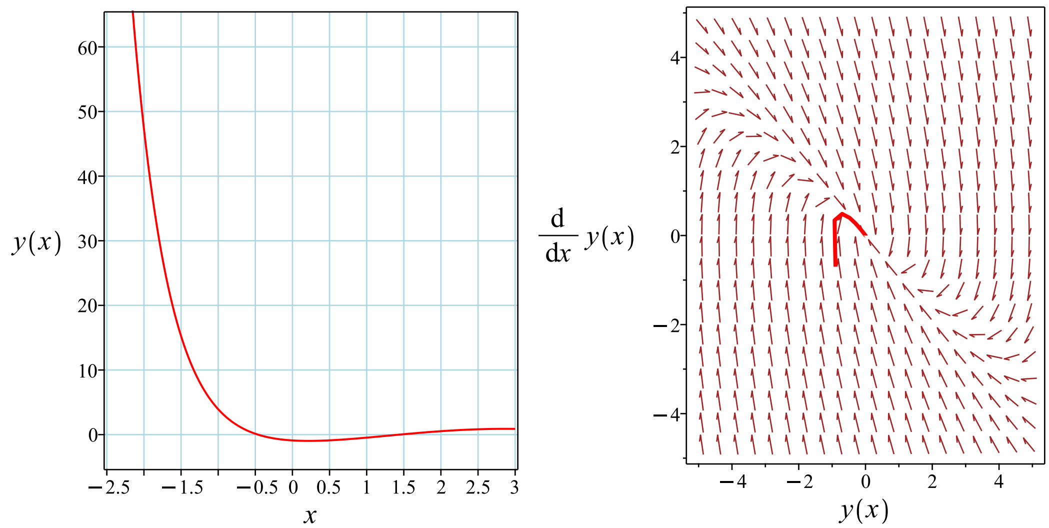
Substituting these values back in above solution results in

$$
y = -\mathrm{e}^{-x} + \mathrm{e}^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}
$$

Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{-x} + \mathrm{e}^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10} \tag{1}
$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = -e^{-x} + e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Verified OK.

4.14.4 Maple step by step solution

Let's solve

$$\left[y'' + 3y' + 2y = 3\sin(x), y(0) = -\frac{9}{10}, y'\Big|_{\{x=0\}} = -\frac{7}{10}\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

$y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3\sin(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -3\,\mathrm{e}^{-2x}\left(\int \sin(x)\,\mathrm{e}^{2x}dx\right) + 3\,\mathrm{e}^{-x}\left(\int \sin(x)\,\mathrm{e}^{x}dx\right)$

- Compute integrals

  $y_p(x) = -\frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$

- Use initial condition $y(0) = -\frac{9}{10}$

  $-\frac{9}{10} = c_1 + c_2 - \frac{9}{10}$

- Compute derivative of the solution

  $y' = -2c_1 \mathrm{e}^{-2x} - c_2 \mathrm{e}^{-x} + \frac{9\sin(x)}{10} + \frac{3\cos(x)}{10}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -\frac{7}{10}$

  $-\frac{7}{10} = -2c_1 - c_2 + \frac{3}{10}$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -1\}$
- Substitute constant values into general solution and simplify

  $y = -e^{-x} + e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$

- Solution to the IVP

  $y = -e^{-x} + e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=3*sin(x),y(0) = -9/10, D(y)(0) = -7/10],y(x), s
```

$$y(x) = e^{-2x} - \frac{9\cos(x)}{10} + \frac{3\sin(x)}{10} - e^{-x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 30

```
DSolve[{y''[x]+3*y'[x]+2*y[x]==3*Sin[x],{y[0]==-9/10,y'[0]==-7/10}},y[x],x,IncludeSingularSo
```

$$y(x) \to -e^{-2x}(e^x - 1) + \frac{3\sin(x)}{10} - \frac{9\cos(x)}{10}$$

4.15 problem 16

Internal problem ID [5158]
Internal file name [`OUTPUT/4651_Sunday_June_05_2022_03_02_43_PM_99856319/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 6y' + 10y = 50x}$$

4.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 6, C = 10, f(x) = 50x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 6y' + 10y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda \,\mathrm{e}^{\lambda x} + 10 \,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 10$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(10)} \\
&= -3 \pm i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -3 + i \\
\lambda_2 &= -3 - i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -3 + i \\
\lambda_2 &= -3 - i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(\cos (x)\, c_1 + c_2 \sin (x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-3x}(\cos (x)\, c_1 + c_2 \sin (x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^{-3x}, \sin(x)\,e^{-3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$10A_2 x + 10A_1 + 6A_2 = 50x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 5x - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-3x}(\cos(x)\,c_1 + c_2 \sin(x))\right) + (5x - 3) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(\cos(x)\,c_1 + c_2 \sin(x)) + 5x - 3 \tag{1}$$

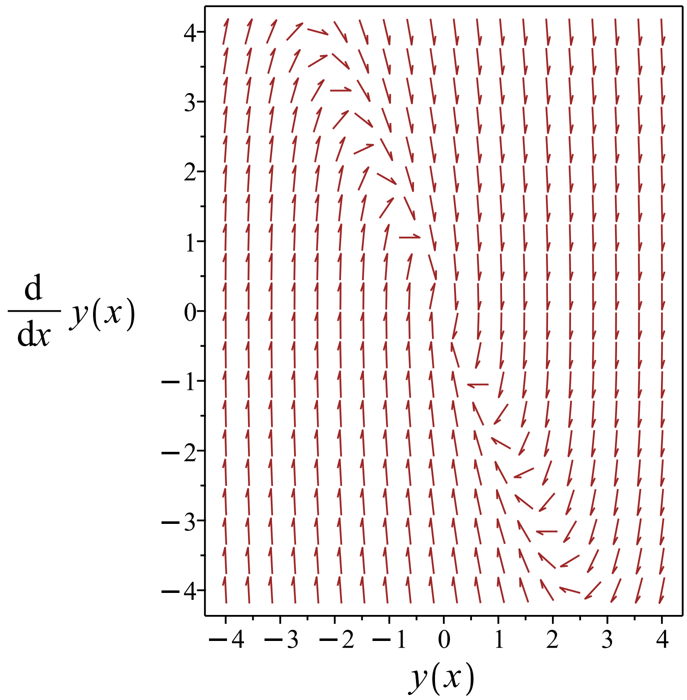


Figure 188: Slope field plot

Verification of solutions

$$y = e^{-3x}(\cos(x) c_1 + c_2 \sin(x)) + 5x - 3$$

Verified OK.

4.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 137: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1\left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(x) \,\mathrm{e}^{-3x}\right) + c_2 \left(\cos(x) \,\mathrm{e}^{-3x} (\tan(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 6y' + 10y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^{-3x} \cos(x) \, c_1 + \mathrm{e}^{-3x} \sin(x) \, c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\cos(x) \,\mathrm{e}^{-3x}, \sin(x) \,\mathrm{e}^{-3x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
10A_2 x + 10A_1 + 6A_2 = 50x
$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3, A_2 = 5]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 5x - 3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-3x} \cos(x)\, c_1 + \mathrm{e}^{-3x} \sin(x)\, c_2\right) + (5x - 3) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{-3x}(\cos(x)\, c_1 + c_2 \sin(x)) + 5x - 3$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}(\cos(x)\, c_1 + c_2 \sin(x)) + 5x - 3 \tag{1}$$

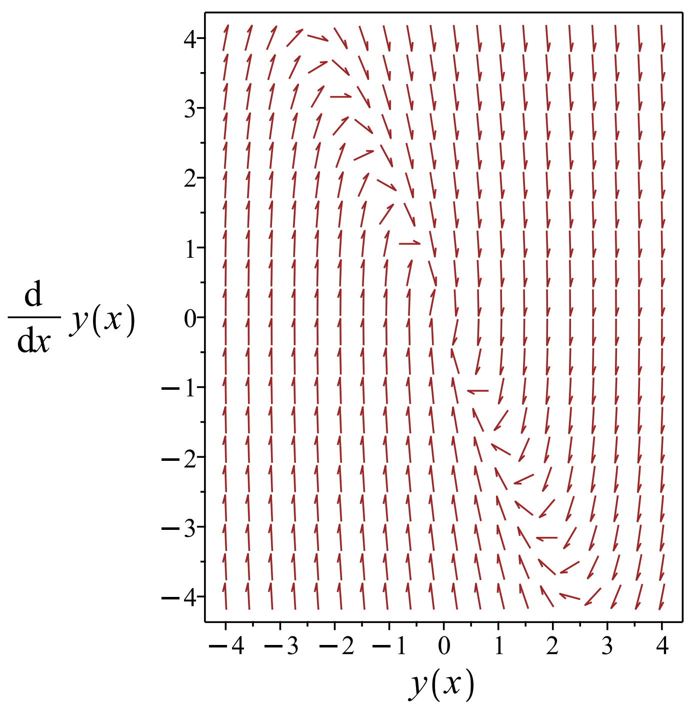


Figure 189: Slope field plot

Verification of solutions

$$y = e^{-3x}(\cos(x) c_1 + c_2 \sin(x)) + 5x - 3$$

Verified OK.

4.15.3 Maple step by step solution

Let's solve

$y'' + 6y' + 10y = 50x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 10 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - \mathrm{I}, -3 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x) e^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x) e^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^{-3x} \cos(x) c_1 + e^{-3x} \sin(x) c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = 50x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) e^{-3x} & \sin(x) e^{-3x} \\ -\sin(x) e^{-3x} - 3\cos(x) e^{-3x} & \cos(x) e^{-3x} - 3\sin(x) e^{-3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{-6x}$
- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -50\,\mathrm{e}^{-3x}\left(\cos(x)\left(\int \sin(x)\, x\, \mathrm{e}^{3x} dx\right) - \sin(x)\left(\int \cos(x)\, x\, \mathrm{e}^{3x} dx\right)\right)$
- Compute integrals

  $y_p(x) = 5x - 3$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^{-3x}\cos(x)\, c_1 + \mathrm{e}^{-3x}\sin(x)\, c_2 + 5x - 3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+10*y(x)=50*x,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-3x}\sin(x)\, c_2 + \mathrm{e}^{-3x}\cos(x)\, c_1 + 5x - 3$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 30

```
DSolve[y''[x]+6*y'[x]+10*y[x]==50*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 5x + c_2 e^{-3x}\cos(x) + c_1 e^{-3x}\sin(x) - 3$$

4.16 problem 17

Internal problem ID [5159]
Internal file name [`OUTPUT/4652_Sunday_June_05_2022_03_02_44_PM_73858023/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 2x' + 2x = 85 \sin(3t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = -20]$$

4.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}
p(t) &= 2 \\
q(t) &= 2 \\
F &= 85 \sin(3t)
\end{aligned}$$

Hence the ode is

$$x'' + 2x' + 2x = 85\sin(3t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 85\sin(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.16.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 2, f(t) = 85\sin(3t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 2x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 2$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 2\lambda\,\mathrm{e}^{\lambda t} + 2\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(2)} \\ &= -1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + i \\ \lambda_2 &= -1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t}(c_1 \cos(t) + c_2 \sin(t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$85 \sin(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t)\,\mathrm{e}^{-t}, \sin(t)\,\mathrm{e}^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 \cos(3t) - 7A_2 \sin(3t) - 6A_1 \sin(3t) + 6A_2 \cos(3t) = 85 \sin(3t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -6, A_2 = -7]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -6\cos(3t) - 7\sin(3t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\mathrm{e}^{-t}(c_1 \cos(t) + c_2 \sin(t))\right) + (-6\cos(3t) - 7\sin(3t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-t}(c_1 \cos(t) + c_2 \sin(t)) - 6\cos(3t) - 7\sin(3t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - 6 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -e^{-t}(c_1 \cos(t) + c_2 \sin(t)) + e^{-t}(-c_1 \sin(t) + c_2 \cos(t)) + 18 \sin(3t) - 21 \cos(3t)$$

substituting $x' = -20$ and $t = 0$ in the above gives

$$-20 = -c_1 + c_2 - 21 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_2 = 7$$

Substituting these values back in above solution results in

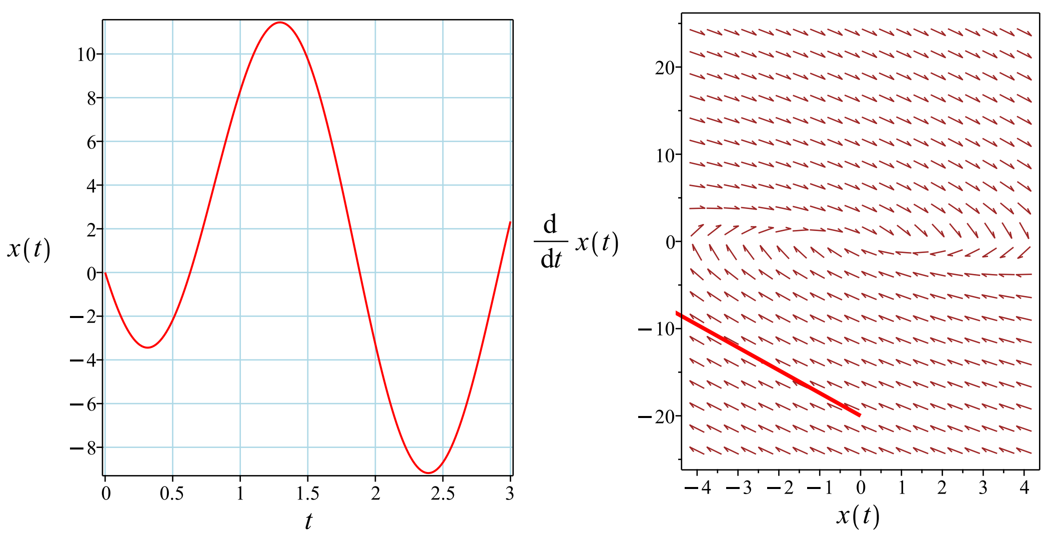
$$x = 6 \cos(t) e^{-t} + 7 \sin(t) e^{-t} - 7 \sin(3t) - 6 \cos(3t)$$

Which simplifies to

$$x = (6 \cos(t) + 7 \sin(t)) e^{-t} - 6 \cos(3t) - 7 \sin(3t)$$

Summary
The solution(s) found are the following

$$x = (6 \cos(t) + 7 \sin(t)) e^{-t} - 6 \cos(3t) - 7 \sin(3t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = (6 \cos(t) + 7 \sin(t)) e^{-t} - 6 \cos(3t) - 7 \sin(3t)$$

Verified OK.

4.16.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 2x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 139: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-t} \\
&= z_1 \left(\mathrm{e}^{-t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \cos\left(t\right) \mathrm{e}^{-t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\, dt}}{\left(x_1\right)^2}\, dt \\
&= x_1 \int \frac{e^{-2t}}{\left(x_1\right)^2}\, dt \\
&= x_1 (\tan\left(t\right))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\cos\left(t\right) \mathrm{e}^{-t}\right) + c_2 \left(\cos\left(t\right) \mathrm{e}^{-t} (\tan\left(t\right))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 2x' + 2x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \cos\left(t\right) \mathrm{e}^{-t} + \mathrm{e}^{-t} \sin\left(t\right) c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$85 \sin(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(t)\,\mathrm{e}^{-t}, \sin(t)\,\mathrm{e}^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-7A_1 \cos(3t) - 7A_2 \sin(3t) - 6A_1 \sin(3t) + 6A_2 \cos(3t) = 85 \sin(3t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -6, A_2 = -7]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -6\cos(3t) - 7\sin(3t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(t)\,\mathrm{e}^{-t} + \mathrm{e}^{-t} \sin(t)\, c_2\right) + (-6\cos(3t) - 7\sin(3t)) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-t}(c_1 \cos(t) + c_2 \sin(t)) - 6\cos(3t) - 7\sin(3t)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-t}(c_1 \cos(t) + c_2 \sin(t)) - 6\cos(3t) - 7\sin(3t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - 6 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -e^{-t}(c_1 \cos(t) + c_2 \sin(t)) + e^{-t}(-c_1 \sin(t) + c_2 \cos(t)) + 18\sin(3t) - 21\cos(3t)$$

substituting $x' = -20$ and $t = 0$ in the above gives

$$-20 = -c_1 + c_2 - 21 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 6 \\ c_2 &= 7 \end{aligned}$$

Substituting these values back in above solution results in

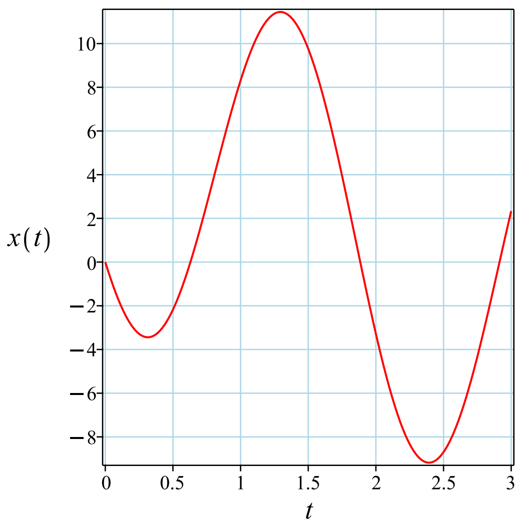
$$x = 6\cos(t)\, e^{-t} + 7\sin(t)\, e^{-t} - 7\sin(3t) - 6\cos(3t)$$

Which simplifies to

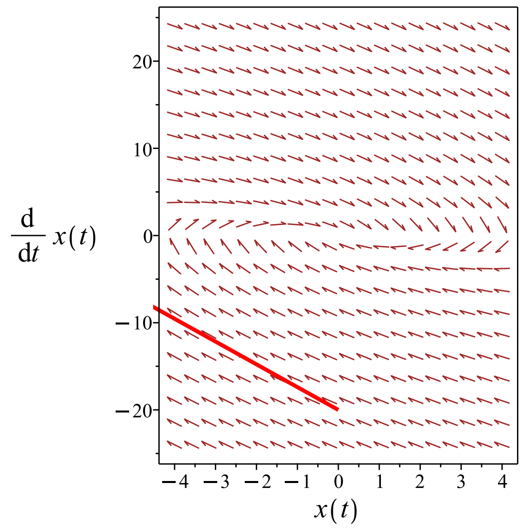
$$x = (6\cos(t) + 7\sin(t))\, e^{-t} - 6\cos(3t) - 7\sin(3t)$$

Summary
The solution(s) found are the following

$$x = (6\cos(t) + 7\sin(t))\, e^{-t} - 6\cos(3t) - 7\sin(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = (6\cos(t) + 7\sin(t))\,\mathrm{e}^{-t} - 6\cos(3t) - 7\sin(3t)$$

Verified OK.

4.16.4 Maple step by step solution

Let's solve

$$\left[x'' + 2x' + 2x = 85\sin(3t)\,, x(0) = 0, x'\Big|_{\{t=0\}} = -20\right]$$

- Highest derivative means the order of the ODE is 2
  x''
- Characteristic polynomial of homogeneous ODE
  $r^2 + 2r + 2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{(-2)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-1 - \mathrm{I}, -1 + \mathrm{I})$
- 1st solution of the homogeneous ODE

$x_1(t) = \cos(t)\,\mathrm{e}^{-t}$

- 2nd solution of the homogeneous ODE

$x_2(t) = \sin(t)\,\mathrm{e}^{-t}$

- General solution of the ODE

$x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

$x = c_1 \cos(t)\,\mathrm{e}^{-t} + \mathrm{e}^{-t} \sin(t)\, c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 85\sin(3t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$W(x_1(t), x_2(t)) = \left[\begin{array}{cc} \cos(t)\,\mathrm{e}^{-t} & \sin(t)\,\mathrm{e}^{-t} \\ -\sin(t)\,\mathrm{e}^{-t} - \cos(t)\,\mathrm{e}^{-t} & \cos(t)\,\mathrm{e}^{-t} - \sin(t)\,\mathrm{e}^{-t} \end{array}\right]$

- ○ Compute Wronskian

$W(x_1(t), x_2(t)) = \mathrm{e}^{-2t}$

- ○ Substitute functions into equation for $x_p(t)$

$x_p(t) = -85\,\mathrm{e}^{-t}\left(\cos(t)\left(\int \sin(t)\sin(3t)\,\mathrm{e}^{t} dt\right) - \sin(t)\left(\int \cos(t)\sin(3t)\,\mathrm{e}^{t} dt\right)\right)$

- ○ Compute integrals

$x_p(t) = -6\cos(3t) - 7\sin(3t)$

- Substitute particular solution into general solution to ODE

$x = c_1 \cos(t)\,\mathrm{e}^{-t} + \mathrm{e}^{-t}\sin(t)\, c_2 - 6\cos(3t) - 7\sin(3t)$

□ Check validity of solution $x = c_1 \cos(t)\,\mathrm{e}^{-t} + \mathrm{e}^{-t}\sin(t)\, c_2 - 6\cos(3t) - 7\sin(3t)$

- ○ Use initial condition $x(0) = 0$

$0 = c_1 - 6$

- ○ Compute derivative of the solution

$x' = -c_1 \sin(t)\,\mathrm{e}^{-t} - c_1 \cos(t)\,\mathrm{e}^{-t} - \mathrm{e}^{-t}\sin(t)\, c_2 + \mathrm{e}^{-t}\cos(t)\, c_2 + 18\sin(3t) - 21\cos(3t)$

- ○ Use the initial condition $x'\Big|_{\{t=0\}} = -20$

$-20 = -c_1 + c_2 - 21$

- Solve for c_1 and c_2

  $\{c_1 = 6, c_2 = 7\}$

- Substitute constant values into general solution and simplify

  $x = (6\cos(t) + 7\sin(t))\,\mathrm{e}^{-t} - 6\cos(3t) - 7\sin(3t)$

- Solution to the IVP

  $x = (6\cos(t) + 7\sin(t))\,\mathrm{e}^{-t} - 6\cos(3t) - 7\sin(3t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 33

```
dsolve([diff(x(t),t$2)+2*diff(x(t),t)+2*x(t)=85*sin(3*t),x(0) = 0, D(x)(0) = -20],x(t), sing
```

$$x(t) = (7\sin(t) + 6\cos(t))\,\mathrm{e}^{-t} - 6\cos(3t) - 7\sin(3t)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 36

```
DSolve[{x''[t]+2*x'[t]+2*x[t]==85*Sin[3*t],{x[0]==0,x'[0]==-20}},x[t],t,IncludeSingularSolut
```

$$x(t) \to 7e^{-t}\sin(t) - 7\sin(3t) + 6e^{-t}\cos(t) - 6\cos(3t)$$

4.17 problem 18

Internal problem ID [5160]
Internal file name [`OUTPUT/4653_Sunday_June_05_2022_03_02_45_PM_56079746/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 3\sin(x)}$$

With initial conditions

$$\left[y(0) = 0, y'\left(\frac{\pi}{2}\right) = 1\right]$$

4.17.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 3\sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos\left(2x\right) + c_2 \sin\left(2x\right))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3 \sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) = 3 \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + (\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\sin(2x) c_1 + 2c_2 \cos(2x) + \cos(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = -2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -\frac{1}{2} \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{\sin(2x)}{2} + \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(2x)}{2} + \sin(x) \tag{1}$$

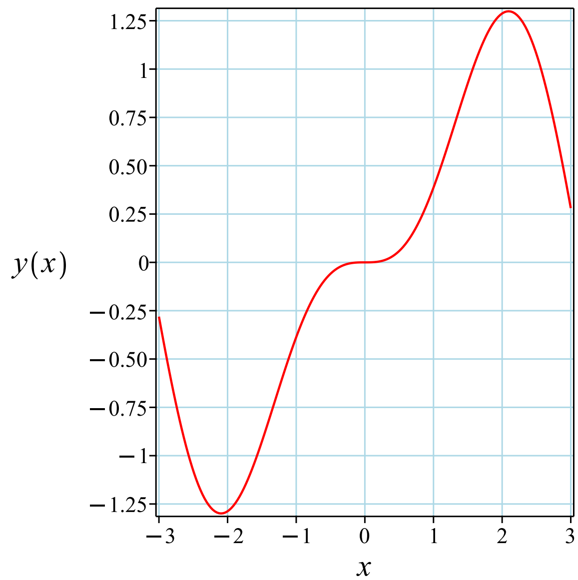


Figure 192: Solution plot

Verification of solutions

$$y = -\frac{\sin(2x)}{2} + \sin(x)$$

Verified OK.

4.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) e^{-\int \frac{B}{2A} dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 141: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\ &= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx\\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \cos(x) + A_2 \sin(x)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) = 3\sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + (\sin(x)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \sin(x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2\sin(2x)\, c_1 + c_2 \cos(2x) + \cos(x)$$

substituting $y' = 1$ and $x = \frac{\pi}{2}$ in the above gives

$$1 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 0 \\ c_2 &= -1 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -\frac{\sin(2x)}{2} + \sin(x)$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(2x)}{2} + \sin(x) \tag{1}$$

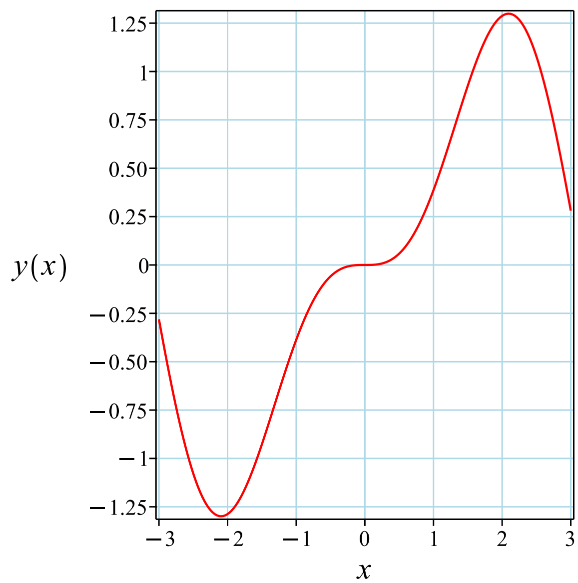


Figure 193: Solution plot

Verification of solutions

$$y = -\frac{\sin(2x)}{2} + \sin(x)$$

Verified OK.

4.17.3 Maple step by step solution

Let's solve

$$\left[y'' + 4y = 3\sin(x), y(0) = 0, y'\Big|_{\left\{x=\frac{\pi}{2}\right\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

y''

- Characteristic polynomial of homogeneous ODE

  $r^2+4=0$
- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-16})}{2}$
- Roots of the characteristic polynomial

  $r=(-2\,\mathrm{I}, 2\,\mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(2x)$
- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(2x)$
- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(2x)+c_2\sin(2x)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=3\sin(x)\right]$
- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x))=\begin{bmatrix}\cos(2x) & \sin(2x)\\ -2\sin(2x) & 2\cos(2x)\end{bmatrix}$
- Compute Wronskian

  $W(y_1(x), y_2(x))=2$
- Substitute functions into equation for $y_p(x)$

  $y_p(x)=-3\cos(2x)\left(\int\cos(x)\sin(x)^2\,dx\right)+\frac{3\sin(2x)\left(\int(\sin(3x)-\sin(x))dx\right)}{4}$
- Compute integrals

  $y_p(x)=\sin(x)$

- Substitute particular solution into general solution to ODE

  $y=c_1\cos(2x)+c_2\sin(2x)+\sin(x)$

- ☐ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x) + \sin(x)$
  - ∘ Use initial condition $y(0) = 0$

    $0 = c_1$
  - ∘ Compute derivative of the solution

    $y' = -2\sin(2x)\, c_1 + 2c_2 \cos(2x) + \cos(x)$
  - ∘ Use the initial condition $y'\Big|_{\{x=\frac{\pi}{2}\}} = 1$

    $1 = -2c_2$
  - ∘ Solve for c_1 and c_2

    $\left\{c_1 = 0, c_2 = -\frac{1}{2}\right\}$
  - ∘ Substitute constant values into general solution and simplify

    $y = -\frac{\sin(2x)}{2} + \sin(x)$
- Solution to the IVP

  $y = -\frac{\sin(2x)}{2} + \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x$2)=3*sin(x)-4*y(x),y(0) = 0, D(y)(1/2*Pi) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{\sin(2x)}{2} + \sin(x)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 13

```
DSolve[{y''[x]==3*Sin[x]-4*y[x],{y[0]==0,y'[Pi/2]==1}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \rightarrow -(\sin(x)(\cos(x) - 1))$$

4.18 problem 19

Internal problem ID [5161]
Internal file name [`OUTPUT/4654_Sunday_June_05_2022_03_02_46_PM_9555196/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{\frac{x''}{2} + 48x = 0}$$

With initial conditions

$$\left[x(0) = \frac{1}{6}, x'(0) = 0\right]$$

4.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 96 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 96x = 0$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 96$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.18.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = \frac{1}{2}, B = 0, C = 48$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\frac{\lambda^2 e^{\lambda t}}{2} + 48\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\frac{\lambda^2}{2} + 48 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = \frac{1}{2}, B = 0, C = 48$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\left(\frac{1}{2}\right)} \pm \frac{1}{(2)\left(\frac{1}{2}\right)}\sqrt{0^2 - (4)\left(\frac{1}{2}\right)(48)} \\ &= \pm 4i\sqrt{6}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +4i\sqrt{6} \\ \lambda_2 &= -4i\sqrt{6}\end{aligned}$$

Which simplifies to

$$\lambda_1 = 4i\sqrt{6}$$
$$\lambda_2 = -4i\sqrt{6}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4\sqrt{6}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0\left(c_1 \cos\left(4t\sqrt{6}\right) + c_2 \sin\left(4t\sqrt{6}\right)\right)$$

Or

$$x = c_1 \cos\left(4t\sqrt{6}\right) + c_2 \sin\left(4t\sqrt{6}\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos\left(4t\sqrt{6}\right) + c_2 \sin\left(4t\sqrt{6}\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{6}$ and $t = 0$ in the above gives

$$\frac{1}{6} = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1\sqrt{6}\sin\left(4t\sqrt{6}\right) + 4c_2\sqrt{6}\cos\left(4t\sqrt{6}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 4c_2\sqrt{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= \frac{1}{6} \\
c_2 &= 0
\end{aligned}
$$

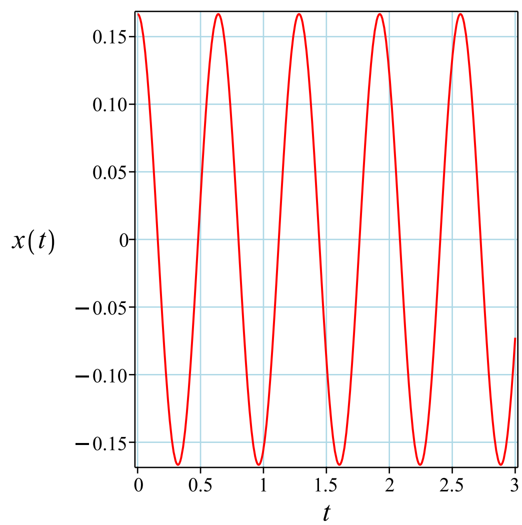
Substituting these values back in above solution results in

$$
x = \frac{\cos\left(4t\sqrt{6}\right)}{6}
$$

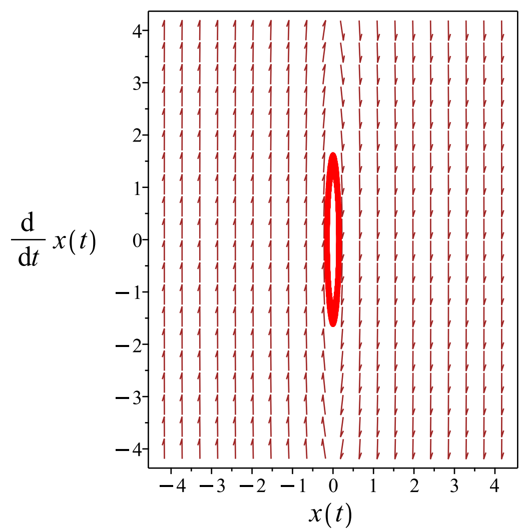
Summary

The solution(s) found are the following

$$
x = \frac{\cos\left(4t\sqrt{6}\right)}{6} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
x = \frac{\cos\left(4t\sqrt{6}\right)}{6}
$$

Verified OK.

4.18.3 Solving as second order ode can be made integrable ode

Multiplying the ode by x' gives

$$\frac{x'x''}{2} + 48x'x = 0$$

Integrating the above w.r.t t gives

$$\int \left(\frac{x'x''}{2} + 48x'x\right) dt = 0$$
$$\frac{x'^2}{4} + 24x^2 = c_2$$

Which is now solved for x. Solving the given ode for x' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$x' = 2\sqrt{-24x^2 + c_1} \tag{1}$$
$$x' = -2\sqrt{-24x^2 + c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{2\sqrt{-24x^2 + c_1}} dx = \int dt$$
$$\frac{\sqrt{6}\,\arctan\left(\frac{2\sqrt{6}\,x}{\sqrt{-24x^2+c_1}}\right)}{24} = t + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{2\sqrt{-24x^2 + c_1}} dx = \int dt$$
$$-\frac{\sqrt{6}\,\arctan\left(\frac{2\sqrt{6}\,x}{\sqrt{-24x^2+c_1}}\right)}{24} = t + c_3$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\sqrt{6}\,\arctan\left(\frac{2\sqrt{6}\,x}{\sqrt{-24x^2+c_1}}\right)}{24} = t + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{6}$ and $t = 0$ in the above gives

$$\frac{\arctan\left(\frac{\sqrt{2}}{\sqrt{-2+3c_1}}\right)\sqrt{6}}{24} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = 2\left(\tan\left(4(t+c_2)\sqrt{6}\right)^2 + 1\right)\sqrt{\frac{c_1}{\tan\left(4\left(t+c_2\right)\sqrt{6}\right)^2+1}} - \frac{2\tan\left(4(t+c_2)\sqrt{6}\right)^2 c_1}{\sqrt{\frac{c_1}{\tan\left(4(t+c_2)\sqrt{6}\right)^2+1}}\left(\tan\left(4\left(t+c_2\right)\sqrt{6}\right)^2 +\right.}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = \frac{2\cos\left(4c_2\sqrt{6}\right)^2 c_1}{\sqrt{c_1\cos\left(4c_2\sqrt{6}\right)^2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\sqrt{6}\,\arctan\left(\frac{2\sqrt{6}\,x}{\sqrt{-24x^2+c_1}}\right)}{24} = t + c_3 \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{6}$ and $t = 0$ in the above gives

$$-\frac{\arctan\left(\frac{\sqrt{2}}{\sqrt{-2+3c_1}}\right)\sqrt{6}}{24} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2\left(\tan\left(4(t+c_3)\sqrt{6}\right)^2 + 1\right)\sqrt{\frac{c_1}{\tan\left(4\left(t+c_3\right)\sqrt{6}\right)^2+1}} + \frac{2\tan\left(4(t+c_3)\sqrt{6}\right)^2 c_1}{\sqrt{\frac{c_1}{\tan\left(4(t+c_3)\sqrt{6}\right)^2+1}}\left(\tan\left(4\left(t+c_3\right)\sqrt{6}\right)^2\right.}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -\frac{2 \cos\left(4c_3\sqrt{6}\right)^2 c_1}{\sqrt{c_1 \cos\left(4c_3\sqrt{6}\right)^2}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

4.18.4 Solving using Kovacic algorithm

Writing the ode as

$$\frac{x''}{2} + 48x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{2} \\ B &= 0 \\ C &= 48 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-96}{1} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -96 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = -96z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1, 2\}$,$\{1, 3\}$,$\{2\}$,$\{3\}$,$\{3, 4\}$,$\{1, 2, 5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -96$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(4t\sqrt{6}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos\left(4t\sqrt{6}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \cos\left(4t\sqrt{6}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \cos\left(4t\sqrt{6}\right) \int \frac{1}{\cos\left(4t\sqrt{6}\right)^2}\,dt \\ &= \cos\left(4t\sqrt{6}\right)\left(\frac{\sqrt{6}\,\tan\left(4t\sqrt{6}\right)}{24}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1\left(\cos\left(4t\sqrt{6}\right)\right) + c_2\left(\cos\left(4t\sqrt{6}\right)\left(\frac{\sqrt{6}\,\tan\left(4t\sqrt{6}\right)}{24}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos\left(4t\sqrt{6}\right) + \frac{c_2\sqrt{6}\,\sin\left(4t\sqrt{6}\right)}{24} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{6}$ and $t = 0$ in the above gives

$$\frac{1}{6} = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1\sqrt{6}\,\sin\left(4t\sqrt{6}\right) + c_2 \cos\left(4t\sqrt{6}\right)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

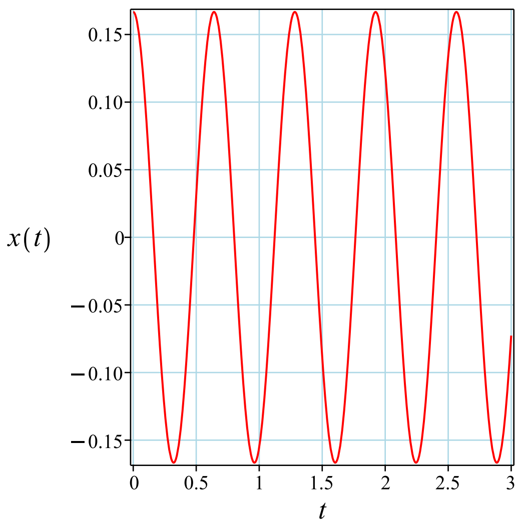
$$\begin{aligned} c_1 &= \frac{1}{6} \\ c_2 &= 0 \end{aligned}$$

Substituting these values back in above solution results in

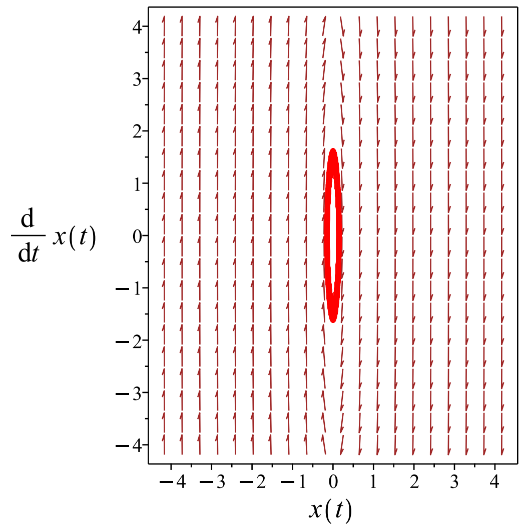
$$x = \frac{\cos\left(4t\sqrt{6}\right)}{6}$$

Summary
The solution(s) found are the following

$$x = \frac{\cos\left(4t\sqrt{6}\right)}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{\cos\left(4t\sqrt{6}\right)}{6}$$

Verified OK.

4.18.5 Maple step by step solution

Let's solve

$\left[\frac{x''}{2} + 48x = 0, x(0) = \frac{1}{6}, x'\Big|_{\{t=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -96x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + 96x = 0$

- Characteristic polynomial of ODE

  $r^2 + 96 = 0$

- Use quadratic formula to solve for r

$r = \frac{0 \pm (\sqrt{-384})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-4\,\mathrm{I}\sqrt{6}, 4\,\mathrm{I}\sqrt{6}\right)$

- 1st solution of the ODE

  $x_1(t) = \cos\left(4t\sqrt{6}\right)$

- 2nd solution of the ODE

  $x_2(t) = \sin\left(4t\sqrt{6}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 \cos\left(4t\sqrt{6}\right) + c_2 \sin\left(4t\sqrt{6}\right)$

□ Check validity of solution $x = c_1 \cos\left(4t\sqrt{6}\right) + c_2 \sin\left(4t\sqrt{6}\right)$

- Use initial condition $x(0) = \frac{1}{6}$

  $\frac{1}{6} = c_1$

- Compute derivative of the solution

  $x' = -4c_1\sqrt{6}\,\sin\left(4t\sqrt{6}\right) + 4c_2\sqrt{6}\,\cos\left(4t\sqrt{6}\right)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = 4c_2\sqrt{6}$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{1}{6}, c_2 = 0\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{\cos\left(4t\sqrt{6}\right)}{6}$

- Solution to the IVP

  $x = \frac{\cos\left(4t\sqrt{6}\right)}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 13

```
dsolve([1/2*diff(x(t),t$2)=-48*x(t),x(0) = 1/6, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{\cos\left(4\sqrt{6}\,t\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 18

```
DSolve[{1/2*x''[t]==-48*x[t],{x[0]==1/6,x'[0]==0}},x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{6}\cos\left(4\sqrt{6}t\right)$$

4.19 problem 20

Internal problem ID [5162]
Internal file name [`OUTPUT/4655_Sunday_June_05_2022_03_02_47_PM_83572764/index.tex`]

Book: Engineering Mathematics. By K. A. Stroud. 5th edition. Industrial press Inc. NY. 2001
Section: Program 25. Second order differential equations. Further problems 25. page 1094
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 5x' + 6x = \cos(t)}$$

With initial conditions

$$\left[x(0) = \frac{1}{10}, x'(0) = 0\right]$$

4.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= 6 \\ F &= \cos(t) \end{aligned}$$

Hence the ode is

$$x'' + 5x' + 6x = \cos(t)$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = \cos(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.19.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 5, C = 6, f(t) = \cos(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 5x' + 6x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 5, C = 6$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 5\lambda \,\mathrm{e}^{\lambda t} + 6 \,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 5\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(6)}\\ &= -\frac{5}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2\\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{(-2)t} + c_2 e^{(-3)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-3t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(t) + 5A_2 \sin(t) - 5A_1 \sin(t) + 5A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-2t} + c_2 e^{-3t}\right) + \left(\frac{\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-2t} + c_2 e^{-3t} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{10}$ and $t = 0$ in the above gives

$$\frac{1}{10} = c_1 + c_2 + \frac{1}{10} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \frac{\sin(t)}{10} + \frac{\cos(t)}{10}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 3c_2 + \frac{1}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{1}{10}$$
$$c_2 = \frac{1}{10}$$

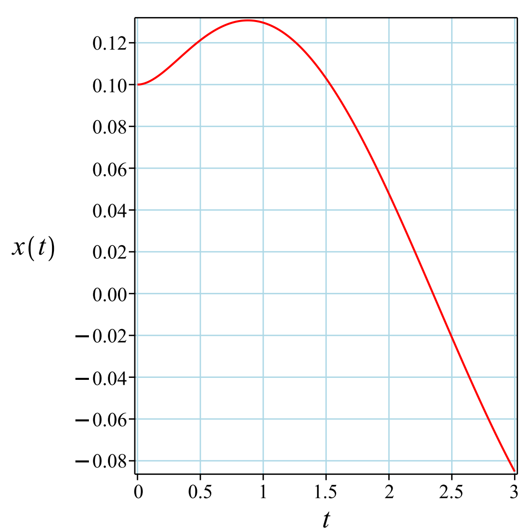
Substituting these values back in above solution results in

$$x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

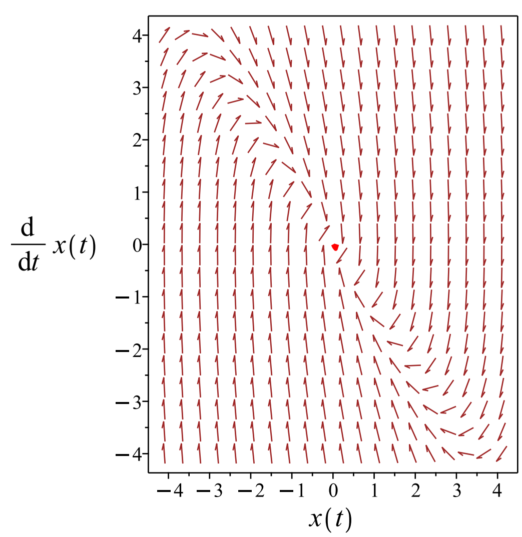
Summary

The solution(s) found are the following

$$x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

4.19.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
x'' + 5x' + 6x &= 0 \tag{1} \\
Ax'' + Bx' + Cx &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1 \\
B &= 5 \tag{3} \\
C &= 6
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = xe^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{1}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 1 \\
t &= 4
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \frac{z(t)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$
x = z(t)\, e^{-\int \frac{B}{2A}\, dt}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 145: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-\frac{t}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-\frac{5t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{5t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-3t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{5}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-5t}}{(x_1)^2}\, dt \\
&= x_1 \left(\mathrm{e}^{t}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-3t}\right) + c_2 \left(\mathrm{e}^{-3t} \left(\mathrm{e}^{t}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 5x' + 6x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-2t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}, e^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(t) + A_2 \sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(t) + 5A_2 \sin(t) - 5A_1 \sin(t) + 5A_2 \cos(t) = \cos(t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{10}, A_2 = \frac{1}{10}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-3t} + c_2 e^{-2t}\right) + \left(\frac{\cos(t)}{10} + \frac{\sin(t)}{10}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-2t} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = \frac{1}{10}$ and $t = 0$ in the above gives

$$\frac{1}{10} = c_1 + c_2 + \frac{1}{10} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \mathrm{e}^{-3t} - 2c_2 \mathrm{e}^{-2t} - \frac{\sin(t)}{10} + \frac{\cos(t)}{10}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - 2c_2 + \frac{1}{10} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

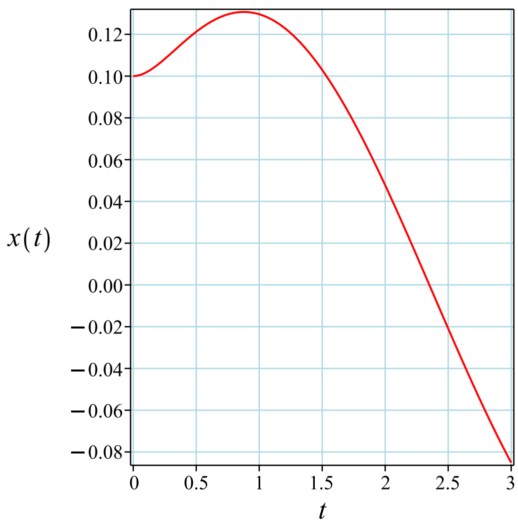
$$c_1 = \frac{1}{10}$$
$$c_2 = -\frac{1}{10}$$

Substituting these values back in above solution results in

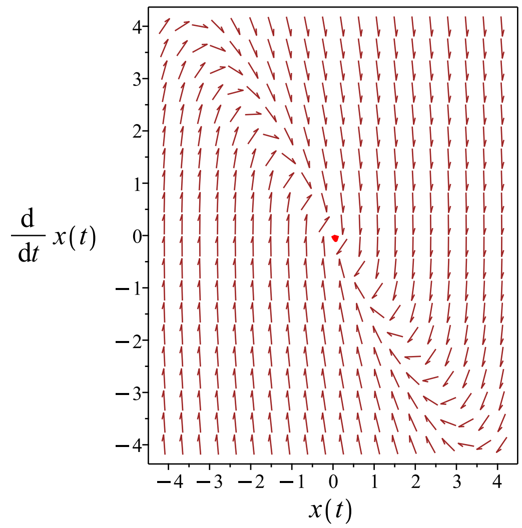
$$x = -\frac{\mathrm{e}^{-2t}}{10} + \frac{\mathrm{e}^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

Summary
The solution(s) found are the following

$$x = -\frac{\mathrm{e}^{-2t}}{10} + \frac{\mathrm{e}^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

Verified OK.

4.19.4 Maple step by step solution

Let's solve

$$\left[x'' + 5x' + 6x = \cos(t), x(0) = \tfrac{1}{10}, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 5r + 6 = 0$

- Factor the characteristic polynomial

  $(r+3)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, -2)$

- 1st solution of the homogeneous ODE

$x_1(t) = e^{-3t}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-2t}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{-3t} + c_2 e^{-2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = \cos(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-3t} & e^{-2t} \\ -3\,e^{-3t} & -2\,e^{-2t} \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = e^{-5t}$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -e^{-3t}\left(\int \cos(t)\, e^{3t} dt\right) + e^{-2t}\left(\int e^{2t}\cos(t)\, dt\right)$

  - Compute integrals

    $x_p(t) = \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-3t} + c_2 e^{-2t} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$

☐ Check validity of solution $x = c_1 e^{-3t} + c_2 e^{-2t} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$

  - Use initial condition $x(0) = \frac{1}{10}$

    $\frac{1}{10} = c_1 + c_2 + \frac{1}{10}$

  - Compute derivative of the solution

    $x' = -3c_1 e^{-3t} - 2c_2 e^{-2t} - \frac{\sin(t)}{10} + \frac{\cos(t)}{10}$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -3c_1 - 2c_2 + \frac{1}{10}$

- Solve for c_1 and c_2

  $\{c_1 = \frac{1}{10}, c_2 = -\frac{1}{10}\}$
- Substitute constant values into general solution and simplify

  $x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$

- Solution to the IVP

  $x = -\frac{e^{-2t}}{10} + \frac{e^{-3t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 25

```
dsolve([diff(x(t),t$2)+5*diff(x(t),t)+6*x(t)=cos(t),x(0) = 1/10, D(x)(0) = 0],x(t), singsol=
```

$$x(t) = \frac{e^{-3t}}{10} - \frac{e^{-2t}}{10} + \frac{\cos(t)}{10} + \frac{\sin(t)}{10}$$

✓ Solution by Mathematica

Time used: 0.059 (sec). Leaf size: 26

```
DSolve[{x''[t]+5*x'[t]+6*x[t]==Cos[t],{x[0]==1/10,x'[0]==0}},x[t],t,IncludeSingularSolutions
```

$$x(t) \to \frac{1}{10}\left(e^{-3t} - e^{-2t} + \sin(t) + \cos(t)\right)$$